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LIE-ADMISSIBLE APPROACH TO THE HADRONIC STRUCTURE

Volume I

NONAPPLICABILITY OF THE GALILEI AND EINSTEIN RELATIVITIES?

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Questo volume è dedicato

alla memoria di mio nonno

RUGGIERO SANTILLI

di cui porto il nome con

devozione ed orgoglio.

" Let us not forget, the problem of interactions is still a mystery"

E. P. WIGNER

Symmetry and Reflections

Indiana University Press, Bloomington Indiana

(1967, page 18)

Ruggero Maria Santilli

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Volume I.

NONAPPLICABILITY OF THE GALILEI AND EINSTEIN RELATIVITIES ?

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P R E F A C E

In these monographs I present the results of my independent and speculative studies on the problem of the structure of the strongly interacting particles (hadrons). Clearly, a problem of this nature goes beyond my capabilities as an isolated researcher . Therefore, my studies must be considered as conjectural, tentative and yet inconclusive not only on physical grounds, as necessary for all current physical studies on the hadronic structure, but also on mathematical grounds, owing to the identification and use of apparently new mathematical tools for the treatment of the strong interactions which , strictly speaking, constitute at this moment open mathematical problems. In essence, I will have achieved my objectives in their entirety if I succeed in

(A) confirming that the problem of the structure of hadron 1-18 is indeed fundamentally open at this time 19 and, as such, of a potentially unpredictable,final solution;

(B) indicating that the strong interactions 20-22 in general, and the strong hadronic forces in particular, may likely result to possess a structure fundamentally more complex than that customarily used in the available studies (local and derivable from a potential); and

(C) expressing my personal belief that Theoretical Physics is a Science which will never admit terminal disciplines. 23

To state it explicitly, these monographs are an expression of my personal belief that the currently available disciplines, such as classical mechanics, quantum mechanics and quantum field theory with underlying relativities (the Galilei, Einstein special and Einstein general relativities) DO NOT constitute, in their currently known form, terminal disciplines for the description of the physical universe.

In this preface I would like to take the liberty of outlining the most significant steps of my journey.

Since the time of my graduate studies at the University of Torino, Italy, (1963-1966) I have been interested in studying the old idea that the strong interactions are representable with local couplings non-derivable from a potential, with particular reference to the problem of the hadronic structure. In essence, the moment I was taught the profound physical differences which exist between the electromagnetic and strong interactions, I had difficulties in accepting for the latter interactions the same basic concepts, laws and principles of the former interactions. The reason was due to the fashinating physical effectiveness of established disciplines for the electromagnetic interactions versus the lack of any comparable physical effectiveness of the same disciplines when applied to the strong interactions. If the strong interactions are represented with couplings analytically equivalent to the electromagnetic interactions, i.e., local and derivable from a potential, I saw no way of escaping from the inflexible laws of established relativity and quantum mechanical disciplines. The representation of strong interactions in terms of couplings still local, but analytically nonequivalent to the electromagnetic interactions, i.e., not derivable from a potential, appeared to me as sufficiently interesting to deserve an inspection prior to the confrontation of more complex models, e.g., in terms of nonlocal couplings not derivable from a potential.

I therefore decided to undertake a long term, detailed and laborious research program aiming at an assessement of the physical relevance of the indicated representation of the strong interactions. These volumes are a report of my studies which I continued independently of the evolution of the theory of strong

interactions 1-22 to which I was in the meantime exposed.

It was not without surprise that a first library search conducted in 1963-64 revealed that the methodology for the treatment of forces non-derivable from a potential had remained virtually ignored in the physical literature already at a classical level, let alone the quantum mechanical level. This accounts for the rather considerable period of time which has passed from the decision to undertake this study and its presentation in these monographs. Indeed, prior to being in a position to study strong interactions as not derivable from a potential, I had to identify at least the rudiments of the methods needed for the treatment of these forces.

My studies were conducted along the following five sequential stages.

Stage 1: Studies on Lagrange's and Hamilton's equations with external terms (Lie-admissible problem). The most natural way of representing systems with (local) arbitrary couplings is that originally conceived by Lagrange and Hamilton, through the use of analytic equations with external terms representing precisely the forces not derivable from a potential. In essence, Lagrange and Hamilton were fully aware that the Newtonian forces are generally nonderivable from a potential. The presence of external terms in their analytic equations was thus essential to avoid an excessive departure from the Newtonian physical reality. Oddly, it has been only since the beginning of this century that Lagrange's and Hamilton's

equations have been "truncated" with the removal of the external terms by acquiring the form currently used in the physical literature. This is not an occurrence of marginal relevance. Instead, it indicates that the virtual totality of our current theoretical knowledge (based on analytic methods) at all the presently known levels, such as classical, quantum mechanical and quantum field theoretical, is restricted to systems with forces derivable from a potential, i.e., those admitting a representation in terms of only the Hamiltonian function (or operator or operator valued density) with the conventional structure $H = T+V$.

Clearly, it was for me mandatory to identify the classical analytic techniques for the studies of forces not derivable from a potential prior to the study of strong interactions realized along these lines. But the Newtonian limit of possible quantum mechanical or quantum field theoretical couplings not derivable from a potential was expected to yield the Newtonian systems of our everyday experience, i.e., non-conservative, such as particles in a physical medium with drag forces, damped and forced oscillators, spinning tops with drag torques, etc. The first stage of my studies was therefore the identification of the analytic methods for the treatment of local nonconservative Newtonian systems.

The same library search conducted in 1963-64 also revealed the existence of a rather sizable methodological gap existing between the analytic equations with and without external terms.

In essence, the study of analytic equations without external terms had developed into a beautiful and articulated body of deeply interrelated methods, such as analytic, algebraic, geometrical, statistical, etc. However, besides their mere formulation, no corresponding method was available for the analytic equations with external terms. Questions for me of vital relevance, such as the Hamilton-Jacobi theory, the algebraic structure and the geometry underlying the analytic equations with external terms had simply not been considered in the available mathematical and physical literature despite my laborious search. To the best of my knowledge, and besides my personal efforts, the situation is still the same as of today.

The objective of this first stage of my studies can now be better formulated. It consisted of the identification of the expected generalizations of the known analytic, algebraic and geometrical methods for Lagrange's and Hamilton's equations without external terms to the case with external terms.

Although not on a full time basis owing to my involvement in other research projects, I have continued to study these problems since the time of their formulation, at the various Institutions I had the pleasure to visit during my academic life. Besides the University of Torino, where I remained until May 1967,[24-25] these studies were continued at the International Center for Theoretical Physics of Trieste, Italy, during the summers of 1966 and 1967,[24] the Center for Theoretical Studies of

the University of Miami, Coral Gables, Florida, during the academic year 1967-1968,[26-28] the Department of Physics of Boston University, Boston, Massachusetts from 1968 until 1976,[29] the Center for Theoretical Physics of the Massachusetts Institute fo Technology, Cambridge, Massachusetts from January 1976 until July 1977,[30] the Instituut voor Theoretische Mechanica of the Rijskuniversiteit of Gent in Belgium and the Institut für Theoretische Physik of the University of Zürich in Switzerland during the summer 1977 and, since September 1977, the Lyman Laboratory of Physics of Harvard University, Cambridge, Massachusetts.[31-34]

A summary of my published, submitted and unsubmitted notes is presented in Sec.3 of ref.[32] and in Vol. II of this series. The central methodological aspect is that the presence of external terms in the analytic equations implies a generalization of the time evolution law whose brackets violate the Lie algebra axioms. A study however indicates that these generalized brackets satisfy the axioms of a broader class of nonassociative algebras called Lie-admissible. The most intriguing profile is due to the fact that these algebras characterize a genuine covering of the Lie algebras in the sense that: (a) the Lie-admissible algebras admit an analytic origin in precisely the same measure as that of the conventional Lie algebras, (b) the Lie-admissible brackets reduce to the Poisson brackets under a clear limiting procedure of clear physical significance, that of null value of the nonconservative forces, and (c) the Lie-admissible algebras contain the Lie algebras in an embedded

algebraic form (technically realized through the transition of universal enveloping algebras from an associative to a non-associative form).

Rather oddly, the available literature in the rather vast discipline known under the name of Abstract Algebras turned out to be of little relevance for my needs. This is due to the fact that the algebras characterized by the brackets of the generalized time evolution laws induced by analytic equations with external terms were not treated in any of the mathematical literature of the period 1950-1965, to the best of my knowledge. However, it was a known fact that the number of non-associative algebras either studied or at least identified by mathematicians is rather large indeed. As a result, the algebras characterized by my generalized analytic framework "had" to be treated in the literature. This forced me into a laborious library search which I conducted from 1965 until 1967 and which, as we shall see, subsequently resulted to be only the first of several searches I had to undertake during my solitary journey. In any case, my determination to pay tribute to previous contributions was rewarded. And indeed I finally succeeded in identifying a paper by A. A. Albert of 1948 in which he introduced the algebras I was interested in, although without any detailed treatment. As a today, to the best of my knowledge, the Lie-admissible algebras are still virtually ignored by textbooks as well as specialized literature in Abstract Algebras as the interested reader is encouraged to verify , with the only exception of the mathematical studies by H.C.Myung.[35]

Owing to the profound role of Lie algebras at virtually all levels of contemporary theoretical physics, the identification of their covering along the indicated analytic lines is, without any doubt, the most intriguing research topic in which I have been involved. For my long term objective of representing strong interactions as not derivable from a potential, this finding essentially opened the possibility of constructing the methodology for strong interactions as a covering of that of the electromagnetic interactions with self-evident predictable implications at the conceptual, theoretical and experimental levels. In the final analysis, my efforts have been devoted to the study whether this possibility is actually feasible.

Stage 2: Studies on Lagrange's and Hamilton's equations without external terms (Inverse Problem). As indicated earlier, my first objective was the identification of the generalized analytic, algebraic and geometrical formulations which were needed for analytic equations with external terms. A mandatory aspect of this program was, of course, the study of the covering of the conventional canonical transformation theory for the analytic equations considered and, more generally, the study of the problem whether Lie's transformation theory admitted a consistent covering of Lie-admissible type.

It was during the study of this problem in 1973 that I stumbled into a major technical difficulty. In essence, the construction of the expected broader transformation theory as a

direct generalization of the conventional formulations (preservation of the value of the fundamental brackets) yielded results which were generally inconsistent with an actual physical situation, such as the transformation theory of the spinning top with drag torques.

This situation forced me into a second stage of studies. My working program was that, if I could represent nonconservative systems also with the conventional analytic equations without external terms, I could construct the intended generalized transformation theory as the image of the conventional one in the transition from the case without to that with external terms. Jointly, this dual approach would have provided a solid basis of consistency of the analysis.

I was therefore forced to study the problem of the representation of arbitrary second order ordinary and partial differential equations in terms of the Lagrange and Hamilton equations without external terms. At a closer inspection, the problem actually consisted of the identification of:

(1) the necessary and sufficient conditions (integrability conditions) for the existence of a Lagrangian or a Hamiltonian representation for given arbitrary second order systems of ordinary and partial differential equations,

(2) the methods for the construction of a Lagrangian or a Hamiltonian, when their existence is guaranteed by said integrability conditions, from the given equations of motion or field equations and, last but not least,

(3) the predictable question: "fine, so what?", namely, what is the significance of the underlying methodology for the problem of interactions.

I soon came to realize that these problems were significant for several branches of physics and engineering, irrespective of my interest for strong interactions as not derivable from a potential. For instance, in Optimal Control Theory, these methods could have filled a sort of missing link. The methodology for the computation of an action functional for arbitrary systems of the indicated type could have clearly broadened the arena of applicability of the discipline considered. Similarly, in Newtonian Mechanics or Space Mechanics, the knowledge of a Lagrangian or a Hamiltonian for nonconservative systems would have allowed the use of rigorous analytic techniques (such as the computation of first integrals via Noether's theorem, the use of canonical perturbation theory, etc.) rather than the semi-empirical techniques currently used nowaday. Also, in High Energy Physics the knowledge of the integrability conditions for the existence of an analytic representation appeared as promising on several counts, such as the study of the structure of unified gauge theories of weak and electromagnetic interactions, the study of soliton or instanton - type solutions of nonlinear equations, etc..

Owing to these possibilities I decided to undertake a detailed study of this second stage of my studies. The field theoretical profile of my studies is presented in papers[30] and their Newtonian counterpart is presented in monographs.[34]

The most time consuming aspect of this second stage of study was the identification of the prior state of the art, conducted during the period from 1973 until 1976. An initial laborious search in the science libraries of the Boston area soon revealed that currently available textbooks in Classical Mechanics, Calculus of Variations, Differential Equations, and other disciplines did not present even a formulation of problems (1) and (2) above, let alone their treatment. A subsequent search in specialized mathematical and physical journals of the last decade proved to be equally fruitless. However, the problem I was interested in was at the very heart of Lagrange's and Hamilton's equations. A problem of this nature "had" also to be treated in the literature. It was not without a further laborious search I came to realize that the problem had indeed been treated in the mathematical literature of the Calculus of Variations (under the name of Inverse Problem) from the end of the past century up to the earlier 40-ties and subsequently ignored to a large extent. A report of all relevant references I have been able to identify is contained in my monographs on this subject.[34] A summary of only certain aspects of the methodology of the Inverse Problem is presented in Chapter 2 of this volume.

The efforts for this second stage of my studies on strong interactions as not derivable from a potential turned out to be fully rewarding on several counts. First of all, the methodology of the Inverse Problem allowed me to solve the problem that originated the study itself: the construction of a consistent generalization of the canonical transformation theory to the case of analytic equations with external terms, as reported in Volume II of this series. The same methodology, however, turned out to be significant for other conceptual and technical aspects.

In essence, for the study of a system with forces derivable from a potential, the analytic equations without external terms are fully sufficient. The situation for the case of systems with forces not derivable from a potential is different. Experience in this latter case suggests the use, whenever possible, of the totality of methodological tools available (i.e., the use of both analytic equations with and without external terms for the representation of the same systems), owing to the complexity of the problems to be confronted. And indeed, this dual methodological profile allows the verification of each result obtained within the context of one type of equations in that of the other type. Also, problems which would appear of difficult initial solution within the context of one type of equations could be more manageable within the context of the other and viceversa.

There is, however, a conceptual aspect of particular significance which recommended the use of this dual methodological

profile, whenever possible. In the final analysis, this dual analytic context provides the mathematical algorithms for the quantitative representation of a dual concept of interactions: for a given system with arbitrary couplings one can study either the system as a whole (in which case the use of conventional analytic equations without external terms is requested-the Inverse Problem) or study each individual element of the system and consider the rest as a external (in which case the use of analytic equations with external terms is requested - the Lie-admissible problem). These two approaches to interactions are clearly complementary to each other and so are the related methodologies. A deeper understanding of the problem of interactions can clearly be achieved by studying both aspects, rather than either single one of them. For instance, if a total physical quantity is conserved, a deeper insight can be gained by knowing whether each individual element of such quantity is conserved or not. Since our ultimate objective is the study of the problem of the structure of the hadrons, this dual perspective is clearly preferable over the conventional single perspective in terms of analytic equations without external terms. To be more specific in this crucial point, consider a hadron as a collection of unidentified constituents. Unequivocal experimental evidence indicates that a number of physical characteristics of this system (intended as isolated) are conserved, such as the rest energy, the charge, the total angular momentum, etc.. As we shall see in due time, one of the central objectives

for an in depth study of this system is to see whether the corresponding kinematical and intrinsic characteristic of the constituents are conserved or not. The joint use of the Inverse-Problem (for the behaviour of the system as a whole) and of the Lie-admissible Problem (for the study of each individual constituent) is clearly preferable over any single methodological choice.

Stage 3: Studies on systems with arbitrary couplings and subsidiary constraints (generalization of the Lie-Admissible Problem and the Inverse Problem to systems with generally nonintegrable subsidiary constraints). As indicated earlier, my approach to the problem of the hadronic structure is based on the assumption that the forces between the constituents are not derivable from a potential, as a local approximation of expected more general nonlocal forces. The mechanics obeyed by such constituents is expected to be of Lie-admissible type under the assumption that this algebraic character is preserved by the quantization of these broader systems in a way similar to the preservation of the Lie albegra structure in the quantization of conventional conservative systems. This is the "hadronic mechanics" of Volumes II and III of this series.

In essence, this was the idea at the time of the identification of the Lie-admissible character of analytic equations with external terms (1965— 66). One would have expected a rapid confrontation of the problem of quantization, once the methodology

of the Lie-admissible problem had been sufficiently well identified.

Such an approach would have been premature for a central problem of consistency. It is an unequivocal experimental evidence, such as daily proved in particle accelerators, that hadrons as a whole under the action of electromagnetic interactions strictly obey the laws of established relativity and quantum mechanical disciplines (which, as it is known, are purely Lie in algebraic structure). In my opinion, no structure model of the hadrons can be consistently attempted with the use of generalized hadronic forces and corresponding generalized methods, unless conventional relativity and quantum mechanical laws for the behaviour of the state as a whole under electromagnetic interactions are recovered in full.

The only way known to me for attempting this compatibility between generalized formulations for the hadronic constituents and conventional formulations for the hadron as a whole is through the imposition of generally nonintegrable subsidiary constraints. In intuitive but effective terms, the idea is that of imposing established relativity and quantum mechanical laws as global subsidiary constraints to generalized structure models, to achieve consistency with the currently available experimental evidence.

This situation called for a third stage of studies, that of the extension of the methodologies of the Lie-Admissible Problem and the Inverse Problem to systems with generally nonintegrable

subsidiary constraints, as a purely classical analysis prior to the confrontation of the problem of quantization.

My first concern of utmost priority was the study of the consistency of these broader systems, i.e., the existence of a physically meaningful classical solution because, in the absence of such consistency, any mathematical manipulation of the equations at hand would have been physically vacuous. Oddly, despite the rather frequent use of constrained systems in physics, such as those of Dirac type[36], this problem of consistency did not appear to be treated in the recent literature with only very few exceptions known to me such as an excellent paper by S. Shanmugadhasan[37]. This called for another laborious search of the state of the art which I conducted from 1975 until recently. My findings are briefly indicated in Appendix 1.C of this volume. The net result, as we shall see, is that consistent generalized structure models which are compatible with established relativity laws (and quantum mechanical laws - see Volume II in this respect) are indeed possible.

My second concern was then the extension of the methodologies of the Lie-Admissible Problem and the Inverse Problem to the case of generally nonintegrable subsidiary constraints. My third and final concern was an understanding of these furtherly generalized formulations for the problem of interactions, with particular reference to a comparative analysis between the electromagnetic and the strong.

In 1976 I had reached sufficient material on these problems

to deliver a seminar course. My (untyped) lecture notes essentially contain:

(a) The study of systems with generally nonintegrable subsidiary constraints with the most powerful analytic techniques known to me, the integrability conditions for the existence of a Lagrangian or, independently, of a Hamiltonian but this time reformulated on the hypersurface of the constraints.

(b) The embedding of the constrained systems of Dirac type into the more general class of systems of Bolza type, as available from the Calculus of variations, and the study of their relationship.

(c) The identification of a yet broader class of systems which is are directly treatable with Lie-admissible techniques but not with Lie techniques.

The following picture essentially emerged from this third stage of studies. Let me recall that the rigorous treatment of electromagnetic interactions demands the use of constraints from relativity considerations. The systems, however, result to be of Dirac type. If the strong interactions are realized in terms of local couplings analytically equivalent to the electromagnetic interactions (i.e., derivable from a potential), there is no need of broader methodological formulations because they are treatable with constrained systems also of Dirac type. However, if the strong interactions are realized in terms of couplings not derivable from a potential, there is a definite

need of broader formulations. These latter interactions, in particular, do not appear to have a single character, as it is the case of the electromagnetic interactions. Instead, there is the emergence of a number of realizations of local strong couplings as not derivable from a potential of increasing complexity, dynamical implications and methodological needs. In their simplest forms they are treatable with the methodology of the Bolza problem of the Calculus of Variations (when the integrability conditions for the existence of an action functional are satisfied). In their general form they are directly treatable only with the Lie-admissible techniques. As a result, the Lie-admissible methods emerge as possessing a direct universality in their applicability to the considered realizations of the strong interactions which is not shared by Lie-type disciplines.

Stage 4: Identification of the relativity which is applicable to strong interactions when local and non-derivable from a potential. The studies of the preceding stages were fundamentally insufficient for my objectives, still at the pure classical level. The reason was due to the need of resolving, at least in a rudimentary form, the true, first,physical problem of my program: the relativity which is applicable to local forces nonderivable from a potential. The preceding stages resulted in essence to be a "methodological homework" for the study of this physical problem.

Newtonian Mechanics proved again to be of invaluable physical guidance. It is know that there exist physical systems which strictly obey the Galilei relativity. This is the case, for instance, of the planetary system under a Newtonian approximation. Indeed, the system

is conservative and, in particular, verifies the ten conservation laws of the Galilei relativity (conservation of the total energy, linear momentum and angular momentum and uniform motion of the center of mass). Also, the equations of motion are form-invariant under the Galilei transformations.

Neverthless, this physical context is the exception and not the rule for the physical systems of our macroscopic environment. Indeed, Newtonian systems, in general, VIOLATE the indicated ten conservation laws of the Galilei relativity and their equations of motion are, in general, FORM-NONINVARIANT under the Galilei transformations. This is the case of the systems of our everyday experience, such as damped oscillators, systems with mechanical holonomic constraints (e.g., the pendulum), spinning tops with drag torques, etc.

Owing to its physical relevance, this point should be stressed more vividly. It is appropriate here to say that, if the Galilei relativity is assumed to be strictly obeyed by physical systems such as the Foucault pendulum, such a pendulum should still be swinging since the time it was put in motion by Foucault in 1851 ! Equivalently, the assumption that spinning tops in our environment verify an exact symmetry under rotation is literally equivalent to the validity of the perpetual motion, trivially, from the conservation of the angular momentum.

This is not a setting of mere semantics. Instead, it indicates that the problem of the relativity which is applicable to the Newtonian systems of our everyday experience is fundamentally open at this moment. After all, a primary function of any relativity is that of reducing physical laws to a form-invariant description of nature, while the Galilei relativity fails to fulfil this function on more than one count. Specifically, the Galilei relativity is unable to provide a

form-invariant description, in general, of nonconservative systems and it is unable to characterize the new physical context, the NON-conservation, in general, of the total energy, linear momentum and angular momentum, as well as the LACK, in general, of the uniform motion of the center of mass, as typical, say, for a Kepler system in a dissipative medium.

In the transition to a relativistic context the situation becomes considerably more nebulous, owing to the lack of well established knowledge on the relativistic extension of the Newtonian nonconservative forces. As a matter of fact, this occurrence is so prominent, to the point that the current discipline known under the name of classical relativistic mechanics (of point particles) may be considered as a covering of only <u>part</u> of Newtonian Mechanics, that with forces verifying the integrability conditions for the existence of a potential. In any case, we should expect the inapplicability of the special theory of relativity to the class of forces under consideration from simple compatibility arguments with Galilei's relativity. It is understood that relativistic extensions of nonconservative forces are not needed for the arena of direct (and unequivocal) applicability of the special relativity: electromagnetic interactions. But it is equally true that the physical context we are referring to (say, motion of massive particles within hadronic matter, such as the core of a neutron star) is fundamentally different than that of the electromagnetic interactions.

But the classical limit of the assumed type of strong forces is precisely of nonconservative type. We reach in this way a most critical point of the analysis of these monographs: the problem whether Galilei's relativity and Einstein's special relativity are applicable or inapplicable, in their currently known form, to the strong interactions.

We assume that the reader is familiar with the fact that, at this moment, each of these two possibilities (validity and invalidity of established relativities) is a mere CONJECTURE for the strong interactions deprived of final and incontrovertible experimental backing.

The line of study based on the validity of established relativities was and still is that of conventional approaches.[1-22] But these studies had been unable to produce an uncontroversial solution of the problem of the hadronic structure, despite the participation of a rather large portion of the community of basic studies, and for a rather extended period of time. As a result, in my solitary studies I was solely interested in a possible invalidity of established relativities for the strong interactions. The idea is that, perhaps, the strong interactions are considerably more complex of what customarily assumed, to the point of demanding the courageous construction of coverings of the established relativities. After all, the current problematic aspects of the quark models of hadron structure (lack of identification of quarks with physically known particles, lack of confinement, etc.) may, at a deeper analysis, result to be only the SYMPTOMS of a much more fundamental problem of consistency at the level of the elemental physical laws: the relativity laws.

Clearly, the best way to study the possible inapplicability of established relativities to strong interactions is to attempt the construction of at least the rudiments of a possible generalization of the Galilei relativity, and then confronting the prediction of the theory with the experimental data. This is a problem to which I have been interested since the time I was taught the role of form-invariance for any relativity. In nontechnical terms, the problem consisted of attempting the construction of a generalization of the Galilei group

which is such to: (1) characterize the violations of the ten Galilei's conservation laws; (2) provide the form-invariance for each arbitrary (but local and of class C^∞) Newtonian force; and (3) recover the conventional Galilei group at the limit of null value of the Galilei symmetry-breaking forces.

As we shall see, each of the dual methodological tools worked out earlier proved to be crucial for the problem considered, although with a different role. In particular, the methods of the Inverse Problem proved to be crucial to identify the mechanisms of breaking of the Galilei relativity in Newtonian Mechanics, although did not exibit a clear capability for a first construction of the desired covering relativity. The methods of the Lie-admissible problem had the complementary role, in the sense that they were insensitive to the structure of the symmetry-breaking forces, but they emerged as possessing a crucial capability for the construction of a generalization of the Galilei relativity which is applicable to ALL Newtonian systems of local (and class C^∞) type, and not only to the subclass of conservative systems.

The results of my efforts for the construction of such a generalization of the Galilei relativity were summarized in paper[32] for a more detailed treatment,as intended for these monographs. As we shall see, the Lie-admissible formulations do offer indeed a realistic hope of achieving a consistent covering of the Galilei relativity for nonconservative systems. Owing to the role of these methods, the proposed generalized relativity is called of Lie-admissible type, i.e., the Galilei-admissible relativity.

My last classical duty was to inspect the implementability of such broader relativity within the yet broader context of systems with nonintegrable subsidiary constraints. The answer to this problem resulted to be in the affermative, for a suitable selection of the

acting forces and constraints. This resulted in the following classical view of a hadron. It consisted of a systems of particles subjected to the broadest possible local forces nonderivable from a potential and subsidiary constraints which are such to violate established relativity laws at the level of the individual constituents, but to verify such laws for the state as a whole. In this way, possible departure from established relativities are possible, by construction, only WITHIN a hadron.

Stage 5: Studies on the quantization of systems with arbitrary local forces and subsidiary constraints, explicit construction of a preliminary model of hadronic structure, and confrontation of the predictions of the models with experimental data. It was only when the classical profile of the problem considered was sufficiently clear in my mind that I finally was in a position to confront the problem I had decided to attack a decade earlier: the construction of an explicit structure model of hadrons based on structure forces nonderivable from a potential. Predictably, the entirety of the conceptual and methodological profiles for the classical treatment of the forces considered entered into this latter stage of studies either in a direct or in an indirect form. The conceptual profile, in particular, turned out to be invaluable. The study of nonconservative forces demands a rather profound counceptual departure from conventional attitudes which, in the final analysis, are focused on forces derivable from a potential, conventional symmetries and conservation laws. Newtonian Mechanics proved again to be the arena of our clearest intuitions. In addition, it provide a rather unique ground of physical effectiveness for the verification of crucial insights, trivially, because on one side the actual structure of the strong forces is unknown, while we

have centuries of knowledge on nonconservative forces.

Of course, before being in a position to construct at least the rudiments of the desired model of hadron structure I had to confront the problem of quantization of the systems considered. Here, again, a dual alternative emerged: validity or invalidity for the strong interactions of established quantum mechanical laws (Schrödinger's equation , Heisenberg's time evolution law, Pauli's exclusion principle, etc.) . The line of study based on (the often tacit assumption of) the validity of these laws is that of current unitary trends.[1-22] In my studies I was solely interested in a possible invalidity of these laws for the strong interactions. We are all familiar with the impact of any relativity in our representation of physical reality, irrespectively of whether actually constructed, or only identified as possible. If nonconservative Newtonian mechanics demands the construction of a covering of the Galilei relativity, this will inevitably demand the construction of a suitable generalization of quantum mechanics as currently known. The results of my efforts for this latter problem are summarized in paper[33] for the intended more detailed presentation of these monographs.

Once I had identified the rudiments of the applicable quantum mechanical formulations, I was finally in a position to initiate the construction of the desired model of structure. My first, uncompromisable objective was to see whether the hadronic constituents can be consistently identified with physical particles, under the assumption of broader forces and laws. My second objective was to see whether such a broader nature of the strong hadronic forces had a clear physical origin. My third objective was the problem of a quantitative representation of at least part of the experimentally established phenomenology of the

particles considered: their intrinsic characteristics, decay modes and related fractions. Predictably, all these problems turned out to be deeply interrelated among themselves.

The primitive idea which I followed was as naive as conceivably possible. An increasing amount of theoretical indications based on the realization of the hadronic forces as derivable from a potential and confirmed by my generalizations, suggest that the hadronic constituents are very lightly bound. The naive attitude was that, under such circumstances, the constituents "must" be produced as free in the "spontaneous" decays. This yielded models of the (unstable) hadrons as a suitably constructed bound state of massive constituents of their sponteneous decays. It was at this point were the need of generalized formulations emerged in its full light. Indeed, no such model is admissible by the established relativity and quantum mechanical laws. A possible physical ground for the considered broader nature of the hadronic forces emerged as consequential. Within the context of the model considered each constituent is forced to move within the charge radius of the other to account for the experimentally established lack of sensible increase of the charge radius of the hadrons with the mass. This creates a motion of a particle within a "hadronic medium", i.e., a framework which, at a primitive Newtonian level, is characterized precisely by the fundamental

systems of my analysis, the nonconservative systems. The problem of a quantitative representation of the hadronic phenomenology was then reduced to a judicious selection of the hadronic forces which essentially emerged as only "partially" nonderivable from a potential, as expected from nuclear models. More specifically, the explicit forms of realizations of the hadronic forces resulted to be constitued by a term derivable from a potential (much along conventional models) plus a term which is not, produces the indicated nonapplicability of established relativity and quantum mechanical laws and motivates the use of the Lie-admissible approach. Also, this realization of the hadronic forces did not result to be unique (as it is the case for the electromagnetic forces). Instead, starting from simple realizations which resulted to be sufficient for the lightest hadrons, such as the π^0, there was the emergence of the need of realizations of increasing complexity, dynamical implications and methodological needs for heavier hadrons, such as the charged pions and the kaons. In the final analysis, a quantum mechanical structure consisting of a hierarchy of increasing dynamical conditions fully parallel to the Newtonian framework emerged.

In intuitive but effective terms, I can say that the conventional "atomic mechanics" (i.e., the ordinary nonrelativistic or relativistic quantum mechanics) was conceived to represent the atomic phenomenology of the time of its inception, the

atomic spectra per each individual atom. The intended "hadronic mechanics" is conceived to represent the corresponding hadronic phenomenology available at this time, the decay modes and fractions per each individual hadron. Although not explicitly stated in this Preface until now, all my efforts, since the very beginning in 1963, have been devoted to this primary objective.

The organization of these monographs is as follows.

This Volume I is devoted to the first essential step: the identification of the arenas of applicability and non-applicability of the conventional Galilei and Einstein relativities through the use of the methodology of the Inverse Problem. This is, in essence, my answer to the question "fine, wo what?" related to the knowledge of the integrability conditions of the fundamental equations of current theoretical physics: Lagrange's and Hamilton's equations without external terms. Besides a concise review of the part of the Inverse Problem which is essential for the problem considered (presented in Chapter 2), I have included in Chapter 1 a presentation of my epistemological arguments which suggest the conduction of the analysis.

Volume II is devoted to the second essential step: the identification of the rudiments of the expected covering of the Galilei relativity for forces not derivable from a potential. This is, in essence, my answer to the question "fine, so what?" related to the knowledge of the

methodology for the analytic equations originally conceived by Lagrange and Hamilton : those with external terms. Besides a review of the Lie-admissible problem, Volume II also contains a brief review of certain essential concepts of abstract algebras.

Volume III is devoted to the last and fundamental issue: the explicit construction of a structure model of the hadrons based on the generalized methodological formulations of the previous two volumes and its comparison with the currently available experimental data. The comparative analysis of this structure model with those currently available will be left to the interested reader. Besides the presentation of my efforts for the quantization, this third volume also includes an indication of some of the problems which remain open. Volume III then ends with suggestions on the future orientation of experimental high energy physics to fulfill the increasing need of a physically meaningful selection among an ever increasing number of models on the hadronic structure.

The reader should be aware that, to contain the size of these volumes to an absolute minimum, a number of technical aspects are referred to the existing literature. Since none of these methods is currently used in high energy theoretical physics, to the best of my knowledge, the study of these aspects is recommended to any researcher with a serious interest in either conducting active research along the lines of these volumes or in attempting a sound comparative analysis of the

proposed structure model of the hadrons with the currently available models. I am here referring in particular to the quoted specialized literature on the methodology of the Inverse Problem, the Lie-admissible problem, the consistency of overdetermined systems of differential equations, the quantization of forces nonderivable from a potential, etc.

Almost needless to say, the amount of efforts I have put into these studies has no necessary bearing on their maturity and a long way remains yet to be covered. In essence, it is hoped that these three monographs will give an effective presentation of the central ideas of the approach for subsequent independent studies by other researchers. The emphasis is then in the selection of the simplest possible, but effective, ways of presentation of these ideas, and in the identification of their plausibility. By no means these monographs are intented to provide mathematical completeness and rigour which, in any case, would have delayed their publication for years. To be more specific in this point, my primary objective is the presentation of apparently new conceptual and methodological insights for a possible resolution of the problem of the hadronic structure and the task of putting my approach into a mathematically rigorous language is left to the interested reader.

But there is a more insidious aspect which should be clearly stated: these monographs deal with nonconservative systems, namely, a type of systems which demands a profound conceptual departure from current, conventional trends. This departure is magnified when considering hadronic issues, and reaches its climax in the intent of exploring the possible non-applicability within a hadron of established relativity and quantum mechanical laws and the consequential possible applicability of generalized formulations, in the true spirit of basic research.

October 24, 1977

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(Updated on June 1978)

CHAPTER 1

THE EPISTEMOLOGICAL ARGUMENT

1.1 AN HISTORICAL PERSPECTIVE

The period of the history of theoretical physics ranging from the awarding of the Davy medal in chemistry to D. I. Mendeléyev* in 1882 to N. H. D. Bohr's reception of the Nobel price in physics of 1922, as is known, 38 was dominated by the problem of identifying methodological tools capable of providing an effective representation of the atomic structure.

It was not without controversy that the need of a radical departure from Newtonian Mechanics finally emerged. In this way a new mechanics, called Quantum Mechanics, was conceived and established by founding fathers such as N. Bohr, M. Born, L. V. de Broglie, P. A. M. Dirac, W. Heisenberg, P. Jordan, M. Planck ; E. Schrödinger and E.P.Wigner, with contributions ranging over the first half of this century.

The outcome was the establishing of one model of the hydrogen atom which proved to be fully consistent with the physical reality. The advent of the (special) Theory of Relativity also resulted to be crucial for bringing the quantum mechanical model of the atomic structure to its utmost maturity as it is known nowadays.

These new methodological perspectives essentially identified the arenas of applicability of different formulations. Newtonian or Relativistic Classical Mechanics emerged as applicable to the macroscopic physical reality, while non-relativistic or relativistic Quantum Mechanics emerged as possessing a more fundamental role, being applicable to the considered microscopic world as

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* Other spellings used in the English literature are Mendeleev, or Mendeléeff, or Mendelejeff.

well as capable of reproducing the conventional classical conceptions for any collection of microscopic systems whose value of the action functional is large compared to Planck's constant.

The aspect of this period of the history of theoretical physics which is relevant for our analysis is that a central objective of the founding fathers of Quantum Mechanics was the interpretation of the stability of the electron orbits in the atomic structure as well as the interpretation of the atomic spectra. We can thus state that Quantum Mechanics was originally conceived to represent conservative systems, such as the two-body Coulomb system, although later on proved to be valid also for more general

The period of the history of theoretical physics ranging from 1922 to E. Fermi's award of the Nobel price in physics of 1938 was dominated, as is also well known,[39] by the problem of the structure of the atomic nucleus. The basic mathematical tool used for this objective was, and still is, Quantum Mechanics. As a result of a large number of contributions, an effective understanding of the nuclear structure was finally achieved.

The aspect of this second period of the history of theoretical physics which is here relevant, is that Quantum Mechanics produced only one model of the hydrogen atom capable of interpreting the entirety of its phenomenological behaviour, while the same discipline, applied to the structure of the atomic nucleus, produced a variety of models, such as the shell model,

the Fermi gas model, the optical model, etc., none of which is individually capable of representing the entirety of the new phenomenological context. To my best knowledge, this situation still persists as of today, despite several progresses of proved physical relevance, such as the nuclear cluster model.[40]

This situation seems to indicate that in the transition from the atomic to the nuclear level, nature suddenly becomes polyhedric, by restricting our capability to effectively represent it to only few of its facets at a time.

The period of the history of theoretical physics ranging from 1938 to now, as it is equally known, has been dominated by the problem of the structure of the strongly interacting particles, the hadrons. The central mathematical tools used in virtually all the available contributions in this topic is again Quantum Mechanics, or its quantum field theoretical extension, but now complemented with new tools, the unitary internal symmetry groups and, most notably, the SU(3) model by Gell-Mann and Zweig, lately implemented to so-called SU(3)-color model.[20-22]

The formidable task confronted by theoretical and experimental physicists during this third period of the history of physics was in actuality twofold. First of all, there was a clear need of producing an effective classification of an ever increasing number of particles into families, i.e., to formulate the "Mendeléyev table" of the subnuclear particles. Secondly, there was the central problem of identifying the structure of

particles which, from mounting experimental evidence, cannot be elementary.

As a result of several brilliant contributions, the SU(3) model has indeed produced an effective "Mendeléyev classification" of the strongly interacting particles of undeniable physical relevance. Therefore, the first of the indicated two objectives can be nowadays considered as accomplished to a major extent.

The same unitary models have been used to jointly represent the structure of the hadrons, again within the context of Quantum Mechanics, or its quantum field theoretical extension, as the primitive discipline. The net outcome has been this time controversial. It would be of questionable scientific mentality of deny that the recent color implementation of the SU(3) structure models of the hadrons constitutes a genuine progress in a formidable problem. But it would be unobjective to deny that the emerging models have a lesser physical effectiveness on a comparative basis with nuclear and atomic models.

The point which is here relevant is that in the transition from the nuclear to the hadronic level there has been a larger proliferation of models which appears to emphasize the departure from the unity of Atomic Physics. It seems as if, in the transition from the nuclear to the hadronic level, nature performs the transition from a polyhedric structure to a structure of undefined topology with consequential loss of guidance for our efforts to effectively represent it.

This, however, is only part of the reasons of controversy indicated earlier. Other reasons are related, in my opinion, to some direct consequences of Quantum Mechanics with respect to the characteristics of the hadronic constituents.

As is known,[3-4] the fundamental representation of the SU(3) group was assumed to represent a physical particle, called quark, and interpreted as the constituent of the hadrons. The inflexible laws of Quantum Mechanics, however, directly imply certain special features of quarks whenever they are interpreted as the hadronic constituents. For instance, in the most familiar model, the quarks possess a fractional elementary charge.

Since no such particle is produced in the spontaneous decays of the unstable hadrons, a sizeable experimental effort has been implemented during the last decade, and lately intensified, aiming at the detection of quarks under high energy inelastic collisions, with no definite result until now.[6] In turn, this situation has stimulated a considerable theoretical effort aiming at the interpretation of an assumed lack of capability of a quark to be individually produced in high energy collisions, namely, the so-called problem of quark confinement.[7-16] Despite equally brilliant efforts in an equally formidable problem, no conclusive model of confinement has been achieved as of now.

But, perhaps, the most problematic aspect of the quark model is only surfacing at this moment. An ever increasing experimental evidence, as vividly presented by Sheldon L. Glashow at the recent MIT symposium in honor of Samuel C. C. Ting, clearly

indicates the possible existence of an infinite number of leptons and hadrons. If this is the case, the quark model could be subjected to an infinite number of sequential implementations, namely, there could exist an infinite number of different quarks. In my opinion, this perspective cast shadow on the same conception of the quarks as the elemental constituents of the hadronic matter.

To summarize this brief historical perspective, the use of Quantum Mechanics as the primitive discipline within the context of the currently identified layers of the microscopic reality

(I) the atomic layer,

(II) the nuclear layer,

(III) the hadronic layer,

has produced:

(I') one single model capable of effectively representing the phenomenological behaviour of the layer for which it was conceived, namely, the atomic structure,

(II') the lack of one single, equally effective model for the new context of the nuclear structure, although the emerging models have an undeniable physical effectiveness, and

(III') a proliferation of models for the subsequent context of the hadronic structure of no physical effectiveness comparable to those of the corresponding atomic or nuclear models, by implying certain characteristics of the hadronic con-

stituents which an increasing number of physicists believe to be in conflict with the physical reality.

This situation clearly calls for an assessement of the current trends in hadronic physics aiming at the identification of conceivable alternatives which deserve a study.

1.2: THE FUNDAMENTAL ASSUMPTIONS OF OUR HADRONIC MODEL

In my opinion, the current status of the art on the problem of the hadronic structure emphasizes more than ever the multiple role of basic research.

There is first the need of continuing the studies along the established trends, such as color unitary models. However, the restriction of the studies along only this line would be contrary to the same spirit of basic research. The need for a joint search of new insights then emerges.

These monographs are devoted to the latter duty of basic research for the hadronic structure.

The first problem which arises in a task of this nature is the identification of the criteria for the selection of valuable alternatives. This objective can be attempted by conducting an anamnesis of the currently available hadronic models with reference to the structure of the atoms as a sound basis of comparison.

We shall here consider three aspects of the current studies on the hadronic structure, the possible dichotomy classification/structure, the role of the structure forces and the methodological context.

As recalled in Section 1.1, the problem of the atomic phenomenology demanded two different but compatible models: the Mendeléyev model of classification and the Bohr model of structure, later generalized to what is currently known as the Thomas-Fermi model for higher structures. Consider, as a

specific reference frame, the palladium group (an octet). The identification of this group belongs to the Mendeléyev classification. Nevertheless, the structure of each individual element of the group belongs to the Bohr-Thomas-Fermi model of structure.

ATOMIC PROBLEM

Classification: Mendelévev model

Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd

Structure: Bohr-Thomas-Fermi model

The two approaches, the classification and the structure, are profoundly different. Nevertheless they are compatible in the sense that the structure model reproduces the classification and, viceversa, the classification produce essential elements for the problem of the structure.

When seen from this profile, the currently available models on the hadronic structure exhibit a rather peculiar feature: they are customarily intended to provide a joint solution of the problem of classification and of the structure. Consider, as a specific reference frame, the family of the "stable" mesons (also an octet). The identification of this family was the result of the SU(3) model.[2] The same model, through the assumption that the fundamental representation of the SU(3) group represents a

physical particle (the quark), is considered as providing a structure model of each individual element of the family as a bound state of a quark and an antiquark.

HADRONIC PROBLEM

Classification: SU(3) model.

π^0	π^+	π^-	K^+	K^-	K_S	K_L	η

Structure: same SU(3) model.

This first aspect of our anamnesis already provides valuable intuitional insight. First of all, there is no a priori reason why the situation which eventually emerged for the atoms (a differentiation between the models of classification and of structure) should also occur within the context of the hadrons. As a result, studies along the SU(3) model of joint classification and structure should continue. However, in the true spirit of basic research, new insights must be attempted. The above comparative analysis clearly indicates that a differentiation between the models of classification and a different but compatible model of structure is also conceivable for the hadrons. We reach in this way the first fundamental assumption of our studies on the hadronic structure.

ASSUMPTION 1: We hereby assume that the problem of the hadrons demands two different but compatible approaches: the established unitary models of classification and a new model of structure.

Notice that in this assumption the currently available unitary models are assumed as valid. We only restrict their arena of

physical effectiveness to the problem of classification. As a matter of fact, most of our subsequent efforts will be devoted precisely to the recovering of the SU(3) framework in much the same way as the efforts for the structure of the atoms were concentrated in recovering the Mendeléyev classification.

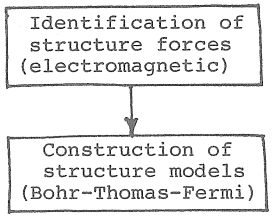
The implications of the above assumption, however, should be better identified. Strictly speaking, the assumption implies that the quark is only a mathematical entity, the fundamental representation of the SU(3) group, and does not represent a physical particle. This is clearly a necessary prerequisite for complete freedom in the construction of a different model of structure along the compatibility conditions with the SU(3) classification indicated earlier.

As we shall see in Volume III, the above assumption has also several other implications. For instance, the concept of multiplet which has such a fundamental role for the classification, need not necessarily be applicable for the problem of the structure in the sense that it should not necessarily acquire a constructive role within this latter context. Similarly, the hadronic correspondents of the atomic concept of valence, such as isospin, hypercharge, etc., do not necessarily have a constructive role at the level of structure, irrespective of their crucial role for the classification. This is the case at the atomic level where the efforts were centered in recovering through a structure model the valence. But such valence had no

constructive role in the structure problem.

We now pass to the second fundamental aspect of our anamnesis, the problem of the nature of the forces responsible for the structure considered. The researchers on the problem of the atomic structure were considerably facilitated in this respect because of the general belief (later on proved to be correct) that the force responsible for the atomic structure is the known Coulomb force. In any case, the studies on the structure of the atoms were conducted by identifiying first the nature of the acting forces and then constructing the structure models.

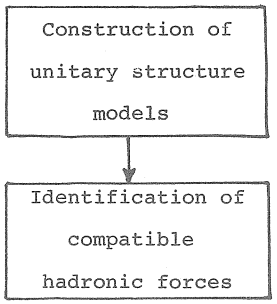
ATOMIC STRUCTURE PROBLEM



The corresponding situation at the hadronic level is considerably more complex owing to the fact that the explicit form of the forces responsible for the hadronic structure is simply unknown at this time.

Neverthless, there is a conceptual attitude in the currently available studies on the hadronic structure which is significant. In essence, from the lack of knowledge of structure forces, the attitude has been that of first constructing structure models, and then studying the compatible realizations of the hadronic forces.

HADRONIC STRUCTURE PROBLEM



In short, a series of circumstances has produced a second considerable departure of the hadronic structure problem from the corresponding study of the atomic structure.

It should be indicated in this respect that recent studies based on the assumption that the hadronic forces are derivable from a potential and a judicious selection of their potential has produced results in excellent agreement with the heavy hadron spectroscopy.[41] This confirms our attitude toward the multiple role of basic research for the problem of the hadronic structure until its definite resolution is achieved: the studies along current trends should continue.

In my opinion, however, the problem of the nature of the forces responsible for the hadronic structure is the truly fundamental problem. This suggests as a valuable alternative the reversing of the order of priority. We are therefore interested in first confronting the problem of the nature of the hadronic forces and then identifying a structure model which

is compatible with the selected nature, rather than the opposite approach.

We reach in this way the second fundamental aspect of the study of these monographs: the selection of a conceivable form of the forces responsible for the hadronic structure.

In order to achieve such selection, a comparative analysis of the nature of the force assumed by unitary structure models and that of the electromagnetic forces is in order.

As is known, the explicit form of the electromagnetic force which produces an excellent agreement with the experimental data for the case, say, of the lightest known atom, the positronium, is a local force derivable from a potential.

Contrary to the previous occurrences, the currently available hadronic structure models this time exhibit a situation equivalent to that of the atomic problem. Consider in this respect the lightest known hadron, the π^o. By ignoring a number of technical implementations which have been subsequently attempted to represent the physical reality (such as the so-called "sea of gluons"), the SU(3) structure model of this particle is close to that of the positronium: a bound state of a particle (quark) and an antiparticle (antiquark). The point which has fundamental relevance for our analysis is that the nature of the force responsible for such bound state is analytically equivalent (in a sense to be technically identified in Chapter 2) to that of the electromagnetic forces of the positronium, i.e., local and derivable from a potential. The differences between the two structures, the

positronium and the π^o, are represented by internal unitary degrees of freedom of the hadronic forces which are absent for the electromagnetic force.

A number of remarks are here in order to assess this situation. First of all it should be indicated that no explicit form of the potential of the hadronic forces capable of producing a quantitative representation of the phenomenology of the light hadrons has been identified until now despite laborious efforts. The previously indicated agreement with the hadronic spectroscopy applies only for heavy hadrons.

Our interpretation of this situation is quite naive and contrary to current trends. It is generally believed that heavy hadrons have a simpler structure than that of lighter hadrons. Let us recall that, at the time of the inception of the structure model of the atoms, the problem of the structure of heavy atoms, such as the palladium, even though not overlooked, was estraneous to the analytic minds of the founders of the "atomic mechanics". Instead, the efforts were concentrated in the lightest atom known at the time, the hydrogen. Our naive attitude is that a similar situation may eventually emerge for the problem of the structure of the hadrons in the sense that the structure of the light hadrons may eventually result to be considerably simpler than that of the heavier hadrons.

In any case, under no circumstance the problem of the nature of the hadronic forces can be considered as resolved until it

produces a quantitative representation of the hadronic phenomenology for all hadrons, irrespective of whether light or heavy.

In conclusion, for the line of study of these monographs, the problem of the structure of the light hadrons is assumed as fundamental and that of the heavy hadrons is assumed as derived in much the same way, again, as it occured at the atomic level.

This assumption produces a better identification of the problem under consideration. Our problem is that of identifying a conceivable nature of the forces responsible for the structure of the light hadrons with the expectation that the same nature, subject to predictable technical implementations, will result equally conceivable for the case of the heavy hadrons.

A comparative analysis of the positronium and the π^o is here essential. These two structures exhibit a number of similarities, such as: they can both be interpreted as a bound state of a particle and an antiparticle; they are both neutral; and they have similar decay modes. However, the two structures exhibit profound physical differences too. Most notably, the charge radius of the π^o ($\sim$ 1F) is of the order of magnitude of that of any other hadron, while the corresponding quantity of the positronium is much larger. Perhaps more significantly, the fraction of the same decay modes for the two structures are different. And last, but not least, the positronium is a strictly electromagnetic state while the π^o

is a strongly interacting particle .

Our problem is the selection of a conceivable nature of the forces responsible for the π^o which, upon due inspection, can account for the above indicated similarities as well as differences with the positronium.

The SU(3)-assumption of representing the π^o structure forces as local, derivable from a potential and possessing unitary internal degrees of freedom is certainly conceivable in light of the indicated similarities. The problem is whether it is sufficient to represent the indicated differences.

Our first contention is that, perhaps, the unitary internal degrees of freedom are not sufficient to represent the indicated differences of the π^o with the positronium. The reason is that, quite simply, those degrees of freedom occur in a mathematical internal space while the indicated differences occur in the physical space of our experimental detection.

This contention better identifies our problem: we are interested in identifying a conceivable structure of the forces responsible for the π^o which is analytically nonequivalent (in a technical sense to be clarified in Chapter 2) to the electromagnetic forces in the assumed physical carrier space, the Euclidean or the Minkowski space. The possible existence of internal degrees of freedom will be ignored throughout our analysis of Volume I and II and considered later on only in Volume III.

Explicitly stated, our contention is that <u>the π^o structure forces are, in general, nonlocal and nonderivable from a potential</u>.

The question which then arises is whether, besides the intent of representing the indicated differentiations, there exist physical grounds for these generalized forces.

With an understanding that no definite answer can be given to this question at this time, a number of remarks is in order.

One of the experimental data which is most significant for the analysis of these monographs is that, contrary to the corresponding situation at the nuclear level, the charge radius of the hadron does not sensibly increase with the mass and it is of the same order of magnitude of any other physical, massive, particle (i.e., $\sim$ 1F).

If the hadronic constituents are also physical, massive and charged particles, a rather peculiar situation occurs for the hadronic structure which has no equivalent at the level of the nuclear and atomic structure: since the charge radius of the constituents is of the same order of magnitude of that of the state as a whole, each constituent is forced to move within the charge radius of the other.

This situation can indeed account for the generalized nature of the hadronic forces as being nonlocal and non-derivable from a potential. And indeed an inspection of the problem by using purely classical methods, such as those of Newtonian Mechanics of, more properly, Continuum Mechanics, yields for the peculiar structure considered an integrodifferential form of the equations (nonlocal forces)

with couplings not derivable from a potential, i.e., a system of integrodifferential equations which, besides satisfying continuity and boundary conditions, is arbitrary (in the sense that does not exhibit the structure of the Lagrange equations). It is in this sense that the generalized hadronic forces here considered can be initially interpreted as being analytically nonequivalent to the electromagnetic forces (for a technical characterization, see, again, Chapter 2).

The conduction of our study based on structure forces which are nonlocal and nonderivable from a potential will clearly be faced with predictable technical difficulties. Our problem is to see whether an acceptable approximation of these forces exists. The answer is soon provided by classical mechanics: local forces nonderivable from a potential are known to be a good approximation of nonlocal force, irrespective of whether derivable from a potential or not. This is typically the case of the systems of our everyday experience, the non-conservative Newtonian systems, such as particles in a physical medium producing drag forces, damped oscillators, spinning tops with drag torques and corresponding systems in Continuum Mechanics. Strictly speaking these systems should be represented with integrodifferential equations of arbitrary (i.e., non-Lagrangian) structure. Nevertheless, it is known that the representation of these systems with ordinary or partial (and thus local) differential equations produces an excellent agreement with the physical reality.

We reach in this way the second fundamental assumption of the study of these monographs.

ASSUMPTION 2: We hereby assume that the strong couplings responsible for the structure of the "stable" mesons are local but not derivable from a potential.

As we shall see, despite the preservation of locality as for the electromagnetic forces, the assumption that the hadronic forces are not derivable from a potential will result in major conceptual and technical implications at each and every level of our analysis.

It should be indicated since now that the terms "not derivable from a potential" should be referred to the considered forces as a whole. This does not exclude that some terms of these forces are derivable from a potential. As a matter of fact, most of our efforts of Volume III will be based on the assumption that the hadronic forces are partially derivable from a potential and partially not. This is suggested by nuclear considerations where the assumption that the nuclear force are entirely derivable from a potential produces excellent agreements with the experimental data. In particular, this profile indicates the possibility of simply implementing existing models with local hadronic forces derivable from a potential with additional local terms which are not. Notice also that, irrespective of whether derivable from a potential or not, the forces considered are not necessarily central (again, without excluding that some of their terms are central)

as it is the case at the nuclear level.

Besides the above considerations related to the conceived structure of the hadrons as bound states of extended particles, there are additional epistemological considerations which suggest a study of our second fundamental assumption.

One of the central problems in the transition from the positronium to the π^0 structure or, more generally, from the electromagnetic to the strong interactions, is the dominance of the latters over the formers. One of the direct consequences of the assumption that the strong interactions are realized in terms of couplings analytically equivalent to the electromagnetic interactions is that this dominance can be achieved with only one remaining possibility: the high value of the strong coupling constant (we here ignore more recent attempts such as asymptotic freedom).[42] In turn this latter circumstance has rather disastrous technical consequences. While quantum mechanical perturbative techniques can be significanly applied to the electromagnetic interactions, they become inapplicable to the strong. The lack of perturbative techniques literally implies the inability of computing physically meaningful numbers.

In my opinion, the significance of this occurrence is of such magnitude that alone is sufficient to justify the study of a nontrivial differentiation between the electromagnetic and the strong interactions, irrespective of any consideration relative to the problem of the hadronic structure.

Our epistemological considerations are centered on the origin of the problem, rather than its ultimate technical form: the means of achieving the dominance of the strong over the electromagnetic interactions.

In essence, a number of alternatives are conceivable. The first is that indicated earlier (analytic equivalence of the strong and electromagnetic couplings, besides internal degrees of freedom) which directly implies the known high value of the strong coupling constant. There is however a second conceivable alternative. The dominance under consideration can also be attempted with a major differentiation of the nature of the electromagnetic and strong forces, again, in the physical (rather than internal) carrier space. A most intriguing aspect of this second alternative is that, while the high value of the strong coupling constant is often[42] necessary for the former possibility, this is not necessarily the case for the second alternative. In other words, the second possibility offers the opportunity of attempting the dominance of the strong over the electromagnetic interactions through their analytic differentiations, rather than differences between the coupling constants. In short, if the second possibility is assumed, there is at least the hope of achieving, in due time, meaningful perturbative (Borel) techniques for the strong interactions while the same hope is excluded, strictly speaking, for the first possibility.

This yields again, although from a different profile, the

considered assumption on the nature of the strong interactions as local and not derivable from a potential. As we shall see in due time, when the strong interactions are assumed of this type, they became so "strong" to actually render unidentifyable with conventianal means the electromagnetic interactions at the level of a Lagrangian, even though the two interactions are obviously identifyable at the level of the equations of motion because additive.

In summary, a number of considerations related to the hadronic phenomenology, with particular reference to the intrinsic charactersitics and interactions (the decay modes will give additional elements, as we shall see in Volume III) suggests the study of the hadronic forces of the indicated type as an approximation of expected, more adequate nonlocal forces.

We reach in this way another central problem for the analysis of these monographs: the methodology which is needed to treat the assumed broader forms of the hadronic forces.

It is here advisable to resume our anamnesis of the SU(3) model on a comparative basis with the corresponding atomic problem.

As it is known, the only disciplines available at the time of the inception of the Mendeléyev classification were classical (as well as nonrelativistic). The problem of the structure of the atom was initially confronted with the use of the same disciplines which had proved to be effective for the classifi-

cation. As recalled in Section 1.1, a number of problems forced later on the acceptance of a radical departure from classical attitudes. The solution of the problem of the atomic structure simply demanded the construction of a new discipline, quantum mechanics, subsequently extended to a series of new disciplines inclusive of relativistic quantum mechanics and quantum field theory.

In conclusion, the problems of the atomic classification and structure, even though can be treated nowadays with the same quantum mechanical techniques, were originally resolved with different but compatible disciplines. With reference to the previously considered group of the palladium we can thus write

ATOMIC PROBLEM

Classification: classical diciplines

Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd

Structure: quantum mechanical disciplines

A comparative analysis of the corresponding situation with the SU(3) models reveals a third central differentiation in attitude of the hadronic versus the atomic problem.

A fundamental idea of all the currently available unitary models of the hadrons is that of implementing with unitary internal symmetry groups the disciplines which have been previously established for the electromagnetic phenomenology,

i.e., quantum mechanics and its relativistic or field theoretical extensions. With reference to the previously considered family of hadrons we can therefore say that the methods used by current trends in hadron physics are the same for both the classification and the structure.

HADRONIC PROBLEM

Classification: quantum mechanical disciplines

π^0	π^+	π^-	K^+	K^-	K_S	K_L	η

Structure: same quantum mechanical disciplines.

A central contention of these monographs is that, perhaps, a differentiation between the established quantum mechanical disciplines for the problem of hadron classification and new but compatible disciplines specifically conceived for the hadronic structure may eventually result to be needed, in much the same way as it occured for the problem of the atomic phenomenology.

A number of epistemological arguments appear to support this different attitude. In essence, the current knowledge of the electromagnetic interactions (of such fascinating physical effectiveness at all its levels – classical, quantum mechanical and quantum field theoretical) is the result not of minor implementations of the previously available methods, but instead of the courageous construction of new disciplines specifically conceived for the reality considered.

Despite these unequivocal successes at the level of the atomic phenomenology, the use of the same disciplines for the new hadronic level of the physical reality has produced results of no comparable physical effectiveness.

It appears as if, in order to attempt a knowledge of the strong interactions of physical effectiveness at least close to that of the electromagnetic interactions, a new generation of theories specifically conceived for the new reality considered must be built.

The study of this possibility is a central objective of these monographs.

The first question which is here pertinent is whether there is any technical ground for these expected new disciplines. Clearly, if the hadronic forces are realized in terms of local couplings derivable from a potential, we are within the arena of unequivocal applicability of the established relativity and quantum mechanical disciplines. However, if the same forces are realized in terms of local couplings not derivable from a potential, the situation is profoundly different.

From a relativity profile, the type of assumed hadronic forces is purely nonconservative at a Newtonian level. It is known that these forces violate the Galilei relativity. A corresponding violation of the Einstein special relativity is then expected from simple compatibility arguments, as we shall see in Chapter 3.

From a quantum mechanical profile it is known that the problem of quantization of nonconservative Newtonian forces is fully open

at this time, despite initial efforts. As we shall see in Volume III, the forces considered are basically incompatible with quantum mechanics as currently known. This produces a dynamical counterpart of a corresponding incompatibility at the relativity level. The incompatibility of other disciplines with the forces considered, such as relativistic quantum mechanics or quantum field theory as presently known, can then be expected.

We can then conclude that, subject to explicit technical verifications to be conducted during the course of our analysis, the realization of the hadronic forces (strong interactions) as local and nonderivable from a potential is expected to be incompatible with the disciplines currently established for the atomic forces (electromagnetic interactions).

This indicates the need of searching for new disciplines.

The next problem which is here pertinent is the identification of the criteria of attempting the construction of these new theories. An inspection of the corresponding situation which occured at the atomic level is again valuable. In essence, the central idea of the founders of Quantum Mechanics and Special Theory of Relativity was that of constructing these disciplines as coverings of the Newtonian Mechanics for conservative forces. For a rigorous treatement of the concept of covering see ref.[23]. For our needs it is sufficient to recall that:

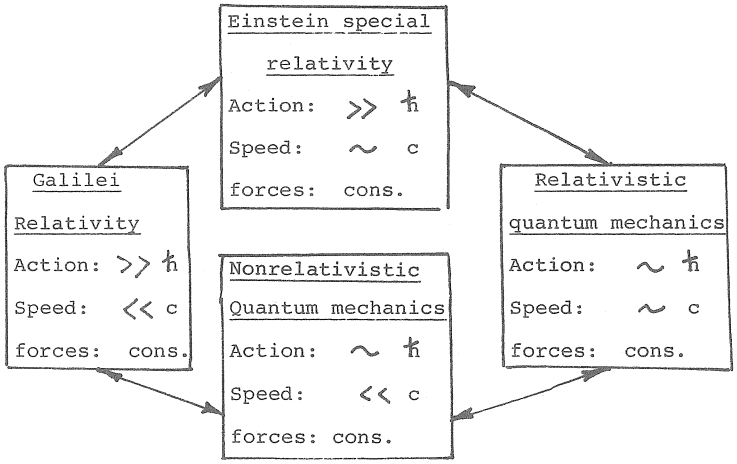
(A) The new and old theory must be compatible in the sense that there must exist clear limiting procedures of clear

physical significance which reduce the new theory to the old, and viceversa.

(B) The new theory must constitute a nontrivial generalization of the old for a nontrivially different physical context.

Quantum Mechanics and Special Theory of Relativity produced two initially distinct coverings of conservative Newtonian mechanics, later on combined into the single, symbiotic, covering known under the name of Relativistic Quantum Mechanics. The following diagram gives a schematic view of the criteria for the verification of conditions (A), and (B) above.

THE CONCEPT OF COVERING FOR THE ATOMIC DISCIPLINES



The appearance of a new third element for the characterization of any one given discipline, the type of considered force, should be here stressed because it is ignored by the currently available literature, to the best of my knowledge. As we have indicated earlier and as we shall see in details in Chapter 3, the inclusion of this third characteristic element for any one of the considered disciplines is crucial in precisely the same measure as that of the values of the action and speed, although on different technical grounds.

The restriction of the table to only conservative forces should be considered as of historical rather than technical character. As we shall see in Chapter 3, the table can be extended to all forces which are variationally selfadjoint. This will include, of course, the Lorentz (velocity-dependent) force.

In summary, the new disciplines of this century which proved to be so effective for the problem of the atomic structure, rather than "destroying" the Galilei relativity and related conservative Newtonian mechanics, simply identified their arena of unequivocal applicability as far as the value of the action and the speed are concerned. One of the objectives of this volume is to complete the identification of this arena as far as the nature of the acting forces are concerned.

Also, the new disciplines were conceived for the intent of being a covering of the Galilei relativity in the sense of criteria (A) and (B).

These informations are sufficient for the problem under consideration. Since the disciplines considered are not expected to apply for the considered realization of the hadronic forces, the new disciplines should be searched for under the condition of being coverings of the existing disciplines. We reach in this way the third fundamental assumption of these monographs which we can write

ASSUMPTION 3: We hereby assume that the Galilei and Einstein relativity as well as the nonrelativistic and relativistic quantum mechanics are inapplicable for the hadronic structure in their currently known form and new coverings of the same disciplines specifically conceived for the strong interactions must be identified.

The next problem which is here pertinent is that of the methods capable of producing these expected coverings. With reference to the comments to this effect in the Preface, the methods which we shall use are those of:

(1) The Inverse Problem, i.e., the methodology underlying the use of analytic equations without external terms for the representation of nonconservative systems,

(2) The Lie-Admissible Problem, i.e., the methodology underlying the use of analytic equations with external terms for the dual (in the sense of Preface) representation of the same systems, and

(3) The quantization through the use of the Inverse Problem and the Lie-Admissible Problem.

As also recalled in the Preface, the Inverse Problem will play a central role for the identification of the arenas of applicability and nonapplicability of established disciplines. This is a central objective of this Volume I. The Lie-Admissible Problem will play a central constructive role for the desired classical generalizations with a dual role of the Inverse Problem whenever applicable. This is a central objective of Volume II. The quantization of the Inverse Problem and the Lie-Admissible problem will play a constructive role for the desired quantum mechanical generalizations. This is a central objective of Volume III.

The reader should be fully aware of the implications of the third fundamental assumption of these monographs. Strictly speaking, the assumption implies the not necessary applicability within the hadron of our quantum mechanical knowledge, such as the Pauli Exclusion Principle, the Spin-Statistics Theorem, and the Wightman's Axioms. As we shall see in due time, all these occurrences are a direct consequence of the assumed broader nature of the hadronic forces. This will confirm our attitude for Assumption 2: the problem of the nature of the hadronic forces is the truly ultimate problem of hadron physics. All the remaining aspects, including the methodology for their treatment and explicit structure models, are consequential.

Notice that conditions (A) and (B) are expected to be satisfied by the coverings of Lie-admissible nature. Trivially, the "clear limiting procedure of clear physical significance"

is in our case given by the null limit of nonconservative forces. As we shall see in Volume III, this will essentially imply (for a two-body case) the transition from the π^0 to the positronium structure. Also "the nontrivially different physical context" is selfevident from the physical differences existing between the atomic and the hadronic structure.

The novelty of the methods which will be used for the construction of the expected new coverings should also be indicated. Indeed, the transition from a Lie algebra structure to a Lie-admissible algebra is estraneous to all covering disciplines constructed until now. This is a valuable element for the nontriviality of the intended coverings.

To conclude this section, one additional problem must be confronted, that of the compatibility between covering disciplines for the hadronic constituents and the conventional disciplines for the behaviour fo the hadron as a whole under electromagnetic interactions. This compatibility is clearly requested by the daily experimental evidence, such as in particle accelerators, that hadrons under electromagnetic interactions obey precisely those disciplines which we have assumed to be nonapplicable to their constituents.

As indicated in the Preface, this compatibility will be attempted through the use of subsidiary constraints. In essence, relativity ideas are nowadays expressed through symmetry considerations. The physical content of the relativity considered, however, remains in the validity of the underlying physical conservation laws which are

(a) for the Galilei relativity: the conservation laws of the total physical energy, linear momentum and angular momentum, as well as the uniform motion of the center of mass, and

(b) for the Einstein relativity: the conservation laws of the energy-momentum and angular momentum tensor.

The idea which we shall study is, first, the imposition of these conservation laws as subsidiary constraints for our classical analysis, and then the imposition of suitable extensions of these constraints for our quantum mechanical analysis.

We reach in this way our concept of hadron as a bound state of massive and charged physical particles whereby the conventional relativity and quantum mechanical laws are violated by conception for its constituents while, also by conception, no deviation from these laws occurs for the state as a whole under ut most electromagnetic interactions.

It is precisely in this dichotomy of methods for the hadronic constituents and the hadron as a whole that relies our hope of achieving a profound differentiation between the electromagnetic and the strong interactions in a way consistent with the physical reality.

In essence, as we shall see in due time, if the conventional relativity and quantum mechanical laws are imposed for the behaviour of both the hadrons as a whole and their constituents, the net result is an analytic equivalence of the strong and the electromagnetic forces which does not appear to be removable by

internal degrees of freedom.

The only possibility known to me at this time of attempting a genuine differentiation between the strong and the electromagnetic interactions is precisely that indicated earlier, the validity of established laws for the hadron as a whole and the validity of covering laws for its constituents, along the indicated lines of compatibility. The net result in this case is the possibility of considering realizations of the strong interactions genuinely nonequivalent to that of the electromagnetic interactions, while being consistent with currently available experimental evidence.

When the problem is seen from this profile, the established relativity and quantum mechanical laws appear in a rather different light: perhaps, they are the major obstacle for a genuine differentiation between the electromagnetic and the strong interactions.

1.3: THE FUNDAMENTAL ASSUMPTIONS OF OUR HADRONIC CONSTITUENTS

Before entering into the realization of our program it is advisable to better identify the concept of hadronic constituent which emerges from the fundamental assumptions of the preceding section. This will later on prove to be valuable for the identification of the technical problems to be confronted. The analysis will be again conducted through a comparative study of the lightest known atomic and hadronic structures, the positronium and the π^0. By ignoring relativity group characterizations, with the term "constituent" or "particle" we mean any microscopic entity which possess definable intrinsic characteristics of mass, charge, spin, electric and magnetic moments and space, charge parity. In line with the assumptions of Section 1.2, all "chemical" characteristics, such as isospin, hypercharge, etc., will be ignored. Thus, the only considered physical quantities are those definable in the physical carrier space of the experimental detection per each individual particle.[43] These quantities will be divided into two classes, the intrinsic characteristics indicated earlier and the kinematical characteristics, namely, the kinetic energy, the linear momentum and the angular momentum.

Let us reinspect again the positronium, but this time for the objective of identifying the behaviour of the kinematical and intrinsic characteristics of its constituents.

It is a known fact that the electromagnetic interactions allow exchange of kinematical quantities. As a result, the kinematical characteristics of the individual positronium

constituents (the energy, linear momentum and angular momentum) are generally nonconserved even though, of course, the corresponding total quantities are conserved.

It is an equally known fact that the electromagnetic interactions do not allow the nonconservation of the instinsic characteristics of the individual constituents of the positronium (mass, spin, charge, etc.).

The following picture of a positronium constituent then emerges. It is a particle which possesses nonconserved kinematical characteristics but conserved intrinsic characteristics. In particular, and most importantly, such a constituent (an electron or a positron) is a fermion and remains a fermion during the entire life of the system.

We consider now the π^0 structure as a bound state of a particle and an antiparticle, first from the viewpoint of currently available unitary structure models and realizations of the structure forces as local and derivable from a potential. In this case the constituents are a quark and its antiquark. Our problem is that of identifying the behaviour of the physical characteristics of these constituents. It is an easy exercise for the interested reader to see that, under the indicated assumption on the nature of the structure forces, the quark (or its antiquark) behaves in a dynamically equivalent way than that of the positronium constituents a part, of course, the different values of the intrinsic characteristics. Explicitly, when seen from this profile, the quark emerges as a constituent which, precisely as it is the case for the

positronium constituent, possesses nonconserved kinematical characteristics but conserved intrinsic characteristics. In particular, the quark is a fermion and remains a fermion during the entire life of the system, again, in precisely the same way as that for the positronium constituents.

Our contention, which we shall elaborate on technical grounds at the appropriate time, is that the above dynamical equivalence of a positronium constituent and a quark is a direct consequence of the assumption of the validity of the same relativity and quantum mechanical laws for both structures, the positronium and the π^0.

Restated in different terms, our contention is that, when the strong and electromagnetic interactions are realized in terms of analytically equivalent forces, only differentiated by internal degrees of freedom,[43] their dynamical implications in the physical space are the same. And indeed, by conception, the internal degrees of freedom of the strong forces of currently available models cannot affect, say, the spin of the constituents.

We now pass to the study of the π^0 constituents (also as a bound state of one particle and its antiparticle) under our Assumption 2 on the nature of the structure forces (local and nonderivable from a potential). As recalled in Section 1.2, these forces are purely nonconservative at a Newtonian level. As a result, these forces will also imply the nonconservation of the kinematical characteristics of the individual π^0

constituent. The central point, however, is the behaviour of the intrinsic characteristics. If we assume that these quantities are conserved as for the positronium constituents, we imply a dynamical equivalence between the strong and the electromagnetic interactions. Our contention is that, perhaps, this is not sufficient to produce a genuine departure from the positronium structure for the characterization of the π^0 structure as demanded by the physical differences of these states.

We reach in this way our fundamental

ASSUMPTION 4: We hereby assume that the hadronic constituents are massive, charged and spinning particles for which all physical characteristics of both kinematical and intrinsic nature are generally nonconserved in such a way that the corresponding total quantities are conserved and quantized according to conventional rules.

As we shall see in due time, the assumption of the non-conservation of the intrinsic characteristics of the hadronic constituents, while prohibited by current relativity and quantum mechanical laws, becomes allowed by their Lie-admissible coverings.

In particular, the assumption implies, strictly speaking, that for our hadronic constituent the conventional statistical character is generally lost, namely, if a constituent is a

fermion (or a boson) at a given instant of time it does not preserve such statistical character during the life of the system. This is sufficient to indicate the profound differences between our concept of hadronic constituent and those currently available.

Of course the problem whether this generalized concept of constituent can be technically realized remains to be seen. For the moment, Assumption 4 serves the purpose of identifying one of the central objective of our Lie-admissible approach: the study whether the transition form Lie algebras to Lie-admissible algebras, with particular reference to the SU(2)-spin case, can produce a technical characterization of the indicated type. This problem will be studied in Volumes II and III.

Somewhat reluctantly owing to the recent proliferation of new words in physics, it appears advisable to introduce a new name for the proposed concept of hadronic constituent for both, concise subsequent references as well as for its necessary differentiation with known concepts such as quarks or partons. After a considerable selection and for reasons to be better clarified later on, I hereby submit the term eleton (as a "mutation" of electron). Thus, the term eleton (or antieleton) stands for a massive charged and spinning particle in such highly dynamical conditions to have all its physical characteristics of both kinematical and intrinsic nature generally nonconserved.

The first question which, of course, arises is whether there

exists any physical ground for the existence of such a particle within a hadron. Again, if the hadron is considered as a collection of point particles subject to local forces derivable from a potential, no such particle can exist. However, if the hadrons are considered as a collection of massive physical particles with a charge radius of the order of magnitude of that of the state as a whole, a picture profoundly different than that emerges. In particular, under such peculiar conditions the concept that the constituents of the hadrons preserve unchanged their intrinsic angular momentum for the entire life of the system simply becomes unappealing on physical grounds. We therefore argue that our concept of motion of each constituent within the charge radius of the other provides intriguing grounds for the physical consistency of the concept of eleton.

With reference to an established structure, the positronium, the fact that its constituents preserve their intrinsic characteristics during the life of the system is fully acceptable on intuitional grounds from the very large relative distances as compared to the charge radius of the individual particles. If, however, under the presence of additional forces not derivable from a potential the constituents are forced to coexist each within the charge radius of the other, the same idea becomes simply questionable.

One aspect of the concept of eleton which will turn out to be quite relevant for the actual construction of a structure

model of the hadrons (Volume III) is that its intrinsic characteristics need not necessarily be all nonconserved, as stated in Assumption 4. In essence, the concept of eleton expresses a hierarchy of particles with increasing dynamical conditions corresponding to an increasing complexity of nonconservative forces according to the major possibilities:

(1) All intrinsic characteristics are conserved while only the kinematical characteristics are nonconserved. In this case the statistical character is the conventional one. The particles, however, are still eletons according to our terminology because subject to local forces not derivable from a potential of a simple but genuine nonconservative form.

(2) The intrinsic angular and magnetic moments and the kinematical characteristics are nonconserved while all remaining intrinsic characteristics are conserved. This is the first nontrivial differentiation with the electromagnetic interactions induced by a suitable generalization of the nonconservative couplings of case (1). In this case the eleton acquires a generalized statistical character by therefore rendering mandatory the identification of generalized quantum mechanical formulations (Volume III).

(3) All physical characteristics of both kinematical and intrinsic type are nonconserved. This is, in essence, the dynamical differentiation between the electromagnetic and

the strong interactions brought to its extreme consequences and realized with the most general possible selection of the nonconservative forces. Notice that in this case not only the intrinsic angular momentum but also the charge and the rest energy are expected to be nonconserved.

As we shall see in due time, the above hierarchy of increasing dynamical conditions (which is physically identified with a corresponding hierarchy of generalized forces-within the hadrons- of increasing complexity) can be represented with a corresponding hierarchy of embeddings of the Lie algebra of the selected relativity (Galilei or Einstein) into Lie-Admissible algebras. Since Lie-admissible algebras can be, as a particular case, Lie algebras, when this is the case for the SU(2)-spin subalgebra only of the relativity algebra we have case (1). If this is not the case for the SU(2)-spin subalgebra we have either case (2) or (3) depending on the selected structure of the embedding.

Another aspect of the concept of eleton is that, by construction, the nonconservation of its characteristics must be consistent with the conservation of the corresponding total quantities. As we shall see in Volume III, this property will turn out to be crucial to conform with the current experimental evidence that the physical characteristics of a hadron (as an isolated state) are not only conserved but quantized according to conventional rule. This additional property will be realized through subsidiary constraints at the quantum mechanical level of the description.

In essence, this latter feature is nothing but a generalization of a feature already present at the level of the positronium, only brought to its extreme possibilities. Consider, for instance, the total energy of a positronium. This quantity is, of course, conserved. Such conservation, however, by no means implies that the energy of its individual constituent must be conserved too. On the contrary, to have a non trivial dynamical state, the best occurrence is when the total energy is conserved in such a way to allow energy-exchanges between the constituents. The concept of eleton, say, according to state (2) is simply a generalization of the above occurrence. Here the total physical quantity of a hadron which is conserved and conventionally quantized is the total angular momentum. Such quantity is the sum of the orbital and intrinsic angular momenta of all constituents. The concept of eleton is conceived to represent the most general possible dynamical conditions whereby, even though the total angular momentum is conserved, its components, irrespective of whether of orbital or intrinsic nature, are nonconserved.

In short, the concept of eleton generalizes the nonconservation of the kinematical characteristics of the constituents of a bound state already admitted by electromagnetic interactions to the case of all physical characteristics, including those of intrinsic type but in such a way that the total quantities are conserved. The additional nonconservation of the intrinsic characteristics is considered as the dynamical

representative of our differentiation between the strong and the electromagnetic interactions in the physical space.

On methodological grounds the extension of these non-conservations to the intrinsic characteristics has several implications. First of all it demands an initial generalization of quantum mechanical formulations to account, for instance, for the new statistical character. Secondly, it demands another generalization for the inclusion of subsidiary constraints.

To understand this second occurrence, let us consider, again, the positronium. The system of equations representing this state is a trivially constrained system in the sense that the conservation laws are first integrals of the equations of motion and, as such, they are not subsidiary constraints. The implementation of the positronium structure with forces not derivable from a potential breaks precisely the latter occurrence in the sense that the conservation laws are now no longer first integrals of the equations. As a result, in our approach, they are imposed as subsidiary constraints.

In Section 1.2 we introduced the initial characterization of our concept of hadron consisting of a bound state of massive, charged and spinning physical particles whereby the conventional relativity and quantum mechanical laws are violated by conception for its constituents while, also by conception, no deviation from these laws occurs for the state as a whole under ut most electromagnetic interactions.

In this section we have introduced a more detailed characterization of this concept. In essence, the idea of the eleton stands precisely to characterize in more specific terms the nonapplicability of conventional relativity and quantum mechanical laws within a hadron and the applicability of the same laws for the state as a whole.

This dichotomy of methodological frameworks can be best represented by introducing a dual concept of interactions consisting of:

(I) The closed approach. In this case the hadron is considered as a whole.

(II) The open approach. In this case each individual constituent of a hadron is considered and the rest of the system is assumed as external.

The closed approach is the primary approach to interactions currently used in physics, at the classical and quantum levels. The emphasis is, as is well known, in the conservation of the total quantities. The analytic equations are Lagrange's and Hamilton's equations without external terms. The structure of a Lagrangian or a Hamiltonian is selected in such a way to exhibit non-trivial couplings. The Inverse Problem essentially allows the extension of this concept to our broader hadronic structure (Chapter 2 of this Volume).

The open approach is a complementary concept of interactions. The emphasis is now in the nonconservation of the physical characteristics of the particles because, trivially,

when all its characteristics are conserved a particle is free. The analytic equations are now Lagrange's and Hamilton's equations with external terms. The nontriviality of the interactions is now expressed precisely by these external terms. Besides conventional couplings, this approach is insensitive to the explicit form of the external terms. This allows our realization of the hadronic forces as not derivable from a potential with the dynamical implications indicated earlier. The methodological context is now that of the Lie-admissible problem (Volume II).

For more details on the closed and open approaches to interactions see Appendices 1.A and 1.B.

We are now in a position to better elaborate our concept of dominance of the strong over the electromagnetic interactions. According to the idea of the eleton, such dominance is associated with the degrees of nonconservation of the physical characteristics of the hadronic constituents : the more nonconservations occur, the stronger the interactions are. When the strong interactions produce only the same nonconservations of the electromagnetic interactions, those of the kinematical quantities, the two interactions are dynamically equivalent. To have a nontrivial dominance, the strong interactions should produce additional nonconservations, those of the intrinsic characteristics. This concept of dominance reaches its maximum when all the physical characteristics of the individual constituents are nonconserved.

Another crucial property of the concept of eleton is that, by construction, it does not exist under electromagnetic interactions only. Indeed, an eletonic state, at the limit of null forces not derivable from a potential, reduces to an ordinary particle with forces derivable from a potential (for which the conventional relativity and quantum mechanical laws strictly apply) plus secondary effects to be identified (Vol. III).

By no means this should be construed as to necessarily imply that a physical particle (as currently detected under electromagnetic interactions) cannot acquire an eletonic state after penetration in a hadronic structure. On the contrary, most of our efforts of Volume III are devoted to the study of precisely this possible occurrence.

In essence we argue that if a physical, massive, spinning and charged particle penetrates into an electromagnetic structure, say, the positronium, the particle will not acquire an eletonic state. However, if such a particle penetrates into a hadronic structure the situation can be profoundly different because, again, of the fact that the charge radius of the particle considered is of the same order of magnitude of that of the hadron.

As a result, we argue that any currently known massive particle can exist in a hierarchy of dynamical conditions, the currently detected state under electromagnetic interactions which obey Lie-type disciplines, and our eletonic states under the additional presence of forces not derivable from a potential

(that is, within a hadron) which obey Lie-admissible disciplines. Of course, the period of time for the existence of the eletonic state is of the order of magnitude of the unity, not of our macroscopic world, but instead of the particle world (i.e., extremely small by our standard). Notice that massless particles, photons and neutrinos, are excluded for these occurrences in the sense that our central concern is restricted to the dynamical conditions of only massive particles.

The possibility that a physical particle can exist either in a conventional or in an eletonic state opens intriguing possibilities for the problem of the identification of the hadronic constituents.

In essence, if the conventional relativity and quantum mechanical laws are assumed as applicable for the hadronic constituents, the unitary models of structure are, in my opinion, the only conceivable models at this time. This has a number of consequences. First of all it implies that the hadronic constituents are the yet unknown quarks. Secondly, the difficulties related to the physical identification of these particles force into the yet unsolved problem of confinement. Thirdly, the structure of the strong interactions is restricted to that in terms of local couplings derivable from a potential with the consequential additional problematic aspects, as indicated during the course of our analisis.

If, on the contrary, the conventional relativity and quantum mechanical laws are assumed as inapplicable for the hadronic

constituents, a profoundly different picture emerges. First of all, Assumption 1 on the dichotomy between the established unitary models of classification and new but compatible models of structure becomes admissible. Secondly, the hadronic constituents can now be physical particles, only in new highly dynamical conditions (the eletons). Thirdly, these constituents can now be produced in the spontaneous decays of the (unstable) hadrons by simply performing the transition from an eletonic to a conventional state with the possible emission of massless particles to reacquire conventionally quantized intrinsic characteristics. Fourthly, there is the consequential elimination of any need of confinement. And last but not least, there is the possibility of reexamining the use of perturbation techniques owing to the emerging broader structure of the Hamiltonian and the not necessary need of high values of coupling constant.

These occurrences can be expressed with our last and, perhaps, most important

ASSUMPTION 5: We hereby assume that the unstable hadrons are an eletonic bound state of suitably selected massive, physical, particles produced in their spontaneous decays.

The central requirement of consistency is here that the particles must be in an eletonic condition. And indeed, as we shall see in more details in Volume III, the above assumption is flatly inconsistent if attempted within the context of established relativity and quantum mechanical laws.

As we shall see in details in Volumes II and III, Assumption 5 ultimately implies that the massive, charged and stable constituents of the hadrons are the ordinary electrons and positrons, but not in their currently known state under electromagnetic interactions, and instead in an eletonic state belonging to a new Lie-admissible mechanics. After all, besides the proton, the electrons and positrons are the only stable particles produced either directly or indirectly in the spontaneous decays of hadrons and, as such, they are the only stable, physically known candidates conceivable at this time.

Therefore, our Lie-admissible approach to the hadronic structure might allow the reduction of the physical universe to only three stable particles: the electron, the photon and the neutrino. This reduction, which is of course prohibited by the currently established Lie disciplines, has such aestetical, conceptual and technical implications to support a study of our Lie-admissible approach to the hadronic structure.

We concluded Section 1.2 with the remark that, perhaps, the established relativity and quantum mechanical laws are the major obstacle for a genuine dynamical differentiation between the electromagnetic and the strong interactions. We can conclude this section by adding that, perhaps, the same laws are also the major obstacle for the identification of the hadronic constituents with physical particles.

1.4: THE FUNDAMENTAL EQUATIONS OF MOTION IN THEIR NEWTONIAN FORM

In order to reduce the fundamental assumptions of the preceding sections to a mathematical language confrontable with the physical reality, several sequential steps will be implemented. First of all, we shall identify the purely Newtonian form of the equations of motion. This is due to several reasons. First of all, as recalled earlier, Newtonian Mechanics constitutes the arena of our best intuitions. The nonconservative nature of the assumed hadronic forces can therefore be subject to a first verification. Secondly, the identification of the Newtonian equations of motion will allow us to identify the classical methods for their treatement. Clearly, this is a necessary prerequisite to the problem of quantization. Thirdly, the Newtonian form of the equations of motion constitutes a reliable basis for its extension to the field theoretical form. A precise knowledge of the mechanisms of nonconservative processes at a purely Newtonian level will then prove to be invaluable for the identification of the corresponding mechanisms in field theory, due to the predictable increase of technical difficulties and decrease of our intuitional capabilities.

The Newtonian form of the equation of motion will be identified in this section. Its field theoretical version will be pointed out in the subsequent section. All the needed

methodological aspects, beginning from the methods to construct a Lagrangian, will be studied thereafter.

There is here a tacit departure from conventional approaches which is important to point out. The starting point of virtually all currently available models is a Lagrangian. The equations of motion are usually ignored because trivially deducible from Lagrange's equations. This approach is not only inapplicable for our program but potentially misleading. It is inapplicable because the acting forces are not derivable from a potential by assumption and, therefore, the problem of the relationship between the equations of motion and their Lagrangian representations is by far nontrivial. It is potentially misleading because, once a Lagrangian for the considered broader class of systems is constructed the implications of the nonconservative nature of the couplings can be more effectively identified at the level of the equations of motion, particularly from the viewpoint of the problem of the physical significance of quantities which are either conserved or not. There are, however, other reasons which suggest the above indicated inverted attitude (identification of the equations of motion first, and then study of their Lagrangian representations). They are related to the fact that the methodology which we shall introduce for the construction of the Lagrangian representations (the Inverse Problem) will not produce only one Lagrangian, but often a family of equivalent Lagrangians which, due to

the novelty of the methods, are not related by conventional means (e.g., an additive function with null Lagrange's derivative). In essence, to recover the physical content of given equations of motion we will be forced to consider all the Lagrangians which produce equivalent self-adjoint forms of the equations of motion, rather than one individual Lagrangian, as customarily done nowadays.

The approach which is here advocated is therefore a return "ad originem": the equations of motion with arbitrary couplings are assumed as the fundamental physical quantities, while the class of equivalent Lagrangians for their analytic representations is assumed as a derived concept of primarily methodological significance.

Let us begin from established grounds. The positronium structure, at its Newtonian limit, is a two-body Coulomb problem in vacuum, as recalled in Appendix 1.A. In their simplest possible form, the equations of motion are

$$m_k \ddot{\underset{\sim}{r}}_k - \underset{\sim}{f}_k(\underset{\sim}{r}) = 0 \,, \quad k=1,2 \,, \; m_1=m_2 \qquad (1.4.1a)$$

$$\underset{\sim}{f}_k = -\frac{\partial V(r)}{\partial \underset{\sim}{r}^k} \,, \quad \underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2 \,, \quad V = V_{coul.} \,. \qquad (1.4.1b)$$

Our first step is that of implementing the above equations into the most general form with forces derivable from a potential, i.e.

$$m_k \ddot{\underset{\sim}{r}}_k - \underset{\sim}{f}_k(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = 0, \quad \underset{\sim}{f}_k \in C^\infty \qquad (1.4.2a)$$

$$\underset{\sim}{f}_k = -\frac{\partial U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial \underset{\sim}{r}_k} + \frac{d}{dt}\frac{\partial U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial \dot{\underset{\sim}{r}}_k}, \qquad (1.4.2b)$$

in order to include the more general Lorentz force.

Next, we shall implement the above equations with the most general form of additive forces $\underset{\sim}{F}_k$ which, by assumption, are local but not derivable from a potential and write [44]

$$\left\{\left[m_k \ddot{\underset{\sim}{r}}_k - \underset{\sim}{f}_k(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\right]_{pos.} - \underset{\sim}{F}_k(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})\right\}_{\Pi^0} = 0, \qquad (1.4.3a)$$

$$\underset{\sim}{F}_k \in C^\infty;$$

$$\underset{\sim}{F}_k \neq -\frac{\partial U'(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial \underset{\sim}{r}_k} + \frac{d}{dt}\frac{\partial U'(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})}{\partial \dot{\underset{\sim}{r}}_k}. \qquad (1.4.3b)$$

The above equations are assumed as a primitive, yet incomplete, Newtonian limit of the structure of the Π^0. Eqs. (1.4.3) contain the Coulomb coupling as a subcase, while they express the most general local form of a nonconservative Newtonian two-body problem. Notice the additive nature of the forces $\underset{\sim}{f}_k$ and $\underset{\sim}{F}_k$. This feature will later on result to be crucial for the identification of the arenas of applicability of known methods and their generalization, as well as for the identification of the limit prodedure which reduces the assumed structure to one of atomic type.

In essence, by keeping in mind the remarks of Section 1.2 on the need to explore other alternatives, the terms within square brackets of Eqs. (1.4.3a) will result as the arenas of applicability of known disciplines, while the full system (1.4.3a), due to the forces not derivable from a potential, will

result as the arena of nonapplicability of the same disciplines. As a matter of fact, most of our subsequent efforts will be devoted precisely to the identification of the rudiments of the generalizations which are needed to treat this broader type of physical system.

Our immediate problem is that of completing the identification of the primitive Newtonian form of our hadronic model. This demands the identification of the implications of the nonconservative force. This problem has a known solution. Most notably, forces not derivable from a potential, irrespective of whether with or without an explicit time dependence, imply the nonconservation of the total physical energy

$$\mathcal{E}_t = T + U = \frac{1}{2}\sum_{k=1}^{2} m_k \dot{\underset{\sim}{r}}_k^2 + U \neq const. \qquad (1.4.4)$$

Secondly, the same forces generally imply the nonconservation of the total linear momentum

$$\underset{\sim}{P}_t = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 = \sum_{k=1}^{2} m_k \dot{\underset{\sim}{r}}_k \neq const. \qquad (1.4.5)$$

Thirdly, along the same lines, the forces $\underset{\sim}{F}_k$ generally imply the nonconservation of the total angular momentum and the lack of uniform motion of the center of mass, i.e.,

$$\underset{\sim}{M}_t = \underset{\sim}{M}_1 + \underset{\sim}{M}_2 = \sum_{k=1}^{2} \underset{\sim}{r}_k \times \underset{\sim}{p}_k \neq const, \qquad (1.4.6a)$$

$$\underset{\sim}{G}_t = \sum_{k=1}^{2} m_k \dot{\underset{\sim}{r}}_k - \underset{\sim}{P}_t \neq const. \qquad (1.4.6b)$$

The verification of these statements is trivial. Conserved quantities are conserved in virtue of the equations of motion. Thus, the problem of their analytic representation can be here ignored. The nonconserved nature of quantities (1.4.4), (1.4.5) and (1.4.6) can be proved by simply computing their time derivative and by using the equation of motion (1.4.3a). This completes the identification of the physical implications of nonconservative forces which is needed at this point.

To better characterize the nonconservative forces we are referring to, let us recall that these forces do guarantee the nonconservation of the total energy, but this is not necessarily the case for the remaining quantities (1.4.5) and (1.4.6). By recalling our intent of bringing the nonconservative nature of the systems considered to its extreme methodological consequences, we here assume that for given forces $\underset{\mu}{f}_k$ derivable from a potential, the additive forces $\underset{\mu}{F}_k$ not derivable from a potential are selected in such a way to guarantee the nonconservation of all physical quantities (1.4.4), (1.4.5) and (1.4.6). Notice that these quantities are total quantities. The nonconservation of their individual components is then guaranteed in line with the assumptions of Section 1.3. In short, all physical characteristics of the two-body conservative problem reviewed in Appendix 1.A are assumed to be nonconserved for its generalization (1.4.3).

Within a purely Newtonian context this type of generalized two-body problem is fully consistent with the physical reality.

Consider, for instance, a two-body central force problem in a physical medium. This necessarily implies the presence of drag forces which are not derivable from a potential. An inspection then reveals that all physical characteristics of this generalized two-body problem are in this case nonconserved.

Despite this Newtonian self-consistency, Eqs. (1.4.3) are insufficient to express the primitive concept of eleton as introduced in Section 1.3. This is due to the fact that, in order to represent the π^o, a bound state of an eleton and an antieleton must necessarily be such that the total physical characteristics are conserved.

Our third and final step is then that of implementing Eqs. (1.4.3) with a system of subsidiary constraints which serve precisely the purpose of inducing the conservation of the total physical characteristics. We reach in this way the following primitive Newtonian form of the equations of motion for a bound state of an eleton and antieleton

$$\left\{ \left[\mu_k \ddot{\underset{\mu}{x}}_k - \underset{\mu}{f}_k(t, \underset{\mu}{x}, \dot{\underset{\mu}{x}}) \right]_{pos.} - \underset{\mu}{F}_k(t, \underset{\mu}{x}, \dot{\underset{\mu}{x}}) \right\}_{\pi^o} = 0, \quad k = 1,2, \tag{1.4.7a}$$

$$\dot{\mathcal{E}}_t = 0, \qquad \dot{\underset{\mu}{\mathcal{P}}}_t = 0, \tag{1.4.7b}$$

$$\dot{\eta}\dot{\underset{\mu}{m}}_t = 0, \qquad \dot{\underset{\mu}{\mathcal{G}}}_t = 0, \tag{1.4.7c}$$

$$\phi_s(t, \underset{\mu}{x}, \dot{\underset{\mu}{x}}) = 0, \quad s = 1,2,\ldots, \tag{1.4.7d}$$

$$\underset{\mu}{f}_k = -\frac{\partial U}{\partial \underset{\mu}{x}^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\underset{\mu}{x}}^k}, \tag{1.4.7e}$$

$$\underset{\mu}{F}_k \neq -\frac{\partial U'}{\partial \underset{\mu}{x}^k} + \frac{d}{dt}\frac{\partial U'}{\partial \dot{\underset{\mu}{x}}^k}, \quad \underset{\mu}{F}_k \in \mathcal{C}^\infty \tag{1.4.7f}$$

where Eqs. (1.4.7a) are the fundamental nonconservative equations of motion, Eqs. (1.4.7b) and (1.4.7c) are the subsidiary constraints for the conservation of the total physical characteristics of the considered system, and Eqs. (1.4.d) is a set of possible additional constraints, in case needed (e.g., for convergence arguments?).

We are now in a position to elaborate on some of the central features of the proposed hadronic model already, and perhaps in a more intuitively effective way, at this purely Newtonian level.

First of all, by construction, Eqs. (1.4.7a) violate the entire symmetry under the Galilei group. This fundamental symmetry, however, is not "destroyed". On the contrary, its physical content is established in full via the imposition of the fundamental conservation laws (1.4.7b)-(1.4.7c) of the two-body system considered as a whole.

In this dualism between the breaking of the Galilei symmetry at the level of the equations of motion and the preservation of its physical content for the system considered as a whole rely precisely the primitive essence of the proposed concept of hadron. Specifically, if the physical content of the Galilei symmetry, namely, the conservation laws (1.4.7b)-(1.4.7c), is introduced, as nowadays customary, through the condition of form-invariance of the equations of motion and, thus, of their Lagrangian representation, the net effect is a substantial

restriction on the allowable form of couplings. Indeed, trivially, this condition of Galilei form-invariance is a condition on the acting forces $\underset{\sim}{F}_k$. As a result, in no way known to me the condition of Galilei symmetry allows the incorporation of couplings which are more general of the Coulomb (or Lorentz) coupling to any significant extent. With a minimum of vision, the reader can now see that the extrapolation of the Galilei invariance at a quantum mechanical level implies the possible lack of an effective departure from the positronium structure which is actually needed to characterize the π^0 structure.

Restated in few words, the breaking of the Galilei symmetry at the level of the equations of motion appears to be a prerequisite for the very existence of nontrivial generalizations of the Coulomb (and Lorentz) force. If, on the contrary, the Galilei symmetry is broken by assumption by the couplings of a two-body system and its physical content recovered through subsidiary constraints, more general forces are allowed and the constituents of such two-body system emerge as being in a highly dynamical (i.e., nonconservative) condition.

The reader must be here alerted that the presence of (non-integrable) subsidiary constraints implies a conspicuous broadening of the methodological context, namely, they extend the context of the theory of ordinary differential equations to that of systems with constraints. The reader should also be

alerted that, to my best knowledge at this time, this type of systems does not appear to be treated in currently available literature in physics, although the methodology for its treatment is indeed available from mathematical disciplines, most notably, the Problem of Bolza of the Calculus of Variations,[45] or, more generally, the Optimal Control Theory.[46] Several readers might be surprised at this statement because a theory of systems with subsidiary constraints, Dirac's theory, is fully known in physical literature. The point is that a prerequistie for the applicability of Dirac's theory[36] is that the Lagrangians be degenerate while the Lagrangians for model (1.4.7) will result to be regular and to remain so also on the hypersurface of the subsidiary constraints. As a result, Dirac's theory of systems with constraints does not directly apply to the proposed model. The reader should be finally alerted that the presence of subsidiary constraints creates the problem of consistency, and thus the existence of physically meaningful solutions. More specifically, when a system of second order ordinary differential equations, such as Eqs. (1.4.7a), is consistent, it is not necessarily so when subject to a system of constraints. This is a potential trap which is common to all theories of systems with subsidiary constraints, including Dirac's theory. More explicitly, the point is here that, unless the consistency problem is confronted, one risks to conduct elaborate treatments of equations which might be vacuous on physical grounds because inconsistent.

To formally complete our Newtonian model, we therefore tacitly imply from here on that the nonconservative couplings, besides satisfying the previously identified requirements, also satisfy the consistency conditions, namely, they characterize a system of equations expressing a physically meaningful motion.

The problem of the explicit functional dependence of the $\underset{\sim}{F}_k$ forces will be considered later on. An example of a consistent generalized two-body problem of type (1.4.7) is given in Appendix 1.C where the methodology of the problem of consistency is also briefly outlined.

1.5: THE FUNDAMENTAL EQUATIONS OF MOTION IN THEIR FIELD THEORETICAL FORM

The fundamental assumption of the eleton field is that i does not transform covariantly under the Poincaré group.

After the analysis of Chapter 3, this assumption will in actuality result to be a consequence of the strong interactions as not derivable from a potential. At this stage of our analysis we assume it for the specific intent of studying the utmost possible differentiation between the electromagnetic and strong interactions. The argument is that the Poincaré covariance would imply the conservation of the intrinsic characteristics of the eleton by therefore rendering the electromganetic and strong interactions as formally equivalent from a dynamical profile.

Let us begin from established grounds. Consider again the simplest bound state produced by the electromagnetic interactions, the positronium. Its field equation, in their simplest form, can be written

$$\begin{pmatrix} -\bar{e}^{\,;}_{\mu}\gamma^{\mu} \\ \gamma^{\mu} e^{;}_{\mu} \end{pmatrix} + m \begin{pmatrix} \bar{e} \\ e \end{pmatrix} + \begin{pmatrix} f_{\bar{e}} \\ f_{e} \end{pmatrix}_{elm} = 0, \quad e^{;}_{\mu} = \frac{\partial e}{\partial x^{\mu}}, \tag{1.5.1a}$$

$$f_{e} = ie\gamma^{\mu} A_{\mu} e(x), \quad f_{\bar{e}} = ie\bar{e}(x)\gamma^{\mu} A_{\mu} \tag{1.5.1b}$$

$$e'(x') = S(\Lambda) e(x), \quad \Lambda \in L^{\uparrow}_{+}, \tag{1.5.1c}$$

where e is the conventional Dirac's field representative of the electrons, $\bar{e}$ is its adjoint and the f-coupling, of course,

is the Lorentz coupling. These fields, as conventional, transform covariantly under the representation $D^{(\frac{1}{2}0, 0\frac{1}{2})}$ of the Lorentz group, Eqs. (1.5.1c). This, in turn, not only characterizes the spin of the Dirac's field, but also the structure of the γ-matrices. The Lagrangian is the familiar form

$$\mathcal{L}_{tot} = \mathcal{L}_{free} + \mathcal{L}^{elm}_{int} = \left[\frac{1}{2}\left(\bar{e}^{\,;}_{\mu}\gamma^{\mu} e - \bar{e}\gamma^{\mu} e^{;}_{\mu}\right) - m\bar{e}e\right]_{free} + \left[-ie\bar{e}\gamma^{\mu} A_{\mu} e\right]^{elm}_{int} \tag{1.5.2a}$$

$$f_{\bar{e}} = -\frac{\partial \mathcal{L}^{elm}_{int}}{\partial e}, \quad f_{e} = -\frac{\partial \mathcal{L}^{elm}_{int}}{\partial \bar{e}}, \tag{1.5.2b}$$

and it is invariant not only under the (identity component of the) Poincaré group, but also under the discrete transformations P, C and T, as well of course (local and global) gauge transformations.

Since the coupling is derivable from an additive term to the free term (according to Definition 1.A.2), the Poincaré symmetry possesses its full physical significance as representative of physical conservation laws. Specifically, the invariance of Lagrangian (1.5.2) under the group of translations implies, via Noether's theorem, the existence of a rank two tensor, the canonical tensor

$$\Theta^{\mu\nu}_{e} = \frac{\partial \mathcal{L}_{tot}}{\partial \varphi^{k;}_{\mu}} \pi^{k;\nu} - g^{\mu\nu}\mathcal{L}_{tot}, \quad \{\varphi^{k}\} = \{e, \bar{e}\} \tag{1.5.3}$$

which is conserved. The physical effectiveness of this mathematical property rests on the fact that this canonical tensor is the physical energy-momentum tensor. Similarly, the

invariance of Lagrangian (1.5.2) under the (identity component of the) Lorentz group implies the existence of a rank three tensor (density)

$$\mathcal{M}_{e}^{\mu\nu\sigma} = x^{\mu}\theta_{e}^{\nu\sigma} - x^{\nu}\theta_{e}^{\mu\sigma} \qquad (1.5.4)$$

which is also conserved. Since this tensor can be proved to represent the physical total angular momentum, this Lorentz symmetry exhibits the direct physical significance of being representative of the physical conservation law of the total angular momentum.

The transition from the Newtonian model (1.4.1) to model (1.5.1) is therefore of dual nature. First it expresses the transition from a non-relativistic to a relativistic framework and, secondly, it expresses the transition from a discrete to a continuous framework. Jointly, all the conservation laws of a two-body central force problem in vacuum under the Galilei symmetry perform the transition to the tensorial conservation laws (1.5.3) and (1.5.4).

By using again Newtonian Mechanics as the arena of our intuitions, we now implement Eqs. (1.5.1) with an additive coupling which, by assumption, is local but not derivable from a potential and write

$$\begin{pmatrix} -\bar{e}^{\,i}_{\mu}\gamma^{\mu} \\ \gamma^{\mu} e^{i}_{\mu} \end{pmatrix} + m \begin{pmatrix} \bar{e} \\ e \end{pmatrix} + \begin{pmatrix} f_{\bar{e}} \\ f_{e} \end{pmatrix} + \begin{pmatrix} F_{\bar{e}} \\ F_{e} \end{pmatrix} = 0 \,, \qquad (1.5.5a)$$

$$F_{\alpha} \neq -\frac{\partial \mathcal{L}_{int}}{\partial \beta} + \frac{d}{dx^{\mu}} \frac{\partial \mathcal{L}_{int}}{\partial \beta^{i}_{\mu}} \,, \quad \alpha, \beta = e, \bar{e} \,. \qquad (1.5.5b)$$

pending the identification of methods for the technical characterization of couplings nonderivable from a potential (Chap.2).

The Newtonian implications of the presence of the nonconservative force $F_{\mu k}$ in model (1.4.3) directly extend to the above field theoretical context. Indeed, the presence of the "nonconservative" coupling (F_e, $F_{\bar{e}}$) in Eqs. (1.5.5) directly implies the nonconservation of the canonical tensor (1.5.3) and the angular momentum tensor (1.5.4). To verify this property there is no need of the knowledge of a Lagrangian density for Eqs. (1.5.5). Irrespective or whether at a Newtonian or a field theoretical level, conserved quantities are conserved in virtue of the equations of motion. The computation of the 4-divergences of tensors (1.5.3) and (1.5.4) and use of Eqs. (1.5.5) then proves the property. The interested reader is here suggested to verify that this property also holds when the nonconservative couplings do not explictly depend on the Minkowski coordinates.

At this point the analogy with the Newtonian formulation of the model ends. Indeed, the implementation of Eqs. (1.5.5) along the lines of model (1.4.7) would be insufficient to characterize the strong interactions according to our assumptions. Suppose that one simply implements Eqs. (1.5.5) with Eqs. (1.5.3) and (1.5.4) interpreted as subsidiary constraints and select the nonconservative couplings in such a way that the overall system is consistent. Then the methodological gain over model (1.5.1) would be mainly formal for the following reasons. As recalled in Section 1.3, it is

an experimentally established fact that electromagnetic interactions allow exchanges of energy, momentum and angular momentum. As a result, the nonconservation of these quantities at the level of the individual constituents is already present in the positronium. If the fields of the generalization of this structure

$$\begin{cases} \begin{pmatrix} -\bar{e}_{;\mu}\gamma^{\mu} \\ \gamma^{\mu} e_{;\mu} \end{pmatrix} + m \begin{pmatrix} \bar{e} \\ e \end{pmatrix} + \begin{pmatrix} f_{\bar{e}} \\ f_{e} \end{pmatrix}_{elm} + \begin{pmatrix} F_{\bar{e}} \\ F_{e} \end{pmatrix} = 0, \\ \theta^{\mu\nu}_{e\ ;\mu} = 0, \qquad M^{\mu\nu\sigma}_{e\ \ ;\mu} = 0, \\ e'(x') = S(\Lambda)\, e(x), \end{cases} \tag{1.5.6}$$

continue to transform covariantly under the Poincaré group in precisely the same form as that of the fields of Eqs.(1.5.1), the dynamical implications of models (1.5.6) and (1.5.1) would be essentially the same. The former model would be nothing but a more complex setting for producing the nonconservations of only the energy, linear momentum and angular momentum of the individual constituents of the latter model. The point is that the intrinsic characteristics of the constituents would be conserved in both models and our assumptions would be unrealized.

According to our hypothesis that the strength of an interaction is proportional to the degrees of nonconservation of the physical characteristics of the individual constituents of a

bound state, model (1.5.6) would not charaterize the presence of an interaction which is "stronger" than that of model (1.5.1). In turn, a model of the π° particle based on Eqs. (1.5.6) would not imply a physically significant departure from the positronium structure (1.5.1). In the language of this paper we can say that Eqs. (1.5.6) do not characterize the bound state of an eleton and an anti-eleton.

After these preliminary considerations we are now in a position to present the proposed simplest possible form of the field equations for the structure of the π° interpreted as a bound state of an eleton and an antieleton. It is given by the following Poincaré symmetry breaking generalization of the Dirac's equations for the positronium

$$\begin{cases} \left\{ \left[\begin{pmatrix} -\hat{\bar{e}}_{;\mu}\gamma^{\mu} \\ \gamma^{\mu}\hat{e}_{;\mu} \end{pmatrix} + m \begin{pmatrix} \hat{\bar{e}} \\ \hat{e} \end{pmatrix} + \begin{pmatrix} f_{\hat{\bar{e}}} \\ f_{\hat{e}} \end{pmatrix}_{elm} \right]_{positronium} + \begin{pmatrix} F_{\hat{\bar{e}}} \\ F_{\hat{e}} \end{pmatrix}_{strong} \right\}_{\pi^{\circ}} & (1.5.7a) \\ \theta^{\mu\nu}_{e\ ;\mu} = 0, & (1.5.7b) \\ M^{\mu\nu\sigma}_{e\ \ ;\mu} = 0, & (1.5.7c) \\ \phi_s(x^{\mu}, \hat{e}_{\mu}, \hat{e}_{\mu;\mu}, A^{\mu}, A^{\mu}_{;\alpha}) = 0, \quad s = 1,2,\ldots & (1.5.7d) \\ m \neq 0, \quad \hat{e}_{\mu} = (\hat{e}(x), \hat{\bar{e}}(x)), \quad \mu,\nu,\sigma = 0,1,2,3, & (1.5.7e) \\ f_{\hat{e}} = ie\gamma^{\mu}A_{\mu}\hat{e}, \quad f_{\hat{\bar{e}}} = ie\hat{\bar{e}}\gamma^{\mu}A_{\mu} & (1.5.7f) \\ \hat{e}'(x') \neq S(\Lambda)\hat{e}(x), \quad x' = \Lambda x, \quad \Lambda \in L_{+}^{\uparrow} & (1.5.7g) \\ \lim \hat{e}(x) = e(x), \quad e'(x') = S(\Lambda)e(x), & (1.5.7h) \\ F_{strong} = 0 & \\ f_{\hat{e}} = -\frac{\partial \mathcal{H}^{elm}_{int}}{\partial \hat{\bar{e}}}, \quad f_{\hat{\bar{e}}} = -\frac{\partial \mathcal{H}^{elm}_{int}}{\partial \hat{e}}, \quad F_{\alpha} \neq -\frac{\partial \mathcal{H}^{strong}_{int}}{\partial \beta} + \frac{d}{dx^{\mu}}\frac{\partial \mathcal{H}^{strong}_{int}}{\partial \beta^{\mu}_{;}} & (1.5.7i) \\ \alpha,\beta = \hat{e}, \hat{\bar{e}} & \end{cases}$$

where: (1) the symbol $\hat{\underset{\sim}{e}}_{\mu} = (\hat{e}, \hat{\vec{e}})$ is representative of the eleton-antieleton field, (2) the field equations (1.5.7a) break the Poincaré symmetry in the sense that the fields do not transform covariantly, Eqs. (1.5.7g), (3) the physical content of the Poincaré symmetry is recovered for the π^o as a whole through the subsidiary constraints (1.5.7b) and (1.5.7c), (4) the additional constraints (1.5.7d) represent the conservation of the total charge as well as any needed additional condition, (5) by assumption, the strong couplings are not derivable from a potential and possess, in general, a dependence on all fields, their first order derivatives as well as, if needed, the Minkowski coordinates, (6) the electromagnetic coupling is the conventional coupling, and (7) at the limit of null strong couplings, Eqs. (1.5.7) recover the equations of the positronium (1.5.1) in full, with the presence of subsidiary constraintes being redundant because guaranteed by the Poincaré and gauge covariance of these equations. The eleton field then reduces to the electron fields, Eq. (1.5,7h).

A few comments are here in order. First of all it should be stressed that we are not in a position to identify at this time the explicit form of the strong couplings. Clearly this selection must be based on inductive phenomenological arguments. But any attempt to realize this program at this point might result later on ineffective because of the lack of the necessary methodology at this time to treat the equations

considered. The criteria for the selection of an explicit form of the strong couplings will then first be of methodological nature and then of phenomenological character.

The reader should be alerted since now that, despite its appearance, the structure of the Dirac's equations as currently known is generally lost for Eqs. (1.5.7a). This is due to the fact the strong couplings will emerge as possessing an essential dependence on the derivatives $(\hat{\vec{e}}^{\,i}_{\mu}, \hat{e}^{\,i}_{\mu})$. As a result, the Dirac's terms $(\hat{\vec{e}}^{\,i}_{\mu}\gamma^{\mu}, \gamma^{\mu}\hat{e}^{\,i}_{\mu})$ loose their dominant role for the characterization of the "velocities". As a matter of fact, rather than the ordinary gamma matrices, when explicitly computed, Eqs. (1.5.7a) result to be in principle characterized by a <u>generalization of the gamma matrices</u> originating from the additive contribution to the derivative terms from the strong couplings. Thus, Eqs. (1.5.7) constitute a "bona fide" generalization of Dirac's equations. The additive nature of the strong couplings, however, allows the possibility of recovering in full, as indicated earlier, the conventional Dirac's equations at the limit (1.5.7h).

The comparative analysis of the proposed model on the structure of the π^o, Eqs. (1.5.7), and the established model on the structure of the positronium, Eqs. (1.5.1), indicate that the former has been constructed as a profound "mutation" of the latter in order to attempt the representation of the profound physical differences between these structures.

This approach also explains the reason for the selected name for the π^o constituents. Indeed, the term "eleton" has

been conceived as a mutation of the term "electron" in this sense.

Several generalizations of model (1.5.7) are conceivable. The most important ones are:

(a) the generalizations through the use of Bethe-Salpeter type equations,

(b) the generalizations to integro-differential equations, and

(c) the generalization to second order equations which recover from (1.5.1) at the limit of null strong couplings.

These generalizations will not be considered at this time.

It should be stressed that model (1.5.7) is here introduced with the specific intent of bringing the implications of the assumed nonconservative nature of the strong couplings to their extreme methodological consequences on grounds which, at the level of these first steps, are mainly conceptual. Thus, model (1.5.7) must be interpreted to be of purely conjectural nature and of yet unassessed physical significance.

APPENDIX 1.A:

THE CONVENTIONAL CLOSED APPROACH TO INTERACTIONS

The conventional approach to the problem of Newtonian interactions can be expressed in terms of the

DEFINITION 1.A.1.: A system of Newtonian particles is interacting when the total Lagrangian is characterized by the kinetic energy L_{free} and a nontrivial additive interaction term L_{int}.

The following comments are in order. The concept of Newtonian "particle" is that of a massive point. An additive interaction term L_{int} is "nontrivial" when the Lagrange's equations in L_{free} and those in

$$L_{tot} = L_{free} + L_{int} \qquad (1.A.1)$$

are not equivalent (to exclude possible equivalence transformations of L_{free}). The term L_{int} is "additive" to the free term in the sense of structure (1.A.1) and it generally depends explicitly on time, coordinates and velocities to allow the representation of, say, the Lorentz force. The quantity $U = -L_{int}$ is the conventional generalized potential of Newtonian Mechanics. The approach is here called "closed" to distinguish it from the complementary approach of the next appendix and to denote the fact that the system is represented in its entirety with one single function, the Lagrangian (or the Hamiltonian) within the context of the conventional analytic equations.

The central aspect which is here relevant is that Definition 1.A.1 with inderlying structure of the Lagrangian (1.A.1) represents a Newtonian system with forces derivable from a potential. It is here tacitly implied that each of the two terms of the total Lagrangian is restricted to its corresponding physical significance indicated above. Indeed, as we shall see later on, if such restriction is lifted, Lagrangian (1.A.1) can represent a system with arbitrary forces in view of the arbitrary functional dependence of the second term.

Definition 1.A.1 characterizes the primitive contemporary concept of interactions. Its extension to classical fields is trivial and reads

DEFINITION 1.A.2: A systems of fields is interacting when the total Lagrangian density is characterized by a term $\mathcal{L}_{free}$ representing the free motion and a nontrivial additive interaction term $\mathcal{L}_{int}$.

The total Lagrangian density now has the known structure

$$\mathcal{L}_{tot} = \mathcal{L}_{free} + \mathcal{L}_{int}. \qquad (1.A.2)$$

If one replaces the terms "Newtonian particles", "Lagrangian" and "Hamiltonian" with "fields", "Lagrangian density" and "Hamiltonian density", respectively, all the previously given definition apply in full. The approach is still closed in the sense indicated above and is restricted, from precise physical

interpretations of the two terms of structure (1.A.2) to represent only fields with couplings derivable from a potential. Of course now $\mathcal{L}_{int}$ is generally restricted to depend explicitly on the fields and their first order partial derivatives and not on the space-time coordinates. This is typical for the transition from the Newtonian to the field theoretical formulation of, say, the Lorentz force where, at a field theoretical level, the dependence of the electromagnetic potential on the space-time coordinates becomes implicit for $\mathcal{L}_{int}$. Although the above definition is more general (i.e., it can represent field equations of elliptic or parabolic type), we shall tacitly assume from now on that it is embodied in a (3+1)-Minkowski space (i.e. it represents field equations of hyperbolic type).

Definition 1.A.2 characterizes the central concept of interaction for virtually all currently available models of classical or quantum field theory, thus, including the current approaches to the structure of the hadrons. The extension from classical to quantum field theory simply demands a reinterpretation of the fields as operator valued distributions, but the concept of interactions according to Definition 1.A.2 applies in full.

One of the most aesthetically appealing, mathematically synthetic and physically effective methodological contexts of the concept of interactions according to Definition 1.A.2 is that of symmetries and conservation laws. Notably, the sym-

metries of the total Lagrangian (1.A.2) characterize, via the Noether theorem, conserved quantities which represent physical laws. In this way, elemental aspects of the considered physical reality are reduced to primitive mathematical concepts.

This synthesis between mathematical tools and the physical reality is possible because, for the considered structure (1.A.2) of the total Lagrangian density under the above specified physical interpretation of its terms, the conserved quantities generated by the symmetries via Noether's theorem possess a direct physical interpretation. Typically, the lack of explicit dependence of Lagrangian (1.A.2) on the Minkowski coordinates implies the invariance of the same under translations in space-time. In turn this symmetry implies the existence, via Noether's theorem, of a rank two tensor, the canonical tensor, which is conserved. Finally, this tensor results to be the physical energy-momentum tensor. A similar situation occurs for all other (connected) symmetries, most notably, the symmetry under the homogeneous Lorentz group.

The symmetries of a microscopic physical system in the closed approach which are significant in contemporary theoretical physics are of several types. Customarily, they are classified into the following four classes with indicated physical significance:

(1) connected space-time (Poincaré) transformations, with related conservation laws of the energy-momentum and angular momentum tensors,

(2) permutation of the coordinates, with related Bose-Einstein and Fermi-Dirac statistics,

(3) discrete transformations of space-time type, i.e. P, T and PT, or of internal type, e.g., C or G-parity, or of mixed space-time and internal type, e.g., PCT,

(4) connected internal transformations, most notably, of unitary type such as the U(1) symmetry for the conservation of the (total) charge, or family number, the SU(2)-isospin symmetry, the SU(3) and SU(3)-color symmetries used in the quark models, etc.

At present, the symmetries of class (1) and (2) are believed to be exact, namely, verified in nature, and those of class (3) and (4) are considered to be generally broken with the most significant possible exception given by PCT.

The present situation of symmetries and conservation laws is summarized in Table 1.A.1.

For subsequent need, it should be here stressed that the conserved quantities of Table 1.A.1 are total quantities and that the table is specifically intended only for interactions derivable from a potential function, namely those according to Definition 1.A.2. It should be also recalled that the data of Table 1.A.1 have been established from a rather considerable amount of experimental evidence, as available at this time.

Notice that the symmetries of the currently available models on the structure of the hadrons are broken at the level of all interactions.

The extension of this closed approach to interactions for the case of forces not derivable from a potential will be introduced in Chapter 2.

Total characteristics of closed approach	Interactions	
	Strong	Electromagnetic
Energy	conserved	conserved
Linear momentum	conserved	conserved
Angular momentum	conserved	conserved
charge	conserved	conserved
Barionic number	conserved	conserved
Parity	conserved	conserved
Charge conjugation	conserved	conserved
Time inversion	conserved	conserved
PCT	conserved	conserved
Isospin	conserved	violated
Strangeness	conserved	conserved
SU(3)-symmetries	violated	violated

Table 1.A.1: The current knowledge on the preservation or violation of physical characteristics or symmetries, under strong and electromagnetic interactions. Indicated physical characterisics such as energy, linear momentum, etc. refer to the corresponding total quantities of the system. The frame-

work is here referred to couplings derivable from a potential (as in unified gauge theories or quantumchromodynamics). The violation of a symmetry is represented according to a number of mechanisms ranging from the classical addition of a symmetry breaking term to the total Lagrangian, to the quantum field theoretical spontaneous symmetry breaking. The fundamental equations of the approach are the conventional Lagrange's equations (without external terms) at all (classical, quantum mechanical and quantum field theoretical) levels.

APPENDIX 1.B:

THE COMPLEMENTARY OPEN APPROACH TO INTERACTIONS

Even within the context of simple forces, the concept of Newtonian interactions according to Definition 1.A.1 is not unique. A different concept can be expressed according to the

DEFINITION 1.B.1: A Newtonian particle is interacting with an external system when at least one of its physical characteristics is not conserved.

And indeed, trivially, when all the physical characteristics of any given particle are conserved, that particle is free.

The approach of Definition 1.B.1 is hereby called "open" to differentiate it from that of Definition 1.A.1 as well as to emphasize the fact that the approach is based on the study of the behaviour of each individual particle of a given system while the rest of the system is considered as external.

The proposed approach is "complementary" to that of Definition 1.A.1 in the sense that both definitions study the same system. Only the conceptual perspective is changed. In particular, while Definition 1.A.1 studies the considered system of particles as a whole, Definition 1.B.1 studies the behaviour of each of its individual constituents.

Definition 1.B.1 characterizes the primitive concept of interactions which will be used in the proposed model on the

structure of the hadrons. Its extension to classical fields is trivial and reads

DEFINITION 1.B.2: A field is interacting with an external system when at least one of its physical characteristics is not conserved.

Here the term "physical characteristics" should be referred to any characteristic of the field, such as energy, linear momentum, angular momentum, spin, charge, magnetic moment, etc.

The transition from Definition 1.A.2 to 1.B.2 is not trivial because it implies a mutation of the conceptual attitude first and then of the methodological context. Specifically, while the emphasis of the conventional approach according to Definition 1.A.2 is on the conservation of the total quantities of the considered system as a whole, the emphasis according to Definition 1.B.2 is on the nonconservation of the individual components of the total quantities. As a result, methodological tools which play a crucial role within the context of Definition 1.A.2 such as symmetries, Noether's theorem, conservation laws, etc. become vacuous within the context of Definition 1.B.2.

Almost needless to say, the nonconservation of the individual physical characteristics of the considered system must always be compatible with the conservation of the corresponding total quantities of the system considered as a whole, and, most importantly, isolated from the rest of the universe. As a matter of fact, this compatibility expresses the ultimate

significance of the concept of complementarity of the two different approaches.

Restated in very simple language, the conservation of the total quantities inherent in Definition 1.A.2 by no means implies the conservation of its individual constituents. On the contrary, if this is tne case, then the system of particles is not interacting or, more technically, the addition of $\mathcal{L}_{int}$ to $\mathcal{L}_{free}$ in structure (1.A.2) implies a mere equivalence transformation of $\mathcal{L}_{free}$. On the contrary, the assumption of nontriviality of $\mathcal{L}_{int}$ as defined in Appendix 1.A implies that, while a total physical characteristic is conserved, its individual constituents are generally nonconserved.

As a simple example, consider the two-body problem in vacuum under a mutual Coulomb force. Assume the system as isolated from the rest of the universe as implicit in Definition 1.A.1. Then the system represented with Lagrangian (1.A.1) exhibits the well known symmetry under the full Galilei group and, more specifically,

(1) the symmetry under translation in time with conservation law of the total energy,

(2) the symmetry under translation in space with conservation law of the total linear momentum,

(3) the symmetry under rotations in space with conservation law of the total angular momentum, and

(4) the symmetry under Galilei boosts with the law of uniform motion of the center of mass.

In the transition to the study of the same system under Definition 1.B.1 all symmetries (1), (2), (3) and (4) become vacuous. Specifically, the energy, linear momentum and angular momentum of the individual constituents of the considered two-body problem are not necessarily conserved (unless one of the two masses is assumed of infinite value), while the concept of uniform motion of the center of mass looses its direct physical significance.

The above simple example has been here indicated to point out that the restriction of the concept of interaction to only one approach, irrespective of whether according to Definition 1.A.1 or 1.B.1, is not recommendable. On the contrary, a deeper insight can be gained by the joint use of both profiles, namely, the closed and open approaches.

In turn, this point provides an initial illustration of the significance of the use of the dual methodological approach postulated in Section 1.2. Indeed, the closed approach to interactions according to Definition 1.A.1 or 1.A.2 is centered on the use of analytic equations without external terms, while the open approach to interactions according to Definition 1.B.1 or 1.B.2 will be centered on the use of analytic equations with external terms. The difference in the structure of the fundamental analytic equations of these two complementary approaches is then an indication of the differences of the related methodological frameworks.

The methodology of this open approach to interactions will be worked out in Volume II.

Individual characteristics of the open approach	Interactions	
	Strong	Electromagnetic
Energy	violated	violated
Linear momentum	violated	violated
Angular momentum	violated	violated
Charge	violated	conserved
Spin	violated	conserved
Magnetic moment	violated	conserved

Table 1.B.1: The central assumptions on the dynamical implications of strong interactions as not derivable from a potential. The table refers to the physical characteristics of the individual constituents of a hadronic structure along the concept of open approach to interactions of Definition 1.B.2. It should be stressed that the assumptions relate only to the column of strong interactions because that on electromagnetic interactions is based on currently available experimental evidence. The above table must be complemented with Table 1.A.1 of Appendix 1.A for the conservation of the corresponding total quantities. The starting point is a hadron either free or subject to electromagnetic interactions only. Its intrinsic characteristics are then quantized according to conventional rules. This situation essentially leaves open the problem of the behaviour of the intrinsic characteristics of its constituents because, when a total physical quantity is

conserved its individual components are not necessarily so. In order to achieve a dynamical differentiation between the electromagnetic and the strong interactions, it is then assumed that for the latter all the physical characteristics are generally nonconserved. As we shall see later on, this is the case when the strong interactions are not derivable from a potential. The assumptions of this table must be considered as conjectural. They are motivated by the desire to attempt a profound differentiation between the electromagnetic and the strong interactions in a way parallel to the corresponding physical differences of the corresponding bound states: the atomic and the hadronic structure.

APPENDIX 1.C

A GENERALIZATION OF THE KEPLER SYSTEM TO NONCONSERVATIVE FORCES

In this appendix we shall study an explicit realization of the generalization of the two-body Kepler system with non-conservative additive couplings and the Galilei conservation laws of total quantities as subsidiary constraints proposed in Section 1.4, i.e.,

$$\begin{cases} m_k \ddot{\underset{\sim}{r}}_{mk} - \underset{\sim}{f}_{mk} - \underset{\sim}{F}_{mk} = 0 \,, \quad k = 1,2 \,, & (1.C.1a) \\ \underset{\sim}{P}_m = m_1 \dot{\underset{\sim}{r}}_1 + m_2 \dot{\underset{\sim}{r}}_2 = 0 \,, & (1.C.1b) \\ \underset{\sim}{G}_m = m_1 \dot{\underset{\sim}{r}}_1 + m_2 \dot{\underset{\sim}{r}}_2 - \underset{\sim}{P}_m t = 0 \,, & (1.C.1c) \\ \underset{\sim}{J}_m = \underset{\sim}{r}_{mi} \times \underset{\sim}{p}_{mi} + \underset{\sim}{R}_m \times \underset{\sim}{P}_m = \underset{\sim}{r}_m \times \underset{\sim}{p}_m = \text{const.} \,, & (1.C.1d) \\ \mathcal{E} = \tfrac{1}{2} \mu \dot{\underset{\sim}{r}}_m^2 + V = \text{const.} \,, & (1.C.1e) \end{cases}$$

$$\underset{\sim}{f}_{m1} = \frac{\partial V}{\partial \underset{\sim}{r}_m^2} = - \underset{\sim}{f}_{m2} \,, \quad V = \frac{k}{r} \,, \qquad (1.C.1f)$$

$$\underset{\sim}{R}_m = \frac{m_1 \underset{\sim}{r}_{m1} + m_2 \underset{\sim}{r}_{m2}}{m_1 + m_2} \,, \quad \underset{\sim}{r}_m = \underset{\sim}{r}_{m1} - \underset{\sim}{r}_{m2} \,, \qquad (1.C.1g)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \,, \quad \underset{\sim}{p}_m = \mu \dot{\underset{\sim}{r}}_m \,. \qquad (1.C.1h)$$

By summing up the equations of motion (1.C.1a), the conservation of the total linear momentum is satisfied if and

only if the nonconservative forces are such that

$$\underset{\mu}{F}_{1} = -\underset{\mu}{F}_{2} . \qquad (1.C.2)$$

In turn, this ensures that the motion is still in a plane as for the conventional Kepler problem. Constraint (1.C.1b) is then verified. Its value is assumed null to treat the system in the center of mass coordinates. Then constraint (1.C.1c) is automatically verified. The system can now be rewritten

$$M \ddot{\underset{\mu}{R}} = 0, \quad \mu \ddot{\underset{\mu}{r}} - \underset{\mu}{f} - \underset{\mu}{F} = 0, \qquad (1.C.3)$$
$$M = \mu_1 + \mu_2, \quad \underset{\mu}{f} = \underset{\mu 1}{f}, \quad \underset{\mu}{F} = \underset{\mu 1}{F} .$$

In view of the property

$$\dot{\underset{\mu}{J}} = \underset{\mu}{r} \times \underset{\mu}{p}^{\cdot} = \underset{\mu}{r} \times \underset{\mu}{f} + \underset{\mu}{r} \times \underset{\mu}{F} = \underset{\mu}{r} \times \underset{\mu}{F}, \qquad (1.C.4)$$

the conservation of the total angular momentum, Eq. (1.C.1d), holds if and only if

$$\underset{\mu}{r} \times \underset{\mu}{F} = 0. \qquad (1.C.5)$$

Similarly, in view of the property

$$\dot{\mathcal{E}} = \dot{\underset{\mu}{r}} \cdot \left(\mu \ddot{\underset{\mu}{r}} - \underset{\mu}{f} \right) = \dot{\underset{\mu}{r}} \cdot \underset{\mu}{F} , \qquad (1.C.6)$$

the conservation of the total energy, Eq. (1.C.1e), holds if and only if

$$\dot{\underset{\mu}{r}} \cdot \underset{\mu}{F} = 0. \qquad (1.C.7)$$

Under restriction (1.C.2), Eqs. (1.C.1) can be written in the equivalent form

$$\begin{cases} M \ddot{\underset{\mu}{R}} = 0 , & (1.C.8a) \\ \mu \ddot{\underset{\mu}{r}} - \underset{\mu}{f} - \underset{\mu}{F} = 0, & (1.C.8b) \\ \underset{\mu}{r} \times \underset{\mu}{F} = 0, & (1.C.8c) \\ \dot{\underset{\mu}{r}} \cdot \underset{\mu}{F} = 0. & (1.C.8d) \end{cases}$$

Our problem is now that of selecting an explicit form of the force $\underset{\mu}{F}$ such that the constrained system is consistent, i.e., it admits a physically significant solution. A simple form of the F-forces which is sufficient for our analysis is given by

$$\underset{\mu}{F} = \begin{pmatrix} g_x \ddot{r}_x \\ g_y \ddot{r}_y \end{pmatrix}, \qquad (1.C.9)$$

where we have ignored the third component because the motion is in a plane. Its role in our model will be studied later on in Section 2.12. Constraint (1.C.8c) then reduces to the single condition

$$r_x F_y - r_y F_x = 0. \qquad (1.C.10)$$

But the equations of motion in the relative coordinate can be written

$$\ddot{r}_x = \frac{r}{\mu - g_x} f_x \,, \quad \ddot{r}_y = \frac{r}{\mu - g_y} f_y \,. \qquad (1.C.11)$$

Thus, along the solution of the system, we can write

$$F_x = \frac{g_x}{\mu - g_x} f_x \,, \quad F_y = \frac{g_y}{\mu - g_y} f_y \,, \qquad (1.C.12)$$

and condition (1.C.10) holds if and only if $g_x = g_y = g$, i.e.,

$$\underset{\sim}{F} = F \hat{\underset{\sim}{r}} \,, \quad \hat{\underset{\sim}{r}} = \frac{\underset{\sim}{r}}{|\underset{\sim}{r}|} \,, \quad F = g \ddot{r} \qquad (1.C.13)$$

Constraint (1.C.8d) then becomes

$$\frac{g}{\mu - g} \dot{\underset{\sim}{r}} \cdot \underset{\sim}{r} = 0 \qquad (1.C.14)$$

and system (1.C.8) can be written in the form

$$\begin{cases} M \ddot{\underset{\sim}{R}} = 0 \,, & (1.C.15a) \\ (\mu - g) \ddot{\underset{\sim}{r}} - \underset{\sim}{f} = 0 \,, & (1.C.15b) \\ \dot{\underset{\sim}{r}} \cdot \underset{\sim}{r} = 0 \,. & (1.C.15c) \end{cases}$$

The solution of the above system is straightforward. By using the method of the Runge-Lenz vector we consider the expression

$$\underset{\sim}{m} = \underset{\sim}{J} \times \underset{\sim}{p} \qquad (1.C.16)$$

for which

$$\dot{\underset{\sim}{m}} = \underset{\sim}{J} \times (\underset{\sim}{f} + \underset{\sim}{F}), \qquad (1.C.17)$$

which yields the generalized form of the conserved Runge-Lenz vector

$$\underset{\sim}{A} = \underset{\sim}{m} + \frac{\mu^2 k}{\mu - g} \hat{\underset{\sim}{r}} \qquad (1.C.18)$$

Thus,

$$\underset{\sim}{A} \cdot \underset{\sim}{r} = A r \cos\theta = -J^2 + \frac{\mu^2 k}{\mu - g} r. \qquad (1.C.19)$$

The trajectory is then given by the conventional expression of a conic

$$\frac{1}{r} = \frac{\mu^2 k}{(\mu - g) J^2} (1 - \varepsilon \cos\theta), \qquad (1.C.20)$$

but with a different coefficient than that of the conventional Kepler problem and the different value of the excentricity

$$\varepsilon = \frac{\mu - g}{k \mu^2} A = \sqrt{1 + \frac{2(\mu - g)^2}{\mu^3 k^2} J^2 E}. \qquad (1.C.21)$$

Constraint (1.C.15c) then restricts the type of admissible conic. Indeed, the only allowed trajectory is the circle (for $F \neq 0$), i.e.,

$$\mathcal{E}=0, \quad E = -\frac{\mu^3 k^2}{2(\mu-g)^2 J}. \qquad (1.C.22)$$

A Lagrangian for the relative motion is then given by

$$L = \frac{\mu-g}{\mu} T - V, \quad T = \frac{1}{2}\mu \dot{\xi}_m^2, \qquad (1.C.23)$$

by exhibiting the presence of a multiplicative term to the kinetic energy which will become crucial after the analysis of Chapter 2.

The above elementary example is sufficient for our objective at this point, namely, to indicate that nontrivial realizations of the generalized Kepler problem (1.C.1) are indeed possible. In particular, the example indicates that such generalizations are consistent under a suitable selection of the additive nonconservative forces. This is clearly a central requirement to avoid the mathematical manipulation of models which might be vacuous on physical grounds.

In essence, Eqs. (1.C.1) with realization (1.C.13) constitutes a "mutation" of the Kepler system in the sense that the only admissible orbit is now a circle and the radius, rather than being given by the familiar form

$$r = \frac{J^2}{\mu k} \qquad (1.C.24)$$

assumes the value

$$r = \frac{(\mu-g)^2}{\mu^2 k} J^2. \qquad (1.C.25)$$

Of course, by performing the limit $g \to 0$ one recovers the conventional Kepler system. The generalization of the model to other types of nonconservative forces is here left to the interested reader.

It should be here indicated that for more general forces not derivable from a potential the model becomes a nontrivial constrained model in the sense that the subsidiary constraints now play a more essential role. To be more specific, the simplicistic procedure which has been used for the reduction of model (1.C.1) to model (1.C.15) has been that of selecting a particular form of the forces $\underset{\sim}{F}_k$ which satisfy most of the subsidiary constraints. In this way, the only surviving constraint is Eq. (1.C.15c) which restrict the conic to be a circle. Suppose that one intends to study more general types of forces $\underset{\sim}{F}_k$. In this case all subsidiary constraints (1.C.1b) through (1.C.1e) are expected to play an essential role in the sense that they are not expected to be reducible to properties of the nonconservative forces. Therefore, model (1.C.15) is one of the simplest conceivable examples of the type of models under consideration with only one surviving constraint. The general model is however model (1.C.1) in which the subsidiary constraints play an essential role and the nonconservative forces may now assume more general forms.

Few comments related to the consistency of these more general models appear to be advisable. From the viewpoint of consistency, models (1.C.1) can be classified into the following three classes of increasing technical difficulties.

(I) Models in which the consistency can be established by selecting the solution of Eqs. (1.C.1a) which satisfies the subsidiary constraints. Model (1.C.15) is precisely of this simplest possible class.

(II) Models in which the consistency can be established by using the Levi-Civita method. In 1906 T. Levi-Civita[47] identified a simple but very effective sufficiency condition for constrained systems to be consistent. It is given by the condition that the subsidiary constraints are particularized first integrals of the equations of motion. The consistency of the system is then self-evident. The effectiveness of the method rests on the fact that it can be applied without the knowledge of a solution. This method allows a first generalization of the class of models (1.C.15). And indeed, rather than selecting the forces F_{ak} in such a way to directly satisfy the constraints, one can select them in such a way that the constraints are particularized first integrals. This allows for more general forces not derivable from a potential. It should be stressed for clarity that the Levi-Civita condition is only sufficient and not necessary and sufficient.

Therefore, more general cases in which the Levi-Civita condition is not satisfied are expected to exist. As an incidental remark it should be here indicated that, despite a recent surge of interest in constrained models (particularly in gauge theories), the Levi-Civita method remained largely ignored until S. Shanmugadhasan[37] brought it to the attention of the physics community by identifying its role in constrained theories.

(III) Models in which the consistency can be proved by using the theory of overdetermined systems. Let us recall that a system is called determined when the number of equations coincide with the number of unknowns (in our case, the coordinates x_a^k), while it is overdetermined when the number of equations exceeds that of the unknown quantities. For determined systems of differential equations the ordinary existence theory of solutions is applicable.[48] This is the case of Eqs. (1.C.1a) "per se", i.e., without subsidiary constraints. When the latter are included, the system becomes overdetermined. In this case the ordinary existence theory is nonapplicable and more general approaches must be used. They are sometimes referred to as the existence theory of overdetermined (ordinary and partial) differential equations. For consistency theorems the reader is referred to the recent literature

and quoted papers.[49] The important point is that this theory allows a further broadening of the models considered.

The generalization of model (1.C.15) along the lines (II) and (III) is here left to the interested reader. The above outline has been given for the intent of indicating that the considered class of generalized Kepler models with subsidiary constraintes (as a Newtonian limit of models on the hadronic structure) allows a rather rich variety of consistent models. The implications of this result for the actual construction of hadronic models will be studied in Volume III.

CHAPTER 2

THE INVERSE PROBLEM

2.1: STATEMENT OF THE PROBLEM

The objective of this chapter is that of reviewing the methodology for the representation of our Newtonian and field theoretical model of the hadronic structure (Sections 1.4 and 1.5) in terms of conventional Lagrange's and Hamilton's equations without external terms, as well as that of pointing out some of its initial implications

To identify this objective in more detail it is here essential to note that the model considered is constituted by a system of equations subject to a system of subsidiary constraints. It is then advisable to consider first the case of unconstrained systems, and then study the extension to the case with subsidiary constraints.

On empirical grounds, the direct problem of Newtonian Mechanics[50] is the conventional approach whereby one first identifies a Lagrangian $L(t,q,\dot{q})$ and then computes the equations of motion, when needed, through Lagrange's equations

$$L_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}$$

$$= \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} \qquad (2.1.1)$$

$$= A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q}) = 0,$$

$$k,i,j = 1,2,\ldots,n$$

Here the q's are the generalized coordinates of conventional analytic mechanics with the understanding that they can also be

interpreted as representing a system of n = 3N Cartesian coordinates $\{x_1^1, \ldots, x_N^N\}$ in a three-dimensional Euclidean space in a given ordering.

The inverse problem of Newtonian Mechanics[34] is precisely the opposite of the direct problem in the sense that now the equations of motion are given and the knowledge of a Lagrangian is requested.

More specifically, with the terms "Inverse Problem of Newtonian Mechanics" we shall refer to the body of formulations consisting of:

(a) the necessary and sufficient conditions (integrability conditions) for the existence of a Lagrangian or, independently, of a Hamiltonian for the analytic representation of Newtonian systems of second-order ordinary differential equations,

(b) the methods for the computation of a Lagrangian or a Hamiltonian, when they exist, from the given equations, and

(c) the application of the underlying methodology to other aspects of the problem of interactions, such as transformation theory, symmetries and first integrals of systems with arbitrary couplings.

An inspection of Eqs. (2.1.1) indicates that, when the equations of motion are given and the Lagrangian is requested, we are dealing with a system of n partial differential equations in only one unknown, the Lagrangian, i.e., an overdetermined system of partial differential equations. This is a type of

system which, irrespective of possible continuity conditions, does not in general admit a solution. From the overdetermined nature of the system one can also see that the identification fo the integrability conditions as well as the computation of at least a solution L (when it exists) is not trivial.

It should be stressed that we have here given an empirical definition of the Inverse Problem which, however, is sufficient for our need. For a technical definition within the context of the Calculus of Variations see references[34]. Notice that aspects (a), (b) and (c) have been formulated for an arbitrary system which has the Newtonian structure, i.e., linear in the second order derivatives, but not necessarily linear in the first and zero order derivatives of the dependent variables (quasilinear systems).

In the transition to Special Theory of Relativity, the Newtonian time t is replaced by the proper time s and the Cartesian coordinates $\underset{\sim}{x}$ are replaced by the Minkowski coordinates $x = (x^0, x^1, x^2, x^3)$, $x^0 = t$ ($c = 1$). On formal grounds, Lagrange's equations for the one-particle case have a structure equivalent to that of Eqs. (2.1.1), i.e.,

$$L_\mu(x) = \frac{d}{ds}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} \qquad (2.1.2)$$

$$= \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}\ddot{x}^\nu + \frac{\partial^2 L}{\partial \dot{x}^\mu \partial x^\nu}\dot{x}^\nu + \frac{\partial^2 L}{\partial \dot{x}^\mu \partial s}$$

$$-\frac{\partial L}{\partial x^\mu} = A_{\mu\nu}(s,x,\dot{x})\ddot{x}^\nu + B_\mu(s,x,\dot{x}) = 0,$$

$$\mu,\nu = 0,1,2,3\ , \quad \dot{x} = \frac{dx}{ds}$$

Thus, we are still dealing, in general, with a type of system of quasilinear type according to our terminology.

"The Inverse Problem of Special Theory of Relativity" can then be formulated as a relativistic generalization of the Newtonian aspects (a), (b) and (c).

Let us recall that, while a Newtonian system can be unconstrained, this is not the case for a system compatible with the Special Theory of Relativity because relativistic requirements demand the use of subsidiary constraints. Thus, Eqs. (2.1.2) must be interpreted as only part of a full relativistic formulation.

The case of more than one-particle will not be considered because of the problematic aspects of the No-Interaction Theorem which, as is known, are still under investigation. For a review of these aspects see reference[51].

The extension from a Newtonian to a field theoretical context can be realized by replacing

(a) the Newtonian independent variable, the time t, with the space-time independent variables x^μ, $\mu = 0, 1,2,3$,

(b) the Newton dependent variables, the generalized coordinates $q^k(t)$ or Euclidean coordinates $\underset{\sim}{r}^k(t)$, with the field components $\phi^k(x^\alpha)$, $k = 1,2,\ldots, m$ and

(c) the Newtonian Lagrangian $L(t,q,\dot{q})$ or $L(t,\underset{\sim}{r},\dot{\underset{\sim}{r}})$ with the Lagrangian density $\mathcal{L}(x^\alpha, \phi^k, \phi^k{}_{;\alpha})$, $\phi^k{}_{;\alpha} = \partial\phi^k/\partial x^\alpha$.

This extension is predictably more laborious, but straightforward. In the general case we are interested in, a system of quasilinear partial differential equations is given and the knowledge of a Lagrangian density is requested according to the identifications

$$\mathcal{A}_k(\phi) = d_\mu \frac{\partial \mathcal{L}}{\partial \phi^{k;}_{\ \mu}} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial \phi^{i;}_{\ \nu}} \phi^{i;}_{\ \mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial \phi^i} \phi^{i;}_{\ \mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial x^\mu} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= \frac{1}{2}\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial \phi^{i;}_{\ \nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \nu} \partial \phi^{i;}_{\ \mu}} \right) \phi^{i;}_{\ \mu\nu}$$

$$+ \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial \phi^i} \phi^{i;}_{\ \mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial x^\mu} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= A^{\mu\nu}_{ki}(x^\alpha, \phi^j, \phi^{j;}_{\ \alpha})\, \phi^{i;}_{\ \mu\nu} + B_k(x^\alpha, \phi^j, \phi^{j;}_{\ \alpha}) = 0 \qquad (2.1.3)$$

$$k, i, j = 1, 2, \ldots, n \ , \ \mu, \nu, \alpha = 0, 1, 2, 3$$

For the case of a system of first order field equations we have in particular

$$\mathcal{A}_k(\phi) = d_\mu \frac{\partial \mathcal{L}}{\partial \phi^{k;}_{\ \mu}} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial \phi^i} \phi^{i;}_{\ \mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}_{\ \mu} \partial x^\mu} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= C^{\mu}_{ki}(x^\alpha, \phi^j)\, \phi^{i;}_{\ \mu} + D_k(x^\alpha, \phi^j) = 0 \qquad (2.1.4)$$

"The Inverse Problem in Field Thoery" can then be formulated as a field theoretical extension of the Newtonian aspects (a), (b) and (c).

It should be indicated that the Lagrange equations used in these latter identifications are not those currently used in the literature of Field Theory (which assume a partial derivative ∂_μ rather than a total derivative d_μ). Instead, they are those of the Calculus of Variations for multiple integral path functionals. For more details see Appendix 2.A.

The objective of this chapter can now be identified. It consists of a review of the central methodological aspects of the Inverse Problem in Newtonian Mechanics, Special Relativity and Field Thoery, with particular emphasis on their application to the considered hadronic structure model. This review appears to be recommendable on several counts. First of all, the methodology of the Inverse Problem is not readily available in the current physical literature for all the indicated branches of physics. As a result, a review of the basic working tools could be useful for the reader interested in independent studies either along our approach to the hadronic structure or along any other area of their applicability (e.g., gauge or nonlinear models). Secondly, the literature of the Inverse Problem as currently available is related to the problem "per sé", without any reference to its application to the considered hadronic forces (local and not derivable from a potential). As a result, a specialization of the basic methodological tools of the Inverse Problem to the considered hadronic structure is essential for

our program. Finally, as indicated in Chapter 1, the methodology here considered will be used for the primary objective of this monograph: the nonapplicability of the Galilei and Einstein (special) relativity to the considered realization of the hadronic forces. As such, its prior identification is essential to avoid possible misrepresentations of the analysis as well as excessive reference to existing literature.

It should be indicated from the outset that no proof of the presented theorems will be given. The reader with a serious interest in learning the techniques of the Inverse Problem is therefore suggested to study the quoted literature. In any case the detailed proof of all theorems is given in my monographs[34] and papers.[30] Rather than putting the emphasis on mathematical completeness and rigour, the pragmatic objective of this chapter is therefore that of only providing the needed working tools in an as concise as possible way.[52]

We are now interested in the basic characteristics of the systems considered to better identify the arena of applicability of the Inverse Problem which will be reviewed.

The first concept which is needed for this objective is that of the region of definition of the equations considered. We shall hereon assume that all considered equations are defined in a region R according to its conventional local meaning, i.e., a point of all the local (independent and dependent) variables and its neighborhood.

The second needed concept is that of continuity. We shall hereon assume the conventional definition of class C^m (R) functions, m=0,1,2,..., i.e., which admit in R continuous partial derivatives up to and including the order m. This implies that a function of class C^0(R) is continuous in R, but its derivatives are not necessarily continuous in R. The above definition also implies that partial derivatives up to the order m "commute", i.e., their order can be arbitrary inverted in the region of definition.

The third needed concept is that of regularity or degeneracy. We shall say that a system of ordinary or partial differential equations is regular or degenerate in its region of definition when its functional determinant is identically nonnull or null in it respectively. For the considered Lagrangian identifications, the functional determinant is given by

$$\mathcal{K}(R) = \left|\frac{\partial L_i}{\partial \ddot{q}^j}\right|(R) = \left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right|(R) = |A_{ij}|(R), \tag{2.1.5a}$$

$$\mathcal{K}(R) = \left|\frac{\partial L_\mu}{\partial \ddot{x}^\nu}\right|(R) = \left|\frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu}\right|(R) = |A_{\mu\nu}|(R), \tag{2.1.5b}$$

$$\mathcal{K}(R) = \left|\frac{\partial \mathcal{L}_i}{\partial \phi^{ij}_{\mu\nu}}\right| = \frac{1}{2}\left|\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^{ij}_{\mu} \partial \phi^{ij}_{\nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{ij}_{\nu} \partial \phi^{ij}_{\mu}}\right)\right| = |A^{\mu\nu}_{ij}|(R) \tag{2.1.5c}$$

and it is often referred to as the Hessian determinant. It should be stressed that a Lagrangian will be called degenerate when its Hessian determinant is identically null not only at a point of the local variables, but also in its neighborhood. It should be also recalled that if a global definition of region is assumed (e.g., an open and connected set), when the Hessian determinant is null at a (finite) number of isolated points, the Lagrangian is still regular.[54] On pragmatic grounds, to check the regularity or degeneracy of a system of equations (or of the Lagrange equations) it is sufficient to compute the functional (Hessian) determinant. If it is nonull as a function, the system is regular, otherwise it is degenerate.

We shall also say that a system of Lagrange's equations or a Lagrangian is totally degenerate when each and every element of the Hessian matrix is identically null, i.e., the Lagrangian is linear in the derivatives. This is the case of structure (2.1.4) or, in general, of Lagrange's equations when they are all of the first order.

We are now in a position to identify a central property of our primitive Newtonian model of the hadronic structure, Eqs. (1.4.7a): they are regular in any local or global region of definition because, trivially, their functional determinant has the constant value m^2. As a result, the arena of applicability

of the methodology of the Inverse Problem which will be considered in this chapter is that of regular unconstrained systems of ordinary or partial second-order equations satisfying the minimal continuity conditions which are needed for the integrability conditions for the existence of a Lagrangian or a Hamiltonian. The case of "bona fide" degenerate systems will not be treated. Nevertheless, we shall consider the Inverse Problem for the totally degenerate field theoretical case, Eqs. (2.1.4).

It should also be indicated that the extension of the methodology of the Inverse Problem to the case of systems with (generally nonintegrable) subsidiary constraints will also not be considered. This is due, first, to the need of containing the size of this monograph to a reasonable limit and, secondly, because this extension is not essential for its objective. In essence, this monograph is devoted to only the initial methodological steps for the study of our hadronic model, those within the context of conventional analytic equations for the study of our classical unconstrained equations. The study of the constrained and quantum mechanical form of the model will be conducted in Volume III.

In any case, the extension under consideration can be conducted with the methodological tools of the Calculus of Variation, with particular reference to the Problem of Bolza and the use of the multiplier rule. This essentially implies the implementation

of systems (2.1.1) through (2.1.4) with generally nonintegrable subsidiary constraintes. The use of the multiplier rule then yields extended Lagrangians which represent the complete constrained system. In the final analysis, this is a technically straightforward generalization of the fundamental step, the methodology of the Inverse Problem for the unconstrained equations considered "per sé", i.e., independently from the existence of the subsidiary constraints. We therefore content courselves at this time by referring the reader to the easily identifiable literature on the variational problem of constrained systems, such as references.[45,46]

It should be here recalled that a class of constrained systems is well known in the current physical literature, that of canonical formulations of degenerate Lagrangians, also called Dirac's Mechanics.[36] This approach is not applicable to our hadronic model because the assumed equations (in their second order form) are regular by construction and they remain regular also in the hypersurface of the constraints. The emerging Lagrangians are therefore regular, also in the sense of the theory of systems with constraints.

In the final analysis, this is the reason why in the canonical formulation of our constrained model of the hadronic structure we do not reach a Dirac's system but instead a system of the Bolza type. The relationship between these two approaches can be indicated as follows. One of the necessary conditions for the existence of an extremal path for a single integral path

functional is that the Lagrangian is regular (i.e., satisfies the Legendre condition in the language of the Calculus of Variations). In principle, this condition can be violated too. Therefore, Dirac's approach can be considered as a particular (i.e., degenerate) case of the more general Bolza's approach.

This differentiation of the methodological role of Dirac's versus Bolza's approach has a rather profound implication within the context of the considered hadronic structure. Dirac's approach is known to be fully applicable to systems with electromagnetic interactions. Its lack of direct[55] applicability to our realization of the hadronic forces is an indication of our intent of achieving a genuine differentiation between the electromagnetic and the strong interactions in the physical space of the experimental detection. The covering nature of Bolza's versus Dirac's approach is then an indication of the broader methodological perspective which is needed for our approach to the hadronic structure.

2.2: THE CONCEPT OF ANALYTIC REPRESENTATION

According to a conventional definition, an analytic representation of a given system of equations of motion in terms of the Lagrange equations occurs when the totality of the solutions of the two systems coincide.

Clearly this definition, even tnough mathematically rigorous, is impractical for our objectives for the simple reason that the systems we are interested in are generally nonlinear in coordinates and velocities and, as such, their solutions are of difficult identification.

Instead, our efforts are devoted to work out the methodology of the Inverse Problem without any knowledge of the solutions of the equations considered. This objective demands its actuation from the very first step, the concept of analytic representations.

30-I, 34-I

DEFINITION 2.2.1: A Newtonian system which is of at least class C^2 and regular in region R of its variables admits an analytic representation in terms of the conventional Lagrange equations in R when either one of the following identities hold

$$\frac{d}{dt}\frac{\partial L(t,q,\dot q)}{\partial \dot q^k} - \frac{\partial L(t,q,\dot q)}{\partial q^k} \equiv h_k^{\ i}(t,q,\dot q)\left[A_{ij}(t,q,\dot q)\ddot q^j + B_i(t,q,\dot q)\right], \qquad (2.2.1a)$$

$$g_k^{\ i}(t,q,\dot q)\left[\frac{d}{dt}\frac{\partial L(t,q,\dot q)}{\partial \dot q^i} - \frac{\partial L(t,q,\dot q)}{\partial q^i}\right] \equiv A_{ki}(t,q,\dot q)\ddot q^i + B_k(t,q,\dot q), \qquad (2.2.1b)$$

$$k,i,j = 1,2,\dots,n, \quad (h) = (g)^{-1} \in C^2(R), \quad |g|(R) \neq 0. \qquad (2.2.1c)$$

The representation is called direct or indirect when the matrix of factor functions is or is not, respectively, the unit matrix and it is called ordered or nonordered when identities

(2.2.1) hold or do not hold, respectively, member by member for all values of the index k = 1,2,..., n in a given ordering.

The above definition ensures the identity of the totality of the solutions of the equations of motion with the Lagrange equations as it can be seen from the fact that the implicit functions of the two systems coincide from the assumed regularity of the factor matrix.

Notice that $h_k^{\ i}(t,q,\dot q)$ is the maximal functional dependence of factor functions. Indeed, a further dependence of these functions in the accelerations would render the system no longer linear in the accelerations and, therefore, neither of Newtonian nor of Lagrangian type.

The first significant aspect of Definition 2.2.1 is that the knowledge of only one function, a Lagrangian, is generally insufficient to characterize an analytic representation because $n^2 + 1$ functions are generally needed, a Lagrangian and n^2 factor functions. For instance, the two-dimensional system

$$\begin{cases} m\ddot q_1 + \omega^2 q_1 = 0, \\ m\ddot q_2 + \omega^2 q_2 = 0, \end{cases} \qquad (2.2.2)$$

can be represented in terms of the functions

$$(h^{(1)}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L^{(1)} = m\dot q_1\dot q_2 - \omega^2 q_1 q_2, \qquad (2.2.3)$$

where the factor functions are nontrivial, even though simple in this case, because they characterize a permutation of the ordering in which the system is represented. As is well known, this is typical of all gauge Lagrangian models in Field Theory.

The second significant aspect of Definition 2.2.1 is that it indicates the fully acceptable case, on analytic grounds, of representing an equivalent system, rather than the system in its original form. This aspect will prove to be crucial for the analytic representations of our hadronic equations.

Since the potential for all the acting forces does not exist by assumption, we can already conclude that their analytic representations as given (i.e., direct in our terminology) do not exist. The indirect representations then acquire central relevance and so is the need of $n^2 + 1$ functions for their characterization.

The third relevant aspect of Definition 2.2.1 is the concept of ordering. In essence, it is needed to characterize the relationship between the considered Lagrangian and the system. For instance, Eqs. (2.2.2) also admit a direct analytic representation in terms of the function

$$(h^{(2)}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L^{(2)} = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} \omega^2 (q_1^2 + q_2^2). \tag{2.2.4}$$

The ordering condition then removes the ambiguities, i.e., it allows each characterization of system (2.2.2) between the two possibilities (2.2.3) and (2.2.4). Notice that system (2.2.2) is regular and so are the Lagrangians $L^{(1)}$ and $L^{(2)}$ owing to the

nonnull value of the functional (Hessian) determinants

$$\left| \frac{\partial^2 L^{(1)}}{\partial \dot{q}^i \partial \dot{q}^j} \right| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad \left| \frac{\partial^2 L^{(2)}}{\partial \dot{q}^i \partial \dot{q}^j} \right| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{2.2.5}$$

As a final remark, notice that the nonuniqueness of a Lagrangian for a given system is implicit in Definition 2.2.1, as illustrated by the existence of the two Lagrangians $L^{(1)}$ and $L^{(2)}$ for system (2.2.2). This is the reason why we have used and we shall continue to use the terms "a" Lagrangian rather than "the" Lagrangian.

Again, strictly speaking, Definition 2.2.1 applies only to systems without subsidiary constraints.

We shall ignore throughout our analysis the study of Lagrange's equations of order higher than the second. The interested reader may consult in this respect refs.[34,53]

2.3: THE INVERSE LAGRANGIAN PROBLEM IN NEWTONIAN MECHANICS

We are now ready to implement our second step, that of constructing an analytic representation of the Newtonian equations of motion (1.4.7a) through the use of the Inverse Problem.

The central methodological tool is the so-called variational approach to selfadjointness.[34] Consider a class C^2 and regular Newtonian system computed along a one parameter family of paths $q^k(t;w)$, $k=1,2,\ldots,n$,

$$F_k(q) = A_{ki}(t,q,\dot{q})\,\ddot{q}^i + B_k(t,q,\dot{q}) = 0. \qquad (2.3.1)$$

These equations uniquely characterize the so-called variational forms

$$M_k(\eta) = \left.\frac{dF_k}{dw}\right|_{w=0} = \frac{\partial F_k}{\partial q^i}\eta^i + \frac{\partial F_k}{\partial \dot{q}^i}\dot{\eta}^i + \frac{\partial F_k}{\partial \ddot{q}^i}\ddot{\eta}^i \qquad (2.3.2)$$

$$= a_{ki}\eta^i + b_{ki}\dot{\eta}^i + c_{ki}\ddot{\eta}^i, \quad \eta^k = \left.\frac{\partial q^k}{\partial w}\right|_{w=0},$$

which, in essence, are the finite part of the first order variations of F_k, i.e., $\delta F_k = M_k(\eta)w$.

When the variational form are put equal to zero, i.e., $M_k(\eta) = 0$, they characterize the so-called system of equations of variations. It should be here indicated that the variational approach to selfadjointness can be formulated in terms of the functions $F_k(q)$ (rather than equations $F_k(q) = 0$) and their variational forms $M_k(\eta)$ (rather then equations of variations $M_k(\eta) = 0$) computed along corresponding arbitrary functions $q^k(t;w) \in \{C^2(R_t), C^1(R_w)\}$ and

$\eta^k(t) \in C^2(R_t)$ called admissible paths and admissible variations respectively. As we shall see, this allows the study of the integrability conditions for the existence of a Lagrangian without any knowledge of the solutions of the system considered.

Under the assumed conditions, the variational forms uniquely characterize the so called adjoint system

$$\tilde{M}_k(\hat{\eta}) = \hat{\eta}^i a_{ik} - \frac{d}{dt}(\hat{\eta}^i b_{ik}) + \frac{d^2}{dt^2}(\hat{\eta}^i c_{ik}), \qquad (2.3.3)$$

through the use of the Lagrange identity

$$\hat{\eta}^i M_i(\eta) - \eta^i \tilde{M}_i(\hat{\eta}) = \frac{d}{dt} Q(\eta,\hat{\eta}), \qquad (2.3.4a)$$

$$Q(\eta,\hat{\eta}) = \hat{\eta}^i b_{ij}\eta^j + \hat{\eta}^i c_{ij}\dot{\eta}^j + \frac{d}{dt}(\hat{\eta}^i c_{ij})\eta^j. \qquad (2.3.4b)$$

The equations of motion (2.3.1) are called selfadjoint when the variational forms coincide with their adjoint system for all admissible variations, i.e.,

$$M_k(\eta) \equiv \tilde{M}_k(\eta), \quad k = 1,2,\ldots \qquad (2.3.5)$$

Explicit calculations then indicate that the necessary and sufficient conditions for Eqs. (2.3.1) to be selfadjoint in a region R of their variables $(t,q,\dot{q})$ is that each and all the following conditions

$$A_{ij} = A_{ji} , \tag{2.3.6a}$$

$$\frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i} , \tag{2.3.6b}$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij} , \tag{2.3.6c}$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right), \tag{2.3.6d}$$

identically hold in R (Theorem 2.2.1, Volume I[34]). If at least one of the above conditions is violated, the system is called nonselfadjoint.

The significance of the above variational approach to self-adjointness is expressed by the following

THEOREM 2.3.1[34-I]: The Lagrange equations

$$L_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \tag{2.3.7}$$

$$= \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^i}\ddot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial q^i}\dot{q}^i + \frac{\partial^2 L}{\partial \dot{q}^k \partial t} - \frac{\partial L}{\partial q^k} = 0$$

in class C^4 and regular Lagrangians are always selfadjoint.

This property, which has a central significance at virtually all levels of Analytic Mechanics, was apparently identified for

time by C. G. Jacobi in 1837 but, regrettably, it had become virtually ignored in contemporary physical literature. Quite properly, the equations of variations of Lagrange's equations

$$J_k(\eta) = \frac{d}{dw} L_k \Big|_{w=0} = \frac{d}{dt}\frac{\partial J}{\partial \dot{\eta}^k} - \frac{\partial J}{\partial \eta^k} = 0 \tag{2.3.8a}$$

$$J = \frac{1}{2}\left(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\dot{\eta}^i \dot{\eta}^j + 2\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\dot{\eta}^i \eta^j + \frac{\partial^2 L}{\partial q^i \partial q^j}\eta^i \eta^j\right) \tag{2.3.8b}$$

are called Jacobi's equations in the literature of the Calculus of Variations. It is an instructive exercise for the interested reader to prove that Jacobi's equations, under the assumed conditions, always coincide with their adjoint system by therefore establishing the selfadjointness of Eqs. (2.3.7). Alternatively, but more tediously, the reader can prove that Eqs. (2.2.7) satisfy all conditions of selfadjointness (2.3.6). For explicit calculations see references.[30-I, 34I]

The most significant consequence of the selfadjointness of Lagrange's equations is the following theorem, called the Fundamental Analytic Theorem for configuration space formulations in reference[34] (see Chapter 3 of Volume I),

THEOREM 2.3.2: Necessary and sufficient condition for a quasi-linear system of second order ordinary differential equations which is well defined, of (at least) class C^2 and regular in a star-shaped region R* of its variables to admit an ordered direct analytic representation in R* in terms of the conventional Lagrange equations is that the system is selfadjoint in R*.

The primitive content of this theorem was a beautiful idea expressed by H. Helmholtz in 1887 on purely intuitional grounds, which, regrettably, had also become virtually ignored in both contemporary physical and mathematical literature. To my reconstruction, Helmholtz's starting point was the previous identification by Jacobi of the selfadjointness of Lagrange's equations. This trivially implies the necessity of the conditions of selfadjointness for the existence of a Lagrangian. Helmholtz then argued that these conditions were also sufficient.

Predictably, the rigorous proof of Helmholtz's intuition did not result to be easy. Several studies were conducted up to the earlier part of this century among which, most notably in my opinion, the thesis by D. R. Davis at the University of Chicago, under the supervision of G. A. Bliss subsequently expanded in two published papers. These studies produced fundamental contributions for the joint proof of necessity and sufficiency although they were not yet conclusive due to an inconsistency as reported in Section 3.5, Volume I[34]

I confronted the problem by using one of the most powerful mathematical tools currently available for the study of integrability conditions in general, the calculus of differential forms with particular reference to the Converse of the Poincaré Lemma. The recent detailed and effective presentation of this topic by D. Lovelock and H. Rund[56] resulted to be crucial for my needs. My efforts at the proof of this fundamental theorem are presented in paper[30-II] for the field theoretical case and in monograph[34-I] for the Newtonian case, It should be here indicated that these are treatments

specifically intended to be understood by the physics audience. A more adequate approach for the mathematics audience is the cohomology or the functional approach, as indicated in footnote.[53]

The restriction of the formulation of Theorem 2.3.2 to a star-shaped (rather than an ordinary) region is a direct consequence of my use of the Converse of the Poincaré Lemma. It basically restricts the considered equations of motion to be well defined, for instance, in a convex neighborhood of the (local) variables q^k and $\dot{q}^k$, $k=1,..,n$. The additional restriction on the formulation of Theorem 2.3.2 to regular systems was introduced on grounds that the extension of this theorem to degenerate systems, even though quite conceivable, demands a specific inspection. Finally, the restriction to systems which are of at least class C^2 was introduced from the fact that the conditions of selfadjointness (2.3.6) contains second order derivatives. For an alternative approach holding for systems of class C^1 only, see Section 2.6. Notice that the above continuity and regularity conditions imply the applicability of the so-called existence theory of (unconstrained) ordinary differential equations. Specifically, such conditions guarantee the consistency of the considered equations of motion, i.e., the existence of a physically admissible solution. On physical grounds, this is reassuring to avoid elaborate treatments of equations which, in the ultimate analysis, might be inconsistent.

Theorem 2.3.2 expresses the central idea of the Inverse Problem as well as of the proposed model on the hadronic structure.

Specifically, the efforts are centered in the construction of the equations of motion with arbitrary couplings and then the problem of their Lagrangian representation is confronted. The reason for this departure from a traditional line of approach is of physical nature, although it will appear in its full evidence at some later stage of our study. Basically, within the context of models with couplings derivable from a potential there is no need of the Inverse Problem because, first of all the problem of the existence of a Lagrangian is trivial and, secondly, most of the physical content of the models is contained in the considered Lagrangian. However, as we shall see, the corresponding situation for couplings which are not derivable from a potential is different.

Despite its selfevident significance, Theorem 2.3.2 was "per sé" insufficient for my needs. The reason is that once the equations of motion are written in such a way as to comply with Theorem 2.3.2, one is still left with the problem of the explicit computation of a Lagrangian. This problem is nontrivial because the Lagrangian structure which emerges from the use of the calculus of differential form is given by the superposition of three terms

$$L(t,q,\dot{q}) = K(t,q,\dot{q}) + D_k(t,q)\dot{q}^k + C(t,q), \quad (2.3.9)$$

where the first term, corresponding to the "kinetic" term, is necessarily nonlinear in the velocities (owing to the assumed

regularity). In particular, the terms K, D_k and C must be a solution of the equations

$$\frac{\partial^2 K}{\partial \dot{q}^i \partial \dot{q}^j} = A_{ij}(t,q,\dot{q}), \quad (2.3.10a)$$

$$\frac{\partial D_i}{\partial q^j} - \frac{\partial D_j}{\partial q^i} = \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right) + \left(\frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} - \frac{\partial^2 K}{\partial \dot{q}^i \partial q^j}\right) \equiv Z_{ij}(t,q), \quad (2.3.10b)$$

$$\frac{\partial C}{\partial q^i} = \frac{\partial D_i}{\partial t} - B_i - \frac{\partial K}{\partial q^i} + \frac{\partial^2 K}{\partial \dot{q}^i \partial t} + \left[\frac{\partial^2 K}{\partial q^i \partial \dot{q}^j} + \frac{1}{2}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right)\right]\dot{q}^j \equiv W_i(t,q). \quad (2.3.10c)$$

An inspection immediately reveals that the above system is an overdetermined system of partial differential equations, namely, a type of systems, which is generally of quite laborious solution. An advantage in the use of the converse of the Poincaré lemma in the proof of the fundamental theorem 2.3.2 is that it provides also a solution of system (2.3.10).[30-II, 34-I] This result in presented below under the heading of a "Theorem" when proved independently from Theorem 2.3.2, otherwise a more appropriate term would be "Corollary" (the same remark will be tacitly applied for most of the subsequent theorems).

THEOREM 2.3.3: A Lagrangian for the ordered direct analytic representation in terms of the conventional Lagrange equations of quasilinear systems of second order ordinary differential equations (2.3.1) which are well defined, of (at least) class C^2, regular and selfadjoint in a star-shaped region, is given by Eq. (2.3.9) where the terms K, D_k and C are a solution of Eqs. (2.3.10) given by

$$K = \dot{q}^i \int_0^1 d\tau \left\{ \left[\int_0^1 d\tau\, \tau A_{ij}(t, q, \tau\dot{q}) \right] \dot{q}^j \right\}(t, q, \tau\dot{q}), \quad (2.3.11a)$$

$$D_i = \left[\int_0^1 d\tau\, \tau Z_{ij}(t, \tau q) \right] q^j \quad (2.3.11b)$$

$$C = q^i \int_0^1 d\tau\, W_i(t, \tau q) \quad (2.3.11c)$$

The significance of the above theorem is represented by the fact that the solution of the overdetermined system (2.3.10) is turned into the solution of the generally simple integrals of Eqs. (2.3.11). Notice that this method of computing a Lagrangian appears to be computerazible.

It should be here indicated that there is no need to verify the consistency of system (2.3.10) under the conditions of Theorem 2.3.2. As a matter of fact, the proof of sufficiency of the conditions of selfadjointness for the existence of a Lagrangian according to Theorem 2.3.2 is precisely centered on the fact that they are the integrability conditions of system (2.3.10). In turn, this illustrates the nontriviality of the same proof of sufficiency.

To summarize, whenever system (2.3.1) satisfies the conditions of Theorem 2.3.2, system (2.3.10) is consistent. Notice that there is also no need to verify the existence of the integrals (2.3.11) under the condition that the system is well behaved in a star-shaped region. As a matter of fact, the restriction of the formulation of Theorem 2.3.2 and 2.3.3 in a star-shaped rather than an ordinary region is precisely centered on the objective of guaranteeing the existence of integrals (2.3.11). Thus, whenever system (2.3.1) satisfies all the requirements of Theorem 2.3.3, integrals (2.3.11) exist.

It should be stressed that the above simple presentation of the central arguments of the Inverse Problem is by no means exhaustive due to several methodological perspectives, insights and methods which we are here forced to ignore. The interested reader is thus referred to the monographs of ref.[34] as well as to the quoted literature. These references, besides containing these additional insights and, of course, the detailed proof of all the statements of this section, also contains several illustrative examples worked out in details, as well as problems and all relevant references known to me.

Despite their selfevident significance for my objectives, the above indicated central methodological aspects of the Inverse Problem were "per sé" insufficient for my needs. The reason is due to the fact that Newton's equations of motion with arbitrary couplings, as derived from Newton's second law, are

always nonselfadjoint (unless the couplings are trivial). Theorem 2.3.2 then implies that a Lagrangian for their ordered direct analytic representation does not exist.

I was therefore forced to enter into a third phase of studies which, again, I conducted jointly at a Newtonian and a field theoretical level. The central idea which I used for the removal of the restrictive character of Theorem 2.3.2 was the use of the totality of degrees of freedom of an action principle. The technical aspect which allowed such removal is the fact that Theorem 2.3.2 specifically treats the case of ordered "direct" analytic representations, namely, representations of Newtonian systems as given (without any equivalence transformation). The procedural approach which I used was first the extension of the concept of direct representations to that of ordered indirect analytic representations (the representations of equivalent systems). Secondly I studied the major types of equivalence transformations which are significant for the problem of the existence of a Lagrangian. The results of my studies are presented in Volume II of reference[34], for the Newtonian profile of the problem, and in papers[30-III,V] for the corresponding field theoretical context.

Consider Hamilton's principle

$$\delta\int_{t_1}^{t_2} dt\, L(t,q,\dot{q})\Big|_{q=\text{actual path}} = -\int_{t_1}^{t_2} dt\, L_k(q)\Big|_{q=\text{actual path}}\delta q^k = 0. \quad (2.3.12)$$

Its known "degrees of freedom" are those induced by the following transformations

(1) Transformations of time variable.

$$t \to t', \quad t' \in C^2(R), \quad \left|\frac{dt'}{dt}\right|(R) \neq 0, \quad (2.3.13)$$

(2) Transformations of the coordinates for a fixed value of the time variable.

$$q^k \to q'^k, \quad q'^k \in C^2(R), \quad \left|\frac{dq'^i}{dq^j}\right|(R) \neq 0 \quad (2.3.14)$$
$$t \to t' \equiv t$$

A third class of "degrees of freedom" of Principle (2.3.12) which is significant for the Inverse Problem was identified in references[30,34] and can be formulated as follows.

(3) Transformations of the variations within a fixed system of the coordinates and time variables. Explicitly, these transformations are of the type

$$\delta q^k \to \delta^* q^k = g_i{}^k(t,q,\dot{q})\,\delta q^i, \quad g_i{}^k \in C^2(R), \quad |g|(R) \neq 0 \quad (2.3.15)$$
$$t \to t' \equiv t, \quad q^k \to q'^k \equiv q^k$$

characterize the following generalization of Hamilton's principle

$$\delta \int_{t_1}^{t_2} dt\, L \Big|_{q=\text{actual path}} = 0 \longrightarrow \delta^* \int_{t_1}^{t_2} dt\, L = -\int_{t_1}^{t_2} dt\, g_K{}^i L_i(q) \Big|_{q=\text{actual path}} \delta q^k = 0, \quad (2.3.16)$$

and induce the following generalized Lagrange's equations

$$g_K{}^i L_i(q) = \left\{ g_K{}^i(t,q,\dot{q}) \left[\frac{d}{dt} \frac{\partial L(t,q,\dot{q})}{\partial \dot{q}^i} - \frac{\partial L(t,q,\dot{q})}{\partial q^i} \right]_{SA}^{C^2,R} \right\}_{NSA}^{C^2,R} = 0 \quad (2.3.17)$$

Let me briefly consider the significance of these degrees of freedom of Hamilton's principle for the problem of the existence of a Lagrangian for systems with arbitrary couplings.

The most visible implications are those related to transformations of the new class (3). This is due to the fact that, as the reader can verify with a simple inspection, generalized equations (2.3.17) are not, in general, self-adjoint. The net effect is a break-down of the very central property of the Inverse Problem in its first formulation, namely the selfadjointness of Lagrange's equations according to Theorem 2.3.1 from which Theorems 2.3.2 and 2.3.3 directly follows. The most visible result is therefore that of now being in a position of directly representing with an action principle nonselfadjoint systems through the concept of

ordered direct analytic representation in terms of the generalized Lagrange's equations [34-II]

$$\left\{ g_K{}^i \left[L_i(q) \right]_{SA}^{C^2,R} \right\}_{NSA}^{C^2,R} \equiv \left(A_{Ki}\, \ddot{q}^i + B_K \right)_{NSA}^{C^2,R} \quad (2.3.18)$$

which is implicit in Definition (2.2.1b), where the upper symbols C^2 and R indicate the assumed continuity and regularity conditions, the lower symbols SA (NSA) represent the selfadjointness (nonselfadjointness), and the symbol $\equiv$ again represents the ordering condition.

The concept expressed by Eqs. (2.3.18) is the central concept of analytic representation which is hereby assumed for the representation of the proposed model on the structure of the hadrons. Its motivation is due to the fact that, by construction, the proposed equations of motion for the eletons are nonselfadjoint. As a matter of fact, the physical assumption of the presence of nonconservative couplings in these equations is mathematically expressed by the fact that these equations are nonselfadjoint. For more details see Section 2.12. It is significant to indicate at this point that, on a comparative basis, the equations of motion of currently available models on the structure of the hadrons, such as the quark or the parton model, are selfadjoint, trivially, because derivable from a conventional Lagrangian structure. There are however other motivations for assuming

structure (2.3.18) rather than the conventional structure as our fundamental concept of analytic representation. They are essentially related to the problem of compatibility of the intended dual methodological approach to the eleton, that of analytic equations without and with external terms. As such, we shall treat this aspect in Volume II of this series.

It should be stressed that the breakdown of Theorems 2.3.1 2.3.2 and 2.3.3 provided by structure (2.3.18) is purely formal. Explicitly, since the matrix (g) of the factor functions of Eqs. (2.3.18) is regular by assumption, it is always invertible. As a result, Eqs. (2.3.18) are trivially equivalent to the more conventional structure

$$\left[L_k(q)\right]^{C^2,R}_{SA} \equiv \left[h_k{}^i\left(A_{ij}\ddot{q}^j + B_i\right)^{C^2,R}_{NSA}\right]^{C^2,R}_{SA} \qquad (2.3.19)$$

$$(h) = (g)^{-1}$$

which, in our language, characterizes an ordered indirect analytic representation in terms of the conventional Lagrange equations. Thus, the problem of the construction of representation (2.3.18) is turned in this way into the full use of Theorem 2.3.3 through the intermediary step (2.3.19).

Before indicating in more details this procedure, let me point out the significance of the other degrees of freedom of Hamilton's principle, those of class (1) and (2). My finding, as reported in references, is that if a system of ordinary

(or partial) differential equations is nonselfadjoint in its original form, it may be turned into a selfadjoint equivalent system through the (invertible, class C^2) transformations of the coordinate and time variables. Restated in different words, the familiar transformations of the time variable and/or of the coordinates do not, in general, preserve the selfadjointness or the nonselfadjointness of a given system. Therefore, they can be used as a tool to induce a selfadjoint structure and, thus, a Lagrangian. For full details, as well as several illustrative examples, see the quoted references.

The emerging methodological context for the problem of the existence of a Lagrangian is therefore that each class of the above indentified degrees of freedom of Hamilton's principle may individually induce a selfadjoint structure and, thus, a Lagrangian. It is significant here to indicate that these classes generally do not "commute" from the viewpoint of self-adjointness. For instance, two independently given transformations of class (1) (transformations of time) and of class (2) (transformations of the coordinates) when applied to the same system in a given order and then in the reverse order produce systems which do not necessarily have the same behaviour from the viewpoint of selfadjointness, even though from the viewpoint of their solution such systems are trivially equivalent by construction. This indicates a somewhat "hidden relation" between transformations which have been believed to

be independent until now, e.g., the space-time and internal transformations of field theory, when seen from the context of the Inverse Problem. We shall elaborate on this concept and give explicit examples in Section 2.11. For the problem whether a Lagrangian "always" exist under these transformations see Section 2.8.

To put this broader methodological profile into a practically usable context, I introduced, also in the quoted reference, the concept of selfadjoint transformation of the equations of motion.[34-II] In essence, this is any equivalence transformation of the equations of motion induced by any combination of the degrees of freedom of Hamilton's principle in any given but fixed ordering which either preserves or induces a selfadjoint structure. Explicitly, we shall write

$$\left(A_{ki}\,\ddot{q}^{\prime i} + B_k\right)^{c^2 iR}_{SA/NSA} = 0 \;\rightarrow$$

$$\rightarrow \left[\, J_k{}^i \left(A'_{ij}\,\ddot{q}^{\prime\prime j} + B'_i\right)\right]^{c^2 iR}_{SA} \qquad (2.3.20)$$

$$= \left(A^*_{kj}\,\ddot{q}^{\prime\prime j} + B^*_k\right)^{c^2 iR}_{SA} = 0$$

where the factor terms $J_k^{\,i}$ are induced not only by the Jacobians of the transformations, but also by possible multiplicative terms of class (3). The selfadjoint transformations were then classified into isotopic or genotopic when they either preserve

or induce a selfadjoint structure, respectively. The reasons for the selection of these terms was algebraic, as we shall see in Section 2.10.

We are finally in a position to present the methodology of the Inverse Problem in a form applicable to the Newtonian equations of motion of the eletons.

[30-III, 34-II]

THEOREM 2.3.4: Necessary and sufficient condition for a class C^2 and regular system of ordinary differential equations (2.3.1) to admit a selfadjoint transformation (2.3.20) is that each and all the following conditions

$$A^*_{ij} = A^*_{ji} \qquad (2.3.21a)$$

$$\frac{\partial A^*_{ik}}{\partial \dot{q}^{\prime j}} = \frac{\partial A^*_{jk}}{\partial \dot{q}^{\prime i}} \qquad (2.3.21b)$$

$$\frac{\partial B^*_i}{\partial \dot{q}^{\prime j}} + \frac{\partial B^*_j}{\partial \dot{q}^{\prime i}} = 2\left\{\frac{\partial}{\partial t'} + \dot{q}^{\prime k}\frac{\partial}{\partial q'^k}\right\} A^*_{ij} \qquad (2.3.21c)$$

$$\frac{\partial B^*_i}{\partial q^j} - \frac{\partial B^*_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t'} + \dot{q}^{\prime k}\frac{\partial}{\partial q'^k}\right\}\left(\frac{\partial B^*_i}{\partial \dot{q}^{\prime j}} - \frac{\partial B^*_j}{\partial \dot{q}^{\prime i}}\right) \qquad (2.3.21d)$$

$$A^*_{ki} = J_k{}^i(t', q', \dot{q}')\; A'_{ij}(t', q', \dot{q}') \qquad (2.3.21e)$$

$$B^*_k = J_k{}^i(t', q', \dot{q}')\; B'_i(t', q', \dot{q}') \qquad (2.3.21f)$$

are identically verified in an image region R' of the original region of definition R.

It should be indicated that in certain practical instances, Theorem 2.3.4 is too general in the sense that the use of only one type of transformations, e.g., those of class (3), is sufficient to produce the desired result, as we shall illustrate later on in this section. The closed analytic approach to the proposed model of the hadronic structure in its primitive Newtonian form without subsidiary constraints can be then expressed by the following

THEOREM 2.3.5: Necessary and sufficient conditions for the Newtonian form of the unconstrained equations of motion for a system of eletons to admit an ordered direct analytic representation in terms of the generalized Hamiltonian principle

$$\delta^* \int_{t_1}^{t_2} dt\, L(t, \xi_\mu, \dot{\xi}_\mu) \Big|_{\xi = \text{act. path}} \qquad (2.3.22)$$

$$= -\int_{t_1}^{t_2} dt \left\{ \left[\mu_k \ddot{\xi}_{\mu k} - \underset{\mu k}{f}(t, \xi_\mu, \dot{\xi}_\mu) \right]_{SA} - \underset{\mu k}{F}(t, \xi_\mu, \dot{\xi}_\mu) \right\}_{NSA} \cdot \delta \xi_\mu^{\,k} = 0$$

$$k = 1, 2, \cdots, n, \quad \xi_\mu = (x_k, t_3, t_2)$$

is that the equations of motion admit a selfadjoint genotopic transformation which is well defined in a star-shaped region of the new variables.

Theorem 2.3.5 is the final characterization of our model in its primitive, unconstrained Newtonian form. It focuses the attention on the selection of models which admit an identifyable or otherwise computable selfadjoint transformation.

The "mechanics" of the practical use of Theorem 2.3.5 is ultimately simple and can be implemented according to the following sequential steps:

(A) select the type of nonconservative coupling for the desired purpose, e.g., the nonconservation of the angular momentum;

(B) construct a selfadjointness inducing (i.e., genotopic) transformation according to Theorem 2.3.4, i.e., solve Eqs. (2.3.21) in the unknown factor functions h_k^i and/or transformations of the variables; when such functions exist,

(C) compute "a" Lagrangian by using Theorem 2.3.3. The last terms needed, the factor terms (g) of Eq. (2.3.18), are trivially given by the inverse of the factor terms (h) of the genotopic transformation.

An explicit example is here in order. Suppose that one wants equations of motion which, at the first stage of the model (1.4.7) without constraints, violate all conservation

laws of a conventional two-body system. The following system will serve this purpose

$$\left(A_{ki}\ddot{z}^{i} + B_{k}\right)^{C^{\infty};R}_{NSA} = 0 \qquad (2.3.23a)$$

$$(A_{ij}) = \begin{pmatrix} 2(\dot{z}_1 + \dot{z}_2) & (2\dot{z}_1 + \dot{z}_2) \\ (\dot{z}_1 + 2\dot{z}_2) & 2(\dot{z}_1 + \dot{z}_2) \end{pmatrix}, \quad (B)_i = \begin{pmatrix} z_2\dot{z}_1 - z_2\dot{z}_1 + z_1z_2 + \frac{1}{2}z_1^2 \\ z_2\dot{z}_2 - z_1\dot{z}_2 + z_1z_2 + \frac{1}{2}z_2^2 \end{pmatrix}$$

$$(m_1 = m_2 = 1) \qquad (2.3.23b)$$

The selected couplings are clearly nonconservative and not derivable from a potential. The problem is formulated in two dimensions for simplicity. Its extension to the true six-dimensional case is left to the interested reader.

The physical interpretation of the fact that the couplings are not derivable from a potential can be translated in the language of the Inverse Problem into the mathematical property that the equations of motion (2.3.23) are nonselfadjoint. A direct verification of this property is here instructive.

The fact that the system is nonselfadjoint means that a Lagrangian for its ordered direct representation, i.e., that according to Theorem 2.3.3, does not exist.

To construct an analytic representation one therefore must identify at least one selfadjointness inducing equivalence transformation according to Theorem 2.3.4. The use of this theorem in its entirety is however redundant for the considered (relatively simple) case in the sense that the use of

only one class of the degrees of freedom of Hamilton's Principle is sufficient. The class which is here selected is class (3). We therefore perform the equivalence transformation

$$\left[h_k{}^{i}\left(A_{ij}\ddot{z}^{j} + B_i\right)^{C^{\infty};R}_{NSA}\right]^{C^{2};R}_{SA} = A^{*}_{kj}\ddot{z}^{j} + B^{*}_{k} \qquad (2.3.24)$$

in (at this time) unknown factor functions h_k^i which are restricted to characterize a regular, class C^2 matrix (h). We then impose the conditions of selfadjointness (2.3.21) on Eqs. (2.3.24) and interpret such conditions as a system of partial differential equations in the unknown factor terms h_k^i. A simple, but tedious calculation yields the desired result which reads

$$(h_i{}^{j}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (2.3.25)$$

Rather than considering the system in its original form (2.3.23) we now consider it in its equivalent form (2.3.24) which, in view of the solutions (2.3.25), explicitly reads

$$\left(A^{*}_{ki}\ddot{z}^{i} + B^{*}_{k}\right)^{C^{\infty};R}_{SA} = 0 \qquad (2.3.26a)$$

$$(A^{*}_{ij}) = \begin{pmatrix} (\dot{z}_1 + 2\dot{z}_2) & 2(\dot{z}_1 + \dot{z}_2) \\ 2(\dot{z}_1 + \dot{z}_2) & (\dot{z}_2 + 2\dot{z}_1) \end{pmatrix}, \quad (B^{*}_{i}) = \begin{pmatrix} z_2\dot{z}_2 - z_1\dot{z}_2 + z_1z_2 + \frac{1}{2}z_2^2 \\ z_1\dot{z}_1 - z_2\dot{z}_1 + z_1z_2 + \frac{1}{2}z_1^2 \end{pmatrix}$$

$$(2.3.26b)$$

We urge at this point the interested reader to verify that the above equivalent system satisfies <u>all</u> the conditions of Theorem 2.3.3 on the computation of a Lagrangian. Specifically, system (2.3.26) is of class C^{∞}, regular, selfadjoint and, last but not least, well behaved for all values fo the variables q^k and $\dot{q}^k$ in the closed interval (0.1). Theorem 2.3.4 then applies in full. Integrals (2.3.11) can be trivially computed and are given by

$$K = \underline{\overline{\dot{r}_1 \dot{r}_2}} \int_0^1 d\tau' \left\{ \int_0^1 d\tau \begin{pmatrix} \tau(\dot{r}_1 + 2\dot{r}_2) & 2\tau(\dot{r}_1 + \dot{r}_2) \\ 2\tau(\dot{r}_1 + \dot{r}_2) & \tau(\dot{r}_2 + 2\dot{r}_1) \end{pmatrix} \right] \begin{pmatrix} \dot{r}_1 \\ \dot{r}_2 \end{pmatrix} \Bigg\} (\tau' \dot{r})$$

$$= \frac{1}{6}\left(\dot{r}_1^3 + \dot{r}_2^3\right) + \dot{r}_1^2 \dot{r}_2 + \dot{r}_1 \dot{r}_2^2$$

$$\begin{pmatrix} D_1 \\ D_2 \end{pmatrix} = \left[\int_0^1 d\tau\, \tau \begin{pmatrix} 0 & Z^*_{12}(\tau r) \\ Z^*_{21}(\tau r) & 0 \end{pmatrix} \right] \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \qquad (2.3.27)$$

$$= \left[\int_0^1 d\tau \begin{pmatrix} 0 & \tau^2(r_2 - r_1) \\ \tau^2(r_1 - r_2) & 0 \end{pmatrix} \right] \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} r_2^2 - r_1 r_2 \\ r_1^2 - r_1 r_2 \end{pmatrix}$$

$$C = \underline{\overline{r_1 r_2}} \int_0^1 d\tau \begin{pmatrix} W^*_1(\tau q) \\ W^*_2(\tau q) \end{pmatrix}$$

$$= \underline{\overline{r_1 r_2}} \int_0^1 d\tau \begin{pmatrix} -\tau^2\left(r_1 r_2 + \frac{1}{2} r_2^2\right) \\ -\tau^2\left(r_1 r_2 + \frac{1}{2} r_1^2\right) \end{pmatrix}$$

$$= -\frac{1}{2}\left(r_1^2 r_2 + r_1 r_2^2\right)$$

A Lagrangian is therefore given by

$$L(r, \dot{r}) = K(\dot{r}) + D_k(r)\dot{r}^k + C(r)$$

$$= \frac{1}{6}\left(\dot{r}_1^3 + \dot{r}_2^3\right) + \dot{r}_1^2 \dot{r}_2 + \dot{r}_1 \dot{r}_2^2 + \frac{1}{3}\left(r_2^2 \dot{r}_1 + r_1^2 \dot{r}_2\right) \qquad (2.3.28)$$

$$- \frac{1}{3} r_1 r_2 (\dot{r}_1 + \dot{r}_2) - \frac{1}{2}\left(r_1^2 r_2 + r_1 r_2^2\right)$$

The reader is here again urged to verify that the above Lagrangian does indeed represent the considered system according to structure (2.3.19).

Representation (2.3.22) then reads

$$\delta^* \int_{t_1}^{t_2} dt\, L(r, \dot{r}) \Big|_{r = \text{act path}} = - \int_{t_1}^{t_2} dt \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_1(r) \\ L_2(r) \end{pmatrix}_{SA} \right]^{TRANS}_{NSA} \begin{pmatrix} \delta r_1 \\ \delta r_2 \end{pmatrix}$$

$$= - \int_{t_1}^{t_2} dt \begin{pmatrix} 2(\dot{r}_1 + \dot{r}_2)\ddot{r}_1 + (2\dot{r}_1 + \dot{r}_2)\ddot{r}_2 + r_1 \dot{r}_1 - r_2 \dot{r}_1 + r_1 r_2 + \frac{1}{2} r_1^2 \\ (\dot{r}_1 + 2\dot{r}_2)\ddot{r}_1 + 2(\dot{r}_1 + \dot{r}_2)\ddot{r}_2 + r_2 \dot{r}_2 - r_1 \dot{r}_2 + r_1 r_2 + \frac{1}{2} r_2^2 \end{pmatrix}^{TRANS}_{NSA} \begin{pmatrix} \delta r_1 \\ \delta r_2 \end{pmatrix} = 0$$

$$(2.3.29)$$

A few comments are here in order. First of all, notice that the simple permutation of the order in which the system is written has been sufficient (for the considered case) to transform it into a selfadjoint form. As a matter of fact this example was selected from those available from references[34] precisely in view of this feature. This occurrence has several implications. First of all it indicates the necessity of the use of the concept of ordering for the proper formulation of the basic theorems of the Inverse Problem. Secondly, it illustrates the sensibility of the conditions of selfadjointness to multiplicative factors of type (2.3.24) because, even a solution of the factor matrices with constant elements can in principle be sufficient to characterize a selfadjoint transformation. Thirdly, it illustrates that "the" Lagrangian is not unique. And indeed, solution (2.3.25) in the "genotopic functions" h_k^i is not necessarily unique. Thus, any other solution h_k^i with or without nontrivial functional dependence in the variables $(t,r,\dot{r})$ (which satisfies the assumed continuity and regularity conditions) will characterize another Lagrangian. This, in turn, indicates the existence of new degrees of freedom in the functional dependence of a Lagrangian which are a direct consequence of the integrability conditions for its existence, the conditions of selfadjointness. This latter aspect will be reviewed in Section 2.11.

2.4: THE INVERSE LAGRANGIAN PROBLEM IN SPECIAL RELATIVITY

We shall now outline the rudiments of the Inverse Problem in Special Relativity for the case of one particle. It should be indicated from the outset that our presentation is, on rigorous grounds, largely insufficient because we are interested at this time only in the Inverse Problem for unconstrained systems, while the full treatment of relativistic systems necessarily demands the use of subsidiary constraints. Nevertheless, the aspectes we shall review will be sufficient for the objectives of this monograph.

Let s be the proper time, and x^μ, $v^\mu = \frac{dx^\mu}{ds}$ and $a^\mu = \frac{dv^\mu}{ds}$ ($\mu = 0,1,2,3$) be the Minkowski coordinate, velocity and acceleration, respectively of the particle considered. A simple reinterpretation of the variational analysis of Section 2.3 then leads to the following

THEOREM 2.4.1: Necessary and sufficient conditions for the covariant class C^2 and regular one-particle system of (local) equations in Minkowski space

$$F_\mu(x) = F_\mu(s, x, v, a) = 0, \quad \mu = 0,1,2,3, \quad (2.4.1)$$

to be slefadjoint in a region R of its variables are that the equations be linear in the four-accelerations, i.e., of the type

$$F_\mu(x) = A_{\mu\nu}(s,x,v)\, a^\nu + B_\mu(s,x,v) = 0, \quad (2.4.2)$$

and all the following conditions

$$A_{\mu\nu} = A_{\nu\mu}\,, \tag{2.4.3a}$$

$$\frac{\partial A_{\mu\rho}}{\partial v^\nu} = \frac{\partial A_{\nu\rho}}{\partial a^\mu}\,, \tag{2.4.3b}$$

$$\frac{\partial B_\mu}{\partial v^\nu} + \frac{\partial B_\nu}{\partial v^\mu} = 2\left\{\frac{\partial}{\partial s} + v^\alpha \frac{\partial}{\partial x^\alpha}\right\} A_{\mu\nu}\,, \tag{2.4.3c}$$

$$\frac{\partial B_\mu}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu} = \frac{1}{2}\left\{\frac{\partial}{\partial s} + v^\alpha \frac{\partial}{\partial x^\alpha}\right\}\left(\frac{\partial B_\mu}{\partial v^\nu} - \frac{\partial B_\nu}{\partial v^\mu}\right), \tag{2.4.3d}$$

$$\mu, \nu, \alpha = 0, 1, 2, 3$$

be identically verified in R.

Eqs. (2.4.2), from the viewpoint of the theory of differential equations, are of quasilinear type. For more familiar forms of the equations of motion we have the

THEOREM 2.4.2: Necessary and sufficient conditions for the covariant, class C^2, one-particle system of (local) equations in Minkowski space

$$m_0 a_\mu - K_\mu(s, x, v) = 0, \tag{2.4.4}$$

to be selfadjoint in a region R of its variables are that the Minkowski force K_μ be linear in the velocities, i.e., of the type

$$K_\mu = \rho_{\mu\nu}(s, x)\, v^\nu + \sigma_\mu(s, v), \tag{2.4.5}$$

and all the following conditions

$$\rho_{\mu\nu} + \rho_{\nu\mu} = 0\,, \tag{2.4.6a}$$

$$\frac{\partial \rho_{\mu\nu}}{\partial x^\sigma} + \frac{\partial \rho_{\nu\sigma}}{\partial x^\mu} + \frac{\partial \rho_{\sigma\mu}}{\partial x^\nu} = 0\,, \tag{2.4.6b}$$

$$\frac{\partial \rho_{\mu\nu}}{\partial s} = \frac{\partial \sigma_\mu}{\partial x^\nu} - \frac{\partial \sigma_\nu}{\partial x^\mu}\,, \tag{2.4.6c}$$

be identically verified in R.

The best illustration of Theorem 2.4.2 is that of a charged particle under the action of the Lorentz force, i.e.

$$m_0 a_\mu - K_\mu(x, v) = 0 \tag{2.4.7}$$

$$K_\mu = e\, F_{\mu\nu}(x)\, v^\nu\,, \quad F_{\mu\nu} = \frac{\partial A_\mu}{\partial a^\nu} - \frac{\partial A_\nu}{\partial a^\mu}\,, \quad c = 1,$$

which, as the reader can verify with a simple inspection, is selfadjoint.[57] The study of other systems (e.g., with the inclusion of retarded and advanced effects) is here left to the interested reader.

The extension of Theorem 2.3.1 is trivial and it is here reported for completeness.

THEOREM 2.4.3: The (unconstrained) Lagrange's equations in class C^4 and regular Lagrangians

$$L_\mu(x) = \frac{d}{ds}\frac{\partial L}{\partial v^\mu} - \frac{\partial L}{\partial x^\mu} \qquad (2.4.8)$$

$$= \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} a^\nu + \frac{\partial^2 L}{\partial v^\mu \partial x^\nu} v^\nu + \frac{\partial^2 L}{\partial v^\mu \partial s} - \frac{\partial L}{\partial x^\mu} = 0$$

are always selfadjoint.

Explicitly, Theorem 2.4.3 states that Eqs. (2.4.8), under the indicated conditions, are selfadjoint for all possible Lagrangians, irrespective of their explicit functional dependence.

It is of some significance to indicate the Jacobi's equations of Special Relativity (i.e., the equations of variation of Lagrange's equations)[34-I] which are given by

$$J_\mu(\eta) = \frac{d}{ds}\frac{\partial J}{\partial \dot\eta^\mu} - \frac{\partial J}{\partial \eta^\mu} = 0, \qquad (2.4.9a)$$

$$J = \frac{1}{2}\left(\frac{\partial^2 L}{\partial v^\mu \partial v^\nu}\dot\eta^\mu\dot\eta^\nu + 2\frac{\partial^2 L}{\partial v^\mu \partial x^\nu}\dot\eta^\mu\eta^\nu + \frac{\partial^2 L}{\partial x^\mu \partial x^\nu}\eta^\mu\eta^\nu\right) \qquad (2.4.9b)$$

$$\eta^\mu = \left.\frac{\partial x^\mu(s;w)}{\partial w}\right|_{w=0}, \quad \dot\eta^\mu = \left.\frac{\partial v^\mu(s;w)}{\partial w}\right|_{w=0} \qquad (2.4.9c)$$

These equations are associated to Eqs. (2.4.8) in the sense that they are (uniquely) characterized by the latter and constitute their necessary complement from the variational approach to selfadjointness. Indeed, Eqs. (2.4.8) are selfadjoint because, strictly speaking, Eqs. (2.4.9) always coincide with their adjoint system for all admissible variations (under the assumed conditions).

Again, Lagrange's equations (2.4.8) are generally nonlinear while their associated Jacobi's equations are always linear.

The continuity conditions of the preceding theorems are selfevident. This is not the case for the condition of regularity. Indeed, a Lagrangian conventionally assumed for system (2.4.7) is given by

$$L = m_0 + e A_\mu v^\mu, \qquad (2.4.10)$$

namely, it is of the first order in the velocity. As such, it is totally degenerate in the terminology of Section 2.1. Nevertheless, it is known that an equally acceptable Lagrangian for the same system is given by

$$L = m_0 v^\mu v_\mu + e A_\mu v^\mu, \qquad (2.4.11)$$

and reduces to Lagrangian (2.4.10) under the relativistic constraint ($c = 1$)

$$v^\mu v_\mu = 1. \qquad (2.4.12)$$

Thus, in the absence of constraints (2.4.12), Lagrangian (2.4.11) is regular. From now on we shall therefore consider the Lagrangian aspect of the unconstrained relativistic motion and then impose the needed subsidiary constraints.

The selfadjointness of Lagrange's equations can also be proved within the context of the relativistic kinematics with subsidiary constraints. This demands the reformulation of the variational approach to selfadjointness on the hypersurface of the constraints. Since this is not essential for our needs, it will not be considered.

The Fundamental Analytic Theorem in Special Relativity, as a direct extension of Thoerem 2.3.2, can be formulated as follows

THEOREM 2.4.4: Necessary and sufficient condition for a quasilinear system

$$F_\mu(s,x,v,a) = A_{\mu\nu}(s,x,v)a^\nu + B_\mu(s,x,v) = 0, \quad (2.4.13)$$
$$\mu = 0,1,2,3$$

which is of at least class C^2, regular and well behaved in a star-shaped region R* of its (local) variables to admit the ordered direct analytic representation

$$\frac{d}{ds}\frac{\partial L}{\partial v^\mu} - \frac{\partial L}{\partial x^\mu} \equiv A_{\mu\nu}a^\nu + B_\mu \quad (2.4.14)$$

is that the system is selfadjoint in R*.

Again, under the indicated restrictions, the conditions of selfadjointness result to be the integrability conditions for the existence of a Lagrangian.

As an alternative method for the computation of a Lagrangian to that of Theorem 2.3.3 we have the following[58]

THEOREM 2.4.5: When all the conditions of Theorem 2.4.4 are verified, a Lagrangian is given by

$$L(s,x,v) \quad (2.4.15)$$
$$= -x^\mu\int_0^1 d\tau\, F_\mu(s,\tau x,\tau v,\tau a) + \frac{d}{ds}\int_0^1 d\tau\int_0^1 d\tau'\,(\tau x^\mu)A_{\mu\nu}(s,\tau x,\tau\tau' v)v^\nu$$

Notice that this alternative method too appears to be computerizable. The specialization of Thoerems 2.4.4 and 2.4.5 to the more familiar form (2.4.4) is here left to the interested reader. The computation of Lagrangian (2.4.11) via Eq. (2.4.15) for system (2.4.7) is trivial and it is here also left to the interested reader.

A reason for our emphasis on the quasilinear form (2.4.2) rather than the conventional form (2.4.4) is that, if the latter form is nonselfadjoint, one can attempt the construction of a Lagrangian through a genotopic transformation, i.e. a transformation to an equivalent selfadjoint form within a fixed system of local variables (Section 2.3). This yields necessarily a quasilinear form according to the structure

$$\left\{A_{\mu\nu}\left[a^\nu - K^\nu\right]^{C^2,R}_{NSA}\right\}^{C^2,R}_{SA} = \left(A_{\mu\nu}a^\nu + B_\mu\right)^{C^2,R}_{SA} = 0. \quad (2.4.16)$$

(The usual way of transforming a covariant into a contravariant vector through the use of the Minkowski metric tensor is here tacitly assumed).

To illustrate this important point consider the equation of motion under the Minkowski force quadratic in the velocities

$$\left(a^\mu - \Gamma^\mu{}_{\rho\sigma}(x)\, v^\rho v^\sigma\right)_{NSA} = 0, \qquad (2.4.17)$$

which is particularly significant for gravitational consideration. And indeed the gravitational field can be interpreted, for the case of Eqs. (2.4.16), as characterized by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial g_{\sigma\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right) \qquad (2.4.18)$$

$$(g^{\mu\nu}) = (g_{\mu\nu})^{-1}$$

where $g_{\mu\nu}$ is now the gravitational potential.

Eqs. (2.4.17) is of form (2.4.4), also called the kinematical form[34-I], and not of the quasilinear form (2.4.2). But one of the necessary conditions for the kinematical form to be selfadjoint is that it is linear in the velocities (Theorem 2.4.2). As a result, Eqs. (2.4.17), as given, cannot be selfadjoint. The only possibility of constructing a Lagrangian (within the context of our approach of considering the subsidiary constraints as a subsequent imposition on the theory) is that of transforming the equations into an equivalent quasilinear form. Indeed, the conditions of selfadjointness for the latter form do not demand

linearity in the velocities or in the coordinates (Theorem 2.4.1).

As an illustrative example in the (flat) Minkowski space, assume that the Γ- tensor admits the decomposition

$$\Gamma^\mu_{\alpha\beta} = A^{\mu\nu}\left(\frac{\partial A_{\nu\alpha}}{\partial x^\beta} - \frac{\partial A_{\alpha\beta}}{\partial x^\nu}\right), \qquad (2.4.19)$$

$$(A^{\mu\nu}) = (A_{\mu\nu})^{-1}$$

The following genotopic transformation then exists.

$$\left\{A_{\mu\nu}\left[a^\nu - \Gamma^\nu_{\alpha\beta} v^\alpha v^\beta\right]_{NSA}\right\}_{SA} = 0, \qquad (2.4.20)$$

Theorem 2.4.5 then trivially yields as an admissible Lagrangian

$$L = v^\mu A_{\mu\nu}(x) v^\nu, \qquad (2.4.21)$$

where the additional need of subsidiary constraints for a geodesic motion must be kept in mind.

For an example of equations appearing naturally in the quasilinear form, consider the known relativistic generalization of the Newtonian forces derivable from a potential

$$m_0 a^\mu + K^\mu(x, v, a)$$
$$= m_0 a^\mu - \frac{\partial \phi}{\partial x^\alpha}\left(g^{\mu\alpha} - v^\mu v^\alpha\right) - \phi a^\mu = 0 \qquad (2.4.22)$$

This equation is quasilinear in our terminology because of the structure

$$A_{\mu\nu}\, a^{\nu} + B_{\mu} = 0, \qquad (2.4.23)$$

$$A_{\mu\nu} = (m_0 - \phi)\, g_{\mu\nu}\,, \quad B_{\mu} = g_{\mu\alpha} \frac{\partial \phi}{\partial x^{\beta}} (g^{\alpha\beta} v^{\alpha} v^{\beta})$$

It is an easy exercise to prove that the above equations satisfy all the conditions of Theorem 2.4.4. In particular, they are selfadjoint. Theorem 2.4.5 then holds, yielding as an admissible Lagrangian

$$L = [m_0 - \phi(x)]\, v^{\mu} v_{\mu}\,. \qquad (2.4.24)$$

Of course, a genotopic transformation within a fixed system of variables does not necessarily exist. This is the case when the conditions of selfadjointness (2.4.3), applied to Eqs. (2.4.19) and interpreted as partial differential equations in the "genotopic factor" $A_{\mu\nu}$ with fixed K-terms, are inconsistent.

When this is the case, a Lagrangian can still exist owing to the possibility of using coordinate transformations. As a matter of fact, in Section 2.8 we shall recall the Universality Theorem of Inverse Problem which, for the case at hand, guarantees the existence of an analytic representation for all class C^{∞}_{4}

regular, quasilinear, systems (2.4.2).

This completes the rudiments of the Inverse Problem in Special Relativity which are needed for our relativity considerations of Chapter 3. The extension of the methodology of this section to the case of many particles is straightforward and it is here left as an exercise for the interacted reader. For the extension of the approach to higher order Lagrange's equations

$$\left[\sum_{k=0}^{m-1} (-1)^{k+1} \frac{d^{(k)}}{ds^{(k)}} \frac{\partial L}{\partial \left(\frac{\partial^{(k)} x^{\mu}}{\partial s^{(k)}} \right)} \right]_{SA} = 0, \qquad (2.4.25)$$

see references[34,53] For instance, third order Lagrange's equations are significant for the analytic representation of the third order Lorentz-Dirac equations[59]

$$F_{\mu}(s, x, v, a, \dot{a}) \qquad (2.4.26)$$

$$= m_0 a^{\mu} - e F^{\mu\nu}_{in} v_{\nu} - F^{\mu}_{ext} - \frac{2}{3} e^2 (\dot{a}^{\mu} - v^{\mu} a_{\alpha} a^{\alpha}) = 0,$$

according to the ordered indirect analytic representation

$$\sum_{k=0}^{2} (-1)^{k+1} \frac{d^{(k)}}{ds^{(k)}} \frac{\partial L}{\partial \left(\frac{\partial^{(k)} x^{\mu}}{\partial s^{(k)}} \right)} \qquad (2.4.27)$$

$$\equiv \left\{ A_{\mu\nu}(s, x, v, a) \left[F^{\nu}(s, x, v, a, \dot{a}) \right]^{C^2, R}_{NSA} \right\}^{C^2, R}_{SA} = 0.$$

The degrees of freedom of the Lagrangian structure will be considered later on in Section 2.10. The reader should keep in mind that, again, this analysis will be conducted for the unconstrained case and that the degrees of freedom of the Lagrangian structure are significantly increased when the systems are implemented with subsidiary constraints. For instance, within the context of unconstrained theories, Lagrangians (2.4.10) and (2.4.11) are nonequivalent. However, when computed on the hypersurface of the constraints (2.4.12) they coincide. Thus, the degrees of freedom of the Lagrangian structure in Special Relativity are characterized by those considered in Section 2.10, plus all the additional degrees of freedom admitted by the constrained nature of the equations. These aspects will be considered in details at some later time.

As a final remark, the reader should be aware that the aspects of the Inverse Lagrangian Problem reviewed in this Section apply specifically to the (flat) Minkowski space. Their extension to a Riemannian space can be formulated, but it demands a predictable technical implementation.

2.5: THE INVERSE LAGRANGIAN PROBLEM IN FIELD THEORY

The transition from the methodology of Section 2.3 to that of classical unconstrained field theories is straightforward, but predictably more laborious. For the reader's convenience we here summarize the main results of references.[30]

Consider first the case of a quasilinear system of second-order partial differential equations

$$F_k(\phi) = \left[A^{\mu\nu}_{ki}(x^\alpha, \phi^j, \phi^j_\alpha)\, \phi^{ij}_{\mu\nu} + B_k(x^\alpha, \phi^j, \phi^j_\alpha) \right]^{C^2;R} = 0, \quad (2.5.1)$$

$$k, i, j = 1, 2, \ldots, n \;, \quad \mu, \nu, \alpha = 0, 1, 2, 3$$

where the symbols C^2 and R again characterize our (minimal) continuity and regularity condition.

The variational forms in this case are

$$M_k(\eta) = \frac{d}{dw} F_k[\phi(x;w)]\Big|_{w=0}$$

$$= \frac{\partial F_k}{\partial \phi^i}\eta^i + \frac{\partial F_k}{\partial \phi^i_\mu}\eta^i_\mu + \frac{\partial F_k}{\partial \phi^{ij}_{\mu\nu}}\eta^{ij}_{\mu\nu} \qquad (2.5.2)$$

$$= a_{ki}\eta^i + b^\mu_{ki}\eta^i_\mu + c^{\mu\nu}_{ki}\eta^{ij}_{\mu\nu}$$

$$\eta^i = \frac{\partial \phi^i}{\partial w}\Big|_{w=0}, \quad \eta^i_\mu = \frac{\partial \eta^i}{\partial x^\mu}, \text{ etc.}$$

The adjoint system is given by

$$\tilde{M}_k(\tilde{\eta}) = \tilde{\eta}^i a_{ik} - \partial_\mu(\tilde{\eta}^i b^\mu_{ik}) + \partial_\nu\partial_\mu(\tilde{\eta}^i c^{\mu\nu}_{ik}), \quad (2.5.3)$$

and is (uniquely) characterized by the Lagrange identity

$$\tilde{\eta}^i M_i(\eta) - \eta^i \tilde{M}_i(\tilde{\eta}) = \partial_\mu J^\mu(\eta, \tilde{\eta}) \qquad (2.5.4a)$$

$$J^\mu = \tilde{\eta}^i b^\mu_{ij}\eta^j + \tilde{\eta}^i c^{\mu\nu}_{ij}\eta^j_\nu - [\partial_\nu(\tilde{\eta}^i c^{\nu\mu}_{ij})]\eta^j \qquad (2.5.4b)$$

Predictably, the total time derivative $\dot{Q}(\eta,\tilde{\eta})$ of Eqs. (2.3.4a) is generalized into a four-divergence of a current density $J^{\mu}(\eta,\tilde{\eta})$. The Green version of the Lagrange identity

$$\int_{R_x} d^4x \left[\tilde{\eta}^i M_i(\eta) - \eta^i \tilde{M}_i(\tilde{\eta})\right] = \int_{\partial R_x} d\sigma_\mu(x)\, J^\mu \qquad (2.5.5)$$

is not used in the Inverse Problem because it implies the lack of uniqueness of the adjoint system.

Again, the system is selfadjoint when the variational forms coincide with their adjoint system for all admissible variations, i.e., $M_i(\eta) \equiv \tilde{M}_i(\eta)$, $i = 1,2,\dots,n$, $\eta^i(x) \in C^2(R_x)$.

The following theorem can then be proved.

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THEOREM 2.5.1: Necessary and sufficient condition for a quasilinear system of second order partial differential equations

$$A^{\mu\nu}_{ki}(x^\alpha, \phi^j, \phi^j{}_{;\alpha})\,\phi^{i;j}_{\mu\nu} + B_k(x^\alpha, \phi^j, \phi^j{}_{;\alpha}) = 0, \qquad (2.5.6a)$$

$$A^{\mu\nu}_{ki} = A^{\nu\mu}_{ki} \qquad (2.5.6b)$$

which is of at least class C^2 and regular in a region R of its variables to be selfadjoint in R is that each and all the following conditions

$$\begin{cases}
A^{\mu\nu}_{ki} = A^{\nu\mu}_{ik}, & (2.5.7a)\\
A^{\nu\alpha;\mu}_{kj\ \ i} + A^{\nu\alpha;\mu}_{ij\ \ k} = A^{\mu\nu;\alpha}_{ik\ \ j}, & (2.5.7b)\\
A^{\alpha\beta;\mu;\nu}_{ki\ j\ \ell} = A^{\alpha\beta;\mu;\nu}_{ji\ k\ \ell}, & (2.5.7c)\\
B_k{}^{;\mu}_{i} + B_i{}^{;\mu}_{k} = 2\left\{\partial_\nu + \phi^{j;}_{\ \nu}\frac{\partial}{\partial\phi^{j;}}\right\}A^{\mu\nu}_{ki}, & (2.5.7d)\\
B_k{}^{;\nu}_{i} - B_i{}^{;\nu}_{k} = \frac{1}{2}\left\{\partial_\nu + \phi^{j;}_{\ \nu}\frac{\partial}{\partial\phi^{j;}}\right\}\left(B_k{}^{;\nu}_{i} - B_i{}^{;\nu}_{k}\right), & (2.5.7e)
\end{cases}$$

$$A^{\mu\nu;\alpha}_{ki\ \ j} = \frac{\partial A^{\mu\nu}_{ki}}{\partial\phi^{j;}_{\ \alpha}}, \quad B_k{}^{;}_{i} = \frac{\partial B_k}{\partial\phi^{i;}}, \text{ etc.}, \qquad (2.5.7f)$$

$$A^{\mu\nu;\alpha}_{ki\ \ j} = B^{\mu\nu;\ \alpha}_{ki;\ j} + B^{\mu\ \alpha;\nu}_{k\ i\ j}, \text{ etc.}, \qquad (2.5.7g)$$

$$A^{\alpha\beta;\mu;\nu}_{ki\ j\ \ell} = A^{\alpha\beta;\mu;\nu}_{ki\ j\ \ell} + A^{\alpha\beta;\mu;\nu}_{k\ell\ j\ i}, \qquad (2.5.7h)$$

$k,i,j,\ell = 1,2,\dots,n$, $\mu,\nu,\alpha,\beta = 0,1,2,3$,

are everywhere satisfied in R.

Notice that, in full analogy with the Newtonian case, a system with the structure (2.5.6a), i.e., quasilinear, is the most general form of field equations which is of the Lagrangian type (2.1.3) and, thus, representable by a Lagrangian density. Symmetry property (2.5.6b) can always be assumed up to redefinition of the terms $A^{\mu\nu}_{ki}$ from the symmetry of the second order derivatives $\phi^{i;j}_{\mu\nu} = \phi^{i;j}_{\nu\mu}$. The following more familiar factorization

$$A^{\mu\nu}_{ki} = A_{ki}\otimes g^{\mu\nu}, \qquad (2.5.8)$$

for which

$$A^{\mu\nu}_{ki}\,\phi^{i;j}_{\mu\nu} + B_k = A_{ki}\,\Box\,\phi^i + B_k = 0, \qquad (2.5.9)$$

$$\Box = \partial_\mu\partial^\mu$$

can in particular hold. The point is that such factorization does not necessarily exist for the considered general class of systems (2.5.6).

This is the first relevant difference between conventional field theoretical models with couplings derivable from a potential and our generalized equations for the eletons. Factorization (2.5.8) always exists for the conventional models while it does not necessarily exist for our models. For more details see Section 2.12.

To recover more familiar models, the approach of Theorem 2.5.1 can be repeated for that of the following

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THEOREM 2.5.2: Necessary and sufficient condition for the semilinear system of second order partial differential equations

$$(\Box + m^2_{(k)})\,\phi_k - F_k(x^q, \phi^j, \phi^{j}{}_{;\alpha}) = 0, \qquad (2.5.10)$$

which are of at least class C^2 in a region R of their variables is that each and all the following conditions on the couplings F_i

$$\begin{cases} F_{i\,;j}{}^{\mu} + F_{j\,;i}{}^{\mu} = 0, & (2.5.11a) \\ F_{k\,;i}{}^{\mu}{}_{;j}{}^{\nu} - F_{i\,;k}{}^{\mu}{}_{;j}{}^{\nu} = 0, & (2.5.11b) \\ F_{i\,;j} - F_{j\,;i} = \frac{1}{2}\left\{\partial_\mu + \phi^{k}{}_{;\mu}\frac{\partial}{\partial \phi^k}\right\}(F_{i\,;j}{}^{\mu} - F_{j\,;i}{}^{\mu}), & (2.5.11c) \end{cases}$$

$$i, j, k = 1, 2, \ldots, n, \quad \mu, \nu = 0, 1, 2, 3$$

are everywhere satisfied in R.

When conditions (2.5.11) are satisfied we shall say that the j's are selfadjoint couplings. If at least one of conditions (2.5.11) is not verified we shall say that the j's are nonselfadjoint couplings. By working out explicitly conditions (2.5.11) we have the

COROLLARY 2.5.2B: Necessary and sufficient conditions for Eqs. (2.5.10) to be selfadjoint are that the couplings j_k possess the structure

$$F_k = \left[\frac{\partial \Gamma^{\mu}_{k}(x^q, \phi^j)}{\partial \phi^i} - \frac{\partial \Gamma^{\mu}_{i}(x^q, \phi^j)}{\partial \phi^k}\right]\phi^{i}{}_{;\mu} + \Lambda_k(x^q, \phi^j), \qquad (2.5.12)$$

and all the following conditions

$$\partial_\mu\left(\Gamma_k{}^{\mu}{}_{;i} - \Gamma_i{}^{\mu}{}_{;k}\right) = \Lambda_{k\,;i} - \Lambda_{i\,;k} \qquad (2.5.13)$$

are everywhere satisfied in the region R.

On a comparative basis between Theorem 2.5.1 and 2.5.2, the central difference is that the condition of selfadjointness necessarily implies the linearity in the derivative couplings for the semilinear form (2.5.10) while this restriction is absent for the quasilinear form (2.5.6). Of course in both cases there is no restriction of linearity in the fields, i.e. polynomial couplings in the fields are admissible by the conditions of selfadjointness (2.5.7) and (2.5.11).

Consider, as an example, the interaction of a complex scalar field with an external electromagnetic field. If the field

equations are written in the ordering $\{\phi\} = (e, \bar{e})$, i.e.,

$$\left(\Box + \mu^2\right)\begin{pmatrix} e \\ \bar{e} \end{pmatrix} - \begin{pmatrix} e^2 A_\mu A^\mu e + 2ieA^\mu e^{;}{}_\mu \\ e^2 A_\mu A^\mu \bar{e} - 2ieA^\mu \bar{e}^{;}{}_\mu \end{pmatrix} = 0, \quad (2.5.14)$$

they are nonselfadjoint. However, if the same equations are written in the reverse ordering, i.e.

$$A_{ki}\left(\Box + \mu^2\right)\phi^i - F_k = 0 \quad (2.5.15a)$$

$$\phi^i = \begin{pmatrix} e \\ \bar{e} \end{pmatrix}, \quad (A_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad F_k = \begin{pmatrix} e^2 A_\mu A^\mu \bar{e} - 2ieA^\mu \bar{e}^{;}{}_\mu \\ e^2 A_\mu A^\mu e + 2ieA^\mu e^{;}{}_\mu \end{pmatrix} \quad (2.5.15b)$$

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they become selfadjoint. This is an indication of the sensitivity of the selfadjointness of the field equations (particularly those of gauge theories) to the assumed ordering. The interested reader is here urged to study the selfadjointness or nonselfadjointness of other familiar models of Field Theory (such as real scalar fields with polynomial couplings, the sine-Gordon equation , etc.).

Theorem 2.3.1 also extends to field theory and we have

THEOREM 2.5.3: Class C^2 and regular Lagrange's equations

$$\mathcal{L}_k(\phi) = d_\mu \mathcal{L}^{;\mu}_{\;k} - \mathcal{L}^{;}{}_k \quad (2.5.16)$$

$$= \frac{1}{2}\left(\mathcal{L}^{;\mu}_{\;k\;;i}{}^{;\nu} + \mathcal{L}^{;\nu}_{\;k\;;i}{}^{;\mu}\right)\phi^{;i}_{\mu\nu} + \mathcal{L}^{;\mu}_{\;k\;;i}\,\phi^{;i}_{\mu} + \mathcal{L}^{;\mu}_{\;k\;;\mu} - \mathcal{L}^{;}{}_k = 0$$

are always selfadjoint in their region of definition.

Notice that if one defined the Lagrange's equation with partial derivatives ∂_μ, (rather than the total derivative d_μ), as currently done in the literature of Field Theory, their central property of selfadjointness is lost. For more details see Appendix 2.A.

The Jacobi equations now reads

$$\mathcal{J}_k(\eta) = \left.\frac{d\,\mathcal{L}_k[\phi(x;w)]}{dw}\right|_{w=0} = d_\mu \mathcal{J}^{;\mu}_{\;k} - \mathcal{J}^{;}{}_k = 0, \quad (2.5.17a)$$

$$\mathcal{J} = \mathcal{J}(x^\alpha, \eta^j, \eta^{j}{}_{;\alpha})$$

$$= \frac{1}{2}\left(\mathcal{L}^{;\mu}_{i\;;j}{}^{;\nu}\eta^{;i}_{\mu}\eta^{;j}_{\nu} + 2\mathcal{L}^{;\mu}_{i\;;j}\,\eta^{;i}_{\mu}\eta^{;j} + \mathcal{L}^{;}{}_{i\;;j}\,\eta^{i}\eta^{j}\right). \quad (2.5.17b)$$

Theorem 2.5.3 can be proved by:(1) establishing that the Jacobi equations coincide with their adjoint system, or (2) by showing that Eqs. (2.5.16) satisfy all conditions (2.5.7) or (3) by proving that the Lagrange current (2.5.3b) for Lagrange's equations is identically null.

Notice that, while Lagrange's equations (2.5.16) are quasilinear, their related Jacobi equations (2.5.17) are always linear in all variables η^k, $\eta^{k;}_{\;\mu}$ and $\eta^{k;}_{\mu\nu}$. The joint use of these two equations is therefore particularly intriguing, e.g., for nonlinear models. Oddly, the Jacobi equations are virtually ignored in the current physical literature while they play a crucial role in the Inverse Problem as well as, more

generally, in the literature of the Calculus of Variations.

The concept of ordering in the definition of the analytic representation essentially implies that all the following identities hold

$$\mathcal{L}_k(\phi) \equiv F_k(\phi) \qquad (2.5.18a)$$

$$\mathcal{J}_k(\eta) \equiv M_k(\eta) \qquad (2.5.18b)$$

$$\tilde{\mathcal{J}}_k(\tilde{\eta}) \equiv \tilde{M}_k(\tilde{\eta}) \qquad (2.5.18c)$$

namely, not only the Lagrange equations coincide with the field equations, but the Jacobi equations coincide with the equations of variations and their adjoint system coincides with the adjoint system of the field equations.

If the ordering condition is relaxed, then the identity of the adjoints can be violated.[30-I,II] This would be the case for system (2.5.14) without ordering restriction.

The Fundamental Analytic Theorem of Classical Field Theory (for the unconstrained case of second order equations) reads

THEOREM 2.5.4[30-II]: Necessary and sufficient condition for a quasilinear system of second order partial differential equations

$$F_k(\phi) = A_{ki}^{\mu\nu}(x^\alpha, \phi^j, \phi^j_{,\alpha})\,\phi^i_{,\mu\nu} + B_k(x^\alpha, \phi^j, \phi^j_{,\alpha}) = 0, \quad A_{ki}^{\mu\nu} = A_{ki}^{\nu\mu} \qquad (2.5.19)$$

$k,i,j = 1,2,\dots,n$, $\mu,\nu,\alpha = 0,1,2,3$

which is well defined, of at least class C^2 and regular in a start-shaped region R* of its variables to admit the ordered direct analytic representation in R*

$$\mathcal{L}_k(\phi) \equiv F_k(\phi), \quad k=1,2,\dots,n, \qquad (2.5.20)$$

is that the system is selfadjoint in R.

As for the corresponding Newtonian case, the proof of the above theorem given in reference[30-II] is based on the use of the Converse of the Poincaré Lemma within the context of the Calculus of Differential forms. This is the reason for the formulation of this theorem in terms of a star-shaped rather than an ordinary region.

As reference[30-II] indicates, the proof of Theorem 2.5.4 is rather laborious. This is partially due to the need of extending the Calculus of Differential Forms to p-forms, p=1,2,..., with mixed indices of internal type, i.e., the k's, and of space-time type, i.e., the μ's and ν's. This latter occurrence is absent in the Newtonian case and therefore the Calculus of Differential forms in its conventional presentation can be used.

In line with the remarks of Section 2.1, Theorem 2.5.4 can be proved with a variety of sophisticated mathematical techniques of disciplines other than that of the Calculus of Differential forms (such as Functional Analysis, Differential Geometry, etc.).[53] The Calculus of Differential Forms has been selected on pragmatic grounds, because it provides one of the most direct approaches not only to the proof of the theorem, but also to the methods for the compuration of a Lagrangian. Indeed, the following theorem directly results from these techniques.

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THEOREM 2.5.5: A Lagrangian density for system (2.5.19) satisfying all the conditions of Theorem 2.5.4 is given by

$$\mathcal{L} = \mathcal{L}(x^\alpha, \phi^j, \phi^{ij}_\alpha) = K(x^\alpha, \phi^j, \phi^{ij}_\alpha) + D^\mu_k(x^\alpha, \phi^j)\phi^{kj}_\mu + C(x^\alpha, \phi^j), \quad (2.5.21)$$

where the 4n+2 densities K, D^μ_k and C are a solution of the generally overdetermined system of partial differential equations as given below

$$\frac{1}{2}\left(K^{;\mu}_{i\ j}{}^{;\nu} + K^{;\nu}_{i\ j}{}^{;\mu}\right) = A^{\mu\nu}_{ij}(x^\alpha, \phi^j, \phi^{ij}_\alpha), \quad (2.5.22a)$$

$$D^\mu_i{}^{;}{}_j - D^\mu_j{}^{;}{}_i = \frac{1}{2}\left(B_i{}^{;\mu}_j - B_j{}^{;\mu}_i\right) + \left(K^{;}_i{}^{;\mu}_j - K^{;}_j{}^{;\mu}_i\right)$$
$$= Z^\mu_{ij}(x^\alpha, \phi^j), \quad (2.5.22b)$$

$$C^{;}{}_i = D^\mu_i{}^{;}{}_\mu - B_i - K^{;}{}_i - K^{;\mu}_i{}^{;}{}_\mu$$
$$+ \left[K^{;}_i{}^{;\mu}_j + \frac{1}{2}\left(B_i{}^{;\mu}_j - B_j{}^{;\mu}_i\right)\right]\phi^{ij}_\mu$$
$$= W_i(x^\alpha, \phi^j), \quad (2.5.22c)$$

$$K = \phi^{ij}_\mu \left\{\int_0^1 d\tau' \left[\int_0^1 d\tau\, A^{\mu\nu}_{ij}(x^\alpha, \phi^j, \phi^{ij}_\alpha)\right]\phi^{ij}_\nu\right\}(x^\alpha, \phi^j, \tau'\phi^{ij}_\alpha), \quad (2.5.22d)$$

$$D^\mu_i = \left[\int_0^1 d\tau\, \tau\, Z^\mu_{ij}(x^\alpha, \tau\phi^j)\right]\phi^j, \quad (2.5.22e)$$

$$C = \phi^i \int_0^1 d\tau\, W_i(x^\alpha, \tau\phi^j). \quad (2.5.22f)$$

The analogy with the corresponding Newtonian case is selfevident. Notice that, again, when the conditions of Theorem 2.5.4 are verified, integrals (2.5.22) exist. In practice they must be computed in a progressive order, i.e., integral (2.5.22d) must be solved first. The knowledge of a solution for K allows to compute the Z-functions and thus integral (2.5.22e). The knowledge of K, D^μ_i (as well as the A- and B-terms of the given system) then allows to compute the W-functions and finally integral (2.5.22f). Presumably, the method is computerizable.

For the case of familiar models with couplings derivable from a potential, we have the

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COROLLARY 2.5.5A: A Lagrangian density for the ordered direct analytic representations of the semi-linear systems of second order equations

$$(\Box + m^2)\phi_k \quad (2.5.23)$$
$$- \left[\Gamma^\mu_k(x^\alpha, \phi^j)^{;}{}_i - \Gamma^\mu_i(x^\alpha, \phi^j)^{;}{}_k\right]\phi^{ij}_\mu - \Lambda_k(x^\alpha, \phi^j) = 0$$

which are well defined, of class C^1 and selfadjoint in a star-shaped region of their variables is given by

$$\mathcal{L} = \frac{1}{2}\left(\phi^{ij}_\mu \phi^{ij\mu} - m^2_{(i)}\phi^i\phi_i\right) - \Gamma^\mu_i(x^\alpha, \phi^j)\phi^{ij}_\mu + C(x^\alpha, \phi^j) \quad (2.5.24)$$

where the C-density is a solution of the integral

$$C = \phi^i \int_0^1 d\tau \left(\Lambda_i - \Gamma^\mu_i{}^{;}{}_\mu\right)(x^\alpha, \tau\phi^j) \quad (2.5.25)$$

As an example of this latter case consider the (self-adjoint) sine-Gordon equation

$$\Box \varphi + \sin \varphi = 0 \qquad (2.5.26)$$

The conditions of Corollary 2.5.5A are satisfied. A Lagrangian is then given by

$$\mathcal{L} = \frac{1}{2} \varphi^{i}_{\ \mu} \varphi_{i}^{\ \mu} + C(\varphi) \qquad (2.5.27)$$

$$C(\varphi) = -\varphi \int_0^1 d\tau \sin \tau\varphi = \cos\varphi - 1$$

As an example of the more general case of Theorem 2.5.5 consider the quasilinear system of field equations

$$G_{ij}(\varphi) \Box \varphi^{j} + G_{ij}(\varphi)^{;}_{\ k} \varphi^{k;j}_{\ \ \mu} \varphi^{;j\mu} - \frac{1}{2} G_{jk}(\varphi)^{;}_{i} \varphi^{j;}_{\ \mu} \varphi^{k;\mu} - R(\varphi)^{;}_{i} = A_{ij} \otimes g^{\mu\nu} \phi^{j;j}_{\ \ \mu\nu} + B_i = 0 \qquad (2.5.28)$$

$$G_{ij} = G_{ji} \in \mathcal{C}^2(R_\varphi), \quad R \in C^2(R_\varphi)$$

It is a laborious but straightforward task to see that this system satisfies all the conditions of Theorem 2.5.5. Integrals (2.5.22) then yield as a Lagrangian

$$\mathcal{L} = \frac{1}{2} \varphi^{i;}_{\ \mu} G_{ij}(\varphi) \varphi^{j;\mu} + R(\varphi) \qquad (2.5.29)$$

i.e., expectedly, a <u>chiral Lagrangian</u>.

We consider now the problem of attempting the construction of a Lagrangian when the field equations, in the given form are nonselfadjoint. The problem of the "indirect universality" for the existence of a Lagrangian, i.e., the problem whether a

Lagrangian for the indirect analytic representation of class C^∞, regular and unconstrained systems "always" exists, will be considered later on in Section 2.8.

As an extension of the corresponding Newtonian case of Section 2.3, we shall define as <u>selfadjoint genotopic transformations</u> of Field Theory all transformations which induce an equivalent selfadjoint structure starting from a non-selfadjoint system. They can be practically realized through the superposition in any given ordering of

(a) the transformations of the Minkowski coordinates (i.e., the space-time transformations)

$$x^\mu \to x'^\mu, \quad |dx'/dx|(R_x) \neq 0 \qquad (2.5.30)$$

(b) the transformations of the field components within a fixed system of Minkowski coordinates (i.e., the internal transformations)

$$\phi^k \to \phi'^k, \quad |d\phi'/d\phi|(R_\phi) \neq 0 \qquad (2.5.31)$$
$$x^\mu \to x'^\mu \equiv x^\mu$$

(c) the transformations of the variations within a fixed system of Minkowski coordinates and field components

$$\delta\phi^k \to \delta^*\phi^k = g_i^{\ k}(x^\nu, \phi^j, \phi^{j;}_{\ \ \alpha}) \delta\phi^i \qquad (2.5.32)$$
$$\phi^k \to \phi'^k \equiv \phi^k, \quad x^\mu \to x'^\mu \equiv x^\mu, \quad g_i^{\ k} \in C^2, \ |g| \neq 0$$

By including all transformations (a), (b) and (c) in the selected order, we shall symbolically write

$$A^{\mu\nu}_{ki} \phi^{i;}_{\ \ \mu\nu} + B_k = 0 \to J_k^{\ i} (A'^{\mu\nu}_{ij} \phi'^{i;}_{\ \ \mu\nu} + B'_i) = A^{*\mu\nu}_{kj} \phi'^{j;}_{\ \ \mu\nu} + B^*_k = 0 \qquad (2.5.33)$$

We have a (selfadjoint) genotopic transformation when: (1) the original system is nonselfadjoint, (2) the transformed system is equivalent to the original system and (3) the transformed system is selfadjoint. Notice that the equivalence condition implies, in particular, that all considered transformations are everywhere invertible in the considered region of definition of the system. Again, with the symbol $J_k{}^i$ we denote the presence not only of the Jacobian of the transformation (a) and (b) but also that of the multiplicative matrixes (g) of transformations (c).

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THEOREM 2.5.6: Necessary and sufficient condition for a quasilinear system of second order field equations

$$A^{\mu\nu}_{ki}(x^\alpha, \phi^j, \phi^{ij}_\alpha)\,\phi^{ij}_{\mu\nu} + B_k(x^\alpha, \phi^j, \phi^{ij}_\alpha) = 0, \quad (2.5.34)$$

which is of class C^2, regular and nonselfadjoint in a region R of its variables to admit a selfadjoint genotopic transformation in an image region R' of the variables

$$\left(A^{\mu\nu}_{ki}\,\phi^{ij}_{\mu\nu} + B_k\right)^{C^2,R}_{NSA} = 0 \;\longrightarrow\; \left(A^{*\mu\nu}_{ki}\,\phi'^{ij}_{\mu\nu} + B^*_k\right)^{C^2,R}_{SA} = 0, \quad (2.5.35)$$

is that each and all the following conditions on the transformed system

$$A^{*\mu\nu}_{ij} = A^{*\mu\nu}_{ji} = A^{*\nu\mu}_{ij}, \quad (2.5.36a)$$

$$A^{*\nu\alpha;\mu}_{kj\ \ i} + A^{*\nu\alpha;\mu}_{ij\ \ k} = A^{*\mu\nu;\alpha}_{ik\ \ j}, \quad (2.5.36b)$$

$$A^{*\alpha\beta;\mu;\nu}_{ki\ \ j\ \ e} = A^{*\alpha\beta;\mu;\nu}_{ji\ \ k\ \ e}, \quad (2.5.36c)$$

$$B^*_{i\ ;j}{}^{;\mu} + B^*_{j\ ;i}{}^{;\mu} = 2\left\{\partial_\nu + \phi^{kj}_{\nu}\frac{\partial}{\partial\phi^k}\right\} A^{*\mu\nu}_{ij}, \quad (2.5.36d)$$

$$B^*_{i\ ;j} - B^*_{j\ ;i} = \frac{1}{2}\left\{\partial_\nu + \phi^{kj}_{\nu}\frac{\partial}{\partial\phi^k}\right\}\left(B^*_{i\ ;j}{}^{;\nu} - B^*_{j\ ;i}{}^{;\nu}\right), \quad (2.5.36e)$$

$$A^{*\mu\nu}_{ij} = J_i{}^k(x'^\alpha, \phi'^k, \phi'^{ik}_\alpha)\, A'^{\mu\nu}_{kj}(x'^\alpha, \phi'^k, \phi'^{kj}_\alpha), \quad (2.5.36f)$$

$$B^*_i = J_i{}^k(x'^\alpha, \phi'^k, \phi'^{ik}_\alpha)\, B'_k(x'^\alpha, \phi'^k, \phi'^{ik}_\alpha), \quad (2.5.36g)$$

are everywhere satisfied in R'.

The specialization of the above theorem to the case

$$A^{\mu\nu}_{ki}\,\phi^{ij}_{\mu\nu} + B_k = \left(\Box + \mu^2_{(k)}\right)\phi_k - j_k(x^\alpha, \phi^j, \phi^{ij}_\alpha) = 0 \quad (2.5.37)$$

is here left to the interested reader. The following theorem is a direct consequence of Theorem 2.5.4.

30-III

THEOREM 2.5.7: Necessary and sufficient condition for a quasilinear system of second order field equations which is of at least class C^2, regular and nonselfadjoint, in a region R of its variables to admit an ordered indirect analytic representation in terms of Lagrange's equations is that the system admits a selfadjoint genotopic transformation which is well defined in a star-shaped image region R'* of the variables.

If the transformation of class (a) and (b) are ignored, the indirect analytic representation of the above theorem reads

$$\mathcal{L}_k(\phi) \equiv \left[h_k{}^i\left(A^{\mu\nu}_{ij}\,\phi^{ij}_{\mu\nu} + B_i\right)^{C^2,R}_{NSA}\right]^{C^2,R}_{SA}. \quad (2.5.38)$$

The factor matrix (h) which allows the above representation will be referred to the matrix of the genotopic functions. Notice that their maximal functional dependence, in a way similar to the corresponding Newtonian case, is given by $h_i^j(x^q, \phi^k, \phi^k_\alpha)$. Notice also that an explicit dependence of the genotopic functions in the Minkowski coordinates cannot be here excluded even when the original system does not exhibit such a dependence. This is a peculiar feature of nonselfadjoint field theoretical models which is, again, parallel to a corresponding situation in Newtonian mechanics. Its implications for symmetries and conservation laws will be discussed later on in Section 2.11.

As a simple example, consider the null mass limit of the $\lambda\varphi^3$ selfcoupled scalar model (which is selfadjoint), implemented with the nonlinear derivative coupling $\varphi^i_\mu \varphi^{i\mu}/\varphi$ (which is nonselfadjoint), i.e.,

$$\left[\left(\Box\varphi + \lambda\varphi^3\right)^{C^\infty, R}_{SA} + \frac{\varphi^i_\mu \varphi^{i\mu}}{\varphi}\right]^{C^\infty, R}_{NSA} = 0. \qquad (2.5.39)$$

Since the system is nonselfadjoint, a Lagrangian for its ordered direct analytic representation does not exist (Theorem 2.5.4). A Lagrangian for its ordered indirect representation can however exist. For simplicity we here use only transformations of type (c), i.e., through a multiplicative integrating factor (the genotopic function in our terminology). In this case the conditions of selfadjointness (2.5.36) for the

equivalent system

$$\left[h(x^q, \varphi, \varphi^i_\alpha)\left(\Box\varphi + \lambda\varphi^3 + \frac{\varphi^i_\mu \varphi^{i\mu}}{\varphi}\right)_{NSA}\right]_{SA} = 0, \qquad (2.5.40)$$

reduce to

$$\frac{\partial h}{\partial \varphi^i_\mu} g^{\alpha\nu} = \frac{1}{2}\left(\frac{\partial h}{\partial \varphi^i_\alpha} g^{\mu\nu} + \frac{\partial h}{\partial \varphi^i_\nu} g^{\mu\alpha}\right), \qquad (2.5.41)$$

$$2\frac{h}{\varphi}\varphi^i_\mu + \frac{1}{\varphi}\varphi^i_\alpha \varphi^{i\alpha}\frac{\partial h}{\partial \varphi^{i\mu}} = \left\{\partial_\mu + \varphi^i_\mu \frac{\partial}{\partial \varphi}\right\} h,$$

with the trivial solution

$$h = \varphi^2, \qquad (2.5.42)$$

yielding the equivalent selfadjoint form

$$\left[\varphi^2(\Box\varphi) + \lambda\varphi^5 + \varphi\,\varphi^i_\mu \varphi^{i\mu}\right]_{SA} = 0. \qquad (2.5.43)$$

This is an illustration of the difference between Theorem 2.5.1 and 2.5.2. System (2.5.39) is of semilinear type. Since its derivative couplings are nonlinear, it cannot be self-adjoint. The only possibility of constructing an equivalent selfadjoint form within a fixed system of variables is through its transformation into an equivalent quasilinear form because for this latter form nonlinear derivative couplings are admissible by the conditions of selfadjointness. And indeed, the equivalent selfadjoint form (2.5.43) is precisely of quasilinear type (owing to the presence of the nontrivial term $A = \varphi^2$). A simple inspection indicates that system (2.5.43),

besides being selfadjoint, is well defined in a star-shaped region. Thus, all conditions of Theorem 2.5.7 for the indirect representation

$$d_\mu \frac{\partial \mathcal{L}}{\partial e^i_\mu} - \frac{\partial \mathcal{L}}{\partial e} = e^2\left[\Box e + \lambda e^3 + \frac{e^i_\mu e^\mu_i}{e}\right] \quad (2.5.44)$$

are satisfied. Theorem 2.5.5 then yields as a Lagrangian

$$\mathcal{L} = \frac{1}{2} e^i_\mu e^2 e^\mu_i - \frac{\lambda}{6} e^6. \quad (2.5.45)$$

As a less trivial example consider the following implementation of system (2.5.15) with selfadjoint electromagnetic couplings through the addition of nonselfadjoint nonlinear couplings

$$A_{ki}\Box\phi^i - j^{em}_k - J^{NSA}_k$$

$$= \left\{\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\Box + \mu^2)\begin{pmatrix} e \\ \bar{e} \end{pmatrix} + e\begin{pmatrix} eA_\mu A^\mu \bar{e} - 2iA^\mu \bar{e}^i_\mu \\ eA_\mu A^\mu e + 2iA^\mu e^i_\mu \end{pmatrix}\right]_{SA} \right.$$

$$\left. + g\begin{pmatrix} (\bar{e}^\mu_i + ieA^\mu\bar{e})(\bar{e}e^i_\mu + \bar{e}^i_\mu e) \\ (e^\mu_i - ieA^\mu e)(e\bar{e}^i_\mu + e^i_\mu \bar{e}) \end{pmatrix}\right\}_{NSA} = 0 \quad (2.5.46)$$

Straightforward (but laborious) calculations show that this system also admits an indirect analytic representation within

a fixed system of variables, i.e., a representation of type (2.5.38). The use of the conditions of selfadjointness (2.5.36) as a system of partial differential equations in the unknown factor functions $h_k{}^i$ for fixed A- and B-terms yields as a solution

$$(h_i{}^j) = \begin{pmatrix} e^{g\bar{e}e} & 0 \\ 0 & e^{g\bar{e}e} \end{pmatrix}. \quad (2.5.47)$$

Theorem 2.5.5 applied to the equivalent selfadjoint system then yields as a Lagrangian (2.5.48)

$$\mathcal{L} = e^{g\bar{e}e}\left[(\bar{e}^i_\mu + ieA_\mu\bar{e})(e^\mu_i - ieA^\mu e) - \mu^2\bar{e}e\right].$$

The most significant difference between systems (2.5.39) and (2.5.46) is that while the former is semilinear the latter is quasilinear owing to the nontrivial value of the A-matrix

$$(A_{ij}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (2.5.49)$$

And indeed, as indicated earlier, if one assumes (A) = (δ_{ij}) the field equations for the interactions of a comples scalar field with an external electromagnetic field are nonselfadjoint, Eqs. (2.5.14).

Thus, for system (2.5.46) we have the transition from a

nonselfadjoint quasilinear form to an equivalent selfadjoint quasilinear form within the same system of variables, simply realised through the multiplication of factor terms (genotopic functions). Notice that the multiplication by these functions is trivial from the viewpoint of partial differential equations because their matrix is always regular by assumption and, thus, invertible. Their presence, however, becomes crucial from the viewpoint of our variational approach to selfadjointness and, in particular, for the construction of a Lagrangian.

As an illustration of the use of transformations of the variables, consider nonselfadjoint system (2.5.39) at the limit $\lambda \to 0$

$$\left(\Box e + \frac{1}{e} e^i_\mu e^{;\mu} \right)_{NSA} = 0. \qquad (2.5.50)$$

The use of the transformations of type (b)

$$x^\alpha \to x'^\alpha \equiv x^\alpha, \quad e \to e' = \frac{1}{2} e^2, \qquad (2.5.51)$$
$$e'^i_\mu = e\, e^i_\mu, \quad \Box e' = e \Box e + e^i_\mu e^{;\mu},$$

yields the selfadjoint genotopic transformation

$$q \left(\Box e + \frac{1}{e} e^i_\mu e^{;\mu} \right)_{NSA} = \left(\Box e' \right)_{SA} = 0, \qquad (2.5.52)$$

with the trivial Lagrangian for the transformed system

$$\mathcal{L}' = \frac{1}{2} e^{i\mu} e^i_\mu . \qquad (2.5.53)$$

The use of the inverse tranformations then yields a Lagrangian in the original system

$$\mathcal{L} = \frac{1}{2} e^i_\mu e^2 e^{;\mu}_i . \qquad (2.5.54)$$

Notice the very peculiar identity (2.5.53) whereby the left hand side is nonselfadjoint while the right hand side is selfadjoint. This is possible if and only if the two members are computed in different systems of (local) variables. There is no contradiction here because if one member of identity (2.5.52) is computed in the "coordinate" system of the other member, the identity of the variational character (i.e., either selfadjoint or nonselfadjoint for both members) is recovered.

Let us also indicate that, for reasons of consistency with the Lie-admissible approach of Volume II, the analytic representations we are primarily interested in are those in terms of the genotopically mapped Hamilton's principle

$$\delta^* \int_{R_x} d^4x \mathcal{L} = - \int_{R_x} d^4x \left\{ g_k{}^i \left[\mathcal{L}_i(\phi) \right]^{e^2;R}_{SA} \right\}^{C^2;R}_{NSA} \Big|^{\delta\phi^k = 0}_{\phi = \text{act. path}} \qquad (2.5.55)$$

which, for the case of system (2.5.46), Lagrangian (2.5.48) and genotopic functions (2.5.47) explicitly reads

$$\delta^* \int_{R_x} d^4x\, \mathcal{L} = -\int_{R_x} d^4x \left[\begin{pmatrix} e^{-g\bar{e}e} & 0 \\ 0 & e^{-g\bar{e}e} \end{pmatrix} \begin{pmatrix} \mathcal{L}_e \\ \mathcal{L}_{\bar{e}} \end{pmatrix}_{SA} \right]^{TR.}_{NSA} \Bigg|_{Act.\ path} \begin{pmatrix} \delta e \\ \delta \bar{e} \end{pmatrix}$$

$$= -\int_{R_x} d^4x \left\{ \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\Box + \mu^2) \begin{pmatrix} e \\ \bar{e} \end{pmatrix} - e \begin{pmatrix} eA_\mu A^\mu \bar{e} - 2iA_\mu \bar{e}^{,\mu} \\ eA_\mu A^\mu e + 2iA_\mu e^{,\mu} \end{pmatrix} \right]_{SA} \right. \quad (2.5.56)$$

$$\left. + g \begin{pmatrix} (\bar{e}^{,\mu} + ieA^\mu e)(\bar{e}e_{,\mu} + \bar{e}_{,\mu} e) \\ (e^{,\mu} - ieA^\mu e)(e\bar{e}_{,\mu} + e_{,\mu}\bar{e}) \end{pmatrix} \right\}^{TR.}_{NSA} \Bigg|_{Act.\ path} \begin{pmatrix} \delta e \\ \delta \bar{e} \end{pmatrix} = 0$$

The underlying analytic equations

$$\left\{ g_{ki} \left[\mathcal{L}_i(\phi) \right]^{C^2,R}_{SA} \right\}^{C^2,R}_{NSA} = 0 \qquad (2.5.57)$$

will be referred to as the genotopically mapped Lagrange's equations. From the viewpoint of the theory of differential equations they are a trivial equivalence transformations of the conventional Lagrange equations. From the viewpoint of our variational approach to selfadjointness they are not trivial because they are nonselfadjoint. In turn, this nonselfadjoint character allows the analytic representations of systems of this type, as given, in a form which will result parallel to the complementary Lie-admissible approach of Volume II (which is also nonselfadjoint).

The problem whether the genotopic transformations of type (a), (b) and (c) are equivalent owing to their invertibility

will be discussed later on in Section 2.8. At this point, we content ourself with the pragmatic significance of these three classes of transformations. Sometimes a solution of the conditions of selfadjointness (2.5.36) in the genotopic functions (when it exists) with, say, method (c) may be of difficult identification. The existence of the method (b) is then useful on pragmatic grounds to see whether the solution of the corresponding system is more manageable. Similarly, one can try the joint use of methods (b) and (c). Method (a) is considerably more delicate to use in practical applications. This is due, first of all, to the fact that the more general transformations (2.5.30) of the Minkowski coordinates (rarely used in physical literature but often used in the Calculus of Variations) are rather delicate to handle. Secondly, any transformation of the Minkowski coordinate, irrespective of whether conventional (i.e., of the type $x^\mu \to x'^\mu = x'^\mu(x)$) or more general, demands the computation of the image of the fields in the new coordinate system. In turn, this demands the knowledge of the transformation laws of the fields under these transformations. For the case of the Lorentz transformations $x^\mu \to x'^\mu = A^\mu_\nu x_\nu$ it is costumary in current physical literature to assume that the fields transform covariantly according to a (generally reducible) transformation of the Lorentz group. This is certainly the case for simple models such as the complex scalar field in interactions with an electromagnetic field, i.e., Eqs. (2.5.15).

But one of the objections of these monographs is precisely that of identifying the limitations of this transformation law for the more general quasilinear systems here considered.

The type of field equations considered until now in this section are often referred to as tensorial field equations. Another type of equations which is particular significant for our analysis is that of spinorial field equations, which, as it is known, are of the first order.

For the reader's convenience, we here outline the methodology of the Inverse Problem for this latter case. A change in attitude must be however indicated. While in contemporary literature of Field Theory first-order field equations on Minkowski space are virtually restricted to have a spinorial character, this restriction is lifted for the Inverse Problem. Explicitly, we are interested in the construction of analytic representations of first order systems of field equations

$$F_k(\phi) = \left[C^{\mu}_{ki}(x^{\alpha}, \phi^j)\, \phi^{i}{}_{;\mu} + D_k(x^{\alpha}, \phi^j) \right]^{C^2} = 0, \qquad (2.5.58)$$

$$k, i, j = 1, 2, \ldots, n \; ; \; \mu, \alpha = 0, 1, 2, 3,$$

in which the covariance properties of the fields ϕ^i are ignored because not essential for the problem considered. As recalled earlier, one of the central objectives of these monographs is precisely the study whether the methods considered can assist in the characterization of the covariance properties of Eqs. (2.5.58)

It is a simple exercise to prove, by generalizing the

variational approach to selfadjointness of Section 2.3, the following

THEOREM 2.5.8: Necessary and sufficient condition for first order systems of field equations (2.5.58) which are of at least class C^1 in a region R of their variables to be selfadjoint in R is that each and all the following conditions

$$\begin{cases} F_i{}^{;\mu}_{j} + F_j{}^{;\mu}_{i} = 0 , & (2.5.59a) \\ F_i{}^{;}_{;j} - F_j{}^{;}_{;i} = d_{\mu} F_i{}^{;\mu}_{j} , & (2.5.59b) \end{cases}$$

or their equivalent form

$$\begin{cases} C^{\mu}_{ij} + C^{\mu}_{ji} = 0 , & (2.5.60a) \\ C^{\mu}_{ij}{}^{;}_{;k} + C^{\mu}_{jk}{}^{;}_{;i} + C^{\mu}_{ki}{}^{;}_{;j} = 0 , & (2.5.60b) \\ C^{\mu}_{ij}{}^{;}_{;\mu} = D_i{}^{;}_{;j} - D_j{}^{;}_{;i} , & (2.5.60c) \end{cases}$$

are identically verified in R.

The Lagrange equations for the representation of system (2.5.58) are formally the same as those for the second order system Eqs. (2.5.16), with the difference that now the Lagrangian densities are totally degenerate, i.e., they depend linearly in the first-order field derivatives. The explicit form of the equations is then given by

$$\left[\mathcal{L}_k(\phi)\right]^{C^1,TD} = \partial_\mu \mathcal{L}^{;\mu}_{k} - \mathcal{L}^{;}_{k} \qquad (2.5.61)$$

$$= \mathcal{L}^{;\mu}_{k\,;i}\,\phi^{i;}_{\ \mu} + \mathcal{L}^{;\mu}_{k\,;\mu} - \mathcal{L}^{;}_{k} = 0$$

THEOREM* 2.5.9: The Lagrange equations in class C^2 and totally degenerate Lagrangian densities are always selfadjoint in their region of definition.

This theorem can be proved by either repeating the variational approach to selfadjointness, or by showing that Eqs. (2.5.61), under the indicated conditions, always satisfy conditions (2.5.59) or (2.5.60).

The Fundamental Analytic Theorem for first order field equations then reads

[60]

THEOREM 2.5.10: Necessary and sufficient condition for a first-order system of field equations which is well behaved and of at least class C^1 in a star-shaped region R* of its variables to admit an ordered direct analytic representation in terms of the totally degenerate Lagrange's equations

$$\left[\mathcal{L}_k(\phi)\right]^{C^1,TD}_{SA} \equiv C^{\mu}_{ki}(x^\gamma, \phi^j)\,\phi^{ij}_{\ \ \mu} + D_k(x^\gamma, \phi^j), \qquad (2.5.62)$$

$k, i, j = 1, 2, \ldots, n$, $\mu, \alpha = 0, 1, 2, 3$,

is that the system is selfadjoint in R*, in which case a Lagrangian density is given by

$$\mathcal{L} = -\phi^k \int_0^1 d\tau\, F_k(x^\gamma, \tau\phi^j, \tau\phi^{ij}{}_\alpha), \qquad (2.5.63)$$

$$F_k = C^{\mu}_{ki}\,\phi^{ij}_{\ \ \mu} + D_k .$$

* The reader should keep in mind the remark at the end of page 156 in relation to the repeated use of the term "theorem".

The proof of this theorem is quite simple. Introduce the quantity

$$G(\tau) = -\phi^k F_k(x^\gamma, \tau\phi^j, \tau\phi^{ij}{}_\alpha). \qquad (2.5.64)$$

Then, by using all conditions of selfadjointness (2.5.59), we can write

$$\partial_\mu G^{;\mu}_{k} - G^{;}_{k}$$

$$= F_k + F_{k\,;i}\,\phi^i - \partial_\mu\left(\phi^{j;\mu}_{\ \ k}\right) \qquad (2.5.65)$$

$$= F_k + F_{k\,;i}\,\phi^i + F_{k\,;\dot{i}}^{\ \ \mu}\,\phi^{\dot{i}j}_{\ \ \mu}$$

$$= \frac{d}{d\tau}\left(\tau F_k\right)$$

By integrating from 0 to 1 we have

$$\int_0^1 d\tau\left(\partial_\mu G^{;\mu}_{k} - G^{;}_{k}\right) = \partial_\mu \mathcal{L}^{;\mu}_{k} - \mathcal{L}^{;}_{k} \qquad (2.5.66)$$

$$= \int_0^1 d\tau\,\frac{d}{d\tau}\left(\tau F_k\right) = F_k, \quad \mathcal{L} = \int_0^1 d\tau\, G(\tau).$$

This establishes the sufficiency of the conditions of self-adjointness as well as Lagrangian (2.5.63). The necessity of the same conditions follows from Theorem 2.5.9.

It is instructive to compare the simplicity of the proof of Theorem 2.5.10 as compared with the rather involved proof of Theorem 2.5.4 for the corresponding second-order case, as given in paper II of reference [30].

This situation is essentially due to the fact that for

Theorem 2.5.10 the so called order of the Lagrangian density, i.e. the order of its maximal field derivative, coincide with the order of the field equations (i.e., the Lagrangians are totally degenerate). For the case of Theorem 2.5.4 the order of the Lagrangian density is still one, but the field equations are of the second order (i.e., the Lagrangian density is regular). It is this difference which implies the indicated differences in the proof of the two theorems.

Theorem 2.5.10 trivially extends to partial differential equations of arbitrary order, thus including the second order. However, this would yield an analytic representation in third order Lagrange's equations and second order totally degenerate Lagrangians, i.e., a representation of the type

$$-\partial_\mu \partial_\nu \mathcal{L}^{i\,\mu\nu}_{\ k} + \partial_\mu \mathcal{L}^{i\,\mu}_{\ k} - \mathcal{L}^{i}_{\ k} \equiv A^{\mu\nu}_{ki}\,\phi^{ij}_{\ \mu\nu} + B_k, \qquad (2.5.67)$$

$$\mathcal{L} = \mathcal{L}(x^\alpha, \phi^i, \phi^{ij}_{\ \alpha}, \phi^{ij}_{\ \alpha\beta}),$$

which is not common in theoretical physics.[60]

When Eqs. (2.5.58) are nonselfadjoint one can attempt the construction of an analytic representation through the use of integrating factors.

THEOREM 2.5.11: Necessary and sufficient condition for a first order system of field equations which is of at least class[1] and nonselfadjoint in a region R of its variables to admit the ordered indirect analytic representation

$$\left(\partial_\mu \mathcal{L}^{i\,\mu}_{\ k} - \mathcal{L}^{i}_{\ k}\right)^{C^1;\,TD}_{SA} \equiv \left\{ h_k^{\ i}(x^\alpha, \phi^j)\left[C^{\ \mu}_{ij}(x^\alpha, \phi^j)\,\phi^{ij}_{\ \mu} + D_i(x^\alpha, \phi^j)\right]\right\}, \qquad (2.5.68)$$

is that the system admits a genotopic transformation which is well defined in star-shaped extension R* of R, i.e., that all the following conditions

$$\begin{cases} C^{*\ \mu}_{ij} + C^{*\ \mu}_{ji} = 0, & (2.5.69a) \\ C^{*\ \mu}_{ij\ \ k} + C^{*\ \mu}_{jk\ \ i} + C^{*\ \mu}_{ki\ \ j} = 0, & (2.5.69b) \\ C^{*\ \mu}_{ij\ \ \mu} = D^{*}_{i\ j} - D^{*}_{j\ i}, & (2.5.69c) \end{cases}$$

$$C^{*\ \mu}_{ij} = h_i^{\ k}(x^\alpha, \phi^j)\, C^{\ \mu}_{ij}(x^\alpha, \phi^j), \qquad (2.5.69d)$$

$$D^{*}_{i} = h_i^{\ k}(x^\alpha, \phi^j)\, D_k(x^\alpha, \phi^j), \qquad (2.5.69e)$$

admit a solution in the factor functions $h_{\cdot}^{\ j}$ and the equivalent system is well defined in R*.

For the case of second order (ordinary or) partial differential equations the existence of genotopic functions for a given system is not guaranteed in the sense that the conditions of selfadjointness, interpreted as a system of partial differential equations in the factor functions, is not necessarily consistent. Surprisingly, the corresponding situation for first order systems of (ordinary or) partial differential equations is different. See in this respect Section 2.8.

As an incidental remark, notice that the conditions of selfadjointness do not need a star-shaped region to be well

defined. This is the case for all conditions of selfadjointness. The conditions that the systems be well defined in a start-shaped region
is requested by our use of the Converse of the Poincaré Lemma for the intent of insuring the existence of the integrals of type (2.5.66). For a different but equivalent restriction see ref.[61]

Notice also that the index "k" of Eqs. (2.5.58) can represent any collection of indeces, including those indicating tensorial character in Minkowski space. Thus, the methodology of the Inverse Problem here outlined is also significant for tensorial field equations, e.g., as occurring in gauge theories.

Besides second-order and first-order field equations, there is a third class of equations which is also significant. It is given by the familiar case of coupling tensorial and spinorial fields and, as such, consists of coupled field equations of second and first order. Surprisingly, this third class of equations is <u>not treatable</u> by the methodology of the Inverse Problem as considered until now.

To understand this rather peculiar occurrence we have to recall the concept of regularity or degeneracy of a Lagrangian. In Newtonian Mechanics a Lagrangian is termed regular or degenerate when its Hessian determinant

$$\mathcal{H}(R) = \left| \frac{\partial L_i}{\partial \ddot{q}^j} \right| (R) = \left| \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right| (R) \qquad (2.5.70)$$

in nonnull or null as a function, repsectively. Section 2.3 has been restricted by assumption to the regular case only. The reason is that when a Lagrangian is degenerate it actually represents a system with subsidiary constraints. Conceivably, the methodology of the Inverse Problem is extendable to the case of degenerate systems, but this demands an explicit investigation. In particular, the same variational approach to selfadjointness and, thus, the integrability conditions for the existence of a Lagrangian, must now be reformulated in the hypersurface of the constraints. This demands the handling of a number of delicate topics, such as the uniqueness of the adjoint system of variational forms. In turn, such uniqueness has a subtle relationship with the consistency of the system considered which, contrary to the regular case, is not guaranteed by continuity properties alone under the conditions of degeneracy.

As it occurs for several methodological aspects, the concept of regularity or degeneracy of the Lagrangian in Newtonian Mechanics has a direct extension to Field Theory. Again, a (first order) Lagrangian density is called regular or degenerate depending on whether the Hessian determinant

$$\mathcal{H}(R) = \left| \frac{\partial \mathcal{L}_i}{\partial \phi^{ij}_{\mu\nu}} \right| (R) = \frac{1}{2} \left| \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^{ij}_{\mu} \partial \phi^{ij}_{\nu}} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{ij}_{\nu} \partial \phi^{ij}_{\mu}} \right) \right| (R) \qquad (2.5.71)$$

is nonnull or null as a function, respectively.* For the case of semilinear second-order systems

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* The problem of the possible existence of isolated zeros (ref.[34] Sec. 1.1) now becomes considerably more involved.

$$\left(\Box + \mu_{(k)}^2\right)\phi_k - \vec{J}_k(x^\alpha, \phi^j, \phi^j{}_\alpha) \qquad (2.5.72)$$

$$= g^{\mu\nu}\phi_k{}_{,\mu\nu} + \mu_{(k)}^2\phi_k - \vec{J}_k(x^\alpha, \phi^j, \phi^j{}_\alpha) = 0,$$

Hessian (2.5.71) reduces to the matrix of the metric tensor, i.e.,

$$\mathcal{H} = \left|\left| g^{\mu\nu} \right| \times \left| \mathbb{1}_{n\times n} \right|\right| = -1, \qquad (2.5.73)$$

and, thus, systems (2.5.72) are always regular. For the special case of quasilinear second order systems

$$A_{ki}(x^\alpha, \phi^j, \phi^j{}_\alpha)\, g^{\mu\nu}\phi^i{}_{,\mu\nu} + B_k(x^\alpha, \phi^j, \phi^j{}_\alpha) = 0, \qquad (2.5.74)$$

Hessian determinant (2.5.71) reduces to

$$\mathcal{H}(R) = -\left|A_{ij}\right|(R), \qquad (2.5.75)$$

and, thus, systems (2.5.73) are regualr when the matrix (A_{ij}) is regualr. Similarly, a general quasilinear second-order system, i.e., Eqs. (2.5.19), is regular when the matrix $(A^{\mu\nu}_{ij})$ is regular.

Consider now a system containing coupled tensorial and spinorial equations such as the conventional representation of the meson-nucleon interactions

$$\mathcal{L} = \frac{1}{2}\left(\phi^i{}_{,\mu}\,\phi^{i,\mu} - \mu^2\phi_i\phi^i\right) + \frac{1}{2}\,G\left[\tilde{\psi}\Gamma_5\tau_i,\psi\right]\phi^i \qquad (2.5.76)$$
$$-\frac{1}{2}\left(i\partial_\mu\tilde{\psi}\gamma^\mu + M\tilde{\psi}\right)\psi - \frac{1}{2}\,\bar{\psi}\left(-i\gamma^\mu\partial_\mu + M\right)\psi$$

This Lagrangian is degenerate because it is linear in a subset of the first order partial derivatives and the Hessian determinant is identidally null.

This aspect, to the best of my knowledge, is practically ignored in current literature of Field Theory as inessential for the customary objectives of this discipline. Nevertheless, the above property becomes significant for the integrability conditions for the existence of a Lagrangian representation of coupled second-order and first-order field equations. It essentially indicates that the first order equations are, strictly speaking, constraint on the second order equations. As such, the integrability conditions for the existence of a Lagrangian should be redefined on the hypersurface of the first-order equations.

The study of degenerate Newtonian and field theoretical systems will not be considered as not essential for our needs.

Notice that the indicated difficulties for degenerate models become absent for totally degenerate models, i.e. for systems of only first order equations, such as Eqs. (2.5.58).

In conclusion, the methodology of the Inverse Problem as reviewed in this section applies to either system of second-order field equations or to systems of first-order field

equations but, strictly speaking, it does not apply to systems of coupled second order and first order field equations because of a number of expected technical implementations.

In conclusion, the methodology of the Inverse Problem identifies a rather fundamental property of the electromagnetic interactions, that their forces or couplings are variationally selfadjoint, and this property extends to the unified gauge theories of weak and electromagnetic interactions (Appendix 2.C). Indeed, starting from the Coulomb force and the Lorentz force in Newtonian Mechanics, and passing to their relativistic extension, as well as extensions in field theory, we have seen in this chapter that all these forces or couplings are variationally selfadjoint. Predictably, this property can be linked to the contemporary notion of particle under (long range) electromagnetic interactions, i.e., as systems transforming irreducibly under the Poincaré group.

The objective of these monographs is that of studying strong interactions realized in terms of forces or couplings which are variationally NON-selfadjoint, to attempt a dynamical differentiation with the electromagnetic interactions, as well as to approximate the forces resulting from the state of penetration of the charge volumes of the particles in interaction (Chap.1). The ultimate objective is that of studying the notion of particle under (short range) strong interactions, as well as electromagnetic interactions, as a prerequisite for the proper formulation of the problem of the hadronic structure.[62]

2.6: THE INDEPENDENT INVERSE PROBLEM FOR HAMILTONIAN FORMULATIONS.

The customary approach to Hamiltonian formulations in contemporary Analytic Mechanics is that of identifying first a Lagrangian and then computing the corresponding Hamiltonian through the Legendre transform

$$H = p_k \dot{q}^k - L = H(t,q,p), \quad p_k = \frac{\partial L}{\partial \dot{q}^k} \qquad (2.6.1a)$$

$$L(t,q,\dot{q}) \in C^2(R), \quad \left|\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right|(R) \neq 0. \qquad (2.6.1b)$$

This approach is, of course, not excluded by the Inverse Problem. And indeed the Legendre transform (2.6.1a) applies irrespective of any functional structure of a Lagrangian (provided that regularity condition (2.6.1b) is satisfied). Therefore, the identification of a Lagrangian for an arbitrary (local) system of the considered type through the use of the Inverse Problem implies the consequential knowledge of a Hamiltonian through the use of transform (2.6.1).

An intriguing aspect of the Inverse Problem is that it allows the computation of a Hamiltonian without any prior knowledge of a Lagrangian. We shall refer to this approach as the Independent, Inverse, Hamiltonian Problem. In essence, this second approach expresses the equivalence of analytic representations in configuration and phase space. If a Hamiltonian is known, a Lagrangian can be computed through the inverse transform

$$L = \dot{q}^k P_k - H = L(t,q,\dot{q}), \quad \dot{q}^k = \frac{\partial H}{\partial P_k}, \qquad (2.6.2a)$$

$$H(t,q,P) \in C^2(\hat{R}), \quad \left|\frac{\partial^2 H}{\partial P_i \partial P_j}\right|(\hat{R}) \neq 0, \qquad (2.6.2b)$$

In the following we shall refer to transforms (2.6.1) and (2.6.2) as the direct and inverse Legendre transforms respectively.

A significance for the identification of a Hamiltonian prior to any knowledge of a Lagrangian is the following. In principle, for a given system of the considered type one can apply both techniques of the Inverse Problem in an independent way, i.e., the identification of a Lagrangian and, independently, of a Hamiltonian. The intriguing result is that, generally, these two functions are not related by a Legendre transform.

But these two Lagrangian and Hamiltonian representations of the same system are equivalent by construction. The net result is the appearance of new aspects of the transformation theory which are relevant for applications, such as, the identification of new independent first integrals via symmetries and Noether's theorem (Section 2.11).

This section is devoted to a summary of the methods of the independent Inverse Hamiltonian Problem[34] for the case of Newtonian systems. Its extension to relativistics and field

theoretical cases will be left to the interested reader. Predictably, the variational approach to selfadjointness, but this time for the case of first order systems of ordinary differential equations, will play a fundamental role.

Consider a Newtonian system in configuration space of the quasilinear second-order n-dimensional form

$$F_k(q) = \left[A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q})\right]^{C^1,R} = 0. \qquad (2.6.3)$$

It can be reduced to an equivalent system of 2n-dimensional first-order equations through the introduction of an arbitrary set of prescriptions for the characterization of n independent variables, say, y_k,

$$G_k(y) = \left[\alpha_{ki}(t,q,y)\dot{q}^i + \beta_k(t,q,y)\right]^{C^1,R} = 0, \qquad (2.6.4)$$

with implicit functions

$$\dot{q}^k = \tilde{g}^k(t,q,y). \qquad (2.6.5)$$

Indeed by applying prescriptions (2.6.5) to system (2.6.3), after elementary manipulations, one reaches the system

$$\left[\Omega_{\mu\nu}(t,a)\dot{a}^\nu + \Gamma_\mu(t,a)\right]^{C^1,R} = 0 \qquad (2.6.6a)$$

$$\mu,\nu = 1,2,\ldots,2n, \quad \{a^\mu\} = \{q^k, y_k\} \qquad (2.6.6b)$$

$$(\Omega_{\mu\nu}) = \begin{pmatrix} A_{ik}(t,q,\tilde{g})\frac{\partial \tilde{g}^k}{\partial q^j} & 0 \\ 0 & \alpha_{ki}(t,q,y) \end{pmatrix}, \quad |\Omega_{\mu\nu}|(\hat{R}) \neq 0, \qquad (2.6.6c)$$

$$(\Gamma_\mu) = \begin{pmatrix} A_{ik}(t,q,\tilde{q})\left(\dfrac{\partial \tilde{q}^i}{\partial y^j}\,\tilde{q}^j + \dfrac{\partial \tilde{q}^i}{\partial t}\right) \\ \\ \beta_k \end{pmatrix}, \qquad (2.6.6d)$$

which we call the general covariant form or quasilinear first-order form of a Newtonian system, for later geometrical precisation.

It should be stressed that prescriptions (2.6.5) are not necessarily canonical. The variables (q,y), therefore, are not necessarily canonically conjugate and they do not necessarily span a phase space. Indeed, the canonicity of the prescriptions for the characterization of the momenta, Eqs. (2.6.1a) is centrally dependent on the knowledge of a Lagrangian which is here absent by assumption.

The independent Inverse Hamiltonian Problem consists precisely in identifying the necessary and sufficient conditions for prescriptions (2.6.5) to be canonical without any knowledge of a Lagrangian. It also consists in the independent computation of a Hamiltonian when its existence is assured by the integrability conditions. Finally it consists of the significance of the underlying methodology for other aspects of the theory, e.g., the transformation theory.

At the foundation of the problem lies the following

34-I

THEOREM 2.6.1: The conventional Hamilton's equations

$$\omega_{\mu\nu}\,\dot{a}^\nu - \frac{\partial H}{\partial a^\mu} = 0, \quad \mu = 1,2,\dots,2n \qquad (2.6.7a)$$

$$\{a^\mu\} = \{q^k, p_k\}, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0_{2\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix} \qquad (2.6.7b)$$

in Hamiltonians which are of at least Class C^2 in a region R of their variables (t,a) are always selfadjoint in R.

The proof of the thoerem is straightforward. By repeating the variational approach in Section 2.3 for Eqs. (2.6.7) one can see that the conditions of variational selfadjointness for this system reduce to

$$\frac{\partial^2 H}{\partial a^\mu \partial a^\nu} - \frac{\partial^2 H}{\partial a^\nu \partial a^\mu} = 0, \quad \mu,\nu = 1,2,\dots,2n, \qquad (2.6.8)$$

and, as such, they are implicit in the assumed continuity conditions.

The reader has eventually identified in form (2.6.7) the formulation of Hamilton's equations in terms of the fundamental symplectic form $(\omega_{\mu\nu})$. Its significance for the conditions of variational selfadjointness will be pointed out in Section 2.7.

The "mechanics" for the independent Inverse Hamiltonian Problem can now be formulated. The matrix $(\Omega_{\mu\nu})$ of the general

covariant form (2.6.6a) is regular by construction. As a result, it always admits a factorization of the covariant symplectic form of the type

$$\Omega_{\mu\nu}(t,a) = g_{\mu}{}^{\rho}(t,a)\,\omega_{\rho\nu} \quad , |g|(\tilde{R}) \neq 0 \qquad (2.6.9)$$

The net effect is that it is always possible to construct from Eqs. (2.6.6a) the system

$$\omega_{\mu\nu}\,\dot{a}^{\nu} - \bar{\div}_{\mu}(t,a) = 0 , \qquad (2.6.10a)$$

$$\bar{\div}_{\mu} = -g^{-1}{}_{\mu}{}^{\rho}\,\Gamma_{\rho} \quad , (g^{-1}) = (g)^{-1} , \qquad (2.6.10b)$$

which we call covariant normal form of a Newtonian system.

THEOREM 2.6.2 [34-I]: Necessary and sufficient condition for a covariant normal form (2.6.10a) which is of at least class C^1 in a region R of its variables to be selfadjoint in R is that each and all the following conditions

$$\frac{\partial \bar{\div}_{\mu}}{\partial a^{\nu}} - \frac{\partial \bar{\div}_{\nu}}{\partial a^{\mu}} = 0 , \mu,\nu = 1,2,\ldots,2n \qquad (2.6.11)$$

are everywhere satisfied in R.

The combination of Theorems 2.6.1 and 2.6.2 yields the Fundamental Analytic Theorem for Canonical Formulations [34-I]

THEOREM 2.6.3: Necessary and sufficient conditions for a covariant normal form (2.6.10a) which is well defined and of at least class C^1 in a star-shaped region R* of its variables to admit an ordered direct analytic representation

in terms of the covariant Hamilton's equations in R*

$$\omega_{\mu\nu}\,\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} \equiv \omega_{\mu\nu}\,\dot{a}^{\nu} - \bar{\div}_{\mu} , \qquad (2.6.12)$$

$$\mu,\nu = 1,2,\ldots,2n ,$$

is that the form be selfadjoint in R*. A Hamiltonian for the analytic representation of the system is then given by

$$H(t,a) = a^{\mu}\int_0^1 d\tau\ \bar{\div}_{\mu}(t,\tau a) . \qquad (2.6.13)$$

Contrary to the rather involved proof of the corresponding theorem for configuration space formulations (Section 2.3), the proof of the above theorem by using the Converse of the Poincaré Lemma is straightforward. For details see Volume I of monographs.[34]

The "mechanics" of the independent Inverse Hamiltonian Problem then results to be quite simple.[63] For a given quasi-linear form (2.6.3) one uses conditions (2.6.11) for the selection of prescriptions (2.6.5) which are canonical, i.e., Eqs. (2.6.11) are used as conditions on the prescriptions (2.6.5) to yield a selfadjoint normal form.

As a simple example consider the particle with a velocity dependent drag force.

$$(\ddot{q} + f\dot{q}) \underset{NSA}{\overset{C^{\infty},R}{=}} 0 , \quad m = 1 . \qquad (2.6.14)$$

This system is nonselfadjoint and, therefore, a direct Lagrangian representation does not exist (Theorem 2.3.2). Introduce the prescriptions

$$\alpha(t,q,y)\,\dot{y} + \beta(t,q,y) = 0, \quad \alpha(\tilde{R}) \neq 0 \qquad (2.6.15)$$

with, at this time, unknown functions α and β. System (2.6.14) can then be written in the equivalent normal form

$$\omega_{\mu\nu}\,\dot{a}^{\nu} - \bar{\div}_{\nu}(t,a) = 0$$

$$(a^{\mu}) = \begin{pmatrix} q \\ y \end{pmatrix}, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (\bar{\div}_{\mu}) = \begin{pmatrix} \tilde{g}_2 \\ \tilde{g}_1 \end{pmatrix} \qquad (2.6.16)$$

$$\tilde{g}_1 = \frac{\beta}{\alpha}, \quad \tilde{g}_2 = \left(\frac{\partial \tilde{g}_1}{\partial q}\,\tilde{g}_1 + \frac{\partial \tilde{g}_1}{\partial t} + \gamma \tilde{g}_1 \right) \Big/ \left(\partial \tilde{g}_1 / \partial y \right)$$

Conditions of selfadjointness in (2.6.11) in this case reduce to

$$\frac{\partial \tilde{g}_2}{\partial y} = \frac{\partial \tilde{g}_1}{\partial q} \qquad (2.6.17)$$

with a solution

$$\alpha = 1, \quad \beta = -e^{P^{(1)}}, \quad P^{(1)} = y \qquad (2.6.18)$$

yielding the selfadjoint normal form

$$\left(\omega_{\mu\nu}\,\dot{a}^{\nu}\right) - \left(\bar{\div}_{\mu}\right) = \begin{pmatrix} -\dot{P}^{(1)} \\ \dot{q} \end{pmatrix} - \begin{pmatrix} \gamma \\ e^{P^{(1)}} \end{pmatrix} = 0. \qquad (2.6.19)$$

The system is also well defined in a star-shaped region of its variables. Theorem 2.6.3 then holds and a Hamiltonian, from Eq. (2.6.13), is given by

$$H^{(1)}(q,P^{(1)}) = a^1 \int_0^1 d\tau\, \bar{\div}_1(\tau a) + a^2 \int_0^1 d\tau\, \bar{\div}_2(\tau a) = e^{P^{(1)}} + \gamma q - 1. \qquad (2.6.20)$$

The use of the inverse Legendre transform (2.6.2) then yields the Lagrangian

$$L^{(1)} = \dot{q}\,\ln \dot{q} - \gamma q, \quad P^{(1)} = \ln \dot{q} + 1, \qquad (2.6.21)$$

(where the additive "gauge" term $(1-\dot{q})$ has been ignored because trivial) with underlying analytic representation

$$\frac{d}{dt}\frac{\partial L^{(1)}}{\partial \dot{q}} - \frac{\partial L^{(1)}}{\partial q} \equiv \left[\frac{1}{\dot{q}} \left(\ddot{q} + \gamma \dot{q} \right)_{NSA} \right]_{SA}, \qquad (2.6.22)$$

which, as expected, is of indirect type.

To complete the illustration of these techniques, let us now apply the Inverse Lagrangian Problem to the same system (2.6.14). This calls for the construction of a genotopic function, i.e., an explicit form of the factor function which render the system selfadjoint

$$\left[h(t,q,\dot{q}) \left(\ddot{q} + \gamma \dot{q} \right)_{NSA} \right]_{SA} = 0. \qquad (2.6.23)$$

Eqs. (2.3.21) in this case reduce to

$$\gamma \frac{\partial}{\partial \dot{q}} (h\dot{q}) = \left\{ \frac{\partial}{\partial t} + \dot{q} \frac{\partial}{\partial q} \right\} h \qquad (2.6.24)$$

with a solution given by

$$h = e^{\gamma t} \qquad (2.6.25)$$

Existence Theorem 2.3.2 is now satisfied for this equivalent form. Theorem 2.3.3 on the construction of a Lagrangian then yields

$$L^{(2)} = e^{\gamma t} \frac{1}{2} \dot{q}^2 \qquad (2.6.26)$$

Finally the use of the direct Legendre transform produces the Hamiltonian

$$H^{(2)} = e^{-\gamma t} \frac{1}{2} P^{(2)},$$
$$P^{(2)} = \frac{\partial L^{(2)}}{\partial \dot{q}} = e^{\gamma t} \dot{q}. \qquad (2.6.27)$$

One can thus see that the Lagrangians and Hamiltonians produced by the independent use of the Inverse Problem for configuration and canonical formulations do not coincide. Neverthless, there is no inconsistency in the two approaches, but simply the emergence of a new aspect of the transformation theory.

Specifically, each of the two Hamiltonians $H^{(1)}$ and $H^{(2)}$ can be constructed with the Inverse Hamiltonian Problem,

trivially, because prescriptions (2.6.27b) are another admissible solution of the conditions of selfadjointness (2.6.17). Similarly and independently, each of the two Lagrangians $L^{(1)}$ and $L^{(2)}$ can be constructed with the Inverse Lagrangian Problem because, trivially, the value $h = 1/\dot{q}$ is also another solution of the condition of selfadjointness (2.6.24). Therefore, the two approaches are consistent.

The point is that the two Lagrangians or Hamiltonians produced by the techniques exhibit rather profound structural differences which are not interpretable with the currently known transformation theory, e.g., that of canonical transformations. We are therefore facing a new type of transformations which is directly induced by the integrability conditions for the existence of a Lagrangian or a Hamiltonian and which will be outlined in Section 2.10.

2.7: ANALYTIC, ALGEBRAIC AND GEOMETRIC SIGNIFICANCE OF THE CONDITIONS OF VARIATIONAL SELFADJOINTNESS

The objective of this section is to identify the significance of the integrability for the existence of a Hamiltonian. This will later on prove to be invaluable, particularly for the Lie-admissible generalizations of Analytic Mechanics of Volume II. This study can be better conducted within the context of quasilinear first-order systems in an even number of dimension, i.e., the general covariant form (2.6.6), but now of class C^∞ for geometrical considerations.

THEOREM 2.7.1 [34-I]: Necessary and sufficient condition for a general covariant form

$$\left[\Omega_{\mu\nu}(t,a)\,\dot{a}^\nu + \Gamma_\mu(t,a)\right]^{C^\infty,R} = 0, \quad \mu,\nu = 1,2,\ldots,2n, \tag{2.7.1}$$

to be selfadjoint in the region $\tilde{R}$ of its definition is that each and all the following conditions

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \tag{2.7.2a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\sigma} + \frac{\partial \Omega_{\nu\sigma}}{\partial a^\mu} + \frac{\partial \Omega_{\sigma\mu}}{\partial a^\nu} = 0, \tag{2.7.2b}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial \Gamma_\mu}{\partial a^\nu} - \frac{\partial \Gamma_\nu}{\partial a^\mu}, \tag{2.7.2c}$$

are everywhere satisfied in $\tilde{R}$.

The objective of this section is that of identifying the significance of conditions (2.7.2). For this purpose it is useful to introduce the contravariant general form

$$(\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1}, \tag{2.7.3}$$

associated to form (2.7.1). The following theorem can then be proved (see refs. [34-I,61] for the early studies of the subject).

THEOREM 2.7.2: Necessary and sufficient condition for a matrix $(\Omega^{\mu\nu}(t,a))$ to be the inverse of a class C^∞ and regular $2n\times 2n$ matrix $(\Omega_{\mu\nu}(t,a))$ whose elements satisfy the subset of conditions of selfadjointness (2.7.2)

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \tag{2.7.4a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\sigma} + \frac{\partial \Omega_{\nu\sigma}}{\partial a^\mu} + \frac{\partial \Omega_{\sigma\mu}}{\partial a^\nu} = 0, \tag{2.7.4b}$$

in a region $\tilde{R}$ of the local variables (t,a), is that all the conditions

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \tag{2.7.5a}$$

$$\Omega^{\mu\rho}\frac{\partial \Omega^{\nu\sigma}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial \Omega^{\sigma\mu}}{\partial a^\rho} + \Omega^{\sigma\rho}\frac{\partial \Omega^{\mu\nu}}{\partial a^\rho} = 0, \tag{2.7.5b}$$

are identically satisfied in $\tilde{R}$. Viceversa, necessary and sufficient condition for a matrix $(\Omega_{\mu\nu}(t,a))$ to be the inverse of a $2n\times 2n$, class C^∞ and regular matrix $(\Omega^{\mu\nu}(t,a))$ satisfying conditions (2.7.5) in a region $\tilde{R}$ of the local variables is that all conditions (2.7.4) are identically satisfied in $\tilde{R}$.

In essence, the above theorem establishe that the subset (2.7.4) of the conditions of variational selfadjointness for quasilinear first-order forms and Eqs. (2.7.5) are equivalent.

I - Analytic significance of the conditions of self-adjointness.

This significance is transparent from the content of Section 2.6: The conditions of variational selfadjointness for class C^∞, quasilinear, forms guarantee the existence of a Hamiltonian, when restricted to normal forms.[34-I] To see it, it is sufficient to note that the tensor $\omega_{\mu\nu}$ of the covariant canonical equations (2.6.7) is one of the simplest possible solution of conditions (2.7.4). The remaining conditions (2.7.2c) under the assumption $\mathcal{R}_{\mu\nu} = \omega_{\mu\nu}$ reduced to

$$\frac{\partial \omega_{\mu\nu}}{\partial t} = 0 = \frac{\partial \dot{\mp}_\nu}{\partial a^\mu} - \frac{\partial \dot{\mp}_\mu}{\partial a^\nu}, \qquad (2.7.6)$$

i.e. to the conditions of selfadjointness of our covariant normal form, Eqs. (2.6.11).[64]

This analytic significance can be seen also from another profile. The tensor $\omega_{\mu\nu}$ admits the inverse

$$(\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1} = \begin{pmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}, \qquad (2.7.7)$$

which is the familiar fundamental ω-symplectic structure.

The two tensors $\omega_{\mu\nu}$ and $\omega^{\mu\nu}$ allow the raising and lowering of the indeces in phase space, i.e., the transition from a covariant to a contravariant form of the equations. In particular, Hamilton's equations, besides their covariant (normal) form (2.6.7) can also be written in the equivalent contravariant form

$$\dot{a}^\mu - \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} = 0. \qquad (2.7.8)$$

The conditions of variational selfadjointness again result to be the integrability conditions for the existence of a Hamiltonian when studied within the context of contravariant forms. And indeed, the ω-symplectic form $(\omega^{\mu\nu})$ is one of the simplest possible solutions of conditions (2.7.5) which, as indicated earlier, are equivalent to Eqs. (2.7.4).

We shall reinspect again these analytic aspects in Volume II within the context of our Lie-amdissible generalization of canonical equations.

II - Algebraic significance of the conditions of selfadjointness.

As is known, the algebraic structure which underlies Hamilton's equations is a Lie algebra expressed in terms of a classical product

$$[A,B]^* = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(t,a)\,\frac{\partial B}{\partial a^\nu}, \qquad (2.7.9)$$

often referred to as the generalized Poisson brackets. When the tensor $\Omega^{\mu\nu}$ reduces to the fundamental symplectic form $\omega^{\mu\nu}$ one recovers the conventional Poisson Brackets

$$[A,B] = \frac{\partial A}{\partial a^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial q^k}\frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial B}{\partial q^k}. \qquad (2.7.10)$$

The Lie algebra axioms, when expressed in terms of brackets (2.7.9),

$$[A,B]^* + [B,A]^* = 0, \qquad (2.7.11a)$$

$$[[A,B]^*,C]^* + [[B,C]^*,A]^* + [[C,A]^*,B]^* = 0, \qquad (2.7.11b)$$

can be written

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0, \qquad (2.7.12a)$$

$$\Omega^{\mu\rho}\frac{\partial \Omega^{\nu\sigma}}{\partial a^\rho} + \Omega^{\nu\rho}\frac{\partial \Omega^{\sigma\mu}}{\partial a^\rho} + \Omega^{\sigma\rho}\frac{\partial \Omega^{\mu\nu}}{\partial a^\rho} = 0 \qquad (2.7.12b)$$

as trivially derivable with the identifications $A = a^\mu$, $B = a^\nu$, $C = a^\sigma$.

The important consequence is that the conditions of variational selfadjointness for class C^∞, regular, quasi-linear first-order forms of Newton's equations of motion guarantee a Lie algebra structure.[34-I] This can be seen from the fact that the contravariant general form (2.7.4) admits the

form

$$\dot a^\mu - \Omega^{\mu\nu}(t,a)\,\bar{\bar{\div}}_\nu(t,a) = 0, \quad \bar{\bar{\div}} = \bar{\bar{\div}}_{SA} \qquad (2.7.13)$$

The analytic brackets of the equations are characterized by the factor tensor $\Omega^{\mu\nu}$. Under the assumption of selfadjointness, this tensor characterizes a generalized bracket (2.7.9) which satisfy the Lie algebra axioms (2.7.11) from Theorem 2.7.2. The case of the conventional Poisson brackets (2.7.10) is simply a subcase of this more general algebraic framework.

This result was, after all, expected. Since the algebraic structure underlying Hamilton's equations is a Lie algebra, the integrability conditions for the existence of a Hamiltonian "must" be related to this algebraic structure. Our variational approach to selfadjointness simply exhibits this relationship in the most direct form, that originating from the Lie algebra axioms, through the equivalence of the algebraic axioms (2.7.11) and the conditions of selfadjointness (2.7.4).

This algebraic significance can be seen also within the context of the covariant form. Consider the factorization

$$\Omega_{\mu\nu}(t,a)\,[\dot a^\nu - \bar{\bar{\div}}^\nu(t,a)] = 0. \qquad (2.7.14)$$

The underlying brackets are now the so-called generalized Lagrange's brackets

$$\{A,B\}^* = \frac{\partial a^\mu}{\partial A}\,\Omega_{\mu\nu}(t,a)\,\frac{\partial a^\nu}{\partial B}, \qquad (2.7.15)$$

and they reduce to the conventional Lagrange brackets.

$$\{A,B\} = \frac{\partial a^\mu}{\partial A}\,\omega_{\mu\nu}\frac{\partial a^\nu}{\partial B} = \frac{\partial q^k}{\partial B}\frac{\partial p_k}{\partial A} - \frac{\partial p_k}{\partial B}\frac{\partial q^k}{\partial A}, \quad (2.7.16)$$

for the particular choice $\Omega_{\mu\nu} = \omega_{\mu\nu}$.

The axioms obeyed by these brackets are

$$\{A,B\}^* + \{B,A\}^* = 0, \quad (2.7.17a)$$

$$\frac{\partial}{\partial A}\{B,C\}^* + \frac{\partial}{\partial B}\{C,A\}^* + \frac{\partial}{\partial C}\{A,B\}^* = 0, \quad (2.7.17b)$$

and can be explicitly written for $A = a^\mu$, $B = a^\nu$, $C = a^\sigma$

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0, \quad (2.7.18a)$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\sigma} + \frac{\partial \Omega_{\nu\sigma}}{\partial a^\mu} + \frac{\partial \Omega_{\sigma\mu}}{\partial a^\nu} = 0, \quad (2.7.18b)$$

yielding precisely the subset (2.7.4) of the conditions of variational selfadjointness. This confirms the algebraic meaning of the conditions considered.

The covariant and contravariant formulations therefore express two aspects of the same underlying algebraic structure characterized by two corresponding tensors $\Omega^{\mu\nu}$ and $\Omega_{\mu\nu}$ whose matrices are the inverse of each other. In turn, these tensors characterize two brackets, the generalized Lagrange and Poisson brackets, which are also the inverse of each other according to the property

$$\sum_{k=1}^{n} [A_i, A_k]^* \{A_k, A_j\}^* = \delta_{ij}. \quad (2.7.19)$$

For each of these two formulations the conditions of variational selfadjointness guarantee the verification of the

corresponding fundamental axioms, i.e., Eqs. (2.7.11) and (2.7.17).

We shall reinspect again these algebraic aspects in Volume II within the context of our Lie-admissible generalization of Lie algebras.

III - Geometrical significance of the conditions of variational selfadjointness.

As is known, the geometry which underlies Hamilton's equations is the symplectic geometry. In elementary language we can say a class C^∞, regular, quasilinear, first-order system of Newton's equations of motion can be interpreted as a vectorfield in a 2n-dimensional manifold M(a) with local coordinates a^μ, $\mu = 1,2,\ldots,2n$. On more technical grounds this manifold is Hausdorff, second countable and ∞-differentiable. The manifold M is called a symplectic manifold (and is here denoted with the symbol $M(a, \Omega)$), when it is equipped with a nonsingular, closed two-form Ω. For the reader's convenience, let us recall that, in our unified notation $a = \{q, p\}$, the two-forms in question can be written

$$\Omega = \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu = \frac{1}{2}\sum_{\mu_1,\mu_2}^{\nu_1\nu_2} \Omega_{\nu_1\nu_2}(a)\, da^{\mu_1} \wedge da^{\mu_2} \quad (2.7.20)$$

where $\wedge$ is the (antisymmetric) exterior product and the δ-symbol represents the generalized Kronecker delta. 56,34-I

The closure condition reads

$$d\,d\Omega = d\left(\frac{1}{2}\,\delta^{\nu_1\nu_2}_{\mu_1\mu_2}\,\Omega_{\nu_1\nu_2}\,da^{\mu_1}\wedge da^{\mu_2}\right) \qquad (2.7.21)$$

$$= \frac{1}{2}\,\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}\,\frac{\partial\Omega_{\nu_1\nu_2}}{\partial a^{\nu_3}}\,da^{\mu_1}\wedge da^{\mu_2}\wedge da^{\mu_3} = 0,$$

and its integrability conditions are given by, for $\Omega_{\mu\nu} = -\Omega_{\nu\mu}$,

$$\frac{1}{2}\,\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}\,\partial\Omega_{\nu_1\nu_2}/\partial a^{\nu_3} \qquad (2.7.22)$$

$$= \frac{\partial\Omega_{\nu_1\nu_2}}{\partial a^{\nu_3}} + \frac{\partial\Omega_{\nu_2\nu_3}}{\partial a^{\nu_1}} + \frac{\partial\Omega_{\nu_3\nu_1}}{\partial a^{\nu_2}} = 0.$$

Thus, the subset (2.7.4) of the conditions of self-adjointness coincides with the integrability conditions for a nondegenerate two-form (2.7.20) to be closed, that is, symplectic.

A (2n+1)-dimensional manifold $M(\hat{a}, \hat{\Omega})$ is called a contact manifold when equipped with a closed two-form of (maximal) rank 2n. By putting

$$\hat{\Omega} = \hat{\Omega}_{ij}\,d\hat{a}^i\wedge d\hat{a}^j, \quad \hat{a} = (q, p, t), \qquad (2.7.23)$$

$$\hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu}, \quad \hat{\Omega}_{0i} = -D_i = -\hat{\Omega}_{i0},$$

$$\Omega_{00} = 0, \quad i = 1, 2, \ldots, 2m, 0,$$

one can then see that the complete system of conditions of selfadjointness (2.7.2) in this case provides the integrability conditions for the two-form (2.7.23) to be closed, that is, of contact type.

In conclusion, the conditions of variational selfadjointness for class C^∞, quasilinear, regular, first-order forms of Newtonian systems guarantee the esistence of a symplectic structure for the autonomous case and a contact structure for the nonautonomous case.

We can thus conclude by saying that the variational approach to selfadjointness provides a symbiotic representation of truly fundamental methodological aspects of classical mechanics, that is, the integrability conditions for the existence of analytic representations, Lie algebras and symplectic (or contact) geometry.

The reader can now appreciate the physical relevance of the property identified earlier, according to which the electromagnetic interactions are realized in terms of selfadjoint forces or couplings at all levels of study. Indeed, this property ultimately constitutes one way for the technical expression of the compatibility with the electromagnetic interactions of established analytic, algebraic and geometrical formulations and, thus, established relativity as well as quantum mechanical approaches.

Jointly, the reader can now understand the rather profound methodological implications of our assumption that the strong interactions are realized, at all levels of study, via NON-selfadjoint forces or couplings. This literally means that the realization of the strong interactions under study must be such to VIOLATE, by central assumption, the integrability conditions for the direct applicability of analytic mechanics, Lie algebra and symplectic (or contact) geometry, as currently understood. As we shall see, this will, on one side, imply the nonapplicability of conventional relativity and quantum mechanical approaches and, on the other side, will stimulate the search for covering formulations, that is, generalized analytic, algebraic and geometrical formulations which directly apply to the broader physical context under consideration, while being able to recover conventional formulations identically under the limit of null nonselfadjoint forces. Once this covering methodological context has been (at least rudimentarily) identified, the construction of the rudiments of conceivable covering relativity and quantum mechanical laws for strong nonselfadjoint interactions is expected to be consequantial.

2.8: THE INDIRECT UNIVERSALITY OF THE INVERSE PROBLEM

In this section we shall review the rather delicate aspect of the limits of applicability of the Inverse Problem. Contrary to a rather general belief that only special cases of differential equations admit an analytic representation, we shall see that the methodology of the Inverse Problem has a rather vast arena of effectiveness. As a matter of fact, the following theorem of indirect universality holds

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THEOREM 2.8.1: Local class C^∞ and regular Newtonian systems always admit an indirect analytic representation.

The origin of possible misrepresentations is the concept of "indirect analytic representation". With these terms we here refer to the representation of any system of differential equations obtained with the equivalence transformations of type (a), (b) and (c) considered in Section 2.3. This section is devoted to an outline of the proof of the above important theorem as well as to the identification of the physical implications of the indirect nature of the considered analytic representations.

The proof of Theorem 2.8.1 is essentially based on the following three steps.

Step 1: Reduction of Newton's equations as derived from the second law to a quasilinear (regular) first-order form. This is achieved by introducing generally noncanonical prescriptions (2.6.4) for the characterization of the new variables

y_k under the condition that they are of at least class C^∞ and regular, i.e., admit the (unique) implicit functions $\dot{q} = g(t,q,y)$. This yields in a natural way a class C^∞, regular quasilinear form

$$\left.\begin{aligned} F_k(t,q,\dot{q},\ddot{q}) &= 0 \\ G_k(t,q,\dot{q},y) &= 0 \end{aligned}\right\} \rightarrow \left[\Omega_{\mu\nu}(t,a)\dot{a}^\nu + \Gamma'_\mu(t,a)\right]^{C^\infty,R}_{NSA} = 0, \quad (2.8.1)$$
$$a = \{q,y\}.$$

Step 2: Reduction of the quasilinear form (2.8.1) to an equivalent selfadjoint form. Unlike the case of second order differential equations, the first order nature of system (2.8.1) is such that it can always be written in an equivalent selfadjoint form through the use of genotopic functions

$$\left(\bar{\Omega}_{\mu\nu}\dot{a}^\nu + \bar{\Gamma}_\mu\right)^{C^\infty,R}_{SA} = \left[g_{\mu\nu}\left(\Omega_{\nu\rho}\dot{a}^\rho + \Gamma_\nu\right)^{C^\infty,R}_{NSA}\right]^{C^\infty,R}_{SA} = 0. \quad (2.8.2)$$

Step 3: Reduction of the selfadjoint quasilinear form (2.8.2) to an equivalent selfadjoint normal form. From Lie-Koening's theorem we know that there always exist a continuity preserving and invertible point transformation under which[61]

$$a^\mu \rightarrow a'^\mu, \quad a'^\mu \in C^\infty(R), \ |da'/da|(R) \neq 0 \quad (2.8.3a)$$

$$\left[\bar{\Omega}_{\mu\nu}(t,a)\dot{a}^\nu + \bar{\Gamma}'_\mu(t,a)\right]^{C^\infty,R}_{SA} = \left[\omega_{\mu\nu}\dot{a}'^\nu - \bar{\bar{\Gamma}}'_\mu(t,a')\right]^{C^\infty,R}_{SA} = 0. \quad (2.8.3b)$$

This yields precisely a selfadjoint normal form as needed for the Fundamental Analytic Theorem 2.6.3 and completes out reviews. For technical details see Volume II of reference.

From a geometrical viewpoint, Theorem 2.8.1 is a cimple consequence of Darboux theorem on existence of a diffeomorphism which reduces a general symplectic or contact structure to the fundamental form. From an algebraic viewpoint the same theorem expresses the known property that Lie's theory is always applicable, up to invertible changes of local variables, to even-dimensional systems of first-order equations.

Our objective is now that of identifying the most important subcases of Theorem 2.8.1 and related physical implications. This can be done with the following classification of Newtonian systems:

CLASS 1 - Essentially selfadjoint Newtonian systems. These are systems which, as derived from the second law, are (of class C^∞, regular and) selfadjoint and we shall write

$$\left[m_k \ddot{x}_{ka} - f_{ka}(t, x_m, \dot{x}_m) \right]_{SA}^{C^\infty, R} = 0 \qquad (2.8.4)$$

$k = 1, 2, \ldots, n$; $a = x, y, z$.

These systems are clearly contained in Theorem 2.8.1, trivially, when prescriptions (2.6.4) are canonical. In this case steps 2 and 3 of the proof of Theorem 2.8.1 are not essential because Eqs. (2.8.1) can be written in the selfadjoint normal form for

which a Hamiltonian, from Theorem 2.6.3, exists.

The terms "essentially selfadjoint" are intended to indicate that the considered systems are selfadjoint in the form derived from the second law, or, equivalently from their direct Lagrangian representations, irrespective of the existence of equivalent nonselfadjoint forms of the same systems. This is due to a rather peculiar occurrence whereby conventional conservative systems can be also written in an equivalent nonselfadjoint form. For instance, the equation for the harmonic oscillator can be written $m\ddot{q} / q + K = 0$ for all values $q \neq 0$ in which case it is nonselfadjoint. Nevertheless, the system considered is essentially selfadjoint in the sense that when written in the Newtonian form $m\ddot{q} + Kq = 0$ is selfadjoint. The class of Newtonian systems here considered, of course, includes the case of charged particles with the conventional Lorentz coupling (which is velocity-dependent). Again, as the reader can verify explicitly, the equations of motion of this system can be written either in a selfadjoint or in a nonselfadjoint form. Nevertheless, the system is essentially selfadjoint because it is so as derived from Newton's second law.[65] The considered class of systems also contains genuine nonconservative systems owing to the admissible velocity-dependence of the acting forces, although such a dependence can at most be linear. (Corollary 2.5.2A). Therefore, even though the class of essentially selfadjoint Newtonian systems is inclusive of conventional conservative systems, it is much broader than that. An

equivalent definition of this class would be that of systems whose acting forces are selfadjoint, i.e., satisfy the (Newtonian limit) of Corollary 2.5.2A.

It is hoped that the selected terms "essentially selfadjoint" here referred to a variational property of ordinary differential equations do not create confusion with the same terms used in the theory of linear operator on vector spaces. As a matter of fact, the indicated terms have been selected precisely in view of a direct parallelism with the latter context. Indeed, under certain conditions, the Lagrange operator $\frac{d}{dt}\frac{\partial}{\partial \dot{q}^k} - \frac{\partial}{\partial q^k}$ is essentially selfadjoint in the operator sense. This point established a close parallelism between the variational and operator approach to selfadjointness. On technical grounds this parallelism can be seen by studying the integrability conditions for the existence of an action functional within a functional (Hilbert space) approach, as originally introduced by Veinberg and subsequently expanded by several authors. A review of this aspect would bring us outside of the objectives of this monograph. We therefore refer the reader to reference.[34-I]

CLASS 2 - Nonessentially nonselfadjoint Newtonian systems. These are systems which, as derived from Newton's second law, are (of class C^∞, regular and) nonselfadjoint but admit genotopic functions within a fixed system of local variables according to the rule

$$\left\{ h_{ka}{}^{jb}(t, x_m, \dot{x}_m) \left[\mu_j \ddot{x}_{jb} - f_{jb}(t, x_m, \dot{x}_m) \right]^{C^\infty, R}_{NSA} \right\}^{C^\infty, R}_{SA} = 0. \quad (2.8.5)$$

In this case step 2 of the proof of Theorem 2.8.1 is needed but step 3 is not essential. Also in this case transformations (b) and (c) of Section 2.3 are equivalent in the sense that, instead of using genotopic functions, i.e.(transformations of class (c)), one could use coordinate transformations of type (b) to induce a selfadjoint structure, construct a Lagrangian in this new coordinate system and then construct a Lagrangian in the original coordinate system through the use of the inverse transformations. The representation in this latter case is still indirect, i.e., through factor functions according to structure (2.3.19). This yields the equivalence of methods (b) and (c) for the class of systems under considerations.

On more technical grounds this class of systems is characterized by the existence of a solution in the genotopic functions of the conditions of selfadjointness (2.3.21) within a fixed system of local variables and the equivalent selfadjoint system is well behaved in a star - shaped region.[34-II]

CLASS 3 - Essentially nonselfadjoint Newtonian systems. These are Newtonian systems which, as derived from the second law, are (of class C^∞, regular and) nonselfadjoint and such that the conditions of selfadjointness on the genotopic functions are inconsistent, i.e., the equations cannot be rendered selfadjoint through the multiplication of a regular matrix of factor functions as in structure (2.8.5). In this case the use of all steps 1, 2 and 3 of the proof of Theorem 2.8.1 are needed with particular reference to the use of the Darboux theorem which now becomes

crucial to construct an equivalent selfadjoint normal form. In particular, the use of coordinate transformations is necessary to achieve this objective.

Our problem is now that of understanding the physical implications of step 3 under the indicated conditions.

For this purpose, let us recall that a <u>canonical transformation</u> is any transformation in phase space which preserves the value of the fundamental cosymplectic (Lie) form, i.e.,

$$a^\mu \longrightarrow a'^\mu = a'^\mu(a), \tag{2.8.6a}$$

$$\omega^{\mu\nu} \longrightarrow \omega'^{\mu\nu} = \frac{\partial a'^\mu}{\partial a^\rho}\,\omega^{\rho\sigma}\frac{\partial a'^\nu}{\partial a^\sigma} = \omega^{\mu\nu}. \tag{2.8.6b}$$

Since this form represents the fundamental Poisson brackets

$$(\omega^{\mu\nu}) = \begin{pmatrix} [x^{ia}, x^{jb}] & [x^{ia}, p_{jb}] \\ [p_{ia}, x^{jb}] & [p_{ia}\ p_{jb}] \end{pmatrix} = \begin{pmatrix} 0_{3n\times 3n} & 1_{3n\times 3n} \\ -1_{3n\times 3n} & 0_{3n\times 3n} \end{pmatrix}, \tag{2.8.7}$$

the physical significance of the above geometrical definition of canonical transformation is selfevident.

If, instead of the fundamental form $\omega^{\mu\nu}$, one has a general symplectic form $\mathcal{A}^{\mu\nu}(t,a)$, we have a <u>generalized canonical transformation</u> when, on similar grounds,

$$a^\mu \to a'^\mu = a'^\mu(a), \tag{2.8.8a}$$

$$\mathcal{A}^{\mu\nu}(a) \to \mathcal{A}'^{\mu\nu} = \frac{\partial a'^\mu}{\partial a^\rho}\,\mathcal{A}^{\rho\sigma}[a(a')]\frac{\partial a'^\nu}{\partial a^\sigma} = \mathcal{A}'^{\mu\nu}(a'), \tag{2.8.8b}$$

To understand step 3, the first point which is in order is that transformations (2.8.3a) are neither canonical nor generalized canonical. And indeed, such transformations must be such that $\mathcal{A}^{\mu\nu} \longrightarrow \omega^{\mu\nu}$.

The second point in order is that the configuration space image of transformations (2.8.3a) has an essential dependence on the velocities. This means that, in order to construct an analytic representation, Newton's equation must be transformed from the original Cartesian coordinate x_μ^k in a three dimensional Euclidean space as constructed from the second law to new coordinate x'^k_μ related to x^k_μ through the transformation laws

$$x'^k = x'^k(t, x_\mu, \dot{x}_\mu) \tag{2.8.9}$$

with an essential dependence in the velocity.

To have an indication of the physical implications of these transformations it is sufficient here to note that transformations (2.8.3a), since they are not canonical, do not preserve the structure of Hamilton's equations. As a result, the configuration-space image of these transformations does not preserve the structure of Lagrange's equations.

As a result, the Newtonian systems we have called essentially nonselfadjoint are truly nonanalytic in structure. They do admit a new coordinate system in which they exhibit an analytic structure, but such a new system produces a profound change in the physical structure of the original system. The nonessentially nonselfadjoint and the essentially selfadjoint systems, on the contrary, exhibit an analytic structure (up to possible multiplication of regular matrices of factor functions) in the original reference system as derived from the second law.

Thus, the analytic representations of essentially selfadjoint and nonessentially nonselfadjoint Newtonian systems preserve the physical reference frame of the systems considered, while those of the essentially nonselfadjoint systems exist only in a mathematical reference frame which is, in general, highly noninertial and non-realizable with experiments (owing to the nonlinear dependence of the new coordinates on the physical velocities).

In conclusion, the analytic representations of Newtonian systems of class 1 and 2 are selfconsistent on both mathematical and physical grounds. However, those of class 3 should be considered, strictly speaking, as selfconsistent on mathematical grounds, but that care should be exercised before attempting physical considerations within such a generalized mathematical setting.

It might be also of some interest to indicate the structure of the equations for representations of class 3.

Let us recall that the fundamental canonical equations of Analytic Mechanics, Hamilton's equations, demand the use of either the fundamental symplectic structure $\omega_{\mu\nu}$ or its contravariant (Lie) form $\omega^{\mu\nu}$ according to the equations

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H}{\partial a^{\mu}} = 0 , \qquad (2.8.10)$$

$$\dot{a}^{\mu} - \omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0 .$$

However, as recalled in Section 2.7, a symplectic form (or, more properly, a canonical form) can also be in the more general structure $\mathcal{R}^{\mu\nu}$. When the transition from the former to the latter structure is assumed, Hamilton's equations are lost. In this case there is the emergence of their generalized forms

$$\mathcal{R}_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} = 0 , \qquad (2.8.11)$$

$$\dot{a}^{\mu} - \mathcal{R}^{\mu\nu}(a)\frac{\partial B}{\partial a^{\nu}} = 0 ,$$

which are often referred to as Birkhoff's equations.[34-I]

Our variational approach to selfadjointness identifies a relationship between Hamilton's and Birkhoff's equations. On similarity grounds, both equations are selfadjoint with the latter equations resulting as the most general structure admitted by the

conditions of selfadjointness of quasilinear first-order systems i.e., Eqs. (2.7.2).[61] As a result, both equations are embodied into a canonical as well as a Lie algebra structure and possess the symplectic geometry as the underlying geometry. On grounds of differences between Eqs. (2.8.10) and (2.8.11), only the former characterize, strictly speaking, a phase space, in the sense that only for the former the variables q^k and p_k are canonically conjugate. Nevertheless Birkhoff's equations should not be overlooked. For instance, they admit a much broader transformation theory which can be significant, e.g., for the problem of symmetries and first integrals. While Hamilton's equations are form-invariant only under the subset of point transformations $a^\mu \to a'^\mu(a)$ which satisfies conditions (2.8.6), Birkhoff's equations are form-invariant under arbitrary (but continuity preserving and invertible) point transformations in the local variable.

With respect to Theorem 2.8.1 for the case of essentially nonselfadjoint systems, notice that step 2 allows a direct analytic representation of Eqs. (2.8.2) in terms of Birkhoff's equations, i.e. (we here ignore the nonautonomous case)

$$\left[\mathcal{R}_{\mu\nu}\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right]_{SA} \overset{C,R}{\equiv} \mathcal{R}_{\mu\nu}\dot{a}^\nu + \Gamma_\mu = 0, \qquad (2.8.12)$$

$$\mu = 1,2,\ldots,6n,$$

as can be proved by using conditions of selfadjointness (2.7.2). The crucial transformations (2.8.3a) are then needed to reduce Birkhoff's equations to the Hamiltonian form, i.e.

$$a^\mu \to a'^\mu = a'^\mu(a), \qquad (2.8.13a)$$

$$\left[\mathcal{R}_{\mu\nu}(a)\dot{a}^\nu - \frac{\partial B}{\partial a^\mu}\right]_{SA} = \left[\omega_{\mu\nu}\dot{a}'^\nu - \frac{\partial H'}{\partial a'^\mu}\right]_{SA} = 0, \qquad (2.8.13b)$$

$$\mathcal{R}_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} = \text{SYMPL. FORM} \qquad (2.8.13c)$$

$$= \text{GENER. SYMPL. FORM}$$

$$H'(a') = B(a) \qquad (2.8.13d)$$

with corresponding transformations for the nonautonomous case[61,34-II]

The relationship between Hamilton's and Birkhoff's equations indicated above is the analytic conterpart, as suggested by the methodology of the Inverse Problem, of the known property in symplectic geometry according to which there always exist a transformation reducing a general symplectic form to its fundamental form, i.e.

$$a^\mu \to a'^\mu = a'^\mu(a), \qquad (2.8.14a)$$

$$\mathcal{R}_{\mu\nu}(a) \to \mathcal{R}'_{\mu\nu} = \frac{\partial a^\rho}{\partial a'^\mu}\mathcal{R}_{\rho\sigma}[a(a')]\frac{\partial a^\sigma}{\partial a'^\nu} = \omega_{\mu\nu}$$

$$= \text{FUNDAM. SYMPL. FORM} \qquad (2.8.14b)$$

This porperty is known under the name of Darboux's theorem in Differential Geometry but it is often referred to as Pauli's theorem in the physical literature.[66] In essence, Pauli's version of the property puts the emphasis in the use of conventional transformation theory, i.e., Eqs. (2.8.14), while Darboux's version puts the emphasis in geometrical considerations.

For completeness it should be here indicated that other types of theorems of indirect universality of the Inverse Problem can be proved. Our efforts have been centered in the use of equivalence transformations which preserve the central characteristics of the system considered, such as the order, dimensionality and regularity, as in steps 2 and 3 of Theorem 2.8.1. But other type of order, dimensionality and regularity changing equivalence transformations are also conceivable. This would yield a further broadening of the concept of indirect analytic representations.

For instance, P. Havas[67] proved a theorem of indirect universality by reducing the second-order equations of motion to an equivalent first order form

$$A_{ki}\,\ddot{q}^{i} + B_k = 0 \;\longrightarrow\; C_{\alpha\beta}(t,z)\,\dot{z}^{\beta} + D_{\alpha}(t,z) = 0, \qquad (2.8.15)$$

$$k = 1,2,\ldots,n \qquad\qquad \alpha = 1,2,\ldots,2n$$

which is representable in terms of totally degenerate Lagrangians, i.e. Lagrangians which are linear in the first order derivatives, (Section 2.5) according to the representation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}^{\alpha}} - \frac{\partial L}{\partial z^{\alpha}} \equiv C_{\alpha\beta}\,\dot{z}^{\beta} + D_{\alpha} \qquad (2.8.16)$$

$$\alpha = 1,2,\ldots,2n$$

As an incidental remark it should be here recalled that Havas' theorem also contains a proof of step 2 above (the property that

first-order systems can always be reduced to equivalent self-adjoint forms through integrating factors) as well as the treatment of differential equations of arbitrary order.

For the case of indirect universality in Field Theory see Appendix A of paper III, reference.[30]

The methodological implications of the Theorem of Indirect Universality of the Inverse Problem are rather substantial. Explicitly stated, Theorem 2.8.1 ensures that strong nonselfadjoint interactions are indeed indirectly treatable with conventional analytic, Lie and symplectic (or contact) methods. Equivalently, we can say that, even though our strong forces ARE NOT derivable from a potential by central assumption, a Hamiltonian always exists (locally, under C^{∞} property as well as in the neighborhood of a regular point). However, special care should be exercised before deriving physical informations via such a mathematical representation, because the admissible coordinates are now noninertial and nonrealizable via experiments. More insidiosly, mathematical algorithms such as "$\underset{\sim}{p}$", "H", "$\underset{\sim}{M}$" do not have their conventional direct physical significance (linear momentum, energy and angular momentum) as a necessary condition for the existence of such a representation (see Appendix 2.B for details).

We can therefore say in figurative terms that the conventional analytic, Lie and geometrical formulations are directly applicable to the electromagnetic interactions with a direct physical significance of the algorithms at hand. The same formulations, on the contrary, are only indirectly applicable to strong nonselfadjoint interactions, i.e., applicable without any direct physical significance of the algorithms at hand ($\underset{\sim}{r}$, $\underset{\sim}{p}$, H, $\underset{\sim}{M}$, etc.). As we shall see, this (and other[68]) properties are at the basis of the need for Lie-admissible formulations.

2.9: USE OF THE INVERSE PROBLEM FOR THE CHARACTERIZATION OF THE LAGRANGIAN STRUCTURE.

We shall now study some implications of the methodology of the Inverse Problem for Newtonian and field theoretical interactions beginning from the problem of the Lagrangian structure.

As is known, the Lagrangian structure currently used in Analytic Mechanics has the familiar form given by the sum of a term representing the free motion and an interaction term, i.e.,

$$L(t, \underset{\sim}{x}_m, \dot{\underset{\sim}{x}}_m) = T(\dot{\underset{\sim}{x}}_m) - U(t, \underset{\sim}{x}_m, \dot{\underset{\sim}{x}}_m)$$

$$= L_{free} + L_{int} = \sum_{k=1}^{N} \sum_{\substack{a= \\ x,y,z}} L_{free}^{(ka)} + L_{int} \qquad (2.9.1)$$

$$= \frac{1}{2m_k} p_{ka} p^{ka} + U$$

This structure has persisted in Field Theory where the structure of a Lagrangian density used in the almost totality of current models, is the sum of a free term representing the decoupled (free) fields plus a term representing interactions i.e.

$$\mathcal{L}(\phi^k, \phi^k_{;\alpha}) = \mathcal{L}_{free} + \mathcal{L}_{int} = \sum_{k=1}^{N} \mathcal{L}_{free}^{(k)} + \mathcal{L}_{int} \qquad (2.9.2)$$

For instance, the currently available models of electromagnetic and strong interactions are based on Lagrangian structures of the type

$$\mathcal{L}_{tot}^{conv} = \mathcal{L}_{free} + \mathcal{L}_{int}^{elm,strong} \qquad (2.9.3)$$

The objective of this section is to identify the Lagrangian structure which originates from the use of the Inverse Problem.

To better identify this objective, let us note that the above Lagrangians represent systems with local couplings derivable from a potential. For instance, Lagrangian (2.9.2) represent the selfadjoint semilinear system

$$\left[(\Box + m_{(k)}^2) \phi_k - j_k(\phi^j, \phi^{j;}_{;\alpha}) \right]_{SA} = 0. \qquad (2.9.4)$$

Structure (2.9.2) is thus equivalent to the assumption that the couplings j_k are derivable from a potential. In the language of the Inverse Problem this is equivalent to the assumption that the couplings j_k are (local and) selfadjoint, i.e., satisfy conditions (2.5.11). Notice also that the semilinearity of the field equations implies that the term $\mathcal{L}_{int}$ can at most be linear in the derivative couplings because conditions (2.5.11) allow at most a linearity in such couplings. (Corollary 2.5.2B). This is a feature shared by currently available models, such as the gauge models. Alternatively, if the term $\mathcal{L}_{int}$ exhibits a nonlinear dependence in the derivative couplings, the underlying field equations are quasilinear, i.e., of the more general type

$$\left[A_{ki}^{\mu\nu}(\phi^j, \phi^{j;}_{;\alpha}) \phi^{i;}_{;\mu\nu} + B_k(\phi^j, \phi^{j;}_{;\alpha}) \right]_{SA} = 0, \qquad (2.9.5)$$

and not of the semilinear type (2.9.4). From now on, unless otherwise stated, all conventional Lagrangians will be assumed to satisfy the condition of at most linearity in the derivative couplings of the additive interaction term.

Our objective can now be identified. It consists in studying the structure of a total Lagrangian for the analytic representation of local systems with arbitrary couplings, i.e., couplings which are not necessarily derivable from a potential or, equivalently, which are not necessarily selfadjoint. The reader should keep in mind that this is precisely the type of systems which, in these monographs, is assumed as representing the hadronic structure.

Let us consider first the Newtonian case. The arena of primary relevance of the inverse problem is that of identifying an analytic representation for the local (unconstrained) Newtonian systems with nonconservative forces, i.e.

$$\left[m_k \ddot{x}_k - F_k(t, x_m, \dot{x}_m) \right]_{NSA} = 0 \qquad (2.9.6)$$

$$k = 1, 2, \ldots, n$$

or, in our language, of Newtonian systems with local nonselfadjoint couplings. Clearly, for these systems structure (2.9.1) is not admissible.

From a mathematical viewpoint the emerging Lagrangian structure

is that characterized by the Converse of the Poicaré Lemma in the proof of the Fundamental Analytic Theorem, 2.3.2, i.e., structure (2.3.9)

$$L^{gen}_{tot} = K(t, x_m, \dot{x}_m) + D_{ka}(t, x_m)\, \dot{x}^{ka} + C(t, x_m). \qquad (2.9.7)$$

The objective of this section is that of reinterpreting this structure from the viewpoint of the interactions.

Assume that system (2.9.6) is nonessentially nonselfadjoint (Section 2.8), i.e., it admits an indirect analytic representation in terms of the conventional Lagrange equations within a fixed system of Cartesian coordinates. The structure of the considered representation is

$$\frac{d}{dt}\frac{\partial L^{gen}_{tot}}{\partial \dot{x}^{ka}} - \frac{\partial L^{gen}_{tot}}{\partial x^{ka}} \equiv \left\{ h_{ka}{}^{jb} \left[m_j \ddot{x}_{jb} - F_{jb} \right]_{NSA} \right\}_{SA} \qquad (2.9.8)$$

with the necessary presence of the multiplicative functions.

It is easy to see that, from the necessary presence of the genotopic functions as well as their functional dependence $h_{ka}{}^{jb}(t, x_m, \dot{x}_m)$, the emerging Lagrangian structure can be written[30,34-I]

$$L^{gen}_{tot}(t, x_m, \dot{x}_m) \qquad (2.9.9)$$

$$= \sum_{k=1}^{n} \sum_{\substack{a = \\ x,y,z}} L^{(ka)}_{int,I}(t, x_m, \dot{x}_m)\, L^{(ka)}_{free}(\dot{x}^{ka}) + L_{int,II}(t, x_m, \dot{x}_m)$$

namely, it is characterized by 3n+1 interaction terms, 3n multiplicative and one additive to the free terms. And indeed, the necessary presence of the genotopic functions h_{ka}^{jb} results in the necessary presence of the multiplicative interaction terms $L_{int,I}^{(ka)}$

Structure (2.9.9) is clearly a covering of the conventional structure (2.9.1) because it is a nontrivial generalization of the latter while at the same time admits the latter as a particular case under the limit

$$\lim_{L_{int,I}^{(k)} \to 1} L_{tot}^{gen} = \lim_{F_{NSA} \to F_{SA}} L_{tot}^{gen} = L_{tot}^{conv}. \qquad (2.9.10)$$

On mathematical grounds Lagrangians (2.9.7) and (2.9.9) are equivalent. However, on physical grounds the latter is preferable over the former because it allows the representation of the underlying couplings with interaction terms,[69] as it is already the case for conventional Lagrangians, i.e., Eqs. (2.9.1).

A few examples are here in order to illustrate this point. One of the simplest nonconservative systems is a Newtonian particle in motion under a velocity-dependent drag force, i.e.

$$\left(m\ddot{r} + \gamma\dot{r}\right)_{NSA}^{c^\infty, R} = 0. \qquad (2.9.11)$$

The methods of Section 2.3 and 2.6 yield a Lagrangian which, after reinterpretation (2.9.9) can be written

$$L_{tot}^{gen} = L_{int,I}\, L_{free} + L_{int,II} \qquad (2.9.12)$$

$$L_{int,I} = e^{\frac{\gamma}{m}t}, \quad L_{free} = \frac{1}{2} m \dot{r}^2, \quad L_{int,II} = 0$$

The underlying analytic representation has the structure

$$\frac{d}{dt}\frac{\partial L_{tot}^{gen}}{\partial \dot{r}} - \frac{\partial L_{tot}^{gen}}{\partial r} \qquad (2.9.13)$$

$$\equiv \left[e^{\frac{\gamma}{m}t} \left(m\ddot{r} + \gamma\dot{r}\right)_{NSA}^{c^\infty, R} \right]_{SA}^{c^\infty, R}.$$

It should be stressed that the presence of the multiplicative interaction term to the free term is necessary to represent the motion. Notice that Lagrangian (2.9.12) has still the dimension of energy.

On similar grounds, the equation of motion for a damped harmonic oscillator

$$\left[m\ddot{r} + \gamma\dot{r} + kr\right]_{NSA}^{c^\infty, R} = 0, \qquad (2.9.14)$$

admits an analytic representation in terms of the generalized structure

$$L_{tot}^{gen} = L_{int,I}\, L_{free} + L_{int,II} \qquad (2.9.15)$$

$$L_{int,I} = e^{\frac{\gamma}{m}t}\,, \quad L_{free} = \frac{1}{2} m \dot{x}^2\,, \quad L_{int,II} = -e^{\frac{\gamma}{m}t}\frac{1}{2} k x^2$$

which also satisfy the indicated covering conditions.

For the case in more than one dimension, consider the system subject to a velocity-dependent drag force and an applied force which is exponentially decaying in time

$$\begin{cases} m\ddot{x} + \gamma_1 \dot{x} = b_1 e^{-\gamma_1 t/m}, \\ m\ddot{y} + \gamma_2 \dot{y} = b_2 e^{-\gamma_2 t/m}. \end{cases} \qquad (2.9.16)$$

The methods of Section 2.3 then yield as an admissible Lagrangian

$$L_{tot}^{gen} = \sum_{a=x,y} L_{int,I}^{(a)}\, L_{free}^{(a)} + L_{int,II} \qquad (2.9.17)$$

$$L_{int,I}^{(a)} = e^{\gamma_a t/m}\,, \quad L_{free}^{a} = \frac{1}{2} m \dot{x}_a^2\,, \quad L_{int,II} = b_a x^a$$

An example which will be particularly significant for our analysis is the generalization of the conventional Kepler system (consisting of the motion of particles in vacuum) to the case of motion in a physical medium which creates drag forces. The conventional Kepler system is characterized by the equations of motion, say, for the two-body case

$$\begin{pmatrix} m_1 \ddot{\underset{\sim}{x}}_1 - \underset{\sim}{f}_1 \\ m_2 \ddot{\underset{\sim}{x}}_2 - \underset{\sim}{f}_2 \end{pmatrix}_{SA} = 0, \qquad (2.9.18)$$

$$\underset{\sim}{f}_k = -\frac{\partial V}{\partial \underset{\sim}{x}^k}\,, \quad \underset{\sim}{f}_1 = -\underset{\sim}{f}_2\,, \quad V = -\frac{k}{r}\,, \quad r = |\underset{\sim}{x}_1|$$

and conventional Lagrangian

$$L_{tot}^{Conv} = \frac{1}{2} m_k \dot{\underset{\sim}{x}}_k \cdot \dot{x}^k - V(r) \qquad (2.9.19)$$

One case of the generalized Kepler system in a physical medium is characterized by the nonconservative system

$$\begin{pmatrix} m \ddot{\underset{\sim}{x}}_1 - \underset{\sim}{f}_1 + \gamma \dot{\underset{\sim}{x}}_1 \\ m \ddot{\underset{\sim}{x}}_2 - \underset{\sim}{f}_2 + \gamma \dot{\underset{\sim}{x}}_2 \end{pmatrix}_{NSA} = 0 \qquad (2.9.20)$$

The methods of Section 2.3 then yield as an admissible generalized Lagrangian

$$L_{tot}^{gen} = L_{int,I} \sum_{k=1}^{2} L_{free}^{(k)} + L_{int,II} \qquad (2.9.21)$$

$$L_{int,I} = e^{\gamma t/m}\,, \quad L_{free}^{(k)} = \frac{1}{2} m \dot{\underset{\sim}{x}}_k^2\,, \quad L_{int,II} = -e^{\gamma t/m} V(r)$$

Another example which will be particularly significant for our analysis is that of the physical spinning top, i.e., the spinning top with drag torques. The conventional conservative approximation

of the spinning top is characterized by the selfadjoint equations of motion in the Euler angles

$$\left(A_{ki}\,\ddot{\theta}^{i} + \dot{B}_{k}\right)_{SA} = 0$$

$$(A_{ij}) = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & (I_1 \sin^2\theta_1 + I_3 \cos^2\theta_2) & (I_3 \cos\theta_1) \\ 0 & (I_3 \cos\theta_1) & I_3 \end{pmatrix}$$

$$(B_k) = \begin{pmatrix} -(I_1 - I_3)\dot{\theta}_2^2 \sin\theta_1 \cos\theta_1 + I_3 \dot{\theta}_2 \dot{\theta}_3 \sin\theta_1 - \mu g R \sin\theta_1 \\ 2(I_2 - I_3)\dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \cos\theta_1 - I_3 \dot{\theta}_1 \dot{\theta}_3 \sin\theta_1 \\ -I_3 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \end{pmatrix}$$

(2.9.22)

Notice that this system is of the quasilinear and not semilinear form. This illustrates the fact that conservative Newtonain systems can also exhibit a quasilinear form of the equations of motion.

The nontrivial structure of the matrix (A_{ij}) then implies the existence of a Lagrangian structure which is generalized in our terminology, in the sense that it contains nontrivial multiplicative

terms to the kinetic terms. Indeed, the conventional Lagrangian for systems (2.9.22) is given by

$$L^{gen}_{tot} = \frac{1}{2}\left[I_1 \dot{\theta}_1^2 + I_2 \dot{\theta}_2^2 \sin^2\theta_1 + I_3 \left(\dot{\theta} + \dot{\theta}_3 \cos\theta_1\right)^2\right] - \mu g R \cos\theta_1 \qquad (2.9.23)$$

by exhibiting multiplicative trigonometric functions to the square of the derivatives of the Euler's angles. This indicates that generalized structure of the type (2.9.9), depending on the selected coordinates, can also represent conservative systems.

As is known, the angular momentum components M_2 and M_3 for system (2.9.22) are conserved. In the physical spinning top, (i.e. the top as it occurs in the physical reality) these angular momenta are not conserved due to the presence of frictional torques. To represent more closely the physical reality, systems (2.9.22) must therefore be implemented into a system of the form

$$\left[\left(A_{ki}\,\ddot{\theta}^{i} + B_{k}\right)_{SA} - T_k\right]_{NSA} = 0, \qquad (2.9.24)$$

where, in general the drag torques T_k have an explicit dependence on time, Euler angles and their derivatives.

Under the assumption that system (2.9.24) is nonessentially nonselfadjoint, the emerging Lagrangian structure is still generalized, but now with a broader functional dependence of the type

$$\hat{L}^{gen}_{tot}(t,\theta,\dot{\theta}) \qquad (2.9.25)$$

$$= \sum_{k=1}^{3} L^{(k)}_{int,I}(t,\theta,\dot{\theta})\,\frac{1}{2} I_k \dot{\theta}_k^{\,2} + L_{int,II}$$

We can therefore conclude that local unconstrained Newtonian systems generally demand the use of a Lagrangian structure of type (2.9.9) which reduces to type (2.9.1) only for simpler cases. The problem of the degrees of freedom of the interaction terms will be studied in the next section.

We consider now the field theoretical case. A conceptual attitude which is essential for the proper understanding of the results should be here stressed. The field theoretical models we are interested in should be conceived as a field theoretical generalization of models which, at the Newtonian level, are purely nonconservative.

This creates the problem whether a nonconservative generalization of Field Theory is conceivable. In the final analysis, this is one of the central objectives of these monographs. At this point let us restrict ourselves to only few epistemological remarks.

An inspection of the available literature indicates that the concept of nonconservative field (the eleton) does not exist in current High Energy Physics. The same concept, however, is fully established on grounds of unequivocal experimental evidence not only in Newtonian Mechanics, but also in the brach of physics which

is technically closer to Field Theory, Continuum Mechanics. And indeed, nonconservative equations are well known, say, in hydrodynamics.

On conceptual grounds all nonconservative systems can be considered as open systems, in the sense that they contain interactions not only between the constituents but also with an external system, say, a physical medium. A closed system, on the contrary, is a system which can be considered as isolated from the rest of the universe, i.e., it contains only mutual interactions between the constituents. The total physical energy can be conserved only for a closed system. For an open system the energy is not conserved precisely because of exchanges with the external part. Thus, open systems are genuinely nonconservative.

Newtonian Mechanics is clearly the best arena for an understanding of these concepts. Consider, for instance, the particle under the presence of a drag force, Eq. (2.9.11). This system is open according to the above definition because of the interaction of the particle with the physical medium. The system is nonconservative because of the dissipation of energy (in this simple case only kinetic) with the external medium. The conventional Kepler system in vacuum, Eqs. (2.9.18), is, on the contrary, closed because the system contains only mutual interactions of the constituents and, by assumption, these constituents move in vacuum. Thus, system (2.9.18) is strictly conservative. In the transition to the case of particles moving in a physical medium, Eqs. (2.9.20), the system becomes open. And indeed, this

generalization of the Kepler system is genuinely nonconservative.

In the transition to Continuum Mechanics the situation becomes predictably more complex, but still subject to an intuitive understanding. For instance the equations of waves in liquids, strictly speaking, characterize an open system. Indeed a simple physical observation indicates that the energy of the waves is not conserved in time but instead it is dissipated in the liquid itself. Thus, the liquid in this case has a dual role: that of allowing the existence and propagation of waves as well as that of the external system for energy dissipation.

In the transition to Field Theory the situation becomes more nebulous. In particular, our intuitional capabilities are in this case lost because of lack of prior knowledge of comparative cases. This, however, does not prohibit from setting up our objective, i.e, the construction of genuine nonconservative classical field theory which, on conceptual grounds, is fully equivalent to Newtonian and Continuum Mechanic. The physical relevance of such theory will be explored for the problem of the hadronic structure, as by now familiar from the arguments of Chapter 1. At this point we would like only to recall

(1) such nonconservative field theory is intended as a local approximation of a more complex nonlocal framework which is expected to occur within the hadronic structure,

(2) the underlying concept of nonconservative field (the eleton) is intended to generalize the conventional concept of

field representative of the motion of a particle in vacuum, to that of the motion of a particle in a hadronic medium (of overall closed nature); and

(3) the underlying field equations are interpreted as representing an open system in the sense indicated above and, thus, of nonconservative character.

Explicitly stated, these assumptions corresponds first to the recovering of the conventional interpretation of the quasi-linear local selfadjoint field equations, i.e., Eqs. (2.9.4), as representative of closed (and, thus, strictly conservative) systems. This is the field theoretical correspondent of, say, the Newtonian Kepler system in vacuum. However, the above assumption implies that the more general local semilinear and nonselfadjoint systems

$$\left[(\Box + \mu_{(k)}^2)\,\phi_k - \bar{J}_k(x^\alpha, \phi^j, \phi^j_{;\alpha}) \right]_{MSA} = 0, \qquad (2.9.26)$$

$$k = 1, 2, \ldots, n,$$

represent open (and, thus, genuine nonconservative) systems. This is the field theoretical correspondent of, say, the Newtonian Kepler system in a physical medium.

The field theoretical extension of the Newtonian Lagrangian (2.9.9) is then straightforward and reads[30]

$$\mathcal{L}^{gen}_{tot}(x^\alpha, \phi^k, \phi^k_{;\alpha}) \qquad (2.9.27)$$

$$= \sum_{k=1}^{n} \sum_{\mu=0}^{3} \mathcal{L}^{(k\mu)}_{int,I}(x^\alpha, \phi^j, \phi^j_{;\alpha})\, \mathcal{L}^{(k\mu)}_{free}(\phi^k_{;\mu}) + \mathcal{L}_{int,II}(x^\alpha, \phi^j, \phi^j_{;\alpha})$$

namely, it also exhibits the presence of both multiplicative and additive terms to the free terms.

To stress the analogy with the arena of our best intuitions, Newtonian Mechanics, consider the closed, conservative anarmonic oscillator

$$\left(m\ddot{x} + kx + \lambda x^2\right)_{SA} = 0 , \tag{2.9.28}$$

$$\mathcal{L}^{conv}_{tot} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 - \frac{\lambda}{3} x^3 .$$

One of its damped and, thus, open (nonconservative) extensions reads

$$\left[\left(m\ddot{x} + kx + \lambda x^2\right)_{SA} + \frac{m}{x}\dot{x}^2\right]_{NSA} = 0 , \tag{2.9.29}$$

$$\mathcal{L}^{gen}_{tot} = \frac{1}{2} m \dot{x} x^2 \dot{x} - \frac{1}{4} k x^4 - \frac{\lambda}{5} x^5 ,$$

by exhibiting a generalized Lagrangian structure.

In full analogy, the selfcoupled scalar field

$$\left[\left(\Box + m^2\right)\varphi + \lambda \varphi^2\right]_{SA} = 0 , \tag{2.9.30}$$

$$\mathcal{L}^{conv}_{tot} = \frac{1}{2} \varphi^{,}_{\mu} \varphi^{,\mu} - m^2 \varphi^2 - \frac{\lambda}{3} \varphi^3 ,$$

is closed. This is our terminology to indicate the fact that the above system is isolated from the rest of the universe and, thus, strictly conservative.

The field theoretical generalization of system (2.9.29), i.e.

$$\left\{\left[\left(\Box + m^2\right)\varphi + \lambda \varphi^2\right]_{SA} + \frac{1}{\varphi} \varphi^{,}_{\mu} \varphi^{,\mu}\right\}_{NSA} = 0 , \tag{2.9.31}$$

$$\mathcal{L}^{gen}_{tot} = \frac{1}{2} \varphi^{,}_{\mu} \varphi^2 \varphi^{,\mu} - \frac{m^2}{4} \varphi^4 - \frac{\lambda}{5} \varphi^5 ,$$

is then interpreted as open and thus, genuinely nonconservative. This is our terminology to indicate the interpretation of system (2.9.31) as interacting with an underlying medium which results in the nonlinear derivative couplings.

It should be stressed that this is the desired interpretation of Eqs. (2.9.31). Of course, different physical interpretations are possible (as it is the case for any differential equations), but, since they are not relevant for the analysis of these monographs, they will be left to the interested reader.

Several other examples will be considered later on during the course of our analysis.

We can therefore conclude that local field equations generally demand a Lagrangian structure of the type (2.9.27) as a covering of the conventional structure (2.9.2) which is recovered at the limit when all couplings reduce to a selfadjoint form. Subject

to a verification of the physical significance, generalized Lagrangian structure (2.9.27) allows an interpretation of the underlying systems as being open, in the sense of being genuinely nonconservative as in the corresponding Newtonian or continuum cases. In this way we can preserve the covering nature of Field Theory over Newtonian Mechanics not only on mathematical but also, and perhaps more singificantly, on physical grounds.

It should be stressed that, by no means, Lagrangian structures (2.9.9) and (2.9.27) are unique. They merely offer a physically si ificant way of interpreting the Lagrangian structure as it emerges from the Inverse Problem.

Another structure which, as we shall see, will be significant (particularly for the study of the relationship of the current unified gauge theories and our approach to the hadronic structure) is given, for the Newtonian case, by[30,34]

$$L^{gen}_{tot}(t, x_\mu, \dot{x}_\mu) = \frac{1}{2}\Big[\dot{x}^{ia} G_{ia\,jb}(t, x_\mu, \dot{x}_\mu)\dot{x}^{jb} + 2\dot{x}^{ia} F_{ia\,jb}(t, x_\mu)\dot{x}^{jb} + x^{ia} E_{ia\,jb}(t, x_\mu) x^{jb}\Big] \tag{2.9.32}$$

with corresponding field theoretical extension

$$\mathcal{L}^{gen}_{tot}(x^\alpha, \phi^k, \phi^k_\alpha) = \frac{1}{2}\Big[\phi^{ij}_{\ \mu} G^{\mu\nu}_{ij}(x^\alpha, \phi^k, \phi^k_\alpha)\phi^{ij}_{\ \nu} + 2\phi^{ij}_{\ \mu} F^{\mu}_{ij}(x^\alpha, \phi^k)\phi^j + \phi^i E_{ij}(x^\alpha, \phi^k)\phi^j\Big] \tag{2.9.33}$$

The Lagrangian structures of the simpler type

$$\mathcal{L}^{gen}_{tot} = \frac{1}{2}\phi^{ij}_{\ \mu} G_{ij}(\phi)\phi^{j\mu} + E(\phi), \quad G_{ij} = G_{ji}, \tag{2.9.34}$$

are called chiral Lagrangians in the current literature in field theory. We shall therefore preserve the name of chiral Lagrangians for structures (2.9.32) and (2.9.33) too.

On comparative grounds between structures (2.9.33) and (2.9.34) the main difference is constituted by the additional functional dependence of the G-tensor of the former on the local Minkowski coordinates and field derivatives. As we shall see in due time, the former dependence will acquire a natural role in our non-conservative interpretation of the underlying system, while the latter dependence is needed to reach a true quasilinear system. Explicitly stated, structure (2.9.34) represents quasilinear systems of the reduced type (2.5.9) while structure (2.9.33) represents quasilinear systems of the type (2.5.1) i.e., the largest possible systems of second order partial differential equations which are representable with Lagrange's equations.

The extension of the analysis of this section to the problem of the Hamiltonian structure is straightforward and will be tacitly assumed hereon.

The reader should also keep in mind the definition of a nontrivial interactions as given in Appendix 2.A. Its extension to the Lagrangian structures here considered simply imply that, to have nontrivial interactions, Lagrange's equations in $\mathcal{L}_{free}$ and those in $\mathcal{L}^{gen}_{tot}$ are nonequivalent.

2.10: USE OF THE INVERSE PROBLEM FOR THE CHARACTERIZATION OF THE DEGREES OF FREEDOM OF A LAGRANGIAN STRUCTURE.

After having reviewed the methodology of the Inverse Problem (Sections 2.2 through 2.8) and identified the emerging Lagrangian structure (Section 2.9), our next objective is that of studying the "degrees of freedom" of a Lagrangian which represents nontrivial interactions.

Specifically, our problem is the identification of the classes of G-, F- and E-tensors of the Newtonian or field theoretical (generalized) chiral Lagrangians

$$L^{gen}_{tot}(t,q,\dot{q}) \qquad (2.10.1a)$$

$$= \frac{1}{2}\left[\dot{q}^i G_{ij}(t,q,\dot{q})\dot{q}^j + 2\dot{q}^i F_{ij}(t,q) q^j + q^i E_{ij}(t,q) q^j\right],$$

$$\mathcal{L}^{gen}_{tot}(x^\nu,\phi^k,\phi^k_{;\alpha}) \qquad (2.10.1b)$$

$$= \frac{1}{2}\left[\phi^{i}_{;\mu} G^{\mu\nu}_{ij}(x^\nu,\phi^k,\phi^k_{;\alpha})\phi^{j}_{;\nu} + 2\phi^{i}_{;\mu} F^{\mu}_{ij}(x^\nu,\phi^k)\phi^j + \phi^i E_{ij}(x^\nu,\phi^k)\phi^j\right],$$

which represent the same system.

Clearly, this is a typical problem of the transformation theory. The following classes of equivalence are well known.

(I) Those induced by continuity preserving and invertible transformations of the local variables, i.e., the Newtonian transformations

$$t \to t', \quad q^k \to q'^k \qquad (2.10.2)$$

and their field theoretical extension

$$x^\mu \to x'^\mu, \quad \phi^k \to \phi'^k \tag{2.10.3}$$

for which

$$L(t,q,\dot q) = L'(t',q',\dot q')\frac{dt'}{dt} \tag{2.10.4a}$$

$$\mathcal{L}(x^\alpha,\phi^k,\phi^k_\alpha) = \mathcal{L}'(x'^\alpha,\phi'^k,\phi'^k_\alpha)\,dx'/dx \tag{2.10.4b}$$

(II) The equivalence transformations induced by the addition of a term with null Lagrange's derivatives, i.e., by the transformations

$$L \to \hat L = L + \dot G(t,q), \quad G \in C^3(R), \tag{2.10.5a}$$

$$\mathcal{L} \to \hat{\mathcal{L}} = \mathcal{L} + d_\mu G^\mu(x^\alpha,\phi^k), \quad G^\mu \in C^3(R), \tag{2.10.5b}$$

which, in view of the identities

$$\frac{d}{dt}\frac{\partial \dot G}{\partial \dot q^k} \equiv \frac{\partial \dot G}{\partial q^k}, \tag{2.10.6a}$$

$$d_\mu \frac{\partial (d_\alpha G^\alpha)}{\partial \phi^k_{;\mu}} \equiv \frac{\partial (d_\alpha G^\alpha)}{\partial \phi^k}, \tag{2.10.6b}$$

leave the underlying equations unaffected, i.e.,

$$\frac{d}{dt}\frac{\partial \hat L}{\partial \dot q^k} - \frac{\partial \hat L}{\partial q^k} \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot q^k} - \frac{\partial L}{\partial q^k}, \tag{2.10.7a}$$

$$d_\mu \frac{\partial \hat{\mathcal{L}}}{\partial \phi^k_{;\mu}} - \frac{\partial \hat{\mathcal{L}}}{\partial \phi^k} \equiv d_\mu \frac{\partial \mathcal{L}}{\partial \phi^k_{;\mu}} - \frac{\partial \mathcal{L}}{\partial \phi^k}. \tag{2.10.7b}$$

One of the most intriguing implications of the methodology of the Inverse Problem is that of allowing the characterization of a new class of equivalence transformations which (a) occurs within a fixed system of variables, (b) produces a profound change in the Lagrangian structure and (c) leaves the underlying dynamics (i.e., the implicit functions and related first integrals) unaffected by construction. They can be characterized by

(III) The class of continuity, regularity and selfadjointness preserving equivalence transformations of Lagrange's equations.

This section is devoted to the study of this latter class of transformations. For simplicity the details of the analysis will be presented for the Newtonian case and the results extended to the field theoretical case.

Consider a nonessentially nonselfadjoint system

$$\left[\ddot q_k - F_k(t,q,\dot q)\right]^{C^2,R}_{NSA} = 0. \tag{2.10.8}$$

As indicated in Section 2.3, the methodology for the construction of a Lagrangian representation of this system consists in solving the conditions of selfadjointness in the genotopic functions (Thoerem 2.3.4). This yields an equivalent system

$$\left\{ h_k{}^i \left[\ddot q - F_k\right]^{C^2,R}_{NSA} \right\}^{C^2,R}_{SA} = 0, \tag{2.10.9}$$

for which a Lagrangian exists and can be computed, resulting in the analytic representation

$$L_k(q) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k} \equiv \left[h_k{}^i \left(\ddot{q}_i - F_i \right)_{NSA}^{c^2,R} \right]_{SA}^{c^2,R} \qquad (2.10.10)$$

$$k = 1, 2, \ldots, n$$

with generalized Lagrangian (3.9.9) or (2.9.32).

However, conditions of selfadjointness (2.3.21) in the genotopic functions are partial differential equations. It is a known fact that, when these systems are consistent, they admit solutions with considerable degrees of freedom. This means that, rather than only one solution in the genotopic functions h_k^i, a family of solutions is in general expected to exist.

Whenever another solution of Eqs. (2.3.21), say, h_k^{*i} (different than h_k^i) has been identified, another Lagrangian, say L*, can be constructed according to the representation

$$L^*_k(q) \equiv \left[h^*{}_k{}^i \left(\ddot{q}_i - F_i \right)_{NSA}^{c^2,R} \right]_{SA}^{c^2,R} \qquad (2.10.11)$$

The transition

$$L(t, q, \dot{q}) \longleftrightarrow L^*(t, q, \dot{q}) \qquad (2.10.12)$$

is an equivalence transformation because, by construction (a) occurs within a fixed system of local variables, (b) implies a nontrivial change in the Lagrangian structure whenever the genotopic matrices (h) and (h*) are functionally different and, last

but not least, (c) leaves the underlying dynamics unaffected. The last property can be seen from the fact that, since the genotopic matrices are regular by assumption, they do not affect the implicit functions and, thus (as we shall see better in Section 2.11), the conserved quantities.

Of course, for each new genotopic matrix we can construct a new Lagrangian. Thus, whenever a family of nonequivalent gentotopic matrices exist for one given system, a family of equivalent Lagrangians satisfying properties (a), (b) and (c) above exist.

We learn in this way that the methodology of the Inverse Problem, rather than providing a Lagrangian for the system considered, in actuality provides the family of all equivalent Lagrangians for the analytic representations of all equivalent selfadjoint forms of the equations of motion.

For algebraic reasons to be indicated later on in this section, we have called mappings (2.10.12) isotopic transformations[30,34] where the terms

"ἰσόω" "τόπος" = "preserve" "configuration"

are interpreted in the sense that the transformations preserve the selfadjointness of the system considered, i.e., are of the type

$$\left(A_{ki} \ddot{q}^i + B_k \right)_{SA}^{c^2,R} = 0 \longrightarrow \left[h_k{}^i \left(A_{ij} \ddot{q}^j + B_i \right)_{SA}^{c^2,R} \right]_{SA}^{c^2,R} = 0. \qquad (2.10.13)$$

On similar grounds we have called genotopic transformations[30,34] the equivalence transformations of the type (Section 2.3)

$$\left(A_{ki}\ddot{q}^{i}+B_{k}\right)^{C^{2},R}_{NSA}=0 \longrightarrow \left[h_{k}{}^{i}\left(A_{ij}\ddot{q}^{j}+B_{i}\right)^{C^{2},R}_{NSA}\right]^{C^{2},R}_{SA}=0 \qquad (2.10.14)$$

where the terms

$$\text{"}\gamma\varepsilon\nu\nu\acute{\alpha}\omega\text{"}\ \text{"}\tau\acute{o}\pi o\varsigma\text{"} = \text{"produce"}\ \text{"configuration"}$$

are now interpreted in the sense that the transformations produce a selfadjoint structure.

At a closer inspection the reader can see that the isotopic transformations can exist also for conservative systems, i.e., the essentially selfadjoint system in our terminology. And indeed, one can start from the conventional selfadjoint form of a conservative system, i.e. that as derived from Newton's second law. A Lagrangian for its ordered direct analytic representation is then known. Nevertheless, Theorem 2.3.4 can still be used to attempt an indirect analytic representation. When such representation exists. This implies the existence of a selfadjointness preserving equivalence transformation

$$\left[h_{k}{}^{i}\left(\ddot{q}_{i}-f_{i}\right)^{C^{\infty},R}_{SA}\right]^{C^{2},R}_{SA}=0 \qquad (2.10.15)$$

i.e., an isotopic transformation in our terminology. Again, a family of isotopically related Lagrangians can, in general, exist, rather than only one Lagrangian.

The intriguing fact is that, starting from a conventional Lagrangian structure for system (2.8.4), a generalized Lagrangian structure is expected for the equivalent system (2.10.15). We therefore recover in more technical terms the fact indicated in Section 2.9 that generalized Lagrangians can also represent conservative (closed) systems.

This reinforces the analytic equivalence of the classes of essentially selfadjoint and nonessentially nonselfadjoint systems indicated in Section 2.8. And indeed, besides the fact that these two classes both admit an analytic representation within the original system of (Cartesian) coordinates, they emerge as being representable with the same generalized Lagrangian structure. The reader can now see the (for us) mandatory nature of the content of this section. At the level of the equations of motion one can easily ascertain whether a given force is conservative or not. This is not the case at the level of a Lagrangian without the explicit computation of the equations of motion, because the assumption of a generalized Lagrangian structure is not sufficient to guarantee a nonconservative system. Since we are primarily interested in nonconservative systems, the implications of this occurrence are selfevident.

The presentation of the isotopic transformations has been here done at the level of the equations of motion. Since for the

systems considered a Lagrangian exists by assumption, the methodology of the isotopic transformations can be formulated at the level of the Lagrange equations. We then have the following

34-II

THEOREM 2.10.1: Necessary and sufficient condition for a class C^4 and regular Lagrangian $L(t,q,\dot{q})$ to possess an isotopic image $L^*(t,q,\dot{q})$ according to the rule

$$L^*_k(q) \equiv \left\{ g_k{}^i \left[L_i(q) \right]^{C^2,R}_{SA} \right\}^{C^2,R}_{SA} \qquad (2.10.16)$$

$k = 1,2,\ldots,n$

is that the equivalent system of Lagrange's equations in L is selfadjoint in a star-shaped region R* of the variables, i.e., that the conditions of selfadjointness

$$\begin{cases} A^*_{ij} = A^*_{ji}, & (2.10.17a) \\ \dfrac{\partial A^*_{ik}}{\partial \dot{q}^j} = \dfrac{\partial A^*_{jk}}{\partial \dot{q}^i}, & (2.10.17b) \\ \dfrac{\partial B^*_i}{\partial \dot{q}^j} + \dfrac{\partial B^*_j}{\partial \dot{q}^i} = 2\left\{ \dfrac{\partial}{\partial t} + \dot{q}^k \dfrac{\partial}{\partial q^k} \right\} A_{ij}, & (2.10.17c) \\ \dfrac{\partial B^*_i}{\partial q^j} - \dfrac{\partial B^*_j}{\partial q^i} = \dfrac{1}{2}\left\{ \dfrac{\partial}{\partial t} + \dot{q}^k \dfrac{\partial}{\partial q^k} \right\} \left(\dfrac{\partial B^*_i}{\partial \dot{q}^j} - \dfrac{\partial B^*_j}{\partial \dot{q}^i} \right), & (2.10.17d) \end{cases}$$

$$A^*_{ij} = g_i{}^k \frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^j}, \qquad (2.10.17e)$$

$$B^*_k = g_k{}^i \left(\frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} \dot{q}^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial t} - \frac{\partial L}{\partial q^i} \right), \qquad (2.10.17f)$$

admit a (nontrivial) solution in the factor functions $g_k{}^i$ (the isotopic functions) which is well defined in R*. The isotopically mapped Lagrangian is then given by

$$L^*(t,q,\dot{q}) = K^*(t,q,\dot{q}) + D^*_k(t,q)\dot{q}^k + C^*(t,q), \qquad (2.10.18a)$$

$$\begin{cases} \dfrac{\partial^2 K^*}{\partial \dot{q}^i \partial \dot{q}^j} = A^*_{ij}(t,q,\dot{q}), & (2.10.18b) \\ \dfrac{\partial D^*_i}{\partial q^j} - \dfrac{\partial D^*_j}{\partial q^i} = \dfrac{1}{2}\left(\dfrac{\partial B^*_i}{\partial \dot{q}^j} - \dfrac{\partial B^*_j}{\partial \dot{q}^i} \right) + \left(\dfrac{\partial^2 K^*}{\partial q^i \partial \dot{q}^j} - \dfrac{\partial^2 K^*}{\partial \dot{q}^i \partial q^j} \right) \equiv Z^*_{ij}(t,q), & (2.10.18c) \\ \dfrac{\partial C^*}{\partial q^i} = \dfrac{\partial D^*_i}{\partial t} - B^*_i - \dfrac{\partial K^*}{\partial q^i} + \dfrac{\partial^2 K^*}{\partial \dot{q}^i \partial t} + \left[\dfrac{\partial^2 K^*}{\partial q^i \partial \dot{q}^j} + \dfrac{1}{2}\left(\dfrac{\partial B^*_i}{\partial \dot{q}^j} - \dfrac{\partial B^*_j}{\partial \dot{q}^i} \right) \right] \dot{q}^j \equiv W^*_i(t,q), & (2.10.18d) \end{cases}$$

$$\begin{cases} K^* = \dot{q}^i \int_0^1 d\tau \left\{ \left[\int_0^1 d\tau\, A^*_{ij}(t,q,\tau\dot{q}) \right] \dot{q}^j \right\}(t,q,\tau\dot{q}), & (2.10.18e) \\ D^*_i = \left[\int_0^1 d\tau\, \tau Z^*_{ij}(t,\tau q) \right] q^j, & (2.10.18f) \\ C^* = q^i \int_0^1 d\tau\, W^*_i(t,\tau q). & (2.10.18g) \end{cases}$$

Notice that, as by now familiar, with the repetition of the continuity and regularity symbols C^2 and R in the fundamental rule (2.10.16) we imply that the matrix (g) is of class C^2 and regular. These are central requirements for the concept of isotopic transformations because, if the matrix (g) is, say, degenerate, the condition of preservation of the underlying dynamics is in general lost.

With the terms "nontrivial isotopic functions" we imply matrices (g) other than multiples of the unit matrix with constant coefficients which would imply the trivial mapping $L \to L^* = NL$, N = cost.

It should be stressed that in Eqs. (2.10.17) all the terms in the derivatives of the original Lagrangian are known and fixed. Thus, Eqs. (2.10.17) are equations in the unknown isotopic functions g_k^i. Also, if a set of these functions has been identified for a given Lagrangian, the same set is not necessarily a solution of the same equations (2.10.17) in a different Lagrangian. Thus, strictly speaking, a system of isotopic functions is isotopic with respect to a given Lagrangian. Of course, one given Lagrangian may admit a family of isotopic functions. This is the case when Eqs. (2.10.17) for a given L admit a family of nontrivially different solutions in the functions g_k^i. For more details see reference. 34-II

The extension of Theorem 2.10.1 to Field Theory is straightforward and it is here reported for the reader's convenience.

30-III

THEOREM 2.10.2: Necessary and sufficient condition for a class C^4 and regular Lagrangian density $\mathcal{L}(x^\alpha, \phi^k, \phi^k_{;\alpha})$ to admit an isotopic image $\mathcal{L}^*(x^\alpha, \phi^k, \phi^k_{;\alpha})$ according to the rule

$$\mathcal{L}^*_k(\phi) \equiv \left\{ g_k{}^i \left[\mathcal{L}_i(\phi) \right]^{C^2_1 R}_{SA} \right\}^{C^2_1 R}_{SA} \qquad k = 1,2,\dots,n \tag{2.10.19}$$

is that the equivalent system of Lagrange's equations in $\mathcal{L}$ is selfadjoint in a star-shaped region R* of its variables i.e., that the conditions of selfadjointness

$$A^{*\mu\nu}_{ij} = A^{*\nu\mu}_{ji} = A^{*\nu\mu}_{ij}, \tag{2.10.20a}$$

$$A^{*\nu\alpha;\mu}_{ik\ j} + A^{*\nu\alpha;\mu}_{jk\ i} = A^{*\mu\nu;\alpha}_{ji\ k}, \tag{2.10.20b}$$

$$A^{*\overbrace{\alpha\beta};\overbrace{\mu;\nu}}_{ij\ k\ e} = A^{*\overbrace{\alpha\beta};\overbrace{\mu;\nu}}_{\underbrace{kj\ i\ e}}, \tag{2.10.20c}$$

$$B^*_{i\ j}{}^{;\mu} + B^*_{j\ i}{}^{;\mu} = 2\left\{ \partial_\alpha + \phi^k_{;\alpha} \frac{\partial}{\partial \phi^k} \right\} A^{*\mu\nu}_{ij}, \tag{2.10.20d}$$

$$B^*_{i\ j}{}^{;\nu} - B^*_{j\ i}{}^{;\nu} = \frac{1}{2}\left\{ \partial_\alpha + \phi^k_{;\alpha} \frac{\partial}{\partial \phi^k} \right\} \left(B^*_{i\ j}{}^{;\alpha} - B^*_{j\ i}{}^{;\alpha} \right), \tag{2.10.20e}$$

$$A^{*\mu\nu}_{ij} = \frac{1}{2} g_i{}^k \left(\mathcal{L}^{;\mu;\nu}_{k\ j} + \mathcal{L}^{;\nu;\mu}_{k\ j} \right), \tag{2.10.20f}$$

$$B^*_k = g_k{}^i \left(\mathcal{L}^{;\mu}_{i\ j;} \phi^{;j}_{;\mu} + \mathcal{L}^{;\mu}_{i\ ;\mu j} - \mathcal{L}_{;i} \right), \tag{2.10.20g}$$

$$\mathcal{L}^{*}_{i} = \frac{\partial \mathcal{L}}{\partial \phi^{i}}\,, \quad \mathcal{L}^{*}_{\mu} = \frac{\partial \mathcal{L}}{\partial x^{\mu}}\,, \quad A^{*\mu\nu;\alpha}_{ij\,k} = \frac{\partial A^{*\mu\nu}_{ij}}{\partial \phi^{k;}_{\alpha}}\,, \text{ etc.,} \tag{2.10.20h}$$

$$A^{*\mu\nu;\overline{\alpha\beta}}_{ij\,k} = A^{*\mu\nu;\alpha}_{ij\,k} + A^{*\mu\alpha;\nu}_{ij\,k}\,, \text{ etc.} \tag{2.10.20i}$$

$$A^{*\mu\nu;\underline{\alpha\beta}}_{ij\,\underline{ke}} = A^{*\mu\nu;\alpha;\beta}_{ij\,k\,e} + A^{*\mu\nu;\alpha;\beta}_{ie\,k\,j}\,, \tag{2.10.20e}$$

admit a nontrivial solution in the factor functions g^{i}_{k} which is well defined in R*. The isotopically mapped Lagrangian is then given by

$$\mathcal{L}^{*}(x^{\nu};\phi^{k},\phi^{k;}_{\alpha}) = K^{*}(x^{\nu};\phi^{k},\phi^{k;}_{\alpha}) + D^{*\mu}_{k}(x^{\nu};\phi^{k})\,\phi^{k;}_{\alpha} + C^{*}(x^{\nu};\phi^{k}), \tag{2.10.21a}$$

$$\begin{cases} K^{*;\mu\,;\nu}_{i\;\;j} = A^{*\mu\nu}_{ij}(x^{\nu};\phi^{k};\phi^{k;}_{\alpha}), & (2.10.21b) \\ D^{*\;\mu}_{i\;;j} - D^{*\;\mu}_{j\;;i} = \frac{1}{2}\left(B^{*}_{i\;;j}^{\;\mu} - B^{*}_{j\;;i}^{\;\mu}\right) & (2.10.21c) \\ \qquad + \left(K^{*}_{;i\;;j}^{\;;\mu} - K^{*}_{;j\;;i}^{\;;\mu}\right) \equiv \mathcal{Z}^{*\mu}_{ij}(x^{\nu};\phi^{k}), & \\ C^{*}_{;i} = D^{*\mu}_{i\;;\mu} - B^{*}_{i} - K^{*}_{;i} - K^{*;\mu}_{i\;;\mu} & (2.10.21d) \\ \qquad + \left[K^{*}_{;i\;;j}^{\;;\mu} + \frac{1}{2}\left(B^{*}_{i\;;j}^{\;\mu} - B^{*}_{j\;;i}^{\;\mu}\right)\right]\phi^{j;}_{\mu} & \\ \qquad \equiv W^{*}_{i}(x^{\nu};\phi^{k}), & \end{cases}$$

$$\begin{cases} K^{*} = \phi^{i;}_{\mu}\int_{0}^{1} d\tau' \left\{\left[\int_{0}^{1} d\tau\, A^{*\mu\nu}_{ij}(x^{\nu};\phi^{k};\quad)\right]\phi^{j;}_{\nu}\right\}(x^{\nu};\phi^{k};\tau'\phi^{k;}_{\alpha}), & (2.10.21e) \\ D^{*\;\mu}_{i\;} = \left[\int_{0}^{1} d\tau\, \tau\, \mathcal{Z}^{*\mu}_{ij}(x^{\nu};\tau\phi^{k})\right]\phi^{j}, & (2.10.21f) \\ C^{*} = \phi^{i}\int_{0}^{1} d\tau\, W^{*}_{i}(x^{\nu};\tau\phi^{k}). & (2.10.21g) \end{cases}$$

The implications of the above theorems from the viewpoint of interactions are expressed by the following

30-34

COROLLARY 2.10.2A: The isotopic transformations represent degrees of freedom of the couplings within a fixed system of local variables, implicit functions and conserved quantities.

Explicitly, the isotopic mappings can be written in one of the forms

$$\mathcal{L}^{\text{gen}}_{\text{tot}} = \sum_{k=1}^{n}\sum_{\mu=0}^{3} \mathcal{L}^{(k\mu)}_{\text{int,I}}\,\mathcal{L}^{(k\mu)}_{\text{free}} + \mathcal{L}_{\text{int,II}} \longleftrightarrow \tag{2.10.22a}$$

$$\longleftrightarrow \mathcal{L}^{*\,\text{gen}}_{\text{tot}} = \sum_{k=1}^{n}\sum_{\mu=0}^{3} \mathcal{L}^{*(k\mu)}_{\text{int,I}}\,\mathcal{L}^{(k\mu)}_{\text{free}} + \mathcal{L}^{*}_{\text{int,II}}$$

$$\mathcal{L}^{gen}_{tot} = \frac{1}{2}\left[\phi^{i;}_{\mu}\, G^{\mu\nu}_{ij}\phi^{j;}_{\nu} + 2\phi^{i;}_{\mu}\, F^{\mu}_{ij}\phi^{j} + \phi^{i} E_{ij}\,\phi^{j}\right] \longleftrightarrow \tag{2.10.22b}$$

$$\longleftrightarrow \mathcal{L}^{*\,gen}_{tot} = \frac{1}{2}\left[\phi^{i;}_{\mu}\, G^{*\mu\nu}_{ij}\phi^{j;}_{\nu} + 2\phi^{i;}_{\mu}\, F^{*\mu}_{ij}\phi^{j} + \phi^{i} E^{*}_{ij}\phi^{j}\right]$$

If the original Lagrangian has the conventional structure we have, for instance,

$$\mathcal{L}^{conv}_{tot} = 1 \sum_{k=1}^{n}\sum_{\mu=0}^{3} \mathcal{L}^{(k\mu)}_{free} + \mathcal{L}_{int,\,II} \longleftrightarrow \tag{2.10.23}$$

$$\longleftrightarrow \mathcal{L}^{*\,gen}_{tot} = \sum_{k=1}^{n}\sum_{\mu=0}^{3} \mathcal{L}^{*\,(k\mu}_{int,I}\, \mathcal{L}^{(k\mu)}_{free} + \mathcal{L}^{*}_{int,II}$$

and similarly for the Newtonian case.

Notice that in the above mappings the free term is not transformed. This is due to the fact that the transformations occur, by construction, within a fixed system of variables. On account of this property we shall sometime symbolically write mappings (2.10.22) and (2.10.23) in the form

$$(\mathcal{L}_{I}\,,\ \mathcal{L}_{II}) \longleftrightarrow (\mathcal{L}^{*}_{I}\,,\ \mathcal{L}^{*}_{II}), \tag{2.10.24}$$

$$(G, F, E) \longleftrightarrow (G^{*}, F^{*}, E^{*}). \tag{2.10.25}$$

A few examples are here in order. Consider the harmonic oscillator

$$\left(\ddot{r} + r\right)^{C^{\infty},R}_{SA} = 0\,, \quad m = 1,\ k = 1, \tag{2.10.25}$$

with conventional Lagrangian

$$L^{conv}_{tot} = \frac{1}{2}\left(\dot{r}^2 - r^2\right) = 1\, L_{free} + L_{II}. \tag{2.10.26}$$

The use of Theorem 2.10.1 then yields the isotopically mapped Lagrangian

$$L^{*\,gen}_{tot} = L^{*}_{I}\, L_{free} + L^{*}_{II},$$

$$L^{*}_{I} = \frac{1}{3}\,\dot{r}\cos t + r\sin t, \tag{2.10.27}$$

$$L_{free} = \frac{1}{2}\dot{r}^2\,, \quad L^{*}_{II} = -r^2\dot{r}\cos t,$$

with underlying analytic representation

$$\frac{d}{dt}\frac{\partial L^{*}}{\partial \dot{r}} - \frac{\partial L^{*}}{\partial r} = \left[(\dot{r}\cos t + r\sin t)\left(\ddot{r} + r\right)^{C^{\infty},R}_{SA}\right]^{C^{\infty},R}_{SA} \tag{2.10.28}$$

Notice that the factor term is nonnull for any value of time. Thus, Lagrangian (2.9.27) is fully admissible from an analytic viewpoint.

As an example of a nonconservative system consider the damped particle

$$\left(\ddot{r} + \gamma\dot{r}\right)^{C^{\infty},R}_{NSA} = 0,\ \dot{r} \neq 0, \tag{2.10.29}$$

with Lagrangian (Section 2.6)

$$L^{gen}_{tot} = e^{\gamma t}\frac{1}{2}\dot{x}^2 = L_I L_{free}. \qquad (2.10.30)$$

The use of Theorem 2.10.1 then yields the isotopically mapped Lagrangian

$$L^{*\,gen}_{tot} = L^*_I L_{free} + L^*_{II}, \qquad (2.10.31)$$

$$L^*_I = \frac{2}{\dot{x}} \ln \dot{x}\,, \quad L_{free} = \frac{1}{2}\dot{x}^2\,, \quad L^*_{II} = -\gamma x\,,$$

with rule (2.10) realized according to

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{x}} - \frac{\partial L^*}{\partial x} = \left[\frac{e^{-\gamma t}}{\dot{x}}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}\right)^{C^\infty,R}_{SA}\right]^{C^\infty,R}_{SA}, \qquad (2.10.32)$$

which is admissible for any nonnull value of the velocity.

For a more general nonconservative system consider the damped oscillator

$$\left(\ddot{x} + \gamma\dot{x} + x\right)^{C^\infty,R}_{NSA} = 0, \qquad (2.10.33)$$

and the Lagrangian

$$L^{gen}_{tot} = e^{\gamma t}\frac{1}{2}\dot{x}^2 - e^{\gamma t}\frac{1}{2}x^2 = L_I L_{free} + L_{II}. \qquad (2.10.34)$$

Theorem 2.10.1 then yields the isotopically mapped Lagrangian

$$L^{*\,gen}_{tot} = L^*_I L_{free} + L^*_{II}\,, \quad L_{free} = \frac{1}{2}\dot{x}^2, \qquad (2.10.35)$$

$$L^*_I = \frac{2\dot{x} + \gamma x}{\omega x \dot{x}} \tan^{-1}\left(\frac{2\dot{x}+\gamma x}{2\omega x}\right), \quad L^*_{II} = -\frac{1}{2}\ln\left(\dot{x}^2 + \gamma x\dot{x} + x^2\right).$$

$$\omega = \sqrt{4-\gamma}$$

Notice the profound structural change in the transition from the original Lagrangians (2.10.26), (2.10.30) and (2.10.34) to their respective isotopic images (2.10.27), (2.10.31) and (2.10.35). This change is bound to have implications for symmetries and conservation laws (Section 2.11). Notice also that while the original Lagrangians have the dimension of the energy, this is no longer the case for their isotopic images. This aspect will have a number of implications in the quantization of nonconservative systems (Volume III).

As an indication of the corresponding situation in Field Theory, consider the free complex scalar field with equations

$$\begin{cases} \left(\Box + \mu^2\right) e = 0\,, \\ \left(\Box + \mu^2\right) \bar{e} = 0\,, \end{cases} \qquad (2.10.36)$$

and their conventional Lagrangian

$$\mathcal{L}_{free} = \bar{e}^{;}_{\mu}\, e^{;\mu} - \mu^2 \bar{e}\, e \qquad (2.10.37)$$

Theorem 2.10.2 then yields the isotopic image

$$\mathcal{L}^*_{free} = \frac{1}{2}\left(\bar{e}^{;}_{\mu}\bar{e}^{;\mu} - \mu^2\bar{e}^2\right) + \frac{1}{2}\left(e^{;}_{\mu}e^{;\mu} - \mu^2 e^2\right) \qquad (2.10.38)$$

with realization of rule (2.10.19) with $(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that Lagrangian (2.10.38) is Hermitean and, thus, fully admissible

on analytic grounds. Also, the isotopic functions, even though constant, are non trivial because not a multiple of the identity (in actuality they characterize a permutation of the field equations).

In the language of Section 2.5 we can say that Lagrangian (2.10.37) produces an indirect representation of the system in the ordering $(\varphi, \bar{\varphi})$ while Lagrangians (2.10.38) produces a direct representation in the same ordering.

We can now add that the fact whether the analytic representation is direct or indirect is implicit in the more general concept of isotopically related Lagrangians.

By no means structure (2.10.38) is the only possible isotopic image of $\mathcal{L}_{free}$. Indeed, another image is given by

$$\left(\mathcal{L}^{*}\right)^{*}_{free} = \frac{1}{2}\left(\bar{\varphi}^{i}_{\mu}\bar{\varphi}^{i\mu} - m^{2}\bar{\varphi}^{2}\right) - \frac{1}{2}\left(\varphi^{i}_{\mu}\varphi^{i\mu} - m^{2}\varphi^{2}\right). \tag{2.10.39}$$

This Lagrangian, however, is skew-Hermitean.

The educated reader has by now already identified a rather profound change in the symmetries of Lagrangians (2.10.37), (2.10.38) and (2.10.39). Nevertheless, as we shall see in the next sections, these changes do not affect the conservation of the charge current.

A less trivial example is given by the complex scalar field in interactions with an external electromagnetic field

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (\Box + m^{2})\varphi \\ (\Box + m^{2})\bar{\varphi} \end{pmatrix} + e\begin{pmatrix} eA_{\alpha}A^{\alpha}\bar{\varphi} - 2iA_{\alpha}\bar{\varphi}^{i\alpha} \\ eA_{\alpha}A^{\alpha}\varphi + 2iA_{\alpha}\varphi^{i\alpha} \end{pmatrix}\right]_{SA} = 0. \tag{2.10.40}$$

The conventional Lagrangian of this system is the (gauge) structure

$$\mathcal{L}^{conv}_{tot} = \left(\bar{\varphi}^{i}_{\mu} + ieA_{\mu}\varphi\right)\left(\varphi^{i\mu} - ieA^{\mu}\varphi\right) - m^{2}\bar{\varphi}\varphi. \tag{2.10.41}$$

Theorem 2.10.2 then gives the isotopic image

$$\mathcal{L}^{*\,gen}_{tot} = \frac{1}{2}\, e^{2ie\varkappa_{\alpha}A^{\alpha}}\left[\bar{\varphi}^{i}_{\mu}\bar{\varphi}^{i\mu} - (m^{2} - e^{2}A_{\mu}A^{\mu})\bar{\varphi}^{2}\right] + \frac{1}{2}\, e^{-2ie\varkappa_{\alpha}A^{\alpha}}\left[\varphi^{i}_{\mu}\varphi^{i\mu} - (m^{2} - e^{2}A_{\mu}A^{\mu})\varphi^{2}\right] \tag{2.10.42}$$

with rule

$$\begin{pmatrix} \partial_{\mu}\dfrac{\partial\mathcal{L}^{*\,gen}_{tot}}{\partial\varphi^{i}_{\mu}} - \dfrac{\partial\mathcal{L}^{*\,gen}_{tot}}{\partial\varphi} \\ \partial_{\mu}\dfrac{\partial\mathcal{L}^{*\,gen}_{tot}}{\partial\bar{\varphi}^{i}_{\mu}} - \dfrac{\partial\mathcal{L}^{*\,gen}_{tot}}{\partial\bar{\varphi}} \end{pmatrix}_{SA} \equiv \left[\begin{pmatrix} 0 & e^{2ie\varkappa_{\alpha}A^{\alpha}} \\ e^{-2ie\varkappa_{\alpha}A^{\alpha}} & 0 \end{pmatrix}\begin{pmatrix} \partial_{\mu}\dfrac{\partial\mathcal{L}^{conv}_{tot}}{\partial\varphi^{i}_{\mu}} - \dfrac{\partial\mathcal{L}^{conv}_{tot}}{\partial\varphi} \\ \partial_{\mu}\dfrac{\partial\mathcal{L}^{conv}_{tot}}{\partial\bar{\varphi}^{i}_{\mu}} - \dfrac{\partial\mathcal{L}^{conv}_{tot}}{\partial\bar{\varphi}} \end{pmatrix}_{SA}\hat{C}^{R}\right]_{SA}\hat{C}^{R} \tag{2.10.43}$$

Since Lagrangian (2.10.42) is Hermitean (and the matrix of the factor functions is regular) is is fully acceptable on analytic grounds. Nevertheless, Lagrangian (2.10.42) implies the breaking of several rules of gauge theories, as we shall see in Section 2.11. And indeed Lagrangian (2.10.42) is not of gauge type but instead of our generalized chiral type (2.10.1b). Notice,

in particular, the appearance of an explicit dependence in the Minkowski coordinates. This is here interpreted as an indication that the system is open. As we shall also see in Section 2.11, system (2.10.40) is genuinely nonconservative despite its formal invariance under Poincaré transformations. This is our first, still embrionic, exposure to systems which appear to formally obey Einstein special relativity but for which the physical content of this relativity (the conservation laws) is lost.

For details in the technical construction of Lagrangian (2.10.42) as well as for alternatives in the isotopic functions see paper IV of reference.[30]

In conclusion the methodology of the Inverse Problem reveals that the class of chiral Lagrangians is rather vast and can be classified as follows:[31]

(A) Trivially chiral Lagrangians. These are the chiral Lagrangians (2.10.1) with the elements of the G-, F- and E-tensors given by numerical constants.

(B) Nonessentially chiral Lagrangians. These are the chiral Lagrangians (2.10.1) with nontrivial G-, F- and E-tensors but which admit an isotopic transformation to a conventional Lagrangian of type (2.9.1) or (2.9.2).

(C) Essentially chiral Lagrangians. These are the chiral Lagrangian (2.10.1) with nontrivial G-, F- and E-tensors which do not admit an isotopic transformation to a conventional Lagrangian structure.

The following property can be proved.

LEMMA 2.10.1: Essentially selfadjoint systems generally admit analytic representations in terms of nonessentially chiral Lagrangians while nonessentially nonselfadjoint systems always admit analytic representations in terms of essentially chiral Lagrangians.

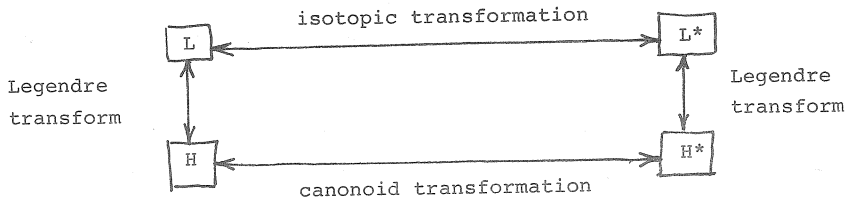
In Newtonian language the above lemma expresses the property that strictly conservative (essentially selfadjoint) systems often admit a generalized Lagrangian structure other than L = T-V, while strictly nonconservative systems possessing a genotopic transformation (nonessentially nonselfadjoint systems) can only be represented with a generalized Lagrangian structure other than L = T-V.

A return "ad originem" is therefore advocated by the Inverse Problem: the assumption of the equations of motion as the quantities of fundamental physical significance and that of their Lagrangian representations as a tool of primarily methodological significance. This is so because the integrability conditions for the existence of an analytic representation essentailly unify conservative and nonconservative systems as far as the Lagrangian structure is concerned. It then follows that the physical nature of a system can be directly and unequivocally ascertained at the level of the equations of motion but not at the level of a Lagrangian. To state it explicitly, whenever a generalized Lagrangian is assigned, such as Eqs. (2.10.27), to ascertain

whether the system is conservative or not it is advisable to compute Lagrange's equations explicitly. This reinstates the equations of motion as the arena of identification of physical content. Additional reasons for this "return ad originem" will be given in the next section within the context of symmetries and first integrals.

We now conclude this section with an outline of the phase space image of the isotopic transformations. This will not only provide the rather intriguing algebraic meaning of the transformations considered, but also the reasons for the selected terminology. For simplicity, the analysis will be restricted to the Newtonian case only.

The phase space image of an isotopic transformation can be constructed in two ways, depending on whether one uses the left-hand-side or the right-hand-side of the fundamental rule (2.10.16). In the former case one essentially has two Lagrangians, L and L*, both within the context of the conventional Lagrange's equations. The use of the Legendre transform then yields the closed and invertible diagram



This produces the first phase space characterization of an isotopic transformation: it is canonoid[70] rather than canonical.

To see this important property, consider a coordinate transformation in phase space which is not canonical, i.e., does not obey rule (2.8.6). A new Hamiltonian, however, can still exist in the new system of coordinates. Let

$$a^\mu \to a^{*\mu} = a^{*\mu}(a) \qquad (2.10.43a)$$

$$a^{*\mu} \in C^2(\hat{R}) \ , \quad \left|\frac{\partial a^{*\mu}}{\partial a^\nu}\right|(\hat{R}) \neq 0 \qquad (2.10.43b)$$

Consider now the time evolution law of $a^{*\mu}$ as a function of a^ν

$$\dot{a}^{*\mu} = \frac{\partial a^{*\mu}}{\partial a^\rho}\dot{a}^\rho = \frac{\partial a^{*\mu}}{\partial a^\rho}\omega^{\rho\sigma}\frac{\partial H}{\partial a^\sigma} . \qquad (2.10.44)$$

Reexpress the right-hand-side in terms of a covariant normal form as a function of the new coordinates

$$\omega_{\mu\nu}\dot{a}^{*\nu} - \overline{\dot{=}}^*_\mu(t, a^*) = 0 , \qquad (2.10.45a)$$

$$\overline{\dot{=}}^*_\mu = \omega_{\mu\nu}\frac{\partial a^{*\nu}}{\partial a^\rho}\omega^{\rho\sigma}\frac{\partial H}{\partial a^\sigma} . \qquad (2.10.45b)$$

When this form is selfadjoint, a new Hamiltonian in the new coordinates exists (Section 2.6). This yields precisely a canonoid transformation.

The first phase space image of Theorem 2.10.1 is then given by

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THEOREM 2.10.3: Necessary and sufficient condition for a noncanonical transformation (2.10.43) to be canonoid with respect to a given Hamiltonian is that each and all the following conditions of selfadjointness

$$\frac{\partial \bar{\bar{\cdot}}^*_\mu}{\partial q^{*\nu}} - \frac{\partial \bar{\bar{\cdot}}^*_\nu}{\partial q^{*\mu}}$$

$$= \frac{\partial}{\partial q^{*\nu}} \left(\omega_{\mu\alpha} \frac{\partial q^{*\alpha}}{\partial q^\rho} \omega^{\rho\sigma} \frac{\partial H}{\partial q^\sigma} \right) \qquad (2.10.46)$$

$$- \frac{\partial}{\partial q^{*\mu}} \left(\omega_{\nu\alpha} \frac{\partial q^{*\alpha}}{\partial q^\rho} \omega^{\rho\sigma} \frac{\partial H}{\partial q^\sigma} \right)$$

$$\mu, \nu = 1, 2, \ldots, 2n$$

are identically verified in a star-shaped region $\tilde{R}^*$ of the variables. In this case the new Hamiltonian is given by

$$H^*(t, q^*) = q^{*\mu} \int_0^1 d\tau \; \bar{\bar{\cdot}}^*_\mu (t, \tau q^*). \qquad (2.10.47)$$

Notice that a canonoid transformation, to be properly defined, must be so with respect to a given Hamiltonian. To illustrate this point, let us recall that the canonical transformations are canonical for all (admissible) systems. Indeed, they can be

defined with rule (2.8.6) without any reference as far as the explicit form of the considered systems (or Hamiltonian) is concerned. This is no longer the case for a canonoid transformation because the integrability conditions (2.10.46) depend explicitly on H. It then follows that a transformation (2.10.43) which satisfies conditions (2.10.46) for a given Hamiltonian does not necessarily satisfy the same condition for a different Hamiltonian. This is precisely the phase space image of the configuration space property that given isotopic functions are isotopic with respect to one Lagrangian and do not necessarily remain so with respect to another Lagrangian.

For completeness it should be also indicated that the phase space image of an isotopic transformation, according to the interpretation here considered, is always canonoid, but not vice-versa. This is so because the class of canonoid transformations is rather vast and includes as a particular case that of the ordinary canonical transformations.[71]

This occurrence can be best seen within the context of the integrability conditions for the existence of a new Hamiltonian via the Converse of the Poincaré Lemma. Eqs. (2.10.46) can, in particular, be satisfied for all Hamiltonians H. This yields precisely the conventional canonical transformations.

We can thus say that, when the canonoid transformations are canonoid with respect to all Hamiltonians they are canonical.

This latter occurrence is excluded by the phase space image of an isotopic transformation because strictly dependent on the

Hamiltonian considered. Also, the transformations

$$L(t,q,\dot{q}) \longleftrightarrow L^*(t,q,\dot{q}),$$
$$q^k \longleftrightarrow q^k,$$
$$p_k = \frac{\partial L}{\partial \dot{q}^k} \longleftrightarrow p^*_k = \frac{\partial L^*}{\partial \dot{q}^k}, \qquad (2.10.48)$$
$$\{a^\mu\} = \{q^k, p_k\} \longleftrightarrow \{a^{*\mu}\} = \{q^k, p^*_k\},$$
$$H(t,q,p) \longleftrightarrow H^*(t,q,p^*),$$
$$H^*(t,a^*) \neq H[t,a(a^*)],$$

do not preserve the fundamental Poisson brackets, Eqs. (2.8.6) unless they trivially reduce to the identity.

This phase space character of the isotopic transformations suggests that their quantum mechanical implications are nontrivial, as we shall indicate in Volume III.

The isotopic transformations can also be subject to a second phase space image through the use of the right-hand side of the fundamental rule (2.10.16). As is known, the canonical prescriptions

$$p_k = \frac{\partial L}{\partial \dot{q}^k} = g_k(t,q,\dot{q}), \qquad (2.10.49)$$

when turned "inside out", i.e., when rewritten in terms of the implicit functions in the velocities

$$\dot{q}^k = \tilde{g}^k(t,q,p) \qquad (2.10.50)$$

coincide with the first set of Hamilton's equations

$$\dot{q}^k = \frac{\partial H}{\partial p_k}. \qquad (2.10.\ 1)$$

The above set, however, can also be equivalently expressed

$$\tilde{g}'_{ki}(t,q,p)\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) = 0 \qquad (2.10.52)$$

with class C^2 and regular but arbitrary matrices $(\tilde{g}'_{i,j})$.

Apply now prescriptions (2.10.49) to the right-hand-side of rule (2.10.16). This yields the second set of Hamilton's equations also with a class C^2 and regular matrix of factor functions

$$\tilde{g}^{ki}(t,q,p)\left(\dot{p}_i - \frac{\partial H}{\partial q^i}\right) = 0 \qquad (2.10.53)$$

The combination of Eqs. (2.10.52) and (2.10.53) yields the following generalized form of Hamilton's equations

$$\left[G_\mu{}^\rho\left(\omega_{\rho\nu}\dot{a}^\nu - \frac{\partial H}{\partial a^\nu}\right)_{SA}^{C^2,R}\right]^{C^2,R} = 0, \qquad (2.10.54a)$$

$$(G_\mu{}^\nu) = \begin{pmatrix} (\tilde{g}^{ki}) & 0 \\ 0 & (\tilde{g}'_{ki}) \end{pmatrix}, \qquad (2.10.54b)$$

which is the canonical counterpart of the isotopically mapped Lagrange's equations

$$\left\{ g_k{}^i \left[L_i(q) \right]_{SA}^{c^2,R} \right\}_{SA}^{c^2,R} = 0. \quad (2.10.55)$$

The main difference is that, while Eqs. (2.10.55) are self-adjoint, Eqs. (2.10.54) are not necessarily selfadjoint. However, it is possible to prove that, for fixed isotopic functions $g_k{}^i$ (i.e., for fixed factor functions in Eqs. (2.10.53)), it is always possible to select functions $G_\mu{}^\nu$ such that the emerging generalized canonical equations are self-adjoint

$$\left[G_\mu{}^\nu \left(\omega_{\nu\rho} \dot{a}^\rho - \frac{\partial H}{\partial a^\nu} \right)_{SA}^{c^2,R} \right]_{SA}^{c^2,R} \quad (2.10.56)$$

$$= \left[\Omega_{\mu\rho}(t,a) \dot{a}^\rho - g_\mu{}^\rho(t,a) \frac{\partial H}{\partial a^\rho} \right]_{SA}^{c^2,R} = 0$$

These equations have been called <u>isotopically mapped Hamilton's equations</u>.[34-II] From a trajectory viewpoint these broader equations are inessential because they leave the actual path unaffected by construction. Nevertheless, Eqs. (2.10.56) are nontrivial from an algebraic viewpoint. This is due to the fact that the underlying analytic brackets can be characterized in terms of the factor tensor $\Omega_{\mu\nu}$. But the conditions of selfadjointness for quasilinear systems guarantee the verification of the Lie-algebra axioms (Section 2.7). Thus, a second phase space image

of the isotopic transformations in configuration space is given by the invertible Lie algebra axiom preserving mapping of the Poisson brackets[72]

$$[A,B] = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \longleftrightarrow [A,B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu} \quad (2.10.57)$$

where the brackets in the left-hand-side (right-hand-side) are the analytic brackets of the original (isotopically mapped) framework.

LEMMA 2.10.2:[34-II] <u>The isotopic transformations can be interpreted as characterizing "degrees of freedom" of the classical realization of the Lie algebra product.</u>

In essence, the Lie algebra axioms do not uniquely characterize their classical product. On the contrary, there exists a family of products all satisfying the Lie algebra axioms which is characterized by all tensors $\Omega^{\mu\nu}(t,a)$ obeying the conditions of selfadjointness (2.7.5), of which the fundamental symplectic tensor $\omega^{\mu\nu}$ (and related conventional Poisson brackets) is only a particular case. This is a class of algebraic degrees of freedom in the sense that the degrees of freedom in the selection of the Ω-tensor and related analytic brackets originate directly from the Lie algebra axioms. The isotopic transformations result as providing an analytic counterpart of these algebraic degrees of freedom. This is the content of Lemma 2.10.2. An example will be

given in the next section.

We are now in a position to indicate the reason for the selected terms "isotopic transformations". In the theory of Abstract Algebras any invertible mapping of the product of an algebra A which preserves its axioms is called <u>isotopic.</u> The emerging new algebra A* which is the same vector space as A, only equipped with the new product, is called <u>isotope</u> of A.[32] Mappings (2.10.57), because invertible and axiom-preserving, are a particularization of this concept of isotopic mapping in Abstract Algebras, only expressed for canonical realizations of Lie algebras.

Regrettably, the concept of isotopic mapping is not treated in recent textbooks in the theory of Abstract Algebras, with only very few exceptions known to me, such as a monograph by R. H. Bruck[73] (for a treatment in specialized mathematical linterature, see reference)[74]. This is regrettable because this concept has rather significant implications, such as for the problem of identification of first integrals of the equations of motion (see next section). Oddly, as R. H. Bruck noticed, the concept of algebraic isotopy is "so natural to creep in un-noticed" (page 56, loc. cit.).

It is hoped that the terms "isotopic transformations", "isotopic functions" etc. used in this analytic context will not create confusion with the high energy physics concept of <u>isotopic spin</u> (<u>isospin</u>). The fact remains that the concept of algebraic isotopy is considerably older than that of spin isotopy. As a

matter of fact it dates back to the earlier stages of set theory (see R. H. Bruck, loc. cit.).

The reader should be aware that the methodology of isotopic transformations presented in this section deals specifically with the case in which the local variables are not changed. The same methodology, however, can be broadened with the inclusion of transformations of local variables, particularly those used in the Universality Theorem 2.8.1.

We would like to close this section with few remarks related to the variationsl principles, such as Hamilton's principle

$$\delta \int_{t_1}^{t_2} dt\, L(t,q,\dot{q}) \Big|_{\substack{q=\text{Act.}\\ \text{path}}} = - \int_{t_1}^{t_2} dt\, L_k(q) \Big|_{\substack{q=\text{Act}\\ \text{path}}} \delta q^k = 0. \qquad (2.10.58)$$

One of the first implications of the Inverse Problem is that of broadening the applicability of Hamilton's principle. And indeed, it emerges that <u>Hamilton's principle can represent, within a fixed system of variables, all systems which possess an equivalent selfadjoint form</u>, according to the generally indirect analytic representations

$$\delta \int_{t_1}^{t_2} dt\, L \Big|_{\substack{q=\text{Act.}\\ \text{path}}} = - \int_{t_1}^{t_2} dt \left[h_k{}^i (A_{ij} \ddot{q}^j + B_i) \right]_{SA} \Big|_{\substack{q=\text{Act}\\ \text{path}}} \delta q^k = 0, \qquad (2.10.59)$$

or, equivalently, according to the genotopically mapped principle (Section 2.3).

$$\delta^* \int_{t_1}^{t_2} dt\, L \Big|_{\substack{q=\text{Act.}\\ \text{path}}} = - \int_{t_1}^{t_2} dt \left\{ g_k{}^i \left[L_i(q) \right]_{SA} \right\}_{NSA} \Big| \substack{\delta q^k = 0 \\ q=\text{Act.}\\ \text{path}} \qquad (2.10.60)$$

If transformations of the local variables are admitted, the Inverse Problem renders Hamilton's principle "universal" in the sense that it is capable of producing an indirect analytic representation of "all" Newtonian systems which are of class C^∞ and regular (Theorem 2.8.1). A further broadening of this "universality" is conceivable with the removal of the condition of regularity. This aspect, however, will not be considered at this time.

The integrability conditions for the existence of a Lagrangian also have other implications for Hamilton's principle. For instance, in a way parallel to principle (2.10.60) we can have the alternative generalization

$$\delta^* \int_{t_1}^{t_2} dt\, L \Big|_{\substack{q=\text{Act.}\\ \text{Path}}} = - \int_{t_1}^{t_2} dt \left\{ g_k{}^i \left[L_i(q) \right]_{SA} \right\}_{SA} \Big| \substack{\delta q^k = 0 \\ q=\text{Act}\\ \text{path}} \qquad (2.10.61)$$

which we shall refer to as the isotopically mapped Hamilton's principle.[34-II] Several modifications of convertional variational approaches are then possible. For instance, we can write

$$\delta^* \int_{t_1}^{t_2} dt\, L = \delta \int_{t_1}^{t_2} dt\, L^*, \qquad (2.10.62)$$

or

$$\begin{aligned} \delta \int_{t_1}^{t_2} dt\, L &= - \int_{t_1}^{t_2} dt\, L_k(q)\, \delta q^k = \\ &= - \int_{t_1}^{t_2} dt\, g_k{}^i \left[L_i(q) \right] g^{-1}{}_j{}^k\, \delta q^j \qquad (2.10.63) \\ &= - \int_{t_1}^{t_2} dt\, L^*{}_k(q)\, \delta_* q^k \\ &= \delta_* \int_{t_1}^{t_2} dt\, L^* \end{aligned}$$

$$\delta_* q^k = g^{-1}{}_j{}^k\, \delta q^j, \quad (g^{-1}{}_i{}^j) = (g_i{}^j)^{-1}$$

or Hamilton's principle can be reformulated to directly exhibit the integrability conditions for the existence of an action functional. For these and other treatments of variational principles (with and without end point contributions) see reference.[34-II]

2.11: USE OF THE INVERSE PROBLEM FOR THE STUDY OF SYMMETRIES AND FIRST INTEGRALS OF SYSTEMS WITH ARBITRARY COUPLINGS.

As by now familiar, one of the central objectives of this monograph is the study of the problem whether established relativities are applicable or not to local systems with arbitrary couplings. But, as is well known, relativity ideas are deeply linked to symmetry considerations. The first relevance of the methodology of the Inverse Problem is then selfevident: it is essential for the conventional formulation of the problem considered. Indeed, without the knowledge of a Lagrangian, the conventional approach to symmetries and first integrals via Noether's theorem cannot be applied.

However, as we shall see, the significance of the methodology of the Inverse Problem for symmetry related topics goes considerably beyond that. It first implies a broadening of the same methodology of symmetries and first integrals, and finally results in specific tools for the identification of the arena of applicability and nonapplicability of established relativities.

This section is devoted to the study of the problem of symmetries and first integrals for local systems with arbitrary couplings within the context of the Inverse Problem, without any reference to relativity considerations. This latter aspect will be considered in Chapter 3. For simplicity, we shall primarily consider the Newtonian case and then indicate the field theoretical extensions of the results.

It is advisable to first review the basic definitions of the problem of symmetries and first integrals to avoid possible misrepresentations of the results due to the existence in the literature of same terms with different meanings. Let us begin with the fundamental concept of symmetry. 75

DEFINITION 2.11.1: A class C^2 and regular Newtonian system in the quasilinear form

$$A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q}) = 0, \quad (2.11.1)$$

$$k = 1, 2, \ldots, n$$

is said to possess a symmetry under the transformations of local variables (2.11.2a)

$$t \to t', \qquad t' \in C^2(R), \left|\frac{dt'}{dt}\right|(R) \neq 0,$$

$$q^k \to q'^k, \qquad q'^k \in C^2(R), \left|\frac{dq'}{dq}\right|(R) \neq 0, \quad (2.11.2b)$$

when it is form-invariant, namely, when the system preserves its functional dependence up to equivalence transformations

$$A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q})$$

$$= J'^i_k(t',q',\dot{q}')\left[A'_{ij}(t',q',\dot{q}')\ddot{q}'^j + B'_j(t',q',\dot{q}')\right] \quad (2.11.3)$$

$$\equiv J'^i_k(t',q',\dot{q}')\left[A_{ij}(t',q',\dot{q}')\ddot{q}'^j + B_i(t',q',\dot{q}')\right].$$

Equivalently, a class C^4 and regular Lagrangian $L(t,q,\dot{q})$ is said to possess a symmetry under transformations (2.11.2) when the new Lagrangian constructed according to the rule

$$L(t,q,\dot{q}) = L'(t',q',\dot{q}')\frac{dt'}{dt}, \qquad (2.11.4)$$

preserves the functional dependence of the original Lagrangians up to additive terms with null Lagrange's derivatives

$$L'(t',q',\dot{q}') = L(t',q',\dot{q}') + \frac{d}{dt}G'(t',q'). \qquad (2.11.5)$$

A few comments are here in order. First of all, notice that the two definitions of symmetry are equivalent because the preservation of the functional dependence of the Lagrangian, rule (2.11.5), is equivalent to the preservation of the functional dependence of the A- and B-terms at the level of the equations of motion, rule (2.11.3). Indeed, from the structure of the Lagrange's equations and rules (2.11.5) and (2.10.6a) we can write

$$L'_k(q') = \frac{\partial^2 L'}{\partial\dot{q}'^k\partial\dot{q}'^i}\ddot{q}'^i + \frac{\partial^2 L'}{\partial\dot{q}'^k\partial q'^i}\dot{q}'^i + \frac{\partial^2 L'}{\partial\dot{q}'^k\partial t'} - \frac{\partial L'}{\partial q'^k}$$

$$= A'_{ki}(t',q',\dot{q}')\ddot{q}'^i + B'_k(t',q',\dot{q}')$$

$$\equiv L_k(q') = \frac{\partial^2 L}{\partial\dot{q}'^k\partial\dot{q}'^i}\ddot{q}'^i + \frac{\partial^2 L}{\partial\dot{q}'^k\partial q'^i}\dot{q}'^i + \frac{\partial^2 L}{\partial\dot{q}'^k\partial t'} - \frac{\partial L}{\partial q'^k} \qquad (2.11.6)$$

$$= A_{ki}(t',q',\dot{q}')\ddot{q}'^i + B_k(t',q',\dot{q}')$$

by reobtaining rule (2.11.3) less the Jacobian.

Thus, Definition 2.11.1 allows the study of the problem of symmetries on two fully equivalent grounds: the equations of motion and their Lagrangian representations. Since the identification of the physical content of the system considered is rather difficult at the level of a Lagrangian (Section 2.10) the significance of this joint treatment is selfevident.

Secondly, the reader should keep in mind the use of velocity dependent transformations, as used, say, in the Calculus of Variations, rather than the conventional forms

$$t \to t' = t'(t), \qquad (2.11.7a)$$

$$q^k \to q'^k = q'^k(q). \qquad (2.11.7b)$$

As we shall see, this broadening of the transformation laws is essential for the problem of symmetries and first integrals of local systems with arbitrary couplings on several counts.

Thirdly, Definition 2.11.1 applies to a Newtonian system in the most general form representable with Lagrange's equations, the quasilinear form (2.11.1) or, equivalently, the most general structure of the Lagrangian, Eq. (2.9.2) or (2.9.32). Thus, Definition 2.11.1 is conceived to apply to the most general local

nonconservative equations of motion or, equivalently, to their Lagrangian representations. The case of conventional conservative systems and Lagrangian structures is of course contained as a trivial subcase.

Fourthly, the Jacobian dt'/dt of Eq. (2.11.4) is introduced, as customary also in the conventional treatment, to ensure the preservation of the value of the action functional, i.e.

$$A = \int_{t_1}^{t_2} dt\, L(t,q,\dot{q}) \equiv \int_{t_1}^{t_2} dt'\, L'(t',q',\dot{q}') \frac{dt'}{dt}. \quad (2.11.8)$$

Fifthly, even though the concepts of symmetry of the equations of motion and that of a Lagrangian for its analytic representation are equivalent, this does not necessarily imply that the class of all symmetries of the equations of motion coincide with that of one given Lagrangian for its analytic representation. This is a direct consequence of the integrability conditions for the existence of a Lagrangian.

In essence, a class C^2 and regular Newtonian system admits a unique form when written in terms of the implicit functions, i.e. the kinematical form

$$\left[\ddot{q}^k - f^k(t,q,\dot{q})\right]^{C^2,R} = 0 \quad (2.11.9)$$

$$k = 1, 2, \ldots, n$$

(where the uniqueness of this form trivially follows from the validity of the theorem on implicit functions) and an infinite variety of equivalent quasilinear forms induced by the multiplication of a continuity and regularity preserving matrix (h_{ki})

$$\left\{h_{ki}(t,q,\dot{q})\left[\ddot{q}^i - f^i(t,q,\dot{q})\right]^{C^2,R}\right\}^{C^2,R} \quad (2.11.10)$$

$$\equiv \left[A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q})\right]^{C^2,R}$$

Let τ_N be the class of all symmetries of the kinematical form (2.11.9) of a given Newtonian system. Suppose that such form is nonessentially nonselfadjoint (Section 2.9). Then, within the class of equivalent quasilinear forms (2.11.10) there exists a subclass which is selfadjoint. The Lagrangian representations for system (2.11.10) exist only for this latter subclass. Let L be one of such Lagrangians and τ_L the class of its symmetries. It is easy to see that $\tau_N \neq \tau_L$. This is the case, for instance, when the kinematical form (2.11.9) does not depend explicitly on time while the corresponding quasilinear selfadjoint form (2.11.10) depends explicitly on time (i.e., the selected solutions of the conditions of selfadjointness in the genotopic functions h_{ki} exhibit such a dependence). The Lagrangian considered also depends explicitly on time from Theorem 2.3.3. It then follows that, while the equations of motion (2.11.9) exhibit a symmetry under translations in time, the selected Lagrangian for

their analytic representation does not. We thus have, as our first result, the following

LEMMA 2.11.1: The class of symmetries of the equations of motion of a Newtonian system does not coincide with the corresponding class of symmetries of each possible Lagrangian for its analytic representation.

To better understand this occurrence we need the following

DEFINITION 2.11.2: A symmetry of the equations of motion is termed manifest or nonmanifest depending of whether it can be identified by simple means (usually a visual inspection) or not.

For instance, a symmetry under the transformation $t \to t' = t+t_0$ is a manifest because identifyable through the lack of explicit dependence on time. On the contrary, a symmetry under a nontrivial transformation $t \to t' = t'(t,q,\dot{q})$ is not manifest because it requires rather elaborate techniques for its identification.

LEMMA 2.11.2: The class of manifest symmetries of the equations of motion is generally larger than that of each individual Lagrangian for its analytic representation.

This lemma can be proved with the following example. Consider the particle under a drag force

$$(\ddot{q} + \gamma \dot{q})_{MSA} = 0 \qquad (2.11.11)$$

It exhibits two manifest symmetries, those under translations in time and space. Consider now the two (isotopically related) Lagrangians for its analytic representation identified in Section 2.10, i.e.

$$L = \frac{1}{2} e^{\gamma t} \dot{q}^2, \qquad (2.11.12a)$$

$$L^* = \dot{q} e^{\gamma t} \dot{q} - \gamma q. \qquad (2.11.12b)$$

A visual inspection of these Lagrangians proves Lemma 2.11.2. And indeed, the Lagrangian L is manifestly invariant under translations in space but not in time. Singificantly, the situation for the Lagrangian L* is the opposite of that, it is manifestly invariant under translations in time but not in space. The important point is that the use of both Lagrangians L and L* is capable of reproducing the two manifest symmetries of the considered system, while each individual Lagrangian is not.

We then have the following

LEMMA 2.11.3: The class of all manifest symmetries of the equations of motion is generally expressed by the class of all isotopically related Lagrangians for the representation of the same system.

The impact of this occurrence is rather profound. In actuality it implies a broadening of the methodology of symmetries and first integrals which is directly induced by the integrability conditions for the existence of a Lagrangian. To understand this point it should be here recalled that the practical applicability of

symmetries and first integrals via Noether's theorem is restricted to manifest symmetries only, owing to known technical difficulties for identifying nonmanifest symmetries. The significance of Lemma 2.11.3 is then selfevident: the use of isotopic transformations allows the computation of first integrals via manifest symmetries which would not be readily identifyable with the same techniques if the analysis is restricted to only one Lagrangian.

In conclusion, the Inverse Problem generates a significant broadening of the methodology of symmetries and first integrals, by extending the applicability of known techniques from only one Lagrangian to the family of all possible isotopically mapped Lagrangians and with consequential capability of identifying new first integrals.

To reach a technical formulation of the latter possibility of practical significance we need further elements.

DEFINITION 2.11.2: A symmetry is called

(1) exact or nonexact where the additive term with null Lagrange's derivatives in the transformation of a Lagrangian, Eqs. (2.11.5), is null or not, respectively,

(2) finite or infinitesimal when Eqs. (2.11.2) represent a finite or infinitesimal transformation, respectively,

(3) connected or nonconnected, when transformations (2.11.2) characterize an m-parameter connected Lie group or not,

(4) infinitesimal of the first order or infinitesimal of higher order p when they are connected and infinitesimal of the indicated order, and

(5) of pure time character, of pure space character, or of mixed space-time character when the transformation are of the type (2.11.7a) only, or of type (2.11.7b) only, or of type (2.11.2), respectively.

Few examples are here useful to illustrate the above terminology. The form invariance of two-dimensional conservative system under rotations

$$\underset{\sim}{r}_\mu \to \underset{\sim}{r}_\mu' = R\,\underset{\sim}{r}_\mu \,, \quad R R^T = R^T R = 1 \qquad (2.11.13a)$$

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad (2.11.13b)$$

is an exact, finite, connected, manifest symmetry of pure space character. A symmetry of the same system under the Galilei boosts

$$\underset{\sim}{r}_\mu \to \underset{\sim}{r}_\mu' = \underset{\sim}{r}_\mu + \underset{\sim}{v}_\mu t \qquad (2.11.14)$$

is on the contrary nonexact and not of pure space character (it is of mixed space-time character according to our definition) but it is still finite, connected and manifest.

For the case of infinitesimal symmetries, the (necessary) prerequisite of connectedness can be realized with the assumption that transformations (2.11.2) are Lie transformations, i.e. besides being of at lease class C^p, $p \geq 2$, in the indicated variables, they are analytic in m parameters, say w^r, r=1,2,...,m

$$\begin{cases} t' = t'(t, q, \dot{q}; w) \\ q'^k = q'^k(t, q, \dot{q}; w) \end{cases} \qquad (2.11.15)$$

and they are such that the identity transformation is recovered at the limit

$$\lim_{w=0} t' = t \,, \quad \lim_{w \to 0} q'^k = q^k \qquad (2.11.16)$$

This ensures that the transformations are well defined in the neighborhood of the identity of the desired order p and allow the expansions

$$t' = t + \frac{1}{1!} \frac{\partial t'}{\partial w^{z}}\bigg|_{w=0} w^{z} + \frac{1}{2!} \frac{\partial^2 t'}{\partial w^{z_1} \partial w^{z_2}}\bigg|_{w=0} w^{z_1} w^{z_2} + \cdots, \qquad (2.11.17a)$$

$$q'^k = q^k + \frac{1}{1!} \frac{\partial q'^k}{\partial w^{z}}\bigg|_{w=0} w^{z} + \frac{1}{2!} \frac{\partial^2 q'^k}{\partial w^{z_1} \partial w^{z_2}}\bigg|_{w=0} w^{z_1} w^{z_2} + \cdots . \qquad (2.11.17b)$$

$w^z \in O_\varepsilon$

The <u>infinitesimal transformations of the first order</u> can then be defined by

$$\hat{\delta}^{(1)} t = \frac{\partial t'}{\partial w^{z}}\bigg|_{w=0} w^{z} \,, \qquad (2.11.18a)$$

$$\hat{\delta}^{(1)} q^k = \frac{\partial q'^k}{\partial w^{z}}\bigg|_{w=0} w^{z} \,, \qquad (2.11.18b)$$

$w^z \in O_\varepsilon$

while those of order p can be defined accordingly by

$$\hat{\delta}^{(p)} t = \frac{1}{p!} \frac{\partial^p t'}{\partial w^{z_1} \cdots \partial w^{z_p}}\bigg|_{w=0} w^{z_1} \cdots w^{z_p}, \qquad (2.11.19a)$$

$$\hat{\delta}^{(p)} q^k = \frac{1}{p!} \frac{\partial^p q'^k}{\partial w^{z_1} \cdots \partial w^{z_p}}\bigg|_{w=0} w^{z_1} \cdots w^{z_p}. \qquad (2.11.19b)$$

$w^z \in O_\varepsilon$

The above expansions apply, trivially, to the rotations (2.11.13) yielding infinitesimal transformations of the desired order. Notice that a connected symmetry refers to the symmetry under the identity component of a Lie group. Thus, <u>discrete transformations</u>, such as space and time inversions, are excluded. This is a typical occurrence of the methodology of symmetries and first integrals according to Noether's theorem (see below). Notice also that a (finite or infinitesimal) transformation of pure space character is <u>contemporaneous</u> in the sense that it occurs at a fixed instant of time. The transformation of pure time or of mixed space-time are on the contrary <u>noncontemporaneous</u> in the sense that they imply a change of the time variable.

From rule (2.11.5) it is possible to prove the following

<u>THEOREM</u> 2.11.1: <u>Necessary and sufficient condition for a class C^2 and regular Lagrangian to possess a symmetry under infinitesimal transformations of the first order is that the following condition</u>

$$\hat{\delta}^{(2)}L + L\frac{d}{dt}\left(\hat{\delta}^{(1)}t\right) + \frac{d}{dt}\left(\hat{\delta}^{(1)}G\right) = 0, \quad (2.11.20)$$

is identically verified, where

$$\hat{\delta}^{(1)}L = \frac{\partial L}{\partial t}\hat{\delta}^{(1)}t + \frac{\partial L}{\partial q^k}\hat{\delta}^{(1)}q^k + \frac{\partial L}{\partial \dot{q}^k}\hat{\delta}^{(1)}\dot{q}^k \quad (2.11.21a)$$

$$L'(t', q', \dot{q}') = L(t', q', \dot{q}') + \frac{d}{dt}\left(\hat{\delta}^{(1)}G\right) + O^{(2)}(w) \quad (2.11.21b)$$

$$\lim_{w\to 0} O^{(2)}(w) = 0 \quad (2.11.21c)$$

For an extension of the above theorem to higher orders infinitesimal symmetries see Volume II of reference.[34] For our needs it is here sufficient to recall that, trivially, when a Lagrangian possesses a symmetry under a finite, connected, Lie, transformation, it also possesses a symmetry under infinitesimal transformations of arbitrary order p. However, the opposite property does not necessarily hold, i.e., the existence of an infinitesimal symmetry of order p does not necessarily imply that of order p+1, e.g., because of lack of sufficient continuity properties.

[34-II]

DEFINITION 2.11.3: A set of functions $I_r(t,q,\dot{q})$, $r = 1,2,\ldots,m$. are called first integrals of the equations of motion when they are conserved along the actual path. A first integral is called a conservation law when it represents a physical conservation law.

Explicitly, the (necessary and sufficient condition) for the functions I_r to be first integrals is that

$$\dot{I}_r = \frac{\partial I_r}{\partial t} + \frac{\partial I_r}{\partial q^k}\dot{q}^k + \frac{\partial I_r}{\partial \dot{q}^k}f^k \equiv 0, \quad (2.11.22)$$

where the f's are the (unique) implicit functions of the system as they occur in the kinematical form (2.11.9). Thus, and this should be stressed, when the functions I_r are conserved, they are so in virtue of the equations of motion and irrespective of any Lagrangian representation. Of course, when the functions I_r are conserved for the equations of motion they are conserved also for their analytic representations.

The distinction between first integrals and conservation laws of Definition 2.11.3 should also be stressed. It will have a crucial impact in the problem of the arenas of applicability and nonapplicability of established relativities, as we shall see in Chapter 3. In essence, this distiction differentiates the mathematical from the physical context. All conservation laws are expressed by first integrals, but first integrals need not be representative of physical conservation laws. It is better to illustrate this crucial concept with explicit examples. Consider, for instance, the Kepler system in vacuum, Eqs. (2.9.18). It possesses a first integral given by

$$H = \frac{1}{2m}\underset{m}{p}_k \cdot \underset{m}{p}^k + V(r). \quad (2.11.23)$$

An inspection indicates that this mathematical occurrence represents the physical conservation law of the total energy of the system. Similarly for other first integrals of the system considered, such as the conservation of the total linear momentum, total angular momentum, etc.

Consider now the damped oscillator (2.10.33). It possesses a first integral given by

$$H = \mu q - \mu \cos(\beta q) - \frac{1}{2}\beta p q . \qquad (2.11.24)$$

But the system considered is genuinely nonconservative. This physical property is expressed by the fact that the first integral (2.11.24) is not representative of a physical law. Thus, the terms "conservation law" are excluded in this case by Definition 2.11.3.

This is, in the ultimate analysis, an occurrence which is intrinsic in the broader class of systems considered. If the problem of symmetries and first integrals is restricted to conservative systems, as customarily done in the current physical literature, there is no need of differentiating between first integrals and conservation laws because all first integrals (identifyable through manifest symmetries - see the later remarks for the Kepler three-body problem) exhibit a direct physical significance. When the same analysis is enlarged to include systems with arbitrary couplings, i.e., genuinely nonconservative, the situation is profoundly different.

The physical nonconservative nature of the systems considered has no mathematical implications for first integrals in the sense that a system of differential equations admits first integrals irrespective of whether conservative or nonconservative. On physical grounds, however, the indicated distinction between first integrals and conservation laws emerges. As a matter of fact there is the opposite occurrence of that of conservative systems: conserved quantities are mathematically well defined as first integrals but they are generally vacuous on physical grounds in the sense that they do not generally represent physical conservation laws. After all, the system considered are nonconservative by assumption.

Hereon we shall use the terms "first integrals" when referring to conserved quantities of unidentified physical meaning. The terms "conservation laws" will be used only when the content of first integrals of being representative of physical laws has been either proved or it is assumed.

We are now equipped to formulate the fundamental theorem of the methodology of symmetry and first integrals, Noether's theorem.

THEOREM 2.11.2: When a class C^3 and regular Lagrangian $L(t,q,\dot{q})$ possesses an exact m-parameter infinitesimal symmetry of the first order, there exist m distinct linear combinations of Lagrange's derivatives of L which

are exact differential of m first integrals $I_r(t,q,\dot{q})$, i.e.,

$$\left[L_k(q)\, \delta^{(1)} q^k - \frac{d}{dt} I_r(q) w^r \right]_{q = \text{Adm. path}} = 0 , \qquad (2.11.25a)$$

$$\dot{I}_r w^r = \frac{\partial L}{\partial \dot{q}^k} \hat{\delta}^{(1)} q^k - \left(\dot{q}^k \frac{\partial L}{\partial \dot{q}^k} - L \right) \hat{\delta}^{(1)} t , \qquad (2.11.25b)$$

$$\left. \frac{d}{dt} I_r(q) \right|_{q = \text{Act. Path}} = 0 , \qquad (2.11.25c)$$

$$\delta^{(1)} q^k = \hat{\delta}^{(1)} q^k - \dot{q}^k \hat{\delta}^{(1)} t = \eta_r^k(t) w^r \qquad (2.11.25d)$$

A few remarks are here in order. First of all, notice that the above theorem (here presented in its most widely accepted form) refers specifically to infinitesimal symmetries of the first-order. For a generalization of this theorem to infinitesimal symmetries of arbitrary ofder p see Volume II of reference.[34]

It should be indicated that Eqs. (2.11.25a) can be computed along an admissible path (Section 2.3). Eqs. (2.11.25c), on the contrary, only hold along the actual path, i.e., the solution of the system considered.

If the symmetry is contemporaneous the η's are the finite part of its transformations. If the symmetry is noncontemporenaeous, from Eqs. (2.11.25d) the η's are the finite part of

the image of the coordinates in the new time variable. If the symmetry is not exact, rule (2.11.25a) is implemented with the term $\frac{d}{dt}(\hat{\delta}^{(1)} G)$ originating from Eqs. (2.11.20).

It should be also noted that Noether's theorem does not guarantee that the m first integrals originating from an m-parameter symmetry are independent. This can be seen from this simple example. Consider the case of a free particle with Lagrangian

$$L = \frac{1}{2} m \dot{\underset{\sim}{x}} \cdot \dot{\underset{\sim}{x}} . \qquad (2.11.26)$$

This Lagrangian exhibits a manifest symmetry under the 10-parameter Galilei group. The use of Theorem 2.11.2 then yields 10 first integrals. However the motion is along a straight line. Thus, the maximum number of independent first integrals is, in this case, two. As a result, eight out of ten first integrals of the Galilei symmetry of Lagrangian (2.11.26) must be dependent. It is an instructive exercise for the interested reader to work out this occurrence in details. This lack of necessary independence of the first integrals has no image at the level of the symmetry in the sense that the Galilei symmetry of Lagrangian (2.11.26) is characterized by transformation laws (the Galilei transformations) which are analytic (and thus Lie) in ten independent parameters. The possible dependence of first integrals is therefore an occurrence identifyiable at the level of the linear combinations

(2.11.25a) of the Lagrange derivatives, and not at the level of the parametric structure of the symmetry group.

There is an aspect of Theorem 2.11.2 which will have a central significance for the problem of the arenas of applicability and nonapplicability of established relativities. Noether's theorem merely establishes the mathematical property that, whenever a Lagrangian exhibits a symmetry under an m parameter connected Lie group, there exist m first integrals. By no means this mathematical occurrence should be interpreted as to imply that the emerging first integrals are necessarily representative of physical conservation laws, nor I am aware of any intent by Emmy Noether to this effect.

An illustration of this crucial occurrence is here in order. The physical conservation law of the total energy, Eqs. (2.11.23), for the Kepler system in vacuum can indeed be derived from the symmetry of the conventional Lagrantian (2.9.19) under translations in time. Thus, in this case, the symmetry under translations in time is representative of a physical law. However, the same symmetry for the case of another Lagrangian can only exhibit a mathematical significance and be vacuous on physical grounds. This is, for instance, the case for the first integral (2.11.24) of the damped oscillator which is also derivable, via Noether's theorem, from the symmetry of Lagrangian (2.10.35) under translations in time.

We reach therefore the conclusion that, when conventional conservative systems are considered and represented in terms of

the conventional Lagrangian structure L = T - V, Noether's theorem can be formulated for practical purposes in terms of conservation laws rather than first integrals. However, when the theorem is applied to (local but) arbitrary Newtonian systems, the use of the terms "conservation laws" could be misleading. In this case the theorem only states a mathematical occurrence (the derivation of first integrals from simmetries) which, in general, is not representative of physical laws.

Our next objective is that of extending the applicability of Noether's theorem with the class of isotopic transformations of a Lagrangian (within a fixed system of local variables), as introduced in Section 2.10, according to the rule

$$L^*_k(q) = \left\{ g_k{}^i \left[L_i(q) \right]^{C^2,R}_{SA} \right\}^{C^2,R}_{SA} \qquad (2.11.27)$$

It is easy to see that the quantities I_r which are conserved for the original Lagrangian L remain conserved for all possible isotopically mapped Lagrangians L^*. Indeed, we can reinterpret Noether's identities (2.11.25a) as follows

$$\begin{aligned} \dot{I}_2 w^2 &= L_k(q) \delta^{(1)} q^k \\ &= \left[g_k{}^i \left[L_i(q) \right] \right\} g^{-1}{}_j{}^k \delta^{(1)} q^j \qquad (2.11.28a) \\ &= L^*_k(q) \, \delta^{(1)}_* q^k \end{aligned}$$

$$\delta^{(1)}_* q^k = g^{-1}{}_j{}^k \delta^{(1)} q^j, \quad (g^{-1}{}_i{}^j) = (q_i{}^j)^{-1} \qquad (2.11.28b)$$

Alternatively, the statement follows from the property indicated earlier that the quantities which are conserved are so in virtue of the equations of motion and, thus, for all admissible Lagrangian representations.

Generalized identities (2.11.28a) also have another singificance. They indicate that, given a Lagrangian L with an m-parameter symmetry of infinitesimal transformations of the first order and related first integrals I_r, the isotopically mapped Lagrangian L* is expected to possess a new symmetry G^*_m also of infinitesimal transformations of the first-order in the same parameters which leads to the original first integrals. Explicitly, this expectation originates from the fact that, according to Eqs. (2.11.28), the first integrals I_r can be associated either with the Lagrangian L and symmetry transformations $\delta^{(1)}q^k$ or with the Lagrangian L*, in which case $\delta^{(1)}_* q^k$ are expected to be the symmetry transformations leading, via Noether's theorem, to I_r.

To put this idea in a proper conceptual perspective we introduce the following

DEFINITION 2.11.4: Two m-dimensional connected Lie groups G_m and G^*_m are called isotopically related when they are symmetry groups of two corresponding, isotopically related Lagrangians L and L* which lead to the same first integrals.

To put the concept of isotopic image of a symmetry group in a proper technical context, the so-called Inverse Noether's Problem is needed. In essence these terms indicate the methods for constructing a symmetry from a given Lagrangian and a given system of first integrals. As such, this problem is the inverse of Theorem 2.11.2. This is precisely the methodological context needed for isotopic images of symmetries. Here, the Lagrangian L* and first integrals I_r are known and the symmetry which leads to these first integrals is requested.

For a review of the status of the art in the Inverse Noether's Problem see Volume II of reference [34] or paper [30-I]. For our needs the following theorem is sufficient

THEOREM 2.11.3: Necessary and sufficient conditions for the generally noncontemporaneous infinitesimal transformations of the first order

$$t \to t' = t + \hat{\delta}^{(1)}_* t \qquad (2.11.29)$$
$$q^k \to q'^k = q^k + \hat{\delta}^{(1)}_* q^k$$

to be an exact symmetry of a class C^3 and regular Lagrangian $L^*(t,q,\dot{q})$ which leads to given first integrals $I_r(t,q,\dot{q})$, r = 1,2,...,m, is that the following conditions

$$\frac{d}{dt}\frac{\partial J^*}{\partial(\delta^{(1)}_* \dot{q}^k)} - \frac{\partial J^*}{\partial(\delta^{(1)}_* q^k)} = 0, \qquad (2.11.30a)$$

$$L^*_k(q)\,\delta^{(1)}_* q^k = I_r w^r, \qquad (2.11.30b)$$

$$\hat{\delta}^{(1)}_{*} q^k - \dot{q}^k \hat{\delta}^{(1)}_{*} t = \delta^{(1)}_{*} q^k, \qquad (2.11.30c)$$

$$\frac{\partial L^*}{\partial \dot{q}^k} \hat{\delta}^{(1)}_{*} q^k - \left(\dot{q}^k \frac{\partial L^*}{\partial \dot{q}^k} - L^*\right) \hat{\delta}^{(1)}_{*} t = I_2 w_1^2 \qquad (2.11.30d)$$

$$\hat{\delta}^{(1)}_{*} L^* + \frac{d}{dt}\left(\hat{\delta}^{(1)}_{*} t\right) = 0, \qquad (2.11.30e)$$

$$J^* = \frac{1}{2}\left(\frac{\partial^2 L^*}{\partial \dot{q}^i \partial \dot{q}^j} \delta^{(1)}_{*} \dot{q}^i \delta^{(1)}_{*} \dot{q}^j + 2\frac{\partial^2 L^*}{\partial \dot{q}^i \partial q^j} \delta^{(1)}_{*} \dot{q}^i \delta^{(1)}_{*} q^j - \frac{\partial^2 L^*}{\partial q^i \partial q^j} \delta^{(1)}_{*} q^i \delta^{(1)}_{*} q^j\right), \qquad (2.11.30f)$$

are identically verified.

On practical grounds, one first solves Jacobi's equations (2.11.30a) associated to Lagrange's equations in L* (Section 2.3) subject to the subsidiary constraint (2.11.30b). This yields the contemporaneous variations $\delta^{(1)}_{*} q^k$. Then the system of algebraic equations (2.11.30c) and (2.11.30d) is solved. This yields the noncontemporaneous variations $\hat{\delta}^{(1)}_{*} q^k$ and $\hat{\delta}^{(1)}_{*} t$. Finally, Eq. (2.11.30e) (which is the condition for these transformations to constitute a symmetry of L^*; Theorem 2.11.1) is inspected. Alternatively, one can study whether the system of algebraic equations (2.11.30b), through (2.11.30e) is consistent with the solutions of Eqs. (2.11.30a).

Clearly, the Inverse Noether's problem does not necessarily admit a solution, namely, given a Lagrangian and a set of first integrals, an infinitesimal symmetry which leads to I_r does not necessarily exist. The situation for isotopically related symmetries is however different. Here a symmetry G_m of L which leads to I_r exists by assumption. The problem is then that of

constructing the image symmetry G^*_m of L* which also leads to I_r. By using the existence of the original symmetry as "integrability conditions" [30-V] it is then possible to prove that conditions (2.11.30) in this case always admit a solution. We therefore have the following

THEOREM 2.11.4: Given a class C^3 and regular Lagrangian $L(t,q,\dot{q})$ with an m-dimensional Lie symmetry group G_m of infinitesimal transformations of the first order and related first integrals $I_r(t,q,\dot{q})$, r=1,2...,m, an isotopically mapped Lagrangian $L^*(t,q,\dot{q})$ always admits a symmetry group G^*_m of infinitesimal transformations of the first order in the same number of independent parameters which leads to the same first integrals I_r.

In simpler terms, we can say that an isotopical image of an infinitesimal symmetry of the first order always exists. The problem whether Theorem 2.11.4 can be extended to higher order symmetries and up to a finite symmetry will not be considered. It essentially demands the extension of both Noether's theorem and its inverse, Theorems 2.11.2 and 2.11.3, to infinitesimal transformations of arbitrary order and then the study of the convergence of the emerging infinite series of type (2.11.17). For our needs it is sufficient to indicate that finite isotopic images exist.

Consider the two dimensional system

$$\ddot{r}_x + r = 0\,, \quad \ddot{r}_y + r_y = 0\,, \qquad (2.11.31)$$

which is manifestly form-invariant under the group SO(2) of rotations (2.11.13). The conventional Lagrangian for the representation of this system

$$L = \frac{1}{2}\left(\dot{r}_x^2 + \dot{r}_y^2\right) - \frac{1}{2}\left(r_x^2 + r_y^2\right), \qquad (2.11.32)$$

also exhibit this manifest symmetry. The infinitesimal transformations of the first order, from Eqs. (2.11.18b), are

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} \to \begin{pmatrix} r'_x \\ r'_y \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} r_x \\ r_y \end{pmatrix} w \qquad (2.11.33)$$

$$\delta^{(1)} r_x = r_y w\,, \quad \delta^{(1)} r_y = -r_x w\,, \quad w \in O_\varepsilon$$

and they induce the conservation of the angular momentum

$$I w = \frac{\partial L}{\partial \dot{\underline{r}}} \cdot \delta^{(1)} \underline{r} = \left(r_x \dot{r}_y - r_y \dot{r}_x\right) w, \qquad (2.11.34)$$

i.e., a physical conservation law. Consider now the simple isotopic mapping[70]

$$L \to L^* = \frac{1}{2}\left(\dot{r}_x^2 - \dot{r}_y^2\right) - \frac{1}{2}\left(r_x^2 - r_y^2\right) \qquad (2.11.35)$$

$$L^*_k(\underline{r}) = g_k{}^i L_i(\underline{r})\,, \qquad (2.11.35b)$$

$$(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; k = x, y. \qquad (2.11.35c)$$

The new Lagrangian (2.11.35a) is no longer invariant under the group SO(2). Nevertheless, the angular momentum (2.11.34) is conserved for the Lagrangian L* too. Our problem is that of identifying the new symmetry SO*(2) of L* which leads to conservation law (2.11.34). For this purpose we first inspect the Jacobi's equations (2.11.30a). Since Eqs. (2.11.31) are linear, they are equivalent to their equations of variations. And indeed we have the identity

$$J^*\left(\delta^{(1)}_* \underline{r}, \delta^{(1)}_* \dot{\underline{r}}\right) = L^*(\underline{r}, \dot{\underline{r}})\, w. \qquad (2.11.36)$$

This yields the transformations

$$\begin{pmatrix} r_x \\ r_y \end{pmatrix} \to \begin{pmatrix} r'_x \\ r'_y \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} + \begin{pmatrix} r_y \\ r_x \end{pmatrix} w, \qquad (2.11.37a)$$

$$\delta^{(1)}_* r_x = r_y w\,, \quad \delta^{(1)}_* r_y = r_x w\,, \qquad (2.11.37b)$$

for which

$$\frac{\partial L^*}{\partial \dot{\underline{r}}} \cdot \delta^{(1)}_* \underline{r} = \left(r_x \dot{r}_y - r_y \dot{r}_x\right) w = I w \qquad (2.11.38)$$

Thus, transformations (2.11.37) are the isotopic image of transformations (2.11.33). By iterating the procedure, in this case with simple calculations, we can identify the isotopic image of infinitesimal transformations of arbitrary order yielding the convergent series

$$\begin{pmatrix} x'_x \\ x'_y \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{1!} \\ \frac{1}{1!} & 0 \end{pmatrix} w + \begin{pmatrix} \frac{1}{2!} & 0 \\ 0 & \frac{1}{2!} \end{pmatrix} w^2 \right. \tag{2.11.39}$$

$$\left. + \begin{pmatrix} 0 & \frac{1}{3!} \\ \frac{1}{3!} & 0 \end{pmatrix} w^3 + \cdots \right] \begin{pmatrix} x_x \\ x_y \end{pmatrix} = \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} \begin{pmatrix} x_x \\ x_y \end{pmatrix}$$

which characterize the Lorentz group SO(1.1), rather than the group of rotations SO(2). Thus, SO(1.1) is the finite, exact, manifest, isotopic image of SO(2). And indeed, a simple inspection reveals that Lagrangian (2.11.35a) is manifestly invariant under SO(1.1) and that this symmetry leads to conservation law (2.11.38). This example of course falls within the class of original contemporaneous transformations (of pure space character).

For the case of a noncontemporaneous symmetry of pure time character we consider the harmonic oscillator

$$\ddot{q} + q = 0 \quad , m = k = 1 \tag{2.11.40a}$$

$$L = \frac{1}{2}\left(\dot{q}^2 - q^2\right) \quad , \tag{2.11.40b}$$

and the manifest symmetry under translations in time

$$t \to t' = t + w \quad , \quad w \in 0_\varepsilon \tag{2.11.41}$$

which leads to the energy conservation

$$I w = -\left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L\right) w = -H w, \tag{2.11.42}$$

$$q(t) \to \hat{q}(t') = \hat{q}(t-w) = q(t) + \hat{\delta}^{(1)} q, \tag{2.11.42b}$$

$$\hat{\delta}^{(1)} q = -\dot{q} w. \tag{2.11.42c}$$

Consider now isotopically mapped Lagrangian (2.10.27), i.e.

$$L^* = \frac{1}{6}\dot{q}^3 \cos t + \frac{1}{2} q^2 \dot{q}^2 \sin t - q^2 \dot{q} \cos t \ , \tag{2.11.43a}$$

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}} - \frac{\partial L^*}{\partial q} = \alpha(t,q,\dot{q})\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right), \tag{2.11.43b}$$

$$\alpha = \dot{q} \cos t + q \sin t, \tag{2.11.43c}$$

Eqs. (2.11.30) in this case yields the isotopic image of transformation (2.11.41)

$$\hat{\delta}^{(1)}_* q = -\frac{\dot{q}}{\alpha} w \ , \tag{2.11.44a}$$

$$\hat{\delta}^{(1)}_* t = -\frac{H + \frac{\partial L^*}{\partial \dot{q}} \frac{\dot{q}}{\alpha}}{\overset{L^*}{\dot{q}}} w \ , \tag{2.11.44b}$$

$$\hat{\delta}^{(1)}_* q = \delta^{(1)}_* q + \dot{q}\, \hat{\delta}^{(1)}_* t \neq -\dot{q}\, \hat{\delta}^{(1)}_* t, \tag{2.11.44c}$$

$$I w = -H w = \frac{\partial L^*}{\partial \dot{q}} \hat{\delta}^{(1)}_* q - \left(\frac{\partial L^*}{\partial \dot{q}} \dot{q} - L^*\right) \hat{\delta}^{(1)}_* t, \tag{2.11.44d}$$

which is clearly not a manifest symmetry of L*.

We therefore reach the conclusion that the mechanism of isotopic transformations can turn a manifest symmetry of the original Lagrangian into a nonmanifest symmetry of the isotopically mapped Lagrangian. In this occurrence lies the practical significance of the isotopic transformations for the computation of new first integrals. Owing to the full analytic equivalence of isotopically related Lagrangians, the mechanism can also turn a nonmanifest symmetry of the original Lagrangian into a manifest symmetry of the isotopically mapped Lagrangian. This yields the possibility of constructing new first integrals via Noether's theorem which would be of difficult if not impossible identification within the context of only one Lagrangian.

To illustrate this possibility consider the case of a particle with damping force, Eq. (2.11.11). The two independent first integrals are $I_1 = \dot{q}\exp(\gamma t)$, $I_2 = \dot{q} + \gamma q$. Consider now Lagrangian (2.11.12a). Its manifest symmetry under translations in space yields first integral I_1 but there is no other manifest symmetry to produce first integral I_2. The isotopically mapped Lagrangian (2.11.12b), however, is manifestly invariant under translations in time and this symmetry produces, via Noether's theorem, first integral I_2. But I_2 is also conserved for L. From Theorem 2.11.4 we therefore infer the existence of a nonmanifest symmetry of L which yields I_2. In this way the isotopic mapping has turned a nonmanifest

symmetry of L (which, as such, is of difficult identification) into a manifest symmetry of L* (which is of trivial identification). An equivalent situation occurs for I_1 and Lagrangians L and L*, although with a reverse role.

On practical grounds, the mechanism of isotopic mapping, when applicable (i.e., when the existence theorems of Section 2.10 are satisfied), is therefore significant for the identification of new first integrals. Since first integrals are useful, e.g., for the solution of the equations of motion by quadratures, the significance of this finding is selfevident.

On methodological grounds the mechanism of isotopic mapping has other intriguing consequences which can be expressed with the following

> THEOREM 2.11.5: Two isotopically related symmetry groups are not necessarily isomorphic. In particular the isotopic transformations do not necessarily preserve the compact or noncompact and space, time or space-time character of the original symmetry.

The above theorem can be proved with the examples given in this section. For instance, mapping SO(2) $\leftrightarrow$ SO(1.1) for the two-dimensional system (2.11.31) proves the first two statements. The last statement is illustrated by the isotopic transformation (2.11.43) (for the conservation of the physical total energy of system) which are no longer of pure time character. And indeed

transformations (2.11.43) incorporate a mixture of time and space transformation, Eq. (2.11.43c). The illustration of other parts of Theorem 2.11.5 is here left to the interested reader.

We therefore reach the following important result.

30-I, 34-II

COROLLARY 2.11.5A: The conservation laws of physical quantities (such as the total energy, total angular momentum, etc.) cannot be uniquely associated with symmetries of unique space, time or space-time character.

To illustrate this corollary it should be here recalled that the conservation law of a physical quantity, such as the total physical energy of a conservative system, is customarily associated with a symmetry of pure time character, translations in time. An implication of the methodology of the Inverse Problem is that of proving that such an association is not unique, in the sense that the conservation law of the same physical quantity, the energy, can also be associated with a symmetry of mixed space-time character. This is a consequence of the fact that all Lagrangians representing equivalent self-adjoint forms of the equations of motion are fully acceptable on analytic grounds and there is no need to restrict the analytic representation of conservative system to only the conventional structure L = T-V. This renders admissible generalized structures of the Lagrangians, as indicated in Section 2.10. This change in the Lagrangian structure implies the

existence of a corresponding change in the symmetries without affecting the physical conservation laws. One of the implications of the Inverse Problem is therefore Corollary 2.11.5A: a unique association of a physical conservation law with a symmetry does not necessarily exist because the latter depends on the selected Lagrangian. In particular, the character of the symmetry yielding a physical conservation law is also not unique. This implies that, say, the conservation law of the angular momentum can be derived with a mixture of space and time transformations as well as, of course, with the familiar rotations. The possible implications of these results for relativity considerations are predictable and they will be indicated in more details in Chapter 3.

In the transition to nonconservative systems the situation is different. For these systems there is in general no physical conservation laws and only first integrals. Thus, Corollary 2.11.5A does not necessarily apply for these systems. Nevertheless the mechanism of isotopic mapping is useful for the identification of first integrals via Noether's theorem. In this case the Lagrangian structure can only be of generalized type. This implies the existence of fewer manifest symmetries on a comparative basis with conservative systems. In turn this implies a reduced possibility of practical computation of first integrals via Noether's theorem if only one Lagrangian is considered. The use of the isotopically mapped Lagrangians, when applicable, increases the capability

of computing first integrals because a nonmanifest symmetry of the original Lagrangian can be turned into a manifest symmetry of its isotopic image. Even when a symmetry of the original Lagrangian is preserved by its isotopic image, Noether's identities yield different first integrals, trivially, from the different Lagrangian structure. Of course, whenever a new first integral is identified with these techniques, the fact whether such quantity is actually independent from previously known first integrals must be proved.

We now give an outline of the canonical formulation of the methodology of symmetries and first integrals within the context of the Inverse Problem.

As is well known, symmetries and first integrals can be formulated in an independent and algebraically effective way within the Hamiltonian formulations. Let $\underline{G}_m$ be the Lie algebra of the symmetry group G_m with generators X_i, $i = 1,2,\ldots,m$. A necessary and sufficient condition for $\underline{G}_m$ to be a symmetry algebra of the considered Hamiltonian H is that all its generators "commute " with H, i.e.,

$$\delta^{(1)} H = [X_i, H]\, w = 0. \qquad (2.11.45)$$

This is the algebraic counterpart of the Lagrangian rule (2.11.5). In particular, the generators X_i are closed under Poisson brackets according to the familiar rule

$$[X_i, X_j] = C_{ij}^{k} X_k, \qquad (2.11.46)$$

where the C's are the structure constants of $\underline{G}_m$.

But in the transition $G_m \rightarrow G_m^*$ the generators remain unchanged because they are the conserved quantities. This creates the rather peculiar need of constructing an algebra $\underline{G}_m^*$ which is generally nonisomorphic to $\underline{G}_m$ (Thoerem 2.11.5) in terms of the same generators.

An occurrence of this type, to my best knowledge, can only be realized by changing the product in an invertible, Lie algebra axiom preserving way, i.e., through the isotopic mapping of the classical realizations of the Lie algebra product (Section 2.10)

$$[A,B] = \frac{\partial A}{\partial a^\mu}\, \omega^{\mu\nu}\, \frac{\partial B}{\partial a^\nu} \rightarrow [A,B]^* = \frac{\partial A}{\partial a^\mu}\, \alpha^{\mu\nu}(a,a)\, \frac{\partial B}{\partial a^\nu}. \qquad (2.11.47)$$

The net effect is that the phase space image of the concept of isotopically related symmetry groups demands the second canonical image of the concept of isotopically mapped Lagrangians, of Eqs. (2.10.56), in which the Hamiltonian is unaffected (as it is the case of all the other generators).

The isotopic image of rule (2.11.45) then reads

$$\delta^{(1)}_{*} H = [X_i, H]^{*} w = 0 \qquad (2.11.48)$$

and the closure conditions of $\underline{G}_m$ now becomes

$$[X_i, X_j]^{*} = C^{*\,k}_{i\,j}\, X_k , \qquad (2.11.49)$$

i.e. they are expressed in terms of the same generators of the original rule (2.11.46) but now with respect to a generalized brackets and generally with $C^{*\,k}_{i\,j} \neq C^{\,k}_{i\,j}$ (Thoerem 2.11.5).

An example is here in order to illustrate these occurrences. Consider the three-dimensional harmonic oscillator with Lagrangian

$$L = \frac{1}{2}\left(\dot{r}_x^{\,2} + \dot{r}_y^{\,2} + \dot{r}_z^{\,2}\right) - \frac{1}{2}\left(r_x^{\,2} + r_y^{\,2} + r_z^{\,2}\right). \qquad (2.11.50)$$

Its manifest symmetry under the group of rotations SO(3) leads in phase space to the conservation of the angular momentum components and the canonical realization of the Lie algebra SO(3) with well known closure conditions

$$[M_x, M_y] = M_z , \; [M_y, M_z] = M_x , \; [M_z, M_x] = M_y , \qquad (2.11.51)$$

where, it should be recalled, the brackets are the conventional Poisson brackets and the ω-structure is given by

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{3\times 3} & \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & 0_{3\times 3} \end{pmatrix} \qquad (2.11.52)$$

Consider now the isotopically mapped Lagrangian

$$L^{*} = \frac{1}{2}\left(\dot{r}_x^{\,2} - \dot{r}_y^{\,2} + \dot{r}_z^{\,2}\right) - \frac{1}{2}\left(r_x^{\,2} - r_y^{\,2} + r_z^{\,2}\right). \qquad (2.11.53)$$

It is no longer invariant under SO(3). Nevertheless, the angular momentum components are still conserved. A generalization of the analysis for example (2.3.11) indicates that the isotopic image of SO(3) is now the Lorentz group SO(2.1). This demands the realization of the Lorentz group SO(2.1) in terms of the angular momentum components. An occurrence of this type can only be realized by changing the brackets of the Lie algebra product, i.e., by assuming the generalized Poisson brackets. The use of the methods of Section 2.10 then leads to the following realization of the SO(2.1) algebra[72]

$$[M_x, M_y]^{*} = M_z , \; [M_y, M_z]^{*} = -M_x , \; [M_z, M_x]^{*} = M_y , \qquad (2.11.54)$$

where the new brackets are characterized by the generalized (but still symplectic) structure

$$(\Omega^{\mu\nu}) = \begin{pmatrix} 0_{3\times 3} & \begin{pmatrix} +1 & 0 \\ 0 & -1 \\ & +1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & +1 \\ & -1 \end{pmatrix} & 0_{3\times 3} \end{pmatrix} \qquad (2.11.55)$$

namely, they are of simple generalized but nontrivial nature. The extension of this approach to other examples is here left to the interested reader. Notice that rules (2.11.46) and (2.11.48) are satisfied by therefore establishing the equivalence of the given canonical realizations of the SO(3) and SO(2.1) algebras for the characterization of the conservation of the angular momentum components.

A few comments are here in order. In essence, the conventional approach to the canonical realizations of Lie algebras is that in terms of the conventional Poisson brackets. If the approach is restricted to only this perspective, it might lead to the erroneous conclusion that the generators of the classical realizations of Lie algebras are unique (up to elements of the center of the enveloping associative algebra). The Inverse Problem focuses the attention of the fact that canonical realizations of Lie algebras can occur in terms of any (nowhere degenerate) Lie algebra axiom preserving brackets. The net effect is that the relationship between a Lie algebra

and its generators is no longer unique because bracket-dependent. This algebraic profile has a role for the problem of symmetries and first integrals. It essentially confirms that the Lie algebra which can be associated with first integrals is not unique because bracket-dependent. In the final analysis, it is precisely this algebraic degree of freedom which allows the mechanism of isotopic mapping to compute new first integral.

For completeness it should be indicated that the analysis of this section has been restricted to the use of isotopic transformations of a Lagrangian within a fixed system of (local) variables. The indicated methods can be extended to the case of isotopic transformations within different systems of coordinates, with particular reference to the use of the indirect representations of the Universality Theorem 2.8.1. This produces a further broadening of the methodology for the computation of new first integrals which is particularly significant, e.g., to study the possibility of identifying the missing six first integrals of the three-body Kepler problem. Studies along these lines are encouraged. Here let us only remark that all manifest symmetries and related first integrals of this (strictly conservative) system within the context of the conventional Lagrangian structure L = T-V have been identified. There is no doubt that if new, independent first integrals of the three-body Kepler problem can be identified, they are associated with nonmanifest

symmetries. It is because of this occurrence that the methodology presented in this section offers at least a new possibility of study. Indeed, our efforts are essentially centered in the study of first integrals associated with nonmanifest symmetries. The practical realization of the approach, however, demands a conceptual departure from conventional lines of study: the attempt to abandon the conventional Lagrangian L = T-V for the considered system and the search instead of a generalized Lagrangian structure for the representation of the same system now becomes the central objective.

Let us close the Newtonian part of this section with an outline of certain peculiar occurrences in the context of symmetries and first integrals which are rendered identifiable by the methodology of the Inverse Problem. This will result to be a valuable basis for the relativity analysis of Chapter 3.

In order to avoid possible misrepresentations of the results, the concept of a "physical quantity" for a nonconservative system should be recalled. Let us start from a conventional, essentially selfadjoint, conservative system

$$\left[\ddot{q}^k - f^k(q)\right]_{SA} = 0 , \qquad (2.11.56)$$

$$k = 1, 2, \dots, n .$$

The terms "physical quantities" for such a system have the

conventional meaning, such as total energy, total linear momentum and total angular momentum. In the implementation of system (2.11.56) with nonconservative forces, i.e. in the transition to a nonselfadjoint system of the type

$$\left\{\left[\ddot{q} - f^k(q)\right]_{SA} - F^k(t,q,\dot{q})\right]_{NSA} = 0 , \qquad (2.11.57)$$

the terms "physical quantities" denotes those of the associated selfadjoint system (2.11.56). The additional presence of nonconservative forces does not alter the definition of a physical quantity such as the energy, the linear momentum and the angular momentum.

One of the best examples is perhaps that of the harmonic oscillator

$$\left(m\ddot{r} + kr\right)_{SA} = 0 \qquad (2.11.58)$$

Its total energy is given by the familiar form E = T+V. In the transition to the damped oscillator

$$\left[\left(m\ddot{r} + kr\right)_{SA} + \gamma\dot{r}\right]_{NSA} = 0 \qquad (2.11.59)$$

the expression of the energy is unchanged and, thus, the quantity E = T+V is still the total physical energy of the nonconservative system. As a matter of fact, the problem is now turned into the study of the variation of E = T+V in time due to the presence of the damping force. For more

details see Appendix 2.B.

We are now equipped to outline the following occurrences.

Occurrence 1. When the physical total energy of a system is conserved, a Lagrangian for its analytic representation can violate the symmetry under translations in time. This occurrence is proved by Lagrangian (2.11.43a) which depends explicitly on time, nevertheless the system represented is strictly conservative (the harmonic oscillator).

Occurrence 2. When the total physical energy is not conserved, a Lagrangian can be invariant under translations in time. This occurrence is proved by Lagrangian (2.10.35) which does not depend explicitly on time, nevertheless the system represented is strictly nonconservative (the damped oscillator).

Occurrence 3. When the physical total linear momentum is conserved, a Lagrangian can violate the symmetry under translations in space. This occurrence is proved by the following system

$$\mu = 1 \begin{cases} \ddot{z}_x + \ddot{z}_y = 0 , & (2.11.60a) \\ \ddot{z}_x - \ddot{z}_y + 2z_x = 0 , & (2.11.60b) \end{cases}$$

where Eqs. (2.11.60a) directly expresses the conservation of the total linear momentum (= $\dot{z}_x + \dot{z}_y$). But a

Lagrangian for the representation of this system is given by

$$L = \frac{3}{2}\dot{z}_x^2 + \dot{z}_x\dot{z}_y + \frac{1}{2}z_y^2 - z_x^2 , \quad (2.11.61)$$

and it is not invariant under translations in space.

Occurrence 4. When the total physical linear momentum is not conserved, a Lagrangian can be invariant under translations in space. The simplest possible way of proving this occurrence is through the particle moving in a physical medium with a velocity-damping force. The physical linear momentum in this case is given by the mass multiplied by the velocity. This quantity, however, is not conserved. As a matter of fact, the problem in this case is the study of the decay in time of the linear momentum. The occurrence is then proved by the fact that this system admits a Lagrangian, Eq. (2.10.31), which is manifestly invariant under translations in space.

Occurrence 5. When the physical total angular momentum is conserved, a Lagrangian can violate the symmetry under the group of rotations. This occurrence is proved by Lagrangian (2.11.53) which is not invariant under SO(3), nevertheless the total angular momentum is conserved.

Occurrence 6. When the physical total angular momentum is not conserved, a Lagrangian can be invariant under rotations.

Consider the following generalization of the harmonic oscillator in two dimension

$$\left[\begin{pmatrix} \ddot{x}+x \\ \ddot{y}+y \end{pmatrix}_{SA} + g\begin{pmatrix} \frac{1}{2}x\dot{x}^2 + y\dot{x}\dot{y} - \frac{1}{2}x\dot{y}^2 + \frac{1}{2}x^3 \\ \frac{1}{2}y\dot{y}^2 + x\dot{x}\dot{y} - \frac{1}{2}y\dot{x}^2 + \frac{1}{2}y^3 \end{pmatrix}_{NSA}\right] = 0 . \qquad (2.11.62)$$

It characterises a highly nonconservative system with nonlinear velocity dependent damping forces. It is a simple exercise to prove that the physical angular momentum

$$M = x\dot{y} - y\dot{x} \qquad (2.11.63)$$

(which is the same as that of the associated selfadjoint system) is not conserved. The use of the methods of Section 2.3 then yields as a Lagrangian the structure

$$L = e^{g(x^2+y^2)}\,\frac{1}{2}\left[(\dot{x}^2+\dot{y}^2) - (x^2+y^2)\right], \qquad (2.11.64)$$

which is manifestly invariant under rotations.

Occurrence 7. The symmetry of a Lagrangian under the Galilei group does not necessarily imply the validity of the physical Galilei conservation laws (energy, linear momentum, etc.). This occurrence is basically a consequence of cases 2,4 and 6. The illustration of the case of lack of uniform motion of the center of mass but invariance under the Galilei boosts is left

to the interested reader. This cocurrence will be particularly useful in Chapter 3 to identify the mechanism of one class of breaking of the Galilei relativity.

The extension of the Newtonian methodology of this section to Field Theory is straightforward. Let us recall that this extensions can be essentially realized with the replacement of the time variable t with the space-time variables x^α and of the coordinates $q^k(t)$ with the fields $\phi^k(x^\alpha)$. This naive rule is sufficient to establish the guideline for the extension considered. Of course, some of the definitions have to be modified. For instance, the Newtonian symmetries of pure time, pure space and mixed space-time character, now become the field theoretical symmetries of pure space-time, pure internal and of mixed space-time and internal character. The explicit details are here left to the interested reader. We here restrict ourself to the identification through the use of the Inverse Problem of certain peculiar occurrences which are typical of Field Theory and, as such, cannot be effectively studied within a Newtonian context.

Occurrence 8. When the physical total charge of a system is conserved, a Lagrangian density is not necessarily invariant under gauge transformations. This occurrence can be simply proved with the free complex scalar field

$$\begin{cases} (\Box + m^2)\, e = 0 , \\ (\Box + m^2)\, \bar{e} = 0 . \end{cases} \qquad (2.11.65)$$

An admissible Lagrangian is given by Eqs. (2.10.38), i.e.,

$$L^* = \frac{1}{2}\left(\bar{e}^i_{,\mu}\,\bar{e}^{i\mu} - m^2\bar{e}^2\right) + \frac{1}{2}\left(e^i_{,\mu}\,e^{i\mu} - m^2 e^2\right), \quad (2.11.66)$$

and it is not invariant under the global (or local) gauge transformations

$$\begin{pmatrix} e \\ \bar{e} \end{pmatrix} \to \begin{pmatrix} e' \\ e' \end{pmatrix} = \begin{pmatrix} \cos we & i\sin we \\ -i\sin we & \cos we \end{pmatrix}\begin{pmatrix} e \\ \bar{e} \end{pmatrix} \quad (2.11.67)$$

$$w \in O_\varepsilon$$

Occurrence 9. The internal symmetries of a Lagrangian density which leads to the conservation of internal physical quantities (such as charge, isospin etc.) are not necessarily of compact type. This occurrence can also be proved with system (2.11.65). The charge current

$$j^\mu = ie\left(\bar{e}\,e^{i\mu} - \bar{e}^{i\mu}e\right) \quad (2.11.68)$$

is of course conserved for Lagrangian (2.11.66). It is here left as an exercise to the interested reader the prove that the symmetry of this Lagrangian whcih leads to the conservation of current (2.11.68) is the (noncompact) Lorentz group in 1+1 dimension, SO(1.1), with transformation laws

$$\begin{pmatrix} e \\ \bar{e} \end{pmatrix} \to \begin{pmatrix} e' \\ \bar{e}' \end{pmatrix} = \begin{pmatrix} \cosh we & +i\sinh we \\ -i\sinh we & \cosh we \end{pmatrix}\begin{pmatrix} e \\ \bar{e} \end{pmatrix} \quad (2.11.69)$$

This should be compared with the SO(2) gauge symmetry group of the equivalent Lagrangian (2.10.37).

Occurrence 9. The conservation laws of internal quantities are not necessarily derivable from symmetries of pure internal type. Consider the essentially selfadjoint extension of system (2.11.65)

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} (\Box + m^2)e \\ (\Box + m^2)e \end{pmatrix}_{SA} + e\begin{pmatrix} eA_\mu A^\mu \bar{e} + 2iA_\mu \bar{e}^{i\mu} \\ eA_\mu A^\mu e - 2iA_\mu e^{i\mu} \end{pmatrix}\right]_{SA} = 0. \quad (2.11.70)$$

The conserved physical charge current in this case is given by

$$j^\mu = ie\left(\bar{e}\,e^{i\mu} - \bar{e}^{i\mu}e\right) + 2eA^\mu \bar{e}e \quad (2.11.71)$$

Consider now Lagrangian (2.10.42) for the representation of this system, i.e.,

$$\mathcal{L}^* = e^{2ix_\alpha A^\alpha}\frac{1}{2}\left[\bar{e}^i_{,\mu}\,\bar{e}^{i\mu} - (m^2 - e^2A_\mu A^\mu)\bar{e}^2\right] \quad (2.11.72)$$
$$+ 2^{-2ix_\alpha A^\alpha}\frac{1}{2}\left[e^i_{,\mu}\,e^{i\mu} - (m^2 - e^2A_\mu A^\mu)\bar{e}^2\right].$$

The field theoretical extention of the methods of this section allow the computation of the (infinitesimal) symmetry of this Lagrangian which leads to the conservation of current (2.11.71). It results to be the superposition of the internal transformations

$$\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} \to \begin{pmatrix} \varphi' \\ \bar{\varphi}' \end{pmatrix} = \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} + w \begin{pmatrix} ie\,\bar{\varphi}\varphi & 2ie\,x_\alpha A^\alpha \\ -ie\,\varphi\varphi & -2ie\,x_\alpha A^\alpha \end{pmatrix} \tag{2.11.73}$$

and the space-time transformations

$$x^\mu \to x'^\mu = x^\mu + \delta^{(1)} x^\mu, \quad \delta^{(1)} x^\mu = \frac{2eA^\mu \bar{\varphi}\varphi}{\mathcal{L}^*} w, \tag{2.11.74a}$$

$$\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} \to \begin{pmatrix} \varphi' \\ \bar{\varphi}' \end{pmatrix} = \begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} \quad \begin{pmatrix} \varphi_{,\mu} \\ \bar{\varphi}_{,\mu} \end{pmatrix} \delta^{(1)} x^\mu, \tag{2.11.74b}$$

and this proves the occurrence. For details in the derivations of Eqs. (2.11.73) and (2.11.74) see paper V of reference.[30] Despite the fact that the electromagnetic field is external (i.e., the system considered is open according to the terminology of Section 2.9), the charge current is in this case conserved because electromagnetic interactions do not occur through charge-exchange processes (second quantization is here ignored).

Occurrence 10. The electromagnetic interactions are not necessarily represented in a Lagrangian density by the minimal coupling rule

$$\varphi_{,\mu} \to \varphi_{,\mu} - ieA^\mu, \quad \bar{\varphi}_{,\mu} \to \bar{\varphi}_{,\mu} + ieA^\mu. \tag{2.11.75}$$

And indeed, this rule is obeyed by the conventional gauge Lagrangian for the representation of system (2.11.70), i.e.,

$$\mathcal{L} = (\bar{\varphi}_{,\mu} + ieA_\mu)(\varphi^{,\mu} - ieA^\mu) - \mu^2 \bar{\varphi}\varphi. \tag{2.11.76}$$

Nevertheless, rule (2.11.76) does not hold for the chiral Lagrangian (2.11.72) for the representation of the same system. As we shall see, this is not an isolated occurrence. It generally occurs for all chiral Lagrangian representing a superposition of conventional electromagnetic couplings and local strong couplings not derivable from a potential. The purpose of pointing it at this time is to indicate that the loss of the minimal coupling rule may occur already for the case of pure electromagnetic interactions.

Occurrence 11. The symmetry of a Lagrangian density under global gauge transformations does not necessarily imply the conservation of the physical total charge. Consider the nonselfadjoint implementation of system (2.11.70)

$$\left\{ \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (\Box + \mu^2)\varphi \\ (\Box + \mu^2)\bar{\varphi} \end{pmatrix}_{SA} + e \begin{pmatrix} -eA_\mu A^\mu \bar{\varphi} + 2ieA_\mu \bar{\varphi}_{,\mu} \\ eA_\mu A^\mu \varphi - 2ieA_\mu \varphi_{,\mu} \end{pmatrix} \right]_{SA} \right.$$
$$\left. + g \begin{pmatrix} (\bar{\varphi}_{,\mu} + ieA^\mu \bar{\varphi})(\bar{\varphi}\varphi_{,\mu} + \bar{\varphi}_{,\mu}\varphi) \\ (\varphi_{,\mu} - ieA^\mu \varphi)(\varphi\bar{\varphi}_{,\mu} + \varphi_{,\mu}\bar{\varphi}) \end{pmatrix} \right\}_{NSA} = 0 \tag{2.11.77}$$

The additive couplings not derivable from a potential are here interpreted as a direct generalization of the Newtonian nonconservative forces in the spirit of Section 2.9. In particular, the physical charge current of the system is here assumed to be still given by quantity (2.11.71). This is supported by the fact that Eqs. (2.11.77) still represent the interaction of a complex scalar field with an external electromagnetic field. But, as the reader can verify with a simple inspection, the charge current (2.11.71) is no longer conserved for system (2.11.77) while a Lagrangian for its representation is given by

$$\mathcal{L}^{gen} = e^{g\bar{e}e}\left[(\bar{e}i_\mu + ieA_\mu)(ei^\mu - ieA^\mu) - \mu^2\bar{e}e\right], \tag{2.11.78}$$

and it is manifestly invariant under global gauge transformations. Of course this symmetry induces a conserved current (as it was the case for occurrences 2,4 and 6). However, according to our interpretation of this framework, this conserved current is not the physical current. On conceptual grounds this is fully equivalent to the nonconservation of the physical total energy of the damped oscillator while a Lagrangian for its representation is manifestly invariant under translations in time. Indeed, both the Newtonian and the field theoretical cases

are represented by systems with couplings not derivable from a potential and they both admit essentially generalized Lagrangian structures.

Occurrence 12. The Poincaré symmetry of a Lagrangian density does not necessarily imply the conservation of the physical energy-momentum and angular-momentum tensors. This is the expected occurrence corresponding to the Newtonian case 7 for the Galilei symmetry. It can be proved at this time with system (2.11.70) and Lagrangian density (2.11.76). Such Lagrangian is manifestly invariant under the Poincaré symmetry. Nevertheless, (as is known since some time) the canonical tensor associated with this symmetry

$$\theta^{\mu\nu}_{canon.} = -\left[\bar{e}i^\mu(ei^\nu - ieA^\nu e) + (\bar{e}i^\nu + ieA^\nu\bar{e})ei^\mu\right] + \alpha g^{\mu\nu} \tag{2.11.79}$$

is not the physical energy momentum tensor which is given instead by

$$\theta^{\mu\nu}_{phys} = -\left[(\bar{e}i^\mu + ieA^\mu\bar{e})(ei^\nu - ieA^\nu e) + (\bar{e}i^\nu + ieA^\nu\bar{e})(ei^\mu - ieA^\mu e)\right] + \alpha g^{\mu\nu} \tag{2.11.80}$$

and satisfied the rule

$$\partial_\mu \theta^{\mu\nu}_{phys} = \mathcal{F}^\nu_{Lorentz} = J^\alpha F_\alpha{}^\nu, \tag{2.11.81}$$

For explicit calculations see reference. 30.V The argument is here quite simple. Unlike the case of the charge (Occurrence 9), electromagnetic interactions do occur through exchanges of energy, linear momentum and angular momentum. Then the physical tensors must necessarily be nonconserved because the electromagnetic field in system (2.11.70) is external (i.e., the system is open). Rule (2.11.81) then verifies this fact.

In conclusion, the methodology of the Inverse Problem, when applied to symmetries and first integrals, produces a definite distinction between the mathematical algorithms considered and their possible physical content. In particular it suggests care in attributing conventional physical singificance to conventional symmetries. This mental attitude is a necessary prerequisite for the proper interpretation of the relativity analysis of Chapter 3.

2.12: USE OF THE INVERSE PROBLEM FOR THE CHARACTERIZATION OF OUR HADRONIC STRUCTURE MODEL

When inspected with the methodology of the Inverse Problem, one of the central characteristics of the unconstrained equations for our structure model of the π^0 is that they are nonselfadjoint, irrespective of whether in their primitive Newtonian form or in their classical field theoretical form. Indeed, Eqs. (1.4.7a) can be characterized as follows

$$\left[\left(m \ddot{\vec{r}}_k - \vec{f}_k \right)^{pos.}_{SA} - \vec{F}_k \right]^{\pi^0}_{NSA} = 0, \quad k = 1,2. \qquad (2.12.1)$$

Similarly, first-order equations (1.5.7a) can be written

$$\left\{ \left[\begin{pmatrix} -\hat{\vec{e}}'_\mu \gamma^\mu \\ \gamma^\mu \hat{\vec{e}}'_\mu \end{pmatrix} + m \begin{pmatrix} \hat{\vec{e}} \\ \hat{e} \end{pmatrix} + \begin{pmatrix} \vec{f}_{\hat{\vec{e}}} \\ f_{\hat{e}} \end{pmatrix}^{elm} \right]^{pos}_{SA} + \begin{pmatrix} \vec{F}_{\hat{\vec{e}}} \\ F_{\hat{e}} \end{pmatrix}^{strong} \right\}^{\pi^0}_{NSA} = 0, \qquad (2.12.2)$$

and, if second-order equations are desired, one can write

$$\left\{ \left[(\Box + m^2) \phi_k - f_k^{elm} \right]^{pos}_{SA} + F_k^{strong} \right\}^{\pi^0}_{NSA} = 0, \qquad (2.12.3)$$

$$\{\phi_k\} = (\vec{e}, e), \quad k = 1,2.$$

We can thus state that the classical unconstrained equations for the eleton and the antieletons are nonselfadjoint by assumption. It is here tacitly implied that this central property persists in the transition from models (2.11.1), 2.12.2) and (2.12.3) to their corresponding many-body extensions.

This property should be compared with the corresponding property of currently available structure models of the hadron. A simple inspection soon reveals that the equations for quarks and antiquarks are selfadjoint, trivially, because their couplings are assumed as derivable from a potential. Similarly for the case of partons.

A better identification of the terms "nonselfadjoint" and "selfadjoint" is here needed. Indeed, we have learned in this chapter that the equations of, say, the conventional conservative Newtonian systems can be written in an equivalent nonselfadjoint form. Viceversa, nonselfadjoint equations a be written in an equivalent selfadjoint form. In turn, these occurrences might hint at a variational equivalence of the equations for the eletons-antieletons with those of the quarks-antiquarks. To properly differentiate these two approaches to the hadronic structure we introduce the following

DEFINITION 2.12.1: The classical unconstrained motion of eletons and antieletons can be characterized by semilinear systems of differential equations provided that their couplings are nonselfadjoint, i.e. violate Theorem 2.5.2.

At the Newtonian (field theoretical) level the term "coupling" is here intended to represent the total acting force. This does not prohibit that a number of terms of this total force do satisfy Theorem 2.5.2. Thus, Definition 2.12.1 is consistent with structures (2.12.1), (2.12.2) and (2.12.3).

Restated in more common language, Definition 2.12.1 implies that the total force characterizing an eletonic motion is made up of terms which are derivable from a potential (potential density) and terms which are not. The primitive Newtonian idea of the concept of eleton is therefore that of a genuine nonconservative system.

Definition 2.12.1 also avoids possible ambiguities with the quark model. Indeed, the total acting force is in this case derivable from a potential (potential density), i.e., satisfies Theorem 2.5.2. The primitive Newtonian idea of the concept of quark is therefore that of a genuine conservative system.

Notice that the total acting force can be expressed in terms of the implicit functions of the system considered. The formulation of Definition 2.12.1 in terms of implicit functions, however, could give rise to ambiguities. To see this point, consider the conservative system

$$\left[\mu_k \ddot{\underset{\sim}{r}}_k - \underset{\sim}{f}_k(r) \right]_{SA} = 0 , \quad k = 1,2,\dots \tag{2.12.4}$$

$$\underset{\sim}{f}_k = - \frac{\partial V(r)}{\partial \underset{\sim}{r}^k} , \quad \frac{\partial^2 V}{\partial \underset{\sim}{r}^i \partial \underset{\sim}{r}^i} \neq 0 , \quad \mu_1 \neq \mu_2 \neq \mu_3 \neq \dots$$

Perform the transition to the equivalent form in terms of the implicit functions

$$\ddot{\underset{\sim}{r}}_k - \frac{1}{\mu_k} \underset{\sim}{f}_k(r) = 0 \tag{2.12.5}$$

Conditions of selfadjointness (2.3.6) for the latter system became

$$\frac{1}{m_i}\frac{\partial^2 V}{\partial x^i_m \partial x^i_m} - \frac{1}{m_j}\frac{\partial^2 V}{\partial x^j_m \partial x^i_m} = 0 \qquad (2.12.6)$$

(no summation on i,j), $i,j = 1,2,\ldots$

and thus, they are violated whenever the masses are different. This illustrates a case whereby the acting force of the system are selfadjoint (i.e., satisfy Theorem 2.5.2) while the corresponding implicit functions are not.

The above remarks also indicate another reason why we have assumed the equations of motion as derived from Newton's second law, e.g., Eqs. (2.12.1), to be our fundamental physical quantities. Definition 2.12.1 is therefore intended to refer specifically to this form.

An equivalent characterization of the unconstrained equations for the eletons can be given in terms of the classification of Newtonian systems of Section 2.8.

> DEFINITION 2.12.2: The classical unconstrained motion of eletons and antieletons can be characterized by quasi-linear systems of ordinary or partial differential equations, provided that they are either nonessentially or essentially nonselfadjoint.

The differentiation with the corresponding systems representing quarks and antiquarks can then be seen from the fact that, also in the language of Section 2.8, they are essentially selfadjoint.

Definition 2.12.2 also provides a classification of admissible models. Indeed, the terms "nonessentially selfadjoint" stands to indicate the assumption that the system admits an equivalent selfadjoint form within the considered set of variables (genotopic transformation) and we shall write for Eqs. (2.12.1)

$$\left\{ h_{ka}{}^{jb} \left[\left(m \ddot{x}_{jb} - f_{jb} \right)^{pos}_{SA} - F_{jb} \right]^{\pi^0}_{NSA} \right\}_{SA} = 0. \qquad (2.12.7)$$

Similarly, for Eqs. (2.12.2) and (2.12.3) we have

$$\left\{ \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{11} \end{pmatrix} \left[\left(\begin{pmatrix} -\hat{\bar{e}}^i_\mu \gamma^\mu \\ \gamma^\mu \hat{e}^i_\mu \end{pmatrix} + m \begin{pmatrix} \hat{\bar{e}} \\ \hat{e} \end{pmatrix} + \begin{pmatrix} f_{\hat{\bar{e}}} \\ f_{\hat{e}} \end{pmatrix}^{elm} \right)^{pos.}_{SA} + \begin{pmatrix} F_{\hat{\bar{e}}} \\ F_{\hat{e}} \end{pmatrix}^{strong} \right]^{\pi^0}_{NSA} \right\}_{SA} = 0. \qquad (2.12.8a)$$

$$\left\{ h_k{}^i \left[\left(\Box + m^2 \right) \phi_i - f^{elm}_i \right]^{pos.}_{SA} - F^{strong}_i \right]^{\pi^0}_{NSA} \Bigg\}_{SA} = 0 \qquad (2.12.8b)$$

It is therefore implicit for Definition 2.12.2 that a Lagrangian for the representation of the nonessentially non-selfadjoint systems within the given set of variables exists.

Theorem 2.3.3 on the computation of a Lagrangian then yields for Eqs. (2.11.7) the structure (we here ignore for simplicity the formulation to a star-shaped region)

$$L_{tot}^{gen} = \sum_{k=1}^{n} \sum_{\substack{a= \\ x,y,z}} L_{I}^{(ka)} L_{free}^{(ka)} + L_{II} \qquad (2.12.9)$$

$$= \frac{1}{2}\left(\dot{z}^{ia} G_{ia\,jb} \dot{z}^{ib} + 2\dot{z}^{ia} F_{ia\,jb} z^{ib} + z^{ia} E_{ia\,jb} z^{ib} \right),$$

which is an essentially generalized Lagrangian in the sense that it does not admit an isotopic transformation to a conventional Lagrangian (i.e., the sum of a free term and an interaction term).

Theorem 2.5.11 and 2.5.4 then yield for Eqs. (2.12.8) field theoretical structures of the type

$$\mathcal{L}_{tot}^{gen} = \sum_{k=1}^{n} \sum_{a=0}^{3} \mathcal{L}_{I}^{(k\mu)} \mathcal{L}_{free}^{(k\mu)} + \mathcal{L}_{II}, \qquad (2.12.10)$$

which are also essentially generalized.

The reader should be aware that, from the results of Appendix 2.C., the above Lagrangians are essentially generalized in the sense that none of the equivalence transformations (2.B.13) and (2.B.14) exists.

We reach in this way the following third possible characterization of our hadronic model

DEFINITION 2.12.3: The classical unconstrained motion of eletons and antieletons can be characterized in terms of generalized Lagrangians or Lagrangian densities, provided that their multiplicative terms to the free term are essential.

Alternatively, we can say that eletons and antieletons can be characterized in terms of essentially chiral Lagrangians. This gives additional elements of differentiation of our model of hadronic structure with currently avialable models. Indeed, to the best of my knowledge, currently available models are based on the use of a conventional Lagrangian structure generally of gauge type, while our model is of chiral type. For a preliminary analysis of the implications of this differentiation see Appendix 2.C.

Definition 2.12.2 also indicates the existence of a hierarchy of models of increasing methodological complexity. Besides the nonessentially nonselfadjoint models there also exist the essentially nonselfadjoint models. For these latter models a Lagrangian for the representation of the systems within the original set of variables does not exist. A Lagrangian can still be computed from the Universality Theorem 2.3.1. But this time it demands the use of derivative-dependent transformations which profoundly affect the physical content of the system considered. As such, these indirect analytic representations should be considered with care from a physical profile.

We now conclude this section with an analysis of the explicit functional dependence of the admissible couplings of Definitions 2.12.1, 2.12.2 and 2.12.3.

Again, Newtonian Mechanics proves to play a crucial conceptual role. Indeed, a task of this nature can be conducted at the Newtonian level with the backing of an extended knowledge of nonconservative forces. The extension of the results to Field Theory will then only demand a technical implementation.

Consider a system of Newtonian particles of mass m_k, $k=1,2,\ldots$ in a three-dimensional Euclidean space with Cartesian coordinates $\underset{\sim}{x}^k$. Their free motion is trivially represented by the equations

$$m_k \ddot{x}_{ka} = 0 \ , \ k=1,2,\ldots \ , \ a=x,y,z. \quad (2.12.11)$$

The most general local, Newtonian form of coupling these particles is given by the superposition of the following three classes of couplings.

(I) Time-dependent generally nonlinear couplings in the coordinates, and we shall write

$$m_k \ddot{x}_{ka} - f_{ka}(t, \underset{\sim}{x}) = 0. \quad (2.12.12)$$

(II) Time-dependent generally nonlinear couplings in the vecolities, and we shall write

$$m_k \ddot{x}_{ka} - F_{ka}(t, \underset{\sim}{\dot{x}}) = 0, \quad (2.12.13)$$

(III) Time, coordinate and velocity dependent linear couplings in the acceleration and we shall write

$$A_{ka\,jb}(t, \underset{\sim}{x}, \underset{\sim}{\dot{x}})\, \ddot{x}^{jb} = 0. \quad (2.12.14)$$

By matching couplings, I, II, and III we then reach the system

$$A_{ka\,jb}(t, \underset{\sim}{x}, \underset{\sim}{\dot{x}})\, \ddot{x}^{jb} + B_{ka}(t, \underset{\sim}{x}, \underset{\sim}{\dot{x}}) = 0 \quad (2.12.15)$$

i.e., the most general form of ordinary differential equations linear in the second order derivatives which we have called of Newtonian, Lagrangian or quasilinear type.

An example is here useful to illustrate these Newtonian concepts. Consider the system of coupled oscillators. The first way of coupling oscillators is in terms of linear couplings in the coordinates which do not depend explicitly on time, i.e.,

$$m_k \ddot{x}_{ka} + a_{ka\,jb}\, x^{jb} = 0 \quad (2.12.16)$$

This system, however, is physically meaningful only for infinitesimal periods of time to render damping effects ignorable.

To remove this restrictive character, one then considers the extension of models (2.12.16) with additive linear couplings in the velocity (damping forces) which also do not depend expicitly on time, i.e.

$$m_K \ddot{z}_{Ka} + b_{Ka\,jb}\,\dot{z}^{jb} + a_{Ka\,jb}\,z^{jb} = 0. \quad (2.12.17)$$

A peculiarity of coupled oscillator is however that of exhibiting the presence also of acceleration couplings. This demands the further extension to the known linear homoneous systems with constant coefficients, i.e.

$$c_{Ka\,jb}\,\ddot{z}^{jb} + b_{Ka\,jb}\,\dot{z}^{jb} + a_{Ka\,jb}\,z^{jb} = 0. \quad (2.12.18)$$

Physical observation then indicates that this system is still restrictive because applicable only for infinitesimal oscillations while the representation of the damping forces with terms linear in the velocities is only a crude approximation of the energy dissipation process. To represent more closely the physical reality, one then removes the linearity character of the coordinates and velocity couplings. The additional dependence of these couplings on time may be demanded by the case at hand. One reaches in this way couplings I and II. Additional physical observation then indicates that the acceleration couplings too may exhibit, in general, an arbitrary dependence on time, coordinates and velocities. One reaches in this way couplings III.

A nonlinear dependence on the acceleration is generally excluded because contrary to the basic postulates of the Newtonian Mechanics (for a review of this latter case, see Volume I of reference[34]). As a result, the general quasilinear system (2.12.15) is representative of the most general way of couplings oscillators.

The above remarks were referred to the physical profile of the problem considered. The mathematical profile is different. Under the assumption that system (2.12.15) is of at least class C^1 and regular, the Theorem on Implicit Functions is applicable yielding the equivalent representation of the system in terms of the (unique) implicit functions

$$\ddot{z}^{Ka} - F^{Ka}(t, z_m, \dot{z}_m) = 0 \quad (2.12.19a)$$

$$F^{Ka} = -A^{Ka\,jb} B_{jb}\,, \quad (A^{Ka\,jb}) = (A_{Ka\,jb})^{-1} \quad (2.12.19b)$$

Thus, from the viewpoint of the theory of ordinary differential equations, the acceleration couplings are inessential to represent the motion under the indicated continuity and regularity conditions.

The analytic profile is still different. Eqs. (2.12.19) are generally nonselfadjoint. As a result, a Lagrangian for their direct analytic representation does not exist. To attempt an analytic representation within the given set of (local) variables, one is then forced into the study of

the possible existence of equivalent selfadjoint forms according to the concept of indirect representation

$$\frac{d}{dt}\frac{\partial L^{gen}_{tot}}{\partial \dot{z}^{ka}} - \frac{\partial L^{gen}_{tot}}{\partial z^{ka}} \qquad (2.12.20)$$

$$= \left\{ h_{kajb}(t, z_m, \dot{z}_m) \left[\ddot{z}^{jb} - F^{jb}(t, z_m, \dot{z}_m) \right]_{NSA} \right\}_{SA}$$

This not necessarily recovers the general quasilinear form (2.12.15) as constructed on grounds of physical considerations, but instead one of its equivalent selfadjoint forms (when it exists).

The net effect is that a necessary condition for the existence of an analytic representation for systems with couplings not derivable from a potential is that the equations of motion possess acceleration couplings. From Theorem 2.3.3 it then follows that the essential multiplicative terms of the generalized Lagrangian structures of Definition 2.12.3 are representative precisely of these acceleration couplings. In other words, all the $3n+1$ interaction terms of generalized Lagrangian (2.12.9) can be interpreted as being representative of Newtonian couplings, and not only the additive term. Specifically, the $3n$ multiplicative terms to the free terms are representative of the couplings of class III, while the remaining additive term is representative of the couplings of class I and II.

We are now equipped to identify the admissible functional dependence of the couplings for our hadronic structure model.

DEFINITION 2.12.4: The classical unconstrained motion of eletons and antieletons can be characterized by a superposition of couplings of class I and II provided that they are nonselfadjoint.

For completeness it should be here indicated that couplings of type I can also be nonselfadjoint, as illustrated by the examples

$$\begin{pmatrix} m_1 \ddot{r}_{m1} + r_{m1} + r_{m2} \\ m_2 \ddot{r}_{m2} + r_{m2} \end{pmatrix}_{NSA} = 0 \qquad (2.12.21)$$

Notice that Definition 1.12.4 implies that the maximal functional dependence of the couplings is in all the local variables with the exception of their second order derivatives. The transition form Eqs. (2.11.1) to the corresponding Lagrangian form (2.12.9) indicates that, under the assumption that the systems are nonessentially nonselfadjoint, they can be equivalently expressed in terms of equations with acceleration couplings.

We reach in this way the last characterization of our hadronic structure model according to the following

DEFINITION 2.12.5: Under the conditions that the integrability conditions for the existence of a Lagrangian

within the considered system of local variables are satisfied, the classical unconstrained motion of eletons and antieletons can be characterized by a self-adjoint collection of couplings I, II and III provided that the acceleration couplings are essential.

For the one-dimensional case (e.g. the radial case for one particle) the above characterization can be written

$$\left[A(t,r,\dot{r})\,\ddot{r} + B(t,r,\dot{r})\right]_{NSA} = 0, \qquad (2.12.$$

$$A, B \in C^{1}(R), \quad A(R) \neq 0,$$

where now the acceleration coupling is in its simplest possible form, i.e. a factor function A to the acceleration. This coupling is essential when the equivalent system

$$\ddot{r} - F(t,r,\dot{r}) = 0, \quad F = -B/A, \qquad (2.12.23)$$

is nonselfadjoint. The generalization to the n-dimensional case is then trivial.

Notice that in the transition to Field Theory, Defi ion 1.12.5 implies the existence of a general quasilinear form

$$A^{\mu\nu}_{Ki}(x^{\alpha}, \phi^{j}, \phi^{j}_{\alpha})\,\phi^{ij}_{\mu\nu} + B_{K}(x^{\alpha}, \phi^{j}, \phi^{j}_{\alpha}) = 0, \qquad (2.12.24)$$

which is not necessarily reducible to the conventional form

$$A_{Ki}(x^{\alpha}, \phi^{j}, \phi^{j}_{\alpha})\left[(\Box + \mu^{2})\,\phi^{i} - J^{i}(x^{\alpha}, \phi^{j}, \phi^{j}_{\alpha})\right] = 0 \qquad (2.12.25)$$

Notice also that Definitions 2.12.4 and 2.12.5 produce two characterizations of the eleton field equations which are generally different although both acceptable. The former may not admit a Lagrangian within the given set of local variables while this condition is always verified for the latter.

This concludes the characterization of our hadronic model of structure within the context of the methodology of the Inverse Problem. From here on we shall use any of the given definitions depending on the profile under consideration.

To put the used methodological context in its proper perspective, the lack of universality of the analytic equations without external terms for the representation of the considered hadronic model in its original variables should be here stressed and compared with the direct universality for the representation of the same system through analytic equations with external terms (Volume II).

This occurrence should not be surprising. The former equations were ultimately conceived for systems with couplings derivable from a potential and the use of the conventional Lagrangian structure L = T-V. The methodology of the Inverse Problem does provide a considerable broadening of the applicability of the equations considered through the use of generalized Lagrangian structures. Nevertheless, they remain limited in their direct representational capabilities. The latter equations, on

the contrary, were conceived for aribtrary couplings and, as such, they enjoy a natural direct universality for the system considered.

However, in line with our dual methodological approach (Appendices 1.A and 1.B) the joint use of analytic equations without and with external terms for the representation of the same system is recommendable, particularly for relativity considerations, as we shall see in the next chapter.

APPENDIX 2.A:

AN ERRONEOUS FORM OF LAGRANGE'S EQUATIONS

OFTEN USED IN THE CURRENT LITERATURE OF FIELD THEORY

In this appendix we would like to point out that the form of Lagrange's equations for continuous systems customarily assumed in the current literature of field theory

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi^{k;}{}_{\mu}} - \frac{\partial \mathcal{L}}{\partial \phi^k} = 0 , \qquad (2.A.1a)$$

$$\partial_\mu = \partial/\partial x^\mu , \quad \phi^{k;}{}_{\mu} = \partial \phi^k / \partial x^\mu , \qquad (2.A.1b)$$

while practically usable for couplings derivable from a potential despite potential confusion, IT IS FUNDAMENTALLY ERRONEOUS . The correct form is given instead by

$$d_\mu \frac{\partial \mathcal{L}}{\partial \phi^{k;}{}_{\mu}} - \frac{\partial \mathcal{L}}{\partial \phi^k} \qquad (2.A.2a)$$

$$= \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}{}_{\mu} \partial \phi^{i;}{}_{\nu}} \phi^{i;}{}_{\mu\nu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}{}_{\mu} \partial \phi^{i}} \phi^{i;}{}_{\mu} + \frac{\partial^2 \mathcal{L}}{\partial \phi^{k;}{}_{\mu} \partial x^\mu} - \frac{\partial \mathcal{L}}{\partial \phi^k}$$

$$= \frac{1}{2} \left(\mathcal{L}^{;\mu;\nu}_{\;\;k\;\;i} + \mathcal{L}^{;\nu;\mu}_{\;\;k\;\;i} \right) \phi^{i;}{}_{\mu\nu} + \mathcal{L}^{;\mu;}_{\;\;k\;\;i} \phi^{i;}{}_{\mu} + \mathcal{L}^{;\mu}_{\;\;k\;;\mu} - \mathcal{L}^{;}_{\;k} = 0 ,$$

$$\mathcal{L}^{;\mu}_{\;\;k} = \frac{\partial \mathcal{L}}{\partial \phi^{k;}{}_{\mu}} , \quad \mathcal{L}^{;}_{\;k} = \frac{\partial \mathcal{L}}{\partial \phi^k} , \quad \mathcal{L}^{;}_{\;\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} , \qquad (2.A.2b)$$

as always used in the literature of the Calculus of Variations for multiple integral path functionals. Similarly, differential

conservation laws are generally written in the literature of Field Theory in the form

$$\partial_\mu J^\mu (x^\alpha, \phi^j, \phi^j_{;\alpha}) = 0 \qquad (2.A.3)$$

while the correct form is that used in the Calculus of Variations, i.e.

$$d_\mu J^\mu = \frac{1}{2}\left(J^{\mu;\nu}_{\ \ k} + J^{\nu;\mu}_{\ \ k}\right)\phi^{k;}_{\mu\nu} \qquad (2.A.4)$$
$$+ J^{\mu;}_{\ \ k}\,\phi^{k;}_{\ \ \mu} + J^{\mu\ ;}_{\ \ \mu} = 0$$

Oddly, few references in the physical literature use the form (2.A.2) and (2.A.4), among which is the celebrated paper by E. L. Hill of 1951[76] on symmetries and conservation laws, and regrettably the use of forms (2.A.1) and (2.A.3) has prevailed.

The fact that Eqs. (2.A.1) are erroneous can be trivially seen from the fact that their Newtonian limit would read

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{x}^k} - \frac{\partial L}{\partial x^k} = 0. \qquad (2.A.5)$$

From the viewpoint of the Inverse Problem the erroneous nature of Eqs. (2.A.1) can be seen from the fact that they are generally nonselfadjoint, while Lagrange's equations are known to be selfadjoint since the time of C. G. Jacobi (1837). More recently, the erroneous nature of Eqs. (2.A.1) can be unequivocally established by using the cohomology or the functional approach. See the references quoted in this respect in footnote[53].

From the viewpoint of the Calculus of Variations the erroneous nature of Eqs. (2.A.1) results from the fact that the variation can have only an implicit dependence on the Minkowski coordinates. In this case the use of the partial derivatives to characterize the variation of the "velocities" is erroneous* and a total derivative must be used. This implies the total derivative of Eqs. (2.A.2). The reader is here urged to inspect the easily identifiable literature on the Calculus of Variations for multiple integrals path functionals.

This peculiar situation should not be underestimated. Besides the breakdown of the fundamental property of self-adjointness of Lagrange's equations and the consequential invalidation of the entire methodology of the Inverse Problem, the use of Eqs. (2.A.1) simply implies the computation of an erroneous form of the field equations for all Lagrangians with a generalized structure although, admittedly, Eqs. (2.A.1) and (2.A.2) produce formally equivalent results for linear equations.

Owing to this peculiar situation it is here useful to inspect the correct form of the Lagrangian representation for the conventional Dirac equation.

Let $e(x)$ and $\bar{e}(x) = (e^\dagger(x))\gamma^0$ be the Dirac field and its adjoint. Let $\mathcal{L}(x^\alpha, e, \bar{e}, e^{;}_{\ \mu}, \bar{e}^{;}_{\ \mu})$ be a Lagrangian (density) in these fields. The correct form of the totally degenerate Lagrange's equations (Section 2.5) are given in this case by

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* Equivalently, Lagrange's equations in the form (2.A.1) with partial derivatives ARE GENERALLY NONDERIVABLE FROM A VARIATIONAL PRINCIPLE.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial e^i_\mu} - \frac{\partial \mathcal{L}}{\partial e} = \frac{\partial^2 \mathcal{L}}{\partial e \, \partial e^i_\mu} e^i_\mu + \bar{e}^i_\mu \frac{\partial^2 \mathcal{L}}{\partial \bar{e} \, \partial e^i_\mu} \qquad (2.A.6a)$$

$$+ \partial_\mu \frac{\partial \mathcal{L}}{\partial e^i_\mu} - \frac{\partial \mathcal{L}}{\partial e} = 0 \,,$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \bar{e}^i_\mu} - \frac{\partial \mathcal{L}}{\partial \bar{e}} = \bar{e}^i_\mu \frac{\partial^2 \mathcal{L}}{\partial \bar{e} \partial \bar{e}^i_\mu} + \frac{\partial^2 \mathcal{L}}{\partial e \, \partial \bar{e}^i_\mu} e^i_\mu \qquad (2.A.6b)$$

$$+ \partial_\mu \frac{\partial \mathcal{L}}{\partial \bar{e}^i_\mu} - \frac{\partial \mathcal{L}}{\partial \bar{e}} = 0 \,,$$

where the row and column nature of the fields has been taken into account. This should be compared with the form currently used

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial e^i_\mu} - \frac{\partial \mathcal{L}}{\partial e} = 0 \qquad (2.A.7a)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \bar{e}^i_\mu} - \frac{\partial \mathcal{L}}{\partial \bar{e}} = 0 \qquad (2.A.7b)$$

Suppose now that the Dirac fields are free. Then, the Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\bar{e}^i_\mu \gamma^\mu e - \bar{e} \gamma^\mu e^i_\mu \right) - m \bar{e} e$$

$$= \frac{1}{2} \left[\left(\bar{e}^i_\mu \gamma^\mu - m \bar{e} \right) e - \bar{e} \left(\gamma^\mu e^i_\mu + m e \right) \right] \qquad (2.A.8)$$

$$= \mathcal{L}_{free}$$

yields the analytic representation

$$\begin{pmatrix} \partial_\mu \frac{\partial \mathcal{L}_{free}}{\partial e^i_\mu} - \frac{\partial \mathcal{L}_{free}}{\partial e} \\ \partial_\mu \frac{\partial \mathcal{L}_{free}}{\partial \bar{e}^i_\mu} - \frac{\partial \mathcal{L}_{free}}{\partial \bar{e}} \end{pmatrix}_{SA} \qquad (2.A.9)$$

$$= \begin{pmatrix} \bar{e}^i_\mu \frac{\partial^2 \mathcal{L}_{free}}{\partial \bar{e} \, \partial e^i_\mu} - \frac{\partial \mathcal{L}_{free}}{\partial e} \\ \frac{\partial^2 \mathcal{L}}{\partial e \, \partial \bar{e}^i_\mu} e^i_\mu - \frac{\partial \mathcal{L}_{free}}{\partial \bar{e}} \end{pmatrix}_{SA}$$

$$= \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \gamma^\mu e^i_\mu + m e \\ - \bar{e}^i_\mu \gamma^\mu + m \bar{e} \end{pmatrix} \right]_{SA} = \begin{pmatrix} F_e(\bar{e}, \bar{e}^i_\mu) \\ F_{\bar{e}}(e, e^i_\mu) \end{pmatrix}_{SA} = 0$$

which recovers the field equations in an inverted ordering as it is typical for all gauge invariant Lagrangians.

To see that Lagrangian (2.A.8) is the only form admissible for the considered type of analytic representation, consider the last identities of Eqs. (2.A.9). The use of Theorem 2.5.10 then yields the Lagrangian

$$\mathcal{L}_{free} = - \left[\int_0^1 d\tau \, F_e(\tau \bar{e}, \tau \bar{e}^i_\mu) \right] e - \bar{e} \left[\int_0^1 d\tau \, F_{\bar{e}}(\tau e, \tau e^i_\mu) \right]$$

$$= - \left[\int_0^1 d\tau \left(- \tau \bar{e}^i_\mu \gamma^\mu + \tau m \bar{e} \right) \right] e - \bar{e} \left[\int_0^1 d\tau \left(\tau \gamma^\mu e^i_\mu + \tau m e \right) \right] \qquad (2.A.10)$$

$$= \frac{1}{2} \left(\bar{e}^i_\mu \gamma^\mu e - \bar{e} \gamma^\mu e^i_\mu \right) - m \bar{e} e$$

which coincides with Eq. (2.A.8).

Consider now Dirac's equation for the positronium in their selfadjoint form, Eqs (1.5.1), i.e.

$$\left[\begin{pmatrix} -\bar{e}\overleftarrow{i\partial_\mu}\gamma^\mu \\ e\, i\partial_\mu \gamma_\mu \end{pmatrix} + m\begin{pmatrix} \bar{e} \\ e \end{pmatrix} + e\begin{pmatrix} i\bar{e}\gamma^\mu A_\mu \\ \gamma^\mu A_\mu e \end{pmatrix}\right]_{SA} = 0 \quad (2.A.11)$$

The use of Theorem 2.5.10 in this case yields the Lagrangian

$$\mathcal{L}^{conv}_{tot} = \mathcal{L}_{free} + \mathcal{L}^{elm}_{int} \quad (2.A.12)$$

$$= \frac{1}{2}\left[(\bar{e}\overleftarrow{i\partial_\mu} + ieA_\mu\bar{e})\gamma^\mu e - \bar{e}\gamma^\mu(e\overrightarrow{i\partial_\mu} + ieA_\mu e)\right] - m\bar{e}e$$

which is unique for the direct analytic representation of system (2.A.11) in the ordering $(\bar{e}\ ,\ e\ )$, i.e., without possible isotopic transformations (Section 2.10).

APPENDIX 2. B:

THE DICHOTOMY OF CANONICAL VERSUS PHYSICAL QUANTITIES FOR NONCONSERVATIVE SYSTEMS.

Surprisingly, one of the most delicate issues for nonconservative systems is the definition of the physical quantities, such as energy, linear momentum and angular momentum. Of course, the concept of, say, the total energy for a conservative system is fully established. However, the concept of the same quantity for the same system under additional nonconservative forces is apparently open to different interpretations.

It is therefore crucial for our program to assume a definition of the physical quantities for the class of systems considered. The reader should be aware that the same system of differential equations can, at least in principle, be subject to a variety of physical interpretations. Thus, the objective of this appendix is not that of resolving the issue under consideration . Instead, our objective is only that of assuming a definition of physical quantities which appears plausible on grounds of our current knowledge. The study of different definitions will be left to the interested reader.

Let us begin with the definition of the total physical energy of a nonconservative Newtonian system. Our attitude in this respect is as naive as conceivably possible. Explicitly stated, we shall assume for the definition of this quantity that

available from undergraduate textboods.

Consider, as an example, the one-dimensional harmonic oscillator

$$\left(m\ddot{x} + kx\right)_{SA} = 0. \qquad (2.B.1)$$

As is well known, the physical total energy, in this case is the sum of the kinetic energy and the potential energy of the elastic force.

System (2.B.1), however, does not exist, strictly speaking, in the Newtonian reality. A part from nonlinear extensions which are needed to consider amplitudes other than infinitesimal, a necessary prerequisite for the equation of motion of the physical oscillator to be closer to the physical reality is that it must represent a damped motion. We reach in this way the simplest possible nonconservative system

$$\left[\left(m\ddot{x} + kx\right)_{SA} + \gamma\dot{x}\right]_{NSA} = 0. \qquad (2.B.2)$$

We hereby assume that the total physical energy of system (2.B.1) and (2.B.2) coincide and are given by the familiar form

$$E = H = \frac{1}{2}\left(m\dot{x}^2 + kx^2\right). \qquad (2.B.3)$$

Indeed, one of the problems of system (2.B.2) is the study of the variation of energy (2.B.3) in time as a result of the presence of the damping force. After all, system (2.B.2) is

nonconservative, where the origin of the term "nonconservative" is precisely referred to the nonconservation of the total physical energy.

When external applied forces are included, the above assumption still holds. Unless energy is supplied, the motion will decay exponentially in time. The supply of energy to the system considered is generally represented with an external force not derivable from a potential, i.e.

$$\left\{\left[\left(m\ddot{x} + kx\right)_{SA} + \gamma\dot{x}\right]_{NSA} - f(t)\right\}_{NSA} = 0. \qquad (2.B.4)$$

The physical total energy of this damped and forced oscillator is still given by Eq. (2.B.3). The study is now reduced to the comparative behaviour of the energy dissipated by the drag force and that supplied by the applied force. As a result, the physical energy of system (2.B.4) can arbitrarily vary in time, i.e.

$$E\big|_{t_1} \gtrless E\big|_{t_2}, \quad t_1 < t_2, \qquad (2.B.5)$$

yielding the concept of dynamical system (for a detailed review see Volume I of reference [34]). In particular, when the energy rate of dissipation is equal to that of supplied energy, quantity (2.B.3) can be constant in time. This is the typical occurrence of physical oscillators with constant amplitudes as they occur, e.g., in engineering.

As another example, consider a charged particle in vacuum under the Lorentz force

$$\left(m\ddot{\underline{r}} - \underline{f}_{Lorentz}\right)_{SA} = 0 \qquad (2.B.6)$$

The conventional Lagrangian is in this case given by

$$L_{tot}^{conv} = T - U = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} - e\left(\varphi - \frac{1}{c}\dot{\underline{r}} \cdot \underline{A}\right), \qquad (2.B.7)$$

and the Hamiltonian is

$$H_{tot}^{conv} = \tilde{T} + \tilde{U} = \frac{1}{2} m \dot{\underline{r}} \cdot \dot{\underline{r}} + e\varphi \qquad (2.B.8a)$$

$$= \frac{1}{2m}\left(\underline{p} - \frac{e}{c}\underline{A}\right)^2 + e\varphi,$$

$$\underline{p} = \partial L_{tot}^{conv} / \partial \dot{\underline{r}} = m\dot{\underline{r}} + \frac{e}{c}\underline{A}. \qquad (2.B.8b)$$

It is a well established knowledge that the above Hamiltonian represents the total physical energy of system (2.B.6). Consider now a nonconservative implementation of system (2.B.6), e.g., the motion of the same charged particle in a physical medium which results in nonconservative drag forces

$$\left[\left(m\ddot{\underline{r}} - \underline{f}_{Lorentz}\right)_{SA} - \underline{F}(t, \underline{r}, \dot{\underline{r}})\right]_{NSA} = 0. \qquad (2.B.9)$$

Our assumption is that the total physical energy of this broader system is still given by the conventional structure (2.B.8a). And indeed, the problem is now turned into that of studying the

variation of this physical energy as a result of the presence of the nonconservative force $\underline{F}$.

These simple nonconservative cases can be extended to an arbitrary n-dimensional nonconservative case with the use of the methodology of the Inverse Problem. Consider the system

$$\left\{\left[m_k \ddot{\underline{r}}_k - \underline{f}_k(t, \underline{r}, \dot{\underline{r}})\right]_{SA}^{c^+,R} - \underline{F}_k(t, \underline{r}, \dot{\underline{r}})\right\}_{NSA}^{c^+,R} = 0 \qquad (2.B.10)$$

$$k = 1, 2, \ldots, n$$

under the assumption that the forces $\underline{f}_k$ ($\underline{F}_k$) are the totality of the forces derivable (not derivable) from a potential. The nonconservative nature of the latter forces is then represented in the language of the Inverse Problem with the fact that the overall system is nonselfadjoint. Let $U(t, \underline{r}, \dot{\underline{r}})$ be the potential of the $\underline{f}$-forces, i.e.

$$\underline{f}_k = -\frac{\partial U}{\partial \underline{r}_k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\underline{r}}_k} \qquad (2.B.11)$$

and let

$$H_{tot}^{conv} = \tilde{T} + \tilde{U} = \sum_{k=1}^{n} \frac{1}{2m_k} \underline{p}_k^2 + \tilde{U} \qquad (2.B.12)$$

be the Hamiltonian for the internal selfadjoint system of Eqs. (2.B.10). Our assumption is that this Hamiltonian represents the total physical energy of the entire nonconservative system (2.B.10).

In conclusion, if we call maximal associated selfadjoint system to the nonselfadjoint system (2.B.10) that within the inner brackets (under the indicated assumption that the $\underset{\sim}{f}$'s represent the totality of forces derivable from a potential), the total physical energy is that of the former system. From the above remarks we learn that such physical quantity can arbitrarily vary in time and it is not necessarily restricted to decay in time.

The case of the total linear physical momentum is more straightforward. It is generally accepted that the linear momentum is the product of the mass multiplied by the velocity. For the system (2.B.4) this yields the quantity

$$\underset{\sim}{p}_{tot}^{phys} = \sum_{k=1}^{n} m_k \dot{\underset{\sim}{r}}_k \,, \qquad (2.B.13)$$

which, as defined, holds for any type of acting force, whether conservative or not.

The relationship of this quantity with the canonical momentum of the maximal associated selfadjoint system

$$\underset{\sim}{p}_{tot}^{can} = \sum_{k=1}^{n} \frac{\partial L_{tot}^{conv}}{\partial \dot{\underset{\sim}{r}}^k} \qquad (2.B.14)$$

demands some care. It should be first recalled that a necessary condition for the system considered to be selfadjoint is that the $\underset{\sim}{f}$-forces are at most linear in the velocities (the Newtonian limit

of Theorem 2.5.2). This implies that the generalized energy $U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$ can at most be linear in the velocities (see also Appendix 1.A).

As a result, the canonical momentum of the maximal associated selfadjoint system can be written

$$L_{tot}^{conv} = T(\dot{\underset{\sim}{r}}) - U(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}}) = \frac{1}{2} m_k \dot{\underset{\sim}{r}}_k \cdot \dot{\underset{\sim}{r}}^k - D_k(t, \underset{\sim}{r}) \cdot \dot{\underset{\sim}{r}}^k - V(t, \underset{\sim}{r}), \qquad (2.A.15a)$$

$$\underset{\sim}{p}_{k\ can.}^{conv.} = \frac{\partial L_{tot}^{conv}}{\partial \dot{\underset{\sim}{r}}^k} = m_k \dot{\underset{\sim}{r}}_k - D_k(t, \underset{\sim}{r}). \qquad (2.A.15b)$$

The relationship with the physical momentum is then given by

$$\underset{\sim}{p}_k^{phys} = \underset{\sim}{p}_{k\ can}^{conv} + D_k, \qquad (2.A.16)$$

as clearly illustrated by the case of the Lorentz force, Eq. (2.B.8b).

We shall therefore assume as the definition of the total physical linear momentum of an arbitrary local Newtonian system either one of the following quantities

$$\underset{\sim}{p}_{tot}^{phys} = \sum_{k=1}^{n} m_k \dot{\underset{\sim}{r}}_k = \sum_{k=1}^{n} \left(\underset{\sim}{p}_{k\ can}^{conv} + D_k \right). \qquad (2.A.17)$$

The case of the total physical angular momentum is then consequential. Indeed, it is generally assumed that this quantity is given by

$$\underset{\sim}{M}_{tot}^{phys} = \sum_{k=1}^{n} \underset{\sim}{r}_k \times \underset{\sim}{p}_k^{phys} = \sum_{k=1}^{n} \underset{\sim}{r}_k \times m_k \dot{\underset{\sim}{r}}_k \qquad (2.A.18)$$

Again, quantities (2.B.17) and (2.B.18) are the total physical quantities for the general system (2.B.10) even though defined for the maximal associated selfadjoint system.

These assumptions can be represented with the following

DEFINITION 2.B.1: The individual or total physical quantities of a local nonselfadjoint system are the conventional physical quantities of the maximal associated selfadjoint system.

Notice, however, that Definition 2.B.1 makes no references as to whether the systems considered are characterized by ordinary or partial differential equations. Thus, it is intended to apply for both Newtonian as well as field theoretical systems. This implies that for the system in real fields

$$\left\{\left[\left(\Box + \mu_{(k)}^2\right)\phi_k - j_k\left(x^\alpha, \phi^j, \phi^{j}_{,\alpha}\right)\right]_{SA} - J_k\left(x^\alpha, \phi^j, \phi^{j}_{,\alpha}\right)\right\}_{NSA} = 0 \qquad (2.B.19)$$

where the currents j_k represent the totality of currents derivable from a potential, the physical total energy-momentum and angular momentum tensors are the corresponding physical tensors of the maximal associated selfadjoint system.

For the case of complex field ϕ^k, k=1,2,...,n with conventional electromagnetic interactions and additive couplings not derivable from a potential we shall instead write

$$\left\{\left[A_{ki}\left(\Box + \mu_{(i)}^2\right)\underline{\Phi}^i - \underline{j}^i\left(\underline{\Phi}, \underline{\Phi}^{i}_{,\alpha}\right)\right]_{SA} - \underline{J}^i\left(x^\alpha, \phi^j, \phi^{j}_{,\alpha}\right)\right\}_{NSA} = 0 \qquad (2.B.20a)$$

$$\left\{\underline{\Phi}^i\right\} = \left\{\phi^1, \bar{\phi}^1, \ldots, \phi^n, \bar{\phi}^n\right\}, \quad \underline{j}^i = \underline{j}^{i\,elm} \qquad (2.B.20b)$$

where the A-terms represent permutations of the fields $\{\phi^k, \bar{\phi}^k\}$ because if the same equations are written in the ordering $\{\phi_1, \bar{\phi}_1, \ldots, \phi_n, \bar{\phi}_n\}$ they are nonselfadjoint (Section 2.5). The total physical energy-momentum and angular momentum tensors are then again given by the corresponding physical tensors of the maximal associated selfadjoint system.

These definitions are crucial for the study of these monographs. One of our central objectives is that of studying the possibility that the hadronic forces are not only nonconservative, but also of such nonconservative character to imply the nonconservation of all physical characteristics of the hadronic constituents (Section 1.3). At the level without subsidiary constraints here considered, a problem of this nature necessarily demands the nonconservation of the physical tensors of the field equations. Thus, Definition 2.B.1 is conceived as to imply that not only the individual components, but also the total physical quantities of nonselfadjoint field theoretical models are not conserved.

Admittedly, this field theoretical definition of total physical quantity is somewhat nebulous on conceptual grounds

(but not on mathematical grounds as characterized by the Inverse Problem). This is not a rare occurrence in Field Theory. Our guidance is, again, Newtonian Mechanics where the underlying concepts can be subject to a direct confrontation with the physical reality, backed by centuries of knowledge on nonconservative forces. Then, our field theoretical definition of physical quantity is a direct generalization of the corresponding definition in Newtonian Mechanics, as need for the former context to be a consistent covering of the latter. The study of different assumptions will be left to the interested reader.

Under Definition 2.B.1 the following property will play a fundamental role for the problem of quantization of nonconservative forces as treated in Volume III of this series.

LEMMA 2.B.1: A necessary condition for the existence of a canonical representation of nonessentially nonselfadjoint systems is that the canonical quantities do not represent physical quantities.

The assumption that system (2.B.10) is nonessentially nonselfadjoint implies that it admits an indirect analytic representation within the same variables in terms of the generalized Lagrangian (Section 2.9)

$$L^{gen}_{tot}(t, \underset{\sim}{z}_m, \dot{\underset{\sim}{z}}) = \sum_{k=1}^{M} \sum_{\substack{a= \\ x,y,z}} L^{(ka)}_{I} L^{(ka)}_{free} + L_{II} \qquad (2.B.21)$$

The canonical momentum is then given by

$$\underset{m\,k\,Can}{p}{}^{gen} = \frac{\partial L^{gen}_{tot}}{\partial \dot{z}^{k}_{m}} \qquad (2.B.22)$$

Similarly, the canonical Hamiltonian is given by structures of the type

$$H^{gen}_{tot}(t, \underset{\sim}{z}_m, \underset{\sim}{p}_m) \qquad (2.B.23)$$

$$= \frac{1}{2}\Big[p_{ia} \tilde{G}^{iajb}(t, \underset{\sim}{z}, \underset{\sim}{p})\, p_{jb} + 2\, p_{ia} \tilde{F}^{ia}{}_{jb}(t, \underset{\sim}{z}_m)\, z^{jb} + z^{ia} \tilde{E}_{ia\,jb}(t, z)\, z^{jb} \Big]$$

and the canonical angular momentum is of the form

$$\underset{m\,k\,Can}{m}{}^{gen} = \dot{\underset{m\,k}{z}} \times \underset{m\,k\,Can}{p}{}^{gen} \qquad (2.B.24)$$

with corresponding generalizations (in terms of tensors) for the field theoretical case. The proof of the Lemma 2.B.1 is then trivial. If the above canonical quantities coincide with the physical quantities of Definition 2.B.1, this necessarily implies the lack of couplings not derivable from a potential (i.e., the system is essentially selfadjoint).

The physical plausibility of Lemma 2.B.1 is of immediate verification at the Newtonian level. For instance, the canonical momentum (2.B.22), when explicitly computed, is an arbitrary function of the local variables, e.g.,

$$\underset{\sim}{P}^{gen}_{can} = \underset{\sim}{r}\, \mathrm{arctg}^{-1}\, \dot{\underset{\sim}{r}}^{2} \qquad (2.B.25)$$

and, as such, it cannot be the physical linear momentum.

But canonical quantities are known to be generators of canonical transformations. Irrespective of their functional structure, the canonical Hamiltonian, momentum and angular momentum are the generators of translations in time, translations in space and rotations, respectively.

Another implication of Definition 2.B.1 is that the corresponding physical quantities cannot be the generators of physical transformations. It should be here recalled that any (class C^2) function of the canonical variables can be the generator of a canonical transformation. Thus, one can also construct canonical transformations by using the physical energy, linear momentum and angular momentum expressed as functions in the phase space coordinates $\underset{\sim}{r}^k$ and $\underset{\sim}{P}_k$. The point is that another necessary condition for these quantities to be the physical quantities is that they do not induce translations in time, translations in space and rotations, respectively.

We reach in this way the following dichotomy of canonical versus physical quantities which will be crucial for our program.

CORROLLARY 2.B.1A: Nonessentially nonselfadjoint systems demand a differentiation between canonical and physical quantities: the former are the generators of physical

transformations, while the latter are representatives of physical quantities. At the limit of null couplings not derivable from a potential, the canonical and physical quantities may coincide, namely, the same quantity may reacquire the dual meaning of being the generator of a physical transformation as well as the representative of a physical quantity.

An example is here useful illustrate this occurrence. Consider, again, the damped oscillator (2.B.2), Its canonical and physical quantities can be written

$$L^{gen}_{tot} = e^{\gamma t/m}\, \frac{1}{2}\left(m\, \dot{\underset{\sim}{r}}^{2} - k\, \underset{\sim}{r}^{2}\right), \qquad (2.B.26a)$$

$$\underset{\sim}{p}^{phys} = \underset{\sim}{p}^{conv}_{can} = m\, \dot{\underset{\sim}{r}}, \quad \underset{\sim}{P}^{gen}_{can} = e^{\gamma t/m}\, m\, \dot{\underset{\sim}{r}}, \qquad (2.B.26b)$$

$$H^{gen}_{tot} = e^{-\gamma t/m}\, \frac{1}{2m}\, \underset{\sim}{P}^{2} + e^{\gamma t/m}\, k\, \underset{\sim}{r}^{2}, \qquad (2.B.26c)$$

$$\underset{\sim}{M}^{phys} = \underset{\sim}{r} \times m\, \dot{\underset{\sim}{r}}, \quad \underset{\sim}{M}^{gen}_{can} = \underset{\sim}{r} \times \underset{\sim}{P}, \qquad (2.B.26d)$$

$$L^{conv}_{tot} = \frac{1}{2}\left(m\, \dot{\underset{\sim}{r}} - k\, \underset{\sim}{r}^{2}\right) = e^{-\gamma t/m}\, L^{gen}_{tot}, \qquad (2.B.26e)$$

$$H^{conv}_{tot} = \frac{1}{2}\left(m\, \dot{\underset{\sim}{r}} + k\, \underset{\sim}{r}^{2}\right) = \frac{1}{2m}\left[\left(\underset{\sim}{p}^{phys}\right)^{2} + k\, \underset{\sim}{r}^{2}\right] = \frac{1}{2m}\left[e^{-2\gamma t/m}\, \underset{\sim}{P}^{2} + k\, \underset{\sim}{r}^{2}\right]. \qquad (2.B.26f)$$

The generator of translations in space is, of course, $\underset{\sim}{P}$ and, as such, it does not coincide with the physical linear momentum $\underset{\sim}{p}$.

If the latter quantity is represented in terms of canonical variables we have $p_\mu = e^{-\gamma t/m} P_\mu$. This quantity can certainly be used as a generator of a canonical transformation. The point is that this transformation is not a translation in space. At the limit of null damping force ($\gamma = 0$) the physical and canonical momentum coincide. The same quantity has then the dual role of being representative of a physical quantity as well as the generator of a physical transformation. But this joint property occurs only for the conservative approximation of the system and it is lost when more realistic nonconservative models are considered. The proof that the same situation occurs for all the other physical versus canonical quantities is here left to the interested reader.

We remain with the illustration of the fact that at the limit of null forces not derivable from a potential the canonical and physical quantities do not necessarily coincide, as expressed in Corollary 2.B.1A with the terms "may coincide". This is, in essence, a direct consequence of the possible existence of isotopic transformations. For instance, Lagrangian (2.B.26a) is not unique. Consider then the isotopic image (2.30.35). It is easy to see that the identity of the canonical and physical quantities at the limit of null damping force is lost for the latter representation. For instance, the Hamiltonian is now given by $\left(\frac{r}{m} \to r,\ m=1,\ k=1\right)$

$$H = \frac{1}{2} r - \frac{1}{2} \cos(\omega r P) - \frac{1}{2} \gamma r P \qquad (2.B.27)$$

$$\omega = \sqrt{4 - \gamma^2}$$

and it does not recover the physical Hamiltonian at the limit $\gamma = 0$.

We learn in this way, that among the class of all possible isotopically mapped Lagrangians for the representation of the same system there can exist one Lagrangian which satisfies the limiting property of Corollary 2.B.1A while, in general, this property is not verified.

The significance of the analysis of this appendix for the problem of quantization of nonconservative forces in general and of nonconservative hadronic forces in particular, is now predictable and will be treated in details in Volume III. At this point it is useful to only indicate a quantum mechanical aspect which supports the statement of Section 1.2 to the effect that the problem of quantization of the forces considered should be considered open at this time, despite the existing attempts.

The simplest, but most fundamental rules of quantization are given by the familiar transitions

$$P_\mu \to \frac{\hbar}{i} \nabla_\mu \quad , \quad H \to i\hbar \frac{\partial}{\partial t}, \qquad (2.B.28)$$

applied to the canonical momentum and Hamiltonian respectively. For the case of forces derivable form a potential (essentially selfadjoint systems) the emerging quantum mechanical operators have the joint meaning of being the generators of physical unitary transformations and their expectation values or eigenvalues

are representative of physical quantities. The crucial confrontation of the prediction of a quantum mechanical model with the experimental data is thus done in terms of expectation values or eigenvalues of the indicated canonical operators. In particular, the approach satisfies the Correspondence Principle (in the sense that, at the limit of high values of the action compared with the Planck's constant, the Schrödinger equation recovers the Hamilton-Jacobi equation in H).

To the best of my knowledge, all currently available efforts at the quantization of generalized Hamiltonian structures (particularly in nuclear physics where the study of dissipative processes has been considerably increased in recent times) are centered in the either explicit or implicit use of rules (2.B.28). This yields a quantum mechanical model for which the generalized canonical operators are the generators of physical unitary transformations and the generalized Schrödinger equation satisfies the fundamental requirement of the Correspondence Principle. Thus, on mathematical grounds the approach appears to be consistent. On physical grounds, however, the perspective is now profoundly changed. Indeed, any comparison of the expectation values or eigenvalues of the generalized canonical operators with the experimental data is now, in my opinion, vacuous. Explicitly stated, I do not believe that the expectation value of, say the operator $\underset{\sim}{P} = \frac{\hbar}{i} \underset{\sim}{\nabla}$ can be meaningfully compared with experimental data on the

physical linear momentum when the quantity $\underset{\sim}{P}$ is representative of classical expression of type (2.B.25). Similarly for the expectation values or eigenvalues of the generalized canonical Hamiltonian and angular momentum.

It is hoped that these remarks will indicate that the problem of the quantization of nonconservative forces should be considered open at this time and suggest care in the physical interpretation of the mathematical algorithms of generalized quantum mechanical models.

Our approach to the quantization of the systems considered will be based in the identification of a dichotomy of canonical versus physical quantum mechanical operators which is fully parallel to the classical dichotomy identified in this appendix. This is motivated by our need of attempting the construction of a genuinely nonconservative generalization of the conventional quantum mechanics which is fully parallel to the considered Newtonian context and capable of recovering conventional quantum mechanical rules at the limit of null forces not derivable from a potential. This problem will be considered in Volume III of this series.

APPENDIX 2.C:

USE OF THE INVERSE PROBLEM FOR THE STUDY OF THE ANALYTIC STRUCTURE OF UNIFIED GAUGE THEORIES AND THEIR POSSIBLE GENERALIZATIONS.

In this appendix we shall use the methodology of the Inverse Problem to study the analytic structure of the unified gauge theories of weak and electromaghetic interactions. The analysis will then be used to identify the conceivable alternatives for the problem of the inclusion of the strong interactions.

As we shall see, Newtonian Mechanics will prove again to be the best arena for our primitive intuitions. Thus, we shall here first study the Newtonian limit of unified gauge theories and then extend the results to a field theoretical context.

Consider the following gauge Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(D_\mu \phi\right)_i \overline{\left(D_\mu \phi\right)_i} + \frac{1}{4} F_{\alpha\mu\nu} F_\alpha^{\mu\nu} + P(\phi), \quad (2.C.1a)$$

$$F_{\alpha\mu\nu} = \partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu} - C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu}, \quad (2.C.1b)$$

$$\left(D_\mu \phi\right)_i = \partial_\mu \phi_i - i \left(\theta_\alpha\right)_{ij} \phi_j A_{\alpha\mu}, \quad (2.C.1c)$$

where spinorial fields have been ignored because they render the system degenerate (Section 2.5). This restriction, however,

is not a drawback for the aspects we are interested in.

Assume, also for simplicity but without loss of generality, that Lagrangian (2.C.1) contains three fields (ϕ_1, ϕ_2, ϕ_3) and their conjugate $(\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3)$ and perform the Newtonian limit

$$\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial t}, \quad (\phi_1, \phi_2, \phi_3) \to z_{\mu 1}, \quad (\bar{\phi}_1, \bar{\phi}_2, \bar{\phi}_3) \to z_{\mu 2} \quad (2.C.2)$$

$$A_\alpha \to \frac{b}{2} = \text{const}, \quad i\theta \to -1, \quad C \to 0$$

Lagrangian (2.B.1) then becomes

$$L = \frac{1}{2}\left[\dot{z}_{\mu 1} \cdot \dot{z}_{\mu 2} + \frac{b}{2}\left(z_{\mu 1} \cdot \dot{z}_{\mu 2} - \dot{z}_{\mu 1} \cdot z_{\mu 2}\right) - b^2 z_{\mu 1} \cdot z_{\mu 2}\right] - P(z_\mu). \quad (2.C.3)$$

As predictable from the antisymmetric nature of the electromagnetic field in Minkowski space, the Newtonian limit of model (2.C.1) sees no representative of "Maxwell's equations." Again, this is not a drawback for the analysis we have in mind.

Our objective is to conduct a variational analysis of the system represented by Lagrangian (2.C.3) and then extend the results to field theory. For this purpose the following five aspects are relevant.

(1) The underlying system of equations of motion is nonselfadjoint, when written in the "natural" ordering $(z_{\mu 1}, z_{\mu 2})$ as originating from Newton's second law. Indeed, the system considered is given by

$$\frac{1}{2}\begin{pmatrix} \ddot{x}_{\mu 1} + b\dot{x}_{\mu 1} + b^2 x_{\mu 1} + 2\partial P/\partial x^1_\mu \\ \ddot{x}_{\mu 2} - b\dot{x}_{\mu 2} + b^2 x_{\mu 2} + 2\partial P/\partial x^2_\mu \end{pmatrix}_{NSA} = 0. \quad (2.C.4)$$

It is a simple exercise to see that the above system is nonselfadjoint because it violates conditions (2.3.6). This rather surprising occurrence indicates a simple, but nontrivial similarity with our model of the hadronic structure which, by assumption, is nonselfadjoint (Section 2.12). In any case, the nonselfadjoint nature of system (2.C.4) demands the identification of its origin.

(2) Some of the acting forces are purely nonconservative. This occurrence can be seen within the context of the Inverse Problem by writing system (2.C.4) in the form

$$\frac{1}{2}\left[\begin{pmatrix} \ddot{x}_{\mu 1} + b^2 x_{\mu 1} + 2\partial P/\partial x^1_\mu \\ \ddot{x}_{\mu 2} + b^2 x_{\mu 2} + 2\partial P/\partial x^2_\mu \end{pmatrix}_{SA} + \begin{pmatrix} +b\dot{x}_{\mu 1} \\ -b\dot{x}_{\mu 2} \end{pmatrix}\right]_{NSA} = 0 \quad (2.C.5)$$

The additive couplings $(+b\dot{x}_{\mu 1}, -b\dot{x}_{\mu 2})$ satisfy all the requirements for being nonconservative. On physical grounds they are velocity-dependent and not derivable from a potential, while on mathematical grounds they break the conditions of selfadjointness,

i.e. (the Newtonian limit of) Theorem 2.5.2. Notice in particular that the force $(+b\dot{x}_{\mu 1}, -b\dot{x}_{\mu 2})$ is not dissipative because of the different signs in its terms.

This point has a considerable conceptual significance for the analysis of these monographs. Let us recall that our central hypothesis is that the strong interactions are local and not derivable from a potential. We here learn that this type of forces is already present at the level of unified gauge theories of weak and electromagnetic interactions. This finding therefore provides an additional argument for the realization of strong interactions along more general non-conservative lines.

In any case this second rather peculiar occurence demands an inspection aiming at the identification of its origin as well as limits of applicability.

(3) The analytic representations of gauge Lagrangians are indirect. This point can be seen from the analytic representation

$$\begin{pmatrix} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^1_\mu} - \frac{\partial L}{\partial x^1_\mu} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^2_\mu} - \frac{\partial L}{\partial x^2_\mu} \end{pmatrix} \quad (2.C.6)$$

$$\frac{1}{2}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} \ddot{x}_{\mu 1} + b^2 x_{\mu 1} + 2\frac{\partial P}{\partial x^1_\mu} \\ \ddot{x}_{\mu 2} + b^2 x_{\mu 2} + 2\frac{\partial P}{\partial x^2_\mu} \end{pmatrix}_{SA} + \begin{pmatrix} +b\dot{x}_{\mu 1} \\ -b\dot{x}_{\mu 2} \end{pmatrix}_{NSA}\right]\right\}_{SA}$$

The reader is here urged to verify that system (2.C.4) in the reverse ordering ($\ddot{r}_{m2}, \ddot{r}_{m1}$) is selfadjoint.

We therefore reach in this way a first understanding of the analytic structure of gauge theories: starting from nonselfadjoint equations of motion in their "natural" ordering, gauge Lagrangians produce an equivalent selfadjoint form (i.e., a genotopic transformation in the language of Section 2.3) characterized by a simple permutation of the ordering or the system.

It is still unclear, however, how this structure allows the presence of purely nonconservative forces, while the structure of Lagrangian (2.C.3) is of the conventional type (i.e., the sum of a term representing the free motion and an additive interaction term). This third peculiar occurrence (which at a first analysis might result to be in contradiction with the approach of Chapter 2) can be resolved through the following property.

(4) <u>The gauge Lagrangian for the free term is an isotopic image of the conventional Lagrangian</u>. In Newtonian Mechanics free particles are customarily represented with the trivial Lagrangian

$$L_{free}(\dot{r}_m) = \frac{1}{2}\left(\dot{r}_{m1}^2 + \dot{r}_{m2}^2\right) \qquad (m=1) \quad (2.C.7)$$

The corresponding Lagrangian appearing in structure (3.C.3), however, is not of this type. Instead, it is of the equivalent

type

$$L^*_{free} = \dot{r}_{m1} \cdot \dot{r}_{m2} \qquad (3.C.8)$$

The crucial point is that the transition from Lagrangian (2.C.7) to its equivalent form (2.C.8) is an isotopic transformation realized according to rule (2.10.16), i.e.

$$\begin{pmatrix} \frac{d}{dt}\frac{\partial L^*_{free}}{\partial \dot{r}^1_m} - \frac{\partial L^*_{free}}{\partial r^1_m} \\ \frac{d}{dt}\frac{\partial L^*_{free}}{\partial \dot{r}^2_m} - \frac{\partial L^*_{free}}{\partial r^2_m} \end{pmatrix} = \left[\begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{pmatrix} \begin{pmatrix} \frac{d}{dt}\frac{\partial L_{free}}{\partial \dot{r}^1_m} - \frac{\partial L_{free}}{\partial r^1_m} \\ \frac{d}{dt}\frac{\partial L_{free}}{\partial \dot{r}^2_m} - \frac{\partial L_{free}}{\partial r^2_m} \end{pmatrix}_{SA} \right]_{SA} \qquad (2.C.9)$$

Notice that this is a nontrivial isotopic transformation owing to the nontrivial change of the Lagrangian structure. In particular, it occurs within a fixed system of (local) variables and it is not recoverable with the addition of a term with null Lagrange's derivatives, i.e.

$$L^*_{free} - L_{free} \neq \frac{d}{dt} G(t, r_m) \qquad (2.C.10)$$

The invertible mapping

$$L_{free} \Longleftrightarrow L^*_{free} \qquad (2.C.11)$$

is therefore a simple, (but nontrivial) example of the new class of equivalent transformations induced by the integrability

conditions for the existence of a Lagrangian (the isotopic transformations of Section 2.10).

It is precisely this occurrence which allows the rather peculiar property whereby gauge theories, even though representing couplings which at a Newtonian level are purely nonconservative, still admit a conventional Lagrangian structure as a sum of a term representing the free motion and an additive interaction term. Such structure, however, to resolve the considered apparent inconsistency, must be written in the form

$$L_{tot}^{gauge} = L_{free}^{*} + L_{int} \qquad (3.B.12)$$

namely, the free term is the isotopic image (2.C.8) rather than the conventional term (2.C.7). Indeed, if the latter is used instead of the former, the conventional gauge structure is lost.

We therefore learn from gauge theories an aspect of crucial methodological significance. When considering conventional Lagrangian structures, the free term need not be the conventional one. Instead, it can be any of its nontrivial equivalent form, i.e., any of its isotopic images.

Newtonian Mechanics turns out to be again crucial for our primitive intuitions. The rather peculiar occurrences here considered (the nonconservative nature of the acting forces,

the conventional structure of the gauge Lagrangian and the isotopic character of the free term) have a clear physical interpretation at a Newtonian level, but they become conceptually nebulous at the higher level of field theory (although fully definable on mathematical grounds). In particular, the isotopic nature of the free term of gauge models implies a better identification of one class of chiral models, those characterized by nonessentially chiral Lagrangians (Section 2.10), i.e., the generalized Lagrangian structure admitting an isotopic transformation into a conventional Lagrangian structure. We here learn that this class of chiral models is considerably larger than what originally suspected because of the possibility that a conventional Lagrangian structure is characterized by an isotopically mapped free term. Explicitly, the nonessentially chiral Lagrangians are not only those admitting the isotopic mapping

$$L_{tot}^{gen} = \frac{1}{2}\left[\dot{r}^{ia} G_{iajb} \dot{r}^{jb} + 2\dot{r}^{ia} F_{iajb} r^{jb} + r^{ia} E_{iajb} r^{jb}\right] \qquad (2.C.13)$$

$$\longleftrightarrow \quad L_{tot}^{conv} = L_{free} + L_{int}$$

but also those admitting the alternative mappings

$$L^{gen}_{tot} = \frac{1}{2}\left(\dot{z}^{ia} G_{ia\,jb} \dot{z}^{jb} + 2\dot{z}^{ia} E_{ia\,jb} z^{jb} + z^{ia} E_{ia\,jb} z^{jb}\right) \tag{2.C.14}$$

$$\longleftrightarrow \quad L^{conv}_{tot} = L^{*}_{free} + L_{int}$$

To summarize, when seen from a variational approach to selfadjointness and, particularly, whithin a Newtonian context, the gauge models exhibit a rather peculiar structure. The equations of motion in their natural form (as written from Newton's second law) are nonselfadjoint owing to the presence of strictly nonconservative velocity couplings. Nevertheless, the Lagrangian structure is the sum of a free and an interacting term. This apparent contradiction is resolved by the rather unique analytic structure of gauge models whereby the assumed Lagrangian produces a representation of the equivalent selfadjoint form of the system induced by the permutation of the ordering. In turn, this representation is made possible by the fact that the free term is not of the conventional form (2.C.7) but instead of the isotopically mapped form (2.C.8).

It is here tempting to express the personal opinion that, perhaps, the successes of gauge theories for the unification of weak and electromagnetic interactions (within the context of established disciplines) is due precisely to this capability of incorporating generalized nonselfadjoint couplings in a conventional Lagrangian structure.

It is here equally tempting to express the personal opinion that, perhaps, the limitations of unified gauge models for a physically effective inclusion of the strong interactions are due to the need of possible further generalizations of the couplings considered. From occurrence (2) this means a generalization of the couplings along more general nonconservative lines, i.e., along more general nonselfadjoint forms. From the content of Section 2.12 this implies the transition from gauge models to essentially chiral models, i.e. generalized Lagrangian structures which do not admit isotopic transformations (2.C.13) and (2.C.14).

To put this possibility in its proper methodological context an additional property of the gauge models is needed.

(5) The gauge Lagrangians have a simple but nontrivial chiral structure. This fifth (also peculiar) occurrence can be seen by writing gauge Lagrangian (2.C.3) in the chiral form

$$L^{gauge}_{tot} = \frac{1}{2}\left(\dot{z}^{ia} G_{ia\,jb} \dot{z}^{jb} + 2\dot{z}^{ia} F_{ia\,jb} z^{jb} + z^{ia} E_{ia\,jb} z^{jb}\right) = L^{chiral}_{tot} \tag{3.B.15a}$$

$$(G) = \begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{pmatrix}, \quad (F) = \frac{1}{4}\begin{pmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{pmatrix}, \quad \frac{1}{2}(z E z) = -\frac{1}{2} b^2 z_{m1} \cdot z_{m2} - P(z_m) \tag{3.B.15b}$$

which clearly exhibits simple (but nontrivial values of the chiral tensors.

This occurrence provides the desired methodological perspective for the implementation of unified gauge theories of weak and electromagnetic interactions for the inclusion of strong interactions.

From a variational approach to selfadjointness only the following three possibilities can occur (for local couplings).

NEWTONIAN ALTERNATIVE I: Implementation of gauge models with selfadjointness preserving couplings. At the level of the equations of motion this implies the study of systems of the type

$$\left\{\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} \ddot{x}_{m_1} + b^2 x_{m_1} + 2\frac{\partial P}{\partial x_{m_1}} \\ \ddot{x}_{m_2} + b^2 x_{m_2} + 2\frac{\partial P}{\partial x_{m_2}} \end{pmatrix}_{SA} + \begin{pmatrix} +b\dot{x}_{m_1} \\ -b\dot{x}_{m_2} \end{pmatrix}_{NSA}\right)\right]^{\text{"elem"}}_{SA} + \begin{pmatrix} f_{m_1} \\ f_{m_2} \end{pmatrix}^{\text{"strong"}}_{SA}\right\}_{SA} = 0 \qquad (2.C.16)$$

At the Lagrangian level, Theorem 2.3.3 yields the preservation of the structure (2.C.12) with only the addition of a term representing the strong interactions

$$L^{\text{gauge}}_{\text{tot}} = L^{*}_{\text{free}} + L^{\text{"elem"}}_{\text{int}} + L^{\text{"strong"}}_{\text{int}}, \qquad (2.C.17)$$

or, equivalently, a trivially chiral Lagrangian of type (2.C.15).

ALTERNATIVE II: Implementation of gauge models with selfadjointness breaking couplings which admit a genotopic transformation. At the level of the equations of motion this

alternative approach implies the study of systems of the type

$$\frac{1}{2}\left\{\left[\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left(\begin{pmatrix} \ddot{x}_{m_1} + b^2 x_{m_1} + 2\frac{\partial P}{\partial x_{m_1}} \\ \ddot{x}_{m_2} + b^2 x_{m_2} + 2\frac{\partial P}{\partial x_{m_2}} \end{pmatrix}_{SA} + \begin{pmatrix} +b^2\dot{x}_{m_1} \\ -b^2\dot{x}_{m_2} \end{pmatrix}_{NSA}\right)\right]^{\text{"elem"}}_{SA} - \begin{pmatrix} F_{m_1} \\ F_{m_2} \end{pmatrix}^{\text{"strong"}}_{NSA}\right\}_{NSA} = 0 \qquad (2.C.18)$$

where the additive couplings violate the conditions of selfadjointness (the Newtonian limit of Theorem 2.5.2). Under the assumption that these systems admit a genotopic mapping to a selfadjoint form according to Theorem 2.3.4 (i.e. they are nonessentially nonselfadjoint in the language of Section 2.10) the emerging Lagrangian structure is, from Theorem 2.3.3, of the essentially generalized type

$$L^{\text{chiral}}_{\text{tot}} = \sum_{k=1}^{2} \sum_{\substack{a= \\ x,y,z}} L^{(ka)}_{I} L^{(ka)}_{\text{free}} + L_{II}, \qquad (2.C.19)$$

i.e., an essentially chiral Lagrangian.

NEWTONIAN ALTERNATIVE III: Implementation of gauge models with selfadjointness breaking couplings which do not admit a genotopic transformation. This is the case of models (2.C.18) where the additive strong couplings are such that a genotopic transformation to an equivalent selfadjoint system within a fixed system of variables does not exist. In this case, the computation of a Lagrangian (within the context of conventional analytic equations without external terms), even though it exists from Universality Theorem 2.8.1, necessarily demands non-

trivial velocity-dependent transformations which profoundly affect the physical content of the system considered. As we shall see in Volume II, the use of the Lie-admissible techniques (within the context of the analytic equations with external terms) in this case does allow the direct representations of the systems considered without any change of variables and by therefore preserving the physical content in its entirety.

This concludes our variational analysis of unified gauge thoeries at a primitive Newtonian level.

The extension of the above results to (classical) field theory is straightforward and can be expressed with the following

LEMMA 2.C.1: Consider a class C^{∞}, regular, quasilinear system of field equations for a unified gauge model of weak and electromagnetic interactions

$$\left[A_{ki}\otimes\left(\Box+m^2_{(i)}\right)\bar{\Phi}^i+B_k^{elm,weak}(\Phi^j,\Phi^{ij}_{\alpha})\right]_{SA}=0, \tag{2.C.20}$$

$$\{\Phi^i\}=\{A^{\alpha},\phi^k,\bar{\phi}^k\},$$

$$i=1,2,\ldots,n+4\ ;\ \alpha=0,1,2,3\ ,\ k=1,2,\ldots,n$$

with conventional gauge Lagrangian

$$\mathcal{L}^{gauge}_{tot}=\frac{1}{4}F_{\alpha\mu\nu}F_{\alpha}{}^{\mu\nu}-\frac{1}{2}\left(D_{\mu}\phi\right)_i\left(\overline{D^{\mu}\phi}\right)_i+P(\phi), \tag{2.C.21a}$$

$$F_{\alpha\mu\nu}=\partial_{\mu}A_{\alpha\nu}-\partial_{\nu}A_{\alpha\mu}-C_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}, \tag{2.C.21b}$$

$$\left(D_{\mu}\phi\right)_i=\partial_{\mu}\phi_i-i\left(\theta_{\alpha}\right)_{ij}\phi_j A_{\alpha\mu}. \tag{2.C.21c}$$

The implementation of this model for the inclusion of strong interactions with additive local strong couplings

$$\left[A_{ki}\otimes\left[\Box+m^2_{(i)}\right)\bar{\Phi}^i+B_k(\Phi^j,\Phi^{ij}_{\alpha})^{elm,weak}\right]_{SA}+C_k(\Phi^i,\Phi^{ij}_{\alpha})^{strong}=0 \tag{2.C.22}$$

can only be done according to the following three possibilities:

(I) The additive strong couplings are selfadjointness preserving. In this case the conventional gauge structure can be preserved by the new Lagrangian.

(II) The additive strong couplings are selfadjointness breaking, but such that Eqs. (2.C.22) admit an equivalent selfadjoint form within the considered system of variables. In this case the conventional gauge Lagrangian structure is lost and an essentially chiral Lagrangian structure is necessary to represent the field equations.

(III) The additive strong couplings are selfadjointness breaking and such that Eqs. (2.C.22) do not admit an equivalent selfadjoint form within a fixed system of variables. In this case, both, the gauge and chiral

Lagrangian structures for the analytic representation of Eqs. (2.C.22) in terms of Lagrange's equations without external terms and within the given system of variables are insufficient.

A few comments are here in order. On conceptual grounds the most important point is the starting point. The fundamental physical quantities of Lemma 2.C.1 are the field equations and not a Lagrangian for their analytic representation. The three possibilities of the lemma then follows with a simple application of the techniques of the Inverse Problem in Field Theory as reviewed in Section 2.5. This position is also suggested by the existence of isotopic transformations of gauge Lagrangians with consequential general loss of gauge invariance (Section 2.11). If Eqs. (2.C.20) are assumed as the fundamental physical quantities and the problem of their analytic representation is subsequently confronted by using the Inverse Problem, the possible nonuniqueness of the Lagrangian (2.C.21) and of the gauge symmetry for the characterization of the system is then consequential. In turn, this possible loss of a gauge structure already in the absence of strong couplings appears to be useful for the identification of the three indicated alternatives.

The first physical implications of Alternatives I, II, and III can be expressed with the following

COROLLARY 2.C.1A: The strong couplings are analytically equivalent (nonequivalent) to the weak and electromagnetic couplings for Alternative I(II and III).

This property trivially follows from the preservation of the selfadjointness of the field equations for the inclusion of strong couplings along Alternative I and the consequential possibility of preserving the gauge structure in the new Lagrangian. The possible presence of unitary internal degrees of freedom for the strong couplings of Alternative I is here ignored because it affects only the explicit form of the strong couplings but not its selfadjoint (and, thus analytic) character.

Of course, a fundamental property of Lagrangian (2.C.21) is that of being invariant under a nonAbelian (unitary) gauge group. But, for Lemma 2.C.1 we have the following

COROLLARY 2.C.1B: The form-invariance of field equations (2.C.22) under (local or global and Abelian or nonAbelian) gauge transformations is a restriction on the class of admissible strong couplings.

This property is known and only its emphasis is now different. The chiral Lagrangians emerging from Alternative II are generally non-invariant under gauge transformations. Owing to the rooted use of gauge symmetries in current high energy physics, one might be tempted to impose such a symmetry also for

Alternative II. But the Lagrangian structure is now of chiral rather than gauge type. Thus, the gauge symmetry would this time result in a major restriction on the allowed forms of strong couplings. Restated in different terms, the conventionally used gauge symmetries have a fundamental role for the unified theories according to Alternative I. When the more general Alternatives II and III are used, the use of the same symmetries results to be excessively restrictive, assuming that the technical difficulties of constructing a nontrivial chiral Lagrangian invariant under a gauge group can be resolved. We therefore conclude that the gauge symmetry is generally lost for Alternatives II and III already at the level of the field equations. Alternatively, we can say that there is no final evidence available at this time that strong couplings, when assumed to be local but not derivable from a potential, should transform covariantly under gauge transformations.

This is only part of the departure from gauge theories implied by Alternatives II and III. Another significant aspect is the general loss of the minimal coupling rule. This can be seen from the loss of this rule already for the case without strong interactions, e.g., in the transition to from gauge Lagrangian (2.11.75) to its isotopic image of chiral type (2.11.72). The loss of the minimal coupling rule then persists after transition to more general chiral structures for the inclusion of the strong interactions along the alternatives considered.

This situation can be expressed with the following

COROLLARY 2.C.1C: The electromagnetic, weak and strong interactions are (are not) distinguishable at the level of the Lagrangian representation for Alternative I (Alternatives II and III).

This corollary essentially expresses our idea or dominance of the strong over the weak and electromagnetic interactions when realized as local couplings not derivable from a potential. In figurative terms we can say that under this assumption, the strong interactions become so "strong" to dominate the structure of a Lagrangian to the point of rendering unidentifyable with conventional means the electromagnetic and weak interactions.

It should be stressed that for all Alternatives I, II and III, the weak, electromagnetic and strong interactions remain always fully distinguishable at the level of field equations (2.B.22), trivially, because of the additive nature of their couplings. This is the main reason why we have advocated a return "ad originem": the assumption of the field equations as the fundamental physical quantities and that of their Lagrangian representation as a tool of primary methodological significance.

It should be here recalled that the objective of these monographs is not that of constructing a unified theory. Instead, the objective is restricted to the study of only a conceivable

realization of the strong forces responsible for the hadronic structure. This is so for a number of reasons. First of all there is the need of gaining some methodological insight for the treatment of local strong couplings as not derivable from a potential prior to the confrontation of the problem of a unified theory. Secondly, there is the need of achieving some understanding of the expected hierarchy of strong couplings of the indicated type (Sections 1.2 and 1.3) prior to confronting the problem of the role of weak interactions. And last, but not least, when considering unified theories of type II or III the need of gravitational considerations becomes more pressing, on a comparative basis with the corresponding situation for unified theories of type I, as indicated by the same presence of the G-tensor of the chiral Lagrangians.

In closing, the unequivocal physical effectiveness of the unified gauge theories of weak and electromagnetic interactions, with particular reference to the Weinberg-Salam SU(2) x U(1) model, should be here recalled. To the best of my knowledge, all currently available unified models for the inclusion of the strong interactions are along Alternative I. The lack of a comparable physical effectiveness for these extended models, the ultimate nonconservative nature of the gauge couplings and our independent study of the nature of the forces responsible for the hadronic structure, indicate the need of conducting a comparative study of unified theories along all conceivable alternatives, and not only Alternative I.

CHAPTER 3

ARENA OF NONAPPLICABILITY OF THE GALILEI AND EINSTEIN RELATIVITIES

3.1: STATEMENT OF THE PROBLEM

The objective of this chapter is to indicate, through the use of the methodology of the Inverse Problem, that

> The hadronic forces, when realized in terms of local couplings not derivable from a potential, are incompatible with the Galilei and Einstein relativities.

Since the nature of the hadronic forces is unknown at this time, this objective alternatively consists in indicating that

> On grounds of our current experimental and theoretical knowledge, the established relativity laws do not appear to be necessarily applicable within a hadron.

Predictably, it is very unlike that an issue of this nature can be resolved in unequivocal terms on theoretical grounds alone. A further alternative formulation of our objective can be, therefore, stated as follows.

> The central objective of this monograph is that of stimulating, in due time, experimental efforts aiming at the verification of either the validity or the invalidity within a hadron of established relativity laws in their currently known form.

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Clearly, any experimental resolution of the issue, irrespective of whether positive or negative, would have far reaching implications. In the final analysis, our studies reveal a deep relationship between the relativity laws and the nature of the hadronic forces. If these forces are realized with couplings analytically equivalent to those of the electromagnetic interactions (in the sense of Corollary 2.C.1A), the applicability of the established relativity laws is consequential. Their nonapplicability emerges only when the hadronic forces are realized in terms of couplings analytically nonequivalent to the electromagnetic interactions.

As a result, if the validity of established relativity laws within a hadron can be experimentally proved, this will provide valuable evidence on the nature of the hadronic forces as being derivable from a potential. The currently available unitary structure models of the hadrons would then emerge, quite likely, as the best conceivable structure models.

On the contrary, if the invalidity of the established relativity laws within a hadron can be experimentally established, this will provide valuable evidence that the hadronic forces are not derivable form a potential. In turn, this will indicate the need of an approach to the hadronic structure different than that of the currently available unitary models. The arena of physical relevance of these latter models would then be restricted to that of the hadron classification only, in line with our analysis of Chapter 1. The hypothesis that the quarks are the elemental constituents of hadrons, in this latter case, would likely be ruled out in the final form.

Predictably, an experimental resolution of the problem considered of clear and unequivocal nature (i.e., without a complex topology of assumptions) is not an easy task. But this does not mean that such a task cannot be, in due time, confronted and resolved. To put it in intuitive but effective terms, the technical difficulties for a genuine experimental verificantion or disproof of the esablished relativity laws within a hadron is comparable to the technical difficulties which existed during the first part of this century for the experimental verification of the same laws for the motion of massive particle at relativistic speed.

A few comments are in order to avoid possible misrepresentations of our objectives.

As indicated in Chapter 1, the validity of established relativity laws for the behaviour of hadrons as a whole under electromagnetic interactions is nowadays established in unequivocal experimental and theoretical grounds. Thus, under no circumstance the content of this chapter should be referred to this context.

However, the validity of established relativity laws under these conditions is, by no means, evidence that the same laws are also applicable for the hadronic constituents. Therefore, the content of this chapter is solely referred to this latter context.

It is here appropriate to recall that Einstein's Special Theory of Relativity was conceived for the electromagnetic

phenomenology. In particular, it resulted to be strictly applicable to the atomic structure.

But, as stressed throughout our analysis, the electromagnetic and strong interactions exhibit a profound physical difference in the physical space of the experimental detection, the Euclidean or the Minkowski space, which, as such, does not appear to be necessarily representable only with additional degrees of freedom of the strong interactions in the mathematical space of the unitary internal groups.

Thus, on grounds of our current experimental and theoretical knowledge, the fascinating physical effectiveness of the Special Theory of Relativity for the electromagnetic interactions is, by no means, evidence that the same relativity is also applicable to the strong interactions.

This raises the delicate question whether our studies are also applicable to the nuclear structure. Let us recall in this respect that, according to a wealth of experimental data, the nuclear and hadronic structures exhibit profound physical differences, such as those related to mean life and decay processes. However, the difference between the nuclear and the hadronic structure which is here considered most significant is that the charge volume of the former sensibly increases with the mass (by being approximately proportional to the number of nucleon constituents), while the charge volume of the latter does not (for instance, a quantitatively similar

differentiation between the charge volume of the pions and that of the nucleons does not occur, despite the sizable differences in the masses of these particles).

These differences are here interpreted as an indication of a possible differentiation of the nuclear and hadronic forces.[84] The net effect is that our studies do not appear to be necessarily applicable also to the nuclear structure in the same quantitative form.

In conclusion, the available experimental knowledge establishes that the three currently identified layers of the microscopic reality, the atomic, nuclear and hadronic layers, possess profound structural differences. On theoretical grounds, these differences can be interpreted as an indication of corresponding differences on the nature of the acting forces. As a result, the problem of the applicable relativity laws should be specifically studied for each individual layer of the considered microscopic reality.

And indeed, our primitive idea of the structure of the hadrons is that of massive, charged and spinning particles forced to move each within the charge radius of the other (Section 1.3). This creates a structure which is profoundly different than that of not only the atomic structure (from the very large distances of the constituents as compared to their charge radius) but also that of the nuclear structure (precisely from the approximate proportionality of the charge volume to the number of nucleons). In turn, this provides

conceptual backing to the assumption of the hadronic forces as being local but not derivable from a potential (as an approximation of expected more complex forces).

The role of the methodology of the Inverse Problem should be here identified. As is well known, the Galilei and Einstein relativities are deeply related to symmetry considerations. In turn, owing to the role of Noether's approach in contemporary physics, Lagrangian representations are used. But the nature of the forces considered is that of not being derivable from a potential. The first role of the methodology of the Inverse Problem is that of allowing a Noether-type study of the problem considered through the computation of an analytic representation. However, as we shall see, the methodology under consideration has also several other implications. In the final analysis, the integrability conditions for the existence of a Lagrangian allow the identification of an arena of nonapplicability of established relativity laws, as well as of the mechanisms of their breakings.

The reader should be aware that, besides incidental remarks, gravitational aspects will not be considered at this time. Thus, the relativistic analysis of this chapter is mainly restricted to a flat Minkowski space.

The reader should be also aware that the analysis we are here referring to is purely classical and, as such, it constitutes only part of our studies. The corresponding quantum

mechanical analysis of the problem will be conducted later on in Volume III. As we shall see, the two independent studies will result to be deeply interrelated and their results compatible.

The conceptual attitude which is implemented should also be indicated. After all, theoretical physics is a quantitative characterization through mathematical algorithms of primitive ideas.

Our first attitude is that any relativity problem, whether related to the microscopic or the macroscopic physical reality, should first be studied within a classical setting. After all, the currently available relativity ideas were conceived within a purely classical context and the problem of their applicability within a quantum mechanical setting was only subsequently confronted. This is the reason why we have differentiated the classical study of the problem of this chapter from its quantum mechanical counterpart of Volume III.

Our second attitude is that of putting primary emphasis on the problem of the Galilei relativity and then implementing the results to the Einstein relativity. This attitude is motivated by several reasons. It is known that these two relativities are related by expansion-contraction techniques and, most importantly, the latter is a genuine covering of the former. Thus, results within the context of the Galilei relativity must admit a consistent image within the context of

the Special Relativity. If this is not the case, the central requirement of compatibility for the latter relativity to qualify as a covering of the former would be violated.

But we have other reasons for putting the primary emphasis on the Galilei rather than the Einstein relativity. As recalled several times during the course of our analysis, Newtonian Mechanics constitutes the arena of our best intuitions. Also, primitive ideas at the Newtonian level can be directly tested in unequivocal terms. Both these aspects become considerably weaker at a relativistic level. Specifically, the effectiveness of our intuitional capabilities becomes much weaker at a relativistic level on a comparative basis with the corresponding Newtonian framework, while the limitations of our current experimental knowledge on the nature of the forces acting in the microscopic world restrict the possibility of a direct verification of any new relativity insights.

Our third (and perhaps most important) attitude is that of subjecting the current status of our theoretical knowledge to an analysis as critical as possible. Therefore, in line with our last remarks of the Preface, these monographs are not written for the reader with a firm personal conviction that Einstein's Relativity is the terminal relativity of Nature.

3.2: ARENA OF NONAPPLICABILITY OF GALILEI'S RELATIVITY IN NEWTONIAN MECHANICS.

The first part of our objective, the indication that the Newtonian limit of the hadronic forces as local and not derivable from a potential is incompatible with the Galilei relativity, can be easily accomplished.

Local Newtonian forces not derivable from a potential are nonconservative. Centuries of knowledge on these forces establishes that they generally violate all Galilei conservation laws (of the total physical energy, linear momentum and angular momentum as well as uniform motion of the center of mass). This produces a violation of the direct physical contents of the Galilei relativity. In addition, the forces considered are generally form-noninvariant under the Galilei transformations and this produces the violation of the complementary symmetry contents. As a result, local, nonconservative , Newtonian systems are incompatible with the Galilei relativity, as currently known.

Our task is thus reduced to a technical characterization of the Galilei relativity violation as well as the identification of the mechanism of relativity breaking.

The first part of this task can be accomplished with the methodology of the Inverse Problem. As indicated in Chapter 2, the class of systems under consideration can be characterized, from the viewpoint of the variational approach to selfadjointness, as follows

$$\left\{\left[m_k \ddot{x}_{mk} - f_{mk}(t, x_m, \dot{x})\right]_{SA} - F_{mk}(t, x_m, \dot{x})\right\}_{NSA} = 0, \quad (3.2.1)$$
$$k = 1, 2, \ldots, n,$$

where the system within the inner brackets is the maximal associated selfadjoint system. We then have the following

CONTENTION 1: An arena of nonapplicability of the Galilei relativity is constituted by class C^∞, regular, nonconservative, local, Newtonian systems in the space and time coordinates of their experimental detection which are either non-essentially or essentially nonselfadjoint.

It should be stressed that, by no means, this characterization of the nonapplicability of the Galilei relativity is exhaustive. Contention 1 merely characterizes that arena of nonapplicability which is sufficient for our needs. The study of the extension of this arena is here left to the interested reader. Most instructive is the prove that also nonlocal Newtonian couplings not derivable form a potential generally imply the violation of the Galilei conservation laws. In turn, this is significant to indicate that our results persist for realizations of the hadronic force of the latter type.

Contention 1 has been here introduced from only one profile: the fundamental aspect of any relativity of being representative of physical conservation laws. Clearly, the same lemma must also be established from the viewpoint of the transformation theory.

This second aspect of the problem demands the study of the mechanism of Galilei's relativity breaking. The following classes of breakings can be identified.

I-CANONICAL BREAKING. It is known that the Newtonian systems under consideration do not generally transform form-invariantly under the Galilei transformations. We have a canonical breaking (CB) when this is the case but the system is nonessentially nonselfadjoint, and we shall write

$$\left\{ \left[m_k \ddot{\vec{r}}_k - \vec{f}_k \right]_{SA} - \vec{F}_k \right\}_{NSA}^{CB} = 0 \quad . \qquad (3.2.2)$$

The reason for the selection of the terms "canonical breaking" is the following. As indicated in Section 2.8, nonessentially nonselfadjoint systems admit, by assumption, an analytic representation within the original set of variables. Thus, a Lagrangian for the representation of the nonconservative systems (3.2.2) exists and a Hamiltonian can be computed with the (direct) Legendre transform. The net result is that for systems (3.2.2) Lie's techniques in general and the canonical formalism in particular, are fully applicable.

It then follows that, even though the systems do not transform form-invariantly under the Galilei transformations, the physical total energy, linear momentum, angular momentum and Galilei boosts (according to Definition 2.B.1) can still be assumed as generators of canonical transformations. The point, however, is that under these conditions, the physical generators of the Galilei Lie algebra(energy, linear momentum, etc.) do not

induce the conventional canonical transformations (translations in time, translations in space, etc.). This is the reason for the selected terminology.

The mechanism of the considered class of Galilei relativity breakings is thus essentially the following. The canonical formalism is applicable as a methodological tool to the systems considered. However, the breakings occur at the level of the canonical formalism of the Galilei relativity, as necessary from the lack of form-invariance of the equations of motion under the Galilei transformations.

This class of Galilei relativity breakings includes the systems of our everyday experience, such as particles under dissipative and applied forces, trajectory problems in our Earthly atmosphere, spinning tops with drag torques, etc.

On methodological ground, the breakings under consideration should not be interpreted as an occurrence of marginal significance. Instead, they should be interpreted at the utmost of their conceptual, technical and physical implications. The best way to emphasise this point is through the breaking of a central aspect of the Galilei relativity, the group of rotations. The best example to illustrate this breaking is the physical spinning top. The conventional approach to this system is that for which the group of rotations is an exact symmetry with underlying conservation laws of the physical angular momentum. This system, however, does not exist in the physical reality of our environment. And indeed, unequivocal

experimental evidence indicates that the angular momentum of the spinning top is not conserved. The equations of motion, to represent the physical system, must then include drag torques responsible for the angular momentum nonconservation. In particular, these drag torques are generally form-non-invariant under rotations and induce a nonselfadjoint system, Eqs. (2.9.24). Under the condition of the existence of a Lagrangian within the original coordinates (the Euler angles) we then have a canonical breaking of the group of rotation. This literally implies the loss of the group of rotation as a methodological tool. The conceptual, technical and physical implications of this breaking are then identifyiable from the fundamental role of the group of rotations for all of contemporary theoretical physics.

II-SEMICANONICAL BREAKING. It occurs when Eqs. (3.2.1) are form-invariant under the Galilei transformations, but the systems are still nonessentially nonselfadjoint and we shall symbolically write

$$\left\{ \left[m_k \ddot{\underline{r}}_k - \underline{f}_k \right]_{SA} - \underline{F}_k \right\}_{NSA}^{SCB} = 0 . \quad (3.2.3)$$

This case constitutes one of the most potentially misleading occurrences of the methodology of symmetries and first integrals, which is rendered identifyable as well as treatable by the

methodology of the Inverse Problem (Section 2.11). If one restricts the analysis to conventional means, one might be tempted to conclude that the Galilei relativity applies to the systems considered because of their form-invariance under the Galilei transformations. This conclusion would be erroneous because the systems considered are genuinely nonconservative. The violation of the physical Galilei conservation laws for these systems is therefore established irrespective of any symmetry consideration. This is, incidentally, a reason why in our approach to the problem of Galilei relativity in Newtonian Mechanics we have put primary emphasis on the physical content of this relativity, the physical conservation laws.

The resolution of this dichotomy (Galilei form-invariance versus Galilei relativity breaking) can be achieved by using the application of the Inverse Problem to symmetries and first integrals (Section 2.11) with particular emphasis on the identification of the physical significance of the algorithms at hand (see also Appendix 2.B).

In essence, the existence of the symmetry of the equations of motion under the ten-parameter (identity component of) the Galieli group implies, via the identification of a Lagrangian exhibiting this symmetry and the use of Noether's theorem, the existence of ten first integrals. The resolution of the dichotomy under consideration is then simply given by the fact that the Galilei symmetry by no means guarantees that these

first integrals are representative of physical conservation laws. This is the occurrence of the class of systems under consideration: the physical conservation laws of the Galilei relativity are violated because the system is nonconservative, nevertheless the Galilei symmetry is verified.

The mechanism of the considered second class of Galilei relativity breakings is thus essentially the following. Not only the canonical formalism in general, but also the canonical formalism of the Galilei group is applicable to the systems considered in the sense that the generators of the Lie algebra of the Galilei group characterize physical transformations (e.g., the generalized Hamiltonian is the generator of translations in time, etc.). Nevertheless, the physical content of the Galilei relativity is lost because the conservation laws of the total physical quantities are violated. This is the reason for the selected terms "semicanonical breaking".

In the final analysis, the existence of the class of semicanonical breakings is a direct consequence of the dichotomy of canonical versus physical quantities which, as indicated in Appendix 2.B, is typical of nonconservative systems.

On comparative grounds between the canonical and semicanonical breakings, both breakings occur at the physical level of the relativity, the conservation laws, owing to the essential presence of nonconservative forces. Also, Lie's techniques in general and the canonical formalism in particular, are directly

applicable to both breakings. Nevertheless, the canonical formalism of the Galilei group is preserved for the semicanonical breakings while it is lost for the canonical breakings.

One of the best examples of semicanonical breakings is again given by the physical spinning top. As recalled earlier, to represent more closely this system to the physical reality, the presence of drag torques inducing the nonconservation of the angular momentum is essential. Often, these dissipative torques are form-invariant under rotations (trivially, when they do not depend explicitly on the Euler angles). In this case the equations of motion are form-invariant under rotations, but the conservation law of the physical angular momentum is lost. Nevertheless, there exists the canonical angular momentum (Appendix 2.B) which is conserved and whose components characterize the generators of the physical rotations. We therefore have a semicanonical breaking: the canonical realization of the group of rotations remains fully significant as a methodological tool, but the conservation of the physical angular momentum is lost. The verification of these statements with explicit examples is straightforward and will be left as an exercise to the interested reader.

For completeness, it should be here indicated that the semicanonical breaking of the full Galilei group is a rather rare occurrence in Newtonian Mechanics. What is more frequent is a semicanonical breaking of a given subgroup of the Galilei group, e.g, the group of rotations, the group of space-time

translations or the group of Galilei boosts.

The net effect is that a breaking of the Galilei group can also be of mixed canonical and semicaninical type, e.g., the symmetry under translations in time is broken by the equations of motion (canonical breaking), but the symmetry under rotations is preserved (semi-canonical breaking).

As we shall see in Volume II and III, this latter possibility has quite deep methodological implications. For our model of hadronic structure it will be crucial to achieve the concept of a hierarchy of dynamical conditions according to the hierarchy of nonconservations of the kinematical and intrinsic characteristics of the eletons as postulated in Section 1.3.

III- NONSELFADJOINT BREAKING. This breaking occurs when Eqs. (3.2.1) are essentially nonselfadjoint irrespective of their transformation properties, and we shall symbolically write

$$\left\{ \left[\mu_k \ddot{x}_{mk} - \frac{f}{m_k} \right]_A - \frac{F}{m_k} \right\}_{NSA}^{NSB} = 0. \quad (3.2.4)$$

This is the most "powerful" breaking of the Galieli relativity of which I am aware. The equations of motion are not only nonconservative, but such to prohibit the validity of the integrability conditions for the existence of an analytic representation whithin the considered Euclidean variables (essentially nonselfadjoint systems). This indicates the

reason for the selected terms "nonselfadjoint breaking"

On a comparative basis with the canonical breakings, the nonselfadjoint breakings are considerably more "powerful". In the former case the canonical formalism is still directly applicable and only the canonical formalism of the Galilei group is lost. In the latter case the entire canonical formalism is not directly applicable (in the sense of Section 2.8) because of the lack of existence of an analytic representation within the considered set of variables.

It is here appropriate to indicate that the class of nonselfadjoint breakings will be used in Volume II as the fundamental class of direct applicability of our Lie-admissible covering approach. The same approach will be proved to be also directly applicable for the classes of canonical and semicanonical breakings. In essence, this is the primitive idea of the direct universality of our Lie-admissible approach as compared to the lack of direct universality of the conventional Lie appraoch for the class of systems considered.

Nonselfadjoint breakings are not, of course, immune from possible misrepresentations. This is a consequence of the Theorem of Indirect Universality of Section 2.8.

To identify this aspect in more detail , let us recall that, despite the lack of existence of an analytic representation within the original set of variables, Theorem 2.8.1

indicates that an analytic representation in a new set of variables always exists for the class of systems considered. The origin of possible misrepresentations is due to the fact that the physical Galilei conservation laws can be satisfied in the new system of variables.

The resolution of this apparent inconsistency lies in the nature of the transformations needed to induce an analytic structure : they profoundly alter the physical content of the original system. Starting from a system without a Lagrangian structure, they must be such to induce a Lagrangian structure. Therefore, even though the two systems are equivalent on mathematical grounds, they are different from a physical profile.

An example is here useful to illustrate this point. Consider the system

$$\ddot{r} + \frac{1}{r}\dot{r}^2 = 0 \quad , m = 1 . \qquad (3.2.5)$$

In the given Cartesian coordinate r , the equation represents the motion of a particle subject to a nonlinear drag force. As such, all the physical Galilei conservation laws are violated.

System (3.2.5) is nonselfadjoint, i.e., it does not possess a Lagrangian structure as derived from Newton second law. We consider now the problem of the construction of an analytic representation under the transformation $r \to r' = r'(r)$.

Clearly, if this transformation is linear, the non-Lagrangian structure will be preserved. We therefore expect the existence of a nonlinear coordinate transformation which induces the desired Lagrangian structure. One solution is given by

$$r \to r' = \frac{1}{2}r^2, \qquad (3.2.6)$$
$$\dot{r}' = r\dot{r} \ , \quad \ddot{r} = r\ddot{r} + \dot{r}^2 ,$$

for which the new system is given by

$$\ddot{r}' = 0 , \qquad (3.2.7)$$

by reaching in this way a trivially selfadjoint system for which the Galilei relativity applies. The point is that systems (3.2.5) and (3.2.7) are <u>profoundly different on physical grounds</u> because, trivially, the former is non-conservative, while the latter represents a free motion.

This illustrates the reason why the transformations of Theorem 2.8.1, even though fully defined on mathematical grounds, should be considered with care when used for physical applications.

Notice that coordinate transformations to systems which obey the Galilei relativity exist also for systems which break the Galilei relativity in a canonical or semicanonical way.

As a matter of fact, system (3.2.5), strictly speaking, does not characterize a nonselfadjoint breaking, but instead a

mixture of canonical breakings (translations in space and Galilei boosts) and semicanonical breakings (translations in time and rotations) because the system is nonessentially nonselfadjoint. As an incidental remark, no motion along a straight line can characterize a nonselfadjoing breaking because, as proved by Darboux since 1891, an analytic representation for these systems within the original coordinates always exists. Thus, to reach a true nonselfadjoint breaking one must consider nontrivial equations in more than one dimension (i.e., motion not along a straight line).

Conceivably, the respect for Galilei's relativity is so rooted in all of us, that one might be tempted at the idea of preserving this relativity for the considered breakings via transformations to relativity obeying systems. This attitude, however, would be physically inconsistent. To illustrate this point, consider two systems in the physical Euclidean space of our experimental detection, one which obeys Galilei's relativity, (i.e., it is conservative) and the other which violates it (i.e., it is nonconservative). The attitude of transforming the relativity violating (nonconservative) system to a new system in a new set of variables which (is conservative and) obeys Galilei's relativity is entirely equivalent to the attitude of transforming the relativity preserving (conservative) systems into a new system in a new set of coordinates which (is nonconservative and) violates Galilei's relativity.

A unified characterization of the canonical and semicanonical breakings within the context of the Inverse Problem can be expressed as follows.

LEMMA 3.2.1: A sufficient condition for the canonical and the semicanonical breakings of the Galilei relativity in Newtonian Mechanics is that any admissible Lagrangian possesses essential multiplicative interaction terms to the free terms.

As we learned in Chapter 2, the essential presence of acceleration couplings is equivalent to the essential presence of multiplicative interaction terms to the free Lagrangian terms and both of them are equivalent to the presence of forces not derivable from a potential (i.e. nonconservative).

IV- SELFADJOINT BREAKING. This is the conventional approach to the (classical) breaking of any symmetry, inclusive of the Galilei symmetry. One starts with a symmetry preserving (SP) Lagrangian and adds a symmetry violating (SB) term, i.e.,

$$L^{SP}_{tot} = (L_{free} + L_{int})^{SP} \longrightarrow L^{SB}_{tot} = (L_{free} + L_{int})^{SP} + L^{SB}_{int} . \qquad (3.2.8)$$

Within the context of the Inverse Problem this approach is trivial. One starts with a symmetry-preserving system and adds a force which is symmetry-violating, but such that the

integrability conditions for the existence of a Lagrangian are preserved, i.e.,

$$\left(\mu_{k\mu k}\ddot{x}_{\mu k} - f_{\mu k}\right)_{SA}^{SP} = 0 \rightarrow \left[\left(\mu_{k\mu k}\ddot{x}_{\mu k} - f_{\mu k}\right)_{SA}^{SP} - F_{\mu k}\right]_{SA}^{SB} = 0. \quad (3.2.9)$$

The aspect which is important for us at this point is that, unless the symmetry breaking term L_{int}^{SB} is trivial (i.e., it possesses identically null Lagrange's derivatives), the quantities which are conserved for the original system are no longer conserved for the new system. Thus, the conventional symmetry breaking mechanism does not in general preserve the underlying dynamics (actual path, implicit functions and first integrals).

V-ISOTOPIC BREAKING. One contribution of the methodology of the Inverse Problem to symmetries and first integrals which we have identified in Sections 2.10 and 2.11 is that a symmetry can be broken also by selfadjointness preserving multiplicative terms (the isotopic factors) according to the rule

$$\left(\mu_k \ddot{x}_{ka} - f_{ka}\right)_{SA}^{SP} = 0 \rightarrow \left[h_{ka}{}^{ib}\left(\mu_i \ddot{x}_{ib} - f_{ib}\right)_{SA}^{SP}\right]_{SA}^{SB} = 0 \quad (3.2.10)$$

and corresponding Lagrangians

$$L_{tot}^{SP} = \left(\sum_{k=1}^{n}\sum_{\substack{a=\\x,y,z}} L_{free}^{(ka)} + L_{int}\right)^{SP} \rightarrow \quad (3.2.11)$$

$$\rightarrow L_{tot}^{SB} = \left(\sum_{k=1}^{n}\sum_{\substack{a=\\x,y,z}} L_{int,I}^{(ka)} L_{free}^{(ka)} + L_{int,II}\right)^{SB}.$$

The original symmetry is broken by the new Lagrangian in this mechanism. This breaking, however, is of rather peculiar nature in as much it leaves the underlying dynamics unaffected.

When the isotopic breaking is applied to the Galilei symmetry, the mathematical aspect is profoundly altered, but the physical content is unchanged. We therefore argue that this breaking is a Galilei symmetry breaking but not a Galilei relativity breaking.

To illustrate this peculiar occurrence it is sufficient to recall that (a) the variables are unaffected by construction, (b) the physical conservation laws remain unaffected also by construction and (c) the symmetry breaking occurs through the multiplication of a regular matrix of factor functions. On mathematical grounds this implies the transition from the original conventional Lagrangian to a nonessentially chiral Lagrangian and the transition from the Galilei symmetry group to an isotopic image, i.e., an isotopically mapped Galilei group. The construction of this group (when the integrability conditions for its existence are satisfied for the finite case) can be attempted with the Inverse Noether's Problem as recalled in Section 2.11. The point is that this new group is generally nonisomorphic to the conventional Galilei group for all nontrivially isotopic Lagrangians. This illustrates the change in the mathematical formulation of the Galilei relativity. The physical profile

is, on the contrary quite simple: by construction the isotopically mapped Galilei group (when it exists in a finite form) leads to the original physical Galilei conservation laws. Therefore, the Galilei relativity is fully preserved in this case, and only expressed in a mathematically different language.

As an example for a subgroup of the Galilei group, the isotopic mapping (2.11.53) replaces the group of rotation SO(3) with the nonisomorphic Lorentz group SO(2.1). However, these two symmetries are nothing but two different languages for the characterization of the same physical law: the conservation of the angular momentum.

From the semicanonical breaking we learned that the symmetry of a Lagrangian under the Galilei group does not guarantee the validity of the Galilei relativity. From the isotopic breaking we therefore learn the complementary property that the violation of the symmetry under the Galilei group by a Lagrangian does not guarantee the invalidity of the Galilei relativity.

These occurrences suggest that the formulation of the arena of applicability of the Galilei relativity we are interested in should be conducted at the level of our fundamental physical quantities, the equations of motion, rather than at the level of one of their possible Lagrangian representations. We reach in this way the following

PROPOSITION 3.2.1: An arena of applicability of the Galilei relativity is constituted by class C^∞,

regular, unconstrained, local Newtonian systems in the space and time coordinates of their experimental detection which are form - invariant under the Galilei transformations and for which the physical Galilei conservation laws are first integrals of the equations of motion.

From Sections 2.3 and 2.11 and Appendix 2.B it then follows that the arena under consideration is that of self-adjoint couplings which are form-invariant under the Galilei transformations.

3.3: ARENA OF NONAPPLICABILITY OF EINSTEIN'S SPECIAL RELATIVITY IN FIELD THEORY

We shall now begin the second part of our task, the indication that the field theoretical representation of the hadronic forces as local and not derivable form a potential is incompatible with Einstein's special relativity.

It should be indicated from the outset that our analysis is based on the fundamental concept that Field Theory is a consistent covering of Newtonian Mechanics of dual nature, in the sense that Galilei's relativity is replaced with Einstein's special relativity and the Newtonian concept of particle (as a massive point) is replaced by the concept of field (as a function of Minkowski coordinates). This occurence implies a deep relationship between the physical quantities of Newtonian Mechanics and Field Theory in the sense that any tensor of the latter theory, to qualify as a physical tensor, must yield at a Newtonian limit a physical quantity as identifiable by our everyday experience. This relationship between physical quantities of the two theories is expressed by their unified Definition 2.B.1.

In this way Newtonian Mechanics is assumed as our fundamental theory not only on conceptual but also on technical grounds. Out task is then reduced to the identification of an arena for which the conservation laws of the physical energy-momentum and angular momentum tensors are violated.

Definition 2.B.1 and the analysis of the preceding section then imply the following

CONTENTION 2: An arena of nonapplicability of the Einstein Special Relativity in Field Theory is constituted by class C^{∞}, regular (or totally degenerate), unconstrained, local, field equations in the Minkowski coordinates and field components of their experimental identification which are either nonessentially or essentially nonselfadjoint, i.e.,

$$\left\{\left[\left(\Box + \mu^2_{(\kappa)}\right)\phi_\kappa - j_\kappa(\phi^i, \phi^i_\alpha)\right]_{SA} - J_\kappa(x^\gamma, \phi^i, \phi^i_\alpha)\right\}_{NSA} = 0. \tag{3.3.1}$$

The above lemma expresses the central idea of the field representative of the eleton as a genuine nonconservative field, in the sense that its physical energy-momentum and angular momentum tensors are not conserved owing to energy, linear momentum and angular momentum exchanges with the hadronic medium.

Contention 2 is essentially introduced on grounds of the inability of Einstein's Relativity of being representative of physical conservation laws, in a way parallel to the corresponding Newtonian case of Section 3.2. Notice that the variational approach to selfadjointness provides a unified characterization of the Newtonian and field theoretical context.

The complementary analysis of the problem from the viewpoint of Poincaré covariance leads to the following classes of Einstein's

Relativity breakings in Field Theory.

I - CANONICAL BREAKING. Assume that Eqs. (3.3.1) are nonessentially nonselfadjoint. This means that they admit an analytic representation within the given variables. The emerging Lagrangian density, as we learned from Chapter 2, is of essential chiral type. In particular, this Lagrangian can be of the type

$$\mathcal{L}^{gen}_{tot}(\phi^k, \phi^k_{;a}) \qquad (3.3.2)$$

$$= \frac{1}{2}\left[\phi^{i;}_{\mu}\, G^{\mu\nu}_{i;j}(\phi^k)\,\phi^{j;}_{\nu} + 2\phi^{i;}_{\mu}\, F^{\mu}_{ij}(\phi^k)\phi^j + \phi^i E_{ij}(\phi^k)\phi^j\right],$$

where a possible explicit dependence of the chiral tensors on the Minkowski coordinates (as well as a dependence of the G-tensor on the field derivatives) has been ignored for reasons to be indicated below.

It is customarily believed that Lagrangian (3.3.2) is manifestly invariant under the Poincaré transformations because not explicitly dependent on the Minkowski coordinates and written in a form which appears to be manifestly invariant under the Lorentz transformations.

Our objective is to indicate that this belief is not necessarily true.

A fundamental requirement for any Lagrangian to be truly invariant under the Lorentz transformations is that its field transform covariantly under a (generally reducible) representation of the Lorentz group, i.e.

$$\phi'(x') = S(\Lambda)\phi(x), \quad x' = \Lambda x, \quad \Lambda \in \mathcal{L}^{\uparrow}_{+}. \qquad (3.3.3)$$

If this is the case, then the explicit structure of the Lagrangian considered can be inspected.

In particular, transformation law (3.3.3) severely restricts the fields $\phi(x)$. Most notably, the law implies that the fields have a definite spin consistent with the representations of the Lorentz group, i.e., they are scalars (spin zero), tensors (integer spin) or spinors (half-odd-integer spin).

Our objective is to indicate that the fields of essentially chiral Lagrangians do not necessarily transform covariantly under the Lorentz transformations. The existence of only one occurrence of this type is sufficient to establish the existence of the canonical breaking of Einstein's Relativity in Field Theory. Indeed, when transformation law (3.3.3) is broken, Lagrangian (3.3.2) is not invariant (or, equivalently, the field equations do not transform covariantly) under the Lorentz group in which case we shall write

$$\left\{\left[(\Box + \mu^2_{(k)})\phi_k - j_k\right]_{SA} - J_k\right\}^{CB}_{NSA} = 0, \qquad (3.3.4)$$

yielding a class of breaking which is in full analogy with the corresponding Newtonian class of Eqs. (3.2.2).

The existence of fields which violate the fundamental law (3.3.3) can be best studied by "linearizing" Lagrangian (3.3.2) or, equivalently, Eqs. (3.3.1). Again, the methodology of the inverse Problem is useful for this objective.

Consider for simplicity the case n = 1 and assume that the selfadjoint couplings of Eqs. (3.3.1) are the conventional electromagnetic couplings. As is well known, the linearization of the selfadjoint subsystem of Eqs. (3.3.1) leads to the conventional (selfadjoint) Dirac equation in the presence of electromagnetic interactions, Eqs. (1.5.1). The linearization of the full equations (3.3.1) then (necessarily) demands the implementation of these equations with additive couplings not derivable from a potential.

We reach in this way the nonselfadjoint equations

$$\left\{\left[\begin{pmatrix} -\hat{\bar{e}}_{,\mu}\gamma^{\mu} \\ \gamma^{\mu}\hat{e}_{,\mu} \end{pmatrix} + m\begin{pmatrix} \hat{\bar{e}} \\ \hat{e} \end{pmatrix} + \begin{pmatrix} f_{\hat{\bar{e}}} \\ f_{\hat{e}} \end{pmatrix}\right]_{SA} + \begin{pmatrix} F_{\hat{\bar{e}}} \\ F_{\hat{e}} \end{pmatrix}\right\}_{NSA} = 0, \quad (3.3.5)$$

where the f-terms are the usual electromagnetic couplings and the $\hat{\bar{e}}$ and $\hat{e}$ fields are conjugate in a way dependent on the selected representation of the γ matrices.

Our problem is now reduced to show that there exists nonselfadjoint F-couplings for which the fields $\hat{\bar{e}}$ and $\hat{e}$ violate transformation law (3.3.3).

By again using Newtonian Mechanics as the arena of our best intuition, we recall that the simplest possible form of

nonconservative force is that linearly dependent on the velocities. The field theoretical covering of this concept reads

$$F_{\hat{\bar{e}}} = \hat{\bar{e}}_{,\mu}\,\bar{\Gamma}^{\mu}(x^{\alpha}, \hat{e}, \hat{\bar{e}}, A), \qquad F_{\hat{e}} = \Gamma^{\mu}(x^{\alpha}, \hat{e}, \hat{\bar{e}}, A)\,\hat{e}_{,\mu} \quad (3.3.6a)$$

$$\Gamma^{\mu} \neq \bar{\Gamma}^{\mu}, \quad \mu = 0,1,2,3. \quad (3.3.6b)$$

where the Γ's are suitably selected matrices and condition (3.3.6b) is introduced to ensure the requirement of non-selfadjointness.

Our task can now be accomplished. The simplest possible choice for the Γ-matrices is that they are proportional to the conventional Dirac matrices and their possible explicit dependence is that on scalar combinations of the fields. We therefore reach the following realization of the nonselfadjoint couplings

$$\Gamma^{\mu} = g(\hat{\bar{e}}\cdot\hat{e})\gamma^{\mu}, \quad \bar{\Gamma}^{\mu} = \bar{g}(\hat{\bar{e}}\cdot\hat{e})\gamma^{\mu}. \quad (3.3.7)$$

It is easy to see that, for realization (3.3.7), the solutions of Eqs. (3.3.5) violate the fundamental transformation law (3.3.3). For instance, the equations considered imply the transition from the conventional spin tensor $\sigma^{\mu\nu}$ to a functionally generalized form of the type

$$\sigma^{\mu\nu} = \frac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) \rightarrow \quad (3.3.8)$$

$$\rightarrow \Gamma^{\mu\nu}(\hat{\bar{e}}\cdot\hat{e})$$

This clearly excludes all possible representations $S(\Lambda)$ of the Lorentz group for law (3.3.3) (whether reducible or not and finite or infinite dimensional) because all these representations are defined in a way independent from the fields to which they are applied, while any valid transformation law for the model considered (necessarily) demands a dependence of the transformation rules on the fields, i.e.

$$\phi'(x') = \hat{S}(\Lambda,\phi)\phi(x). \qquad (3.3.9)$$
$$x' = \Lambda x \ , \ \Lambda \in L_+^\uparrow$$

For more details see Appendix 3.A. We can now comment on the selected type of chiral Lagrangian (3.3.2). In essence, we have selected a structure without an explicit dependence on the Minkowski coordinates to emphasize the fact that a canonical breaking of the Poincaré symmetry can occur even under this circumstance. In general, however, an explicit functional dependence in the Minkowski coordinates of Eqs. (3.3.2), and, thus, of their Lagrangian, cannot be excluded. This yields a canonical breaking of the symmetry under the group of translations . The preceeding argument then yields a canonical breaking of the remaining components of the Poincaré group, the Lorentz group.

For additional details the interested reader is referred to Appendix 3.A and papers[32,33]. Of course, the breaking of the Poincaré symmetry we are here considering will appear in its proper light after the identification of the covering Lie-admissible treatment of Volume II.

II- SEMICANONICAL BREAKING. This breaking occurs when the field equations are nonessentially nonselfadjoint and transform covariantly under the Poincaré group, in which case we shall write, in analogy with the corresponding Newtonian case (3.2.3),

$$\left\{\left[\left(\Box + \mu_{(k)}^2\right)\phi_k - j_k\right]_{SA} - J_k\right\}_{NSA}^{SCB} = 0. \qquad (3.3.10)$$

It should be stressed that this breaking demands the validity of the fundamental transformation law (3.3.3). In turn, this stands to indicate that an essentially chiral Lagrangian does not necessarily imply a canonical breaking of the Lorentz symmetry. To be more specific in this point, given an essentially chiral Lagrangian, its fields can either transform covariantly under the Lorentz group or not, depending on the nature of the acting coupling. In the former case we have a semicanonical breaking, while in the latter we have a canonical breaking.

A simple case of semicanonical breaking is given when the nonselfadjoint couplings (3.3.7) do not depend on the local variables, i.e.

$$\Gamma'^\mu = g\gamma^\mu \ , \ \bar{\Gamma}'^\mu = \bar{g}\gamma^\mu \qquad (3.3.11)$$

This essentially implies the "rescaling" of the spin tensor

$$\sigma^{\mu\nu} \longrightarrow \Gamma'^{\mu\nu} = \text{Const.}\,\sigma^{\mu\nu} \qquad (3.3.12)$$

in which case the solutions of Eqs. (3.3.5) satisfy law (3.3.3).

Another example of semicanonical breaking is given when the field equations (3.3.10) do not depend explicitly on the Minkowski coordinates, while such a dependence appears at the level of their Lagrangian from a corresponding dependence of the genotopic functions of Theorem 2.5.7. This confirms the breaking of the conservation law of the physical energy-momentum tensor due to the nonselfadjoint couplings, in which case, as it is known, the breaking of the conservation law of the angular momentum tensor follows. Thus, the physical conservation laws of the Einstein relativity are in this case lost. Nevertheless, the field equations transform covariantly under the Poincaré group. This is precisely the structure of the semicanonical breaking of Einstein relativity in a way fully parallel to the corresponding Galilean case of Section 3.2.

A third example of semicanonical breaking is given by the complex scalar field in interaction with an external electromagnetic field, Eqs. (2.11.70). The equations are known to transform covariantly under the Poincaré group. Nevertheless, as recalled in Section 2.11, it is equally known that the physical conservation laws are in this case violated owing to the external nature of the electromagnetic field. The validity of this example for the case at hand is illustrated by the fact that the field equations, when written in the "natural ordering" $\{\psi, \bar{\psi}\}$, i.e., Eqs. (2.5.14), are nonselfadjoint.

In turn, this confirms that the gauge models exhibit the structure of our hadronic model, although in one of its simplest possible forms, as indicated in Appendix 2.C.

In line with the preceding remarks, Eqs. (2.11.70) do not depend explicitly on the Minkowski coordinates. Nevertheless a Lagrangian exhibiting such a dependence exists and is given by Eqs. (2.11.72). This confirms the semicanonical nature of the Poincaré symmetry breaking.

Finally, it should be recalled that this gauge invariant system is open because of the external nature of the electromagnetic field. This is also a peculiar aspect of our nonselfadjoint models, in the sense that they are used to characterize the field equations of the eletons conceived as open systems owing to the exchanges of energy, linear momentum and angular momentum with the hadronic medium. To put it in intuitive terms, the open nature of the nonselfadjoint gauge model (2.5.14) is due to the external character of the electromagnetic field while the open nature of our nonselfadjoint equations for the eletons is due to the assumed nature of the strong couplings.

The generalization of Lemma 3.3.1 for the unified treatment of the semicanonical or canonical breakings is trivial and reads

LEMMA 3.3.1: A sufficient condition for the canonical and the semicanonical breaking of Einstein's special relativity in field theory is that any admissible Lagrangian density is of essentially chiral type.

III - NONSELFADJOINT BREAKING . This breaking occurs when the field equations (3.3.1) are essentially nonself-adjoint, in which case a Lagrangian representation within the given variables does not exist. For this class of breaking the entire canonical formalism (and, thus, the canonical representation of the Poincaré group) cannot be introduced within the given field components, and transformation law (3.3.3) becomes undefinable. The clarification of this latter occurrence demands the analysis of Volume II in which we study the direct applicability for the class of systems considered of the Lie-admissible algebras. The covering nature of these algebras with respect to the Lie algebras, including that of the Poincaré group, identifies the "mechanics" of this class of breakings.

To avoid possible misrepresentations, it should be recalled that the Universality Theorem 2.8.1 is expected to extend to Field Theory. If this is the case, a Lagrangian for the representation of the systems considered exists, but necessarily in new variables. In particular, these new systems can be fully consistent with Einstein Special Relativity. This possible occurrence can be illustrated by transformations which map essentially nonselfadjoint systems in the field $\phi(x)$ into free field equations in new fields $\phi'(x')$.

The point is that the existence of an equivalent system of partial differential equations which is consistent with Einstein Special Relativity by no means should be construed as to imply that the original essentially nonselfadjoint system is

also compatible with the same relativity. And indeed, the same argument would imply that free field equations are incompatible with Einstein special relativity because of the existence of equivalent nonselfadjoint systems in new fields which violate such relativity.

Again, the nonapplicability (to the original system) of the relativity analysis of the equivalent system is due to the fact that the transformations which are needed to achieve an analytic structure profoundly alter the physical content of the original system ; while being noninertial and nonrealizable with experiments (Section 2.8).

On a comparative basis among the classes of breaking I, II, and III, the nonselfadjoint breakings are again the most "powerful" in the sense that in this case the entire canonical formalism cannot be introduced within the original variables. The semicanonical breakings are the "weakest", in the sense that in this case the canonical formalism of the Poincaré group can be fully defined and characterizes physical tranformations, but the physical conservation laws are lost. The canonical breakings are an intermediate form in the sense that the applicability of the canonical formalism within the given variables is guaranteed by the existence of an analytic representation, however, the canonical formalism of the Poincaré group is inapplicable.

In contrast to these breakings, a simple generalization of the arguments of Proposition 3.2.1 leads to the following

PROPOSITION 3.3.1: An arena of applicability of the Einstein special relativity in Field Theory is constituted by class C^∞, regular, unconstrained, local field equations in the Minkowski coordinates and field components of their experimental identification which are covariant under the Poincaré transformations and verify the conservation laws of the physical energy-momentum and angular-momentum tensors.

Within the context of the Inverse Problem this necessarily demands that the field equations are essentially selfadjoint or, equivalently, the couplings are selfadjoint according to Theorem 2.5.2.

IV - SELFADJOINT BREAKING. This is the conventional approach to symmetry breaking in (classical) Field Theory, i.e.,

$$\mathcal{L}^{SP}_{tot} = \left(\mathcal{L}_{free} + \mathcal{L}_{int}\right)^{SP} \rightarrow \mathcal{L}^{SB}_{tot} = \left(\mathcal{L}_{free} + \mathcal{L}_{int}\right)^{SP} + \mathcal{L}^{SB}_{int}, \tag{3.3.13a}$$

$$\left[\left(\Box + \mu^2_{(k)}\right)\phi_k - j_k\right]^{SP}_{SA} = 0 \rightarrow \left\{\left[\left(\Box + \mu^2_{(k)}\right)\phi_k - j_k\right]^{SP}_{SA} - j'_k\right\}^{SB}_{SA} = 0, \tag{3.3.13b}$$

and implies the transition from an essentially selfadjoint system to a new system which is also essentially selfadjoint. As such, we remain within the arena of applicability of Einstein's Special Relativity. Indeed, breakings (3.3.13) are primalily used for internal symmetries. Nevertheless, they cannot be excluded within the context of the Poincaré symmetry, e.g., when

the additive term $\mathcal{L}^{SB}_{int}$ has nonnull Lagrange's derivatives and an explicit dependence on the Minkowski coordinates.

V - ISOTOPIC BREAKING. This breaking can be written in field theory

$$\mathcal{L}^{SP,\,covr.}_{tot} \longrightarrow \mathcal{L}^{SB,\,Gen.}_{tot}, \tag{3.3.14a}$$

$$\left[\left(\Box + \mu^2_{(k)}\right)\phi_k - j_k\right]^{SP}_{SA} = 0 \rightarrow \left\{h_k{}^i\left[\left(\Box + \mu^2_{(k)}\right)\phi_k - j_k\right]^{SP}_{SA}\right\}^{SB}_{SA} = 0, \tag{3.3.14b}$$

yielding (when it exists) a nonessentially chiral Lagrangian (Section 2.10). The breaking is only formal in the sense that the conservation laws of the physical energy-momentum and angular momentum tensors are preserved. Nevertheless the symmetry of the Lagrangian leading via Noether's theorem to these conservation laws is mapped from the original Poincaré group to a different group, the isotopically mapped Poincaré group. The study of this structure, (as well as the conditions for its existence in a finite form), is not essential for our needs and it is therefore left to the interested reader.

For incidental considerations related to a possible incompatibility of models (3.3.1) with currently available theories of gravitation (for the interior case only) see Section 3.4. For the incompatibility of the same models with gauge symmetries see Appendix 3.B.

Almost needless to say, the breaking of Einstein's Special Relativity is expected to persist after (second) quantization. Thus, the analysis of this section has a number of implications

in Quantum Field Theory. For instance, it indicates from purely classical arguments that the models under considerations (say, of the essentially nonselfadjoint type) are outside of the arena if applicability of the Wightman axioms.[85] It then follows that fundamental tools, such as the PCT Theorem or the Spin-Statistics Theorem are inapplicable to the same class of models, because all these quantum field theoretical aspects are crucially dependent on the Poincaré symmetry (the PCT and Spin Statistics Theorems cannot be proved under the breaking of the Poincaré symmetry, because of the lack of existence of an analytic extension of the n-point functions of the Bargmann-Hall-Wightman type). All these rather crucial implications at the quantum field theoretical level can be inferred from the mere classical setting. As we shall indicate in Volume III, there appear to exist even deeper difficulties for the applicability of conventional quantum field theoretical tools to nonselfadjoint couplings. Indeed, irrespective (or as a complement of ?) of these symmetry considerations, the fundamental canonical commutation rules appear to be in question under nonself-adjoint couplings.

In short, according to all the indications available to me at this time, it appears that Quantum Field Theory, while so effective for selfadjoint couplings (including those of the unified gauge theories -- see App. 2.c), must be rebuilt literally "ab initio" for the broader class of local nonselfadjoint couplings.

3.4: ARENA OF NONAPPLICABILITY OF EINSTEIN'S SPECIAL AND GENERAL RELATIVITY IN PARTICLE DYNAMICS.

As is well known, Galilei's relativity was identified as incompatible with optics and electromagnetism in general, and with Maxwell's equations in particular, since the early stages of these disciplines.

This motivated the construction by A. Einstein of a covering of the Galilei relativity specifically conceived for the electromagnetic phenomenology, the Special Theory of Relativity, which subsequently proved to be fully consistent with the considered layer of the physical reality. As C. W. Misner, K. S. Thorne and J. A. Wheeler put it (reference,[86] page 304),

> « Of all the foundations of physics, none is more firmly established than special relativity; and of all the lessons of special relativity none stand out with greater force than these.
>
> (1) Spacetime, far from being stratified, is homogeneous and isotropic throughout any region small enough ("local region") that gravitational tide-producing effects ("spacetime curvatures") are negligible.
>
> (2) No local experiment whatsoever can distinguish one local inertial frame from another.
>
> (3) The speed of light is the same in every local inertial frame.

(4) It is not possible to give frame-independent meaning to the separation in time ("no Newtonian stratification").

(5) Between every event and every nearby event there exists a frame-independent, coordinate-independent spacetime interval ("Riemannian Geometry").

(6) Spacetime is always and everywhere locally Lorentz in character ("local Lorentz character of this Riemannian geometry"). >>

As by now familiar, a central contention of these monographs is that, despite its successes for the electromagnetic phenomenology, the Special Theory of Relativity is, in turn, incompatible with the strong interactions in general and the hadronic forces in particular, if these forces are local and nonderivable from a potential. It is understood that full compatibility is achieved if the hadronic forces are local and derivable from a potential. The complete lack of direct experimental verification at this time of the Special Relativity within a hadron, as well as of direct experimental informations on the nature of the hadronic forces, suggest the implementation of the above statements with the following

(7) The problem of the validity of properties (1) through (6) within a hadron is open at this time on both experimental and theoretical grounds.

The objective of this section is that of providing the

arguments which support the addition of statement (7). The analysis demands the study of the problem of particle dynamics under local but arbitrary forces and can be considered as a complement of that of Section 3.3 on the field theoretical profile of the problem.

This statement of the problem, however, is yet incomplete.

As is also well known, A. Einstein was aware of the incompatibility of his Special Relativity with gravitation. This motivated his studies of a covering of the Special Relativity specifically conceived for this broader layer of the physical reality, the General Theory of Relativity. The physical effectiveness of this broader relativity has been equally impressive. As the above quoted authors put it (reference 86, page 1131)

<< general relativity has emerged from each of its tests unscathed - a remarkable 1973 tribute to the 1915 genius of A. Einstein. >>

With an understanding that the problem under consideration demands a detailed separate treatment, gravitational aspects cannot be ignored in our analysis on several counts. In principle, the invalidity of the Special Relativity within a hadron could also be a desired feature, e.g., from the viewpoint of the General Relativity. After all, matter is essentially an aggregate of hadrons. It is therefore not inconceivable that gravitational considerations could play a central role for the problem of the hadronic structure.

Our objective therefore consists of the study of the compatibility or incompatibility of the assumed nature of the hadronic strong forces with both the Special and the General Theory of Relativity.

A central concept of the geometrical treatment of any relativity is that of geodesic (or geodesic motion). It is simply the extremal path of an arc-length functional, i.e.,

$$\delta a = \delta \int_{P_1}^{P_2} ds = 0 . \qquad (3.4.1)$$

The geodesic character of the Galilei relativity within its arena of applicability (as identified in Section 3.2) is expressed by Maupertius's Principle. For more details see Appendix 3.C.

In Special Relativity we have

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu , \qquad (3.4.2)$$

where $\eta_{\mu\nu}$ is the metric tensor of the (flat) Minkowski space. Principle (3.4.1) then yields the trivially selfadjoint geodesic equation

$$\left(\frac{d^2 x^\mu}{ds^2} \right)_{SA} = 0 , \quad \mu = 0,1,2,3, \qquad (3.4.3)$$

which, for consistency, must be complemented with an invariant concept of separation of the events. In momentum configuration

this yields the known mass hyperboloid

$$p^\mu p_\mu - m^2 = 0 , \quad p^\mu = m \frac{dx^\mu}{ds} . \qquad (3.4.4)$$

Thus, the geodesic of Special Relativity are the straight lines in Minkowski space which satisfy the condition of invariant separation of the events, i.e.,

$$x^\mu = s K^\mu + x_0^\mu \qquad (3.4.5a)$$

$$K^\mu K_\mu = 1 , \quad c = 1 \qquad (3.4.5b)$$

The admissible transformations are the inhomogeneous Lorentz (Poincaré) transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + x_0^\mu \qquad (3.4.6a)$$

$$\Lambda^\mu{}_\nu \eta_{\mu\lambda} \Lambda^\lambda{}_\rho = \eta_{\nu\rho} \qquad (3.4.6a)$$

In General Relativity we have

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \qquad (3.4.7)$$

where $g_{\mu\nu}(x)$ is now the metric of a Riemannian space. Principle (3.4.1) then yields the selfadjoint geodesic equations (Section 2.3)

$$\left[g_{\nu\mu} \frac{d^2 x^\nu}{ds^2} + \frac{1}{2} \left(g_{\nu\mu,\rho} + g_{\rho\mu,\nu} - g_{\nu\rho,\mu} \right) \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} \right]_{SA} = 0$$

$$g_{\nu\mu,\rho} = \partial g_{\nu\mu} / \partial x^\rho \qquad (3.4.8)$$

which, by introducing the Christoffel symbols (of the second kind)

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right), \quad (g^{\mu\nu}) = (g_{\mu\nu})^{-1}, \tag{3.4.9}$$

can be written in the equivalent nonselfadjoint form

$$\left[\frac{d^2 x^{\mu}}{ds^2} + \Gamma^{\mu}_{\rho\sigma}(x) \frac{dx^{\rho}}{ds} \frac{dx^{\sigma}}{ds} \right]_{NSA} = 0. \tag{3.4.10}$$

The admissible coordinate transformations now become arbitrary (provided that they are differentiable and everywhere invertible)

$$x^{\mu} \to x'^{\mu} = x'^{\mu}(x), \tag{3.4.11a}$$

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g'_{\mu\nu} dx'^{\mu} dx'^{\nu} \tag{3.4.11b}$$

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} g_{\rho\sigma} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \tag{3.4.11c}$$

This property expresses the covering nature of the General versus the Special Relativity as well as the reason for the term "General".

The fundamental nature of the concept of geodesic is expressed by the fact that the curvature can be associated with the deviation of one geodesic from a nearby geodesic. By using established technical procedures, this yields the Riemannian, Ricci and Einstein curvature tensors given respectively by

$$R^{\mu}{}_{\nu\alpha\beta} = \Gamma^{\mu}_{\nu\beta,\alpha} - \Gamma^{\mu}_{\nu\alpha,\beta} + \Gamma^{\mu}_{\rho\alpha} \Gamma^{\rho}_{\nu\beta} - \Gamma^{\mu}_{\rho\beta} \Gamma^{\rho}_{\nu\alpha}, \tag{3.4.12a}$$

$$R^{\mu}{}_{\nu} = R^{\alpha\mu}{}_{\alpha\nu}, \quad R = R^{\mu}{}_{\mu}, \tag{3.4.12b}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{3.4.12c}$$

In this way the geodesic equations (3.4.10) can be considered at the foundation of Einstein's equations for a massive body with null electromagnetic phenomenology (null total charge, moments and radiation)

$$\text{Exterior Problem}: \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{3.4.13a}$$

$$\text{Interior Problem}: \quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{\text{Matter}}_{\mu\nu}, \tag{3.4.13b}$$

with underlying Hilbert Action Principle (for the exterior problem)

$$\delta A = \delta \int_V dx \, R \sqrt{-g}. \tag{3.4.14}$$

Consider now the motion of a charged particle under the action of an electromagnetic field in Minkowski space. Eqs. (3.4.3) must be modified according to the selfadjoint extension (Section 2.3)

$$\left[m \frac{d^2 x^{\mu}}{ds^2} - e F^{\mu\alpha}(x) \frac{dx_{\alpha}}{ds} \right]_{SA} = 0. \tag{3.4.15}$$

Again, the Special Relativity is strictly valid for this broader system, as experimentally established on several independent grounds. For our needs it is here sufficient to recall that the transition from Eqs. (3.4.3) to (3.4.15) implies the preservation of an invariant separation of events which can be written in momentum space

$$(P^\mu - eA^\mu)(P_\mu - eA_\mu) - m^2 = 0 \qquad (3.4.16)$$

P^μ = Canonical Momentum

The applicability of Special Relativity then follows from the invariance of Eqs. (3.4.16) under Lorentz transformations.

In a Riemannian space Eqs. (3.4.15) are implemented into the nonselfadjoint form

$$\left[m \frac{d^2x^\mu}{ds^2} - eF^{\mu\nu}(x)\frac{dx_\nu}{ds} + m\Gamma^\mu_{\rho\sigma}(x)\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right]_{NSA} = 0 \qquad (3.4.17)$$

which is known to be derivable from an action principle within the set of variables x^μ. Thus, Eqs. (3.4.17) are nonessentially nonselfadjoint in our language, in precisely the same way as that of Eqs. (3.4.10).

By also using established techniques, this broader context yields Einstein's equations under the presence of electromagnetic fields

$$\text{Exterior Problem}: \quad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi T^{elm}_{\mu\nu}, \qquad (3.4.18a)$$

$$\text{Interior Problem}: \quad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi \left(T^{Matter}_{\mu\nu} + T^{elm}_{\mu\nu}\right), \qquad (3.4.18b)$$

We are now equipped to study the problem whether strong interactions in general and the hadronic strong forces in particular are compatible with the Special and General Relativity or not.

Consider first the current assumption that these forces are local and derivable from a potential. This implies the implementation of Eqs. (3.4.15) according to the selfadjoint extension *

$$\left\{\left[m\frac{d^2x^\mu}{ds^2} - eF^{\mu\nu}(x)\frac{dx_\nu}{ds}\right]^{elm}_{SA} - \left[S^\mu\left(x, \frac{dx}{ds}\right)\right]^{strong}_{SA}\right\} = 0 \qquad (3.4.19)$$

It is easy to see that these models are compatible with both the Special and the General Relativity. From Theorem 2.3.2 we know that the most general form of the strong forces, under the indicated assumption of selfadjointness, is that of being analytically equivalent to that of the electromagnetic forces, i.e.

$$S^\mu = S^{\mu\nu}(x)\frac{dx_\nu}{ds}, \qquad (3.4.20a)$$

$$S^{\mu\nu} + S^{\nu\mu} = 0, \qquad (3.4.20b)$$

$$S^{\mu\nu},_\rho + S^{\nu\rho},_\mu + S^{\rho\mu},_\nu = 0, \qquad (3.4.20c)$$

$$S^{\mu\nu} = B^{\mu},_\nu - B^{\nu},_\mu . \qquad (3.4.20d)$$

---

* Of course, models in which this additive nature of the strong and electromagnetic interactions at the level of the equations of motion is lost, are certainly conceivable. However, they will not be studied.

This implies the preservation of an invariant separation of the events, i.e., the implementation of Eq. (3.4.16) according to

$$(P^\mu - eA^\mu - B^\mu)(P_\mu - eA_\mu - B_\mu) - m^2 = 0, \quad (3.4.21)$$

which is fully invariant under the Lorentz transformations. The reformulation of this context in a Riemannian space then yields the compatibility with the General Relativity.

It should be here indicated, for completeness, that the currently assumed realization of the strong interactions is even more manifestly compatible with the Special and General Relativity because the strong forces are assumed to be velocity independent, but local and derivable from a potential (self-adjoint). This yields in momentum space, the preservation of the invariant (3.4.16) also for Eqs. (3.4.19), rather than invariant (3.4.21).

Suppose now that the hadronic strong forces are local but not derivable from a potential (nonselfadjoint). Then Eqs. (3.4.15) must be implemented according to the nonselfadjoint extension

$$\left\{\left[m\frac{d^2x^\mu}{ds^2} - eF^{\mu\nu}(x)\frac{dx_\nu}{ds}\right]^{em}_{SA} - \left[S^\mu\left(s,x,\frac{dx}{ds}\right)\right]^{strong}\right\}_{NSA} = 0 \quad (3.4.22)$$

where the Lorentz force is preserved under the assumption that the hadronic constituents are charged particles.

Eqs. (3.4.22) are incompatible with the Special Theory of Relativity. This can be seen from the fact that the separation of the events is not necessarily Lorentz invariant.

Indeed, by using canonical formulations, one reaches in momentum space an implementation of Eq. (3.4.16) according to

$$(P^\mu - eA^\mu)(P_\mu - eA_\mu) - m^2 = S\left(s,x,\frac{dx}{ds}\right), \quad (3.4.23)$$

where S is not necessarily a Lorentz scalar.

The incompatibility of the Special Theory with the hadronic models of type (3.4.22) can also be proved on other grounds here left to the interested reader. We restrict ourselves to few epistemological considerations which can be of assistance in the study of these additional arguments.

The fundamental difference between Eqs. (3.4.15) and (3.4.22) lies on their underlying physical assumptions. The former equations represent the motion of charged particles in the physical vacuum (intended as the absence of matter.) This context has proved to be fully consistent with properties (1) through (6) recalled at the beginning of this section. The latter equations are conceived to represent the motion of charged particles (our eletons) in a physical medium (our hadronic medium) which is precisely the origin of the nonselfadjoint forces. This different physical context is bound to produce deviation from the Special Relativity.

As stressed throughout our analysis, a nonselfadjoint coupling (force) is a genuine nonconservative force. This concept must apply irrespective of whether the considered context is relativistic or nonrelativistic because is requested on one side for the Special Relativity to be a consistent covering of the Galilei relativity, and on the other side by the unified character of the methodology of the Inverse Problem for these frameworks. In conclusion, the hadronic strong forces of model (3.4.22) must be interpreted, for consistency with their Newtonian limit, as genuinely nonconservative. This implies the lack of preservation of the conventional conservation laws of Special Relativity in a way fully parallel to the Newtonian analysis of Section 3.2, as the reader can verify by inspection. In turn, the loss of the customary conservation laws of physical quantities has a number of implications for Relativistic Dynamics for, say the relative character of simultaneity, the superposition of velocity, the relationship between mass and energy, etc. These aspects are here also left to the interested reader and they will be considered in more detail in Volume II when introducing our Lie-admissible approach for Eqs. (3.4.22).

There is, however, an epistemological argument which is more fundamental. Our concept of hadronic structure as constituted by charged particles moving in a physical medium allows the possibility of introducing "priviledged" reference frames which are at rest with respect to the hadronic medium, contrary

to the central assumption of the Special Relativity of the lack of existence of any priviledged frame of reference. Once technically implemented, the existence of priviledged frames of reference in our hadronic structure is sufficient to render inapplicable properties (1) through (6) above. Besides, by conception, our hadronic medium is expected to be nonhomogeneous and anisotropic, trivially, owing to the "geometry" of the structure: a finite number of particles of finite overlapping size, all within a finite volume. This adds further reasons for a distinction between the reference frames at rest with the hadronic medium and all other possible inertial systems in the physical vacuum outside the considered hadronic structure.

To put it in intuitive but effective terms, the concept of motion within a physical medium invalidates a central historical experiment of the Special Theory of Relativity, the Michelson-Morley experiment. Indeed, this experiment was centered on the motion in the physical vacuum (and resulted in the proof of the lack of existence of the ether conceived as a physical medium), while our hadronic context is centered on the existence of a physical medium which is responsible for forces more general than the Lorentz force.

Our next problem is to see whether Eqs. (3.4.22) are compatible or not with the General Theory of Relativity.

Rather intriguingly, there exists indeed a class of models (3.4.22) which is compatible with the General Theory. A

sufficient condition for a local force to violate the self-adjointness condition of Theorem 2.4.2 is that it is nonlinear in the velocities. Assume that this dependence is quadratic and the model is nonessentially nonselfadjoint. We reach in this way the model

$$\left\{\left[\mu \frac{d^2x^\mu}{ds^2} - e F^{\mu\nu}(x) \frac{dx_\nu}{ds}\right]^{elm}_{SA} - \left[\mu \Gamma^\mu_{\rho\sigma}(x) \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds}\right]^{strong}_{NSA}\right\} = 0, \tag{3.4.24}$$

where the Γ's can be interpreted as the Christoffel symbols, yielding a full equivalence with Eqs. (3.4.17).

The above result is quite intriguing on technical grounds. Eqs. (3.4.24) indicate that the force

$$S^\mu = \mu \Gamma^\mu_{\rho\sigma}(x) \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} \tag{3.4.25}$$

can be interpreted as a fully admissible realization of our nonselfadjoint couplings for the hadronic strong forces. As such, the hadronic structure model (3.4.24) invalidates the Special Relativity in favor of a full applicability of the General Relativity.

This result should not be considered as an occurrence of marginal significance. It confirms the expected existence of field theoretical models which are outside the arena of applicability of the Special Relativity but for which the General

Relativity applies. We are here referring, for instance, to models (3.3.5) with couplings (3.3.7) where $g = g(x)$, or to essentially chiral Lagrangians with G-tensor $G^{\mu\nu}_{ij} = \delta_{ij} \otimes g^{\mu\nu}$ in which case the fields are expected to transform covariantly in the sense of the General Theory.

On conceptual grounds model (3.4.24) is also intriguing. And indeed, it confirms the expectation that, in the final analysis, gravitational consideration cannot be ignored for the problem of the hadronic structure because, as recalled earlier, hadrons are the primary sourse of the gravitational field of any massive body.

We remain now with the question whether all possible models (3.4.22) are compatible with the General Theory of Gravitation. At this point the hierarchy of hadronic forces of increasing dynamical implications postulated in Section 1.3 should be recalled. As we shall see in Volume III, while the motion of particles under an electromagnetic field admits a unique form of the acting force (the Lorentz force), the motion of the same particle under the additional precence of hadronic forces (as local and not derivable from a potential) demands a variety of realizations of the hadronic strong forces depending on the state considered ($\pi^0, \pi^\pm, K^\pm$, etc.). This is essentially due to the increase in complexity of the structure with the mass, by keeping constant the charge radius of the particle as a whole according to the available experimental evidence.

This situation implies that model (3.4.24) has mainly a qualitative significance for our approach to the hadronic structure

and it is insufficient, if considered "per sé".

It is not difficult to see that our (discrete or field theoretical) model on the hadronic structure is, in general, incompatible with the General Theory of Relativity, as well as with the Unified Theories of conventional, supersymmetric or gauge type, unless a number of technical difficulties are resolved. Some of the problems can be formulated as follows.

(a) Eqs. (3.4.22) can be written in the form

$$\left\{\left[m\frac{d^2x^\mu}{ds^2} - eF^{\mu\nu}(x)\frac{dx_\nu}{ds}\right]^{em}_{SA} - \left[S^\mu_{\rho\sigma}\left(s,x,\frac{dx}{ds}\right)\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds}\right]^{strong}_{NSA}\right\} = 0 \tag{3.4.26}$$

Assuming that this form is expressible in terms of a "metric tensor", this tensor will exhibit also a dependence on proper time and velocities, contrary to the strict dependence of the Riemannian tensor of the Einstein General Relativity in the spacetime coordinates only. The field theoretical image of this occurrence is expressed by the fact that the G-tensor of an essentially chiral Lagrangian, Eqs. (2.9.33), is given by $G^{\mu\nu}_{ij}(x^\alpha, \phi^k, \phi^{k;}{}_\alpha)$, yielding a type of space which is outside the arena of applicability of Riemannian geometry as conventionally referred to.

(b) The conventional concept of curvature as deviation of a geodesic from a nearby geodesic is lost because the non-selfadjoint forces (or couplings) produce the lack of realization of the concept of geodesic in its conventional

form (3.4.10). More explicitly, if the Lorentz force of Eqs. (3.4.26) is put equal to zero, the emerging equations are not geodesic in the sense of Eqs. (3.4.10). This occurrence is after all expected from our central assumption that we have motion in a physical medium (see also Appendix 3.C).

(c) There is the loss of the conventional conservation laws of physical quantities. This is the case also for the charge in a field theoretical setting, as indicated in Appendix 3.B, irrespective of whether the equations are covariant or not under gauge transformations (semicanonical or canonical breaking of gauge symmetries). This creates a number of technical difficulties for any attempt to render applicable within the considered hadronic structure the conventional General Relativity, as well as its generalizations, say, of gauge type.

(d) The dominance of the nonselfadjoint over the selfadjoint couplings is so "strong" to render unidentifyable at the level of the integrand of an action functional (when it exists) the electromagnetic and the short range interactions. This occurrence is typical of our essentially chiral Lagrangian structures and creates additional technical difficulties, e.g., for the separation of the matter versus the electromagnetic energy-momentum tensor of Eqs. (3.4.18b). Last, but not least,

(e) The ultimate nature of our model is that of <u>not</u> being derivable from an action principle within the considered set of variables, contrary to the direct derivability from an action principle of the available models of gravitation (including unified, gauge and supersymmetric extensions).

As we shall see in Volume II through our Lie-admissible approach, it is the latter occurrence which indicates in the clearest possible terms the incompatibility of our hadronic structure with Einstein's relativity ideas.

But, the consistency of the General Relativity with available experimental data is unequivocal, as recalled earlier. The net effect is that one could attempt the invalidation of our hadronic model from its incompatibility with the General Relativity.

The fact that such an approach would be inconsistent on both theoretical and experimental grounds can be seen as follows. Consider a massive body with non-null total electromagnetic phenomenology, such as the star of our solar system, and inspect the applicable equations, Eqs. (3.4.18). They are divided into two sets of equations, one for the exterior and one for the interior problem.

Our contention is that the only <u>final</u> experimental knowledge which supports the validity of the Einstein-Maxwell equations is that for the exterior problem because, trivially, there is no final experimental evidence on the validity of

the corresponding equations, say, for the interior of a star.

This occurrence should not be underestimated. In actuality it opens the way to a possible dichotomy on the problem of gravitation which is compatible with currently available experimental evidence: the construction of generalized nongeodesic models of gravitation for the interior problem and the recovering of the conventional geodesic model for the exterior problem through the use of the Einstein-Maxwell equations (3.4.18a) as subsidiary constraints. In this way, the General Relativity would be violated by conception for the interior problem owing to the difficulties indicated earlier (lack of conservation laws, lack of direct derivability from an action principle, nongeodesic nature of the motion, lack of the concept of curvature as geodesic deviation, etc.). Nevertheless, under the assumption of consistency, the model would be in agreement with the currently available experimental data owing to the use of the Einstein equations as subsidiary constraints which, as such, reproduce the conventional structure for the exterior case (global conservation laws, global direct derivability from an action principle, geodesic nature of the exterior motion of test particles, curvature of the exterior problem as geodesic deviation, etc.). <u>On purely indicative grounds</u>, we consider equations of the type

$$\left\{\left[\mu\frac{d^2x^\mu_k}{ds^2}+\mu\,\Gamma^{\mu ij}_{k\rho\sigma}(x)\frac{dx^\rho_i}{ds}\frac{dx^\sigma_j}{ds}-eF^{\mu\nu}_{ki}(x)\frac{dx^i_\nu}{ds}\right]_{NE\text{-}NSA}\right.$$

$$\left.-S^{\mu ij}_{k\rho\sigma}\left(s,x,\frac{dx}{ds}\right)\frac{dx^\rho_i}{ds}\frac{dx^\sigma_j}{ds}\right\}_{E\text{-}NSA}=0 \qquad (3.4.27a)$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T^{eem}_{\mu\nu}, \qquad (3.4.27b)$$

$$\phi_z\left(s, x, \frac{dx}{ds}\right) = 0 \qquad (3.4.27c)$$

$$\mu,\nu,\rho,\sigma = 0,1,2,3 \; ; \; i,j,k = 1,2,\ldots,n \; ; \; z = 1,2,\ldots,m$$

where: (1) Eqs. (3.4.27a) are essentially nonselfadjoint (E-NSA) and represent the motion of a system of eletons, i.e., the motion of charged particles in a physical medium (the hadron), (2) the Ricci and Einstein tensors of Eqs. (3.4.27b) are the tensors of the associated nonessentially nonself-adjoint (NE-NSA) subsystem of Eqs. (3.4.27a), and (3) Eqs. (3.4.27c) represent any needed additional constraint (e.g., for the construction of a "global" metric for the exterior problem ?).

Of course, model (3.4.27) can be written in a variety of equivalent ways through the implementation of available techniques. This study is here left to the interested reader because, as indicated earlier, these monographs are not devoted to the technical treatment of the gravitational aspect of the problem of the structure of the hadrons. Nevertheless, a few concluding remarks of epistemological nature are in order.

The indicated dichotomy which is rendered conceivable by the lack of experimental verification of the General Relativity for the interior problem is, in the final analysis, the geometrical formulation of our concept of hadronic structure. Indeed, Eqs. (3.4.27) are the geometrical counterpart of our

primitive Newtonian equations for the hadronic structure, Eqs. (1.4.7), according to the following sequential steps: (A) relativistic generalization of the equations of the positronium, Eqs. (3.4.15), (B) further extension of the positronium structure in a Riemannian geometry, Eqs. (3.4.17), (C) representation of the hadronic forces through a further implementation of the equations into an essentially non-selfadjoint form, (D) departure from the Riemannian geometry as a result of the (additive) strong hadronic forces, and (E) recovering of the Riemannian geometry for the global exterior behavior (only) through the use of Einstein's equations as subsidiary constraints. In this way, by conception, the model should produce no deviation from established gravitational laws for the exterior problem. The limit to a flat Minkowski space than implies the lack of deviation from Special Relativity for the behaviour of the hadron as a whole under electromagnetic interactions, jointly with a departure from this relativity for the interior problem, to account for physical difference between the electromagnetic and the strong interactions. The extension of this approach to field theoretical models is straingtforward and can be technically implemented by using known techniques for the field theoretical version of steps (A) through (E) above.

We cannot close this section with a few remarks in relation to the problem of the unified theory of gravitation, electromagnetic and short range interactions. To the best of my knowledge, the currently available unified theories are based on the assumption

that the gravitational field is physically distinct from the electromagnetic field.

In paper[87] I studied this problem for the intent of indicating the existence of an alternative approach consisting of the geometrical identification of the gravitational field of a massive body with the fields of its elementary constituents, the electrons for the peripheral atomic clouds and the hadronic constituents for the nuclear structure. More specifically, the model consisted of the geometrical identification of the source tensor of the gravitational field with the energy-momentum tensor of the electromagnetic (and short range) fields of the elementary constituents, irrespective of whether the body has a null or a nonnull total electromagnetic phenomenology.

The analysis of paper[87] was primarily restricted to (a classical approximation of) the π^o. The reason was that (lacking an experimentally established structure model of the nucleons) any massive body can be considered as at a naive approximation as a collection of a large number of π^o with a rather widely accepted structure as a bound state of a charged particle and its antiparticle. Also, the π^o is the lightest known "hadronic image" of a massive body with null total electromagnetic phenomenology. As a result, if the problem of unified theory can be resolved within the context of the π^o, it is conceivably extendable to charged particles, nuclei and, thus, massive body.

The mechanics of our unified model of the π^o was quite simple. One first constructs the conventional gravitational source tensor for this object by using conventional means (mass density), and

then identifies the energy-momentum tensor of the fields responsible for the π^o structure which produces the same gravitational field. In this way the model is fully within the arena of available experimental evidence. Only the physical interpretation of the symbol of the gravitational source is changed.

It is here significant to recall that, in order to achieve this geometrical identification of the gravitational field of the π^o with its structure fields, its constituents had to be very light. And indeed, paper[87] contains my first formulation of the hypothesis that the ordinary electrons and positrons are conceivable candidates for the π^o constituents, although in a "mutated" form (footnote[10], paper page 150). It is therefore of some significance to point out that the gravitational considerations of paper[87], even though of admittedly qualitative character, arrived at the conclusion that the massive particle of spontaneous decays are conceivable candidates for the hadronic constituents along a line of study independent from that of these monographs.

This line of study is clearly significant for the problem of a unified theory. The primary purpose of paper[87] was to point out that, if one assumes that the gravitational field is physically different than the electromagnetic field, one approach for the problem of the unified theory is requested. However, if one assumes that the gravitational field can be geometrically identified with the structure fields, a different and nonequivalent approach is clearly needed. Even though the two approaches are not differentiated on quantitative grounds by construction,

they differ on conceptual grounds. In the transition from the former to the latter approach there is the shift from the problem of the "unified theory" that that of the "origin" of the gravitational field.

Consider the (geodesic) motion of a test particle in the neighborhood of a massive object. Assume, at a naive approximation, that this object is the π^o. The problem of the "origin" of the gravitational field can be essentially formulated as consisting of the physical nature of the energy-momentum tensor which produce the gravitational field of the object considered and, thus, the curvature in its environment. Since the object has null total electromagnetic phenomenology, the conventional approach is that of assuming that this tensor is the energy-momentum tensor of matter as characterized by the mass density. But any mass term is a technical expedient to avoid the problem of structure. When the structure of the considered object is taken into account, a different perspective emerges. Since this object is a bound state of a charged particle and an antiparticle, by central assumption of the Einstein theory the electromagnetic field of these constituents contributes to the gravitational field of the object as a whole.

Paper [87] was then devoted to the classification and initial study of all conceivable alternatives which are called

(A) Strong assumption, it occurs when the electromagnetic fields of the π^o constituents are entirely responsible for the gravitational field of this particle. Intriguingly, the possible presence of a "mutation" of the electromagnetic field within the π^o was argued from the null total electromagnetic phenomenology.

This is in line with the "mutation" (3.A.12) of the electromagnetic field for the same structure.

(B) Weak assumption, it occurs when the electromagnetic and the short range fields of the π^o constituents are responsible for the gravitational field of this structure. Notice that for both the strong and the weak assumption the origin of the gravitational field is associated with the field of the hadronic constituents without any contribution from mass terms.

(C) Standard assumption. Since the π^o has null total electromagnetic phenomenology, this is the conventional assumption that the electromagnetic field of the constituents do not contribute to the gravitational field of the state as a whole.

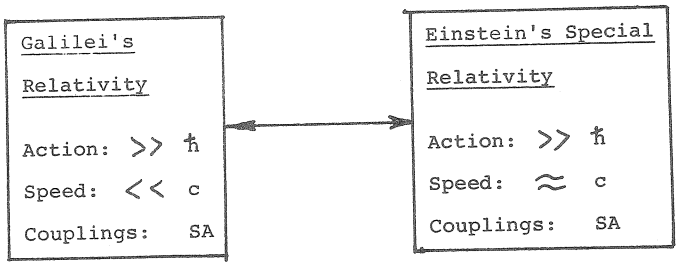
The paper concluded with a number of problematic aspects for the standard assumption because it appears to contradict the central idea of the Einstein-Maxwell theory of gravitation according to which "any" electromagnetic field and, thus, including that of the hadronic constituents, is a source of the gravitational field.

In conclusion, what we are advocating is the study of the next conceivable step for the problem of gravitation: the abandonment of mass terms in the equations for the interior problem and the confrontation instead of the structure of matter. This inevitably brings into focus the problem of the compatibility or incompatibility of current knowledge on gravitation with the strong interactions. The intent of this section was to indicate that there is incompatibility already when the strong interactions are realized in terms of local but nonselfadjoint forces, with an understanding that this incompatibility is selfevident in case the strong interactions necessarily demand nonlocal realization nonderivable from a potential.

3.5: CONCLUDING REMARKS

By looking in retrospect, one of the objectives of this monograph was the completion of the identification of the arena of applicability of Galilei's and Einstein's relativities according to the schematic diagram:

ARENA OF CLASSICAL APPLICABILITY OF ESTABLISHED RELATIVITIES



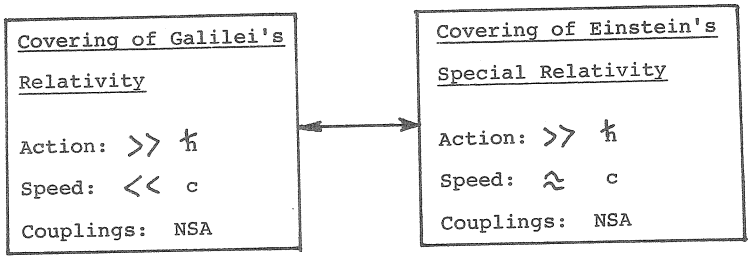
where $\hbar$ is Planck's constant, c is the speed of light, SA stands for couplings which are selfadjoint, and the arrows indicate the known expansion-contraction techniques.

The first element of characterization, the value of the action functional, is introduced, as is known, to identify a classical context. The second element, the admissible value of the speed, is introduced to differentiate the arena of applicability of Galilei's versus Einstein's relativity, as is also well known. The third element, the nature of the acting couplings (or forces), is introduced to identify an arena in which the relativity considered is representative of physical conservation laws (Lemmas 3.2.3 and 3.3.3). To the best of my

knowledge, this third element of characterization is introduced in paper[32] apparently for the first time .

The second objective of this monograph was to establish the need of searching for broader relativity ideas for the case of nonselfadjoint couplings according to the schematic diagram:

ARENA OF CLASSICAL APPLICABILITY OF POSSIBLE COVERING RELATIVITIES



where the couplings are now nonselfadjoint (NSA).

The terms "nonapplicability of established relativities" used in this nomograph deserve a comment to avoid possible misrepresentation. We here assume that the reader is familiar with the significance of the Galilei and Poincaré transformations for inertial considerations. This significance remains entirely unaffected by our analysis: the Galilei and Poincaré transformations are and remain the applicable transformations for the transition from one inertial frame (observer) to another.

Our efforts have been devoted instead to the study of the behaviour of given physical systems under the relativity transformations. If the couplings are (covariant and) selfadjoint

the established relativity transformations enjoy the additional significance of being representative of form-invariant physical laws. If the couplings are nonselfadjoint and the breaking is semicanonical the Galilei and Poincaré transformations can still be applied to the system considered as it is the case for any class C^{∞}, invertible transformation of the local variables. However, they now loose their physical meaning as being representative of (form-invariant) conservation laws. Finally, if the breaking is canonical or nonselfadjoint, the Galilei and Poincaré transformations exhibit the additional loss of their conventional meaning as characterizing relativity laws. Conventional gravitational coverings result to be also generally inapplicable.

This situation establishes the need of searching for the possible existence of covering relativities (first for a flat carrier space) for the case of nonselfadjoint couplings. This problem will be considered in Volume II.

With respect to our model of the hadronic structure, the problem of the possible existence of covering relativities is clearly of fundamental nature.

The primitive origin of the problem considered has been associated to the assumption that the hadronic structure is constituted of charged, massive and spinning particles each moving within the charge radius of the other, thus creating the motion in a physical medium (our hadronic medium). This framework should be compared with that of conventional relativities, which

essentially deal with the motion of particles in the physical vacuum under actions at a distance. The different nature of the physical medium of our hadronic structure is then sufficient "per sé" to suggest the search for covering relativities, as indicated in Section 3.4.

The first and simplest formulation of the problem is of Newtonian character and consists of the search for a group of transformations $\hat{G}(3.1)$ which: (A) leaves form-invariant local nonconservative systems

$$\hat{G}(3.1): \frac{dq^{\mu}}{dt} - \tilde{\dot{p}}^{\mu}(t,q) = 0 \rightarrow \frac{d\hat{q}^{\mu}}{d\hat{t}} - \tilde{\dot{p}}^{\mu}(\hat{t},\hat{q}) = 0$$

$$(q^{\mu}) = \begin{pmatrix} x_k \\ p_k \end{pmatrix}, \quad (\tilde{\dot{p}}^{\mu}) = \begin{pmatrix} p_k/m \\ f_k^{SA}(x) + F_k^{NSA}(t,x,p) \end{pmatrix} \tag{3.5.1}$$

(B) characterizes, via its behaviour in the neighborhood of the identity, the nonconservative nature of the systems and (C) recovers the Galilei group G(3.1) identically at the limit of null value of the Galilei symmetry-breaking forces

$$\lim_{F^{NSA} \to 0} \hat{G}(3.1) = G(3.1) \tag{3.5.2}$$

The search for this covering form-invariant description of nonconservative Newtonian Mechanics and for the underlying covering relativity is the primary objective of Volume II. The problem of the construction of the the expected relativistic extension will be considered as sequential.

In conclusion, the contents of this first volume can be expressed with the following most crucial

CONTENTION 3: The Galilei relativity, the Einstein special relativity and the Einstein general relativity for the interior problem are incompatible with the strong interactions.

I hope that this contention is not subjected to misrepresentations. First of all, it is not intended to deny the validity of the relativities considered as a first approximation of the context here referred to, which would be contrary to the same spirit of Theoretical Physics. In any case, as I shall elaborate in detail in Volume III, the relativities considered can be interpreted as applicable under the approximation of point-like constituents of hadrons.

I have provided my best efforts to present Contention 3 and treat it throughout this volume in the traditional spirit of the pursuit of fundamental knowledge, that is, to stimulate the experimental resolution of the issue, in due time, either in favor or against established relativities. Also, I have attempted to present Contention 3 in the traditional way of pursuing such fundamental knowledge, that is, via criticisms of experimentally unverified, basic, physical laws.

In any case, Contention 3 is the expression of my deep conviction that the imposition to the strong interactions of the same relativities for the electromagnetic interactions implies a form of dynamical equivalence between these interactions which does not appear to be conform to nature. Equivalently, Contention 3 is an expression of my deep conviction that the strong interactions cannot be represented via the simplest conceivable forces, that is, local and derivable from a potential, but demand instead more complex forces which are more adequate for the complexity of the physical reality to be represented. Also equivalently, Contention 3 is an expression of my belief, as indicated in the Preface, that Theoretical Physics is a Science which will never admit terminal disciplines.

APPENDIX 3.A:

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GENERALIZATION OF DIRAC'S EQUATION TO VARYING SPIN

In this appendix we shall study in more details one of the simplest possible generalizations of Dirac's equations with "nonconservative" couplings, Eqs. (3.3.5), i.e.

$$\left\{\left[\begin{pmatrix} -\hat{e}^{\dagger}_{;\mu}\gamma^{\mu} \\ \gamma^{\mu}\hat{e}^{\dagger}_{;\mu}\end{pmatrix} + m\begin{pmatrix}\hat{e}^{\dagger}\\ \hat{e}\end{pmatrix} + \begin{pmatrix} f_{\hat{e}}\\ f_{\hat{e}}\end{pmatrix}\right]_{SA} + \begin{pmatrix} F_{\hat{e}}\\ F_{\hat{e}}\end{pmatrix}\right\}_{NSA} = 0 \quad (3.A.1)$$

The type of couplings we are interested in are

$$F_{\hat{e}} = -g^{\mu}\hat{e}^{\dagger}_{;\mu}\gamma^{\mu}, \quad F_{\hat{e}} = g^{\mu}\gamma^{\mu}\hat{e}_{;\mu} \quad (3.A.2)$$

where the g's are free parameters independent of the Minkowski coordinates and the γ's are the conventional Dirac's matrices. The f's are the conventional electromagnetic couplings, Eqs. (1.5.1b).

Under assumption (3.A.2), Eqs. (3.A.1) can be written

$$\left[\begin{pmatrix} -\hat{e}^{\dagger}_{;\mu}\Gamma^{\mu} \\ \Gamma^{\mu}\hat{e}_{;\mu}\end{pmatrix} + m\begin{pmatrix}\hat{e}^{\dagger}\\ \hat{e}\end{pmatrix} + \begin{pmatrix} f_{\hat{e}}\\ f_{\hat{e}}\end{pmatrix}\right]_{SA} = 0 \quad (3.A.3a)$$

$$\Gamma^{\mu} = (1+g^{\mu})\gamma^{\mu} \quad (\text{no summation}) \quad (3.A.3b)$$

and, from Theorem 2.5.10, they admit the Lagrangian density

$$\mathcal{L}^{gen}_{tot} = \frac{1}{2}\left(\hat{e}^{\dagger}_{;\mu}\Gamma^{\mu}\hat{e} - \hat{e}^{\dagger}\Gamma^{\mu}\hat{e}^{\dagger}_{;\mu}\right) - m\hat{e}^{\dagger}\hat{e} - ie\,\hat{e}^{\dagger}\gamma^{\mu}A_{\mu}\hat{e} \quad (3.A.4)$$

An inspection indicates that this Lagrangian is of a simple but generalized nature. In particular, it satisfies the requirements of Definition 2.12.3 for the characterization of the eletonic field. Notice that the minimal coupling rule is no longer applicable in its conventional form. And indeed, Lagrangian (3.A.4) can be explicitly written

$$\mathcal{L}^{gen}_{tot}$$

$$= \frac{1}{2}\left\{\left[(1+g^\mu)\,\hat{\bar{e}}\,i\partial_\mu + ieA^\mu\right]\gamma_\mu\hat{e} - \hat{\bar{e}}\,\gamma^\mu\left[(1+g_\mu)\,i\partial_\mu - ieA_\mu\right]\right.$$
$$\left. - m\,\hat{\bar{e}}\,\hat{e}\right\} \qquad (3.A.5)$$

Rather than considering it as a drawback, this occurrence is interpreted as essential to achieve a nontrivial differentiation between the electromagnetic and the strong interactions according to the assumptions of Sections 1.2 and 1.3.

Our interpretation of Eqs. (3.A.3) is that they represent Dirac's particles after penetration into a hadronic medium. The F-couplings are then interpreted as a simple but nontrivial realization of the action of the hadronic medium on Dirac's particles (strong couplings). Our problem is that of identifying some of the most important implications of these couplings.

A direct way of achieving this objective is through the identification of the second order differential equations

satisfied by the eletonic fields. To strengthen the similarity with the conventional case, we introduce the "scaled Minkowski coordinates"

$$y^\mu = \frac{x^\mu}{1+g^\mu} \qquad (3.A.6)$$

and write for the $\hat{e}$- field

$$\left[\left(\gamma^\mu\frac{\partial}{\partial y^\mu} + ie\gamma^\mu A_\mu\right) - m\right]\left[\left(\gamma^\mu\frac{\partial}{\partial y^\mu} + ie\gamma^\mu A_\mu\right) + m\right]\hat{e} \qquad (3.A.7a)$$
$$= \left[\left(\frac{\partial}{\partial y^\mu} + ieA^\mu\right)\left(\frac{\partial}{\partial y_\mu} + ieA_\mu\right) + \frac{1}{2}e\sigma^{\mu\nu}\hat{F}_{\mu\nu} - m^2\right]\hat{e} = 0$$

$$\hat{F}_{\mu\nu} = \frac{\partial A_\mu}{\partial y^\nu} - \frac{\partial A_\nu}{\partial y^\mu}\,, \quad \sigma^{\mu\nu} = \frac{1}{2i}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \qquad (3.A.7b)$$

An equivalence case occurs for the $\hat{\bar{e}}$- field.

The crucial term is given by

$$\frac{1}{2}e\sigma^{\mu\nu}\hat{F}_{\mu\nu} = \hat{\underline{\mu}}\cdot\underline{H}_m + \hat{\underline{m}}\cdot\underline{E}_m \qquad (3.A.8)$$

and it indicates the presence of the <u>modified forms of the magnetic and electric dipole moments</u> given respectively by

$$\hat{\mu}^k = -(1+g^k)\frac{e\hbar}{2mc}s^k\,, \qquad \hat{m}^k = i(1+g^k)\frac{e\hbar}{2mc}\alpha^k\,; \qquad (3.A.9a)$$

$$(s) = \begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix}, \quad (\alpha) = \begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix}. \qquad (3.A.9b)$$

This occurrence is here considered as an indication that Eqs. (3.A.3) are simple but genuine characterization of the eleton field. And indeed, Eqs. (3.A.9) inevitably imply a departure of the spin of Dirac's field characterized by the transition

$$\sigma^{\mu\nu} = \frac{1}{2i}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \to \hat{\sigma}^{\mu\nu} = \frac{1}{2i}\left(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu\right). \tag{3.A.10}$$

Under the assumption that the g's are arbitrarily varying parameters, the generalized spin tensor $\hat{\sigma}^{\mu\nu}$ no longer characterized a spin 1/2, despite the fact that the γ-matrices of Eqs. (3.A.3) are the conventional Dirac's matrices. Eqs. (3.A.3) then characterize a canonical breaking of Einstein relativity according to the definition of Section 3.3. In particular, at the limit when the g's are all equal among themselves and constant, the breaking becomes semicanonical as indicated in Section 3.3.

In the final analysis, the loss of the conventional concept of spin is expected from purely Newtonian considerations, such as the loss of the group of rotation as a methodological tool for the physical spinning top.

As an incidental remark, notice that under the "scaled coordinates" (3.A.6), Lagrangian (3.A.4) can be written

$$\mathcal{L}^{geu}_{tot} = \frac{1}{2}\left[\left(\frac{\partial\hat{\bar{e}}}{\partial y^\mu} - ieA_\mu\right)\gamma^\mu\hat{e} - \hat{\bar{e}}\gamma^\mu\left(\frac{\partial\hat{e}}{\partial y_\mu} + ieA_\mu\right)\right] - m\hat{\bar{e}}e, \tag{3.A.11}$$

by formally exhibiting no representative of the considered realization of the strong couplings.

Alternatively, one can introduce the "scaled electromagnetic potentials"

$$\hat{A}_\mu = \frac{A_\mu}{1+g^\mu} \tag{3.A.12}$$

for which Lagrangian (3.A.4) becomes

$$\mathcal{L}^{geu}_{tot} = \frac{1}{2}\left[\left(\hat{\bar{e}}^{;}_{\mu} - ie\hat{A}_\mu\right)\Gamma^\mu - \hat{\bar{e}}\Gamma^\mu\left(\hat{e}^{;}_{\mu} + ie\hat{A}_\mu\right)\right] - m\hat{\bar{e}}\hat{e} \tag{3.A.13}$$

The implementation of the model considered with the subsidiary constraints to attempt a structure model of the π^o along the lines of Eqs. (1.5.7) is here left to the interested reader.

APPENDIX 3.B:

ARENA OF NONAPPLICABILITY OF GAUGE SYMMETRIES AS REPRESENTATIVE OF PHYSICAL LAWS.

As is well known, the charge conservation law in Field Thoery is customarily associated to the symmetry of a Lagrangian under (global or local and Abelian or nonAbelian) gauge transformations. It is therefore important to identify the arenas of applicability and nonapplicability of this symmetry as representative of a physical law.

The first part of this task is easily accomplished.

> An arena of applicability of the gauge symmetries as representative of the conservation law of the total physical charge is constituted by class C^{∞}, regular (or totally degenerate), unconstrained, local, essentially selfadjoint complex field equations in the Minkowski coordinates and field components of their experimental identification which are covariant under the gauge transformations and verify the differential conservation law of the physical charge current.

Again, this arena of applicability of gauge symmetries is by no means exhaustive. It is merely sufficient for our needs. Notice the absence of the concept of Lagrangian owing to the fact identified in Section 2.11 that, when the physical total

charge is conserved, an admissible Lagrangian may violate the symmetry under gauge tranformations. The existence of at least one Lagrangian which is invariant under gauge transformations and leads via Noether's theorem to the differential conservation laws of the physical charge current is however implicit from the analysis of Sections 2.5 and 2.11. The terms "arena of applicability" should not be confused with "arena of effectiveness", in the sense that the conditions considered do not guarantee the integral conservation of the total physical charge, but only the differential conservation law of the charge current. Thus, the arena considered implies that the gauge symmetries are applicable but not necessarily effective under the indicated conditions. The same remarks hold also for 3.2 and 3.3. The terms "field equations in the Minkowski coordinates and field components of their experimental identification" are intended to exclude the study of equivalent systems under transformations of the local variables which alter the physical content of the system, e.g., $\phi^F \to \phi'^K(\phi^i, \phi^i{}_{,\alpha})$. Finally, the condition of regularity has been introduced mainly for consistency with the variational approach to selfadjointness presented in Section 2.5 and, on strict grounds, should be dropped because electromagnetic interactions of complex fields are known to produce degenerate Lagrangians. This step is here left to the interested reader. The arena considered is sufficient for our needs, despite the restriction of regularity, because electro-

magnetic interactions (at the classical level) do not occur through charge exchange processes and, thus, we can consider the degeneracy inducing terms (the Maxwell equations) as external. Under these assumptions, the field equations admitted virtually include all currently available models of gauge theories.

The identification of the arena of nonapplicability of the gauge symmetries is considerably more delicate owing to the apparently unsolved problem of the identification of the "physical total charge" for nonselfadjoint systems of the type

$$\left\{\left[A_{ki}\,(\Box + m^2)\,\underline{\Phi}^i + B_k\right]_{SA} + C_k\right\}_{NSA} = 0 \qquad (3.B.1a)$$

$$(A_{ki}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \{\underline{\Phi}^i\} = \{\varphi, \bar{e}\} \qquad (3.B.1b)$$

For consistency with the case of the Poincaré symmetry we here assume Definition 2.B.1, i.e., the physical total charge of system (3.B.1) is that of the maximal associated selfadjoint system. The remaining nonselfadjoint couplings are then responsible for the lack of conservation of this quantity.

An arena of nonapplicability of gauge symmetries as representative of the conservation

law of the total physical charge is constituted by class C^∞, regular (or totally degenerate) unconstrained, local, complex field equations in the Minkowski coordinates and field components of their experimental identification which are either nonessentially or essentially nonselfadjoint.

This is illustrated by Eqs. (2.11.77) which represent an implementation of the conventional selfadjoint equations of complex scalar fields with nonselfadjoint couplings according to structure (3.B.1).

Again, the latter context is introduced on grounds of the nonconservation of the charge current. The analysis, however, can be complemented with the study of the problem from the viewpoint of the transformation theory. This yields the canonical, semicanonical, nonselfadjoint, selfadjoint and isotopic breakings of gauge symmetries, in full analogy with the corresponding case of the Poincaré symmetry. This is here left the interested reader.

For the case of nonessentially nonselfadjoint systems we again have an essentially chiral Lagrangian which may or may not be invariant under gauge transformations. In both cases the gauge symmetries are not representative of physical conservation laws.

The context of this appendix complements that of Sects.3.1-3.5 for the characterization of the concept of eletonic field as a genuine nonconservative field. The latter indicates the nonconservation of the

physical energy-momentum and angular momentum of the field owing to exchanges with the hadronic medium. The former completes this framework with the nonconservation of the physical charge current owing to charge exchange processes with the hadronic medium which now are rendered identifyiable already at the classical level.

Finally, notice that the arenas of nonapplicability of the Poincaré and gauge symmetreis as representative of physical laws coincide.

APPENDIX 3.C:

THE NONGEODESIC NATURE OF OUR HADRONIC STRUCTURE MODEL

Consider a conservative system in the Euclidean space of its experimental detection with Cartesian coordinates $\underset{\mu}{r}^k$. Let L^{conv}_{tot} (H^{conv}_{tot}) be the Lagrangian (Hamiltonian) for the representation of this system in $\{\underset{\mu}{x}^k\}$. Among all the admissible varied paths $\underset{\mu}{r}^k \rightarrow \underset{\mu}{r}'^k = \underset{\mu}{r}^k + \Delta \underset{\mu}{r}^k$ select those for which

$$\Delta \underset{\mu}{x}^k = \frac{\partial H^{conv}_{tot}}{\partial \underset{\mu}{p}_k} \Delta t \,, \quad \Delta \underset{\mu}{p}_k = -\frac{\partial H^{conv}_{tot}}{\partial \underset{\mu}{x}^k} \,, \quad \underset{\mu}{p}_k = \frac{\partial L^{conv}_{tot}}{\partial \dot{\underset{\mu}{x}}^k} \,, \tag{3.C.1}$$

$$\Delta \underset{\mu}{x}^k(t_s) = 0 \,, \; s = 1,2 \,, \quad \Delta \dot{\underset{\mu}{x}}^k = \frac{d}{dt} \Delta \underset{\mu}{x}^k \,, \quad \Delta H^{conv}_{tot} = \frac{\partial H^{conv}_{tot}}{\partial t} \Delta t$$

Then, Maupertius' principle

$$\Delta \int_{t_1}^{t_2} dt \; \underset{\mu}{p}_k \cdot \dot{\underset{\mu}{x}}^k = \Delta \int_{t_1}^{t_2} dt \left(H^{conv}_{tot} - L^{conv}_{tot} \right) = 0 \tag{3.C.2}$$

holds. The geometrical meaning of this principle is expressed by the Jacobi's version

$$\Delta \int_{t_1}^{t_2} dt \; \underset{\mu}{p}_k \cdot \dot{\underset{\mu}{x}}^k = \Delta \int_{t_1}^{t_2} dt \; 2T = \Delta \int_{P_1}^{P_2} ds \; \dot{s} \tag{3.C.3}$$

which essentially indicates that the trajectories are geodesics.

Consider now a nonconservative system in the same representation space. Suppose that it is nonessentially nonselfadjoint and let L^{gen} (H^{gen}) be the Lagrangian (Hamiltonian) for the analytic representation in the given Cartesian variables. The variations (3.C.1) can still be introduced because, trivially, they are independent of the explicit functional structure of the functions considered, i.e.,

$$\Delta \underset{\mu}{r}^{k} = \frac{\partial H^{gen}_{tot}}{\partial \underset{\mu}{P}_{k}} \Delta t, \quad \Delta \underset{\mu}{P}_{k} = -\frac{\partial H^{gen}_{tot}}{\partial \underset{\mu}{r}^{k}} \Delta t, \quad \underset{\mu}{P}_{k} = \frac{\partial L^{gen}_{tot}}{\partial \dot{\underset{\mu}{r}}^{k}},$$

$$\Delta \underset{\mu}{r}^{k}(t_s) = 0, \; s = 1,2, \quad \Delta \dot{\underset{\mu}{r}}^{k} = \frac{d}{dt} \Delta \underset{\mu}{r}^{k}, \tag{3.C.3}$$

$$\Delta H^{gen}_{tot} = \frac{\partial H^{gen}_{tot}}{\partial t} \Delta t .$$

As a result, the arena of applicability of Maupertius' principle (as well as, in general, of any variational principle) within the given variables is that of all nonessentially nonselfadjoint systems, including nonconservative systems, and we shall write

$$\Delta \int_{t_1}^{t_2} dt \; \underset{\mu}{P}_{k} \cdot \dot{\underset{\mu}{r}}^{k} = \Delta \int_{t_1}^{t_2} dt \left(H^{gen}_{tot} - L^{gen}_{tot} \right) = 0. \tag{3.C.4}$$

The motion, however, is not in general geodesic. This can be seen from the fact that, in general, $(H^{gen} - L^{gen})$ is not expressible as an arc-length. However, in particular, one could have

$$H^{gen}_{tot} - L^{gen}_{tot} = \frac{1}{2} P_{ia} \, g^{iajb}(r) \, P_{jb} \tag{3.C.5}$$

As a result, nonconservative systems may have, in special circumstances, a geodesic nature, but in the mathematical Riemannian space with metric tensor $g^{iajb}(\underset{\mu}{r})$, rather than the Euclidean space of the experimental analysis. This point indicates the possible existence of particular realizations of the nonselfadjoint, hadronic, strong forces which are compatible with the General Theory of Relativity (Section 3.4).

The general nongeodesic nature of the nonconservative systems can also be seen from another profile. Let $\underset{\mu k}{F}$ be the nonconservative forces and let $W = \underset{\mu k}{F} \cdot \Delta \underset{\mu}{r}^{k}$ be their virtual work under the variations (3.C.4). Then principle (3.C.3) becomes

$$\Delta \int_{t_1}^{t_2} dt \; \underset{\mu k}{p} \cdot \dot{\underset{\mu}{r}}^{k} = \Delta \int_{P_1}^{P_2} ds \, \dot{s} = \int_{t_1}^{t_2} dt \; \underset{\mu k}{F} \cdot \Delta \underset{\mu}{r}^{k} \neq 0 \tag{3.C.6}$$

namely, the trajectory cannot be geodesic in the physical Euclidean space of their experimental detection, precisely in view of the presence of the forces not derivable from a potential.

By recalling the fundamental role of the concept of geodesic motion for all the currently available relativity ideas (Section 3.4), this is a geometrical reason[88] for the incompatibility of our hadronic structure model with the Galilei, the Special and the General Theory of Relativity.

REFERENCES AND FOOTNOTES

(finalized on June 1978)

1. The models on the structure of hadrons currently receving the majority of the consensus are those known under the name of quark models (or unitary models) as originally proposed in ref.[2] For a review of the early stages of the quark models see, for instance, ref.[3]. For reviews of recent developments see, for instance, refs.[4,5]. For a recent review of the search for quarks see, for instance, ref.[6] For contributions on the search for quark confinement see reviews[7-8] and papers[9-16]

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18. The reader interested in additional reviews may consult the series of Ettore Majorana Lecture Notes or of other Summer Schools.

19. According to the Encyclopedia Britannica, Vol. 17, page 420, the answer to the question what is a hadron or, more generally, a particle,

"would be provided by a quantitative theory which as yet does not exist."

20. The theory of strong interactions currently receiving the majority of the consensus goes under the name of quantum chromodynamics. This is, in essence, a symbiotic generalization of quantum electrodynamics and of the quark models, in the sense that it employes the fundamental relativity , quantum mechanics and gauge laws of the former, while including the so-called color extension of the (fractionally charged) quarks and (massless) gluons. See, in this respect, for instance, papers[21-22] and quoted references.

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38. See, for instance, R. MOORE, Niels Bohr: the Man, His Science and the World they changed.

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40. See, for instance, K. WILDERMUTH and Y.C.TANG, A Unified Theory of the Nucleus, Academic Press, N.Y. (1977).

41. E. EICHTEN and K. GOTTFRIED, Phys. Letters 66B, 286 (1977) and W. CELMASTER, H. GEORGI and M. MACHACEK, Phys. Rev. D17, 874 and 879 (1978) and quoted papers.

42. According to quantumchromodynamics, the effective coupling costant of the strong interactions under the assumption of asymptotic freedom is small. We have excluded this case from our considerations in the text, which is therefore devoted to the "old" (i.e. pre-QCD) approach . The reason is due to the rather involved and diversified topology of assumptions of truly fundamental character, none of which has achieved an enequivocal experimental verification, which are needed to achieve such a "small value of the effective coupling costant". I am here referring to:the assumption that Einstein's special relativity applies to the strong interactions in general and to the hadronic structure in particular; the assumption that Pauli's exclusion principle and the spin-statistics theorem hold for the hadronic constituents; the assumption that such constituents are the quarks; the assumption that they confine; the assumption that they verify asymptotic freedom; etc. It should be here stressed that we do not claim these assumptions to be necessarily invalid. As stressed numerous times in the text, we encourage the continuation of studies along these assumptions.

We are simply interested in studying an approach to strong interactions which is fundamentally different than the conventional one.

43. This is a crucial point for the practical realization of the assumptions of Section 1.2. The background idea is that the notion of particle multiplet, while crucial for the classification of hadrons, prohibits the proper formulation and treatment of the problem of structure of each individual member of the multiplet. This approach is radically different than that of current trends in hadrons physics (virtually dominated, in my view , by the concept of particle multiplets at both the classification AND the structure level). The notion we are here introducing can therefore be best illustrated at the atomic, rather than the hadronic level. Consider the atomic dichotomy classification/structure. The notion of atomic multiplet can in this case be expressed via that of valence. Indeed, as well known, the valence plays a fundamental role for the classifi cation of atoms. In the transition to the structure problem, the situation (as far as the valence is concerned) is fundamentally diffe- rent. Indeed, the idea of attempting the construction of a structure model of atoms in which the valence had a fundamental constructive role was estraneous to the minds of the Founders of the atomic mechanics and remains inconsistent these days. To state it in different terms, the fundamental bulding blocks of the structure of atoms resulted to be basically different than those for the classification. Our attitude as far as the corresponding hadronic context is concerned is as naive as conceivably possible. We assume that the hadronic equivalent of the concept of atomic valence is that of unitary multiplet (e.g., an isospin multiplet). We then argue that such

notion is crucial for the classification of hadrons, e.g., as for the case of the isotopic triplet of pions. In the transition to the problem of structure of each individual element of a given multiplet of classification, we believe that the situation is fundamentally different. To state it explicitly, we believe that the repeated use of exactly the same notion of multiplet used for the classification renders the problem of structure questionable in its very formulation. Indeed, there is no a priori reason why the structure, say, of π^0, should depend in a rather essential way, from that of, say π^+, as it is the case when the notion of unitary multiplet is assumed at the foundation of the problem of structure. Notice that it is this assumption that leads to the sequential assumption that quarks are the constituents of hadrons. In order to attempt a model of structure along the lines of Section 1.2, including the achievelent of full compatibility with the established unitary classification of hadrons, we therefore ignore all "chemical" numbers at the structure level. This literally implies that, in the study of the structure of say, the π^0 and $\pi^\pm$ particles, the fact that they possess the values I = 0 and I = $\pm$1 will be ignored. There is an additional reason indicated in the text for the desired ignorance of these numbers at the structure level of each individual particle. It is given by the fact that, when <u>one given</u> hadron is detected in laboratory, say the π^+, what is measured is the set of quantities such as mass, charge, linear momentum, angular momentum, spin, electric and magnetic moments and charge and space parity. The other "chemical" numbers <u>cannot</u> be detected in this way and demand the use of <u>other experiments</u>. For instance, the identification

of the value I= 1 of π^+ demands experiments on <u>all three pions</u> π^0, π^+ and π^-, trivially, in order to identify the existence of an isotopic <u>triplet</u>. But <u>THIS IS EXACTLY A TRANSITION FROM THE PROBLEM OF EACH INDIVIDUAL HADRON TO THAT OF HADRONIC CLASSIFICATION</u>, as intended in this monograph. The reader, therefore, should not be surprised if, say, the charge is accepted at the structure level and the hypercharge not. The reason is that the charge can be measured for each individual particle, and the hypercharge not, in the sense that it demands additional particles and, as such, it demands the study of the classification. For further elaboration on this point the interested reader may consult ref.[33] To stress the crucial nature of these remarks, we can say that one of the primary objectives of these monographs is to indicate that, perhaps, the hadronic constituents can indeed be identified with established, experimentally known particles, <u>PROVIDED THAT THE NOTION OF MULTIPLET IS EXCLUDED IN ITS ENTIRETY AT THE STRUCTURE LEVEL.</u> It is understood that the structure must reproduce the unitary classification in exactly the same way as the established structure model of the atoms reproduces the valence. Notice also that the terms "internal degrees of freedom" of current use in the contemporary literature are reinterpreted in these monographs as "classification degrees of freedom", when referred to quantity characterized by SU(3)-type models.

44. Throughout out analysis, unless explicitly stated, we shall assume Newtonian forces which are of class C^∞ in

the considered region of their variables. See Chapter 2 for more details.

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53. The reader interested in mathematical rigour may consult the cohomology approach to the integrability conditions for variational principles as provided by G.W.HORDNESKI, Tensor N.S. 28, 303 (1974) and 29, 21 (1975), as well as the functional approach by M.M. VAINBERG in Variational Methods for the Study of Nonlinear Operators, Holden Day, San Francisco (1964). For the reader interested in the pragmatic need of tools for the explicit construction of a Lagrangian it is here appropriate to recall that, according to R.W. ATHERTON and G.M.HOMSEY (Studies in Appl. Math. 54, 31 (1975)), Vainberg's

approach is so abstract to "remain inaccessible to many applied mathematicians and engineers". A similar comment is also appropriate for the cohomology approach. As I indicate in ref.[34], Part I, all these different methodological approaches to the integrability conditions for the existence of a Lagrangian ultimately produce either the same conditions obtained via the simpler variational approach (as used in Chap. 1) or equivalent forms. It is appropriate here to stress that the above remarks are intended only for the pragmatic need of the explicit form of the integrability conditions in questions. The reader with a serious interest in studying strong interactions as nonderivable from a potential is urged to study the variational, as well as the cohomology and the functional approach to the Inverse Problem. Indeed, these latter approaches provide subtle but valuable insights in a number of technical problems, from the construction of the relativity which is applicable to the forces considered, to the geometric quantization.

54. The reader interested in more details may consult ref.[34], Part I, Sec. 1.1.

55. Systems with arbitrary (class C^∞) couplings and subsidiary constraints can be treated in more than one methodological approach. It is then expected that these approaches are linked by given equivalence procedures. Thus, the possibility that the regular constrained systems here considered can be turned into an equivalent degenerate system cannot be excluded. In conclusion, the indirect applicability of Dirac's methods to our approach to strong interactions remains, in principle, conceivable. But this would require rather complex equivalence transformations which are here excluded.

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57. Almost needless to say, this crucial property of the Lorentz force in special relativity is preserved in the reduction to the Newtonian limit. For an explicit proof, see ref.[34-I], Example 2.7, page 105. More generally, we can say that the forces or couplings of the electromagnetic interactions are variationally selfadjoint at all levels of formulations, that is, in Newtonina mechanics, special relativity, field theory and general theory or gravitation. In contrast, the realization of the strong interactions we are interested in for these monograph is that of being variationally NON-selfadjoint, also at all levels of representation. This is precisely the central idea of attempting an analytic differentiation between the electromagnetic and the strong interactions, but now quantitatively expressed via the methods of the Inverse Problem.

58. Theorem 2.4.5 is a relativistic reformulation of the results by E. ENGELS, Hadronic J. 1, 465 (1978) for the Newtonian case.

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60. For the first order case of the Inverse Problem for multiple integral path functionals one may consult G.W.HORDNESKI (see the quotation of footnote[53]). As elaborated later on in the text, his proof, however, does not extend to the second order systems of customary use in physics, that is, the second-order Lagrange's equations via first order Lagrangians. Instead, it applies to systems of arbitrary order, but represented via Lagrangians of

the same order, although totally degenerate in our terminology. For additional references, the reader may consult refs.[34]

61. W. SARLET and F. CANTRIJN, Hadronic J. 1, 101 (1978).

62. It might be of some assistance for the interested reader to elaborate on this crucial point already at this introductory stage In turn, this may be useful to better identify the objectives of these monographs. To state it explicitly, the current approach to the problem of the hadronic structure, that of the quark models, in the final analysis might result to be problematic in the very formulation of the problem. In Chapter 1 we have initiated the elaboration of this point by noting the possible d chotomy classification/ structure and the possibility that the fundamental notion of the quark models,that of unitary multiplet, prohibits the proper formulation and treatment of the problem of the structure of each individual element of the multiplet (while the same notion of multiplet is, of course crucial for classification). The point we are here beginning to express in a technical language is that, a part from these difficulties, the notion of particle(more specifically, that of constituent) used in the quark models might result to be inconsistent with physical reality, in which case, of course, the formulation of the problem of hadronic structure according to the quark models is in need of profound revisions. The argument is as naive as conceivably possible. First of all, by "particle" I here mean a physically detected particle(i.e., interacting - I ignore the case of free particles). Secondly, permit me to stress that we possess nowadays a notion of particle under electromagnetic interactions which is incontrovertibly conform to physical reality. This is due to the fact that the funda-

mental tools for the characterization of such a notion of particle, Einstein's special relativity and quantum mechanics, are incontroveribly consistent with the electromagnetic interactions. The result is that the current formulation of the problem of the atomic structure is consistent because we possess a notion of constituent which is consistent with the interactions of the system considered. Thus the statement "the hydrogen atom is a bound state of a proton and an electron" possesses a rather impressive experimental backing not only as far the constituents per sé are concerned, but, more specifically, in relation to the tools used for a quantitative characterization of the notions of electrons and proton.

In the transition to the problem of the hadronic structure, the situation becomes fundamentally different. As we all know, the current approach to strong interactions is based on the truly crucial assumption that the fundamental relativity and quantum mechanical laws which have proved so effective for the electromagnetic interactions identically apply (i.e., apply as currently known) to the new layer of the reality considered, and they are simply implemented with unitary symmetries. As a result, the hypothesis: "the quarks are the constituents of hadrons" is vitally dependent on the validity of the Einstein's special relativity and conventional quantum mechanics for the <u>strong</u> interactions. Indeed, these tools are vital for the very definition of a quark as a constituent particle.

The first crucial remark of these monographs is that nowadays the validity of Einstein's special relativity and quantum mechanics for the strong interactions is a mere <u>ASSUMPTION</u>, deprived of any final, incontrovertible, experimental backing. This literally means

that the contemporary notion of constituent under strong interactions does not possess an experimental backing, whether of quark or non-quark inspiration.

The second crucial remark of these monographs is that there exists serious doubts on the validity considered under the conditions considered. These doubts, as by now familiar, originate from (A) the fact that massive particles are extended objects (all with approximately the same radius of 10^{-13}) and <u>NOT</u> point-like-objects, as in most of the current treatments of strong interactions; (B) the strong interactions are short-range and , more significantly, of a range precisely equal to the charge radius of physical particles, while the electromagnetic interactions are long range; and this renders a point-like approximation acceptable for the latter but unacceptable for the former; and (C) when the strong interactions are subjected to a mental view closer to the physical reality, that is, according to a state of penetration of extended objected, the presence of generalized forces and generalized dynamical conditions becomes conceivable. This is so to the point of rendering questionable the preservation of the atomic notion of spin for, say, an electron created in the core of a neutron star.[33] By ignoring all the other considerations of this (and of the subsequent) volumes, the need for a more general notion of spin is sufficient alone to render inapplicable both Einstein's special relativity and quantum mechanics for the intended physical framework.[32,33]

The third crucial remark of these monograph is that the problem of the hadronic structure should be the very last in the order of priorities and utmost, fundamental priority should be given to the study of the disciplines which are applicable to the strong interactions. This remark applies first of all for the experimental profile and then for

the theoretical setting. This is the reason why, as the reader can begin to see, our primary efforts have been devoted to the identification of tools for the quantitative treatment of more general forces, first at the classical and then at the quantum mechanical level, and the problem of the actual construction of a structure model of hadrons will be considered at the very last step of our program.

The reader interested in the identification of the status of the project at the end of Section 2.5 might desire to know that we have implemented only a very first step: the identification of mathematical tools for the quantitative treatment of the notion of strong interactions as local nonderivable from a potential, which is now reformulated into the conditions that the Newtonian, "relativistic" or field theoretical forces or couplings representing strong interactions are variationally nonselfadjoint. In particular, we have identified the "analytic NON-equivalence" of these interactions with the electromagnetic ones, in the sense that the latter interactions are instead selfadjoint.

A truly long way remains to be covered. Our immediate objectives are to bring the mathematical tools of the variational approach to selfadjointness up to a sufficiently diversified level which is suitable for the next important physical objective: to use these mathematical tools for the study of the mechanism of VIOLATION of established relativities for the considered realization of the strong interactions. This is clearly crucial to attempt the construction of conceivable generalizations.

The final objective remains the search for a quantitative treatment of the notion of particle under strong nonselfadjoint interactions,

via generalization of both conventional relativity and quantum mechanical treatment, as a necessary prerequisite for the intended formulation of the problem of hadronic structure, let alone its treatment.

63. It should be stressed at this point that a solution does not necessarily exist because the underlying system is overdetermined. We shall elaborate better on this aspect later on during the course of the analysis.

64. In actuality, the conditions of variational selfadjointness identify a generalization of Hamilton's equations, known under the name of Birkhoff's equations, which preserve the Lie algebra and symplectic geometry of the conventional canonical equations. For a detailed treatment see paper[32,61] or monographs[34]. This broader approach to anlytic mechanics is not needed for the problem under study and, thus, it has not been here considered. However, Birkhoff's equations will play a rather crucial function for the program of Volume II (the construction of generalization of the conventional analytic, Lie and symplectic formulations of Lie-admissible character). See also the role of these equations for the problem of universality of Section 2.8.

65. This ambiguity arises at the level of the equations of motion, owing to possible factor terms, and not at the level of the Lorentz force which is selfadjoint.

66. R. JOST, Rev. Mod. Phys. 36, 572 (1964).

67. P. HAVAS, Acta Phys. Austr. 38, 145 (1973).

68. The reader should keep in mind that sometimes, for given strong nonselfadjoint forces, the computation of a Hamiltonian is so complex, to discourage the most devoted Lie scolar, assuming that it exists in a closed form. Indeed, as the reader will recall from the outline of the proof of Theorem 2.8.1, the computation of a Hamiltonian demands first the solution of a (semilinear system of first order partial differential equations in the genotopic functions of Eqs. (2.8.2) and then the solution of a nonlinear system of partial differential equations, that of Darboux's reduction (2.8.14). Under the assumed conditions, these systems are consistent ; but the solution do not necessarily exist in a closed form and, in any case, they are often of truly difficult solution, even for simple forms of strong nonselfadjoint forces and for low dimensionality of the system of particles. In conclusion, this part of my studies on the Inverse Problem has been conducted for the intend of indicating that conventional analytic, Lie and geometrical techniques can be indeed applied to strong interactions when (local and)nonderivable from a potential. However, I have devoted an equal amount of efforts in studying the limitations on physical grounds of this approach, as well as the technical difficulties for its realization.

69. Of course, such a representation is not unique, in the sense that there exists a number of interaction terms $\alpha^{(ka)}_{int,I}$ for one given system. This problem belongs to the study of the degrees of freedom of a Lagrangian, and will be considered later on.

70. D.G.CURRIE and E.J.SALETAN, J. Math. Phys. 7, 967 (1966); G. MARMO and E.J.SALETAN, Nuovo Cimento 40B, 67 (1977).

71. E.J.SALETAN and A.H.CROMER, Theoretical Mechanics, Wiley, New York (1971).

72. I have learned from WILLY SARLET a more direct method for constructing the generalized Lie tensor Ω. It is given by the transform of the fundamental tensor under the canonoid transformation of the coordinates, i.e.,

$$\Omega^{\mu\nu} = \frac{\partial a'^{\mu}}{\partial a} \omega^{\rho\sigma} \frac{\partial a'^{\nu}}{\partial a^{\sigma}}$$

$$(a^{\rho}) = \begin{pmatrix} q \\ p \end{pmatrix}, \quad (a'^{\mu}) = \begin{pmatrix} q \\ p^{*} \end{pmatrix}, \quad p^{*} = \frac{\partial L^{*}}{\partial \dot{q}}$$

73. H.R.BRUCK, A Survey of Binary Systems, Springer-Verlag, Berlin (19 (1958).

74. K. Mc.CRIMMON, Pacific J. Math. 15, 925 (1965).

75. The basic definitions used in this section are in accordance with generally accepted versions. The presentation of the content of this section follows that of paper[30-V] and Chap. 3 of ref.[34-II]. For a comprehensive list of literature on the subject the interested reader may also consult ref.[20] of paper[32].

76.E.L.HILL, Rev. Modern Phys. 23, 253 (1951).

77. For theoretical proposals on the verification of Einstein's special relativity at small distances see refs.[78-80]. For an initial experiment via cosmic rays, see ref.[81]. Unfortunately, this experiment was inconclusive owing to the errors in the experimental data. For the problem of the experimental verification of Einstein's general relativity in the interior of massive objects, the interested reader may consult ref.[82]

78. L.B.REDEI, Phys. Rev. 162, 1299 (1967)

79. L.E.LUNDEBERG and L.B.REDEI, Phys. Rev. 169, 1012 (1968)

80. D.Y.KIM, Lettere al Nuovo Cimento 2, 591 (1975)

81. M. DARDO, G. NAVARRA and P. PENENGO, Nuovo Cimento 61A, 219 (1969)

82. C. WILL, Astrophys. J. 165, 409 (1971)

83. We sometimes use the vague terms "within a hadron" to emphasize the vague nature of our current knowledge of the hadronic structure.

84. For preliminary studies on this differentiation see paper[33]. It essentially emerges that nuclear forces might have a very small term nonderivable from a potential. This is sufficient to activate the Lie-admissible formulations as an "infinitesimal deformation" of Lie formulations. In turn, this implies very small departures of the nuclear structure from established quantum mechanical laws. One of the primary objectives of paper[33] was indeed the proposal of the experimental verification of basic physical laws for the strong interactions restarting at the nuclear level. In particular, paper[33] proposed, apparently for the first time, the experimental verification in $_7N^{16}$ as well as in other nuclei

whether Pauli's exclusion principle is exactly valid or very small deviations are experimentally detectable. The inapplicability of Pauli's principle would then default the entire body of laws known under the terms "quantum mechanics", owing to its truly fundamental character. As not unusual for the physical contents of these monographs, the argument is as naive as conceivably possible. Let us inspect the following experimental data: (1) nucleons have a well defined charge radius which (has been measured rather accurately and it) is of the order of 10^{-13}cm; (2) the nuclear volume is approximately proportional to the total number of nucleons; and (3) there exist nuclei with volume smaller than that predicted by this proportionality rule. In the latter case, the only conceivable situation is that nucleons spend part of their time in a state of small penetration of their charge volumes. In turn, this is sufficient to activate the entirety of the methodological lines we are studying in these monographs, only referred to small values of the nonselfadjoint forces. In conclusion, the experimental verification of basic physical laws at the nuclear level is essential not only per sé, but also as an intermediate step prior to the predictably more complex experimental verification of the same laws, but at the level of the hadronic constituents. Indeed, the studies under consideration essentially indicate that possible small deviation from established laws at the nuclear level are only the first indication of major deviations at the hadronic level, owing precisely to the differences in the geometry of penetration of the charge volumes of the constituents in the considered two layers of the physical reality.

Permit me to recall from paper[33] that Pauli's principle IS NOT expected to be "violate" at the nuclear level. Instead, it is expected, structly speaking, to be inapplicable to the nucleons in an exact form. This is due to the fact that nonselfadjoint forces produce the modification of a crucial statistical characteristics, the spin, or equivalently, the forces under study produce a breaking of the SU(2)-SPIN symmetry. In simple language, nucleons have spin 1/2 while within a nucleus (according to this view) only when their charge volume does not overlap with that of other nucleons. When this overlapping occurs, the nucleons are subjected to a small "mutation" of their value of the spin. As a result, they are no longer Fermions and Pauli's principle is inapplicable. This is, in essence, the mechanism of inapplicability (rather than invalidity) of Pauli's principle. Almost needless to say, such a mechanism occurs for very brief periods of time or, equivalently, has a very small statistical weight, or, also equivalently, the deviations τ from the totally antisymmetric wave functions of two identical nucleons while part of a nuclear structure

$$\Psi = \frac{1}{\sqrt{2}} \left(\psi_1(1) \psi_2(2) - (1+\tau) \psi_1(2) \psi_2(1) \right)$$

can at most have a very small value.

These quantum mechanical effects of nonselfadjoint forces will be studied in Volume III. The significance for the content of this volume is that the indicated mutation of the spin under nonselfadjoint forces (or, equivalently, the breaking of the exact SU(2)-spin symmetry) also implies, in the final analysis, the inapplicability of Einstein's special relativity. This is due, on one side, to the truly fundamental role of the SU(2)-spin symmetry in the Poincaré symmetry and, on the other

side, the inability of such symmetry to quantitatively treat these mutations of spin.

These remarks also stress the fundamental character of Pauli's exclusion principle for the virtual entirety of our current quantum mechanical knownedge, to the point that its invalidation would imply, either in a manifest and direct way, or in a subtle indirect way, the invalidation of the entire methodological construction.

In conclusion, an experimental verification which is also pressing and unprocrastinable is that of the problem whether Pauli's exclusion principle is exactly valid in nuclear physics (that is, it is valid in the same quantitative measure as that of atomic physics), or very small deviations are experimentally detectable, particularly for the nuclei whose volume is lower than that predicted by the proportionality rule with the total number of nucleons.[33]

85. See, for instance, R.F.STREATER and A.S. WIGHTMAN, PCT, Spin and Statistics and all That, Benjamin, New York (1964) and N.N.BOGOLUBOV, A.A.LOGUNOV and I.T.TODOROV, Introduction to Axiomatic Quantum Field Theory, Benjamin, New York (1975).

86. C.W.MISNER, K.S.THORNE and J.A.WHEELER, Gravitation, Freeman, San Fracisco (1970).

87. R.M.SANTILLI, Ann. Phys. 83, 108 (1974).

88. Additional nongeodesic characters will be provided in Volume II

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