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ISOTOPIC LIFTING OF HEISENBERG'S UNCERTAINTIES FOR GRAVITATIONAL SINGULARITIES

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In this paper we study a paradox of quantum mechanics called the *"paradox of Heisenberg's uncertainties at the limit of gravitational singularities"*, according to which at that limit, Heisenberg's uncertainties should recover classical determinism because gravitational singularities verify classical determinism and so must all particles in their interior. The paradox is resolved via the use of the recently introduced isotopes of contemporary algebras, geometries and mechanics, essentially consisting of the generalization of the trivial unit of contemporary relativistic formulations, $I = \text{diag.}\ (1, 1, 1)$, into the most general possible, integro-differential units $\hat{I}(s, x, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial^\dagger\psi, ...)$ verifying the original axioms of I. The decomposition of conventional Riemannian metrices $g(x) = T(x)\eta$, where η is the Minkowski metric, and the lifting $I \Rightarrow \hat{I} = [T(x)]^{-1}$ then imply the generalized Heisenberg's uncertainties $\Delta x\, \Delta p \geq \frac{1}{2}\, |< [x \hat{,} p] >| = \frac{1}{2}\, |< (xTp - pTx) >|$, called *isouncertainties* which possess conventional expectation values for matter in ordinary condition, yet recover classical determinism at the limit of gravitational singularities for which $T \rightarrow 0$. The analysis therefore confirms the celebrated argument by Einstein, Podolsky, Rosen et al on the lack of terminal character of quantum mechanics.

1. Introduction

In this paper we study a paradox of quantum mechanics, here called *"paradox of Heisenberg's uncertainties at gravitational singularities*'' which can be expressed as follows. Consider an astrophysical body (say, a large star) undergoing gravitational collapse. Since the object is large even for macroscopic standards, its center of mass (say, at the origin $r = 0$) must obey classical determinism, which evidently persists for gravitational collapse all the way to a singularity at $r = 0$.

Consider now elementary particles in the interior of such a collapsing star. They are customarily assumed to verify conventional quantum mechanics, thus including Heisenberg's uncertainties. Now during the collapsing phase we have no means to ascertain any derivation owing to the interior conditions of the particles. However, at the limit of gravitational collapse into a dimensionless singularity at $r = 0$, all interior particles must necessarily acquire classical determinism which is not predicted by Heisenberg's uncertainties.

In this paper we shall therefore study a generalization of the quantum uncertainties which, while recovering the conventional form for particles in vacuum, become generalized for particles in interior conditions, while recovering classical determinism at the limit of gravitational singularities. The analysis will be patterned much along the historical argument by Einstein, Podolsky and Rosen (see, e.g. [1]) on the lack of completion of quantum mechanics.

On mathematical grounds, we now possess methods structurally more general than those used in quantum mechanics, known under the name of *isotopies of contemporary algebras geometries and mechanics*[2-4], which are based on the axiom-preserving isotopy $I \Rightarrow \hat{I}$ of the conventional trivial unit $I = \text{diag.}(1, 1, 1, ..., 1)$ of contemporary formulations into a quantity $\hat{I}$, called *isounit*, with the most general possible nonlinear and nonlocal dependence on all variables and their derivatives (including wavefunctions and their derivatives, as illustrated below).

On physical grounds, we know that Heisenberg's uncertainties for space coordinates x and momenta p,

$$\Delta \mathbf{x}\, \Delta \mathbf{p} \geq \frac{1}{2} \mid < [\mathbf{x}, P] > \mid = \frac{1}{2} \mid < (\mathbf{xp} - \mathbf{px}) > \mid = \frac{1}{2}, \qquad (1.1)$$

are valid for the arena of their original conception, that is, for a particle in vacuum under action-at-a-distance interactions (also known as the *exterior dynamical problem*).

Nevertheless, when the same particle is immersed within hyperdense hadronic matter, such as in the interior of nuclei, hadrons or stars (conditions also known as the *interior dynamical problem*), we have the emergence of *nonlocal interactions* due to the deep mutual penetration and overlapping of wavepackets, as well as *nonpotential-nonhamiltonian effects* due to motion of wavepackets within the medium composed by other wavepackets, which render conventional quantum mechanics and related uncertainties inapplicable (and not violated) on numerous independent counts, such as topological, analytic, algebraic, etc. (see [5, 6] for details).

These more general conditions are representative via the isotopies of quantum mechanics known as *hadronic mechanics*[2b,6], and related isotopy of the unit $I \Rightarrow \hat{I}$, resulting in the following *isouncertainties*[5,6]

$$\Delta \mathbf{x}\, \Delta \mathbf{p} \geq \frac{1}{2} |< [\mathbf{x} \hat{,} \mathbf{p}] >| = \frac{1}{2} |< (\mathbf{x} T \mathbf{p} - \mathbf{p} T \mathbf{x}) >|, \quad \hat{I} = T^{-1} \qquad (1.2)$$

where the interior, nonlocal and nonhamiltonian effects are represented precisely by the operator $T(t, \mathbf{x}, p, \dot{p}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \ldots)$, called *isotopic element*, and the product $\mathbf{x}Tp - pT\mathbf{x}$ is Lie-isotopic [2a], that is, it verifies Lie's axioms, although it is less trivial than the simplest possible product $\mathbf{x}\mathbf{p} - px$ of conventional use.

Besides the treatment of nonlocal and nonlagrangian effects, the same methods also permit a novel quantization of (conventional, local and Lagrangian) theories of gravitation, rudimentarily submitted in [7] under the name of *isoquantization of gravity*. It is based on the decomposition of a gravitational metric $g(x)$ on a (3+1)-dimensional Riemannian space into the form

$$g(x) = T(\mathbf{x})\, \eta; \qquad (1.3)$$

here η is the Minkowski metric, $\eta = \text{diag.}\ (1, 1, 1, -1)$, and the lifting the trivial unit $I = \text{diag.}\ (1, 1, 1, 1)$ of conventional relativistic quantum theories into the gravitational isounit $\hat{I} = [T(x)]^{-1}$,

$$I = \text{diag.}\ (1, 1, 1, 1) \Rightarrow \hat{I}(x) = [T(x))]^{-1}. \qquad (1.4)$$

The novel approach therefore consists of achieving an operator version of gravitation not via its Hamiltonian (which is identically null for Einstein gravitation, as well known), but rather via the well defined unit of the theory, thus avoiding the predictable problematic aspects of a quantum theory whose classical limit has a null Hamiltonian.

This results in the *isotopies of relativistic quantum mechanics*[6,7] for the description of the conventional interior gravitational problem whose uncertainties are also given by equations (1.2) where $T = T(x)$ is now the gravitational element of decomposition (1.3).

At the limit of gravitational collapse of a star all the way to a singularity at $\mathbf{x} = 0$, as well known from the Schwartzchild's line elements, the space components of $T(x)$ are identically null, thus implying the regaining of classical determinism

$$\Delta\mathbf{x}\,\Delta\mathbf{p}\,|_{\mathbf{x}} = 0 \Rightarrow \mathrm{Lim}_{\mathbf{x}=0}\,\frac{1}{2}\,|<(\mathbf{x}T\mathbf{p} - \mathbf{p}T\mathbf{x})>| = 0. \qquad (1.5)$$

On epistemological grounds, the following hierarchy of physical conditions and related uncertainties emerge from this study. First, we have the *exterior* problem of point-like particles moving in vacuum for which conventional quantum mechanics and related uncertainties (1.1) were conceived and subsequently resulted to be valid according to an overwhelming amount of experimental evidence.

Second, we have the *interior* conditions of extended particles/wavepackets moving within hyperdense hadronic media in which the consequential nonlocal and nonhamiltonian effects render inapplicable conventional quantum mechanics and related uncertainties (1.1) in favor of isouncertainties (1.2). This includes the operator formulation of *interior gravitational theories* their conventional, local and Lagrangian formulations.

Third, we have the *limit interior case of a star collapsing all the way to a singularity at* $x = 0$, in which case all particles in its interior must evidently have the same classical determinism as the collapsed star itself. This latter case is manifestly not treatable by conventional quantum mechanics and related uncertainties (1.1), but is quantitatively treatable as a limit of isotopic uncertainties (1.2).

The analysis of this paper therefore confirms the vision of de Broglie, Einstein and others mentioned earlier, by offering rigorous methods for a possible completion of quantum mechanics and the identification of the related uncertainties for generalized physical conditions beyond those of their original conception and established applicability.

2. Isotopies of Classical Formulations

The *isotopies* of contemporary mathematical structures[2,3] are essentially given by axiom-preserving, nonlinear, nonlocal and noncanonical realizations of any given structure.

Consider an n-dimensional manifold $M(x, F)$ with local coordinates x on a field F (hereinafter assumed of characteristic zero and essentially restricted in this paper to the field of real numbers $\mathcal{R}$ and complex numbers C). The basic isotopy is the lifting of the trivial n-dimensional unit I = diag. (1, 1, ..., 1) of $M(x, F)$ in to the most general possible quantity $\hat{I}(t, x, \dot{x}, \ddot{x}, ...)$ with an arbitrary , nonlinear and nonlocal dependency on all possible variables and their derivatives with respect to an invariant qualify s, which, to quality as an *isounit* or isotope of I, must preserve the original axioms of I[2a,3a], i.e. ,

$$I = \text{diag}\,(1, 1, ..., 1) \Rightarrow \hat{I} = \hat{I}\,(t, x, \dot{x}, \ddot{x}, ...,) \qquad (2.1\text{a})$$

$$\det I \neq 0,\, I = I^{\dagger},\; I > 0 \Rightarrow \det \hat{I} \neq 0,\; \hat{I} = \hat{I}^{\dagger},\; \hat{I} > 0. \qquad (2.1\text{b})$$

The isotopic lifting $I \Rightarrow \hat{I}$ requires the corresponding isotopy of the basic field F into the *isofields*[3a]

$$F = \{n : n = n1\} \Rightarrow \hat{F} = \{\hat{n} : \hat{n} = n\,\hat{I},\, n \in F\} \qquad (2.2\text{a})$$

$$n_1 + n_2 \Rightarrow \hat{n}_1 + \hat{n}_2 = (n_1 + n_2)\,\hat{I}, \qquad (2.2\text{b})$$

$$n_1\, n_2 \Rightarrow \hat{n}_1 * \hat{n}_2 = \hat{n}_1\, T\, \hat{n}_2 = (n_1 n_2)\, \hat{I}, \qquad (2.2c)$$

$$\hat{I} = T^{-1} \qquad (3.2d)$$

where one can recognize the preservation of the conventional additive unit 0 and related sum, and the generalization of the conventional associative product ab of arbitrary quantities a, b into the isoassociative form $a * b = aTb$, where the quantity T, called *isotopic element,* is fixed. Then, under the condition $\hat{I} = T^{-1}$, $\hat{I}$ is the correct left and right unit of the theory

$$\hat{I} * a \equiv a * \hat{I} \equiv a. \qquad (2.3)$$

The generalizations of the basic unit and fields then imply, for evident compatibility, a corresponding isotopy of (pseudo) metric spaces $S\,(x, g, \mathcal{A})$ into the *isospace*[3a]

$$S\,(x, g, \mathcal{A}) \Rightarrow \hat{S}\,(x, g, \mathcal{A}), \quad \hat{g} = Tg, \quad \mathcal{A} = \mathcal{A}\hat{I}, \quad \hat{I} = T^{-1} \qquad (2.4)$$

We should recall that the basis of a metric (or pseudo-metric) space is unchanged by isotopies (Proposition 3.1[3a], p. 181). Note also that each given conventional space admits an infinite number of different, although geometrically equivalent isotopic images, evidently because of the infinitely possible different isounits $\hat{I}$.

The isotopic generalization of the unit, fields and metric or psedo-metric spaces then imply corresponding, compatible isotopes of Lie theory (universal enveloping algebras, Lie algebras, Lie groups, representation theory, etc.[2a,3a,4], conventional geometries (symplectic, affine and Riemannian geometries[3b,8]), as well as classical[2a,8b] and quantum[2b,6] Hamiltonian mechanics. Regrettably, we cannot review here these formulations for brevity, and must limit ourself to a review of only the basic notions needed for the treatment of the uncertainty. Nevertheless, a technical knowledge of the isotopic formulations is a prerequisite for a true understanding of this paper.

The first property of the isotopic formulations needed for the isoquantization of gravity is given by the following

LEMMA 1 [**3a**] *Under the conditions* $\hat{g} = Tg$, $\hat{\mathcal{R}} \approx \mathcal{R}\hat{I}$, $\hat{I} = T^{-1} > 0$ *(as well as sufficient smoothness), all infinitely possible isospaces* $\hat{S}(x, \hat{g}, \hat{\mathcal{R}})$ *are locally isomorphic to the original space* $S(x, g, F)$, $\hat{S}(x, \hat{g}\, \hat{\mathcal{R}}) \approx S(x, g, \mathcal{R})$,

The most important isospaces for this analysis are given by the *isominkowskian spaces* [3a,8b]

$$\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}}), \hat{\eta} = T\eta, \eta \in M(x, \eta, \mathcal{R}), \hat{R} \approx R\hat{I}, \hat{I} = T^{-1}, \qquad (2.5)$$

and the *isoriemannian spaces*[3a, 8b]

$$\hat{R}(x, \hat{g}, \hat{\mathcal{R}}), \hat{g} = Tg, g \in R(x, g, \mathcal{R}), \hat{\mathcal{R}} \approx \mathcal{R}\hat{I}, \hat{I} = T^{-1}, \qquad (2.6)$$

where $M(x, \eta, \mathcal{A})$ and $R(x, g, \mathcal{R})$ are the conventional Minkowski and Riemannian spaces, respectively, over the reals $\mathcal{A}$.

The above isospaces are naturally set to represent directly the *interior dynamical problems* indicated in the introduction (i.e. motion of extended-deformable particles within physical media). In fact, owing to the arbitrariness of the isotopic element $T(t, x, \dot{x}, \ddot{x}, ...)$, the isometries $\hat{g} = Tg$ can represent the inhomogenuity (due to the local variation of the density), anisotropy (due to an intrinsic angular momentum of the medium considered) and generally integral nature of the system (due to motion of an extended object within a physical medium, classically and quantum mechanically).

This paper is specifically devoted to the study of *conventional* gravitational theories. We shall therefore ignore the latter generalizations and approximate the interior gravitational problem as being homogenous, isotropic and local-differential. The following property has particular value for our analysis.

LEMMA 2 [**3a**] *A conventional* (3+1)*-dimensional Riemannian space* $R(x, g, \mathcal{R})$ *is locally isomorphic to the isominkowski space* $\hat{M}(x, \hat{\eta}, \mathcal{R})$ *under the factorizations and identifications*

$$R\,(x, g, \mathcal{R}) \approx \hat{M}\,(x, \hat{\eta}, \hat{\mathcal{R}}), \tag{2.7a}$$

$$g\,(x) = T\,(x)\,\eta, \quad \eta \in M\,(x, \eta, \mathcal{R}), \ \hat{\eta} = T\,(x)\,\eta \equiv g\,(x), \ \hat{\mathcal{R}} \approx \mathcal{A}\hat{I}, \ \hat{I} = T^{-1}. \tag{2.7b}$$

Note that all conventional exterior gravitational theories including Einstein's gravitation, admit the factorization of the Riemannian metric $g\,(x) = T\,(x)\,\eta$ verifying the positive-definiteness condition $T\,(x) > 0$ (because necessary to admit the local Minkowski topology). Therefore, Lemma 2 holds for all possible Riemannian spaces. But, from Lemma 1, the isospaces are locally isomorphic to the original spaces. We therefore have the chain of local isotopic isomorphisms, or *iso-isomorphisms*

$$R\,(x, g, \mathcal{R}) \approx \hat{M}\,(x, \hat{\eta}, \hat{\mathcal{R}}) \approx M\,(x, \eta, \mathcal{R}), \tag{2.8}$$

namely, isomorphisms based on the isotopy of the unit.

In conclusion, Lemma 2 illustrates a central point of the analysis of this paper, the fact that *the geometric structure of the Minkowski space persists under any functional generalization of its unit $I \Rightarrow \hat{I}$, provided that $\hat{I} > 0$. This permits the axiom-preserving embedding of the isotopic gravitational element $T\,(x)$ in the isounit of the space, $\hat{I} = [T\,(x)]^{-1}$, that is, the treatment of a genuinely curved space via the Minkowski axioms*.

This property can be seen in a number of other ways independent from Lemma 2, such as the fact that the local topology of the Minkowski space (say, the Zeeman topology) is not changed by the generalization of the unit $I \Rightarrow \hat{I}\,(x)$, provided $\hat{I}\,(x) > 0$, because all topologies are insensitive to the structure of their own units, once positive-definite.

3. Isotopes of Operator Formulations

Analytic formulations on the isominkowski space $\hat{M}\,(x, \hat{\eta}, \mathcal{R})$ can be based on the following *isovariational principle* with subsidiary constraint[9]

$$\delta \hat{A} = \delta \int ds\ [p * dx - Hds] = \delta \int ds\ [p_\mu\ \hat{\eta}^{\mu\nu}\ \dot{x}_\nu - H] = 0 \quad (3.1a)$$

$$ds^2 = dx^\mu\ \hat{\eta}_{\mu\nu}\ dx^\nu = -1 \quad (3.1b)$$

where H is a conventional relativistic Hamiltonian (although properly writtten in isospace); the isotopic element T has the most general possible dependence subject to the positive-definiteness condition $T > 0$; the integrand $p * dx$ is a *one-isoform* of the isosymplectic geometry[3b], and equation (3.1b) is the isotope of the conventional relativistic subsidiary constraint.

As studied in **[8b]** at the Newtonian, relativistic and gravitational levels, the transition from the conventional canonical variational principle to the isotopic generalization essentially permits the transition from the exterior to the interior problem, that is, the transition from local and canonical equations (representing motion of a point-like particle in vacuum) to their nonlocal and noncanonical generalizations (representing motion of an extended particle within a physical medium).

Principle (3.1) then yields an isotopic generalization of Hamilton's equations here ignored for brevity, as well as the following *isotopic Hamilton-Jacobi equations*[8b]

$$\frac{\partial \hat{A}}{\partial t} + H = 0, \quad \frac{\partial \hat{A}}{\partial x^\mu} = P_\alpha\ t^\alpha_\mu, \quad \frac{\partial \hat{A}}{\partial p_\mu} = 0. \quad (3.2)$$

needed for the following analysis on isouncertainties.

The simplest way to construct the isotopes of conventional relativistic quantum mechanical formulations is via the so-called *naive isoquantization*[9, 5, 6] . It is based on the property that the action $\hat{A}$ on $\hat{M}(x, \hat{\eta}\ \hat{\mathcal{R}})$ is structurally more general than the conventional canonical action A on $M(x, \eta, \mathcal{R})$ and, as such, must be subjected to a mapping more general than the conventional one $A \Rightarrow -i\, I \log | \psi >$,

$I = \text{diag.}\,(1, 1, 1, 1), n = 1$. Since the transition from A to $\hat{A}$ is an isotopy, the same must hold for the latter mapping, yielding the expression [loc. cit]

$$\hat{A} \Rightarrow -i\,\hat{I}\,(s, x, p, \dot{p}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, ...)\log|\psi> \qquad (3.3)$$

where the isounit $\hat{I}$ is, in general, $a\ 4 \times 4$, nowhere degenerate, Hermitean matrix on $\hat{M}\,(x, \hat{\eta}, \hat{\mathcal{R}})$ such that $\hat{I} = (\hat{I}^{\nu}_{\mu}) = (\hat{I}^{\mu}_{\nu}) > 0$.

Then under mapping (3.3), the iso-Hamiltonian-Jacobi equations are mapped into the structures[5,6]

$$\tilde{H} * |\psi> \stackrel{\text{def}}{=} (H - i\frac{\partial\hat{I}}{\partial s}\log|\psi>)\,T\,|\psi> = i\frac{\partial}{\partial s}|\psi> \qquad (3.4a)$$

$$\tilde{p}_{\mu} * |\psi> \stackrel{\text{def}}{=} (p_{\mu} + i\,\hat{I}^{\nu}_{\mu}\,(\frac{\partial\hat{I}}{\partial x^{\nu}}\,log|\psi>)\,T\,|\psi> = -i\,\hat{I}^{\nu}_{\mu}\frac{\partial}{\partial x^{\mu}}|\psi>$$

$$(3.4b)$$

$$ds^2 = dx^{\mu}\,\hat{\eta}_{\mu\nu}\,dx^{\nu} = -1, \hat{I} = T^{-1} \qquad (3.4c)$$

where $\tilde{H}$ and $\tilde{p}$ are the *effective Hamiltonian and linear momentum*, repectively, Note that $H \equiv \tilde{H}$ for isounits $\hat{I}$ independent from the invariant quantity s, while $p_{\mu} \equiv \tilde{p}_{\mu}$ for all theories in flat isospaces.

Along similar lines one can construct the isotopic lifting of other relativistic formulations, e.g., one with variational principles based on the measure d^4x rather than ds, by reaching equivalent results.

The most salient aspect of operator equations (3.4) is that they imply the generalization of the conventional, right modular, associative action $H|\psi>$ of an operator H on a state $|\psi>$ into the generalized action $H * |\psi> = HT|\psi>$ which is still to the right, modular and associative, yet of isotopic type, thus confirming the achievement of a consistent lifting of the axiomatic structure of quantum mechanics.

At a deeper inspection equations (3.4) result to be a particular form of the so-called *isorelativistic hadronic mechanics*[6], that is, the isotopes of conventional relativistic quantum mechanics based on the following main structures:

(1) *the enveloping isoassociative operator algebra* $\hat{\xi}$ *with elements* $A, B, \ldots$, *isotopic product*

$$A * B \stackrel{\text{def}}{=} A\,T\,B, \quad T > 0 \text{ fixed} \tag{3.5}$$

and left and right isounit $\hat{I}$, $\hat{I} * A = A * \hat{I} = A\,, \quad \forall A \in \xi$;

(II) the *isofields* of reals $\hat{\mathcal{R}} = \mathcal{R}\hat{I}$ or complex numbers $\hat{C} \approx C\hat{I}, \hat{I} = T^{-1}$; and

(III) the *isohilbert spaces* $\hat{\mathcal{H}}$ with states (a) $|\psi|>$ and *isoinner product* in $\hat{C}$

$$\hat{\mathcal{H}}: \langle \phi \hat{|} \psi \rangle \stackrel{def}{=} \langle \phi | T | \psi \rangle \hat{I} \in \hat{C} \tag{3.6}$$

Note that $\hat{\xi}$ is still associative, $\hat{C}$ is still a field, and $\hat{\mathcal{H}}$ is still Hilbert. The isotopic character of hadronic versus quantum mechanics is then established by the local isomorphisms

$$\hat{\xi} \approx \xi, \; \hat{C} \approx C, \; \hat{\mathcal{H}} \approx \mathcal{H} \tag{3.7}$$

The preservation of the original axioms is then assured by the fact that hadronic and quantum mechanics coincide at the abstract level, because at that level all the following structures coincide

$$\hat{I} \approx I, A * B \approx AB, H * | \psi \rangle \approx H | \psi \rangle, \hat{C} \approx C, \langle \phi \hat{|} \psi \rangle \approx \langle \phi | \psi \rangle, \text{ etc.} \tag{3.8}$$

Each and every aspect of conventional quantum mechanics admits a (mathematically) consistent, axiom-preserving, isotopic generalization. This is the case for all axioms and properties of quantum mechanics, including all operators on a Hilbert space, causality, superposition principle, etc. These results are so stringent that, if a generalization of a given quantum mechanical law does not preserve the original abstract axioms, then, either that generalization is not an isotopy, or the original law is not true axiom of the theory.

Regrettably, we are not in a positon to review relativistic hadronic mechanics in the necessary details, and must limit ourselves to an indication of only those aspects essential for the treatment of the uncertainties of the structures.

The first property the reader should be aware of is that *an operator which is Hermitean in quantum mechanics remains Hermitean in hadronic mechanics as defined above*. Thus the notion of observability is preserved in its entirety. The notion of unitary of an operator is instead lifted into the *isounitary law*.

$$U * \hat{U^{\dagger}} = \hat{U^{\dagger}} * U = \hat{I} \tag{3.9}$$

while the *isoexpectation values* acquire the form

$$\langle \hat{A} \rangle = \hat{I} \langle \psi | *A* | \psi \rangle = \hat{I} \langle \psi | T A T | \psi \rangle$$

$$= \hat{I} \int dv \, \psi^{+} T A T \psi \in \hat{C},$$

$$\langle \psi | T | \psi \rangle = 1 \tag{3.10}$$

where the isounit $\hat{I}$ as a factor can be ignored for practical applications because the isomultiplication of an isonumber $\hat{c} = c\hat{I}$ by any quantity Q coincides with the conventional product, $\hat{c} * Q = cQ$. For the isotopic lifting of all other operations on a Hilbert space, including determinant, trace, etc., we must refer the interested reader to[6]

Heisenberg's equations of motion are generalized into the *isoheisenberg equations* in their infinitesimal form

$$i \dot{A} = [A \hat{,} H]_{\xi} = A * H - H * A =$$

$$= AT (s, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger} \partial\psi, \partial\psi^{\dagger}, \dots)$$

$$H - HT (s, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi \, \partial\psi^{\dagger}, \dots) A \tag{3.11}$$

submitted in the original proposal to build hadronic mechanics[2b], where one can see the Lie-isotopic character of the brackets $[A \hat{,} H]_\xi = ATH - HTA$, i.e., their preservation of the original Lie axioms, although with a product less trivial than the simplest conceivable form $[A, H]_\xi = AH - HA$. The finite form of Equations (3.11) is of the isounitary type[2b]

$$A\,(t) = e_{|\xi}^{\;iHt} * A\,(0) * e_{|\xi}^{\;-itH} = \hat{I}\, e^{it*H} * A\,(0) * e^{-iH*t}\, \hat{I}, \tag{3.12}$$

where $e_{|\hat{\xi}}$(or e) is the isoexponentiation in the generalized iso envelope $\hat{\xi}$ (or the conventional exponentiation).

The axiom-preserving isotopies of all remaining aspects of quantum mechanics then follow[6]. Particularly intriguing is the *isotopic measure theory* which is a conventional theory essentially referring measures to a quantity other than the trivial unit 1[5].

One should recall that equations (3.12) have been proved to be "directly universal[66], that is, capable of admitting as particular case all possible, linear or nonlinear, local or nonlocal and Hamiltonian or nonhamiltonian operator equations ("universality"), directly in the frame of the observer ("direct universality").

Hadronic mechanics is therefore naturally set for the representation of the most general known interior particle problems, that is, the study of an ordinary particle when immerses within a *hadronic medium* (the hyperdense hadronic matter existing in the interior of nuclei hadrons and stars). In particular, said representation occurs via a covering of conventional quantum mechanics. In fact, hadronic mechanics recovers quantum mechanics identically whenever the particle considered exits the hadronic medium and returns to move in vacuum, in which case $\hat{I} \equiv I$.

Needless to say, hadronic mechanics is at its first infancy and so much remains to be done to reach a final appraisal of its physical effectiveness, while its mathematical consistency is nowadays sufficiently established. A number of applications of hadronic mechanics to specific cases, and their confrontation with available experimental data have been conducted with sufficiently encouraging results to warrant additional studies[6].

We are here referring to typical applications where nonlocal and nonpotential internal effects are expected from deep mutual overlapping of the wavepackets of particles, such as Bose-Einstein correlation, quark confinement, behaviour of the meanlife of unstable hadrons with speed, Cooper pairs in superconductivity, and other applications reviewed in [**6**].

4. Isotopes of Heisenberg's Uncertainties.

We are now sufficiently equipped to define the *isouncertainty* ΔA of a quantity A via the expression[5].

$$(\Delta\hat{A})^2 = (\Delta A\,\hat{I})\,T\,(\Delta A\,\hat{I}) = (\Delta A)\,(\Delta A)\,\hat{I} = (\Delta A)^2\,\hat{I}$$

$$= \hat{I}\int \psi^*\,T\,(A - \langle A\rangle)^{\hat{2}}\,T\,\psi = \int \psi^*\,T\,(A - \langle A\rangle)\,T\,(A - \langle A\rangle)\,T\,\psi. \quad (4.1)$$

where, $\Delta\hat{A} = \Delta A\,\hat{I}$ is an isonumber while ΔA is an ordinary number; we assume the isonormalization

$$\langle\,\psi\,\hat{|}\,\psi\,\rangle = \langle\,\psi\,|\,T\,|\,\psi\,\rangle\,\hat{I} = \hat{I}, \quad \text{or} \int dv\,\psi^\dagger\,T\,\psi = 1. \qquad (4.2)$$

and set $h = 1$ hereon.

The isotopic generalizations of the familiar quantum mechanical procedure to derive Heisenberg's uncertainties, including the use of the *isoschwartz inequality* for two functions $\hat{f}$ and g

$$(\int dv\,|\,f\,|^{\hat{2}}) * (\int dv\,|\,g\,|^{\hat{2}} \geq |\int dv\,\bar{f}\,g\,|^{\hat{2}} \qquad (4.3)$$

yield the following result for two operators A and B[5]

$$[(\Delta\hat{A}) * (\Delta\hat{B})]^{\hat{2}} = (\Delta A\,\hat{I})\,T\,(\Delta A\,\hat{I})\,T\,(\Delta B\,\hat{I})\,T\,(\Delta B\,\hat{I}) = [(\Delta A)\;(\Delta B)]^2\,\hat{I}$$

$$\geq |\hat{I} \int dv\, \psi^{\dagger}\, T\,(A - \langle A \rangle)\, T\,(B - \langle B \rangle)\, T\, \psi\, |^{\hat{2}}$$

$$\geq |\frac{1}{2}\hat{I} \int dv\, \psi^{\dagger}\, T\,\{(A - \langle A \rangle) \hat{,} (B - \langle B \rangle)\}\, \psi\, |^{\hat{2}} + |\frac{1}{2}\hat{I} \int dv\, \psi^{\dagger}$$

$$T\,[(A - \langle A \rangle) \hat{,} (B - \langle B \rangle)]\, T\, \psi |^{\hat{2}}$$

$$\geq |\frac{1}{2}\hat{I} \int dv\, \psi^{\dagger}\, T\,[A \hat{,} B]\, T\, \psi\, |^{\hat{2}} = |\frac{1}{2} \int dv\, T\,[A \hat{,} B]\, T\, \psi\, ||^{2}\, \hat{I}, \tag{4.4}$$

that is,

$$\Delta A \;\; \Delta B \geq \frac{1}{2} | \int dv\, T\,[A \hat{,} B]\, T\, \psi\, | = \frac{1}{2} | \langle\, [A \hat{,} B]\, \rangle | \,. \tag{4.5}$$

Consider now the case of the position r and momentum p operator in one dimension. Then, their isouncertainties are given by[5]

$$\Delta r\, \Delta p \geq \frac{1}{2} | \langle\, [r \hat{,} p]\, \rangle | \,. \tag{4.6}$$

From equation (3.4b), the momentum operator has the realization

$$\tilde{p} * | \psi \rangle \overset{\text{def}}{=} (p + i\, \hat{I}\, (\frac{\partial \hat{I}}{\partial r} \log | \psi \rangle\,)\, T\, | \psi \rangle = -\, i\, \hat{I} \frac{\partial}{\partial r} | \psi \rangle \tag{4.7}$$

thus yielding the isocommutation rules

$$[r \hat{,} p] = r\, T\, p - p\, T\, r = i\, \hat{I}. \tag{4.8}$$

The *isouncertainties in one space dimension* are then given by[5]

$$\Delta r\, \Delta p \geq \frac{1}{2} | \langle\, [r \hat{,} p]\, \rangle | = \frac{1}{2} | \langle \hat{I} \rangle | = \frac{1}{2} \int dr\, \psi^{\dagger}\, T\, \hat{I}\, T\, \psi\, | = \frac{1}{2}, \tag{4.9}$$

and we have proved the following:

LEMMA 3 [5] *For the nonsingular case of one space dimension, the numerical value of the uncertainties*

$$\Delta r \, \Delta p \geq \frac{1}{2} h \tag{4.10}$$

remains invariant under isotopies (3.3) *of the quantization mapping with consequential isotopies* (3.4b) *of the momentum operator.*

To put it differently, we can say that the numerical value $\frac{1}{2}h$ of the uncertainties is a true axiom of quantum mechanics in one space dimension, precisely because invariant under isotopies.

The case in more than one dimension is different, with the appearance of "hidden" degrees of isotopic freedom evidently absent in quantum mechanics. Consider in this respect the interior problem of any gravitational theory on a Riemannian space $R(x, g, \mathcal{R})$ reformulated into the equivalent isominkowski form on $\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}})$

$$R(x, g, \mathcal{R}) \approx \hat{M}(x, \hat{\eta}, \hat{\mathcal{R}}), \quad g(x) = T(x)\, \eta \equiv \hat{\eta}, \quad \hat{\mathcal{R}} \approx \mathcal{R}\hat{I}, \quad \hat{I} = [T(x)]^{-1}. \tag{4.11}$$

As recalled in Section 1, the *isotopic quantization of gravity* consists of the lifting of the trivial unit I = diag. (1, 1, 1, 1) of conventional relativistic quantum theories on Minkowski space $M(x, \eta, \mathcal{R})$ into isounit (1.4), that is

$$I = \text{diag.}\,(1, 1, 1, 1) \Rightarrow \hat{I} = [T(x)]^{-1} = (\hat{I}^{\mu}_{\nu}) = (\hat{I}^{\mu}_{\nu}),\ g(x) = T(x)\, \eta\,, \tag{4.12}$$

Recall now the isoquantization of the linear momentum, equations (3.4b), where $\tilde{p}$ is the effective linear momentum operator. Then the, *Fundamental isocommutator rules* are given by

$$[x^{\mu} \hat{,} \tilde{p}_{\nu}] \equiv [x^{\mu} \hat{,} p_{\nu}] = i\, \hat{I}^{\mu}_{\nu}\,, \tag{4.13}$$

namely, for all practical purposes related to uncertainties we can assume in a curved isospace that $p_{\mu} \equiv \tilde{p}_{\mu}$.

As a specific application to a gravitational model, let us consider the familiar Schwartzchild's line element

$$ds^2 = [r/(r-2M)]\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\theta^2 - (1-2M/r)\, dt^2 \,. \quad (4.14)$$

It is now best to consider the space components Δx and Δp separate from the time one in order to avoid old problems, such as the lack of operator character of time. Then, in spherical polar coordinates, the isotopic element T has the explicit diagonal form

$$T = \text{Diag.}\ \{T_1^1, T_2^2, T_3^3\} = \text{diag.}\ \{r/(r-2M),\ r^2, r^2 \sin^2\theta\}. \quad (4.15)$$

Outside singularities, the isouncertainties are then given by

$$\Delta x^k\, \Delta p_k \geq \frac{1}{2} \langle \hat{I}_k^k \rangle = \frac{1}{2} \mid \int dx\, \psi^\dagger\, T\, \hat{I}_k^k\, T\, \psi\,, \quad (4.16)$$

with normalization

$$\int dx\, \psi^\dagger\, T\, \psi = 1\,. \quad (4.17)$$

For the case of singularities, equations (4.16) cannot be directly applied because one must reinspect the entire algebraic structure of the theory which is at the foundation of quantization rule (3.4) from which equations (4.16) follow. It is easy to see that at the Schwartzchild's horizon $r = 2M$, the isotopic element T diverges, that is, we have for all the modular actions of operators on states, such as $H * |\psi\rangle \Rightarrow \infty$, $p * |\psi\rangle \Rightarrow \infty$, etc., thus implying the following

LEMMA 4 *The isouncertainties of the space coordinates and momenta at the Schwartzchild horizon of the gravitational collapse diverge to infinity*

$$\Delta x\, \Delta p\, |_{r \Rightarrow 2M} \geq \frac{1}{2}\, \text{Lim}_{T \Rightarrow \infty} \mid \int dx\, \psi^\dagger\, T\, (\boldsymbol{x}\, T\, \boldsymbol{p} - \boldsymbol{p}\, T\, \boldsymbol{x})\, T\, \psi \mid = \infty\,.$$

(4.18)

We remain now with the problem of primary interest for this paper, which is that of gravitational collapse beyond the horizon all the way to a geometric singularity at $x = 0$. It is easy to see from Equation (4.15) that the isotopic element T becomes identically null at $x = 0$. This implies the null values of all modular actions of operators on isostates, $H * | \psi \rangle = 0$, $p * | \psi \rangle = 0$, etc., resulting in the following

LEMMA 5 *At the isotopic limit of gravitational collapse all the way to a singularity at* $x = 0$ *, the space coordinates and momenta reacquire their full deterministic character, i.e.*

$$\Delta x \, \Delta p \,|_{r \Rightarrow 0} \geq \frac{1}{2} \, \mathrm{Lim}_{T \Rightarrow 0} \,|\int dx \, \psi^\dagger \, T \, (x \, T p - p \, T x) \, T \psi \,| = 0. \quad (4.19)$$

Intriguingly, in a one-dimensional theory with isotopic element $T = (r - 2M))/r$ the uncertainties on time t and energy E behave in way opposite to that of the space coordinates and momenta, i.e.

$$\Delta t \, \Delta E \,|_{r \Rightarrow 2M} \geq 0 \,, \quad (4.20a)$$

$$\Delta t \, \Delta E \,|_{r \Rightarrow 0} = \infty \,, \quad (4.20b)$$

which rather intriguing geometric implications, particularly when studied from the viewpoint of the *isodual isospaces* of [**8b**].

Needles to say, the preceding results should be considered in the spirit in which they have been submitted, that of mainly epistemological character, and should not be taken *ad litteram*. In fact, we merely intended to study *geometric limiting conditions* in the universe, without any claim that they actually exist. This is due to several reasons, beginning with the known problematic aspects of Einstein's gravitation itself at the purely classical level [10,11], and the existence of theories (see, e.g. [**11, 12**]) which, even though admitting gravitational collapse, do not permit the achievement of a true singularity all the way to a dimensionless point at $x = 0$.

5. Concluding Remarks

This paper has been inspired by the historical doubts of Einstein, Podolisky, Rosen, et al. (see review [**1**]) on the lack of terminal character of quantum mechanics. Far from aiming at the ultimate resolution of the issue, the paper is essentially intended to present the following main aspects.

(1) Bring to the attention of the scientific community the distinction between the *exterior* problem in vacuum and the *interior* problem within a physical medium which played an important role in the early part of this century, but was subsequently abandoned (see, e.g. the care Schwartzchild had in presenting, first, his historical solution for the exterior problem in **[13a]** and, then, his little known solution for the interior problem in the *separate* paper **[13b]**).

(2) Indicate that the isotopes of quantum mechanics characterized by the lifting of the eigenvalue equation $H | \psi \rangle \Rightarrow H * | \psi \rangle = H T | \psi \rangle$ provide a rigorous methods for the study of interior problems at large (including interior gravitational problems), where the deviation of the Hermitean operator T from the unit I represents the physical differences between the interior and exterior problem (including the differences between a curved and a flat operator theory).

(3) Submit the simple epistemological observation that, at the limit of gravitational collapse all the way to a singularity, the (space component of the) isotopic element T becomes identically null, thus permitting the regaining of classical determinism for the space coordinates and momenta.

As a result, the isotopic methods permit explicit, quantitative realizations of the vision of de Broglie, Einstein and others on the lack of completion of quantum mechanics. Preliminary studies on the isotopic treatment of the Einstein-Podolsky-Rosen argument, Bell's inequality and the theory of "hidden variables" were conducted in **[14]**. A more recent account along the same lines can be found in the adjoining paper[15] .

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