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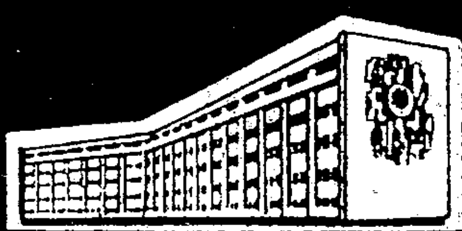
R. M. SANTILLI

# ELEMENTS of HADRONIC MECHANICS

## Volume I:

## Mathematical Foundations

1993



Київ

# EXCERPTS FROM THE REVIEWS

**A. Jannussis** (Univ. of Patras, Greece): *"Hadronic Mechanics supersedes all theories to date."*
(opening address of the International Conference on the Frontiers of Physics. Olympia, Greece, 1993)

**H. P. Leipholz** (Univ. of Waterloo, Canada): *"Santilli's studies are truly epoch making."*

**J. V. Kadeisvili** (Intern. Center of Phys., Kazakhstan): *"Santilli's Lie-isotopic and Lie-admissible generalizations of the algebraic, geometric and analytic foundations of Lie's theory are of clear historical proportions."*

**A. U. Klimyk** (Inst. for Theor. Phys., Ukraine): *"The three books on Hadronic Mechanics are the most authoritative for a study of the Lie-isotopic and Lie-admissible generalizations of Lie's theory and their many applications."*

**D. F. Lopez** (Univ. of Campinas, Brasil): *"Santilli succeeded, first, in reaching a structural generalization of the available mathematics as a pre-requisite for his generalization of current physical theories. These achievements are unprecedented in the history of physics."*

**A. O. E. Animalu** (Univ. of Nsukka, Nigeria): *"Because of its beauty, mathematical consistency and range of applicability vastly beyond quantum mechanics, if we deny the historical character of Hadronic Mechanics we exit the boundaries of science."*

**T. L. Gill** (Howard Univ., Washington, D. C.): *"The three volumes on Hadronic Mechanics represent the most important contribution to physics in the last fifty years."*

ACADEMY OF SCIENCES OF UKRAINE
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# ELEMENTS of HADRONIC MECHANICS

## Volume I:

## MATHEMATICAL FOUNDATIONS

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P. O. Box 1577
Palm Harbor, FL 34682, USA

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*dedicated to the memory of*

***ENRICO FERMI***

*because of his inspiring doubts on the exact validity of quantum mechanics for the nuclear structure.*

See, e.g., E. Fermi, *Nuclear Physics*, Univ. of Chicago Press (1950), the beginning of Chapter VI, page 111, when referring to the applicability of quantum mechanics for the treatment of nuclear forces:

***"..... there are some doubts as to whether the usual concepts of geometry hold for such small region of space."***

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# FOREWORD

These three volumes are devoted to a structural generalization of contemporary theoretical physics known under the name of *Hadronic Mechanics* (HM). Volume I presents a generalization of contemporary mathematical structures, including the theory of numbers, vector spaces, Lie algebras and groups, contemporary geometries, functional analysis, etc.

Volume II begins with a generalization of classical Lagrangian and Hamiltonian mechanics and then, after a suitable lifting of conventional quantization procedures, presents a step-by-step generalization of nonrelativistic and relativistic quantum mechanics capable of representing the most general known systems, while admitting of traditional mechanics and systems as particular cases.

Finally, Volume III presents a variety of novel and refreshing physical applications and experimental verifications in nuclear physics, particle physics, astrophysics, superconductivity and other unexpected fields such as conchology.

In short, Hadronic Mechanics concerns such a wide class of phenomena, that we can use for brevity the word *Nature.*

The Author's main idea consists of a generalization of the fundamental *constants* of contemporary physics into *variables* of the most general possible form representing their dependence on local physical conditions of the so-called *interior dynamical problem.* Mathematical and theoretical structures are then reconstructed in such a way to treat consistently these generalized notions.

The motivations for the consideration of these variable "constants" is rather natural. For instance, the speed of light in a physical medium is variable. Additional considerations then lead to the variable character of other "constants" in interior conditions, such as in the interior of a star. For instance, the coupling constant of quantum electrodynamics depends on quantum corrections and changes with the scale [1]. Contributions of integral character or the possible fractal structure of space-time then lead to a locally variable Planck's "constant". [2].

The transition from contemporary theoretical physics to the covering theories presented in these volumes can be expressed via a nice concept of M. P. Bronstein (1906–1938) on the so-called three-dimensional *Space of Physical Theories* (SPT) with axes characterized by Planck's constant $\hbar$, the gravitational constant G an the inverse of the speed of light 1/c (see ref. [3]). Conventional theories are characterized by the following *points* in this SPT:

($\hbar$, 0, 0) = nonrelativistic quantum mechanics;

(0, G, 0) = Newtonian mechanics;

(0, 0, 1/c) = special relativity;

($\hbar$, 0, 1/c) = relativistic quantum mechanics; and

(0, G, 1/c) = general relativity.

Because of the local dependence of the "constants" on density, temperature, pressure, etc., Santilli's covering theories fill up Bronstein's entire space.

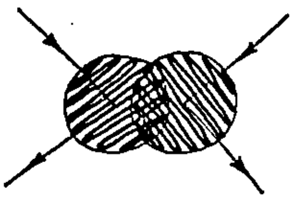
**Nugzar V. Makhaldiani**
Joint Institute for Nuclear Research
Dubna, Russia
October, 1993

1. N. N. BOGOLIUBOV and D. V. SHIRKOV, *Introduction to the Theory of Quantized Fields*, Wiley and Sons, New York (1980)
2. N. V. MAKHALDIANI, Number field dynamics and compactification problem in the theory of fields and strings, JINR Communication No. P2-88-916 (1988)
3. G. VENEZIANO, Fundamental constants in field and string theory, CERN preprint TH-6725/92 (1992)

# PREFACE

These volumes are the first books written on a nonlinear, nonlocal and noncanonical, axiom-preserving generalization of quantum mechanics called *hadronic mechanics*, proposed by the author back in 1978 when at Harvard University under support from the U. S. Department of Energy, and subsequently studied by a number of mathematicians, theoreticians and experimentalists.

The main objective is a systematic and quantitative study of the historical, open legacy of the nonlocality of the strong interactions at large, and of the structure of hadrons in particular, due to mutual overlapping of the wavepackets/wavelength/charge-distributions of hadrons,



in such a way as to preserve causality, measurement theory, and other basic features of quantum mechanics.

The scope of this first volume is the study of the mathematical foundations of the new mechanics. The main working hypothesis is *the generalization of Planck's constant into an integro-differential operator*

$$\hbar = 1 \rightarrow \hat{\hbar} = \hbar \hat{1}$$

under the condition of verifying the needed smoothness, boundedness and regularity properties. The lifting of the unit then requires the following corresponding generalization of the associative product AB among generic quantum mechanical quantities A, B

$$A B \rightarrow A * B = A T B, \quad T = \text{fixed}, \quad \hat{1} = T^{-1},$$

in which case $\hat{1}$ is the correct right and left generalized unit of the theory, $\hat{1} * A =$

$A * \hat{1} = A$.

The main idea is that the exchanges of energy are indeed discrete for particles moving in vacuum under action-at-a-distance interactions, such as for an electron in an atomic cloud. However, when the same particle is immersed within a hyperdense medium, such as for an electron in the core of a collapsing star, we expect integral contributions in the exchanges of energy due to the total immersion of the wavepacket of the particle within those of the surrounding particles.

The need for the generalization of the unit, and of the corresponding associative product, originates from the fact that the nonlocal interactions due to wave-overlappings, whether in electron pairing in superconductivity, or in deep inelastic scattering, or in other events, are of "contact" type; that is, of a type which does not admit a potential energy. Conventional Hamiltonians $H = K + V$ can therefore represent the kinetic energy K and all possible action-at-a-distance interactions with potential V. However, the contact interactions due to mutual wave-penetration, by conception, cannot be represented with the Hamiltonian H and, in this sense, they are called "nonhamiltonian". The alternative studied in these books is then their representation via the generalized unit of the theory for certain algebraic, geometric and analytic reasons presented in the text.

These preliminary ideas are sufficient to indicate the axiomatic structure of hadronic mechanics, and its connection with *all* existing generalizations of quantum mechanics. In fact, in Ch. 7 of this volume we show that hadronic mechanics is *directly universal;* that is, capable of representing all possible nonlinear, nonlocal, nonhamiltonian, continuous or discrete, inhomogeneous and anisotropic generalizations of quantum mechanics (universality), directly in the frame of the experimenter (direct universality). In the appendix of Ch. 7 we then outline the connection between hadronic mechanics and other generalized theories.

Consider, the generalizations of quantum mechanics known under the name of *q-deformations,* e.g., of the type

$$AB \rightarrow A * B = q\,A\,B\,, \quad q = \text{fixed} \neq 0,$$

(where q is a number). As we shall see, hadronic mechanics can be interpreted as an axiomatic *reformulation* of q-deformations which is invariant under its own time evolution and holds for arbitrary integro-differential deformations. This is essentially achieved via the redefinition of the unit

$$1 \rightarrow \hat{1} = q^{-1}.$$

and consequential reformulation of the entire structure of the theory (numbers, fields, metric spaces, Lie's theory, etc.). By keeping in mind the *mathematical* consistency of the current treatment of the q-deformations (that at a fixed time in which the basic unit is not generalized), the above reformulation also resolves some of the *physical* problematic aspects emerging under time evolution, such as lack of the basic unit, inapplicability of the measurement theory, general loss of Hermiticity of the Hamiltonian, and others.

Similarly, numerous *nonlinear generalizations* of Schrödinger's equation (those with a nonlinearity in the wavefunctions $\psi$) have been proposed in the literature. As it is the case for the q-deformations, they are mathematically correct, but are afflicted by a number of problematic aspects of physical consistency, such as the general lack of exponentiation of an algebra to the corresponding group, the inequivalence of the Heisenberg-type and Schrödinger-type equations (due to the so-called *Okubo's No Quantization Theorem),* and others. Hadronic mechanics can be interpreted as an axiomatic *reformulation* of these studies into a form admitting nonlinearity in the wavefunctions $\psi$ *and* their derivatives of arbitrary order $\partial\psi$, $\partial\partial\psi$, .... This axiomatization also permits a quantitative identification suitable for tests of the *deviations* from quantum mechanical formalisms implied by the nonlinearity itself.

Also, *nonlocal generalizations* of quantum mechanics for the study of wave-overlappings can be traced back to the very inception of that discipline. They were also treated via conventional quantum mechanical methods, thus leading to a number of problematic aspects still under study, such as causality. Hadronic mechanics preserves the abstract axioms of quantum mechanics and realizes them in a more general way, by therefore ensuring the preservation of causality *ab initio.* Hadronic mechanics is therefore ideally suited for an axiomatic *reformulation* of these studies into a causal description admitting all possible nonlocal-integral generalizations of quantum mechanics.

A number of *discrete generalizations* of quantum mechanics, such as those with a discrete structure in time, have been proposed in the literature although the elaborations continue to be based on conventional units and methods. These theories too are deeply linked to hadronic mechanics because the discreteness of time implies the alteration of the basic unit of time, thus requiring generalized methods for their treatment. Hadronic mechanics can be interpreted as providing an axiomatization of these generalizations by embedding the discrete structure of time in the generalized unit $\hat{1}$ of the theory. Intriguingly, hadronic mechanics shows that such discrete structure is ultimately compatible with the abstract axioms of quantum mechanics itself, when properly realized. Finally, discrete theories emerge as being compatible with conventional experimental data because (as shown in Vol. II) the appropriate expectation value of a discrete unit recovers the conventional unit, $<\hat{1}> = \hbar = 1$.

Numerous additional generalizations of quantum mechanics exist in the literature, some of which will be studied in the appendix of Ch. 7, and others in the subsequent Volumes II and III. All these theories are independent from hadronic mechanics, yet exhibit intriguing connections with the latter whose study is beneficial to both theories.

It is evident that, in a scientific horizon of this type, I could not provide a comprehensive review of all existing generalizations without avoiding a prohibitive length. In these volumes I shall therefore limit myself to a review and re-elaboration of only some representative generalizations for each of the above classes. Nevertheless, I would be grateful to colleagues who care to bring to my attention (at the address below) studies directly or indirectly related to hadronic mechanics which I should quote in a possible future edition.

Judging from discussions and correspondence with various colleagues over the years, the primary difficulty for a first inspection of the field is of *mathematical* nature. The nonlinear-nonlocal-noncanonical generalization of the basic unit of quantum mechanics demands, for various technical reasons, a suitable generalization of the *totality* of the mathematical structure of quantum mechanics, beginning with a generalization of the contemporary notion of *number*, such as $\hbar = 1$, into a structurally more generalized notion called *isonumbers*, such as $\hat{\hbar} = \hbar\hat{1}$. In turn, generalized units, products and numbers demand a suitable generalization of the notions of field, vector spaces, transformation theory, enveloping algebras, Lie algebras, Lie groups, symmetries, symplectic, affine and Riemannian geometries, Lagrange and Hamilton mechanics, etc.

In short, the studies reported in these volumes indicate that, in the same way as the full understanding of the structure of atoms required a revision of the mathematical foundations of classical mechanics, further basic advances in the structure of hadrons require a similar revision, this time, of the mathematical foundations of quantum mechanics.

Difficulties in communicating with colleagues therefore emerge whenever hadronic mechanics is approached (and appraised) via the use of old quantum mechanical knowledge, without the awareness of numerous ensuing inconsistencies which generally remain undetected.

The author has therefore no words to recommend that colleagues seriously interested in inspecting the advances reported herein acquire a technical knowledge of the novel mathematical methods prior to any judgment and, above all, prior to setting up the mind along old lines. After all, the new mathematical methods are quite easy to understand, as one can see.

Technically, the topic of these books is in the field of the *isotopies* and *genotopies* of contemporary mathematical and physical theories proposed by the Author back in 1978, which essentially are nonlinear, nonlocal-integral and nonpotential-nonhamiltonian liftings of given mathematical or physical structures capable of preserving the original axioms at the abstract, realization-free level (isotopies), or induce new covering axioms (genotopies).

As we shall see, the study of the fundamental hypothesis on the integral generalization of Planck's unit requires suitable nonlinear-nonlocal-nonhamiltonian isotopies and genotopies of the totality of mathematical methods used in quantum mechanics, including Hilbert spaces and all that.

The physical relevance of isotopic and genotopic methods is well established and consists in permitting quantitative studies of the transition:

a) from the *exterior dynamical problem,* characterized by motion of point-like particles within the homogeneous and isotropic vacuum;

b) to the *interior dynamical problem,* characterized by motion of extended and therefore deformable particles within inhomogeneous and anisotropic physical media, resulting in the most general known dynamical equations.

In particular, the isotopies preserve the original, abstract, algebraic, geometric and analytic axioms, thus achieving a unity of physical and mathematical thought in the treatment of both problems.

The isotopies are used when interior structural problems are studied as a whole with conserved conventional total quantities under a generalized interior structure. The genotopies are instead used to characterize one individual constituent while considering the rest of the system as external, thus resulting in the nonconservation of its physical quantities, of course, in a way compatible with total conservation laws.

The *classical isotopies and genotopies* are the classical realizations of the isotopies and genotopies of contemporary algebras, geometries, mechanics, symmetries and relativities. They have been sufficiently well identified in preceding monographs (quoted in the text), with a number of applications to Newtonian, relativistic and gravitational systems of our interior classical reality.

These volumes are the first books on the corresponding *operator isotopies and genotopies,* that is, the *axiom-preserving isotopies and axiom-inducing genotopies of quantum mechanics* originally proposed under the name of *hadronic generalization of quantum mechanics,* or *hadronic mechanics* for short, and today also known as *isotopic completion of quantum mechanics, isolocal realism,* and similar terms.

The operator isotopies and genotopies are far from being as developed as the corresponding classical counterparts. Despite that, I decided to write these first books for the following reasons:

1) the mathematical consistency of hadronic mechanic is now established, thus allowing rigorous quantitative treatments of interior particle problems in a form suitable for experimental tests;

2) we have today a number of experimental verifications which, even though evidently preliminary, nevertheless confirm the predictions of the covering mechanics quite clearly; and

3) hadronic mechanics suggests a number of *novel* experiments that is, experiments on internal nonlinear-nonlocal-nonhamiltonian effects simply beyond the descriptive and predictive capacities of conventional theories, which deserve a serious consideration by the experimental community owing to their

seemingly fundamental character.

Above all, a primary reason for writing these books is to point out for young minds of all ages that hadronic mechanics identifies the apparent existence of a new technology I tentatively called *hadronic technology,* because emerging from mechanisms in the structure of individual hadrons, while the current technologies emerge from mechanisms in the structure of molecules, atoms and nuclei. The societal implications of these possibilities, e.g., for possible new forms of energy, new approaches to cold fusion, new computer modeling, new medical applications, etc., have warranted this first identification of the state of the art in the conceptual, mathematical, theoretical and experimental foundations of hadronic mechanics.

**Ruggero Maria Santilli**
Dubna, Russia,
Kiev, Ukraine and
Palm Harbor, U.S.A.
Summer of 1993

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However, I am solely responsible for these volumes owing to the numerous changes and expansions of the final version.

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# 1: INTRODUCTION

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## 1.1: STATEMENT OF THE PROBLEM

The discipline today known as *quantum mechanics* (see, e.g., ref. [1] for a historical account and ref. [2] for a contemporary account) was originally conceived for the structure of the atoms and the electromagnetic interactions at large, for which it subsequently emerged as being exact according to an overwhelming amount of experimental evidence.

Whether in its nonrelativistic, relativistic or field theoretical versions, quantum mechanics was subsequently applied to the study of the nuclear structure (see, e.g., ref.s [3,4]), to the strong interactions at large (see, e.g., ref. [5]), as well as, more recently, to the unified gauge theories (see, e.g., ref. [6]), with equally impressive results and experimental verifications (see the recent experimental review [7]).

But physics is a discipline that will never admit final theories. No matter how effective and fundamental a theory is, the construction of a more general theory for a deeper understanding of physical reality is only a matter of time.

Despite its undeniable achievements and verifications, quantum mechanics possesses well identifiable limitations essentially expressed by the characteristics of its original conception and the point-like approximation of particles, as inherent in its essential local-differential structure.

Thus, in all physical conditions in which particles, their wavepackets, and/or their charge distributions can be well approximated as being point-like (as in the case for the electrons in an atomic structure), we expect quantum mechanics to be *exactly valid*. However, there exists an open issue in the literature, at times known as *the legacy of Fermi,* [1] *Blochintsev and others* (see Efimov's monographs on nonlocality [8] and historical references therein), according to which wavepackets and/or charge distributions in conditions of

[1] See the dedication of this first volume

deep mutual immersion and overlapping experience *short range, internal, nonlocal–integral interactions* (see Fig. 1.1.1).

**THE FUNDAMENTAL INTERACTIONS OF HADRONIC MECHANICS**

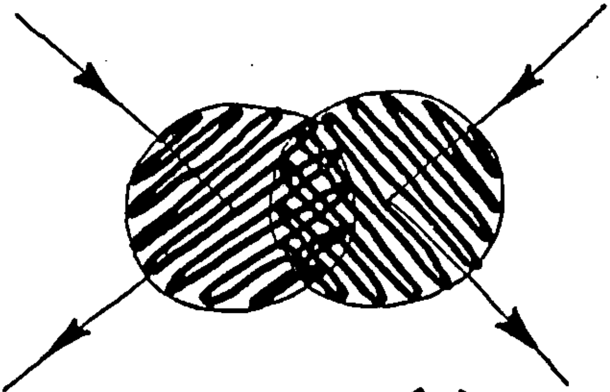

FIGURE 1.1.1: Quantum mechanics was conceived for the study of action–at–a–distance interactions among particles which, as such, are representable by ~~the~~ a potential. The interactions are therefore *local–differential;* that is, representable with differential equations defined over a finite set of isolated points. Hadronic mechanics was conceived for the study of the additional *nonlocal–integral* interactions due to mutual wave–overlapping as schematically depicted in this figure. We are here referring to interactions which, by central conception, are defined over an *entire volume* and, as such, cannot be effectively approximated via their abstraction into a finite number of isolated points. As we shall see, quantitative studies of the latter interactions, which are ~~evidently~~ not possible with quantum mechanics, permit deeper insights into existing knowledge (e.g., how can two seemingly noninteracting electrons obey Pauli's exclusion principle), and predict new knowledge (such as the apparent ~~cold fusion~~ of particles in conditions to total mutual penetration).

After lingering in the literature for decades, this legacy has recently been subjected to quantitative mathematical, theoretical and experimental studies with rather encouraging results. In fact, recent studies have identified clear theoretical and experimental information supporting the ultimate nonlocal structure of matter, thus suggesting the need for a suitable covering of quantum mechanics specifically conceived for the representation of the most general known interactions which are:

1) *nonlinear* in the coordinate x and wavefunctions $\psi$, $\psi^\dagger$ as well as their derivatives of arbitrary order $\dot{x}$, $\ddot{x}$, $\partial\psi$, $\partial\psi^\dagger$, ...;

2) *nonlocal–integral* in all variables;

3) *nonpotential–nonhamiltonian* , i.e., violating the integrability conditions for the existence of a Hamiltonian, the so–called *conditions of variational selfadjointness* [9];

4) *inhomogeneous* (e.g., because of a local variation of physical quantities such as density $\mu$, temperature $\tau$, index of refraction $n$, etc.); and

5) *anisotropic* (e.g., because of the presence of an intrinsic angular momentum which, as such, creates a preferred direction in the interior physical medium, with the understanding that the background space is and remains homogeneous and isotropic).

Since the above interactions cannot be represented with a Hamiltonian by central assumption, the fundamental hypothesis studied in these volumes is to represent them via a suitable, nonlinear, nonlocal and noncanonical generalization of the fundamental unit of quantum mechanics, Planck's constant $\hbar = 1$, into an integro–differential operator $\hat{1}$ with the indicated most general possible functional dependence[2]

$$\hbar = 1 \quad \Rightarrow \quad \hat{\hbar} = \hbar\,\hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots) , \qquad (1.1.1)$$

verifying certain smoothness, boundedness and regularity conditions identified later on.

A primary objective of these ~~books~~ is to review the mathematical, theoretical and experimental studies conducted on this historical legacy at this writing (Spring 1993) and propose additional mathematical, theoretical and experimental research.

This first volume is solely devoted to the mathematical foundations of hadronic mechanics. Theoretical profiles are presented in Vol. II, while applications and experimental verifications are studied in Vol. III.

## 1.2: LIMITATIONS OF QUANTUM MECHANICS

For the "young minds of all ages" indicated in the Preface, there is no need to conduct experiments in order to identify the limitations of quantum mechanics, but simply observe (and admit) physical evidence.

First, let us observe a classical event, such as a space–ship during re–entry

---

[2] The dependence on the accelerations is absent for exterior problems, but it is a peculiarity of interior problems. As we shall see in Vol.s II and III, hadronic bound states clearly exhibit acceleration dependent forces which ~~explains~~ in part their novelty.

into our atmosphere. The *No-Reduction Theorems* [11,12] establish that such a classical system *with monotonically decaying angular momentum* cannot be consistently reduced to a finite collection of ideal elementary particles verifying the quantum mechanical rotational symmetry, thus being all in stable orbits *with conserved angular momentum* (a similar situation holds for the Lorentz symmetry as we shall see in Volume II).

Viceversa, an ensemble of quantum mechanical particles each with conserved angular momentum simply cannot yield, under any rigorous limiting procedure, a macroscopic object whose center of mass has a continuously decaying angular momentum ~~as established by visual experimental evidence~~.

Since the macroscopic object is a concrete, visual evidence, while the reduction to elementary constituents is an academic abstraction, we must expect insufficiencies in the *quantum* theory. In fact, at ~~a~~ the deeper analysis the macroscopic object has classical interactions precisely of the nonlinear, nonlocal and nonhamiltonian type[3] which are absent in quantum mechanics.

The studies presented in these volumes can therefore be first seen as identifying a certain generalization of quantum mechanics which permits the recovering under a suitable limiting procedure of *all* possible ~~Newtonian~~ Classian systems, that is:

a) the *conservative* New *systems* of point-particles moving in the homogeneous and isotropic vacuum which are admitted by quantum mechanics (the *exterior dynamical problem* of the Preface), as well as

b) all *nonconservative* New *systems* representing motion of extended, and therefore deformable bodies within inhomogeneous and anisotropic physical media (the *interior dynamical problem* of the Preface).[4]

Alternatively and equivalently, the mechanics studied in these volumes is constructed under the condition of being the operator-image, not of Hamiltonian mechanics, but of a covering mechanics which achieves the "direct universality" of the Preface, here referred to the entirety of ~~Newtonian~~ Classian mechanics. We are here referring to the the so-called *Birkhoffian mechanics* [10], where: "universality"is the capability to represent all possible Newtonian systems (whether conservative or not) verifying certain continuity and regularity conditions; and the "direct

[3] Missiles in atmosphere have nowadays drag forces depending up to the *tenth* power of the speed and more, thus being manifestly nonselfadjoint [9], In addition, they have forces characterized by *integrals* over their surface $\int \sigma$ because their shape directly affects the trajectory. This provides a primitive classical example of the type of interactions studied in these volumes, intentionally selected to void any hope of "finding a Lagrangian or a Hamiltonian" in favor of structurally more general theories (see later on Fig. 1.4.1 for a first explicit example)

[4] For historical accounts on the classical distinction between exterior and interior problems beginning with Lagrange, Hamilton, Jacobi and other founders of analytic dynamics, the reader may inspect ref.s [9,11].

universality" is their representation directly in the frame of the experimenter, without transforming the systems into more manageable frames (see ref. [10], in particular, Theorem 4.5.1, p. 54, of "direct universality" of the Birkhoffian mechanics).

The need for the latter condition is dictated by a number of rather insidious physical aspects. In fact, the *Lie-Koening theorem* [loc. cit.] does indeed ensure the possibility of constructing Hamiltonian representations for all systems that are local-differential and verify certain regularity and continuity conditions. However, the transformations are *noncanonical* because the original systems are nonhamiltonian by assumption. As a result, the transformed frames are not generally realizable with experiments owing to their nonlinearity, besides implying the loss of contemporary relativities owing to their noninertial character.

The above occurrences illustrate the emphasis throughout these volumes of studying methods which are "universal" (rather than representing only a subclass of possible systems), and then "direct", that is, *first* admitting of representations in the frame of the experimenter (*prior* to any use of the transformation theory).

An inspection of the physical reality at the particle level without a preset mental attitude to preserve as much as possible current knowledge (which would not be scientific anyhow) reveals the existence of clear insufficiencies of quantum mechanics also at the particle level. This is due to the experimentally established existence of particle systems which simply cannot be derived from the strict implementation of first quantum mechanical axioms.

The first case that comes to mind is the *Cooper pair in superconductivity* (see general presentation [13] and the most recent review [14]). Clear experimental evidence establishes that ordinary electrons with negative elementary charge $-e$ can bound each other in a singlet state at small distances in high $T_c$ superconductivity. Scientific objectivity demands the admission that an *attractive* interaction among two *identical* electrons simply cannot be derived from first quantum mechanical axioms, whether nonrelativistic, relativistic or field theoretical, evidently because of the increase of the repulsive character of the Coulomb interactions with the decrease of the mutual distance. Similar insufficiencies can be seen in numerous other occurrences implying short range superposition of electrons, all the way to the very notion of *valence*.

Numerous models have been evidently proposed to interpret the attractive interaction of electron pairs in superconductivity (see ref. [14], pp. 19–20 and quoted literature) which, in the absence of more basic methods, have a clear scientific value. Nevertheless, objectivity demands the admission that these models do not resolve *the basic inability of quantum mechanics to represent the attractive interaction among the electrons of the same charge directly from first axioms*, rather than via semiphenomenological attempts.

At a deeper analysis we find a situation in electron pairing analytically equivalent to that of the classical space-ship during re-entry. In fact, experimental evidence establishes that, in the above pairing, electrons are in conditions of mutual penetration of their wavepackets, that is, in condition which can only be quantitatively treated via *integral* representations (Fig. 1.1.1). Moreover, these are contact interactions for which the notion of potential has no physical or mathematical meaning, thus implying their *nonhamiltonian* character. The "inapplicability" (and not the "violation") of quantum mechanics[5] for quantitative treatments of electron pairs in superconductivity is then beyond reasonable doubts.

In reality, the insufficiencies of quantum mechanics are much deeper than the above because they are of *geometric* nature much along Fermi's vision.[1] In fact, a predominant experimental evidence in electron pairing is their *anisotropy* (see, e.g., ref. [13,14]). The derivation of the event from first axioms therefore requires a theory which is structurally anisotropic. The insufficiencies of quantum mechanics are then clear also from a geometric viewpoint owing to the fact that isotropy is a fundamental pillar of all its structures, from the Euclidean and Minkowski spaces, to the Galilean and Poincaré symmetries.

Thus, the studies presented in these volumes can be seen as efforts to construct a generalization of quantum mechanics capable of a quantitative derivation of the attractive interaction of electron pairs in superconductivity from first axioms. Such a pairing will then be assumed as the origin of the *cold fusion* in nature, that at the ultimate level of elementary particles, where the term "cold" stands to indicate that the bound state is enhanced at low energy (low temperatures).

Once the cold fusion of electrons of the same charge is truly understood, one can readily predict the existence of similar cold fusions in particle physics, beginning with the cold fusion at short distances of electron–positron of charges $-e$ and $+e$ for which, after all, the Coulomb interactions are attractive. Once this second cold fusion is established, the extrapolation to the cold fusion of elementary particles at large is then predictable. These possibilities are mentioned here for quantitative studies later on, to indicate that the insistence on preserving old knowledge as much as possible in face of an increasing complexity of physical reality *de facto* implies the suppression of new knowledge.

Yet another particle event identifying the limitations of quantum mechanics quite clearly is the so-called *Bose–Einstein correlation* (see, e.g., review [15] and references quoted therein), e.g., as occurring for the proton–antiproton annihilation at high energy, in which the particles coalesce into a state called a *fireball* which then decays in a variety of modes whose final products are correlated bosons (see later on in this section Fig. 1.2.1).

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[5] Quantum mechanics was strictly conceived for the exterior particle problem in vacuum in which has resulted to be exact. Quantum mechanics is therefore "inapplicable" for fundamentally different physical conditions and the use of the term "violation" would be scientifically inappropriate.

In the opinion of this author, the Bose–Einstein correlation has a particularly fundamental value for the studies here considered inasmuch as it constitutes the most representative, complex and diversified manifestation of the strong interactions. As such, it touches the very foundation of the historical legacy recalled earlier.

Again, scientific objectivity requires the admission that the Bose–Einstein correlation is not *exactly* derivable from first quantum mechanical axioms, whether nonrelativistic, relativistic or field theoretical. Needless to say, there are numerous phenomenological models providing a sufficient representation of experimental data via apparent quantum mechanical techniques [15]. The issue is that, at a closer scrutiny, these models do imply a departure from one or the other of quantum mechanical axioms.

This is due to a variety of conceptual and technical reasons studied in details in Vol. III. At this point we merely indicate that the fireball of the $p-\bar{p}$ annihilation is composed of two hadrons in conditions of total mutual penetration. But hadrons are not "ideal empty spheres" with "points" in them. Instead, hadrons are the densest objects measured in the laboratory until now with well defined and experimentally measured wavepackets/wavelengths/charge distributions of the order of 1 fm. The total mutual penetration of these particles, one inside the other then demands, for scientific objectivity, the expectation that in its interior we have the most general conceivable interactions of nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic type.

At any rate, there is a rather general consensus that, when the physical conditions are strictly local–differential, there is no correlation. The insufficiency of quantum mechanics and the crude character of all its possible representations of the event is then beyond credible doubts.

Thus, the methods presented in this volume are aimed at constructing a covering of quantum mechanics capable of a direct representation of the Bose–Einstein correlation from nonlinear–nonlocal–noncanonical first axioms. This will then provide similar representational capabilities for strong interactions at large, and the structure of hadrons in particular, via their realization as the most general conceivable interactions in Nature.

Numerous additional limitations can be identified under the indicated open mind. The best way to see them is to *identify physical conditions as <different> as possible from those of the original conception of quantum mechanics* . As another example, quantum mechanics was conceived for the characterization of particles in stable orbits under generally long range interactions verifying conventional conservation laws. To identify the limitations of the theory, one should then consider particles under *interactions which <maximize> the instability of the orbit and/or the effects expected from their extended character.* If we consider instead physical conditions approaching as much as possible those of the original conception of the theory, no deviation should be expected, and, in fact, no deviation has been measured until now under these premises [7].

A further class of phenomena in which the limitations of quantum mechanics are also clear, is given by effects expected from the *inhomogenuity and anisotropy of physical media in which particles and/or electromagnetic waves move* . Consider an electron when a member of the atomic structure. Then, the particle moves in the homogeneous and isotropic vacuum, in which case quantum mechanics is exact.

Consider now the same electron when moving in the medium inside a collapsing star, or, for that matter, the medium inside a hadron, called *hadronic medium* [10]. Then, the particle moves within a medium which is manifestly *inhomogeneous* and *anisotropic* .

Theoretical and experimental questions then arise as to whether such inhomogenuity and anisotropy have any measurable effect in the dynamical evolution of the particle considered. We are here referring to measurable effects in the intrinsic characteristics of particles such as their rest energy, the behaviour of their meanlife with speed, the behaviour of their Doppler frequencies, etc.

Customarily, these quantities are treated by Minkowskian methods. But their ~~basic~~ geometric pillars are the homogeneity and isotropy of space. The insufficiency of Minkowskian methods for inhomogeneous and anisotropic physical media must then be admitted in order not to exit the boundaries of science. The open nature of the problem herein considered then follows.

It is recommendable to identify the *origin* of the limitations of quantum mechanics in some detail so as to have a guideline during the subsequent analysis.

First, quantum mechanics is strictly *local-differential* in its topological structure, which prevents any mathematically consistent treatment of nonlocal • int interactions, whether in electron pairing in superconductivity, or in the Bose-Einstein correlation, or in the strong interactions at large.[6]

Second, quantum mechanics is structurally of *potential-Hamiltonian* type; namely, it can only represent in an established way action-at-distance interactions described by a potential. On the contrary, as indicated earlier, nonlocal effects due to mutual penetration of wavepackets are well known to be of contact type without any potential. As such, contact-nonlocal interactions are conceptually, topologically and analytically outside the representational capabilities of quantum mechanics.[7]

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[6] The reader should be warned against the (not unusual) simplistic attitude of adding a <nonlocal-integral potential> to the Hamiltonian because it can be proved to be mathematically and physically inconsistent on various grounds. To begin, such an addition is in violation of the local-differential topology of quantum mechanics and carries rather serious consequences, such as the inapplicability of *Mackey imprimivitivy theorem* [16] with consequential loss of conventional relativities [17]. Additional inconsistencies will be pointed out shortly.

In order to understand better these insufficiencies, let us review the essential structural lines of quantum mechanics [1,2]. The central notion of quantum mechanics is *Planck's quantum of energy*

$$\hbar = h/2\pi = 1.054589 \times 10^{-34} \text{ joule second} \tag{1.2.1}$$

The primary mathematical structure of the theory is given by:

A) The *universal, enveloping, associative, operator algebra* $\xi$ with elements A, B, ... (say, matrices or local-differential operators) and product given by the familiar multiplication of matrices or operators AB, verifying the familiar *associativity law*

$$(A\,B)\,C \equiv A\,(B\,C), \tag{1.2.2}$$

under which Planck's constant in the form

$$\hbar = I = \text{diag.}\,(1, 1, \ldots\ldots, 1), \tag{1.2.3}$$

assumes the meaning of the *left and right unit* of the theory

$$\xi: \quad A\,B = \text{assoc.}, \qquad I\,A \equiv A\,I \equiv A \quad \forall\ A \in \xi; \tag{1.2.4}$$

B) The *field* F of real numbers R or of complex numbers C.

C) The *Hilbert space* $\mathcal{H}$ with states $|\psi>$, $|\phi>$,..., and inner product

$$\mathcal{H}: \qquad <\psi\,|\,\phi> = \int dr'\, \psi^\dagger(t, r')\, \phi(t, r') \in C; \tag{1.2.5}$$

All familiar formulations of quantum mechanics can be derived from the above primitive mathematical structures either in a direct or an an indirect way. As an example, the fundamental *Heisenberg's equation* for the time evolution of a quantity Q in terms of a (Hermitean) Hamiltonian H

$$i\,\dot{Q} = [Q, H]_\xi = Q\,H - H\,Q, \tag{1.2.6}$$

---

[7] As discussed more technically in subsequent sections, the addition of a potential to a given Hamiltonian implies the underlying tacit assumption of granting potential energy to the interactions considered. For conventional action-at-a-distance interactions this is evidently correct. However, the granting of a potential energy to contact interactions due to the mutual penetration of wavepackets, has no physical sense, and results in a dynamical evolution which has no connection with that in the physical reality. As we shall see later, this is a motivation for representing internal nonlocal effects of strong interactions with the generalized unit of the theory; that is, *with a quantity other than the Hamiltonian.*

is characterized by the antisymmetric brackets $[\ ...,\ ...]_\xi$ attached to the enveloping algebra $\xi$.

Similarly, *Schrödinger's equation*

$$i \frac{\partial}{\partial t} | \psi > \ = \ H | \psi > = E | \psi >, \tag{1.2.7}$$

is a consequence of the original associativity of the envelope $\xi$ which results in the action $H | \psi >$ of the operator H on the state $| \psi >$ as being *right, modular and associative*, i.e., such that

$$A\,B\,C | \psi > \ = \ A\,(B\,C | \psi >) \ = \ (A\,B\,)\,C | \psi > \ = (A\,B\,C\,) | \psi >. \tag{1.2.8}$$

Finally, we recall that the *exponentiation* of Eq.s (1.2.6) into a finite Lie group is a power series expansion in the envelope $\xi$, namely, it is technically permitted by the infinite-dimensional basis in $\xi$ with familiar expansion

$$e_{|\xi}^{i\alpha X} \ = \ I + i\,\alpha\,X\,/\,1! \ + \ (i\,\alpha\,X)\,(i\,\alpha\,X)\,/\,2! \ + \ \ldots,\ \alpha\ \in F,\ \ X \ = X^\dagger \in \xi, \tag{1.2.9}$$

under which the infinitesimal form (1.2.6) can be exponentiated to the Lie group of finite time-evolution

$$Q(t) \ = \ e_{|\xi}^{iHt}\ Q(0)\ e_{|\xi}^{-itH}. \tag{1.2.10}$$

This means that the above group too is fundamentally dependent on the assumption of the unit $\hbar = 1$.

The ultimate essence of quantum mechanics is embodied in the celebrated *Dirac's δ-function* ,

$$\delta(r) \ = \ (2\pi)^{-1} \int_{-\infty}^{+\infty} dz\ e_{|\xi}^{\,i\,z\,r} \ , \tag{1.2.11}$$

verifying the familiar properties under sufficient smoothness conditions

$$\delta(r) = \delta(-r)\,, \qquad \delta(r - r') \ = \int_{-\infty}^{-\infty} dz\ \delta(r - z)\ \delta(z - r')\,, \tag{1.2.12a}$$

$$f(r') \ = \int_{-\infty}^{-\infty} \delta(r - r')\ f(r)\ dr \ \ , \tag{1.2.12b}$$

In fact, the δ-function characterizes the point-like structure of particles and their inherent local-differential topology.

All other aspects of quantum mechanics, such as linear operations on $\mathcal{H}$, Heisenberg's uncertainty principle, Pauli exclusion principle, transformation theory, etc., can be constructed via a judicious use of formulations derivable from or compatible with the above fundamental structures ξ, R (or C) and $\mathcal{H}$.

We are now in a position to identify in more details the following representative limitations of quantum mechanics.

**Limitation 1: Lack of direct representation of extended nonspherical charge distributions of individual hadrons.** The above structural lines show that the topological, geometrical and algebraic structures underlying quantum mechanics are strictly *local-differential.* As a result, *quantum mechanics cannot effectively represent the actual charge distributions of hadrons which are extended as well as nonspherical (e.g., oblate spheroidal).*

Admittedly, the *extended* character of the hadrons can be represented via the so-called *second quantization* [5,6]. However, such approach provides only the remnants of the actual shape via the so-called *form factors.* The insufficiency here considered then becomes evident by nothing that an effective theory must represent the *actual generally nonspherical shape* of the charge distributions of hadrons. In fact, assuming that form factors can represent the shape, considered, that shape must [redacted] be perfectly spherical in order not to violate a pillar of the discipline: the rotational symmetry.

At any rate, the basic unit of the Euclidean space is the trivial unit matrix I = diag. (1, 1, 1) which, as such, can only geometrize the perfect sphere (or the homogeneous and isotropic vacuum).

As an example, there are indications that, as it is the case for all spinning objects in nature, the shape of the charge distribution of a nucleon is not perfectly spherical, but is instead an oblate spheroidal ellipsoid along, say, the z-axis with values for the semiaxes for the proton [18]

$$b^2_x = b^2_y = 1, \quad b^2_z = 0.60, \qquad (1.2.13)$$

which provides one (not necessarily unique) explanation of the anomalous magnetic moments of the nucleons based on the shape alone, that is, without any consideration of any nature on the structure and its constituents.

In regard to the Bose-Einstein correlation, there is clear experimental evidence that the fireball is not perfectly spherical, but a highly prolate spheroidal ellipsoid oriented along the direction of the original $p-\bar{p}$ collision (see Fig. 1.2.1). The above limitation then implies *the inability of quantum mechanics to represent the highly prolate shape of the correlation fireball* ,with evident limitations in the quantitative description of the phenomenon considered.

**Limitation 2: Lack of representation of the deformation of extended charge distributions.** Once the need of representing the actual shape of a charge distribution is understood, one can see that *quantum mechanics is intrinsically unable to represent the possible* [redacted] *deformations of given charge distributions, whether spherical or not, under sufficient external forces or* *collisions* . This is again prohibited by the underlying rotational symmetry.

The ~~need for~~ an actual and direct representation of possible deformations of the charge distribution of hadrons is evident for advancements ~~as a pre-requisite for the experimental resolution of the issue~~. In fact, the admission of the extended character of hadrons implies their deformability under sufficient conditions, for perfectly rigid objects are admitted in academic abstractions, but they do not exist in the physical reality. Thus, the only scientific issue is the *amount* of deformation of the charge distribution of a given hadron under given conditions, but its *existence* is beyond credible doubts.

As a well established example, it is known that the fireball of the Bose–Einstein correlation expands immediately after its formation and alters its shape under sufficiently intense external fields. The above limitation therefore implies *the inability of quantum mechanics to represent the evolution and deformations of the ~~correlation~~ fireball* (see Fig. 1.2.1).

Equivalently, we can say that *quantum mechanics can only represent fireballs which, besides being perfectly spherical , are also perfectly rigid* . The ensuing limitations of the theory are then evident. In the final analysis, the rotational symmetry is taught since undergraduate courses in physics to be solely applicable to *rigid bodies* .

**Limitation 3: Lack of representation of nonlocal nonpotential interactions.** Above all, a most basic limitation is *the inherent inability of quantum mechanics to represent the nonlocal terms expected in the strong interactions at large, as well as under appreciable overlappings of the wavepackets of particles (including leptons as in the Cooper pair in superconductivity).*

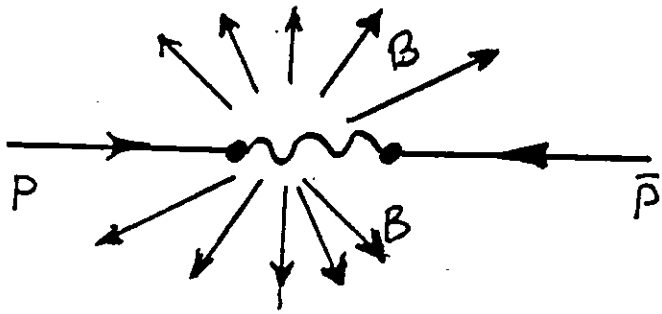
In regard to the Bose–Einstein correlation, this implies *the inability of quantum mechanics to reach a quantitative representation of the expected very origin of the correlation, the nonlocal interactions.* In fact, as recalled earlier, interactions of particles which can be effectively approximated as being point-like show no known correlation, while the boson correlation appears to be due precisely to the nonlocality of the interactions in the interior of the fireball, as we shall see in details in Vol. III.

The experimental data on the Bose–Einstein correlation therefore have fundamental significance because, in the final analysis, they can result in being the first experimental evidence on the historical legacy on the ultimate nonlocal structure of matter.

**Limitation 4: Lack of representation of a number of physical systems from first principles.** To illustrate the case for the Bose–Einstein correlation, consider a system of n particles represented with the symbol k = 1, 2, ..., n, each one possessing correlated and uncorrelated components represented with the

**QUANTUM MECHANICAL APPROXIMATION OF THE BOSE-EINSTEIN CORRELATION**

(1.1)




**A MORE REALISTIC DESCRIPTION OF THE CORRELATION**

(1.2)

(1.3)

(1.4)

(1.5)

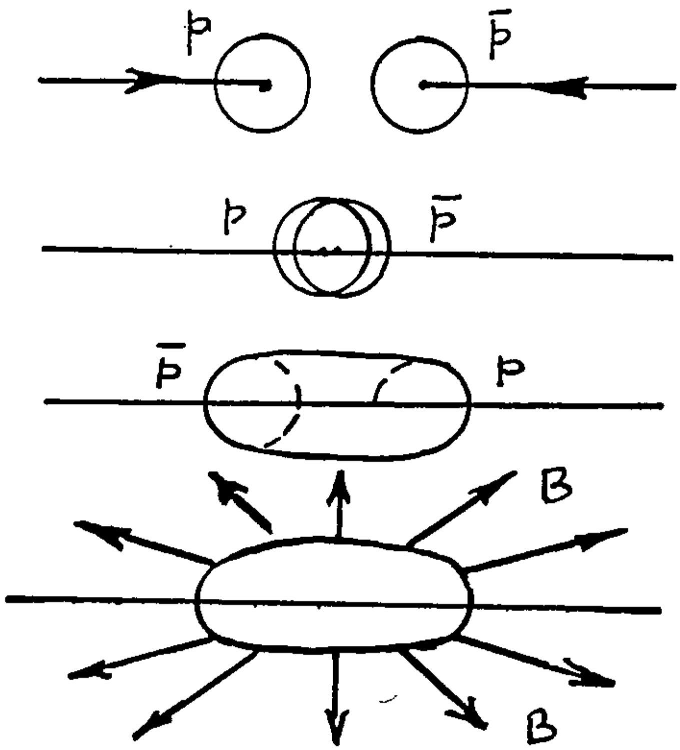



FIGURE 1.2.1: a schematic view of the quantum mechanical approximation of the Bose–Einstein correlation (Diagram 1.1), and a more realistic description suggested by available experimental information (Diagrams 1.2–1.5). In the quantum mechanical case the original proton and antiproton are represented as

points. The correlation and production of the emitted bosons B is then reducible to virtual, action-at-a-distance exchanges, resulting in Limitations 1–5 pointed out in the text. In the physical reality, the proton and antiproton are extended charge distributions of radius ~ 1 fm (Diagram 1.2). Under very high energy, they annihilate in conditions of total mutual penetration and compression of their wavepackets (Diagram 1.3). This creates the fireball which is a highly prolate spheroidal ellipsoid oriented toward the original $p-\bar{p}$ direction (Diagram 1.4). This fireball rapidly expands and decays into particles whose final product is the set of correlated bosons B (Diagram 1.5). A satisfactory representation of the Bose-Einstein correlation must therefore be in a position to provide a quantitative representation of phases 1.2–1.5, as well as resolve Limitations 1–5 of the text from basic axioms.

symbol a and b, respectively. Let the states be given by $|k,a\rangle \times |k,b\rangle$, k = 1, 2, ..., n. According to quantum mechanics the axiomatic characterization of the correlation probability is that based on the conventional expectation values, i.e.

$$C_n = \langle 1,a|\langle 1,b|\ldots\ldots\langle n,a|\langle n,b| \begin{pmatrix} |1,a\rangle \\ |1,b\rangle \\ \ldots \\ |n,a\rangle \\ |n,b\rangle \end{pmatrix} =$$

$$= \sum_k (\langle k,a|k,a\rangle + \langle k,b|k,b\rangle), \qquad (1.2.14)$$

The point is that the above expression lacks exactly the cross terms $\langle k, a | k, b\rangle$ representing the correlation. Admittedly, these cross terms are introduced via a number of artificial expedients (see review [15]). However, for scientific objectivity we must admit that these semiphenomenological models are, strictly speaking, beyond the capability of the axiom of expectation value, thus confirming the inability of quantum mechanics to derive the event from first principles.

A fully similar situation occurs for various other systems, as we shall see.

**Limitation 5: Loss of basic space-time symmetries under nonlinear, nonlocal and nonhamiltonian interactions.** The historical open legacy of Fermi, Blochintsev and others on the ultimate nonlocality of the strong interactions has profound epistemological, theoretical and mathematical implications, because it implies the inapplicability of all conventional space-time symmetries for a number of independent reasons studied in details in volumes [9–12], such as:

a) the homogeneous and isotropic character of the basic medium of conventional relativities, empty space, as compared to the generally inhomogeneous and anisotropic character of all physical media of interior problems, whether of classical or operator type;

b) the Lie–Hamiltonian character of the conventional relativities, as compared to the nonhamiltonian structure of the interactions considered;

c) the local–differential character of the underlying topology (e.g., the Zeeman topology of the special relativity), as compared to the nonlocal–integral nature of the events considered; and others.

Yet another objective of the studies presented in these volumes is to show that, under appropriate generalizations of the methods, we do have a consistent operator image of a property already established at the classical level [11,12], namely, that conventional linear, local and canonical symmetry *transformations* are manifestly inapplicable, but the basic space–time and internal *symmetries* themselves remain exact at the isotopic level.

In conclusion, the viewpoint submitted in this volume is that:

*I) Quantum mechanics does indeed provide an exact description of the physical conditions for which it was conceived, that of particles admitting an effective point–like approximation moving in the homogeneous and isotropic vacuum. This includes all electromagnetic interactions and a large class of additional conditions, such as the approaching phase of the* $p$–$\bar{p}$ *constituents of the Bose–Einstein correlation;*

*II) Quantum mechanics provides a description of particles events in condition of deep mutual overlapping (Fig. 1.1.1), such as the Boson correlation after the creation of the fireball, which can only be valid in first approximation, and*

*III) A more accurate, quantitative description of nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic interactions, as expected in the Cooper pair, the Bose–Einstein fireball, the structure of hadrons, and the strong interactions at large, requires a structural generalization of quantum mechanics itself, perhaps similar to the generalization of classical mechanics that resulted to be necessary for the final understanding of the atomic structure [1].*

## 1.3: CONCEPTUAL FOUNDATIONS OF HADRONIC MECHANICS

In an attempt to resolve the limitations of quantum mechanics, this author submitted in a memoir [19] [8] of 1978 the proposal to construct the so-called *isotopies and genotopies of the conventional Lie's theory,* under the name of *Lie-isotopic and Lie-admissible theories,* respectively. In the subsequent memoir [20] of the same year, this author ~~then~~ proposed the construction of the *isotopies and genotopies of quantum mechanics* under the name of *hadronic mechanics.*

The term <isotopy> was suggested from the Greek "ισοζ τοποζ", meaning "preserving configuration" and interpreted as "axiom preserving".

The basic *isotopic equations* proposed in ref. [20], p. 752 for the time evolution of a physical quantity Q in terms of a (Hermitean) Hamiltonian H on a conventional Hilbert space, are given by the following generalization of Heisenberg's equation (1.2.6)

$$i\,\cancel{\dot{Q}}\,\frac{dQ}{dt} = [Q\,\hat{,}\,H] \;:=\; Q\,T\,H - H\,T\,Q = \tag{1.3.1}$$

$$= QT(t, x, \dot{x}, \ddot{x}, \psi, \cancel{\psi^\dagger}, \partial\psi, \cancel{\partial\psi^\dagger}, \mu, \tau, n, ..)H - HT(t, x, \dot{x}, \ddot{x}, \psi, \cancel{\psi^\dagger}, \partial\psi, \cancel{\partial\psi^\dagger}, \mu, \tau, n, ..)Q,$$

with exponentiated form

$$Q(t) = e^{iHTt}\, Q(0)\, e^{-itTH}, \tag{1.3.2}$$

which admit quantum mechanics as a particular case because

$$[A\,\hat{,}\,B]_{T=I} \equiv [A\,,B] = A\,B - B\,A, \tag{1.3.3}$$

¶ By assuming the generalized unit (1.1.1), to be

$$\hat{1} = T^{-1} = \hat{1}^\dagger, \tag{1.3.4}$$

then the quantity $\hat{1}$, called *isounit,* results to be the correct left and right unit of the new theory, as we shall see in detail in the next chapter. The above formulations were called "Lie-isotopic" [19] because the brackets $[A\,\hat{,}\,B] = A\,T\,B - B\,T\,A$ preserve the original Lie axioms and, in this sense, the lifting $[A\,,B] \rightarrow [A\,\hat{,}\,B]$ is an isotopy.

The term <genotopy> was proposed by the author in ref. [19] from the Greek "γενναω τοποζ" meaning "inducing configuration" and interpreted as "axiom inducing", that is, altering of the original axioms in favor of covering axioms.

The basic *genotopic equations* proposed in ref. [20], p. 746 are given by the following generalization of Eq.s (1.2.6) *and* (1.3.1)

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[8] When at Harvard University under support from the U.S.Department of Energy, contract numbers ER-78-S-02-4742, AS02-78ER-4742, and DE-AC02-8-ER10651.

$$i\dot{Q} = (Q \hat{,} H) := Q R H - H S Q = \qquad (1.3.5)$$

$$= QR(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, ..)H - HS(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, ..)Q$$

with exponentiated form

$$Q(t) = e^{i H S t} \, Q(0) \, e^{-t R H} , \qquad R^\dagger = S , \qquad (1.3.6)$$

which also admit quantum mechanics as a particular case because

$$(A \hat{,} B)_{R = S = 1} \equiv [A , B] = A B - B A . \qquad (1.3.7)$$

The most dominant aspect of the latter formulations is the existence of *two generalized units*, called *genounits*, one for motion forward in time, denoted with $1^>$, and the second for motion backward in time, denoted $^<1$ which can be identified with the inverse of two operators R and S of Eq. (1.3.5)

$$1^> = R^{-1} , \ ^<1^\dagger = S^{-1} . \qquad (1.3.8)$$

The above more general theory was called "genotopic" because the generalized brackets $(A , B) = A R S - B S A$ violate this time the Lie algebra axioms in favor of covering algebras called *Lie-admissible algebras* as proposed by Albert [21] back in 1948 at the abstract level.

In fact, the brackets (A , B) characterize an explicit realization of the Lie-admissible algebras because their attached antisymmetric algebras are Lie-isotopic

$$(A , B) - (B , A) = [A \hat{,} B] = A T B - B T A , \quad T = R - S . \qquad (1.3.9)$$

The Lie-isotopic formulations were then studied by the author in monographs [9–12], while the more general Lie-admissible formulations were studied in monographs [22,23].

From the completely unrestricted functional dependence of the generalized unit, it is evident that the above formulations have a clear capability to represent nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic systems. In effect, the equations were subsequently proved to be "directly universal" for the systems considered, as we shall see in Ch. 7 of this volume.

The physical differences of the isotopies and genotopies were also identified in the original proposal [20]. In fact,

*The* ***isotopic formulations*** *characterize closed-isolated systems with conserved total Hamiltonian* H *and other total physical quantities under* *the most general possible nonlinear, nonlocal and nonhamiltonian internal forces represented by the operator* $T$ *because, from the totally antisymmetric character of the brackets, we have*

$$i\dot{H} = [H \hat{,} H] = HTH - HTH \equiv 0. \qquad (1.3.10)$$

On the contrary:

*The* ***genotopic formulations*** *characterize open-nonconservative systems, such as one particle under the most general known nonlinear, nonlocal and nonhamiltonian external interactions represented by the operators R and S because, from the lack of anticommutativity of the brackets, we have*

$$i\dot{Q} = (H, H) = H(R-S)H \neq 0. \qquad (1.3.11)$$

The physical differences between the isotopic and genotopic formulations can also be effectively seen from the viewpoint of time-reversal invariance. In fact, one can see from the Hermiticity of the T operator T that *isotopic formulations are structurally reversible, that is, they are reversible for a time-reversal invariant Hamiltonian*.

On the contrary, it is equally easy to see from the lack of Hermiticity of the R and S operators that *genotopic formulations are structurally irreversible; that is, they are irreversible even for all time-reversal invariant Hamiltonians*.

The above occurrences suggested the characterization of the genotopic formulations with the arrow of time, the operator R characterizing motion forward in time, while the operator S characterizes the motion backward in time. Thus,

*The basic conceptual structure of hadronic mechanics has essentially remained that of the original proposal [19,20]: the integral generalization of Planck's unit, Eq. (1.1.1), of two primary types:*

*A) a first type of Hermitean-reversible character for motion in both forward and backward direction in time, which characterizes axiom-preserving generalizations of quantum mechanics, and*

*B) a second type requiring two different generalized units, one for motion forward in time and another for the motion backward in time, which characterize a generalization of the axiomatic structure of quantum mechanics.*

Proposal [20] concluded with the illustration of the <novel> capabilities of hadronic mechanics; that is, applications beyond the technical capacities of quantum mechanics. In fact, in Sect. 5, it was shown that the above isotopic and genotopic formulations provide a first quantitative representation precisely of the cold fusion of two electrons indicated in Sect. 1.2.

As we shall see in details later on, the originally proposed mechanism of achieving a bound state of electrons at short distances was based on the absorption and consequential elimination of the Coulomb interactions by internal nonlinear, nonlocal and nonhamiltonian effects because of mutual wave-overlappings. That mechanism subsequently resulted in being correct and in agreement with experimental data in superconductivity.

**THE TWO BRANCHES OF HADRONIC MECHANICS**

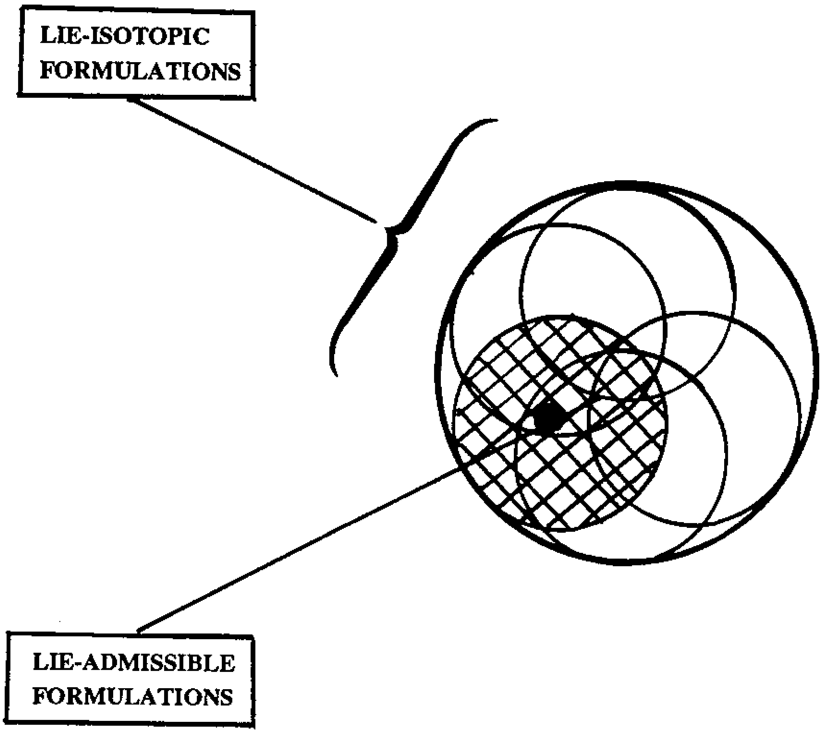


FIGURE 1.3.1: A schematic view of the main branches of hadronic mechanics, the Lie–isotopic branch describing closed–isolated systems verifying conventional total conservation laws under nonlinear–nonlocal–nonhamiltonian internal forces, and the more general Lie–admissible branch describing the most general possible open–nonconservative systems or, more specifically, one component of a Lie–isotopic system when considering the rest as external. It is generally believed that the conservation of the total energy, $\dot{H} = 0$, can only occur under conservative internal forces or, more technically, for systems called *closed variationally selfadjoint* [11], such as planetary or atomic systems. This belief was disproved in memoir [20] of 1978 by showing that the total energy can also be conserved under contact,

nonhamiltonian internal forces. In the latter case we merely have internal exchanges of energy and other physical quantities but always such to balance each-other and result in conserved total quantities. These studies identified a new class of physical systems called *closed variationally nonselfadjoint* studied in detail at the classical level in monograph [12] of 1983, and at the operator level in memoirs [35] of 1989. A classical example is provided by Jupiter in which one can visually see in telescopes its global stability in a way compatible with irreversible, unstable interior processes, such as vortices with continuously varying angular momenta. A particle example is given by a neutron star, which is also manifestly stable at the global level, yet the orbits of the individual neutrons in its interior are generally unstable precisely because of the interactions studied in these volumes due to motion of the particle under penetration within the medium composed by other particles. We can therefore say that

*A) Quantum mechanics is an operator formulation of closed variationally selfadjoint systems, i.e., isolated systems with only local-differential-potential internal forces, in which case one formulation only of Lie type is sufficient for both the system as a whole as well as its individual constituents. Global stability is achieved in this case via the stability of each constituent; while*

*B) Hadronic mechanics is an operator formulation of closed variationally nonselfadjoint systems, i.e., isolated systems with local-differential-potential, as well as nonlocal-nonpotential internal effects, in which case two mutually compatible formulations are generally needed, one for the description of the stable system as a whole (in which case the isotopic brackets of the time evolution must be totally antisymmetric), and one for the description of the individual constituents in unstable orbits (in which case the genotopic brackets of the time evolution cannot be antisymmetric). Global stability is generally achieved in this latter case under the maximal possible instability of each constituents.*

Intermediate cases have also been identified, i.e., systems which are closed-isolated and variationally nonselfadjoint because of contact internal forces, yet the orbits of all constituents are stable. This is generally the case of strongly interacting systems with a relatively small number of constituents, such as few-body nuclei and the structure of hadrons. In this latter case the Lie-isotopic formulations are sufficient for the representation of both, the system as a whole and each individual constituent.

Moreover, the proposal was for the "hadronic bound state" (today called "cold fusion") of two electrons of arbitrary elementary charges $\pm e$. This implies the existence of the electron-electron pairing as established in superconductivity today, plus a <novel> bound state, that of electron-positron at short distances which was capable of a quantitative representation of the totality of the characteristics of the $\pi^{\circ}$ particle in our space-time, that *without* unitary symmetry and, thus, without any possibility of even defining quarks.

Significantly, the latter model was and is generally dismissed by orthodox scholars because it is "not predicted by quantum mechanics" or "not in line with current quark theories". However, if physical evidence has already established the existence of the electron–electron pairing in superconductivity under a highly repulsive Coulomb interactions [13,14], in order not to exit the boundary of science we must expect the existence of the pair electron–positron at short distances because much facilitated by highly attractive Coulomb interactions. The identification of the latter with the $\pi^{\circ}$ (again, in our space–time) is then the only logical possibility, not only because of the numerical representation of its characteristics, but also because the $\pi^{\circ}$ produces spontaneously the electron–positron constituents in one of its decay modes, which can therefore result to be a tunnel effects of the constituents.
This brings us into the central physical purposes for which the entirety of the classical and operator, Lie–isotopic and Lie–admissible studies were and continue to be conducted, as clearly stated in the original proposal [20]:

> *Attempt the identification, within the context of a covering of quantum mechanics, of the hadronic (or quark) constituents with massive, physical, ordinary particles which are freely produced in the spontaneous decays.*

As we shall see, nonlinear, nonlocal, nonhamiltonian, inhomogeneous and anisotropic internal effects which have permitted a quantitative formulation of the electron–electron pair in superconductivity can be be readily extended not only to the electron–positron pairing at short distance, but also to all *massive* particles, thus permitting the cold fusion (also called "chemical synthesis"[9]) of unstable hadrons, i.e., their reduction to hadronic bound states of lighter hadrons suitable selected in the decay modes.

Such a "chemical synthesis" is demonstrably impossible with quantum mechanics, as we shall see, precisely because of its abstraction of particles as dimensionless points, with consequential abstraction of hadrons as "ideal spheres with points in them".

However, if hadrons are represented as they actually are in the physical reality beginning from the very first level of discrete nonrelativistic treatment, they emerge as being the densest, inhomogeneous and anisotropic media measured in laboratory until now. Then, the hadronic structure emerges in a fundamentally different light, because now the constituents move within a hyperdense medium, rather than moving within ideal empty spheres.

In turn, these structural generalizations of the basic geometry produces structural alterations of the particles under which the above objective becomes

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[9] The author would like to thank Prof. A. N. Sissakian, Deputy Director of the J. I. N. R., Dubna, Russia, for the suggestive terms "chemical synthesis".

quantitatively possible in a form verifiable with experiments.

In the final analysis, this is precisely the reason for the construction (and the name) of the covering "hadronic mechanics".

As we shall see in Vol. III, compatibility of the above "model of hadronic structure in our space-time only" with the "unitary model of hadronic classification" [22] is readily achieved when our space-time in enlarged to include unitary spaces. As a matter of fact, the new structure model offers genuine possibilities of basic advances for the conventional quark models themselves, such as:

1) the identification of quark constituents with ordinary massive particles;
2) the achievement of a "true confinement" with an identically null transition probability for free quarks beginning at the discrete non-relativistic level;
3) perturbation series which are convergent under isotopies or genotopies when conventionally divergent; and others.

The primary purpose for writing these first books in the field is to solicit the participation of the scientific community at large for their confirmation or dismissal, not in marginal conversations in academic corridors, but in the only way physical knowledge progresses: via publications.

What is at stake here is the apparent existence or lack of existence of a new technology, tentatively called by this author "hadronic technology" with far reaching possibilities in all aspects of science, all depending on one central, basic issue:

*The capability, resulting from a covering mechanics, to admit ordinary massive particles as hadronic (or quarks) constituents and, therefore, produce them free either in spontaneous decays or in stimulated processes .*

## 1.4: GUIDE TO THE LITERATURE

The above lines of inquiry of 1978 were subsequently subjected to systematic studies by numerous authors, as indicated by the following meetings:

1) Five *Workshops on Lie-Admissible Formulations* held in Cambridge, MA, from 1978 to 1983;
2) Five *Workshops on Hadronic Mechanics* held from 1983 to 1989 in various Countries;
3) The *First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment* held at the Université

d'Orléans, France, in 1982;

and other meetings (see Proceedings [24] and references therein).

In a situation of this type, in this introductory section I can only indicate the most significant steps. Specialized advances will be reviewed and quoted in the subsequent chapters. This presentation, however, is and will remain incomplete in the review and quotation of all contributions to avoid a prohibitive length. Also, contributions on other lines of inquiry cannot possibly be quoted (if nothing else, because of their shear number) unless they study *a structural generalization of current theories*, such as: the *quantum groups* (see, e.g., ref. [25] and quoted papers); the so-called *q-deformations* (see, e.g., ref. [26] and quoted paper) which are particular cases of the Lie-admissible formulations and, thus, particular cases of hadronic mechanics; the studies on *nonlocality* by Russian colleagues (see, e.g., monographs [8]); the *discrete formulation* of space-time; and other true generalizations.

The most salient advances in the studies of isotopies and genotopies of quantum mechanics can be summarized as follows. The original proposal [20] of 1978 suggested the formulation of Eq.s (1.3.1)–(1.3.5) on a conventional Hilbert space, a formulation which subsequently proved to be correct, yet insufficiently general.

In fact, in papers [27,28] of 1982 on the Lie-isotopic and Lie-admissible formulations, Myung and Santilli achieved a first mathematically rigorous formulation over the isotopies $\hat{\mathcal{H}}_T$ of a Hilbert space $\mathcal{H}$ with inner product $< \psi \mid T \mid \phi >$, $T > 0$ (defined over a generalized field reviewed in the next Chapter) where the operator $T$ is the same as that in Eq.s (1.3.1).

As we shall see, the liftings $\mathcal{H} \rightarrow \hat{\mathcal{H}}_T$ have the fundamental implication of preserving Hermiticity under isotopies, as a result of which *the observable of quantum mechanics remain observable in hadronic mechanics.*

Subsequent studies [29] by Mignani, Myung and Santilli of 1983 showed that the preceding formulation [27,28] even though correct, were themselves insufficiently general because Eq.s (1.3.1) and (1.3.5) can also be consistently defined on an isotopic Hilbert space $\hat{\mathcal{H}}_G$ with inner product $< \psi \mid G \mid \phi >$, $G > 0$ (also over a generalized field), where the operator G can be independent of R and S or T.

The primary results of these studies are clear and deal with the largest possible "degrees of freedom" of isotopic and genotopic theories. As well known, a system in quantum mechanics is characterized by only one operator $H = K + V$. The corresponding system in the "isotopic branch" of hadronic mechanics is characterized by three independent operators, the Hamiltonian H, characterizing the potential forces, the isotopic operator T characterizing the nonpotential forces, and the operator G characterizing additional degrees of freedom of the underlying Hilbert space, while in the "genotopic branch" a system is characterized by four operators, H and G as well as R and $S = R^{\dagger}$.

Thus, the algebraic part of hadronic mechanics, that of Heisenberg-type based on generalized equations (1.3.1) and (1.3.5), had reached sufficient mathematical maturity by 1983. The additional advances since that time have been in the technical knowledge of Lie-isotopic algebras, Lie-admissible algebras, isotopic Hilbert spaces, and their applications.

In fact, in 1983 we already had the isotopic generalization of Wigner's theorem on unitary symmetries [30] and a structural generalization of the Lorentz symmetry O(3.1) of isotopic type [31] which, while locally isomorphic to O(3.1) (for all $T > 0$), produced a generalization of the conventional linear-local-canonical Lorentz transformations of the desired, most general possible nonlinear, nonlocal, noncanonical, inhomogeneous and anisotropic type. The other developments and applications were merely consequential.

The studies on the Scrödinger-type formulations equivalent to the preceding Heisenberg-type ones resulted to be considerably more laborious than the above, to such an extent to require a further generalization of the already generalized classical studies, including an integral isotopic lifting of the local-differential Birkhoffian generalization of Hamiltonian mechanics [10].

In essence, Myung and Santilli [27] identified the following isotopic generalization of Schrödinger's equation on the isotopic Hilbert space $\hat{\mathcal{H}}_T$

$$i \frac{\partial}{\partial t} | \psi > = H\, T \, | \psi > , \tag{1.4.1}$$

which resulted to be equivalent to Eq.s (1.3.1) under the applicable unitary-isotopic transformations (except for scalar factors subsequently resulted to be important for an overall consistency of the theory). It should be mentioned that Eq.s (1.4.1) had also been independently identified by Mignani [32], although without Hilbert space treatment.

Also, Animalu and Santilli [33] identified the following isotopy of the naive quantization called *naive isoquantization*

$$A \rightarrow -i\, I \log | \psi > \quad \Rightarrow \quad \hat{A} \rightarrow -i\, \hat{I} \log | \hat{\psi} > \tag{1.4.2}$$

under which the Birkhoffian form of the Hamilton-Jacobi equations [10] was uniquely and unambiguously mapped exactly into the hadronic equation (1.4.1).

However, subsequent studies indicated that Eq. (1.4.1) was not compatible with the relativistic isotopic formulations [31]. More specifically, the isotopic generalization of the conventional field equations characterized by the Lorentz-isotopic symmetry of ref. [31] admitted the following generalization of the plane-waves (for the simple case here considered of generalized units not dependent on coordinates and time, but dependent on the velocities and all other variables)

$$\hat{\psi}(t, r) = N\, e^{\, i ( p\, T\, r - E\, T_t\, t )} , \tag{1.4.3}$$

which permitted a quantitative interpretation of the geometric alterations of electromagnetic waves when propagating within inhomogeneous and anisotropic media via the isotopies of the Minkowski space of ref. [31], exactly as desired.

On the contrary, Eq. (1.4.1) admitted the simpler "plane-wave"

$$\hat{\psi}(t, r) = N e^{i(p T r - E t)}, \qquad (1.4.4)$$

without the generalized element $T_t$ in the energy term, thus resulting not to be compatible with relativistic form (1.4.3).

Also, Eq. (1.4.4) prevented the achievement of a consistent expression for the isotopic linear momentum operator, which in fact was completely lacking at that time (mid 1980's). In turn, the lack of such consistent isotopic forms literally precluded the construction of most applications, which had to be conducted instead at the abstract level (as done for that reason in ref.s [30,31]).

The resolution of these basic deficiencies required this author to conduct again a laborious effort at the purely classical level because of the evident need to reach the isotopic form of the linear momentum operator via isoquantization of corresponding well defined Hamilton-Jacobi equations, as a covering of conventional quantum derivations.

Now, as recalled earlier, a step-by-step generalization of Hamiltonian mechanics of Birkhoffian type has been proposed in memoir [19] of 1978 as a first application of the Lie-isotopic theory, and then studied in monograph [10] of 1983.

The difficulties here mentioned are due to the fact that, while the Heisenberg-type image of Birkhoff's equations has been reached since the original proposal [20] of 1978, the achievement of a Schrödinger-type version of Birkhoffian mechanics escaped all efforts for a number of technical problems, including: an excessively general "wave functions" $\psi(t, r, p)$ with an essential dependence also in the momenta $p$; lack of any practically usable expression for the isotopic linear momentum; the nonlinear and noncanonical, yet strictly local-differential character of Birkhoffian mechanics as compared to the general nonlocal character of hadronic mechanics; and others.

These occurrences forced this author to reinspect the classical generalized theories *ab initio*, and to conduct a second, step-by-step generalization of Hamiltonian mechanics, this time, of the so-called *Birkhoff-isotopic type* reviewed later on in Volume II which admitted the most general possible nonlinear and noncanonical, as well as nonlocal-integral systems. This novel mechanics, was presented for the first time in memoirs [34] of 1988 jointly with the corresponding, compatible isotopies of the symplectic, affine and Riemannian geometries for interior problems (only). Memoirs [34] were then expanded in monographs [11,12] for a detailed treatment of this classical profile.

The final form of the basic (nonrelativistic) equations of hadronic mechanics

$$i\, T_t^{-1}\, \dot{Q} = [Q \hat{,} H] = Q\, T\, H - H\, T\, Q, \tag{1.4.5a}$$

$$i\, T_t^{-1} \frac{\partial}{\partial t} | \psi > = H\, T\, | \psi >, \tag{1.4.5b}$$

$$P_k\, T\, | \psi > = - i\, \hat{1}_k{}^i\, \frac{\partial}{\partial x^i}\, | \psi >, \tag{1.4.5c}$$

where $T_t$ is the *time isotopic element* (different than the *space isotopic element* T), including the final expression of the operator linear momentum so vital for practical applications, was achieved in memoirs [35] of 1989 via the isoquantization of generalized Hamilton–Jacobi equations, exactly as desired.

Relativistic equations were then achieved via isotopies of the conventional relativistic equations, as we shall see in Vol. II, and they resulted to be fully compatible with basic nonrelativistic equations (1.4.5).

Classical studies [34] also set the basis for the novel topology of hadronic mechanics (see Fig. 1.4.1).

Two further aspects deserve a mention for advance guidance in the following analysis. The construction of theories based on a generalized unit $\hat{1}$ permit the identification the antiautomorphic map

$$\hat{1} \;\rightarrow\; \hat{1}^d = -\hat{1}, \tag{1.4.6}$$

called by this author *isoduality*, with corresponding *isodual isotopic formulations* which were identified in ref.s [31], and studied in more details in memoirs [35].

The studies on isoduality essentially permitted *a novel interpretation of antiparticles based on theories with negative–definite generalized units.*

In fact, antiparticles originate from the negative–energy solutions of relativistic equations, although these solutions behave unphysically when conventionally interpreted, that is, when the negative energies are interpreted as having the conventional unit $\hbar = I > 0$, thus forcing the conjecture of an "infinite sea" of virtual particles and other assumptions.

In the above indicated studies, this author essentially showed that the negative–energy solutions behave in a fully physical way in empty space when characterized as having *a negative–definite unit* $I^d = -I$, without any need of conjecturing infinite seas of particles, or passing to second quantization. A fully similar situation emerged for antiparticles within physical media characterized by the *isodual isounit* $\hat{1}^d = -\hat{1}$.

The isotopic and isodual formulations then emerged as possessing intriguing interconnections from the finite transition probabilities existing in conventional relativistic equations between positive- and negative-energy solutions.

In these volumes we shall therefore study isotopic formulations for positive-definite generalized units $\hat{1} > 0$, and the other for negative-definite units $\hat{1}^d < 0$. Other cases will be evidently studied too (see next section).

The last aspect deserving an advance mention regards gravitation. As we shall see, isotopic and isodual formulations, including those of the Riemannian geometries [11,12,34] permit truly remarkable and diversified advances in gravitation, including the identification of a hitherto unknown "isodual universe" for antimatter.

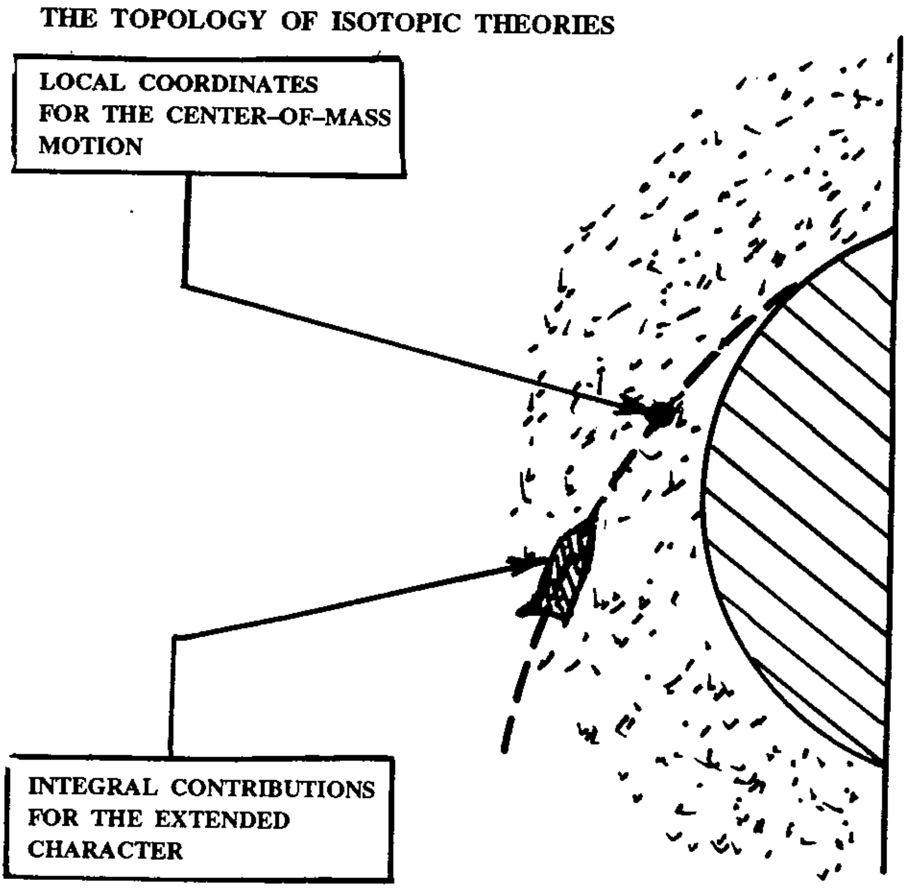


FIGURE 1.4.1. A conceptual view of the topology of hadronic mechanics. Nonlocal systems are of notoriously difficult treatment because they demand the so-called *integral topologies* which are some of the most complex mathematical constructions, particularly for physical applications. The solution proposed by this author to by-pass these difficulties is so simple to appear trivial, yet it is effective for physical applications, as we shall see.

The main ideas are the following: 1) preserve the conventional local-differential variables x for the description of the trajectory of the center-of-mass of the particle in interior conditions; 2) consider all nonlocal-integral contributions as corrections to the local-differential description; and 3) embed all nonlocal terms in the isounit of the theory. By recalling that topologies are insensitive to the functional dependence of their own unit when positive-definite, one can therefore see that *all classical and operator isotopic theories admit the conventional local-differential topology everywhere except in the unit.* Such a generalized topology, hereon referred to as *integro-differential topology, carries* subtle but important theoretical implications, such as the achievement of a fully *causal* description of *nonlocal* interactions, as well as experimental implications; e.g., the capability to test the nonlocal contribution as distinct from the conventional local ones.

An illustration is given by the space-ship during re-entry of this figure, whose shape directly affects the trajectory x(t) of the center of mass, as well known, resulting in two forces, a variationally selfadjoint (SA) force which is local-differential and derivable from a potential V(x), and a variationally nonselfadjoint (NSA) force which is generally nonlinear (in all variables), nonlocal and nonpotential. We therefore have classical equations of motion of the type

$$m\,\ddot{x} \;=\; F^{SA}(x) \;+\; F^{NSA}(t,\,x,\,\dot{x},\,\ddot{x},\,...),\quad F^{NSA} = -\,\gamma\,\dot{x}^{\,2}\int_\sigma d\sigma\;\mathcal{F}(\sigma)$$

where $\sigma$ is the shape of the satellite, with analytically equivalent operator counterparts of Fig. 1.1.1 (where the surface integral is generally replaced by a volume integral representing the overlapping of wavepackets).

In these volumes we shall by-pass the notorious difficulties in the practical application of integral topologies via the representation of the local-differential part in terms of the conventional Hamiltonian $H = K(\dot{x}) + V(x)$ and the embedding of all nonlocal-NSA forces in the isounit $\hat{1}$ of the theory.

Rigorous mathematical studies on this integro-differential topology have been conducted by Tsagas and Sourlas [40], to whom we refer for all technical details.

The aspect warranting advance notice regards gravitational collapse. The isotopies permit the factorization of all Riemannian metrics into the firm $g(x) = T(x)\,\eta$, where $\eta$ is the Minkowski metric and the embedding of the isotopic part $T(x)$ truly representing gravitation in the generalized unit via the rule [34]

$$\hat{1} \;=\; [\,T(x)\,]^{-1}\,,\qquad g(x) \;=\; T(x)\;\eta\,. \tag{1.4.7}$$

This permits *an alternative formulation of curvature via the generalization of the unit of the conventional Minkowski space,* i.e., via the isotopies of the Minkowski space originally proposed in ref. [31] as the structural foundations of the isotopies of the special and general relativities for nonlocal interior conditions.

Among the various novel possibilities (such as the geometrically unified treatment of the special and general relativities for the exterior problem in vacuum), we see in this way that the generalized unit has a very special meaning when singular. In fact, *the limit* $\hat{1}(x) \rightarrow \infty$, *as we shall see, can represent a gravitational singularity at x*.

Thus, in addition to the two cases $\hat{1} > 0$ and $\hat{1} < 0$, the third case $\hat{1} \rightarrow \infty$ also has intriguing physical interest and should be kept in mind during the analysis of these volumes. Two additional significant cases of the isounit will be identified in the next section.

Among the numerous researchers who have contributed to the development application and test of hadronic mechanics at this writing (summer of 1993), we mention

**Animalu, Aringazin, Bartzis, Baskoutas, Borghi, Brodimas, Caldirola, Cardone, Dall'Olio, Eder, Fronteau, Gasperini, Gill, Giori, Ioannidou, Jannussis, Kadeisvili, Kalnay, Kamiya, Karayannis, Kliros, Klimyk, Kobussen, Lin, Lopez, Mignani, Mijatovic, Myung, Nishioka, Papadoupoulos, Papaloukas, Papatheou, Rauch, Sourlas, Skaltzas, Streclas, Tsilimigras, Veljanoski, Vlahos, Tellez Arenas, Tsagas, Weiss, Wolf,**

and others we shall review in these volumes. The understanding is that we are referring to *contributions specifically dealing with the generalization of the unit.* Generalizations of quantum mechanics based on the conventional unit are considered elsewhere (see Appendix 7.A and Vol.s II, III).

Independent reviews of the classical studies are available in monographs [36,37], while comprehensive mathematical presentations of the isotopies of Lie's theory is available in monographs [38,39].

As indicated in the Preface, all existing nonlinear, nonlocal, discrete and other generalizations of quantum mechanics are directly or indirectly related to hadronic mechanics. We regret the inability to quote them here because of their large numbers, although we shall review in due course representative examples of existing generalizations (see Appendix 7.A and Vol.s II, III).

## 1.5: CLASSIFICATION OF HADRONIC MECHANICS

Hadronic mechanics is nowadays a rather diversified discipline with structurally different mathematical methods in different branches. In a situation of this type, it is recommendable to assume a classification from the beginning of the studies, because it can prove to be later on a valuable guide.

First, the hadronic mechanics is divided into the two main branches identified in the preceding section:

**A - The Lie-isotopic branch for closed-isolated NSA systems,** which is characterized by Hermitean generalized units $\hat{1} = \hat{1}^\dagger$ for both motions forward and backward in time, and

**B - The Lie-admissible branch for open-nonconservative NSA systems,** which is characterized by two different generalized units $\hat{1}^>$ and $^<\hat{1}$, for motions forward and backward in time, respectively.

Next, each branch admits a classification depending on the main structural characteristics of the generalized unit. In this volume we shall assume the classification introduced by Kadeisvili [38] for the isotopies of functional analysis, here called *Kadeisvili's classification*, which divides the isotopic branch into the following five classes:

**Class I: Isotopic formulations properly speaking,** holding when the generalized unit $\hat{1}$ is sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive–definite. This is the class of primary interest in these volumes for the study of particles in interior conditions.

**Class II: Isodual isotopic formulations,** holding when the generalized unit has the same characteristics of Class I, except that it is negative–definite. This is the class of primary relevance for the study of antiparticles in interior conditions.

**Class III: Indefinite isotopic formulations,** holding for generalized units with the same characteristics of Class I and II except that they have an indefinite signature, with the possibility of being either positive–definite or negative–definite. This class has primary mathematical relevance, e.g., for the unified treatment of Class I and II.

**Class IV: Singular isotopic formulations,** holding for generalized units that are divergent. As we shall see, this class is useful for the study of gravitational singularities.

**Class V: General isotopic formulations,** holding for generalized units of arbitrary structure, thus including discrete structures, distributions, discontinuous functions, etc. This last class is useful for the study of fundamentally novel mathematical notions, such as a discrete group defined over a continuously varying unit (and viceversa) and, except for isolated remarks, will not be considered in these volumes for brevity.

Evidently a corresponding distinction into Classes I–V holds for the Lie–admissible/genotopic branch of hadronic mechanics with the understanding that the condition of Hermiticity and positive– or negative–definiteness are referred only to the Hermitean part of the nonhermitean operators R and S.

In conclusion, hadronic mechanics is a generalization–covering of quantum mechanics which possesses *ten* topologically different classes, and this begins to illustrate the rather broad character of the new discipline from which its "direct universality" follows (Ch. I–7).

Unless otherwise stated, the mathematical foundations studied in this Volume I specifically treats the isotopic formulations of Class I (for particles) and II (for antiparticles), with comments on the construction of the remaining isotopic formulations of Class III, IV and V. The genotopic formulations will be reviewed in the appendices and in Ch. 7.

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# 2: ISONUMBERS

## 2.1: STATEMENT OF THE PROBLEM

We indicated in the Preface that the primary difficulty in addressing (and appraising) hadronic mechanics is the prior knowledge of its novel mathematical structure, because even the conventional numbers and their operations are inapplicable. The understanding is that, when inspected, such novel mathematical structure soon emerges to be simple and intriguing.

The best way to illustrate this aspect is by noting that the traditional statement "two multiplied by two equals four" is at best mathematically incomplete, because it lacks the identification of the underlying unit and of the operation of multiplication, and is generally inapplicable under isotopies.

In fact, we shall first show in this chapter that, by assuming, say, for generalized unit $\hat{1} = 3^{-1}$, "two multiplied by two equals twelve" and then we shall show that the *the numbers themselves* and their operations are in general, of *integral* character as necessary from basic assumption (1.1.1).

The use under isotopies of the conventional mathematical structure of quantum mechanics therefore leads to a host of generally undetected inconsistencies.

We shall ~~therefore~~ study in this chapter the generalized numbers needed for hadronic mechanics, and then study in subsequent chapters the ~~the~~ generalized structures built on them.

As well known, the *theory of numbers* received momentous advances in the past century, thanks to the contributions of famed scholars such as Gauss [1], Abel [2], Hamilton [3], Cayley [4], Galois [5] and others (see review [6] in the early part of this century, and ref.s [7–9] for contemporary presentations).

Additional important advances in number theory were made during this century, including the axiomatic formulation, the theory of algebraic numbers, etc. (see, e.g., ref.s [10] and contributions quoted therein).

The "numbers" significant for these volumes are the *real numbers, complex numbers, quaternions and octonions.* The topic is therefore the classification of all *normed algebras with identity over the reals* according to the studies, e.g., by Hurwitz [11], Albert [12] and (N.) Jacobson [13] (see also reviews [7,8]). The main properties can be expressed via the following

**Theorem 2.1.1** (ref. [8], p. 122): *All possible normed algebras with multiplicative unit over the field of real numbers are given by algebras of dimension 1 (real number), 2 (complex numbers), 4 (quaternions) and 8 (octonions).*

The fundamental notions under study in this chapter are therefore fields and normed algebras with unit.

During a talk at the conference *Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980[10], this author submitted an axiom-preserving generalization of numbers, today known as *isotopic numbers* or *isonumbers* for short. The generalizations are induced by the so-called *isotopies* of the conventional multiplication of numbers introduced in ref.s [14,15], with consequential generalization of the basic multiplicative unit.

The isonumbers received a formal treatment in ref. [16], and first applications in ref. [17] for the isotopic lifting of unitary symmetries, in ref. [18] for the lifting of the Lorentz symmetry, and in ref.s [19.20] for the general isotopies of Lie symmetries. Subsequent studies were conducted in ref.s [21,22]. A theorem on the unification of different isonumbers (studied in Sect. 2.7] was presented in ref. [23]. The presentation of this chapter follows ref. [24] which is the comprehensive mathematical study on the theory of isonumbers on record at this time, with the understanding that the studies are still at the beginning, and so much remains to be done.

The author also submitted in ref.s [19,20] a new conjugation, under the name of *isoduality* which yields an additional class of numbers, today known as *isodual isonumbers.* Recent presentations of isodual isonumbers can be found in ref.s [22,24].

The isonumbers were motivated by the specific physical need of a quantitative representation of the transition from the exterior to the interior dynamical problem, as discussed in Chapter 1. The isodual isonumbers were constructed for a representation of antiparticles (see Vol. II and ref. [29]).

The isonumbers and their isoduals are at the foundations of the Lie-isotopic formulations but they are inapplicable for more general theories such as the Lie-admissible formulations.

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[10] Thanks to a kind invitation by Prof. H.-D. Doebner which is here gratefully acknowledged.

For this reason the author identified in ref. [21] an additional class of numbers under the name of *genotopic numbers*, or *genonumbers* for short. The primary difference between isonumbers and genonumbers is that *isonumbers have a unique left and right generalized unit* or, equivalently, *the multiplication of the isonumbers applies to both left and right operations.*

By comparison, *genonumbers have two different generalized units, one for the multiplication from the right and a different one for the multiplication from the left,* $a<b = b<a$, $a>b = b>a$, $a<b \neq a>b$, in which case they indeed result to be at the foundation of the Lie–admissible formulations.

To avoid excessive initial complexities, we shall proceed in stages. In this and in the following chapters we shall solely study the isonumbers, their isoduals and their Lie–isotopic formulations. The primary objective of this chapter is therefore the study of the isotopies and isodualities of the notions of numbers, fields and normed algebras with unit.

The minimal mathematical knowledge needed for hadronic mechanics is that of isoreal and isocomplex numbers and their isoduals studied in Sect.s 2.5 and 2.6. The isoquaternions, isooctonions and their isoduals of Sect.s 2.7 and 2.8 are needed for a more technical knowledge of the topic. The more general (and complex) theory of genonumbers and related Lie–admissible formulations will be studied in Ch. 7.

For a recent independent study of the field, including elements of isotopies, we suggest the monograph by Lôhmus, Paal and Sorgsepp [28]. Applications of the the generalized numbers of this chapter to classical mechanics can be found in monograph [29].

The author would like to thank David Ring of Dunedin, FL, for bringing to his attention the fact that the Egyptians have been the first in recorded history to change the value of their basic unit, called *finger*, in the transition from the sides of a right triangle to the hypothenuse.

## 2.2: ISOUNITS AND THEIR ISODUALS

Studies [14–27] (and references quoted therein) have shown that the transition

A) from the local–canonical *exterior problem* in vacuum,

B) to the nonlocal–noncanonical *interior problem* within physical media,

can be effectively represented via an axiom–preserving *isotopic generalization* of the conventional multiplication of numbers a, b (or functions or operators).

We are here referring to the generalization of the current, simplest possible multiplication "ab" (here often denoted $a \times b$ for notational convenience), into the *isotopic multiplication*, or *isomultiplication* for short, introduced in ref. [14], p. 332,

$$a * b \ := aTb \equiv a \times T \times b, \tag{2.2.1}$$

which will be symbolically denoted $* \equiv \times T \times$, where $T$ is a fixed, invertible quantity ~~for all possible elements a, b~~ called *isotopic element.* The lifting $\times \rightarrow *$ is isotopic because (for nondegenerate elements T) it preserves all the original operations among ordinary numbers as seen in more details in the next section.

The conventional, right and left multiplicative unit I of current theories, $I \times a = a \times I \equiv a$, is then lifted into the form

$$\hat{I} * a = a * \hat{I} \equiv a, \qquad \hat{I} := T^{-1}, \tag{2.2.2}$$

called the *multiplicative isounit*, or *isounit* for short.

Under the condition that $\hat{I}$ preserves all the axioms of I the lifting $I \rightarrow \hat{I}$ is an *isotopy*, that is, the conventional unit I and the isounit $\hat{I}$ (as well as the conventional product $a \times b$ and its isotopic form $a * b$) coincide at the abstract level by conception.

The *isonumbers* can be first introduced as the generalization of conventional numbers when characterized by isoproduct (2.2.1) with respect to the generalized isounit $\hat{I} = T^{-1}$.

As one can see, the isounits have a completely unrestricted functional dependence, thus admitting the most general possible integro-differential structure of type (1.1.1),

$$\hat{\hbar} = \hbar \hat{I} = \hat{I}(t, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \mu, \tau, n, \ldots), \quad \hbar = 1. \tag{2.2.3}$$

A necessary condition for a quantity $\hat{I}$ to be an isounit, that is, a joint left and right generalized unit, is that it is Hermitean. Then, isomultiplication (2.2.1) is the same for both right and left operations.

As indicated in Sect. 1.5, in these volumes we shall use *Kadeisvili's classification* into:

**Class I: Isounits** properly speaking, when they are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite;

**Class II:Isodual isounits** , when they are as in Class I, except that they are negative-definite;

**Class III:Indefinite isounit**, when they are as in Class I except that they have an indefinite signature with local values which can be either positive-definite or negative-definite;

**Class IV:Singular isounits** , when they diverge at at least one given value of their variables;

**Class V:General isounits**, when they have an unrestricted structure, e.g., given by discrete forms, distributions, step functions, etc.

In this we shall study isounits of Classes I and II with a few comments on those of Class III. The theory of isonumbers for Classes Iv and V is vastly unexplored at this writing.

We should note that the most important functional dependence of the isounits is that of *integral* type. Thus, the isotopies $1 \rightarrow \hat{1}$ characterize a new form of *integro-differential topology* in which all integral terms are embedded in the isounit, while the rest of the structure is conventionally local-differential (see Fig. 1.4.1 and ref. [15]). As an example, in the isotopies of Minkowskian spaces, this novel structure permits the preservation of conventional topologies (e.g., the Zeeman topology) everywhere, except for the interior of the isounit itself.

The integral generalization of the unit is the conceptual, mathematical and physical foundation of hadronic mechanics, because it permits a quantitative treatment of the *integral generalization of Planck's constant* $\hbar \rightarrow \hat{1}$ discussed in Ch. 1.

As we shall see better in applications presented in subsequent chapters, the isounits of hadronic mechanics generally have a matrix representation with considerable degrees of freedom in their elements. As such, they permit a geometrization of inhomogeneous and anisotropic physical media, in such a way to preserve the axioms of the homogeneous and isotropic vacuum and admit the latter as a particular case.

The *isodual isounits* are given by

$$\hat{\hbar}^d = \hbar \hat{1}^d = -\hat{1}, \ \hbar = 1, \tag{2.2.4}$$

and are based on the following antiautomorphic conjugation of multiplication (2.2.1)

$$a * b \rightarrow a *^d b := a\,T^d\,b = -a\,T\,b = -a * b, \ T^d = -T, \tag{2.2.5}$$

under which $\hat{1}^d$ (but not $\hat{1}$) is the correct left and right generalized unit of the theory,

$$\hat{1}^d *^d a = a *^d \hat{1}^d \equiv a, \tag{2.2.6}$$

The map characterized by liftings (2.2.4) and (2.2.6) was called *isoduality* [20] and this terminology will be kept in these volumes. As we shall see, these liftings are significant inasmuch as they can be applied to each aspect of the Lie-isotopic formulations, yielding the *isodual Lie-isotopic theory*.

The *isodual isonumbers* were constructed via isodual multiplication (2.2.5) with respect to the the multiplicative isodual isounit $\hat{1}^d$.

Note that the notion of isoduality first applies to *conventional* numbers. In fact, the expressions

$$\hat{T}^d = -1, \quad \hat{1}^d = 1^d := -1, \tag{2.2.7}$$

characterize *isodual numbers*. This means that the conventional formulations, such as Lie's theory, Riemannian geometry, etc., admit hitherto unknown images given by *the isodual Lie theory, the, isodual Riemannian geometry,* etc., which are constructed in such a way to admit everywhere the isodual unit $1^d = -1$.

One can now see the necessity of lifting the product $\times \rightarrow *$ for the very conception of isodual numbers, isodual isonumbers, and related formulations. The restriction of the studies in number theory to the conventional multiplication $\times$ may therefore be a reason why isodual numbers, isodual Lie formulations, isodual Riemannian geometry, etc. have escaped detection until recently.

The author also studied the problem whether isomultiplication (2.2.1) exhausts all isotopies of the conventional product of numbers. The issue is important because any new isotopy of the associative product characterizes a new realization of the theory of isonumbers and, therefore, a new mechanics, with new Heisenberg-type equations and all that.

Only three isotopies of the multiplication ab were found [15]:

A) The *scalar isotopy*

$$a * b := a\, c\, b, \quad T = c = \text{number}, \tag{2.2.8}$$

B) The *operator isotopy*

$$a * b := a\, T\, b, \quad T = \text{operator}, \tag{2.2.9}$$

C) The *idempotent isotopy*

$$a * b := W\, a\, W\, b\, W, \quad W^2 = W = \text{operator}, \tag{2.2.10}$$

and any of their combinations.

The problem whether the above liftings exhaust all possible isotopies of the multiplication is unknown at this writing.

These volumes are based on the fundamental condition that *any acceptable generalization of quantum mechanics must possess a well defined unit, which is evidently needed, e.g., to permit the formulation of a consistent measurement theory, etc.*.

Along these lines, isotopies (2.2.8) and (2.2.9) are acceptable, while isotopies

(2.2.10) are not because the product $a*b = WaWbW$ does not admit a consistent, left and right unit for all elements a, b.

## 2.3: ISOFIELDS, PSEUDOISOFIELDS AND THEIR ISODUALS

Let us introduce the following definition of isofields:

**Definition 2.3.1** [24]: *Let* $F = F(a,+,\times)$ *be a "field" as conventionally understood (see, e.g., ref. [8], p. 101), here referred to a ring with elements* a, b, c, ..., *which is commutative with respect to the operation of addition* $+$ *and associative under both the addition* $+$ *and multiplication* $\times$ *with corresponding additive unit 0 and multiplicative unit* 1. *Then, the infinite family of "isotopic images" of* $F(a,+,\times)$, *called "isofields" and denoted* $\hat{F} = \hat{F}(\hat{a}, +, \hat{\times})$, *are given by elements* $\hat{a}$, $\hat{b}$, $\hat{c}$, ..... *characterized by all possible one–to–one and invertible maps* $a \rightarrow \hat{a}$ *of the original elements* $a \in F$ *equipped with two operations* $(+, \hat{\times})$, *the conventional addition* $+$ *of* F *and a new multiplication, called "isomultiplication", with corresponding conventional additive unit* 0 *and a generalized multiplicative unit* $\hat{1}$, *called "multiplicative isounit", which are such to satisfy all axioms of the original field* F, *i.e.:*

*1) Axioms of addition:*

*1.A) The set* $\hat{F}$ *is closed under addition,*

$$\hat{a} + \hat{b} \in \hat{F} \qquad \forall\, \hat{a}, \hat{b} \in \hat{F}, \tag{2.3.1}$$

*1.B) The addition is commutative for all elements* $\hat{a}, \hat{b} \in \hat{F}$

$$\hat{a} + \hat{b} = \hat{b} + \hat{a}\,; \tag{2.3.2}$$

*1.C) The addition is associative for all* $\hat{a}$, $\hat{b}$, $\hat{c}$, $\in$ F,

$$\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}\,; \tag{2.3.3}$$

*1.D) There is an element 0, the "additive unit", such that for all elements* $\hat{a} \in \hat{F}$

$$\hat{a} + 0 = 0 + \hat{a} \equiv \hat{a}\,; \tag{2.3.4}$$

*1.E) For each element* $\hat{a} \in \hat{F}$, *there is an element* $-\hat{a} \in \hat{F}$, *called the "opposite of* $\hat{a}$", *which is such that*

$$\hat{a} + (-\hat{a}) = 0 \tag{2.3.5}$$

*2) Axioms of isomultiplication:*
*2.A) The set* $\hat{F}$ *is closed under isomultiplication,*

$$\hat{a} \hat{\times} \hat{b} \in \hat{F}, \quad \forall \quad \hat{a}, \hat{b} \in \hat{F}, \tag{2.3.6}$$

*2.B) The multiplication is generally non-isocommutive, i.e.,* $\hat{a} \hat{\times} \hat{b} \neq \hat{b} \hat{\times} \hat{a}$, *but "isoassociative", i.e., it satisfies the law for all elements* $\hat{a}$, $\hat{b}$, $\in \hat{F}$

$$\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{c}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{c} \; ; \tag{2.3.7}$$

*2.C) There exists a quantity* $\hat{1}$, *the "multiplicative isounit", which is such that, for all elements* $\hat{a} \in \hat{F}$,

$$\hat{a} \hat{\times} \hat{1} = \hat{1} \hat{\times} \hat{a} \equiv \hat{a}, \tag{2.3.8}$$

*2.D) For each element* $\hat{a} \in \hat{F}$, *there is an element* $\hat{a}^{-1} \in \hat{F}$, *called the "isoinverse", which is such that*

$$\hat{a} \hat{\times} (\hat{a}^{-1}) = (\hat{a}^{-1}) \hat{\times} \hat{a} = \hat{1}. \tag{2.3.9}$$

*3) Properties* [11] *of joint addition and isomultiplication:*
*3.A) The set* $\hat{F}$ *is closed under joint isomultiplication and addition,*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} \in \hat{F}, \; \forall \; \hat{a}, \hat{b}, \hat{c} \in \hat{F} \tag{2.3.10}$$

*3.B) All elements* $\hat{a}$, $\hat{b}$, $\hat{c} \in \hat{F}$ *verify the right and left "isodistributive laws"*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) = \hat{a} \hat{\times} \hat{b} + \hat{a} \hat{\times} \hat{c}, \; (\hat{a} + \hat{b}) \hat{\times} \hat{c} = \hat{a} \hat{\times} \hat{c} + \hat{b} \hat{\times} \hat{c}, \tag{2.3.11}$$

*The elements* $\hat{a}$ *of isofields* $\hat{F}(\hat{a},+,\hat{\times})$ *are called "isonumbers". When there exists a least positive integer p such that the equation*

$$p \hat{\times} \hat{a} = 0, \tag{2.3.12}$$

*admits solution for all elements* $\hat{a} \in \hat{F}$, *then* $\hat{F}$ *is said to have*

[11] Property (2.3.10) is generally derived from axioms 1A and 2A. Nevertheless, we shall encounter in Sect. 2 (see the comments after Proposition 2.3.3) a case in which Axioms 1A and 2A are verified, but property (2.3.10) is not.

*"isocharacteristic p". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero".*

A few additional properties are needed before we can select the realization of isonumbers used in these volumes. First, we should indicate that *only isofields of isocharacteristic zero will be used throughout the our studies* . Nevertheless, we thought that an exposure of physicists to isofields of isocharacteristic p is warranted because of their potential physical relevance for a number of applications, ranging from string theory to gravitational collapse, particularly when inspected from an isotopic viewpoint[12].

The dominant mathematical aspect here is the isotopy. In fact, the lifting $F(a,+,\times) \rightarrow \hat{F}(\hat{a},+,\hat{\times})$ preserves all original axioms by construction. The realizations of the isonumbers must then be selected in such a way to preserve such basic isotopic character.

In this respect, we note that the liftings $a \rightarrow \hat{a}$, and $\times \rightarrow \hat{\times}$ can be used jointly or individually. The following property is then important for our analysis.

**Proposition 2.3.1** [24]: *Necessary and sufficient condition for the lifting (where the multiplication is lifted but the elements are not)*

$$F(a, +, \times) \;\rightarrow\; \hat{F}(a, +, \hat{\times}), \quad \hat{\times} = \times T \times, \quad \hat{1} = T^{-1} \tag{2.3.13}$$

*to be an isotopy is that the lifting* $\times \rightarrow *$ *is a scalar isotopy (2.2.8), i.e.,* T *is a non-null element of the original field* F.

In fact, the laws of addition are unchanged under lifting (2.3.13), while the multiplication and distributive laws can be readily verified to hold. The closure of the original set under the addition is evident because that operation is not changed. We then remain with the closure under the isomultiplication,

$$a * b \;=\; a \times T \times b = a\,T\,b \;\in \hat{F}, \;\; \forall\, a, b \in \hat{F}, \tag{2.3.14}$$

which does indeed hold when $T \in F$, by therefore establishing the sufficiency of the condition. Its necessity follows from simple contrary arguments.

**Proposition 2.3.2** [24]: *The lifting (in which both the multiplication and the elements are lifted)*

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[12] It should be mentioned in this respect that the classification of all possible simple Lie algebras (Cartan's classification) is known only over fields of characteristic zero, while the classification over fields of characteristic p is far from being complete at this writing. As we shall see, isonumbers of isocharacteristic zero will produce no new Lie algebras, but only new, nonlinear–nonlocal–noncanonical *realizations* of known abstract Lie algebras. However, isonumbers of characteristic p may permit the identification of *new Lie algebras.*

$$F(a, +, \times) \rightarrow \hat{F}(\hat{a}, +, *), \quad \hat{a} = a \times \hat{1} \equiv a\hat{1}, \quad * = \times T \times, \quad \hat{1} = T^{-1}, \quad (2.3.15)$$

*constitutes an isotopy even when the multiplicative isounit* $\hat{1}$ *is not an element of the original field* F *, e.g., when the lifting* $\times \rightarrow *$ *is an operator isotopy (2.2.3).*

In fact, one can readily verify for lifting (2.3.15) the validity of all axioms of a field, and closure under addition. Closure under multiplication readily holds because

$$\hat{a} \hat{\times} \hat{b} = (a \times b) \times \hat{1} = c \times \hat{1} = \hat{c} \in \hat{F}, \quad \forall a, b, c = a \times b \in F, \qquad (2.3.16)$$

The above mathematically simple proposition expresses the physically fundamental capability of generalizing Planck's unit $\hbar = 1$ of quantum mechanics into an integro-differential operator $\hat{1}$ for a quantitative treatment of nonlocal interactions.

In fact, basic assumption (1.1.1) requires, by conception, an isounit which is outside the original field. The realization we shall adopt throughout these volumes is therefore form (2.3.16) with the understanding that more complex realizations are possible (see later on).

The implications of the above realization are evidently fundamental for hadronic mechanics. One implication deserving advance mention is that *the "numbers" predicted by hadronic mechanics for measurements are ordinary numbers*.

In fact, the above realization implies that the isomultiplication of an isonumber $\hat{a}$ by any quantity Q coincides with the conventional multiplication

$$\hat{a} * Q \equiv a\,Q. \qquad (2.3.17)$$

Thus, *the isoeigenvalues of hadronic mechanics can be made to coincide with ordinary numbers*

$$H * |\psi> = \hat{E} * |\psi> \equiv E\,|\psi>, \quad \hat{E} \in \hat{F}, \quad E \in F. \qquad (2.3.18)$$

The numerical predictions of the theory are then ordinary numbers E and not isonumbers $\hat{E}$.

It should be noted that the mathematically correct expression in hadronic mechanics is the form $H*|\psi> = \hat{E}*|\psi>$. Nevertheless, since $\hat{E}*|\psi> = \hat{E}T|\psi> \equiv E|\psi>$, ordinary eigenvalues E can be used in practical calculations.[13]

[13] It should be noted that the lifting of eigenvalues is far from being trivial. In fact, as we shall see in more details Vol. II, if an operator H has the conventional eigenvalue $E°$, $H\,|\psi> = E°\,|\psi>$, it admits a *different* eigenvalue E under isotopy, $H * |\psi> = H\,T\,|\hat{\psi}>$

Evidently, all conventional operations depending on the multiplication are altered under lifting to isofields. Let us consider the isofields $\hat{F}(a,+,*)$ of Proposition 2.3.1 under the condition that the isounit $\hat{1}$ commutes with all elements of a. Then, the "square" $a^2 = a\,a$ is lifted into the *isosquare* $a^{\hat{2}} = a * a = a\,T\,a$, with *n–th isopower*

$$a^{\hat{n}} = a\,T\,a\,T\,a\,......T\,a \quad \text{(n times)} \tag{2.3.19}$$

Recall that the conventional square root can be defined as the quantity $a^{\frac{1}{2}}$ such that $(a^{\frac{1}{2}})(a^{\frac{1}{2}}) = a$. Then, the *isosquare root* is given by

$$a^{\hat{\frac{1}{2}}} := a^{\frac{1}{2}}\,\hat{1}^{\frac{1}{2}}, \quad a^{\hat{\frac{1}{2}}} * a^{\hat{\frac{1}{2}}} = a^{\hat{\frac{1}{2}}}\,T\,a^{\hat{\frac{1}{2}}} = a\,. \tag{2.3.20}$$

The *isoinverse* is given by

$$a^{-\hat{1}} := \hat{1}\,a^{-1}\,\hat{1}, \quad a * a^{-\hat{1}} = \hat{1}. \tag{2.3.21}$$

The *isoquotient* can then be defined by

$$a\,\hat{/}\,b := (a\,/\,b)\,\hat{1} = c \qquad c\,T\,b = a\,. \tag{2.3.22}$$

The reader can then compute all other isooperations.

In the transition to the realization of Proposition 2.3.2 we have instead

$$\hat{a}^{\hat{n}} = \hat{a} * \hat{a} * \hat{a} * ....... * \hat{a} = a^n\,\hat{1}\,. \tag{2.3.23}$$

The reformulation of the remaining operations is then follows, as the reader is encouraged to work out to acquire familiarity with the theory of isonumbers.

Recall that a primary objective of hadronic mechanics is the integro–differential generalization of Planck's constant $\hbar = 1 \rightarrow \hat{\hbar} = \hbar\hat{1} = \hat{1}$. It is therefore important to understand that the new unit $\hat{1}$ preserves all axiomatic properties of the original unit 1. In fact, *the isounit* $\hat{1}$ *is idempotent of arbitrary (finite) order* n *as the original unit* 1

$$\hat{1}^{\hat{n}} = \hat{1} * \hat{1} * .....* \hat{1} = \hat{1} \quad \text{(n times)}\,, \tag{2.3.24}$$

*the isosquare root of the isounit is the isounit itself*,

$$\hat{1}^{\hat{\frac{1}{2}}} = \hat{1}\,, \tag{2.3.25}$$

---

$= E\,|\hat{\psi}>$, $E \neq E^{\circ}$. Thus, *the isotopies of numbers imply an alteration of the eigenvalues of conventional quantum mechanical operators*. This mechanism, called *mutation* [15], is at the foundation of the capabilities of hadronic mechanics to represent the cold fusion of particles and other novel applications.

*the isoquotient of the isounit by itself is the isounit*,

$$\hat{1} / \hat{1} \equiv \hat{1}, \tag{2.3.26}$$

the *isoinverse of the isounit is the isounit itself*,

$$\hat{1}^{-\hat{1}} = \hat{1}, \tag{2.3.27}$$

etc. This confirms the axiom-preserving character of the lifting $\hbar = 1 \rightarrow \hat{\hbar} = \hat{1}$ when realized via the isotopies. Note that the above properties hold for the most general possible *integral* representation of $\hat{\hbar} = \hat{1}$.

Note the different between the two elements 1 and $\hat{1}$, because the element 1 remains an element of an isofield, although now $1 * a \neq a$ and $1 * \hat{a} \neq \hat{a}$.

Note also that the set of purely imaginary numbers $S = \{in\}$, n real, *is not* a field, evidently because it is not closed under multiplication, $(in) \times (im) = -nm \notin S$. However, the isotopy $\hat{F}(\hat{n},+,*)$ of real numbers n equipped with the purely imaginary isounit $\hat{1} = i$, and isoproduct $* = \times T \times$, $T = i^{-1} = -i$, do form indeed an isofield, that is, they verify all axioms of a field. This illustrates the possibility offered by the isotopies according to which, given a set S of numbers which do not form a field, there may exist an isotopic lifting $\hat{S}$ under which $\hat{S}$ is indeed a field.

Note that, according to Hamilton [3] original conception, *the quaternions constitute a field* because their multiplication is noncommutative, but associative. On the contrary, according to Cayley [4] original conception, *octonions are not generally considered to constitute a field* because their multiplication is not associative, but verifies the weaker *right and left alternative laws*

$$(a\,b)\,b = (a\,b)\,b\,, \qquad (a\,a)\,b = a\,(a\,b)\,. \tag{2.3.28}$$

This is the reason for assuming a more general definition of field in ref. [24] which is based on the above alternative laws and, as such, it includes as "fields" the octonions. Also, in this way all "fields" coincide with all "normed algebras with unit" of Sect. 2.1.

In these volumes we shall follow for simplicity the conventional definition of fields [8]. Nevertheless, for completeness, we shall consider the isotopy of octonions with the understanding that, according to Definition 2.3.1, they form a weaker form of fields based on the alternative law.

We now pass to the studies of a further new class of numbers, called *isodual isonumbers*. Owing to their importance for these studies as well as for clarity, it is best to present them according to the following separate definition.

**Definition 2.3.2** [24]: *Let* F(a, +, ×) *be a conventional field as per Definition 2.3.1. Then, the "isodual field"* $F^d(a^d, +, \times^d)$ *is constituted by elements called "isodual numbers"*

$$a^d = a \times 1^d = -a, \qquad (2.3.29)$$

*defined with respect to the "isodual multiplication" and related "isodual unit"*

$$\times^d = \times 1^d \times = -\times, \quad 1^d = -1. \qquad (2.3.30)$$

*Let* $\hat{F}(\hat{a}, +, \hat{\times})$ *be an isofield as per Definition 2.3.1. Then, the "isodual isofield"* $\hat{F}^d(\hat{a}^d, +, \hat{\times}^d)$ *is given by "isodual isonumbers"*

$$\hat{a}^d = a^c \hat{1}^d = -a^c \hat{1}, \qquad (2.3.31)$$

f.c. → H.c.

*where* $a^c$ *is the conventional conjugation of* F *(e.g., complex conjugation), defined in terms of the "isodual isomultiplication"*

$$\hat{\times}^d := \times \hat{T}^d \times = -\hat{\times}\, \hat{T}^d = -\hat{T}. \qquad (2.3.32)$$

Again one can see that the isodual unit $1^d$ is idempotent of arbitrary degree n, that the isodual square root of $1^d$ is $1^d$ and the isodual quotient of $1^d$ by itself is $1^d$, with similar occurrences for $\hat{1}^d$.

The reader has noted our insistence in leaving the conventional sum unchanged, and lifting only of the multiplication. The underlying reason warrants a few comments because, as indicated earlier, any generalization of conventional operations implies a new mechanics. A possible generalization of the operation of addition would therefore imply a further generalization of hadronic mechanics.

In essence, in addition to the lifting (2.2.1) of the multiplication, this author also inspected in ref. [21] (see ref. [24] for more technical studies) the lifting of the addition of the form

$$+ \quad \rightarrow \quad \hat{+} = +\hat{K}+, \quad \hat{K} = K \times \hat{1}, \qquad (2.3.33)$$

with consequential redefinition of the conventional additive unit

$$0 \quad \rightarrow \quad \hat{0} = -\hat{K}. \qquad (2.3.33)$$

However, unlike the isotopy of the multiplication $\times \rightarrow *$, the lifting of the addition $+ \rightarrow \hat{+}$ has the following implication:

**Proposition 2.3.3** [21,24]: *The liftings*

$$F(a, +, \times) \quad \rightarrow \quad \hat{F}(\hat{a}, \hat{+}, \hat{\times}), \tag{2.3.35a}$$

$$\hat{a} = a \times 1 \, , \hat{+} \; = + \hat{K} + , \hat{0} \; = - \hat{K} \; = - K \times \hat{1}, \hat{\times} = \times \, T \times, \hat{1} = T^{-1} \, , \tag{2.3.35b}$$

*where* K ∈ F *and* T *is invertible, is not an isotopy for all nontrivial values of the quantity* K ≠ 0, *because it preserves all axioms of Definition 2.3.1, except the distributive law (2.3.11).*

In fact, all axioms (2.3.1)–(2.3.11) can be readily verified to be preserved under liftings (2.3.34). On the contrary, for the right distributive law we have

$$\hat{a} \, \hat{\times} \, (\hat{b} \, \hat{+} \, \hat{c}) \; = \; a \times (b \; + \; K \; + \; c\,) \times \hat{1} \; = (\,a \times b \; + \; a \times K \; + a \times c\,) \times \hat{1} \; \neq$$

$$\neq \; \hat{a} \, \hat{\times} \, \hat{b} \; \hat{+} \; \hat{a} \, \hat{\times} \, \hat{c} \; = \; (\,a \times b \; \; + \; K \; + \; a \times c\,) \times \hat{1} \, , \tag{2.3.36}$$

with similar lack of identities for the left isodistributive law. Note that the set $\hat{F}$ in lifting (2.3.35) is closed under isoaddition for K ∈ F (but not for K ∉ $\hat{F}$), and, separately, under isomultiplication for an arbitrary isounit $\hat{1}$ outside the original set F. The same results hold for the lifting $F(a, +, \times) \rightarrow \hat{F}(a, \hat{+}, \hat{\times})$, $\hat{+} = + K +$, K ∈ F, K ≠ 0.

The implications of Proposition 2.3.3 are so deep to prevent its use in physics. A central notion of quantum mechanics is that of unitary transformations $UU^{\dagger} = U^{\dagger}U = I$, with the exponential representation in terms of a Hermitean operator X and parameter w

$$U \; = \; I \; + \; i \, w \, X \, / \, 1 ! \; + \; (i \, w \, X\,)(\,i \, w \, X\,) \, / \, 2 ! + \; .... \; = \; e^{\,i \, w \, X} \tag{2.3.37}$$

Now, as we shall see in Chapter 4, the isotopy of the multiplication implies a fully consistent isotopic generalization of the above notion which is convergent into a finite form

$$\hat{U} \; = \; \hat{1} \; + \; i \, w \, X \, / \, 1 ! \; + \; (i \, w \, X\,) * (\,i \, w \, X\,) \, / \, 2 ! + \; .... \; = \hat{1} \, e^{\,i \, w \, T \, X} \tag{2.3.38}$$

resulting in this way in the fundamental isotopies of these volumes, those of Lie's transformation groups studies.

The point is that the isotopies of conventional unitary transformations under the lifting of the addition are divergent,

$$U \; = \; I \; \hat{+} \; i \, w \, X \, / \, 1 ! \; \hat{+} \; (i \, w \, X\,)(\,i \, w \, X\,) \, / \, 2 ! \, \hat{+} \; .... \; \Rightarrow \; \pm \, \infty \tag{2.3.39}$$

thus precluding the achievement of finite forms of the time evolution and other fundamental physical laws.

A property expressed by Proposition 2.3.3 is that *the lifting of the addition is not an isotopy* because one of the original axioms is not preserved. We shall then use the following notion

**Definition 2.3.3** [24]: *An "isotopy" is any lifting of a given mathematical or physical structure preserving the original axioms. A "pseudoisotopies" is a lifting which preserves only part of the original axioms .*

As we shall see, the difficult task is in the identification of which property is a true axiom of a given conventional formulation and which is not. As a matter of facts, the isotopies can help precisely in the identification of true axioms and their separation from other algorithms which do not have a truly essential character.[14]

It is important to classify the new numbers and the related new fields identified until now as the foundation of the Lie–isotopic formulations. In this section we have studied the lifting of the multiplication $\times \rightarrow *$ and/or of the addition $+ \rightarrow \hat{+}$ *which do not require ordering*, that is, the action to the right is the same as that to the left (see Ch. 2.7 for the introduction of ordering and a further generalization of isonumbers for the Lie–admissible formulations). This results in the following two groups of generalized fields and related numbers:

**1) Isofields** $\hat{F}(\hat{a},+,\hat{\times})$, which are characterized by the lifting of the multiplication $\times \rightarrow \hat{\times} = *$ while keeping the conventional addition to ensure the preservation of the distributive law (2.3.11). They can be classified in the same way as the isounits resulting in:

**Isofields** properly speaking (Class I),
**Isodual isofields** (Class II),
**Indefinite isofields** (Class III),
**Singular isofields** (Class IV) and
**General isofields** (Class V).

The isotopic branch of hadronic mechanics is based on the following four fundamental types of numbers [24]:

**1.a) Ordinary numbers:** *real numbers* $R(n,+,\times)$, *complex numbers*

[14] As an example, the isotopies of the Riemannian geometry show that all familiar properties are indeed true geometric axioms because preserved under isotopies, *except Einstein's tensor* $G_{\mu\nu} = R_{\mu\nu} \dot{-} g_{\mu\nu}R$, which emerges as being "geometrically incomplete", that is, lacking a certain term to be invariant under isotopies, with intriguing possibilities of resolving at least some of the vexing open problems of gravitation [25].

$C(c,+,\times)$, *quaternions* $Q(q,+,\times)$ and *octonions* $O(o,+,*)$;

**1.b) Isodual numbers:** *isodual real numbers* $R^d(n^d,+,\times^d)$, *isodual complex numbers* $C^d(c^d,+,\times^d)$, *isodual quaternions* $Q^d(q^d,+,\times^d)$, and *isodual octonions* $O^d(o^d,+,\times^d)$;

**1.c) Isonumbers:** *isoreal numbers* $\hat{R}(\hat{n},+,*)$, *isocomplex numbers* $\hat{C}(\hat{c},+,*)$, *isoquaternions* $\hat{Q}(q,+,*)$ and *isooctonions* $\hat{O}(\hat{o},+,*)$.;.

**1.d) Isodual isonumbers:** *isodual isoreal numbers* $\hat{R}^d(\hat{n}^d,+,*^d)$, *isodual isocomplex numbers* $\hat{C}^d(\hat{c}^d,+,*)$, *isodual isoquaternions* $\hat{Q}^d(\hat{q}^d,+,*^d)$, and *isodual isooctonions* $\hat{O}^d(\hat{o}^d,+,*^d)$.

**2) Pseudoisofields** $\hat{F}(\hat{a},\hat{+},\hat{\times})$, which are characterized by the further lifting of the addition $+ \rightarrow \hat{+}$ with consequential loss of distributive law (2.3.11), as a result of which they do not possess a known physical significance at this time. Nevertheless they are mathematically intriguing and also admit Kadeisvili's five Classes I, II, III, IV and V with realizations of type 1.a–1.d.

The above classification is sufficient to illustrate the rather broad and diversified character of the theory of isonumbers and pseudoisonumbers, as well as the broad character of the mechanics and other formulations build on isonumbers, such as the Lie–isotopic theory.

Except for marginal comments, in the remaining parts of this chapter we shall study the generalized numbers at the foundation of hadronic mechanics, which are the isonumbers of types 1.a–1.d above.

## 2.4: ISONORMED ISOALGEBRAS

A further notion needed for the study of explicit realizations of isonumbers is the applicable definition of algebra. In fact, conventional numbers constitute normed algebras with unit, as recalled in Sect. 2.1. It is then important to identify the corresponding notion under isotopies.

**Definition 2.4.1** [8,14,24]: *Let* U *be a conventional algebra (see, e.g., ref. [8]) with elements* A, B, C *(say, matrices) and (abstract) product* $A \odot B$ *(say, the associative product* AB *or some nonassociative form) over a field* $F(a,+,\times)$ *with elements a, b, .. operations* $a + b$ *and* $a \times b$ *and related units* 0 *and* 1 *satisfying the basic scalar and distributive laws*

$$(a \times A) \odot B = A \odot (a \times B) = a \times (A \odot B), \qquad (2.4.1a)$$

$$(A \times a) \odot B = A \odot (B \times a) = (A \odot B) \times a, \qquad (24.1b)$$

$$A \odot (B + C) = A \odot B + A \odot C, (B + C) \odot A = B \odot A + C \odot A. \quad (2.4.1c)$$

*The algebra* U *is called a "division algebra" when the equation* $A \times x = B$ *always admit a solution in* U *for* $A \neq 0$. *The algebra* U *is said to admit a unit when there is a quantity* I *such that*

$$I \odot A = A \odot I = A, \quad (2.4.2)$$

*for all* $A \in U$. *Finally, the algebra* U *is "normed" when it admits a norm* $|A|$ *satisfying the basic axiom*

$$|A \odot B| = |A| \times |B|. \quad (2.4.3)$$

*The infinitely possible "isotopic images"* $\hat{U}$ *of* U, *called "isoalgebras" for short, are given by the original elements* A, B, C, ... *now equipped with the isomultiplication* $A \hat{\odot} B$ *over an isofield* $\hat{F}(a,+,*)$ *of elements* a, b, c, (without lifting) *with operations* + *and* $* = \times T \times$, *and related units* 0 *nd* $\hat{1} = T^{-1}$ *under the condition of preserving the original axioms of* U, *i.e., of verifying the following left and right "isoscalar and isodistributive laws"*

$$(a * A) \hat{\odot} B = A \hat{\odot} (a * B) = a * (A \hat{\odot} B), \quad (2.4.4a)$$

$$(A * a) \hat{\odot} B = A \hat{\odot} (B * a) = (A \hat{\odot} B) * a, \quad (2.4.4b)$$

$$A \hat{\odot} (B + C) = A \hat{\odot} B + A \hat{\odot} C, \quad (B + C) \hat{\odot} A = B \hat{\odot} A + C \hat{\odot} A, \quad (2.4.4c)$$

*for all elements* A, B, C $\in \hat{U}$ *and* a, b, c $\in \hat{F}$. *The isoalgebra* $\hat{U}$ *is called an "isodivision algebra" when the equations* $A * x = B$ *always admit a solution for* $A \neq 0$. *An isoalgebra* $\hat{U}$ *is said to admit an isounit* $\hat{1}$ *when*

$$\hat{1} \hat{\odot} A = A \hat{\odot} \hat{1} = A, \quad (2.4.5)$$

*for all* $A \in \hat{U}$. *Finally, the isoalgebra* $\hat{U}$ *is said to be "isonormed", when it admits an isotopic image* $\hat{|} A \hat{|}$ *of* $|A|$ *which verifies the axioms*

$$\hat{|} A \hat{\odot} B \hat{|} = \hat{|} A \hat{|} * \hat{|} B \hat{|} \in \hat{F}, \quad \hat{|} n * A \hat{|} = \hat{|} n \hat{|} * \hat{|} A \hat{|} \in \hat{F} \quad (2.4.6)$$

*The "isodual algebra"* $U^d$ *is then the image of* U *under the isodual field* $F^d(a^d,+,\times^d)$, *while the "isodual isoalgebra"* $\hat{U}^d$ *is the image of* $\hat{U}$ *under the isodual isofield* $\hat{F}^d(a^d,+,*^d)$.

Note the differentiation, in general, between the isomultiplication $A \hat{\odot} B$ of the elements of the isoalgebras, which are, say, matrices, from the isomultiplication of the elements of the isofields $a*b$, which can be ordinary numbers. However, one should keep in mind that, when the elements of $\hat{U}$ and $\hat{F}$ coincide, the two multiplications coincide too, $\hat{o} \equiv *$, as it is the case when isonormed algebras are realized in terms of isonumbers (see subsequent sections).

A realization of the isonorm is the following. Let $\hat{e}_k$ be an "isobasis" of $\hat{U}$ over the isofield $\hat{F}(a,+,*)$ of Proposition 2.3.1, i.e., such that a generic element $A \in \hat{U}$ can be written

$$A = \sum_{k=1,...,m} n_k \hat{*} \hat{e}_k , \qquad n_k \in \hat{F}. \tag{2.4.7}$$

and $\hat{e}^2 = \sum_k \hat{e}_k \hat{o}\, \hat{e}_k = \hat{1}$. The *isonorm* of $\hat{U}$ in the isobasis considered is then given by

$$\hat{|} A \hat{|} : = ( \sum_{k=1,...,m} n_k^{\hat{2}} )^{\frac{1}{2}} \times \hat{1} = ( \sum_{k=1,...,m} n_k * n_k )^{\frac{1}{2}} \times \hat{1} \in \hat{F} . \tag{2.4.8}$$

The extension of the above notions to isofields $\hat{F}(\hat{a},+,*)$ of Proposition 2.3.2 is trivial and, as such, it will be ignored.

The isoalgebra $\hat{U}$ is said to be *isoassociative* when it satisfies the isoassociative law

$$A \,\hat{o}\, ( B \,\hat{o}\, C ) = ( A \,\hat{o}\, B ) \,\hat{o}\, C , \quad \forall A, B, C \in \hat{U} ; \tag{2.4.9}$$

and it is said to be *isoalternative* when it verifies the isoalternative laws

$$A^{\hat{2}} \,\hat{o}\, B = A \,\hat{o}\, ( A \,\hat{o}\, B) , \quad A \,\hat{o}\, B^{\hat{2}} = (A \,\hat{o}\, B) \,\hat{o}\, B . \tag{2.4.10}$$

By recalling that ordinary numbers are associative and alternative only under the inclusion of the octonions, in this chapter we are primarily interested in isoassociative normed isoalgebras with isounit $\hat{1}$, with the extension to isoalternative algebras when the inclusion of isooctonions is desired.

Note that in the conventional case the unit I of the algebra U and the unit 1 of the field F are generally different. In fact, in the former case I is, in general, the n–dimensional unit matrix, while in the latter case 1 is the ordinary unit number. According to Definition 2.4.1, the two units can be assumed to coincide under isotopies, trivially, because the isounit of the isofield can now be a matrix.

As well known, the scalar and distributive laws (2.4.1) are basic axioms for any structure to characterize an "algebra" as commonly understood [7–10]. The image of an algebra U under the lifting to an isofield $\hat{F}(a,+,\hat{\times})$ or $\hat{F}(\hat{a},+,*)$ is then a true algebra because it preserves axioms (2.4.1) by central assumption. However, the image of U (and $\hat{U}$) under the pseudoisofield $\hat{F}(a,\hat{+},\hat{\times})$ of Proposition 2.3.3 (in which the addition is also lifted) implies the loss of the distributive laws and, for this reason, it is no longer an algebra as commonly understood. We shall then call it a *pseudoisoalgebra* [24].

As we shall see, the isotopies of the operations with numbers require, for mathematical consistency, corresponding compatible isotopies of all other operations on algebras.

A case deserving advance mention because needed in the subsequent sections is the notion of determinant of a matrix A which is applicable to an isonormed isoalgebra. The conventional notion is inapplicable under isotopies and must be replaced by the *isodeterminant* [16,21]

$$\hat{Det}\, A \ := \ [\, Det_F\,(A \times T)\,] \times \hat{1} \ , \tag{2.4.11}$$

where $Det_F A$ represents the conventional determinant computed in the conventional field F.

In fact, Det A violates the basic axioms under isotopies, e.g.,

$$(\,Det\ A) * (\,Det\ B) \neq Det\ AB \text{ or } \neq Det\,(A * B),\ \ Det\ A^{-1} \neq (\,Det\ A)^{-1}, \tag{2.4.12}$$

However, $\hat{Det}\, A$ does preserve the above axioms because

$$\hat{Det}\,(A * B\,) = (\,\hat{Det}\ A\,) * (\,\hat{Det}\ B\,),\ \ \hat{Det}\,(\,A^{-1}\,) = (\hat{Det}\ A\,)^{-1} . \tag{2.4.13}$$

The corresponding *isodual isodeterminant* is given by [21,24]

$$\hat{Det}^d\, A \ := \ [\, Det_F(\,A \times T^d\,)\,] \times \hat{1}^d \ ; \tag{2.4.14}$$

which is now computed in $F^d$.

The isotopies of various operations (e.g., on a Hilbert space) will be studied in details in Ch. 6.

## 2.5: ISOREAL NUMBERS AND THEIR ISODUALS

By following ref. [24], we shall now study in more details explicit realizations of the isoreal numbers and their isoduals.

**2.5.A: Realization of ordinary real numbers.** Let us recall for completeness and notational convenience (see, e.g., ref. [7]) that conventional real numbers $n \in R(n,+,\times)$ are realized on the one–dimensional real Euclidean space $E_1(x,\delta,R(n,+,\times))$, which essentially represents a straight line with origin at 0, local coordinates x, metric $\delta = 1$, additive unit 0 and multiplicative unit 1. In fact, the *dilations*

$$x' = n\, x = n \times x \qquad n \in R(n,+,\times), \qquad x, x' \in E_I(x,\delta,R)\ , \qquad (2.5.1)$$

characterize an isomorphism of the reals $R(n,+,\times)$ into the commutative one-dimensional *group of dilations* $G(1)$.

The trivial basis is $e = 1$, with norm given by the familiar positive-definite expression

$$|n| \;=\; (n \times n)^{\frac{1}{2}} \;>0\,, \qquad (2.5.2)$$

verifying axioms (2.4.3)

$$|n \times n'| \;=\; |n| \times |n'|\,. \qquad (2.5.3)$$

This shows that *real numbers constitute a one-dimensional normed associative and commutative algebra* $U(1)$ [7].

**2.5.B: Realization of isodual real numbers.** Isodual real numbers $n^d \in R^d(n^d,+,\times^d)$ are conventional numbers n, although defined with respect to the isodual unit $1^d = -1$. The isodual conjugation for real numbers can then be written

$$n \;=\; n \times 1 \quad\rightarrow\quad n^d \;=\; n \times 1^d \;=\; -\,n\,. \qquad (2.5.5)$$

Thus, *all numerical values change sign under isoduality.* One should however keep in mind that such a sign inversion occurs only when the isodual real numbers are projected in the field of conventional real numbers.

As a specific example, the negative integer number $-3$ referred to negative unit $-1$ is fully equivalent to the positive integer $+3$ referred to the positive unit $+1$.

The representation of $R^d(n^d,+,\times^d)$ constitutes the first occurrence in our analysis requiring a generalized notion of space. In fact, the one-dimensional Euclidean space is evidently inapplicable because the underlying field is now the isodual field $R^d(n^d,+,\times^d)$.

The identification of the generalized space applicable under isotopies was done in ref. [18], as reviewed in details in the next chapter. In the simple case here considered, it is given by the one-dimensional, real, isodual, Euclidean space $E^d(x,\delta^d,R^d(n^d,+,\times^d))$, which is also a straight line, although with conventional additive unit 0, isodual multiplicative unit $1^d = -1$, and isodual metric $\delta^d = -\,\delta = -1$. The *isodual dilations* are then given by

$$x' \;=\; n^d \times^d x = \; n \times x \qquad (2.5.6)$$

They establish an isomorphism between $R^d(n^d,+,\times^d)$ and the *isodual group of dilations* $G^d(1)$, i.e., the conventional group $G(1)$ reformulated with respect to the multiplicative unit $1^d$ (see Chapter 4 for details).

Note that $E_1(x,\delta,R)$ and $E_1^d(x,\delta^d,R^d)$ are anti-isomorphic and the same property holds for $G(1)$ and $G^d(1)$. Note that isodual dilations coincide with the conventional ones, and this could be a reason for the lack of detection of isodual numbers until ref.s [19,20].

The *isodual basis* is

$$e^d = 1^d\,, \tag{2.5.7}$$

and the *isodual norm* becomes now *negative definite*

$$|n|^d := (n \times n)^{\frac{1}{2}}\, 1^d = |n| \times 1^d = -|n| < 0\,, \tag{2.5.8}$$

although preserving the basic axioms (2.4.6),

$$|n^d \times n'^d|^d = |n^d|^d \times^d |n'^d|^d\,. \tag{2.5.9}$$

The above results show that *isodual real numbers constitute a one-dimensional isodual, associative and commutative normed algebra* $U^d(1)$ *which is anti-isomorphic to* $U(1)$ [24].

**2.5.C: Realization of isoreal numbers.** We consider now the isoreal numbers $\hat{n} = n \times \hat{1}$ as elements of an isofield of Class I, $\hat{R}_I(\hat{n},+,*)$ with isomultiplication $* = \times T \times$, and multiplicative isounit $\hat{1} = T^{-1} > 0$ generally outside the original set $R(n,+,\times)$, as requested for basic assumption (1.1). Their representation requires the lifting of the original Euclidean space into a form compatible with the basic isofield $\hat{R}_I(\hat{n},+,*)$, which is given by the *isoeuclidean spaces* [18] of Class I, $\hat{E}_{I,1}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$, with metric $\hat{\delta} = T\delta$ over $\hat{R}(\hat{n},+,\hat{\times})$ (see next chapter for details).

One should keep in mind that $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$ is a simple, yet bona-fide nonlinear-nonlocal and noncanonical generalization of the original space, because the original one dimensional metric $\delta = 1$ is now lifted into the expression

$$\hat{\delta} \;=\; T(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots)\,\delta\,; \tag{2.5.10}$$

Thus, the one-dimensional isospace $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$ represents a generalization of the conventional straight line, here called an *isostraight line*, because of its intrinsically nonlinear, nonlocal and noncanonical metric $\hat{\delta}(t, x, \dot{x}, \ddot{x}, \ldots)$ with multiplicative isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \ldots)$, yet it preserves the original axioms of the straight line as ensured by the isotopies (see Ch. 5 for more details on this feature of isogeometries).

$\hat{R}_I(\hat{n},\hat{+},\hat{\times})$ can then be realized via the *isodilations* on $\hat{E}_{I,1}(x,\hat{\delta},\hat{R})$

$$x' = \hat{n} * x = n\,x\,, \tag{2.5.11}$$

which, again, coincide with the original dilations, as it is the case for the isodual dilations, thus providing a reason for the lack of detection of the isoreal numbers until recently.

Isodilations (2.5.11) characterize an isomorphism of the isoreal numbers with the one-dimensional *group of isodilations* $\hat{G}(1)$, i.e., the group $G(1)$ realized with respect to the isounit $\hat{1}$ (see Ch. 4 for details). The local isomorphism $E(x,\delta,R(n,+,\times)) \approx \hat{E}_{I,1}(x,\hat{\delta},\hat{R}(\hat{n},+,*)$ holds for all positive-definite isounits (see next chapter) and readily implies $\hat{G}(1) \approx G(1)$.

The *isobasis* is now given by

$$\hat{e} = \hat{1}, \tag{2.5.12}$$

while the *isonorm* can be defined by

$$\uparrow\hat{n}\uparrow \;:=\; (n \times n)^{\frac{1}{2}}\hat{1} \;=\; |n|\,\hat{1}\,, \tag{2.5.13}$$

namely, by the conventional norm, only rescaled to the new unit $\hat{1}$, which is the essence of the transition from real number n to their isotopes $\hat{n} = n\times\hat{1}$.

In particular, axioms (2.4.6) trivially hold,

$$\uparrow\hat{n} * \hat{n}'\uparrow \;=\; \uparrow\hat{n}\uparrow * \uparrow\hat{n}'\uparrow\,, \tag{2.5.14}$$

with the same product inside and out because referred to the same elements. One can see that the *isoreal numbers constitute a one-dimensional, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}(1) \approx U(1)$ [24].

**2.5.D: Realization of isodual isoreal numbers.** We consider now the isodual isonumbers of Class II

$$\hat{n}^d = n\;\hat{1}^d \;=\; -\,\hat{n}\;\in \hat{R}_{II}{}^d(\hat{n}^d,+,*^d)\,. \tag{2.5.15}$$

In this case we need the one-dimensional, *isodual isoeuclidean space* of Class II, $\hat{E}_{II,1}{}^d(x,\hat{\delta}^d,\hat{R}^d)$, and the *isodual isodilations*

$$x' \;=\; \hat{n}^d *^d x \;=\; n\,x\,, \tag{2.5.16}$$

which also coincide with the conventional dilations, by characterizing an isomorphism of the isodual isoreal numbers with the one-dimensional *isodual group of isodilations* $\hat{G}^d(1)$, i.e., the image of G(1) under the isodual isounit $\hat{1}^d = -\hat{1}$. The evident underlying isomorphism

$$E_{II,1}{}^d(x,\delta^d,R_{II}{}^d(n^d,+,\times^d)) \approx \hat{E}_{II,1}{}^d(x,\hat{\delta}^d,\hat{R}_{II}{}^d(\hat{n}^d,+,\hat{\times}^d)), \qquad (2.5.17)$$

then implies $\hat{G}^d(1) \approx G^d(1)$.

The *isodual isobasis* is now given by

$$\hat{e}^d = \hat{1}^d, \qquad (2.5.19)$$

with *isodual isonorm*

$$\uparrow \hat{n}^d \uparrow^d := (n \times n)^{\frac{1}{2}} \times \hat{1}^d = -\uparrow \hat{n} \uparrow, \qquad (2.5.20)$$

which is also negative-definite, yet verifying basic axiom (2.4.6),

$$\uparrow \hat{n}^d *^d \hat{n}'^d \uparrow^d = \uparrow \hat{n}^d \uparrow^d *^d \uparrow \hat{n}'^d \uparrow^d, \qquad (2.5.21)$$

Thus, *the isodual isoreal numbers are a realization of the one-dimensional isodual, isonormed, isoassociative and isocommutative isoalgebra* $\hat{U}^d(1) \approx U^d(1)$ [24].

The extension of the above results to the case of pseudoisoreal numbers and their isoduals is left to the interested reader.

## 2.6: ISOCOMPLEX NUMBERS AND THEIR ISODUALS

**2.6.A: Realization of ordinary complex numbers.** Let us recall for completeness (see, e.g., ref. [7]) that conventional complex numbers

$$c = n_0 + n_1\, i \in C(c,+,\times), \quad n_0\,, n_1 \in R(n,+,\times), \qquad (2.6.1)$$

where i is the imaginary unit and $n\, i = n \times i$, are represented in a *Gauss plane* [1], which is essentially a realization of the two-dimensional Euclidean space $E_2(x,\delta,R(n,+,\times))$ with basic separation

$$x^2 = x^t\, \delta\, x = x_i\, \delta_{ij}\, x_j = x_1{}^2 + x_2{}^2 \in R(n,+,\times). \qquad (2.6.2)$$

Its group of isometries, the one-dimensional *orthogonal group* O(2), is the invariance of the circle (2.6.2), as well known. For this reason, complex number can be represented via the fundamental representation of O(2) (see below).

The correspondence between complex numbers $c = n_0 + n_1\ i$ and the Gauss plane with points $P = (x_1, x_2)$ is then made one–to–one by the *dilative rotations*

$$z' = (x_1 + x_2\ i)' = \dot{c} \odot z = (n_0 + n_1\ i) \odot (x_1 + x_2\ i), \qquad (2.6.3)$$

with multiplication rule

$$c \odot z = (n_0, n_1) \odot (x_1, x_2) = (n_0 \times x_1 - n_1 x_2, n_0 x_2 + n_1 x_1)\ . \ (2.6.4)$$

which is known to preserve all properties to characterize a field, thus establishing a one–to–one correspondence between complex numbers and points in the Gauss plane. Transformations (2.6.3) also form a two–dimensional group of dilations G(2) in one to one correspondence with C(c,+,×).

Complex numbers also admit the matrix representation

$$c := n_0\ 1_o + n_1\ i_1 = \begin{pmatrix} n_0 & n_1\ i \\ n_1\ i & n_0 \end{pmatrix} \qquad (2.6.5a)$$

$$1_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad (2.6.5b)$$

which are the identity and fundamental representation of O(2), respectively, as expected.

The *norm* is then given by the familiar expression

$$|c| = |n_0 + n_1 \times i| := (\mathrm{Det}\ c)^{\frac{1}{2}} = (\bar{c} \times c)^{\frac{1}{2}} = (n_0^2 + n_1^2)^{\frac{1}{2}}. \ (2.6.6)$$

which readily verifies axioms (2.4.6)

$$|c \odot c'| = |c| \times |c'| \in R, \quad c, c' \in C. \qquad (2.6.7)$$

where now we have different products because referred to different elements.

Finally, the identification of the basis in terms of matrices (2.6.5b)

$$e_1 = 1_0, \quad e_2 = i_1, \qquad (2.6.8)$$

implies the well known result that *complex numbers constitute a two–dimensional, normed, associative and commutative algebra* U(2) [7].

**2.6.B: Realization of isodual complex numbers.** We now consider the isodual complex numbers from Definition 2.3.2 [24]

$$C^d = \{ (c^d,+, \times^d ) \,|\, \times^d = - \times;\ 1^d = - 1;\ c^d = \bar{c} \times 1^d = - \bar{c},\ \bar{c} \in \bar{C} \}, \qquad (2.6.9)$$

where $\bar{c}$ is the usual complex conjugation. Thus, given a complex number $c = n_0 + n_1 i$, its isodual is given by

$$c^d = -\bar{c} = n_0{}^d + n_1{}^d\, \bar{i} = -n_0 - n_1 \bar{i} = -n_0 + n_1 i \in C^d. \qquad (2.6.10)$$

In this case we need the two-dimensional *isodual Euclidean space* $E_2{}^d(x,\delta^d,R^d(n^d,+,\times^d))$ with basic invariant

$$x^{2d} = x^t \delta^d x = x_i \delta^d{}_{ij} x_j = x_1{}^{2d} + x_2{}^{2d} =$$

$$= x_1 \times^d x_1 + x_2 \times^d x_2 = -x_1{}^2 - x_2{}^2 \in R^d(n^d,+,\times^d) \qquad (2.6.11)$$

whose group of isometries is the one-dimensional *isodual orthogonal group* $O^d(2)$ [20], i.e., the image of O(2) under the lifting $I = \mathrm{diag.}\,(1,1) \to I^d = \mathrm{diag.}\,(-1,-1)$ (see Ch. 4 for details). We then expect isodual complex numbers to be characterized by the representation of $\hat{O}(2)$.

We now introduce the *isodual Gauss plane* [21] as the image of the conventional plane under isoduality. The correspondence between isodual complex numbers and the isodual Gauss plane with points $P = (x_1, x_2)$ is then made one-to-one by the *isodual dilative rotations*

$$z' = (x_1 + x_2 i)' = c^d \odot^d z = (-n_0 + n_1 i) \odot^d (x_1 + x_2 i), \qquad (2.6.12)$$

with multiplication rules

$$c^d \odot^d z = (-n_0, n_1) \odot^d (x_1, x_2) =$$

$$= (-n_0 x_1 + n_1 x_2, -n_0 x_2 + n_1 x_1), \qquad (2.6.13)$$

which can be easily shown to preserve all properties to characterize a field. Also isodual transformations (2.6.13) form an isodual group $G^d(2)$ antiisomorphic to G(2). We therefore see that, as expected, *the one-to-one correspondence between complex numbers and the Gauss plane persists under isoduality*.

Isodual complex numbers also admit the matrix representation

$$c^d := n_0{}^d\, 1_0{}^d + n_1{}^d\, i_1{}^d = \begin{pmatrix} -n_0 & n_1 i \\ n_1 i & -n_0 \end{pmatrix} \qquad (2.6.14a)$$

$$1_0{}^d = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i_1{}^d = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \qquad (2.6.14b)$$

which are the isodual unit and isodual representations of $O^d(2)$, respectively.

The *isodual norm* is now given by

$$|c^d|^d := [\det_R(c^d \times T^d)]^{\frac{1}{2}} \times 1_0{}^d = (\bar{c}^d \times c^d)^{\frac{1}{2}} \times 1_0{}^d, \tag{2.6.15}$$

can be written

$$|c^d|^d = (c \times \bar{c})^{\frac{1}{2}} \times 1_0{}^d = (n_0{}^2 + n_1{}^2)^{\frac{1}{2}} \times 1_0{}^d. \tag{2.6.16}$$

and also verifies axiom (2.4.6),

$$|c^d \odot^d c'^d|^d = |c^d|^d \times^d |c'^d|^d \in R^d, \quad c^d, c'^d \in C^d. \tag{2.6.17}$$

Finally, the identification of the *isodual basis* in terms of matrices (2.6.14)

$$e_1{}^d = 1_0{}^d, \quad e_2{}^d = i_1{}^d, \tag{2.6.18}$$

implies that *isodual complex numbers constitute a two-dimensional, isodual, normed, associative and commutative algebra* $U^d(2)$ *which is anti-isomorphic to* U(2) [24]

**2.6.B: Realization of isocomplex numbers.** By following again ref. [24], we consider now the isofield of isocomplex numbers from Definition 2.3.1

$$\hat{C} = \{ (\hat{c}, \hat{+}, \hat{\times}) \mid \hat{\times} = \times T \times, \hat{1} = T^{-1}, \hat{c} = c \times \hat{1}, c \in C(c,+,\times) \}, \tag{2.6.19}$$

with generic element $\hat{c} = \hat{n}_0 + \hat{n}_1$ i. In this case we need the two-dimensional isoeuclidean space of Class I, $\hat{E}_{1,2}(x,\hat{\delta},\hat{R}(\hat{n},+,*))$. Their realization most used in the physical literature is that with diagonalized and positive-definite isotopic element and isounit as discussed in more details in the next chapter

$$T = \mathrm{diag.}(b_1{}^2, b_2{}^2), \quad \hat{1} = \mathrm{diag.}(b_1{}^{-2}, b_2{}^{-2}), \quad b_k > 0, \ k = 1, 2, \tag{2.6.20}$$

with basic isoseparation

$$x^2 = (x^t \hat{\delta}\, x)\hat{1} = (x_i \hat{\delta}_{ij} x_j)\hat{1} = (x_1 b_1{}^2 x_1 + x_2 b_2{}^2 x_2)\hat{1} \in \hat{R}(\hat{n},+,\hat{\times}), \tag{2.6.21}$$

whose group of isometries is the one-dimensional *isoorthogonal group* $\hat{O}(2) \approx O(2)$ (see Ch. 4 for details), i.e., the group O(2) constructed with respect to the multiplicative isounit $\hat{1} = \mathrm{diag.}(b_1{}^{-2}, b_2{}^{-2})$, which provides the invariance of all possible ellipses with semiaxes $a = b_1{}^{-2}$, $b = b_2{}^{-2}$ as the infinitely possible deformation of the circle [20]. We then expect that isocomplex numbers are characterizable via the fundamental isorepresentation of $\hat{O}(2)$.

We now study the *isogauss plane*, first introduced in ref. [21] which is the set of points $P = (\hat{x}_1, x_2)$ on $\hat{E}_{1,2}(x,\hat{\delta},\hat{R}(\hat{n},+,\hat{\times}))$ for the characterization of isocomplex numbers $\hat{c} = (\hat{n}_0, \hat{n}_1)$.

The correspondence between the isocomplex numbers $\hat{C}(\hat{c},+,\hat{\times})$ and the isogauss plane can be made one-to-one by the *isodilative isorotations*

$$z' = (x_1 + x_2\, i)' = \hat{c} \,\hat{\odot}\, z \tag{2.6.22}$$

with isomultiplication rule

$$\hat{c} \,\hat{\odot}\, z = (\hat{n}_0, \hat{n}_1) \,\hat{\odot}\, (x_1, x_2) =$$

$$= \{[(n_0 x_0)\hat{1} - \Delta^{\frac{1}{2}}(n_1 x_2)\hat{1}], [(n_0\, x_2)\hat{1} + (n_1 x_1)\hat{1}]\}, \tag{2.6.23a}$$

$$\Delta = \mathrm{Det}\, T = b_1^2\, b_2^2, \tag{2.6.23b}$$

where the appearance of the $\Delta^{\frac{1}{2}}$ factor will be justified shortly, and confirmed later on for the case of isoquaternions and isooctonions studied in Appendices 4.A and 4.B.

It is easy to see that the isogauss plane preserves all axioms to characterize an isofield. In particular, isotransformations (2.6.22) form a two-dimensional *isodilation isogroup* $\hat{G}(2) \approx G(2)$. As expected, the one-to-one correspondence between complex numbers and points in the Gauss plane is preserved under isotopy.[15]

The implications are however nontrivial, as illustrated by a number of properties, such as the lack of existence of unitary transformations

$$c' = U \odot c \odot \hat{U},\ U \odot U^{\dagger} = U^{\dagger} \odot U = I = \mathrm{diag.}\,(1, 1), \tag{2.6.24}$$

mapping the matrix representation of complex numbers into their isotopic form. The understanding is that a transformation does indeed exist, but it is of the more general isotopic type

$$\hat{c} = \hat{U} \,\hat{\odot}\, c \,\hat{\odot}\, \hat{U}^{\dagger}, \qquad \hat{U} \,\hat{\odot}\, \hat{U}^{\dagger} = \hat{U}^{\dagger} \,\hat{\odot}\, \hat{U} = \hat{1}. \tag{2.6.25}$$

Another way to see the nontriviality of the isotopy is by noting that *the conventional trigonometry is inapplicable to the isogauss plane.* In fact, conventional functions such as cos $\alpha$, sen $\alpha$, etc. which are well defined in the Gauss plane, have no mathematical meaning in our isogauss plane, as discussed in Appendix 2.A.

The reader should be aware that, by no means, realization (2.6.23) is unique, owing to the intriguing "degrees of freedom" of the isotopic formulations studies later on.

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[15] Note that the notion of *point* in the isoeuclidean plane can be introduced despite its nonlocal-integral character thanks to its integro-differential topology (Fig. 1.4.1). In fact, the isogauss plane is everywhere local-differential except at the isounit.

Isocomplex numbers also admit the following two-by-two matrix representation

$$\hat{c} = \hat{n}_0 \times \hat{1}_0 \hat{+} n_1 \times \hat{1}_1 = \begin{pmatrix} n_0 \times b_1^{-2} & i \times n_1 \times b_1^{2} \times \Delta^{-\frac{1}{2}} \\ i \times n_1 \times b_2^{2} \times \Delta^{-\frac{1}{2}} & n_0 \times b_2^{-2} \end{pmatrix} \qquad (2.6.26a)$$

$$\hat{1} = \hat{1}_0 = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix}, \ \hat{1}_1 = i\,\hat{1} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i \times b_1^{2} \\ i \times b_1^{2} & 0 \end{pmatrix}, \qquad (2.6.26b)$$

$$\Delta = \text{Det. } T = b_1^2\, b_2^2, \qquad (2.6.26c)$$

which verify rule (2.6.23) and characterize the isounit and the fundamental isorepresentation of $\hat{O}(2)$ respectively (see Ch. 4, and subsequent confirmation via the fundamental isorepresentation of the isotopic $S\hat{U}(2)$ group for the isoquaternions and isooctonions).

Then, the set $\hat{S}(\hat{c},\hat{+},\hat{\times})$ of matrices (2.6.26a) is closed under addition and isomultiplication, each element possesses the isoinverse

$$\hat{c}^{-\hat{1}} = \hat{c}^{-1} \times \hat{1}, \qquad (2.6.27)$$

where $\hat{c}^{-1}$ is the conventional inverse. Thus, $\hat{S}(\hat{c},+,\hat{\times})$ is an isofield. The local isomorphism $\hat{S}(\hat{c},+,\hat{\times}) \approx \hat{C}(\hat{c},+,\hat{\times})$ is then consequential.

The *isonorm* is defined, from Eq.s (2.4.7) and (2.4.10) by

$$\hat{|}\, \hat{c}\, \hat{|} = [\, \text{Det}_R(\, \hat{c} \times T)\, ]^{\frac{1}{2}} \times \hat{1}_0 = (\, n_0^2 + \Delta\, n_1^2\, )^{\frac{1}{2}} \times \hat{1}_0, \qquad (2.6.28)$$

and readily verifies axioms (2.4.6),

$$\hat{|}\, \hat{c} \,\hat{\circ}\, \hat{c}'\, \hat{|} = \hat{|}\, \hat{c}\, \hat{|} \,\hat{\times}\, \hat{|}\, \hat{c}'\, \hat{|} \in \hat{R}, \quad \hat{c}, \hat{c}' \in \hat{C}. \qquad (2.6.29)$$

Finally, the *isobasis*

$$\hat{e}_1 = \hat{1}_0, \quad \hat{e}_2 = \hat{\imath}, \qquad (2.6.30)$$

show that *isocomplex numbers constitute a two-dimensional, isonormed, isoassociative and isocommutative isoalgebras over the isoreals* $\hat{U}(2) \approx U(2)$, a result first achieved in ref. [24].

**2.6.C: Realization of isodual isocomplex numbers.** We consider now the isodual isocomplex numbers

$$\hat{C}^d = \{ (\hat{c}^d,+,\hat{\times}^d) \mid \hat{c}^d = -\bar{c}\,\hat{1}^d, \quad \hat{\times}^d = \times\, T^d \times,\ T^d = -T,\ \hat{1}^d = T^{d\,-1},\ c \in C(c,+,\times) \}, \tag{2.6.31}$$

with generic element

$$\hat{c}^d = \hat{n}^d + \hat{n}_1{}^d\, i^d = -\hat{n}_0 + \hat{n}_1\, i\,. \tag{2.6.32}$$

In this case we need the two-dimensional *isodual isoeuclidean space* of Class II, $\hat{E}_{II,2}{}^d(x,\hat{\delta}^d,\hat{R}^d(n^d,+,\hat{\times}^d))$ with realization

$$T^d = \text{diag.}\,(-b_1{}^2, -b_2{}^2)\ ,\quad \hat{1}^d = \text{diag.}\,(-b_1{}^{-2}, -b_2{}^{-2}),\quad b_k > 0,\ k = 1, 2, \tag{2.6.33}$$

and basic isodual isoseparation

$$x^{\hat{2}d} = (x^t\,\hat{\delta}^d\,x)\,\hat{1}^d = (x_i\,\hat{\delta}^d{}_{ij}\,x_j)\,\hat{1} =$$

$$= (-x_1\,b_1{}^2\,x_1 - x_2\,b_2{}^2\,b_2)\,\hat{1}^d \in \hat{R}^d(\hat{n}^d,+,\hat{\times}^d), \tag{2.6.35}$$

whose group of isometries is the *isodual isoorthogonal group* $\hat{O}^d(2) \approx O^d(2)$ [20].

The *isodual isogauss plane* (identified for the first time in ref. [21]) is then the set of points $P = (\hat{x}_1, x_2)$ on $\hat{E}_{I,2}{}^d(x,\delta^d,\hat{R}^d(\hat{n}^d,+,\hat{\times}^d))$ for the characterization of isodual isocomplex numbers $\hat{c} = (-\hat{n}_0, \hat{n}_1)$.

The correspondence between the isodual isocomplex numbers $\hat{C}^d(\hat{c}^d,+,\times^d)$ and the isodual isogauss plane can be made one-to-one by the *isodual isodilative isorotations*

$$z' = (x_1 + x_2\ i)' = \hat{c}^d\,\hat{\odot}^d\,z \tag{2.6.36}$$

with multiplication rule

$$\hat{c}\,\hat{\odot}^d\,z = (\hat{n}_0, \hat{n}_1)\,\hat{\odot}^d\,(x_1, x_2) =$$

$$= \{[(-n_0\,x_0)\,\hat{1} + \Delta^{\frac{1}{2}}\,(n_1\,x_2)\,\hat{1}], [(-n_0 \times x_2)\,\hat{1} + (n_1\,x_1)\,\hat{1}]\}, \tag{2.6.37a}$$

$$\Delta = \text{Det}\,T = b_1{}^2\,b_2{}^2, \tag{2.6.37b}$$

It is easy to see that the isodual isogauss plane preserves all axioms to characterize an isodual isofield. Also, isodual isotransformations (2.6.36) form an *isodual isodilation isogroup* $\hat{G}^d(2) \approx G^d(2)$. As expected, *the one-to-one correspondence between complex numbers and Gauss plane is also preserved under isodual isotopy.*

Isodual isocomplex numbers also admit the two-by-two matrix representation

$$\hat{c}^d = \hat{n}_0{}^d \times \hat{1}_0{}^d + n_1{}^d \,\hat{\imath}^d = \begin{pmatrix} -n_0\, b_1{}^{-2} & i\, n_1\, b_1{}^2\, \Delta^{-\frac{1}{2}} \\ i\, n_1\; b_2{}^2\, \Delta^{-\frac{1}{2}} & -\, n_0\, b_2{}^{-2} \end{pmatrix} \qquad (2.6.38a)$$

$$\hat{1}_0{}^d = \begin{pmatrix} -b_1{}^{-2} & 0 \\ 0 & -b_2{}^{-2} \end{pmatrix}, \; \hat{\imath}^d = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & -\, i\, b_1{}^2 \\ -\, i\, b_2{}^2 & 0 \end{pmatrix} \qquad (2.6.38b)$$

which satisfies isomultiplication rule (2.6.37), which characterize the isodual isounit and fundamental representation of $\hat{O}^d(2)$, respectively.

Then, the set $\hat{S}^d\,(\hat{c}^d,+,\hat{*}^d)$ of matrices (2.6.38) is closed under addition and isomultiplication, each element possesses the isodual isoinverse

$$\hat{c}^{-\hat{1}d} = (\,\hat{c}^d\,)^{-1}\, 1^d\,. \qquad (2.6.39)$$

Thus $\hat{S}^d(\hat{c}^d,+\times^d)$ is an isofield. The local isomorphism $\hat{S}^d(\hat{c}^d,+,\hat{\times}^d) \approx \hat{C}^d(\hat{c}^d,+,\times^d)$ is then consequential.

The *isodual isonorm* is defined by

$$\uparrow \hat{c}^d \uparrow^d \;=\; [\,\mathrm{Det}_R(\,\hat{c}^d\, T^d\,)\,]^{\frac{1}{2}} \times \hat{1}_0{}^d \;= (\, n_0{}^2 \;+ \Delta\, n_1{}^2)^{\frac{1}{2}} \times \hat{1}_0{}^d \qquad (2.6.40)$$

and readily verifies axioms (2.4.6),

$$\uparrow \hat{c}^d\, \hat{o}^d\, \hat{c}'^{\,d} \uparrow^d = \,\uparrow \hat{c}^d \uparrow^{d} \hat{*}^{d} \uparrow \hat{c}'^{\,d} \uparrow^d \;\in \hat{R}^d\,, \quad \hat{c}^d, \hat{c}'^d \;\in \hat{C}^d\,. \qquad (2.6.41)$$

Finally, the *isodual isobasis*

$$\hat{e}_1{}^d = \hat{1}_0{}^d\,, \quad \hat{e}_2{}^d = \hat{1}_1{}^d\,, \qquad (2.6.42)$$

shows that *isodual isocomplex numbers constitute a two-dimensional, isodual, isonormed, isoassociative and isocommutative isoalgebras over the isodual isoreals isoreals* $\hat{O}^d(2) \approx \hat{U}^d(2)$ (a result first proved in ref. [24]).

In conclusion, *the "numbers" used in hadronic mechanics are characterized by the lifting of conventional real numbers* n *or complex numbers* c *into the most general known integro-differential expressions* $\hat{n} = n\hat{1}$ *and* $\hat{c} = c\hat{1}$, *respectively, with an integral dependence on all possible local quantities and their derivatives*

$$n \;\Rightarrow\; \hat{n} = \hat{n}(t,\, x,\, \dot{x},\, \ddot{x},\, \psi,\, \psi\dagger,\, \partial\psi,\, \partial\psi\dagger,\, \mu,\, \tau,\, n,\, \ldots)\,, \qquad (2.6.43a)$$

$$c \;\Rightarrow\; \hat{c} = \hat{c}(t,\, x,\, \dot{x},\, \ddot{x},\, \psi,\, \psi\dagger,\, \partial\psi,\, \partial\psi\dagger,\, \mu,\, \tau,\, n,\, \ldots)\,, \qquad (2.6.43b)$$

*as a direct way to represent integro-differential generalizations of*

*Planck's unit, Eq. (1.1.1).*

Moreover, the generalizations are nontrivial inasmuch as they are not unitarily equivalent to the conventional numbers. We finally note that, even under the condition

$$\Delta = b_1{}^2 b_2{}^2 = 1, \tag{2.6.44}$$

realized for

$$b_1 = b_2{}^{-1} = \lambda, \tag{2.6.45}$$

isocomplex numbers preserve their nontrivially generalized form

$$\hat{c} = \hat{n}_0 \times \hat{1}_0 \mathbin{\hat{+}} n_1 \times \hat{i}_1 = \begin{pmatrix} n_0 \times \lambda^{-2} & i \times n_1 \times \lambda^2 \\ i \times n_1 \times \lambda^{-2} & n_0 \times \lambda^2 \end{pmatrix} \tag{2.6.46}$$

because the "hidden quantity" $\lambda \neq$ has an unrestricted functional dependence, $\lambda(x, \dot{x}, \ddot{x}, \ldots)$. As we shall see in Vol.s II and III, a number of intriguing physical applications originate precisely from the above "hidden degree of freedom" $\lambda$.

## 2.7: ISOQUATERNIONS AND THEIR ISODUALS

**2.7.A: Realization of quaternions.** Recall (see, e.g., ref.s [7,8] and quoted literature) that quaternions $q \in Q(q,+,\times)$ admit a realization in the complex Hermitean Euclidean plane $E_2(z,\delta,C)$ with separation

$$E_2(z,\delta,C): \quad z^\dagger z = \bar{z}_i\, \delta_{ij}\, z_j = \bar{z}_1 z_1 + \bar{z}_2 z_2, \quad \delta^\dagger \equiv \delta, \tag{2.7.1}$$

whose basic (unimodular) invariant is SU(2). Thus, quaternions can be characterizable via the fundamental (adjoint) representation of SU(2), i.e., by Pauli's matrices, as reviewed below.

Quaternions can be first realized via pairs of complex numbers, $q = (c_1, c_2)$, $q \in Q$ and $c_1, c_2 \in C$ with multiplication $\odot$ (see below). A *Hermitean dilative rotation* on $E_2(z,\delta,C)$, i.e., one leaving invariant $z^\dagger z$, is given by

$$z'_1 = c_1 \odot z_1 + c_2 \odot z_2, \qquad z'_2 = -\bar{c}_2 \odot z_1 + \bar{c}_1 \odot z_2, \tag{2.7.2}$$

where the dilation is represented by the value $\bar{c}_1 \odot c_1 + \bar{c}_2 \odot c_2 \neq 1$. Again, transformations (2.7.2) form a group G(4), this time associative but noncommutative, which is in one-to-one correspondence with quaternions.

Rule (2.7.2) characterizes the following matrix representation of quaternions $Q(q,+,\times)$ over the field of complex numbers $C(c,+,\times)$

$$q = \begin{pmatrix} c_1 & c_2 \\ -\bar{c}_2 & \bar{c}_1 \end{pmatrix} \tag{2.7.3}$$

which is also one-to-one. By assuming

$$c_1 = n_0 + n_3\, i\,, \quad c_2 = n_1 + n_2\, i, \tag{2.7.4}$$

matrix (2.7.3) admits the representation

$$q = n_o\, l_o + n_1\, i_1 + n_2\, i_2 + n_3\, i_3\,, \tag{2.7.5}$$

where the i's are the celebrated two-dimensional *Pauli's matrices* plus the two-dimensional identity,

$$l_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, i_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; i_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.7.6}$$

with fundamental properties

$$i_n\, i_m = -\epsilon_{nmk}\, i_k\,, \quad n \neq m\,, \qquad n, m = 1, 2, 3\,, \tag{2.7.7}$$

where $\epsilon_{nmk}$ is the conventional totally antisymmetric tensor of rank three. The algebra A of Pauli's matrices is closed under commutators, and characterize the fundamental representation of the $su(2)$ Lie algebra

$$[\, i_n\,, i_m\,] = i_n\, i_m - i_m\, i_n = -2\,\epsilon_{nmk}\, i_k\,, \tag{2.7.8}$$

with Casimir invariants $l_o$ and $i^2 = \sum_{k=1,2,3} i_k{}^2$,

$$[l_o\,, i_k\,] = [\, i^2, i_k\,] = 0\,, \quad k = 1, 2, 3\,, \tag{2.7.9}$$

and eigenvalues on a two-dimensional basis $\psi$ with normalization $\psi\dagger\times\psi = 1$

$$\textstyle\sum_{k=1,2,3} i_k{}^2 \times \psi = \sum_{k=1,2,3} i_k \times i_k \times \psi = -3 \times \psi, \tag{2.7.10}$$

By noting that

$$q\dagger = n_o\, l_o - n_1\, i_1 - n_2\, i_2 - n_3\, i_3\,, \tag{2.7.11}$$

the *norm* of q can be written

$$|q| = (q\dagger q)^{\frac{1}{2}} = (\textstyle\sum_{k=0,1,2,3} n_k^{\,2})^{\frac{1}{2}}, \tag{2.7.12}$$

and also satisfies axioms (2.4.6),

$$|q \odot q'| = |q| \times |q| \in R, \quad q, q', \times \in Q. \tag{2.7.13}$$

The *basis*

$$e_1 = 1_0, \qquad e_{k+1} = i_k, \quad k = 1, 2, 3, \tag{2.7.14}$$

then establishes that *quaternions constitute a normed, associative, noncommutative algebra of dimensions 4 over the reals* U(4) [7,8].

**2.7.B: Realization of the isodual quaternions.** We consider now the isodual quaternions $q^d \in Q^d(q^d,+,\times^d)$ [24] which can be represented via the isodual complex Hermitean Euclidean space

$$E_2{}^d(z,\delta^d,C^d\, c^d,+,\times^d)): \ (\bar{z}_i\, \delta^d{}_{ij}\, z_j)\, 1^d = (-\bar{z}_1 z_1 - \bar{z}_2 z_2)\, 1^d \in R^d, \tag{2.7.15}$$

in which case they can be realized via pairs of isodual complex numbers (Sect. 2.6) $q^d = (c_1{}^d, c_2{}^d)$, $q^d \in Q^d$, $c^d{}_1$, $c^d{}_2 \in C^d$. An *isodual Hermitean dilative rotation* on $E_2{}^d(z,\delta^d,C^d(c^d,+,\times^d))$, i.e., one leaving invariant $z\dagger\delta^d z$, is given by

$$z'_1 = c_1{}^d \odot^d z_1 - \bar{c}_2{}^d \odot^d z_2, \qquad z'_2 = c_2{}^d \odot^d z_1 + \bar{c}_1{}^d \odot^d z_2, \tag{2.7.16}$$

where the dilation is represented by the value $\bar{c}_1{}^d \odot^d c_1{}^d + \bar{c}_2{}^d \odot^d c_2{}^d \neq -1$. Again, transformations (2.7.16) form an associative but noncommutative isodual group $G^d(4)$, which is in one-to-one correspondence with isodual quaternions $Q^d(q^d,+,\times^d)$.

Rule (2.7.16) characterizes the following matrix representation of isodual quaternions over the field of isodual complex numbers $C^d(c^d,+,\times^d)$

$$q^d = \begin{pmatrix} c_1{}^d & -\bar{c}_2{}^d \\ c_2{}^d & \bar{c}^d \end{pmatrix} \tag{2.7.17}$$

By assuming

$$c_1{}^d = -n_0 + n_3\, i, \quad c_2{}^d = -n_1 + n_2\, i, \tag{2.7.18}$$

and by recalling that $-\bar{c}^d = c$, we have the representation

$$q^d = n_o{}^d + n_1{}^d\, i_1{}^d + n_2{}^d\, i_2{}^d + n_3\, i_3{}^d =$$

$$= \; - n_o \, 1_o \; + \; n_1 \; i_1 \; + \; n_2 \, i_2 \; + \; n_3 \, i_3 \, , \tag{2.7.19}$$

where the i's are the Pauli's matrices reviewed above. We learn in this way that the Pauli's matrices change sign under isoduality although their product with isodual numbers is isoselfdual.

By using the results of Sect. 2.4, the *isodual norm* is then defined by

$$|q^d|^d = [\, \mathrm{Det}_C (q^d \times T^d \,)\,] \times 1^d = \; (\, - \textstyle\sum_{k=0,1,2,3} n_k^{\,2}\,)^{\frac{1}{2}} \times 1^d , \tag{2.7.20}$$

with property

$$|q^d \odot^d q'^d|^d = |q^d|^d \;\times^d\; |q'^d|^d \in R^d, \quad q^d, q'^d \in Q^d . \tag{2.7.21}$$

The use of the *isodual basis*

$$e^d{}_1 = 1^d{}_0 , \quad e^d{}_{k+1} = i_k \;, k = 1, 2, 3, \tag{2.7.22}$$

then shows that *isodual quaternions constitute an isodual four-dimensional, normed, associative and noncommutative algebra over the isodual reals* $U^d(4)$, *which is antiisomorphic to* U(4) [24]

**2.7.C: Realization of isoquaternions.** To study the isoquaternions $\hat{q} \in \hat{Q}(\hat{q},+,\hat{\times})$ [24], we need the *two-dimensional, complex Hermitean isoeuclidean space* of Class I, $\hat{E}_{I,2}(z,\hat{\delta},\hat{C})$ on the isofield $\hat{C}(\hat{c},+,\hat{\times})$ with separation (see next chapter for more details)

$$z\dagger \,\hat{\delta}\, z = \bar{z}_i \,\hat{\delta}_{ij}\, z_j = \bar{z}_1 \, b_1^{\,2} z_1 \;\hat{+}\; \bar{z}_2 \, b_2^{\,2} \, z_2 \, , \quad \hat{\delta}\dagger \equiv \hat{\delta} > 0 \, , \tag{2.7.23}$$

basic isotopic element and isounit

$$T = \mathrm{Diag.}\,(\, b_1^{\,2} , b_2^{\,2}\,) , \qquad \hat{1} = \mathrm{Diag.}\,(\, b_1^{\,-2} , b_2^{\,-2}\,) , \tag{2.7.24}$$

whose (unimodular) invariance group is the Lie-isotopic group $S\hat{U}(2)$ (see Ch. I.4 and II.6). Isoquaternions can therefore be characterized via the fundamental isorepresentation of the $s\hat{u}(2)$ algebra.

A *Hermitean isodilative isorotation* on $\hat{E}_{I,2}(z,\hat{\delta},\hat{C}(\hat{c},+,\hat{\times}))$, i.e., one leaving invariant $z\dagger\hat{\delta}z$, is given by

$$z'_1 = \hat{c}_1 \,\hat{\circ}\, z_1 + \hat{c}_2 \,\hat{\circ}\, z_2 \;, \qquad z'_2 = -\hat{c}_2 \,\hat{\circ}\, z_1 + \bar{c}_1 \,\hat{\circ}\, z_2 \, , \tag{2.7.25}$$

where the dilation is represented by the value $\bar{\hat{c}}_1 \,\hat{\circ}\, \hat{c}_1 + \bar{\hat{c}}_2 \,\hat{\circ}\, \hat{c}_2 \neq \hat{1}$.

The map of isoquaternions into two-by-two matrices on $\hat{C}(\hat{c},\hat{+},\hat{\times})$ must now be characterized by the isorepresentations of the Lie-isotopic algebra $S\hat{U}(2)$ first studied in ref.s [21] (see also ref.s [23,27]), which can be expressed in terms of the basic isounit

$$\hat{1} = \hat{1}_o = \begin{pmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{pmatrix} \tag{2.7.26}$$

and the *fundamental isorepresentation of* $\hat{su}(2)$

$$\hat{1}_1 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & i\,b_1^2 \\ i\,b_2^2 & 0 \end{pmatrix}, \hat{1}_2 = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & b_1^2 \\ -b_2^2 & 0 \end{pmatrix}, \hat{1}_3 = \Delta^{-\frac{1}{2}} \begin{pmatrix} i\,b_2^2 & 0 \\ 0 & -i\,b_1^2 \end{pmatrix} \tag{2.7.27}$$

called *regular isopauli matrices*, which were first introduced in ref. [21] and then studied in various articles (see ref.s [25–27], Sect. I.4.7 and Ch. II–6). As expected, the $\hat{1}$–matrices verify the isotopic image of properties (2.7.7), i.e.,

$$\hat{1}_n * \hat{1}_m = -\Delta^{\frac{1}{2}} \epsilon_{nmk} \hat{1}_k\,, \quad n \neq m\,, \; n, m = 1, 2, 3\,, \Delta = b_1^2\, b_2^2, \tag{2.7.28}$$

and are therefore closed under isocommutators (as a necessary condition to have an isotopy), resulting the Lie–isotopic $\hat{su}(2)$ algebra

$$[\hat{1}_n \hat{,} \hat{1}_m] := \hat{1}_n * \hat{1}_m - \hat{1}_m * \hat{1}_n = -2\,\Delta^{\frac{1}{2}} \epsilon_{nmk} \hat{1}_k\,, \tag{2.7.29}$$

with *isocasimir invariants* and generalized eigenvalues equations studied in Ch. II.6. For alternative realizations with $\Delta = 1$ see Sect. I.4.7.

Note the complete abstract identity of the isotopic $\hat{su}(2)$ with the conventional su(2) algebra. Nevertheless, Pauli's matrices and their isotopic covering are not unitarily equivalent.

Note also that the isoinvariance $\hat{O}(2)$ of the isocomplex numbers (Sect. 2.6) is a subgroup of $S\hat{U}(2)$ characterizable by $\hat{1}_1$, thus confirms the matrix isorepresentation of isocomplex numbers.

Isoquaternions can therefore be written in the form (apparently presented in ref. [24] for the first time)

$$\hat{q} = n_0 \hat{1}_0 + n_1 \hat{1}_1 + n_2 \hat{1}_2 + n_3 \hat{1}_3 =$$

$$= \begin{pmatrix} (n_0\, b_1^{-2} + \Delta^{-\frac{1}{2}}\, i\, n_3\, b_2^2) & \Delta^{-\frac{1}{2}}\,(-n_2 + i\; n_1)\, b_1^2 \\ \Delta^{-\frac{1}{2}}\,(n_2 + i\, n_1)\, b_2^2 & (n_0\, b_2^{-2} - \Delta^{-\frac{1}{2}}\, i\, n_3\, b_1^2) \end{pmatrix} \tag{2.7.30}$$

It is straightforward to show that the set $\hat{S}(\hat{q},+,\hat{\times})$ of all possible expression (2.7.30) preserves the axioms of the original set $S(q,+,\times)$. In fact, the set $\hat{S}(\hat{q},\hat{+},\hat{\times})$ is a four–dimensional vector space over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ which is closed under the operation of conventional addition and isomultiplication, thus being an isofield.

The isomorphism $\hat{S}(\hat{q},+,\hat{\times}) \approx \hat{Q}(\hat{q},+,\hat{\times})$ then follows.

The *isonorm* of the isoquaternions is given by

$$\hat{\uparrow}\hat{q}\hat{\uparrow} = [\,\mathrm{Det}_R(\hat{q}\ T)\,]^{\frac{1}{2}}\,\hat{1}_0 = (\,\hat{q}^\dagger \hat{\circ}\,\hat{q}\,)^{\frac{1}{2}}\,\hat{1}_0\,, \tag{2.7.31}$$

and can be written

$$\hat{\uparrow}\hat{q}\hat{\uparrow} = [\,n_0^2 + \Delta(\,n_1^2 + n_2^2 + n_3^2)\,]\,\hat{1}_0\,, \tag{2.7.32}$$

which should be compared with expression (2.7.12) for the ordinary quaternions. Isonorm (2.7.31) also verifies the basic rule

$$\hat{\uparrow}\hat{q}\,\hat{\circ}\,\hat{q}'\hat{\uparrow} = \hat{\uparrow}\hat{q}\hat{\uparrow}\,\hat{\circ}\,\hat{\uparrow}\hat{q}'\hat{\uparrow} \in \hat{R}\,, \quad \hat{q}, \hat{q}', \hat{\circ} \in \hat{Q}\,. \tag{2.7.33}$$

The *isobasis*

$$\hat{e}_1 = \hat{1}_0\,, \quad \hat{e}_{k+1} = \hat{1}_k\,, \quad k = 1, 2, 3\,, \tag{2.7.34}$$

then establishes that *isoquaternions constitute a four-dimensional, isonormed, isoassociative, non-isocommutative isoalgebras over the isoreals* $\hat{U}(4) \approx U(4)$ [24].

**2.7.D: Realization of isodual isoquaternions.** The *isodual isoquaternions* $\hat{q}^d \in \hat{Q}^d(q^d,+,\hat{\circ}^d)$ can be characterized via the two-dimensional isodual complex Hermitean isoeuclidean space of Class II over the isodual isocomplex field,

$$\hat{E}^d_{II,2}(z,\hat{\delta}^d,\hat{C}^d(c^d,+,\hat{\times}^d)):\ \ z^\dagger\,\hat{\delta}^d\,z = \bar{z}_1\,\hat{\times}^d\,z_1 + \bar{z}_2\,\hat{\times}^d\,z_2 = -\,\bar{z}_1\,b_1^{\,2}\,z_1 - \bar{z}_2\,b_2^{\,2}\,z_2\,, \tag{2.7.35}$$

with basic isodual isotopic element and isodual isounit

$$T^d = \mathrm{Diag.}\,(\,-b_1^{\,2}, -b_2^{\,2}\,)\,, \quad \hat{1}^d = \mathrm{Diag.}\,(\,-b_1^{\,-2}, -b_2^{\,-2}\,)\,, \tag{2.7.36}$$

whose (unimodular) invariance is now that of the isodual Lie-isotopic group $\hat{SU}^d$ (see Ch. I.4). An *isodual Hermitean isodilative isorotation* on $\hat{E}^d_{II,2}(z,\hat{\delta}^d,\hat{C}^d(\hat{c}^d,+,\hat{\times}^d))$, i.e., leaving invariant $z^\dagger\hat{\delta}z$, is given by

$$z'_1 = \hat{c}^d_1\,\hat{\circ}^d\,z_1 - \bar{\hat{c}}^d_2\,\hat{\circ}^d\,z_2\,, \qquad z'_2 = \hat{c}^d_2\,\hat{\circ}^d\,z_1 + \bar{\hat{c}}^d_1\,\hat{\circ}^d\,z_2\,, \tag{2.7.37}$$

where the dilation is represented by the value $\bar{\hat{c}}^d{}_1\hat{\circ}^d\hat{c}^d{}_1 + \bar{\hat{c}}^d{}_2\hat{\circ}^d\hat{c}^d{}_2 \neq \hat{1}^d$.

Isoquaternions then admit a realization in terms of the isodual isorepresentation of $\hat{su}^d(2)$ which can be written

$$\hat{q}^d = \hat{n}_0{}^d + \hat{n}_1{}^d\,\hat{1}_1{}^d + \hat{n}_2{}^d\,\hat{1}_2{}^d + \hat{n}_3{}^d\hat{1}_3{}^d =$$

$$= -\hat{n}_0 + \hat{n}_1\,\hat{1}_1 + \hat{n}_2\,\hat{1}_2 + \hat{n}_3\hat{1}_3 =$$

$$= \begin{pmatrix} (-n_0 b_1^{-2} + \Delta^{-\frac{1}{2}}\, i\, n_3 b_2^2) & \Delta^{-\frac{1}{2}}\,(-n_2 + i\; n_1)\, b_1^2 \\ \Delta^{-\frac{1}{2}}(n_2 + i\, n_1)\, b_2^2 & (-n_0 b_2^{-2} - \Delta^{-\frac{1}{2}}\, i\, n_3 b_1^2) \end{pmatrix} \tag{2.7.38}$$

It is again easy to show that the set $\hat{S}^d(\hat{q}^d,+,\times^d)$ of all possible matrices (2.7.38) is an isofield. The isomorphism $\hat{S}^d(\hat{q}^d,+,\times^d) \approx \hat{Q}^d(\hat{q}^d,+,\hat{\times}^d)$ then follows.

The *isodual isonorm* is now given by

$$\uparrow\hat{q}^d\uparrow^d = [\,\mathrm{Det}_R(\hat{q}^d\ T^d)\,]^{\frac{1}{2}}\,\hat{1}_0^d = (\,\hat{q}\dagger^d \times^d \hat{q}^d\,)^{\frac{1}{2}}\,\hat{1}_0^d =$$

$$= [\,-n_0^2 - \Delta(\,n_1^2 + n_2^2 + n_3^2)\,]\,\hat{1}_0^d\ , \tag{2.7.39}$$

and also verified the basic rule

$$\uparrow\hat{q}^d\,\hat{\circ}^d\,\hat{q}'^{\,d}\uparrow^d = \uparrow\hat{q}^d\uparrow^d\,\hat{\circ}^d\,\uparrow\hat{q}'^d\uparrow^d \in \hat{R}^d, \quad \hat{q}^d, \hat{q}'^d, \hat{\circ}^d \in \hat{Q}^d\ . \tag{4.A.40}$$

The *isodual isobasis* is now given by

$$\hat{e}^d{}_1 = \hat{1}^d{}_0, \quad \hat{e}^d{}_{k+1} = \hat{1}^d{}_k, \quad k = 1, 2, 3, \tag{2.7.41}$$

and proves that *isodual isoquaternions constitute a four–dimensional, isodual, isonormed, isoassociative, non–isocommutative isoalgebra over the isodual isoreals* $\hat{U}^d(4) \approx U^d(4)$ [24].

In summary, the isotopy of the conventional quaternions permits the introduction of nontrivial degrees of freedom represented by the diagonal elements of the isotopic element $T = \mathrm{diag.}\,(b_1^2, b_2^2)$, owing to their unrestricted functional dependence $b_k(t, x, \dot{x}, ...) \neq 0$. The "isotopic degrees of freedom" persist even under condition (2.6.44), (2.6.45) under which the regular isopauli matrices (2.7.26)

$$\hat{1}_1 = \begin{pmatrix} 0 & i\lambda^2 \\ i\lambda^{-2} & 0 \end{pmatrix}, \quad \hat{1}_2 = \begin{pmatrix} 0 & \lambda^2 \\ -\lambda^{-2} & 0 \end{pmatrix}, \quad \hat{1}_3 = \begin{pmatrix} i\lambda^{-2} & 0 \\ 0 & -i\lambda^2 \end{pmatrix}, \tag{2.7.42}$$

called standard isopauli matrices [25–27] (see Ch. II.6 for their detailed study).

It should be indicated for completeness that in this section we have studied the isotopies and isodualities only of the fundamental form of quaternions. For additional forms of quaternions for which no isotopies and isodualities have been studied until now, such as the *spit quaternions, antiquaternions and semiquaternions*, we refer the interested reader to monograph [28].

## 2.8: ISOOCTONIONS AND THEIR ISODUALS

For completeness, we also present realizations of octonions, isodual octonions, isooctonions and isodual isooctonions, which follow very closely the construction of isoquaternions and their isoduals from isocomplex numbers and their isoduals.

**2.8.A: Realization of octonions.** Recall (see, e.g., ref. [7,8] and contributions quoted therein), that the octonions $o \in O(o,+,\times)$ can be realized via two quaternions, $o = (q_1, q_2)$, with composition rules

$$o \odot o' = (q_1, q_2) \odot (q'_1, q'_2) = (q_1 \odot q'_1 + q_1 \odot q'_2, -\bar{q}_1 \odot q'_2 + \bar{q}_1 \odot q_2), \qquad (2.8.1)$$

The antiautomorphic conjugation of an octonion is given by

$$\bar{o} = (\bar{q}_1, -q_2). \qquad (2.8.2)$$

It is then possible to introduce the *norm*

$$|o| := (\bar{o} \odot o)^{\frac{1}{2}} = |q| + |q_2|, \qquad (2.8.3)$$

which also verifies the basic axiom

$$|o \odot o'| = |o| \times |o'| \in R, \quad o, o' \in O. \qquad (2.8.4)$$

We finally recall that *the octonions form an eight dimensional normed, nonassociative and noncommutative, alternative algebra* U(8) *over the field of reals* R(n,+,×) [7,8]

**2.8.B: Realization of isodual octonions.** The *isodual octonions* are defined via the isoconjugation

$$o^d = (q_1{}^d, q_2{}^d) \qquad (2.8.5)$$

this time, over the isodual reals $R^d(n^d,+,\times^d)$, and are therefore different than the conventional conjugate octonions $\bar{o}$, Eq. (2.8.2). Their isodual multiplication is

$$o^d \odot^d o'^d = (q_1{}^d, q_2{}^d) \odot^d (q'_1{}^d, q'_2{}^d) =$$

$$(q_1{}^d \odot^d q'_1{}^d - \bar{q}_1{}^d \odot q'_2{}^d,\ q_1{}^d \odot^d q'_2{}^d + \bar{q}_1{}^d \odot^d q_2{}^d), \qquad (2.8.6)$$

the isodual antiautomorphism is then given by

$$\bar{o}^d = (\bar{q}^d, -q_2^d). \tag{2.8.7}$$

It is then possible to introduce the *isodual norm*

$$|o^d|^d := (\bar{o}^d \odot^d o^d)^{\frac{1}{2}} I^d = |q_1^d|^d + |q_2^d|^d, \tag{2.8.8}$$

which also verifies the basic axiom

$$|o^d \odot^d o'^d| = |o^d|^d \times^d |o'^d|^d \in R^d, \quad o^d, o'^d, x^d \in O^d. \tag{2.8.9}$$

Thus *the isodual octonions form an eight dimensional isodual, normed, nonassociative, alternative and noncommutative algebra* $U^d(8)$ *over the isodual real numbers* $R^d(n^d,+,\times^d)$ [24].

**2.8.C: Realization of isooctonions.** Isooctonions [24] $\hat{o} \in \hat{O}(\hat{o},+,\hat{\times})$ can be defined as the pair of isoquaternions, $\hat{o} = (\hat{q}_1, \hat{q}_2)$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ with multiplication rules

$$\hat{o} \,\hat{\odot}\, \hat{o}' = (\hat{q}_1, \hat{q}_2) \,\hat{\odot}\, (\hat{q}'_1, \hat{q}'_2) = (\hat{q}_1 \,\hat{\odot}\, \hat{q}'_1 + \hat{q}_1 \,\hat{\odot}\, \hat{q}'_2, -\bar{\hat{q}}_1 \,\hat{\odot}\, \hat{q}'_2 + \bar{\hat{q}}_1 \,\hat{\odot}\, \hat{q}_2), \tag{2.8.10}$$

It is then easy to see that the lifting $o \rightarrow \hat{o}$ is an isotopy, thus preserving all original axioms of $o$. In fact, we have the antiautomorphic conjugation

$$\bar{\hat{o}} = (\bar{\hat{q}}, -\hat{q}_2), \tag{2.8.11}$$

and the *isonorm*

$$\hat{|}\hat{o}\hat{|} := (\bar{\hat{o}} \odot \hat{o})^{\frac{1}{2}} \times \hat{1} = \hat{|}\hat{q}_1\hat{|} + \hat{|}\hat{q}_2\hat{|} \tag{2.8.12}$$

with property

$$\hat{|}\hat{o} \,\hat{\odot}\, \hat{o}'\hat{|} = \hat{|}\hat{o}\hat{|} \,\hat{\times}\, \hat{|}\hat{o}'\hat{|} \in \hat{R}, \quad \hat{o}, \hat{o}' \in \hat{O}. \tag{2.8.13}$$

It is then easy to see that *isooctonions form an eight dimensional isonormed, non-isoassociative, non-isocommutative, isoalternative isoalgebra* $\hat{U}(8) \approx U(8)$ *over the isoreals* $\hat{R}(\hat{n},+,\hat{\times})$ [24].

**2.8.D: Realization of isodual isooctonions.** The notion of isoduality applies also to the isooctonions yielding the isodual isooctonions $\hat{o}^d = (\hat{q}_1^d, \hat{q}_2^d)$ with composition rule

$$\hat{o}^d \hat{\circ}^d \hat{o}'^d = (\hat{q}_1{}^d, \hat{q}_2{}^d) \hat{\circ}^d (\hat{q}'_1{}^d, \hat{q}'_2{}^d) =$$

$$(\hat{q}_1{}^d \hat{\circ}^d \hat{q}'_1{}^d - \bar{q}_1{}^d \hat{\circ}^d \hat{q}'_2{}^d,\ q_1{}^d \hat{\circ}^d q'_2{}^d + \bar{q}_1{}^d \hat{\circ}^d q_2{}^d), \qquad (2.8.14)$$

Then we have the isodual isoantiautomorphism

$$\bar{o}^d = (\bar{q}^d, -\hat{q}_2{}^d). \qquad (2.8.15)$$

the *isodual isonorm*

$$\uparrow \hat{o}^d \uparrow^d := (\bar{o}^d \hat{\circ}^d \hat{o}^d)^{\frac{1}{2}} \times 1^d = \uparrow \hat{q}_1{}^d \uparrow^d + \uparrow \hat{q}_2{}^d \uparrow^d \qquad (2.8.16)$$

which also verifies the basic axiom

$$\uparrow \hat{o}^d \hat{\circ}^d \hat{o}'^d \uparrow = \uparrow \hat{o}^d \uparrow^d \hat{\times}^d \uparrow \hat{o}'^d \uparrow^d \in \hat{R}^d, \quad \hat{o}^d, \hat{o}'^d \in \hat{O}^d. \qquad (2.8.17)$$

It is then possible to prove that *isodual isooctonions form an eight dimensional isodual, isonormed, non–isoassociative, non–isocommutative, but isoalternative isoalgebra* $\hat{O}^d(8) \approx U^d(8)$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,\times^d)$ [24].

We close this appendix by suggesting caution in the use of octonions and their isotopies as fields because of the loss of associativity and, thus, the loss of enveloping associative algebras of Lie algebras, in favor of alternative algebras. In turn, such a loss has fundamental physical implications we shall see in Vol. II, such as the loss of the equivalence between Heisenberg's and Schrödinger's representations.

In this section we have studied the isotopies and isodualities of the conventional notion of octonions. For additional forms of octonions (e.g., the *sedenions*) and the construction of their representations, we suggest the consultation of ref. [28] and literature quoted therein.

## 2.9: ISOTOPIC UNIFICATION OF CONVENTIONAL NUMBERS

One additional property of isonumbers is important for the subsequent analysis, and it is given by their capability to unify different conventional numbers into one single, abstract notion of isonumber.

This property, called "isotopic unification" (first identified in ref. [23]) has the following three important applications.

**Number theory:** According to contemporary formulations (see ref.s [7–12]), real numbers, complex numbers and quaternions are considered to be different mathematical entities, possessing different properties and structures. This conception is surpassed by the isonumber theory because, as shown below in this section, one single entity, the abstract notion of isoreals, can unify all above indicated conventional numbers evidently because of the degree of freedom offered by the isounit, with evident mathematical and physical advances (for both cases of characteristic zero or p).

**Lie's theory:** In the contemporary formulation of Lie's theory, nonisomorphic simple Lie groups of Cartan's classification of the same dimension, such as O(3) and O(2.1), or O(4), O(3.1) and O(2.2), etc. are generally considered to be different entities possessing different structures and properties. As shown for the first time in ref. [18], this approach too is surpassed by isotopic theories which offer the possibility of unifying all simple Lie groups of the same dimensions into one single, abstract Lie-isotopic group. An evident pre-requisite for such unification is precisely the unification of all fields studied in this section.

**Quantum mechanics on quaternionic fields:** Even though the most dominant use of field in contemporary quantum mechanics is restricted to real and complex fields,, the generalization of quantum mechanics over a quaternionic field has been recently studied by various authors (see ref.s [7–9] and literature quoted therein). In these volumes we shall show that this approach too is superseded by isotopic techniques because quantum mechanics on a quaternionic field is a particular case of hadronic mechanics on an isoreal field.

The existence of an isotopic unification of all numbers had been conjectured by the author in various publications throughout the years, but it was proved only recently by Kadeisvili, Kamiya and Santilli in ref. [21]. Their main result is the following

**Theorem 2.7.1 :** *Let* R, C, Q *be the fields of real numbers, complex numbers and quaternions, respectively,* $R^d$, $C^d$, $Q^d$ *the isodual fields,* $\hat{R}$, $\hat{C}$, $\hat{Q}$ *the isofields, and* $\hat{\mathfrak{R}}^d$, $\hat{C}^d$, $\hat{Q}^d$ *the isodual isofields as defined in preceding sections. Then all these fields can be constructed with the same methods for the construction of* $\hat{R}$ *from* R, *under the relaxation of the condition of positive-definiteness of the isounit, thus achieving a unification of all fields, isofields and their isoduals into the single, abstract isofield of Class III, hereon denoted* $\mathfrak{R}$.

**Proof:** The field of real numbers R is a trivial particular case of $\hat{\mathfrak{R}}$ when the isotopy is the identity, $\mathfrak{R}_{1=\hat{1}} \equiv R$. The fact that the field of complex numbers C is a subcase of $\hat{\mathfrak{R}}$ can be proved as follows. Introduce the binary (Cayley–Dickinson) realization [28] of the elements of $\hat{\mathfrak{R}}$, $\hat{a} = (a_1, a_2)$, where $(a_1, 0)$ and $(0, a_2)$ represent the real (Re) and imaginary (Im) parts, respectively, with the following isomultiplication

$$(a_1, a_2)\hat{\times}(b_1, b_2) := (a_1 \times b_1 - b_2 \times a_2, a_1 \times b_2 + b_2 \times a_1), \qquad (2.9.1)$$

where $\times$ represents the conventional multiplication, and introduce the additional multiplication for elements of Im C

$$(0, a)\hat{\times}_2(0, b) := (0, a)\hat{\times}(0, -1)\hat{\times}(0, b). \qquad (2.7.2)$$

Then, $\mathfrak{K}$ can be decomposed into the tensorial product of the following two parts

$$\mathfrak{K}_1 = \{(a, 0) \mid a \in R, \hat{1} = (1, 0)\}, \qquad (2.9.3)$$

$$\mathfrak{K}_2 = \{(0, a) \mid a \in R, \hat{1} = (0, 1)\}. \qquad (2.9.4)$$

The local isomorphism $\mathfrak{K}_1 \approx$ Re C is trivial. The fact that $\mathfrak{K}_2 \approx$ Im C follows from the expressions $\hat{1} = (0, 1)$, $T = \hat{1}^{-1} = (0, -1)$. Thus, the multiplication in $\mathfrak{K}_2$ is characterized by

$$\hat{a}\hat{\times}\hat{b} := a \times (0, 1)\hat{\times}_2 b \times (0, 1) = a \times (0, 1)\hat{\times}(0, -1)\hat{\times} b \times (0, 1). \qquad (2.9.5)$$

Moreover,

$$(0, 1)\hat{\times}\hat{b} = \hat{b}\hat{\times}(0, 1) \equiv \hat{b}, \qquad (2.9.6a)$$

$$(0, a)\hat{\times}(0, a^{-1}) = (0, 1), \qquad \exists \quad a \neq 0 \qquad (2.9.6b)$$

and this proves that $\mathfrak{K}_2 \approx$ Im C. Thus, in the above binary realization and multiplications (2.9.6a) and (2.7.6b), $\mathfrak{K}$ coincides with C.

The proof that the field of quaternions Q is a subcase of $\mathfrak{K}$ can be done via the quaternary realization $\mathfrak{K} \approx C \hat{\times} C$ with isomultiplication

$$(a_1, a_2) * (b_1, b_2) := (a_1 b_1 - \bar{b}_2 a_2, a_1 \bar{b}_1 + b_2 a_1), \qquad (2.9.7)$$

for all $a_1, a_2, b_1, b_2 \in C$ and $\bar{a}$ denoting conventional complex conjugation in C.

Then $\mathfrak{K}$ can be decomposed into the following parts

$$\mathfrak{K}_1 = \{(a, 0) \mid a \in C\}, \quad \mathfrak{K}_2 = \{(0, b) \mid b \in C\}. \qquad (2.9.8)$$

The product for $\mathfrak{K}_2$ can be defined as

$$(0, a) *_2 (0, b) := (0, a) * (0, -1) * (0, b) = (0, b\,a). \qquad (2.9.9)$$

By making use of these products we readily obtain that $\hat{\mathfrak{R}}_1 \approx C$. To identify the role of $\hat{\mathfrak{R}}_2$ we note that

$$\hat{a} * \hat{b} = a \times (0, 1) * (0, -1) * (0, 1)\ b = (0, b\, a) = a\, b\, (0, 1). \qquad (2.9.10)$$

This implies that, in the above quaternary realization of the elements with multiplications (2.9.9), $\hat{\mathfrak{R}}$ then coincides with Q.

The inclusion in $\hat{\mathfrak{R}}$ of all isotopes $\hat{R}$, $\hat{C}$ and $\hat{Q}$ readily follows from the lifting of all trivial unit 1 into isotopic form $\hat{1}$ with corresponding lifting of the related operations. The inclusion of isodual fields and isodual isofields follows from the the assumption of Class III which includes positive-definite, as well as negative-definite isounits **q.e.d.**

The following property is also implicit in the above proof.

**Corollary 2.7.1.A** [23]: *If the isofield* $\hat{R}$ *is such that* $\hat{R} = \{ (0, x) \mid x \in R, \hat{1} = (0, 1) \}$, *then* $\hat{R} \approx$ Im C *with respect to product (2.7.9) and (2.9.10).*

For completeness we point out that the octonions O are locally isomorphic to the realization $\hat{\mathfrak{R}} = Q * Q$ essentially along the lines for $\hat{\mathfrak{R}} = C * C \approx Q$. Consider again the binary realization of the elements, $\hat{a} = (a_1, a_2)$, although now $a_1$ and $a_2$ represent quaternions, and introduce the isomultiplication in $\hat{\mathfrak{R}}$

$$(a_1, a_2) * (b_1, b_2) := (a_1\, b_1 - \bar{b}_2\, a_2,\ a_2\, \bar{b}_1 + b_2\, a_1), \qquad (2.9.11)$$

where $a_k, b_k \in Q$, $k = 1, 2$, with the additional multiplication for the elements (0, a)

$$(0, a) *_2 (0, b) := [(0, a) * (0, -1)] * (0, b). \qquad (2.9.12)$$

Then, as it was the case for quaternions, $\hat{\mathfrak{R}}$ can be decomposed into the tensorial product of the following two parts

$$\hat{\mathfrak{R}}_1 = \{ (a, 0) \mid a \in Q, \hat{1} = (1, 0) \},\ \ \hat{\mathfrak{R}}_2 = \{ (0, a) \mid a \in Q, \hat{1} = (0, 1) \}. \ (2.9.13)$$

The local isomorphism $\hat{\mathfrak{R}}_1 \approx Q$ is trivial. To identify the role of $\hat{\mathfrak{R}}_2$ note that

$$\hat{a} \hat{*}_2 \hat{b} := a\ (0, 1) *_2 b\ (0, 1) = [a\ (0, 1) * (0, -1)] * b\, (0, 1). \qquad (2.9.14)$$

Moreover, also as in the case of quaternions,

$$(0, 1) *_2 \hat{b} = \hat{b} \hat{*}_2 (0, 1) \equiv \hat{b},\ (0, a) \hat{*}_2 (0, a^{-1}) = (0, 1),\ \exists\ a \neq 0 \quad (2.9.15),$$

and

$$[\,(0, a)*_2(0, b)\,] *_2 (0, c) \;=\; (0, a) *_2 [\,(0, b) *_2 (0,c) \;=\; (\,0, c\, b\, a)\,. \qquad (2.9.16)$$

Thus, $\mathfrak{A}$ in the above considered realization with isomultiplication (2.9.15) and (2.7.16) is locally isomorphic to the octonions.

## APPENDIX 2.A: "HIDDEN NUMBERS" OF DIMENSION 3, 5, 6, 7

Historically, the conventional numbers were studied via the solution of the following problem (see, e.g., ref. [8])

$$(\,a_1{}^2 + a_2{}^2 + .... + a_n{}^2\,) \times (\,b_1{}^2 + b_2{}^2 + ....+ b_n{}^2\,) \;=\; A_1{}^2 + A_2{}^2 + \,....\, + A_n{}^2, \quad (2.A.1a)$$

$$A_k \;=\; \textstyle\sum_{r,s} c_{krs}\, a_r\, b_s\,. \qquad (2.A.1b)$$

where the a's, b's and c's are elements of a conventional field $F(a,+,\times)$ with familiar operations $+$ and $\times$. As well known, the only possible solutions of problem (2.A.1) studied by Gauss [1], Abel [2], Hamilton [3], Cayley [4], Galois [5], Albert [12], Jacobson [13] and others are of dimension 1, 2, 4, 8 (Theorem 2.1.1).

The isotopies and pseudoisotopies of the theory of numbers creates the problem of the possible existence of *"hidden numbers"*, that is, new solutions of dimension different than 1, 2, 4, 8 which are hidden in the operations $\times$ and/or $+$ . This problem, studied for the first time in ref. [24], essentially asks whether the classification of Theorem 2.1.1 persists under isotopies, pseudoisotopies and their isodualities, or it is incomplete.

It is easy to see that the reformulation of problem (2.A.1) under the isotopies of the multiplication $\times \to * = \times T \times$, $1 \to \hat{1} = T^{-1}$, *does not* lead to new solutions. In fact, Problem (2.A.1) under lifting $\times \to *$ is given by

$$(\,a_1{}^{\hat{2}} + a_2{}^{\hat{2}} + .... + a_n{}^{\hat{2}}\,) * (\,b_1{}^{\hat{2}} + b_2{}^{\hat{2}} + \,....\, + b_n{}^{\hat{2}}\,) \;=\; A_1{}^{\hat{2}} + A_2{}^{\hat{2}} + \,....\, + A_n{}^{\hat{2}}, \quad (2.A.2a)$$

$$A_k \;=\; \textstyle\sum_{r,s} c_{krs} * a_r * b_s\,, \qquad (2.A.2b)$$

where the a's, b's and c's now belong to an isofield of the type $\hat{F}(a,+,*)$, in which case $\hat{1}$ is an element of the original field F (Proposition 2.3.1). Problem (2.A.2) can then be written in conventional operations

$$(\,a_1{}^2 + a_2{}^2 + .... + a_n{}^2\,)\;(\,b_1{}^2 + b_2{}^2 + \,....\, + b_n{}^2\,) \;= T^{-2}\,(\,A_1{}^2 + A_2{}^2 + \,....+ A_n{}^2), \quad (2.A.3a)$$

$$A_k \;=\; T^2\, c_{krs}\, a_r\, b_s\,, \qquad n \leq 8 \qquad (2.A.3b)$$

The substitution of of the latter expression into the former, then recovers Problem (2.A.1 identically for liftings of Class I, II, and III. The reformulation in the isofield $\hat{F}(\hat{a},+,*)$ is also equivalent to the original one. We can therefore summarize the studies of this section with the following generalization of Theorem 2.1.1:

> **Theorem 2.A.1** [24]: *All possible isonormed isoalgebras with multiplicative isounit over the isoreals are the isoalgebras of dimension 1 (isoreals), 2 (isocomplex), 4 (isoquaternions) and 8 (isooctonions), and the classification persists under isoduality.*

Nevertheless, there exists a third formulation of pseudoisotopic type (Proposition 2.3.3 and Definition 2.3.3) characterized by the further lifting of the addition

$$+ \;\rightarrow\; \hat{+} \;=\; +\;\hat{K}, \quad 0 \;\rightarrow\; \hat{0} \;=\; -\hat{K}\,,\; \hat{K} \;=\; K \times \hat{1} \qquad (2.A.4)$$

under which problem (2.A.2) can be rewritten over the pseudoisofield $\hat{F}(\hat{a},\hat{+},*)$

$$(\,\hat{a}_1{}^{\hat{2}} \,\hat{+}\, \hat{a}_2{}^{\hat{2}} \,\hat{+} .... \hat{+}\; \hat{a}_n{}^{\hat{2}}\,) * (\,\hat{b}_1{}^{\hat{2}} \,\hat{+}\, \hat{b}_2{}^{\hat{2}} \,\hat{+}\, .... \,\hat{+}\, \hat{b}_n{}^{\hat{2}}\,) \;=\; \hat{A}_1{}^{\hat{2}} \,\hat{+}\, \hat{A}_2{}^{\hat{2}} \,\hat{+}\, ....\hat{+}\;\; \hat{A}_n{}^{\hat{2}}\,, \quad (2.A.5a)$$

$$\hat{A}_k \;=\; \textstyle\sum_{r,s} \hat{c}_{krs} * \hat{a}_r * \hat{b}_s = (\,\sum_{r,s} c_{krs}\, a_r\, b_s\,)\,\hat{1} \;=\; A_k \times \hat{1}\;, \qquad (2.A.5b)$$

and can be rewritten in conventional operations

$$[\,(\,a_1{}^2 + a_2{}^2 + .... + a_n{}^2\,)\hat{1} + (n-1)\,K\,\hat{1}\,]\; T\;\; [\,(\,b_1{}^2 + b_2{}^2 + ... + b_n{}^2\,)\hat{1} \;+ (n-1)\;\, K\,\hat{1}\,] \;=$$

$$=\;\; (\,A_1{}^2 + A_2{}^2 + .... + A_n{}^2\;)\,\hat{1} \,+\, (\,n-1\,)\; K\,\hat{1},\;\; \hat{A}_k \;=\; A_k\,\hat{1}, \qquad (2.A.6)$$

where we have the cancellation of the isotopic element as in the preceding cases, but the preservation of the additive "degree of freedom" K.

The conjecture of the "hidden numbers" was therefore formulated in ref. [24], specifically, *under the pseudoisofield* $\hat{F}(\hat{a},\hat{+},\hat{\times})$, *that is, under the loss of a sufficient number of properties of the original field, such as the loss of the distributive laws* (Proposition 2.3.3).

We here limit ourselves to the following example of hidden number of dimension 3

$$(\,\hat{1}^{\hat{2}} \,\hat{+}\; \hat{2}^{\hat{2}} \;\hat{+}\; \hat{3}^{2}\,) * (\,\hat{5}^{\hat{2}} + \hat{6}^{2} + \hat{7}^{2}\,) \;=\; \hat{12}^{\hat{2}} \;\hat{+}\;\; \hat{24}^{\hat{2}} \;\hat{+}\;\; \hat{30}^{2}, \qquad (2.A.7)$$

Note that the combinations for the elements in the r. h. s. do exist in terms of elements in the l. h. s.

$$12 = 2 \times 6, \qquad 24 = 2 \times 5 + 2 \times 7, \qquad 30 = 3 \times 3 + 3 \times 7. \tag{2.A.8}$$

Problem (2.A.7) can then be written

$$[ ( 1^2 + 2^2 + 3^2 ) \hat{1} + 2 K \hat{1} ] T [ ( 5^2 + 6^2 + 7^2 ) \hat{1} + 2 K \hat{1} ] =$$

$$= ( 12^2 + 24^2 + 30^2 ) \hat{1} + 2K \hat{1}, \tag{2.A.9}$$

which reduces to the following equation in K

$$4 K^2 + 246 K - 80 = 0, \tag{2.A.10}$$

with solution

$$K = 0.325.......... \tag{2.A.11}$$

However, the above solution is not an integer. This implies the loss of closure under isoaddition (see the comments after Proposition 2.3.3). As a result, starting with an original set of integers, one must complete them under isotopies into the field of all real numbers. The issue left open in ref. [24] is therefore the problem whether the above solutions do indeed constitute a pseudoisofield.

To understand the example one should recall that the solution considered *does not* exist for ordinary numbers (because the dimension $n = 3$ is prohibited by Theorem 2.1.1), i.e.,

$$( 1^2 + 2^2 + 3^2 ) ( 5^2 + 6^2 + 7^2 ) \neq 12^2 + 24^2 + 30^2. \tag{2.A.12}$$

The reader can compute solutions of dimension 5, 6, 7.

Note that Problems (2.A.2) and (2.A.5) are restricted to dimensions $n \leqq 8$. This is due to the fact that algebras of dimensions higher than 8 are no longer alternative [8], and such a property is expected to persist under isotopies and pseudoisotopies.

Genonumbers will be studied in Ch. 7. It is possible to show that the results of this section essentially persist in the restriction of the multiplication to be one–sided, and the differentiation of the left and right multiplication.

For further generalizations of conventional numbers via termary operations and other needs, we suggest ref. [28] and literature quoted therein.

Among endless novel problems identified by the isofields which are still open at this writing (Spring 1993), we suggest the study of:

> The novel notion of "number with a singular unit", i.e., the isofields of Class IV which are at the foundations of the isotopic studies of gravitational collapse and are vastly unknown at this writing;

> The study of isofields of isocharacteristic $p \neq 0$, to see whether new fields, and therefore new Lie algebras, are permitted by the isotopies;

> The study of the integro-differential topology characterized by isofields with local-differential structure integral isounits; and others.

## APPENDIX 2.B: INAPPLICABILITY OF TRIGONOMETRY

Trigonometry is a basic tool of quantum mechanics, e.g., because trigonometric functions are fundamental for the characterization of spherical harmonics and, thus, for the study of angular momentum in vacuum (e.g., that of electrons in atomic orbits).

It is important to see that conventional trigonometry is inapplicable in hadronic mechanics, so as to prevent a host demonstrable, yet generally undetected inconsistencies and misjudgments.

To state it differently, because of protracted use, noninitiated researchers often approach the problem of the *interior angular momentum* (e.g., the angular momentum of an electron when in the core of a collapsing star) via the use of conventional trigonometry, related spherical harmonics, and corresponding conventional local-differential formulations (say, in Euclidean space). In so doing, however, they completely ignore the effect to be described caused by the hyperdense medium in the orbital motion, thus *de facto* ignoring the presence of the interior of the star, without any real departure from the original motion of the atomic electron in empty space.

Let us consider the conventional Gauss plane [1] with x- and y-axis (the conventional two-dimensional Cartesian plane). Its trigonometric quantities can be defined via the distance D of a point $P_1(x_1, y_1)$ from the origin

$$D = (x_1^2 + y_1^2)^{\frac{1}{2}} , \tag{2.B.1}$$

the related Pythagorean theorem

$$x_1^2 + y_2^2 = D^2 , \tag{2.B.2}$$

and the cosine of the angle $\alpha$ between two vectors leading from the origin to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$

$$\text{Cos}\ \alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1^2 + y_1^2)^{\frac{1}{2}}(x_2^2 + y_2^2)^{\frac{1}{2}}}. \tag{2.B.3}$$

The above elementary and familiar notions are inapplicable under isotopies.To begin, we have the loss of straight lines in favor of the most general known curvature, that dependent also in velocities and acceleration. Second. the notion of conventional distance is inapplicable, e.g., because the conventional product "x y" now has no mathematical or physical meaning under isotopies. Third, the conventional Pythagorean theorem has no mathematical or geometric sense under isotopies. Thus, the very notion of "angle" between two intersecting "straight lines" in the Gauss plane cannot be preserved for curved lines in our isogauss plane.

The reconstruction of trigonometry under isotopy shall be studied in Ch. 5 after the study of the isogeometries.

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31 - Cryptog
32. Hyperstructures

# 4: ISOSPACES

## 3.1: STATEMENT OF THE PROBLEM

The fundamental spaces of contemporary physics are the 3-dimensional *Euclidean space* E, the (3+1)-dimensional *Minkowski space* M, the (3+1)-dimensional *Riemannian spaces* $\mathfrak{R}$, and others well known spaces.

All these spaces are dependent on the field in which they are defined, the field of real numbers $R = R(n,+,\times)$. The Euclidean space can then be written

$$E = E(r,\delta,R) : r = (x^1, x^2, x^3), \quad \delta = \text{diag.}\,(1, 1, 1)\ , \tag{3.1.1a}$$

$$r^2 = x^i\,\delta_{ij}\,x^j = x^1 x^1 + x^2x^2 + x^3 x^3 \in R = R(n,+,\times)\,, \tag{3.1.1.b}$$

where $i, j = 1, 2, 3$ and $\delta$ is the *Euclidean metric*; the Minkowski space can be written

$$M = M(x,\eta,R), \quad x = (\mathbf{r}, x^4)\,, \quad x^4 = c_0 t, \quad \eta = \text{Diag.}\,(1, 1, 1, -1)\,, \tag{3.1.2a}$$

$$x^2 = x^\mu\,\eta_{\mu\nu}\,x^\nu = x^1 x^1 + x^2x^2 + x^3 x^3 - x^4 x^4 \in R(n,+,\times)\,, \tag{3.1.2b}$$

where $\mu, \nu = 1, 2, 3, 4$, $c_0$ is the *speed of light in vacuum*, ~~and~~ $\eta$ is the *Minkowski metric*; and the Riemannian spaces can be written

$$\mathfrak{R} = \mathfrak{R}(x,g,R), \quad x = (\mathbf{r}, x^4), \quad g = g(x) = g^t, \quad \text{Det.}\,g \neq 0\,, \tag{3.1.3a}$$

$$x^2 = x^\mu\,g_{\mu\nu}(x)\,x^\nu \in R(n,+,\times)\,, \tag{3.1.3b}$$

where $g(x)$ is the *Riemannian metric.*

By inspecting these structures, and as it already emerged from the study of isonumbers of the preceding chapter, it is evident that the isotopic generalization of numbers and related fields implies a corresponding, necessary generalization of all conventional spaces of current use in mathematics and physics.

**EXTERIOR DYNAMICAL PROBLEMS IN VACUUM: EUCLIDEAN, MINKOWSKIAN AND RIEMANNIAN SPACES**

**INTERIOR DYNAMICAL PROBLEMS WITHIN PHYSICAL MEDIA: ISOEUCLIDEAN, ISOMINKOWSKIAN AND ISORIEMANNIAN SPACES**

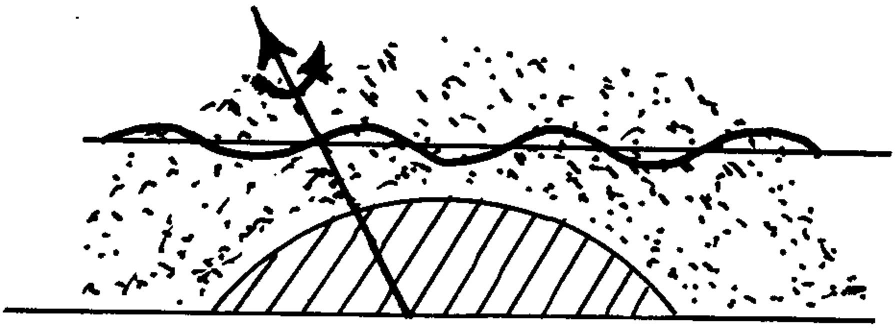

FIGURE 3.1.1. As it is well known, the Euclidean (3.1.1), Minkowskian (3.1.2) and Riemannian spaces (3.1.3) are the foundations of the *Newtonian, relativistic and gravitational, exterior dynamical problems*, respectively. As such, they provide corresponding *geometrizations of the homogeneity and isotropy of empty space (vacuum).* In this chapter we shall study the isotopies of the above spaces under the names of *isoeuclidean, isominkowskian and isoriemannian spaces* which were specifically conceived [1] for the *description, this time, of the respective interior dynamical problems.* As such, they provide corresponding *geometrizations of the inhomogenuity and anisotropy of physical media.* In this figure we illustrate a central objective of isospaces, a quantitative representation of the *deviations* from motion of an electromagnetic wave in empty space caused by motion within a physical medium such as our atmosphere (which is manifestly inhomogeneous, because of the local variation of the density, and anisotropic, because of the intrinsic angular momentum of Earth). A known deviation is the replacement of the constant speed of light in vacuum $c_0$ with a locally varying speed $c = c_0/n$ where n is the local index of refraction. An objective of isospaces is to provide quantitative predictions suitable for experimental tests of additional deviations from motion in empty space expected from the inhomogenuity and anisotropy of the medium itself. Primary emphasis is put in achieving, first, a *purely classical description* of the inhomogenuity and anisotropy ~~here considered~~ studied in this chapter and in Ch. I.5, with

operator descriptions to be considered only thereafter in Vol. II.[16] Another important distinction is that between the isospaces themselves, studied in this chapter, and the isogeometries defined on them, which are studied in Ch. I.5.

At a deeper study, it emerges that, for evident mathematical consistency, *the isotopies of ordinary numbers imply compatible liftings of all mathematical structures used in quantum mechanics.* In fact, the isotopic generalization of conventional spaces implies the necessary, corresponding generalization of the transformations defined on them. In turn, the lifting of the transformations implies that of algebras, groups, geometries, etc., according to the sequence:

$$\begin{aligned}&\text{isonumbers} \rightarrow \text{isofields} \rightarrow \text{isospaces} \rightarrow \text{isotransformations}\\ &\rightarrow \text{isoalgebras} \rightarrow \text{isogroups} \rightarrow \text{isosymmetries} \rightarrow\\ &\rightarrow \text{isorepresentations} \rightarrow \text{isogeometries}, \text{etc.}\end{aligned} \tag{3.1.4}$$

In this chapter we shall study the isotopies of the conventional spaces proposed for the first time in ref. [1] of 1983, under the name of *isotopic spaces* or *isospaces* for short, as the foundations of the isotopic generalization of the Lorentz group O(3.1) and of Einstein's special relativity for interior dynamical problems. The isospaces were then applied in ref.s [2,3] for the construction of the isotopies of the rotational symmetry O(3), as well as for the formulation of a general theorem on symmetries under isotopies. Isospaces were then used in monographs [4,5] for comprehensive applications in classical mechanics.

The *isodual spaces* and *isodual isospaces* were identified for the first time in ref.s [2,3], and then applied in classical mechanics in monographs [4,5]. The first operator applications of isodual isospaces were done in ref. [6] while the most recent advances can be found in ref. [7]. A mathematical presentations is available in memoirs [8,9].

A first experimental verifications of isospaces can be found in ref. [10] which computes a modification of the Minkowski metric in in the interior of pions and kaons via conventional gauge theories in the Higgs sector. Additional independent experimental verifications can be found in ref.s [11,2] on the behaviour of the meanlives of unstable hadrons with speed. Numerous additional applications and experimental verifications will be studied in Vol. II.

[16] This is done to void the predictable attitude of attempting the interpretation of interior conditions via conventional means, such as inelastic scatterings of photons on atoms which, as such, reduce the interior problem to conventional exterior conditions. This attitude is precluded in these volumes because it eliminates the central geometric characteristics to be described, the inhomogeneity and anisotropy of the medium (recall the "No reduction theorems" of Sect. I.2). Needless to say, we shall indeed consider second quantization and related .photons, but only *after* achieving a classical and direct representation of the inhomogeneity and anisotropy of the medium in which the dynamical evolution holds.

The "direct universality" of isospaces was first proved by Aringazin in ref. [13]. Additional studies on isospace were conducted by Lopez [14] in gravitation. An independent mathematical review of isospaces can be found in monograph [15].

In this chapter we shall study isospaces at the purely classical level in Kadeisvili's classification (Sect.s 1.5 and 2.3). The study of pseudoisospaces will be left to the interested readers. The isogeometries built on isospaces will be studied in Ch. 5 also at the classical level. Operator formulations of both isospaces and their isogeometries are studied in Vol. II.

## 3.2: ISOSPACES AND THEIR ISODUALS

Let $F(\alpha,+,\times)$ be a field (Def. 2.3.1) with elements $\alpha$, $\beta$, ..., conventional sum $\alpha + \beta$ multiplication $\alpha\times\beta = \alpha\beta$ and related additive and multiplicative units 0, and I, respectively. A *linear space* $V(a,F)$ (see, e.g., ref.s [16–18] for mathematical studies) is a set of elements a, b, c,... over a field $F(\alpha,+,\times)$ such to verify the following laws for all $a, b, c \in V$ and $\alpha, \beta, \gamma \in F$

$$a + b = b + a; \quad a + ( b + c ) = ( a + b ) + c; \tag{3.2.1}$$

and

$$\alpha ( \beta a ) = ( \alpha \beta ) a; \quad \alpha ( a + b ) = \alpha a + \alpha b; \quad ( a + b ) \alpha = \alpha a + \beta a; \tag{3.2.2}$$

and, for every $a \in V$, there exists an element $-a$ such that

$$a + (-a) = a - a = 0; \tag{3.2.2}$$

From the above structural lines one can ~~see~~ the following:

**Definition 3.2.1** [1,3]: *Given a linear space* $V(a,F)$ *over a field* $F(\alpha,+,\times)$, *the Class I "isotopes"* $\hat{V}(a,\hat{F})$ *of* V *called "isolinear spaces", are the same set of elements* a, b, c,... $\in V$ *although defined over the isofield of Class I* $\hat{F}(\hat{\alpha},+,*)$ (Def. 2.3.1) *with elements* $\hat{\alpha} = \alpha\hat{1}$, $\hat{\beta} = \beta\hat{1}$, *conventional sum* $\hat{\alpha} + \hat{\beta}$ *and isomultiplication* $\hat{\alpha}*\hat{\beta} = \hat{\alpha}T\hat{\beta}$, *additive unit* 0, and multiplicative unit $\hat{1} = T^{-1}$, *such to preserve all original axioms of V, i.e.,*

$$\hat{\alpha} * ( \hat{\beta} * a ) = ( \hat{\alpha} * \hat{\beta} ) * a, \quad \hat{\alpha} * (a + b) = \hat{\alpha} * a + \hat{\alpha} * b, \tag{3.2.4a}$$

$$( \hat{\alpha} + \hat{\beta} ) * a = \hat{\alpha} * a + \hat{\beta} * a, \ \hat{\alpha} * ( a + b ) = \hat{\alpha} * a + \hat{\alpha} * b, \tag{3.2.4b}$$

*for all* $a, b \in \hat{V}$ *and* $\hat{\alpha}, \hat{\beta} \in \hat{F}$. *The "isodual isolinear spaces"* $\hat{V}^d(a,\hat{F}^d)$ *are Class II images of* $\hat{V}(a,\hat{F})$ *under the isoduality*

$$\hat{1} \Rightarrow \hat{1}^d = -\hat{1}, \tag{3.2.5}$$

*and, as such, are defined over an isodual isofield* $\hat{F}^d(\hat{n}^d,+,\hat{*}^d)$ *of Class II.*

Note the lifting of the field, but the elements of the vector space remain unchanged. The interested reader can prove as an exercise a number of properties of isolinear spaces and their isoduals via a simple isotopy of the corresponding properties of linear spaces [16]. One which is particularly relevant for these volumes follows from the invariance of the elements a, b, c, ... under isotopy as well as under isoduality and can be expressed as follows.

**Proposition 3.2.1** [4]: *The basis of a (finite-dimensional) linear space remains unchanged under isotopy up to possible renormalization factors.*

The above property essentially anticipates the fact that, when studying later on the Lie-isotopic algebras and their isoduals, we shall expect no alteration of its basis because a Lie algebra is, first of all, a linear space. In turn, this implies that hadronic mechanics preserves the *conventional* total conservation laws because, as well known, the generator of Lie symmetries are conserved quantities.

Linear spaces V are also called *vector spaces* [16] in which case their elements a, b, c, are called *vectors.* The isotopes $\hat{V}$ are then called *isovector spaces* and $\hat{V}^d$ are called *isodual isovector spaces.* Their elements a, b, c are then called *isovectors* and *isodual isovectors* , respectively [4]. Note the existence of the simpler *isodual vector spaces* $V^d$ with *isodual vectors* .

An important concept here is that, in all these spaces, the elements a, b, c, ... do not change. This means that a given quantity a is a vector, or an isodual vector, or an isovector or an isodual isovector depending on the space in which it is defined.

Finally, note that *the formulation of isospaces via Kadeisvili's Class III unifies: vector, isovector, isodual vector and isodual isovector spaces.*

A *metric space* [16] hereon denoted S(x,g,F) is a (universal) set of elements x, y, z,.. over the fields F = F(n,+,×) equipped with a nonsingular, and Hermitean map (function) g: S x S ⇒ F, such that:

$$g(x, y) \geq 0, \tag{3.2.6a}$$

$$g(x, y) = g(y, x) \quad \forall\, x,y \in S; \quad g(x, y) = 0 \text{ iff } a = 0 \text{ or } b = 0 \text{ or both.} \tag{3.2.6b}$$

$$g(x, y) \leq g(x, z) + g(y, z), \quad \forall\, x, y, z \in S. \tag{3.2.6c}$$

A *pseudo-metric space,* hereon also denoted by S(x,g,F), occurs when the first condition (3.2.6a) is relaxed. Finally, recall that only metric or pseudo-metric spaces over the reals $F = R$ are used in contemporary physics to characterize our physical space-time. Spaces over the complex numbers, such as the complex Hermitean Euclidean spaces $E(z,\delta,C)$ are used for unitary symmetries, such as SU(2) or SU(3).

Suppose that the space $S(x,g,F)$ is n-dimensional, and introduce the components $x = (x^i)$, $y = (y^i)$, $i = 1, 2, ..., n$. Then, the familiar way of realizing the map $g(x, y)$ is that via a (Hermitean) *metric* $g$ of the form

$$g(x,y) = x^i g_{ij} y^j, \qquad \text{Det. } g \neq 0, \quad g = g^\dagger . \tag{3.2.7}$$

The axiom $g(x, y) > 0$ for metric spaces then implies the condition that $g$ is positive-definite, $g > 0$.

A celebrated physical example of metric spaces is the Euclidean space (3.1.1). Pseudo-metric spaces of primary physical relevance are the Minkowski space (3.1.2), and the Riemannian spaces (3.1.3).

The simplest possible way of constructing an infinite family of isotopes of $S(x,g,F)$ is by introducing n-dimensional isounits of Class I

$$\hat{1} = (\hat{1}_i^{\ j}) = (\hat{1}^j_{\ i}), \qquad i, j, r, s, = 1, 2,..., n. \tag{3.2.8}$$

with isotopic elements

$$T = \hat{1}^{-1} = (T_i^{\ j}) = (T^j_{\ i}), \tag{3.2.9}$$

Then, we can introduce the notion of the *isomap* $\hat{g}$: $\hat{S} \times \hat{S} \Rightarrow \hat{F}$ with realization

$$\hat{g}(x, y) = (x^i \hat{g}_{ij} y^j)\hat{1} \in F \tag{3.2.10}$$

where the quantity

$$\hat{g} = T g = (T_i^{\ k} g_{kj}), \tag{3.2.11}$$

is the *isometric* [1].

The *basis* $e = (e_i)$, $i = 1, 2, ..., n$ of an n-dimensional space $S(x,g,F)$ can be defined via the rule

$$g(e_i, e_j) = g_{ij} . \tag{3.2.12}$$

Then, the *isobasis* is characterized by

$$\hat{g}(\hat{e}_i, \hat{e}_j) = \hat{g}_{ij}, \tag{3.2.13}$$

The above isotopic generalizations can be expressed as follows.

**Definition 3.2.2** [1]: *The "isotopic liftings" of Class I of a given, n–dimensional, metric or pseudometric space* $S(x,g,F)$ *over the field* $F = F(\alpha,+,\times)$, *called "isospaces", are given by the infinitely possible "isospaces"* $\hat{S}(x,\hat{g},\hat{F})$ *characterized by: a) the same dimension* $n$ *and the same local coordinates* $x$ *of the original space; b) the isotopies of the original metric* $g$ *into one of the infinitely possible nonsingular, Hermitean "isometric"* $\hat{g} = Tg$ *with isotopic element* $T$ *of Class I depending on the local variables* $x$, *their derivatives* $\dot{x}$, $\ddot{x}$, ... *with respect to an independent variable* $t$, *the local density* $\mu$, *the local temperature* $\tau$, *the local index of refraction* $n$, *as well as any needed additional quantity (such as wavefunctions* $\psi$ *and their derivative for operator formulations)*

$$g \Rightarrow \hat{g} = T\,g, \tag{3.2.14a}$$

$$T = T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)\,, \quad \det T \neq 0, \quad T^{\dagger} = T > 0, \tag{3.2.14b}$$

$$\det.\, g \neq 0, \quad g = g^{\dagger}, \tag{3.2.14c}$$

*and c) the lifting of the field* $F(\alpha,+,\times)$ *into an isotope of Class I* $\hat{F}(\hat{\alpha},+,\hat{\times})$ *whose isounit* $\hat{1}$ *is the inverse of the isotopic element T, i.e.,*

$$\hat{1} = T^{-1}\,, \tag{3.2.15}$$

*with "isocomposition"*

$$(x \,\hat{,}\, y) = (x\,,T\,y)\,\hat{1} = (T\,x,\,y)\,\hat{1} = \hat{1}\,(x,\,T\,y) = (\,x^{i}\,\hat{g}_{ij}\,y^{j}\,)\,\hat{1} \in \hat{F}\,. \tag{3.2.16}$$

*The "isodual isospaces" of Class II* $\hat{S}^{d}(x,\hat{g}^{d},\hat{F}^{d})$ *are given by the image of* $\hat{S}(x,\hat{g},\hat{F})$ *under isoduality and are defined by the map*

$$\hat{g} \Rightarrow \hat{g}^{d} = T^{d}\,g, \;\; T^{d} = -T\,, \tag{3.2.17a}$$

$$\hat{1} \;\; \Rightarrow \;\; \hat{1}^{d} = (\,T^{d}\,)^{-1} = -\,\hat{1} \tag{3.2.17b}$$

*with "isodual isocomposition" in* $\hat{F}^{d}$

$$\begin{aligned}(x \,\hat{,}\, y)^{d} &= (x\,,T^{d}\,y)\,\hat{1}^{d} = (T^{d}\,x,y)\,\hat{1}^{d} = \\ &= \hat{1}^{d}\,(x,T^{d}\,y) = (x^{i}\,\hat{g}^{d}{}_{ij}\,y^{j}\,)\,\hat{1}^{d} \;\in\; \hat{F}^{d}\,.\end{aligned} \tag{3.2.18}$$

A few comments are now in order. The first and geometrically most dominant aspect is that, because of the unrestricted functional dependence of the isotopic element T, the isometrics $\hat{g}$ = Tg are generally of *integral* type.

Thus, *the isotopic liftings* S(x,g,F) → $\hat{S}(x,\hat{g},\hat{F})$ *imply a nonlocal-integral generalization of the original local-differential space.* In particular, isospaces require a suitable integral topology for their rigorous treatment which is vastly unexplored at this time at the pure mathematical level.

However, all integral terms are embedded, by construction, in the isounits $\hat{1}$. On the other hand, topologies are known to be insensitive to the functional dependence of their own units, provided that they are positive-definite. This implies the particular integro-differential topology of hadronic mechanics whereby conventional topologies hold everywhere except at the unit (Fig. 1.4.1).

Moreover, again from the arbitrariness of the functional dependence of the isotopic element T, one can readily see that *the isotopies* S(x,g,F) → $\hat{S}(x,\hat{g},\hat{F})$ *imply nonlinear and noncanonical generalizations of the original spaces,* where the nonlinearity is in all variables and their derivatives.

Finally note from an abstract viewpoint that the distinction in the use of different fields is meaningful in the conventional metric or pseudo-metric spaces. However, at the isotopic level such a distinction cease to exists because of the isotopic unification of all fields and isofields of Theorem 2.9.1.

Isospaces can also be distinguished via Kadeisvili's classification depending on the characteristics of the unit (Sect. 1.5) into:

**Isospaces** properly speaking (Class I),
**isodual isospaces** (Class II)
**Indefinite isospaces** (Class III),
**Singular isospaces** (Class IV], and
**General isospaces** (Class V).

In this section we shall solely study isospaces of Classes I, II and III, with few comments on isospaces of Class IV.

An important property derived from Proposition 3.2.1 is that *the basis of a metric or pseudo-metric space remains unchanged under isotopies (except for renormalization factors).*

As indicated earlier, isospaces are bona-fide nonlinear, nonlocal and noncanonical generalizations of the original spaces. Despite the above differences, we have the following

**Theorem 3.2.1** [1]: *Isospaces of Class I* $\hat{S}(x,\hat{g},\hat{F})$ *(isodual spaces of Class II* $\hat{S}^d(x,\hat{g}^d,\hat{F}^d)$) *are locally isomorphic to the original spaces* S(x,g,$\hat{F}$) *(isodual space* $S^d(x,g,F^d)$).

The above simple mathematical property has fundamental physical implications because, since a given space $S(x,g,F)$ and its isotope $\hat{S}(x,\hat{g},\hat{F})$ are locally isomorphic, so are expected to be the corresponding groups of isometries.

This implies that the isotopies of Class I of space–time symmetries such as the rotation, Lorentz, Poincaré and unitary symmetries will be locally isomorphic to the original symmetries. Nevertheless, the explicit form of the transformations will be generally nonlinear, nonlocal and noncanonical, thus achieving the desired structural generalization of conventional symmetry transformations to represent interior problems, while achieving a geometric unity with the axiomatic structure of the exterior problem.

Note the necessity for these isomorphisms of the *joint* liftings

$$g \;\Rightarrow\; \hat{g} = T\, g \quad \textit{and} \quad F \;\Rightarrow\; \hat{F}, \quad \hat{1} = T^{-1}. \tag{3.2.19}$$

In fact, a lifting of the type $S(x,g,F) \Rightarrow S'(x,\hat{g},F)$, $\hat{g} = T\, g$, without the joint lifting of the base field *is not* an isotopy and the spaces $S(x,g,\mathfrak{R})$ and $S(x,\hat{g},\mathfrak{R})$ are generally *non–isomorphic*.

The same mechanism of joint lifting of the metric and of the field characterizes the isoduality, e.g., for the Minkowski space $M(x,\eta,R) \Rightarrow M^d(x,\eta^d,R^d)$ and for the Riemannian spaces $\mathfrak{R}(x,g,R) \Rightarrow \mathfrak{R}^d(x,g^d,R^d)$, and it is the foundation of our characterization of antimatter [6].

From property (3.2.18) we have the following

**Proposition 3.2.2** [4,5]: *Compositions $(x, y)$ on a given space $S(x,g,F)$ and their isotopes $(x\hat{,}y)$ on isospaces $\hat{S}(x,\hat{g},\hat{F})$ are isoselfdual, i.e., invariant under isoduality*

$$(x\hat{,}y) \;=\; (x^i\, \hat{g}_{ij}\, y^j)\,\hat{1} \;\equiv\; (x\hat{,}y)^d \;=\; (x^i\, \hat{g}^d{}_{ij}\, y^j)\,\hat{1}^d\,. \tag{3.2.20}$$

As we shall see, the above property implies the novel *universal invariance of physical laws under isoduality*, which has been established at the classical level in monograph [5] and studied at the particle level in these volumes.

*Scalar functions* $f(x)$ on isospaces $\hat{S}(x,\hat{g},\hat{F})$ are ordinary functions. An *isoscalar function* $\hat{f}(x)$ on $\hat{S}(x,g,\hat{F})$ is a function with values on the isofield, i.e.,

$$\hat{f}(x) \;=\; f(x)\,\hat{1} \;\in\; \hat{F}\,. \tag{3.21}$$

As it happens for isonumbers, conventional elements of a space can be preserved (although their operations are lifted), or they can be themselves lifted. As we shall see, this implies nontrivial consequences in functional analysis, e.g., the existence of *two*, rather than one, isotopies of Dirac's delta function.

It should be indicated that in Definition 3.2.1 *the local coordinates* $x \in \hat{S}(x,\hat{g},\hat{F})$ *are assumed to be ordinary scalars and not isoscalars.* One can then build an isospace $\hat{S}(\hat{x},\hat{g},\hat{F})$ with *isocoordinates*

$$\hat{x} = x\hat{1} . \tag{3.2.22}$$

in which case the isocomposition is factorizable into the conventional one

$$\hat{x}^{\hat{2}} = \hat{x}^t * \hat{x} = (x^t x)\hat{1} . \tag{3.2.23}$$

The interchange between the isotopic element and the isounit

$$T \Rightarrow \hat{1} \tag{3.2.24}$$

is called *isoreciprocity map* [6].

In summary, we have four different formulations of isospaces per each individual Class, given in self–explanatory notations by

$$\hat{S}(x,\hat{g},\hat{R}(n,+,*)) , \quad \hat{S}(x,\hat{g},\hat{R}(\hat{n},+,*)), \quad \hat{S}(\hat{x},\hat{g},\hat{R}(n,+,*)) \quad \hat{S}(\hat{x},\hat{g},\hat{R}(\hat{n},+,*)) . \tag{3.2.25}$$

By recalling that the basic unit of hadronic mechanics, Eq. (1.1.1) is outside conventional fields, and by recalling Proposition 2.3.1 and 2.3.2, the isospaces of primary relevance for hadronic mechanics are given by the structures $\hat{S}(x,\hat{g},\hat{F})$ of Definition 3.2.1 specialized to the cases of isoreal and isocomplex fields $\hat{F} = \hat{R}$, $\hat{C}$, plus their image under isoduality $\hat{S}(x,\hat{g}^d,\hat{R}^d)$ and under isoreciprocity $\hat{S}(\hat{x},\hat{g},\hat{R})$.

## 3.3: ISOTOPIC UNIFICATION OF SPACES AND ISOSPACES

In Sect. 2.9 we initiated our presentation of the unifying power of isotopic techniques, beginning with the unification of all conventional numbers into the single abstract notion of isoreal numbers of Class III.

We now illustrate this unifying power for spaces and isospaces. We shall then show in Sect. 3.7 that this is not a sterile mathematical properties, because it permits the geometric unification of the special and general relativities which in turn, is at the foundation of their isotopies [5].

The capability of isospaces of unifying all conventional spaces was identified by the author in their original proposal [1]. By using subsequent advances the property can be expressed as follows:

**Theorem 3.3.1** [loc. cit.]: *All possible metric and pseudometric spaces in n–dimension* $S(r,g,F)$ *plus all their possible isotopic images* $\hat{S}(x,\hat{g},\hat{F})$, *their isodual images* $S^d(x,g^d,F^d)$ *and their isodual isotopic images* $\hat{S}^d(x,\hat{g}^d,\hat{F}^d)$

*can be unified into one, single notion, the abstract n-dimensional isoeuclidean space $\hat{\mathfrak{E}}(x,\hat{\delta},\hat{\mathfrak{R}})$ of Class III over the abstract isoreals $\hat{\mathfrak{R}}$.*

In fact, the assumption of Class III implies the relaxation of the positive- or negative-definite character of the isounit. The property then follows from the fact that all possible metric $g$ and isometrics $\hat{g}$, as well as all their possible isoduals $g^d$ and $\hat{g}^d$ can be trivially derived via the isotopies of the Euclidean metric $\delta = \text{diag.}(1, 1, ..., 1)$

$$\delta \quad \Rightarrow \quad \hat{\delta} = T\delta, \quad T = g, \ \hat{g}, \ g^d, \ \hat{g}^d \tag{3.3.1}$$

Thus, from a mathematical viewpoint, there is no need to study the isotopies of individual spaces, because those of the fundamental Euclidean space are sufficient, and inclusive of all others.

Note that all possible distinctions between spaces over the real or complex numbers are lost under Proposition 3.3.1 because all fields and isofields are particular case of the abstract isoreals $\hat{\mathfrak{R}}$ (Theorem 2.9.1).

## 3.4: ISOEUCLIDEAN SPACES AND THEIR ISODUALS

Isospaces are so fundamental for the study of hadronic mechanics to warrant a brief individual study of the most important ones prior to the study of their geometries. We therefore begin with the following:

**Definition 3.4.1** [1]: *The liftings of the conventional n-dimensional Euclidean spaces* $E(r,\delta,R)$ *over the reals* $R$, *Eq.s (3.1.1), into the "isoeuclidean spaces" of Class I are given by*

$$E(r,\delta,R) \ \Rightarrow \ \hat{E}(r,\hat{\delta},\hat{R}), \tag{3.4.1a}$$

$$\delta = I_{n\times n} \Rightarrow \hat{\delta} = T(t, r, \dot{r}, \ddot{r}, \mu. \tau, n, ...)\,\delta, \tag{3.4.1b}$$

$$\det \delta = 1 \neq 0, \ \delta = \delta^{\dagger} \Rightarrow \det.\hat{\delta} \neq 0, \ \hat{\delta} = \hat{\delta}^{\dagger}, \tag{3.4.1c}$$

$$R \Rightarrow \hat{R}, \quad \hat{1} = T^{-1} = \hat{\delta}^{-1} \tag{3.4.1d}$$

$$r^2 := (r, r) = r^i\,\delta_{ij}\,r^j \ \Rightarrow \ r^{\hat{2}} := (r \hat{,} r) = (r, \hat{\delta} r)\,\hat{1} =$$

$$= (\hat{\delta} r, r)\,\hat{1} = \hat{1}\,(r, \hat{\delta} r) = [\, r^i\,\hat{\delta}_{ij}(r, \dot{r}, \ddot{r}, ..)\,r^j\,]\,\hat{1} \in \hat{R} \tag{3.4.1e}$$

*where the isofield* $\hat{R}(\hat{n},+,*)$ *is of Class I. The "isodual isoeuclidean spaces" of Class II are given by the isodual image of the preceding ones*

$$\hat{E}(r,\hat{\delta},\hat{R}) \Rightarrow \hat{E}^d(r,\hat{\delta}^d,\hat{R}^d), \tag{3.4.2a}$$

$$\hat{\delta} = T\delta \quad \Rightarrow \quad \hat{\delta}^d = T^d\delta = -\hat{\delta}, \quad T^d = -T, \tag{3.4.2b}$$

$$\hat{R} \quad \Rightarrow \quad \hat{R}^d \approx R\hat{1}^d, \quad \hat{1}^d = -\hat{1} \tag{3.4.2c}$$

$$r^{\hat{2}} = (r\hat{,}r) = (r^i\hat{\delta}_{ij}r^j)\hat{1} \Rightarrow r^{\hat{2}\,d} = (r\hat{,}r)^d = (r^i\hat{\delta}^d{}_{ij}\hat{\ }r^j)\hat{1}^d \equiv r^{\hat{2}} \tag{3.4.2d}$$

*The n–dimensional complex Euclidean spaces* $E(z,\delta,C)$ *with separation*

$$\bar{z}\,\delta\,z = \bar{z}^i\,\delta_{ij}\,z^j\ , \tag{3.4.3}$$

*is lifted into the "complex isoeuclidean spaces" of Class I*

$$\hat{E}(z,\hat{\delta},\hat{C}) : (\bar{z}^t\hat{\delta}z)\hat{1} = (\bar{z}^i\hat{\delta}_{ij}z^j)\hat{1}, \tag{3.4.4a}$$

$$\hat{\delta} = T\delta, T \equiv T^\dagger, \hat{1} = T^{-1} > 0. \tag{3.4.4b}$$

*where upper bar denotes complex conjugation. The "isodual complex Hermitean isoeuclidean spaces" are instead given by*

$$\hat{E}^d(z,\hat{\delta}^d,\hat{C}^d) : (\bar{z}^t\hat{\delta}^d z)\hat{1}^d = (\bar{z}^i\hat{\delta}^d{}_{ij}z^j)\hat{1}^d, \tag{3.4.5a}$$

$$\hat{\delta}^d = T^d\delta, T^d = -T, \hat{1}^d = -\hat{1}. \tag{3.4.5b}$$

We now outline a few mathematical and physical aspects of isoeuclidean spaces for subsequent more detailed treatment. The applications of isoeuclidean spaces are of three primary types:

**A) Geometric applications.** Recall that the conventional Euclidean metric $\delta = \text{diag.}(1, 1, 1)$ is a geometrization of the perfect rigid sphere with unit radius. From their topological characteristics, isometrics of Class I can always be diagonalized. We can therefore always assume the realization of the isotopic element in the diagonal form

$$T = \text{diag.}(b_1{}^2, b_2{}^2, b_3{}^2), b_k = b_k(t, r, \dot{r}, \ddot{r}, ...) > 0, k = 1, 2, 3, \tag{3.4.6}$$

where the b's are called *characteristics functions of the isospace.*

The first geometrical application of isospaces is therefore that of representing all infinitely possible deformations of the original perfect sphere $\delta = \text{diag.}(1, 1, 1)$ into the ellipsoids with semiaxes

$$\hat{1} = \mathrm{diag.}\,( b_1^{-2}, b_2^{-2}, b_3^{-2} ), \tag{3.4.7}$$

where the functional dependence expresses the physical origin of the deformations as due to local pressures, densities, temperatures, etc. We therefore have the:

**Geometric meaning:** *the isounit permits a direct representation of the actual nonspherical shape of a given body as well as the representation of all its infinitely possible deformations.*

As we shall see, this capability exists at the pure classical level [5] and then simply persists under operator formulations prior to any second quantization or use of form factors.

**B) Analytic applications.** As well known, nonrelativistic exterior dynamical problems are representable via conventional analytic equations, such as Lagrange equations, which are defined on the 3-dimensional Euclidean space $E(r,\delta,R)$ (plus an additional one dimensional space representing time, see below). In this case the trajectory in vacuum is solely characterized by *one single quantity*, the Lagrangian $L = K - V$.

The main objective of the isotopies is the representation of interior dynamical problems with conventional potential forces, plus contact, nonlinear–nonlocal–nonlagrangian[17] forces due to the medium. In this latter case, the system is represented by *two independent quantities*, the Lagrangian $L = K - V$ and the isounit $\hat{1}$. We therefore have the following

**Analytic meaning:** *The isounit permits a direct representation of contact, nonlinear–nonlocal–nonlagrangian forces for interior physical conditions.*

The Lagrangian L must now be properly written in isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$. This results in expression $\hat{L}(\hat{x},\hat{\dot{x}}) = \hat{K}(\hat{\dot{x}}) - \hat{V}(\hat{x})$ which is defined in terms of the conventional coordinates x and velocities $\dot{x}$ although all their operations are now of isotopic character.

In particular, the *isokinetic energy* is given by

$$\hat{K} = K_{\hat{E}} = \tfrac{1}{2}\, m\, \hat{\dot{x}}^{\hat{2}} = \tfrac{1}{2}\, m\, \dot{x} \hat{*} \dot{x} = \tfrac{1}{2}\, m\, \dot{x}\, T\, \dot{x}, \tag{3.4.8}$$

[17] By "nonlagrangian" we mean hereon non–first–order Lagrangians, namely, equations of motion which violate the integrability conditions for their representation via first–order Lagrangians $L = L(t, r, \dot{r})$. Evidently, higher order Lagrangian may exist, e.g., $L = L(t, r, \dot{r}, \ddot{r})$. The point is that, under these latter conditions, there is no (conventional) Hamiltonian. The term "noncanonical" is then used as a synonym of "non–first–order–Lagrangian".

and the *isopotential energy* is given by the isotopic image of the function V(x) (see Ch. 6), e.g., for the case of V(x) depending on the norm x

$$\hat{V}(x): \quad x = (x \hat{*} x)^{\frac{1}{2}}. \tag{3.4.9}$$

The isotopies of the analytic equations in which the above isofunctions are defined with various examples are studied in Ch. 5 of this volume and in Vol. II.

Even though not claimed to be unique, the above isotopic representation has already been established as being effective and "directly universal" in classical mechanics [5]. Its effectiveness and "direct universality" in particle physics will be studied in these volumes.

By adding the preceding geometric meaning of the isounit, one can see that *isospaces provide a direct geometrization of the inhomogenuity and anisotropy of physical media* (Fig. 3.1.1). In fact, the inhomogenuity can be represented in isospaces, e.g., via a dependence of the isometric $\hat{\delta}$ on the locally varying density $\mu$. The anisotropy, e.g., due to the presence of an intrinsic angular momentum along the direction $\bar{n}$, is then representable via a factorization of such a preferred direction also in the isometric, much along the Finslerian geometry, for via the differentiations $b_1 \neq b_2 \neq b_3$.

Note that the representation is "direct" because occurring directly in the isometric itself, without any need of operator formulations or any use of artificial or indirect approaches.

Thus, the transition from exterior to interior conditions is done via a generalization of the basic unit $I \Rightarrow \hat{1}$. A condition the reader should keep in mind to avoid undetected inconsistencies is that in most (but not necessarily all[18]) physical applications the isounits $\hat{1}$ are constructed in such a way to recover the conventional unit identically in the exterior problem.

This condition can be realized by assuming that the entire matter of the medium considered is enclosed in a minimal surface S° with local radius R° and density $\mu$, in which case

$$\hat{1}_{r \geq R^\circ} \equiv I = \mathrm{diag.}(1, 1, 1), \text{ or } \qquad \mathrm{Lim}_{\mu \to 0} \hat{1} \equiv I . \tag{3.4.10}$$

[18] There are cases in which extended particles in vacuum can effectively use isotopic theories, e.g., when they are extended and experience deformations due to external fields, such as a charged sphere in vacuum under the influence of an intense electric field. Even though there are no nonhamiltonian interactions, the generalization of the unit is still effective, as shown in details in ref. [5] because the physical event of deformation of shape is conceptual, geometrically and analytically outside the representational capabilities of the Hamiltonian. The not necessarily unique, yet simple, direct and effective approach used in these studies is the representation of these nonhamiltonian effects via a generalization of the unit.

Note that the 3–dimensional Euclidean "space" is one. On the contrary, there exist infinitely many 3–dimensional isoeuclidean "spaces". This is evidently due to the infinitely possible isometrics $\hat{\delta}$ representing the infinitely possible physical conditions of interior problems.

We finally remark that, when overall notions are needed, that is, the quantities are referred to the physical medium as a whole, the characteristic b–functions can be averaged into constants

$$b^\circ_k = < b_k(t, r, \dot{r}, ...) >, \quad k = 1, 2, 3. \tag{3.4.11}$$

As we shall see, constant isotopic elements T and characteristic b°–quantities will have numerous applications. The point to keep in mind is that such constancy is in actuality an average over a rather complex functional dependence.

**C) Algebraic applications:** Recall that the unit I = diag. (1, 1, 1) of the Euclidean space is the fundamental unit of the related Lie theory, e.g., the unit of the group of isometries of the Euclidean space, the orthogonal group O(3). The following property is then consequential

**Algebraic meaning:** *The isounit constitutes the basic generalized units of the Lie–isotopic theory.*

As we shall see in the next chapter, the isotopies of Lie's theory for the achievement of nonlinear, nonlocal and noncanonical realizations of conventional space–time and unitary symmetries is based precisely on the isotopies $I \Rightarrow \hat{1}$.

The following property is a consequence of Theorem 4.2.1.

**Corollary 3.2.1A:** *Isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{R})$ *of Class I (isodual isoeuclidean spaces of Class II* $\hat{E}^d(r,\hat{\delta}^d\hat{R}^d)$*) are locally isomorphic to the conventional Euclidean spaces of the same dimension* $E(r,\delta,R)$ *(isodual Euclidean spaces of the same dimension* $E^d(r,\delta^d R^d)$*).*

We shall say that, from a geometrical viewpoint, Euclidean spaces and their isotopes are equivalent, as ensured by the preservation of the original axioms, as well as the identity of the two spaces at the abstract level, and we shall write $E(r,\delta,R) \approx \hat{E}(r,\hat{\delta},\hat{R})$.

We close this section with the identification of the isoeuclidean spaces used in nonrelativistic hadronic mechanics, inclusive of a time component.

**Definition 3.4.2** [5]: *The "nonrelativistic isotopic space–time" of hadronic mechanics of Class I is given by the Cartesian product of two isoeuclidean spaces, one representing time and the other representing space with* *corresponding isounits $\hat{1}_t$ and $\hat{1}$, isocomposition*

$$\hat{E}(t,\hat{R}_t) \times \hat{E}(r,\hat{\delta},\hat{R}): \quad t^{\hat{2}} = (t\, T_t\, t)\hat{1}_t \in \hat{R}_t, \; \hat{1}_t = T_t^{-1}, \tag{3.4.12}$$

$$r^{\hat{2}} = (r^t\, T\, \delta\, r)\hat{1} \in \hat{R}, \; \hat{1} = T^{-1}, \tag{3.4.11}$$

*and diagonal realization*

$$T_t = b_4^{\,2}, \quad b_4 = b_4(t, r, \dot{r}, \mu, \tau, n, \ldots) > 0, \tag{3.4. 12a}$$

$$T = \mathrm{diag.}(b_1^{\,2}, b_2^{\,2}, b_3^{\,2}), \; b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0 \tag{3.4.12b}$$

*The "isodual nonrelativistic isotopic space–time" of hadronic mechanics of Class II is then given by*

$$\hat{E}^d(t,\hat{R}^d_t) \times \hat{E}^d(r,\delta^d,\hat{R}^d): \quad t^{\hat{2}\,d} = (t\, T^d_t\, t)\hat{1}^d_t \in \hat{R}^d_t, \tag{3.4.13a}$$

$$r^{\hat{2}\,d} = (r^t\, T^d \delta\, r)\hat{1}^d \in \hat{R}, \tag{3.4.13b}$$

$$\hat{1}^d_t = (T^d_t)^{-1} = -\hat{1}_t, \quad \hat{1}^d = (T^d)^{-1} = -I, \tag{3.4.13c}$$

$$T^d_t = -b_4^{\,2}, \quad b_4 = b_4(t, r, \dot{r}, \mu, \tau, n, \ldots) > 0, \tag{3.4.13d}$$

$$T^d = -\mathrm{diag.}(b_1^{\,2}, b_2^{\,2}, b_3^{\,2}), \; b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0. \tag{3.4.13e}$$

As it is the case for all other quantities, the above definition implies the existence of four distinguishable nonrelativistic times in hadronic mechanics:

**Time,** as the usual element $t$ of the field of real numbers $R(t,+,\times)$;
**Isotime,** the element $\hat{t} = t\,\hat{1}_t \in \hat{R}(\hat{t},+,*)$;
**Isodual time,** the element $t^d = t\, I^d \in R^d(t^d,+,\times^d)$;
**Isodual isotime,** the element $\hat{t}^d = t\,\hat{1}^d_t \in \hat{R}^d(\hat{t}^d,+,*^d)$.

The following property is a consequence of the theory of isonumbers of the preceding chapter.

**Proposition 3.4.1:** *The direction of time (Eddington's "time arrow") changes sign in the transition from our space–time to its isodual.*

In fact, under isoduality, we have the map of our time $t \in R(t, R_t)$

$$t > 0 \quad \Rightarrow \quad t^d = t\, I^d = -t < 0, \tag{3.4.14}$$

and the same result occurs under isotopy.

The isotopies of time have been introduced here via the purely mathematical use of the methods studied until now. Nevertheless, as we shall see in the next section, the time isotopies emerge rather forcefully from the nonrelativistic limit of relativistic isotopic theories.

## 3.5: ISOMINKOWSKI SPACES AND THEIR ISODUALS

We now study the central carrier spaces of the relativistic hadronic mechanics, according to the following:

**Definition 3.5.1** [1]: *The isotopic liftings of the conventional (3+1)-dimensional Minkowski space* $M(x,\eta,R)$ *of Class I over the reals* $R(n,+,\times)$ *are given by the isotopes called "isominkowski spaces"*

$$M(x,\eta,R) \Rightarrow \hat{M}(x,\hat{\eta},\hat{R}), \tag{3.5.1a}$$

$$\eta = \text{diag.}(1,\ 1,\ 1,\ -1) \Rightarrow \hat{\eta} = T(s,\ x,\ \dot{x},\ \ddot{x},\ \tau,\ n,..)\ \eta, \tag{3.5.1b}$$

$$\det\eta = -1 \neq 0,\quad \eta = \eta^{\dagger} \Rightarrow \det\hat{\eta} \neq 0,\quad \hat{\eta}^{\dagger} = \hat{\eta}, \tag{3.5.1c}$$

$$R \Rightarrow \hat{R},\quad \hat{1} = T^{-1}, \tag{3.5.1d}$$

$$x^2 = (x,x) = x^{\mu}\,\eta_{\mu\nu}\,x^{\nu} \quad \Rightarrow \quad x^{\hat{2}} = (x\,\hat{,}\,x) = (x, T\,x)\,\hat{1} =$$

$$(T\,x,\,y)\,\hat{1} = \hat{1}(x,\,T\,y) = [x^{\mu}\,\hat{\eta}_{\mu\nu}(s,\,x,\,\dot{x},\,\ddot{x},...)\,x^{\nu}\,]\,\hat{1},\quad \mu,\nu = 1,2,3,4, \tag{3.5.1e}$$

*with diagonal realization of the isounit and isoseparation*

$$\hat{1} = \text{diag.}(\,b_1^{-2},\,b_2^{-2},\,b_3^{-2},\,b_4^{-2}\,) > 0,\quad b_{\mu} = b_{\mu}(s,\,x,\,\dot{x},\ \dots\ ..) > 0, \tag{3.5.2a}$$

$$x^2 := (\,x^1\,b_1^{\,2}\,x^1 + x^2\,b_2^{\,2}\,x^2 + x^3\,b_2^{\,2}\,x^3 - x^4\,b_4^{\,2}\,x^4\,)\,\hat{1} \in \hat{R}\,. \tag{3.5.2b}$$

*invariant measure*

$$ds^{\hat{2}} = (\,-\,dx^{\mu}\,\hat{\eta}_{\mu\nu}\,dx^{\nu}\,)\,\hat{1}, \tag{3.5.3}$$

*and characteristic functions constants*

$$b^{\circ}_{\mu} = < b_{\mu}(s,\,x,\,\dot{x},\,....) >,\ \mu = 1,2,3,4 \tag{3.5.4}$$

*derived via a given averaging procedure <...>. The "isodual isominkowski spaces" of Class II are given by*

$$\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d):\ \hat{\eta}^d = T^d(x,\dot{x},\ddot{x},\mu,\tau,n,..)\,\eta = -\hat{\eta}, \qquad (3.5.5a)$$

$$T^d = -T, \qquad \hat{1}^d = (T^d)^{-1} = -\hat{1} \qquad (3.5.5b)$$

$$x^{\hat{2}\,d} = (x\,\hat{,}\,x)^d = (x,T^d x)\hat{1}^d = (T^d x, y)\hat{1}^d = \hat{1}^d (x, T^d y) =$$

$$[x^\mu \hat{\eta}^d{}_{\mu\nu}(s,x,\dot{x},\ddot{x},...)\,x^\nu]\,\hat{1}^d \equiv x^{\hat{2}} = [x^\mu \hat{\eta}_{\mu\nu}(s,x,\dot{x},\ddot{x},...)\,x^\nu]\,\hat{1} \qquad (3.5.5c)$$

*with diagonal realization of the isodual isounit and isodual isoseparation*

$$\hat{1}^d = -\,\mathrm{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) > 0, \quad b_\mu = b_\mu(s, x. ......) > 0, \qquad (3.5.6a)$$

$$x^{2\,d} := (-x^1 b_1^2 x^1 - x^2 b_2^2 x^2 - x^3 b_2^2 x^3 + x^4 b_4^2 x^4)\,\hat{1}^d \in \hat{R}^d. \qquad (3.5.6b)$$

*invariant measure*

$$ds^{\hat{2}\,d} = (+\,dx^\mu \eta^d{}_{\mu\nu}\,dx^\nu)\,\hat{1}^d \equiv ds^{\hat{2}}, \qquad (3.5.7)$$

*and characteristic functions averaged into constants*

$$b^\circ{}_\mu = \langle b_\mu(s, x, \dot{x}, ....)\rangle, \ \mu = 1, 2, 3, 4 \qquad (3.5.8)$$

*which coincide with those of the isospace .*

Again, we have four distinguishable types of spaces: the conventional *Minkowski space* $M(x,\eta,R)$, the *isominkowski spaces* $\hat{M}(x,\hat{\eta},\hat{R})$, the *isodual Minkowski space* $M^d(\eta^d,R^d)$, and the *isodual isominkowski spaces* $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$.

The conventional Minkowski space is and will remain the fundamental space for the description of particles and electromagnetic waves in vacuum (Fig. 3.1.1). The primary function of isominkowskian spaces *is* to provide a relativistic geometrization, first, of classical physical media (see also Fig. 3.1.1) and, then of the interior of hadrons, upon suitable operator formulation. The primary geometric task is therefore the representation of the *departures* from the homogeneous and isotropic vacuum expected, classically, from physical media in general, and the deep superposition of the wavepackets of the particles, for the case of hadronic matter at large.

In the latter case we recall that all massive particles have an experimentally established wavepacket/wavelength of the order of 1 fm ($10^{-13}$ cm). But all hadrons have a charge distribution with a radius also of the order of 1 fm. The region of space in the interior of hadrons are then expected to have a nonlinear, nonlocal-intègral and noncanonical-nonhamiltonian structure for which representation the isominkowskian spaces were built for [1].

The isominkowskian characterization of the interior of hadrons has received numerous direct and indirect experimental verifications which will be studied in detail in Vol. III. We here limit ourselves to recall that phenomenological calculations conducted in ref. [10] via the conventional gauge theory in the Higgs sector identify the following modification of the Minkowski metric in the interior of pions and kaons

$$\hat{\eta} = \mathrm{Diag.}\,[\,(1 - \alpha/3), (1 - \alpha/3), (1 - \alpha/3), -(1 + \alpha)\,], \qquad (3.5.9)$$

with

$$\alpha = -\,3.79\times 10^{-3} \text{ for pions and } \alpha = +6.1\times 10^{-4} \text{ for kaons,} \qquad (3.5.10)$$

It is evident that modified metric (3.5.9) is a particular case of the general class (3.5.5.2b) for $b^\circ_1 = b^\circ_2 = b^\circ_3 = b^\circ$

$$\hat{\eta} = \mathrm{diag.}\,(\,b^\circ{}_1{}^2, b^\circ{}_1{}^2, b^\circ{}_3{}^2, -\ b^\circ{}_4{}^2), \qquad (3.5.11a)$$

$$b^{\circ 2} \cong 1 + 1.2\times 10^{-3}, \quad b_4{}^2 \cong 1 - 3.79\times 10^{-3} \text{ for pions}, \qquad (3.5.11b)$$

$$b^{\circ 2} \cong 1 - 2\times 10^{-4}, \quad b_4{}^2 \cong 1 + 6.1\times 10^{-4} \text{ for kaons}, \qquad (3.5.11c)$$

Similarly, the phenomenological studies of ref.s [11] conducted also for the kaons yield the numerical values

$$b^\circ{}_1{}^2 = b^\circ{}_2{}^2 = b^\circ{}_3{}^2 = b^{\circ\,2} \simeq 0.909080 \pm 0.0004, \ b^\circ{}_4{}^2 \simeq 1.002 \pm 0.007\,, \qquad (3.5.12)$$

which are remarkably close to value (3.5.11c) for kaons.

Note the change of value (as well as of sign of the $\alpha$-parameter) in the transition from pions to kaons thus confirming the expectation that the characteristic $b^\circ$-quantities are different for different physical conditions. In fact, the charge radius of hadrons is approximately the same for all particles, thus implying *different densities for different hadrons*, which result in *different interior conditions for different particles* and, therefore, *different hadrons necessarily have different values of the characteristic* $b_\mu{}^\circ$*-constants.*

This point has been here illustrated to prevent the customary tendency of looking for universal constant, which is inapplicable under isotopies because quantities that are constants in quantum mechanics, such as Planck's constant $\hbar$, the speed of light $c_0$, etc., are replaced with *locally varying* values.

By now mean, one should therefore search for the "universal values" of the characteristic $b_\mu{}^\circ$-constants because such constants provide an average of the physical characteristics of the medium considered and, as such, vary from medium to medium.

The primary physical application of the isodual Minkowski space

$M^d(x,\eta^d,R^d)$ is the representation of antiparticles in vacuum via the representation of negative-energy solutions of conventional relativistic field equations (such as Dirac's equation). In fact, it is easy to see that the expression

$$E^d = p^d{}_4 = -E. \tag{3.5.13}$$

The aspect we have to show later on in Volume II is that negative-energy solutions behave in a fully physical way when interpreted via isodual spaces.

Recall that, by putting $x^2 = R^2 = \text{const.}$, the non-relativistic limit of the Minkowski space is the familiar structure

$$\text{Lim}_{R/c_o = 0}\, M(x,\eta,R) = E(t,R_t) \times E(r,\delta,R). \tag{3.5.14}$$

Along the same lines, by assuming $x^2 = R^2 = \text{cost.}$, it has been shown in ref. [5], Ch. VI, that

$$\text{Lim}_{R/c_o = 0}\, \hat{M}(x,\hat{\eta},\hat{R}) = \hat{E}(t,\hat{R}_t) \times \hat{E}(r,\hat{\delta},\hat{R}), \tag{3.5.15}$$

thus recovering the isoeuclidean space of nonrelativistic hadronic mechanics of Definition 3.4.2.

Note that, jointly with the "decoupling" of space and time, we have a corresponding "decoupling" of the space and time components of the isotopic element T and isounit $\hat{1}$. Then the isorelativistic quantity $b_4{}^{-2}$ becomes the nonrelativistic isounit of time.

A further geometric meaning of isominkowski spaces is provided by the realization

$$b_\mu = 1 / n_\mu \quad \mu = 1, 2, 3, 4. \tag{3.5.16}$$

under which the isoseparation becomes

$$x^2 := ( x^1 \frac{1}{n_1{}^2} x^1 + x^2 \frac{1}{n_2{}^2} x^2 + x^3 \frac{1}{n_3{}^2} x^3 - x^4 \frac{1}{n_4{}^2} x^4 ). \tag{3.5.17}$$

where we have ignored the factor $\hat{1}$.

Let us recall that the constant value $n^\circ{}_4$ represents the *index of refraction of light* for transparent, homogeneous and isotropic media such as water. The speed of light is then

$$c = c_o / n_4 < c_o. \tag{3.5.18}$$

When the medium is transparent, but no longer homogeneous and isotropic, the index of refraction $n_4$ acquires a rather complex functional dependence on the local density $\mu$ the local temperature $\tau$ etc., in which case we have the locally varying speed of light as in Fig. 3.1.1

$$c = c(x, \mu, \tau, ....) = c_o / n_4(x, \mu, \tau, ...), \qquad (3.5.19)$$

as experimentally established, say, for light traveling in our atmosphere.

Thus, *a geometric meaning of the characteristic quantity* $b_4 = 1/n_4$ *is that of characterizing the inverse of the local index of refraction of light. Its average* $b°_4 = 1/n°_4$ *then provides the "global" quantity, such as the average speed of light throughout our entire atmosphere* $c = c_o b°_a = c_o/n°_4$.

When the medium is no longer transparent to light, the quantity $b_4$ persists although it acquires a pure geometrical meaning much similar to the term $g_{44}$ in gravitation, without representing any actual physical speed.

The remaining space quantities $b_k$ are an isorelativistic extension of the $b_4$ term. To begin their illustration, consider first the case of the homogeneous and isotropic water. Then simple considerations lead to the identities $b_\mu = 1/n°_4$, $\mu = 1, 2, 3, 4$, with corresponding lifting of the separation

$$x^2|_{\text{empty space}} \Rightarrow x^2|_{\text{water}} = \frac{1}{n°_4{}^2} x^2 . \qquad (3.5.20)$$

This establishes that *the transition from empty space to water is directly representable via the simplest possible isotopy called "scalar isotopy"*, that with $b_\mu = 1/n_4$, $\mu = 1, 2, 3, 4$ (see ref. [5], Ch. IV, for a detailed study).

If the medium is inhomogeneous and anisotropic, such as our atmosphere, we have the inapplicability of the geometric foundations of the special relativity. Nevertheless, its isotopies have shown that the isorelativistic transformation of the locally varying index of refraction $n_4$ yields generally different space values $n_k$, $k = 1, 2, 3$, precisely because of the inhomogeneous and anisotropic structure, thus resulting in isoseparation (3.5.2).

The following property is a corollary of Theorem 3.2.1:

**Corollary 3.2.1B:** *Isominkowskian spaces of Class I* $\hat{M}(x,\eta,\hat{R})$ *(isodual isominkowskian spaces of Class II* $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$*) are locally isomorphic to the conventional Minkowski space* $M(x,\eta,R)$ *(isodual isominkowski space* $M^d(x,\eta^d,R^d)$*).*

Despite the profound differences in functional dependence, conventional and isotopic Minkowski spaces are "geometrically equivalent". In fact, all the original geometric axioms of the space are preserved as a central condition of isotopy.

This implies that certain operations are equivalently done in both spaces.

As an example, one can introduce the *contravariant isometric tensor* $\hat{\eta}^{-1}$ in $\hat{M}(x,\hat{\eta},R(n,+,*))$ with elements[19]

$$\eta^{\mu\nu}(s, x, \dot{x}, \ldots) = ( | \hat{\eta}_{\alpha\beta}(s, x, \dot{x}, \ldots) |^{-1} )^{\mu\nu} . \qquad (3.5.21)$$

Then, the transition from covariant to contravariant indices, and viceversa, is done as in the conventional case

$$x_\mu = \hat{\eta}_{\mu\nu} x^\nu , \qquad x^\mu = \hat{\eta}^{\mu\nu} x_\nu . \qquad (3.5.22)$$

This implies that, by ignoring the multiplicative isounit, the isoseparation can formally be written in a way identical to the conventional one,

$$x_\mu x^\mu {}_{|M} = x^\mu \eta_{\mu\nu} x^\nu \Rightarrow x_\mu x^\mu {}_{|\hat{M}} = x^\mu \hat{\eta}_{\mu\nu}(s, x, \dot{x}, \ldots) x^\nu . \qquad (3.5.23)$$

In this sense, most of the relativistic isotopic formulations are "hidden" in the conventional ones. To identify them, one must identify the basic units and related assumptions.

Despite this "isotopic equivalence", the physical differences between the isotopic and conventional formulations are considerable and experimentally measurable, classically and operationally. In fact, isorelativistic theories can:

A) Directly represent the actual, generally nonspherical shape of the considered hadrons, say, an oblate spheroidal ellipsoid;

B) Can directly represents all infinitely possible deformation of the above original shape due to sufficiently intense external fields or collisions;

C) Directly represent the nonlinear, nonlocal and nonhamiltonian dynamics of the interior particle problems (that is, particles moving inside other particles);

D) Directly geometrize the inhomogenuity and anisotropy of matter; and

E) All the above via a covering of conventional Minkowskian formulations, which admit the latter at the limit $\hat{1} \Rightarrow I$.

We close this section with the following Corollary of Proposition 3.3.1

**Corollary: 3.3.1A:** *The conventional Minkowski space* $M(x,\eta,R)$ *in (3+1) space-time dimensions is an isotope* $\hat{E}(r,\hat{\delta},\hat{R})$ *of the 4-dimensional Euclidean space* $E(x,\delta,R)$ *of Class III characterized by the isotopy of the metric*

$$\delta = I_{4\times4} \Rightarrow \hat{\delta} = T\delta = \eta = \mathrm{diag.}\,(1, 1, 1, -1), \qquad (3.5.24)$$

*under the redefinition of the fields*

[19] The reader who has acquired a technical knowledge of the preceding analysis can see that the contravariant isometric tensor on $\hat{M}(x,\hat{\eta},\hat{R}(\hat{n},+,*)$ is given by $\eta^{-1} = \hat{\eta}^{-1}\hat{1}$

$$R \Rightarrow \hat{R}, \quad \hat{1} = T^{-1} = \eta^{-1} = \eta. \tag{3.5.25}$$

In fact, the way according to which the isominkowskian spaces were derived the first time [1] is given by the *"isotopies of isotopies"*

$$E_4(r,\delta,R) \Rightarrow \hat{E}_{3+1}(x,\hat{\delta},\hat{R}) \approx M(x,\eta,\hat{R}) \Rightarrow \hat{M}(x,\hat{\eta},\hat{R}). \tag{3.5.26}$$

The reader should remember that the isotopy of the field is a feature needed for mathematical consistency, but it does not affect the practical numbers of the theory. In fact, as pointed out in the preceding chapter, the product of an isonumber $\hat{n}$ by a quantity Q in Minkowski space coincides with the conventional product

$$\hat{n} * Q \equiv n\,Q. \tag{3.5.27}$$

Also, as we shall see in the next chapter, the symmetries of $\hat{E}_{3+1}(x,\hat{\delta},\hat{R})$ and those of $M(x,\eta,\hat{R})$ coincide because characterized by the same metric $\hat{\delta} = \eta$.

This essentially means that at the isotopic level of Class III, there is no essential geometrical distinction between the 4-dimensional Euclidean space $E(r,\delta,R)$ and the (3+1)-dimensional Minkowski space $M(x,\eta,R)$. These notions are important pre-requisite for their isotopic liftings [1].

We finally close this section with the following important property proved by Aringazin [15]

**Proposition 3.5.1** [loc. cit.]: *Isominkowski spaces of Class I are "directly universal" for all infinitely possible deformations of the Minkowski metric preserving the signature (+, +, +, –), i.e., they are capable of representing all possible modifications of the class considered, directly in the frame of the experimenter. A similar occurrence holds for the remaining classes with different signatures.*

Aringazin [loc. cit.] illustrated the above property, as we shall review in details in Vol. II, by showing that all generalizations of the Einsteinian expression for the behaviour of the meanlife with speed existing in particle physics are particular cases of the single, unified, geometric expression characterized by the isominkowski space. The difference between one or the other of the existing expressions is merely due to the assumption of different expansion with different parameters, different truncations, etc.

The above property should be kept in mind because other approaches to the interior problem are indeed possible, and their study is indeed encouraged, with the understanding that they are expected to be particular cases of isotopic techniques.

This completes our preliminary presentation of isominkowski spaces. We will have ample opportunities of additional studies during the course of our analysis.

## 3.6: ISORIEMANNIAN SPACES AND THEIR ISODUALS

The additional spaces of particular relevance for isotopic studies can be presented via the following:

**Definition 3.6.1** [1,5]: *The liftings of a given n–dimensional Riemannian or pseudoriemannian space* $\mathfrak{R}(x,g,R)$ *over the reals* $R(n,+,\times)$ *into the infinitely possible isotopes* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *of Class I called "isoriemannian spaces" are given by*

$$\mathfrak{R}(x,g,R) \;\Rightarrow\; \hat{\mathfrak{R}}(x,\hat{g},\hat{R}), \tag{3.6.1a}$$

$$g = g(x) \;\Rightarrow\; \hat{g} \;=\; T(s,\, x,\, \dot{x},\, \ddot{x},\, \mu,\, \tau,\, n,\, ...)\, g(x), \tag{3.6.1b}$$

$$\text{Det. } g \neq 0, \quad g = g^{\dagger} \;\Rightarrow\; \text{Det. } \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\dagger}, \tag{3.6.1c}$$

$$R \;\Rightarrow\; \hat{R}, \quad \hat{1} \;=\; T^{-1}, \tag{3.6.1d}$$

$$\begin{aligned}(x,\, y) \;&= [\, x^{i}\, g_{ij}(x)\, x^{j}\, ]\, I \quad \Rightarrow\; (x \,\hat{,}\, x) \;=\; (x,\, T\, x)\, \hat{1} \;=\; (T\, x,\, x)\, \hat{1} \\ &= \hat{1}\, (x,\, T\, x) \;= [\, x^{i}\, \hat{g}_{ij}(s,\, x,\, \dot{x},\, \ddot{x},\, ...)\, x^{j}\, ]\, \hat{1} \in \hat{R}\, .\end{aligned} \tag{3.6.1e}$$

*with "invariant isoseparation"*

$$ds^{\hat{2}} = [\, -\, dx^{\mu}\, \hat{g}_{\mu\nu}(s,\, x,\, \dot{x},\, \ddot{x},\, ...)\, dx^{\nu}\, ]\, \hat{1}\, . \tag{3.6.2}$$

*The "isodual isoriemannian spaces" of Class II are given by*

$$\mathfrak{R}^{d}(x,\hat{g}^{d},\hat{R}^{d}):\; \hat{g}^{d} \;=\; T^{d}(s,\, x,\, \dot{x},\, \ddot{x},\, \mu,\, \tau,\, n,\, ...)\, g(x) = -\, \hat{g} \tag{3.6.3a}$$

$$\hat{R}^{d}, \quad \hat{1}^{d} \;=\; (\, T^{d}\, )^{-1} \;=\; -\, \hat{1} \tag{3.6.3b}$$

$$(x \,\hat{,}\, x)^{d} \;=\; (x,\, T^{d}\, x)\, \hat{1}^{d} \;=\; (T^{d}\, x,\, x)\, \hat{1}^{d} \;\; \in \hat{R}^{d}\, . \tag{3.6.3c}$$

*with "isodual invariant isoseparation"*

$$ds^{\hat{2}d} \;=\; (\, +\, dx^{\mu}\, \hat{g}^{d}{}_{\mu\nu}(s,\, x,\, \dot{x},\, \ddot{x},\, ...)\, dx^{\nu}\, ]\, \hat{1}^{d}\, . \tag{3.6.4}$$

As now familiar, the above definition characterizes four important spaces:

**Riemannian spaces** $\Re(x,g,R)$;
**Isoriemannian spaces** $\hat{\Re}(x,\hat{g},\hat{R})$;
**Isodual Riemannian spaces** $\Re^d(x,g^d,R^d)$; and
**Isodual isoriemannian spaces** $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$.

The conventional Riemannian spaces are (and will remain) the basic spaces for the representation of gravitation in vacuum because, as stressed in Ch. 1, the Riemannian geometry is *exactly* valid for the *exterior gravitational problem*.

The primary physical application of isoriemannian spaces for which they were built in the first place [4,5], is a more adequate representations of the *interior gravitational problem*, such as the study of the interior of a neutron star and, more specifically, of gravitational collapse.

The interior of the systems considered is composed by a large number of particles/wavepackets/charge distributions, not only in condition of total mutual penetration, but also of compression in large numbers into a small region of space. Under these conditions, the emergence of interior nonlinear, nonlocal and nonlagrangian interactions is beyond credible doubts, and so is the lack of exact applicability of the conventional Riemannian spaces in favor of structurally more general spaces.

The understanding (stressed in Ch. 1) is that the Riemannian spaces are indeed *approximately* valid for nonlocal–nonlagrangian conditions.

The isodual Riemannian spaces are the central tool for our characterization of antimatter in vacuum, while the isodual isoriemannian spaces are used for the characterization of antimatter in interior conditions.

The reader can now begin to see the "hidden universe" mentioned in Ch. 1. In fact, *Riemannian and isodual Riemannian spaces share the same separation by construction*. As such, it has not and cannot be identified with Riemannian techniques, although it can be identified and distinguished from the conventional universe via the isotopic techniques.

As one can see, the functional dependence of the isotopic element remains again totally unrestricted under isotopies. Thus, *isoriemannian spaces are bona-fide nonlinear (in the velocities), nonlocal–integral and nonpotential–nonlagrangian generalizations of the conventional spaces.*

Despite these physical differences, the two spaces are geometrically equivalent, as expressible via the following particular case of Theorem 3.2.1:

**Corollary 3.2.1C:** *A given (3+1)–dimensional Riemannian space* $\Re(x,g,R)$ *(isodual space* $\Re^d(x,g^d,R^d)$*) and all its infinitely possible isotopes of Class I* $\hat{\Re}(x,\hat{g},\hat{R})$ *(isotopes of Class II* $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$*) are locally isomorphic.*

We should again recall that this is possible because of (and only under the) joint liftings

$$g \rightarrow Tg, \qquad I \rightarrow \hat{I} = T^{-1}. \tag{3.6.5}$$

which ensures that all the deviations from the Riemannian spaces (velocity–dependent, etc.) are embedded in the isounit of the theory. In particular, the above mechanism permits the use of the integro–differential topology indicated earlier, with considerable simplifications over the conventional, rather complex integral topology.

As an example, a conventional integral generalization of the Riemannian metric $g \rightarrow \hat{g}$ without the joint lifting of the unit would require a full integral geometry, without any local isomorphism, in general, between the old and new spaces.

Corollary 3.2.1C implies that some of the operations in isoriemannian spaces can be conducted in a way geometrically equivalent to the conventional ones, as it was the case for Minkowski and isominkowski spaces. Nevertheless, as we shall see in Ch. 5, the isoriemannian geometry is structurally different than the conventional Riemannian geometry, evidently because of the explicit dependence in the velocities and accelerations.

Note that, we have the Euclidean "space" and Minkowski "space" because their metric is unique, while we have Riemannian "spaces" because we have an infinite number of different (but geometrically equivalent) metrics g. By the same token, we now have an infinite number of isoriemannian spaces for *each* given Riemannian space. This multiple variety is necessary to represent physical reality. In fact, for each given total gravitational mass M, and, thus, exterior metric g, there exist infinitely different interior conditions depending on size, density, temperature, etc. Thus, each given exterior total gravitational mass M admit an infinite number of interior isometrics $\hat{g}$ for the representation of all its possible physical realizations.

This point is important to understand that, under no condition, one should expect isotopic techniques to predict the numerical values of the isotopic element T because this would be exactly the same as requiring Einstein's gravitation to predict the numerical value of the mass.

On the contrary, a beauty and effectiveness of Einstein's gravitation is that it applies for all infinitely possible masses M whose explicit value in a given case must be obtained from experimental measures. By the same token, the physical effectiveness of isotopic theories is that they apply for all infinitely possible interior conditions whose characteristics must be identified via experiments.

In the final analysis, one should remember that no theory, whether conventional or isotopic, can predict the numerical value of its own unit.

It is best to provide some explicit example of isoriemannian metrics which can later on be of guidance in further studies.

Recall that the Riemannian spaces are locally minkowskian. This property is evidently preserved under isotopies, according to which *isoriemannian spaces are locally isominkowskian* , as evident from the preservation of the signature (+, +, +, –),

As shown in ref. [7], the above property essentially implies that the isotopic element T in gravitation is considerably similar to that in isominkowski space. In fact, owing to their characteristics for Class I, isotopic elements can alway (but, as we shall see, not necessarily) diagonalized into the form

$$T = \text{diag.}\,( b_1^2, b_2^2, b_3^2, b_4^2 ),\ b_\mu = b_\mu(s, x, \dot{x}, \ldots) > 0, \tag{3.6.6}$$

An isoriemannian line element is then the following *isoschwartzschild line element* [7]

$$ds^2 = -b_1^2 (1-2\ M\ /r)^{-1} dr^2 - b_2^2 r^2\ d\theta^2 - b_3^2 r^2 \sin^2 \theta d\phi^2 + b_4^2 (1-2M\ /r)\ dt^2 \tag{3.6.7}$$

As one can see, the characteristic b-quantities essentially represent the deformation of the conventional metric expected from nonlinear, nonlocal and nonlagrangian internal effects. In fact, isoline element (3.6.7) can directly represent the decrease of the speed of light within a physical medium while the conventional element evidently cannot because conceived for exterior conditions in vacuum. The novel, experimentally testable and intriguing predictions of isotopies of type (3.6.7) will be studied in Vol. II and III.

Moreover, the characteristic b-quantities can be effectively averaged for all "global" treatments, such as the speed of light throughout our entire atmosphere

$$T^\circ = < T > = \text{diag.}\,( b^\circ{}_1^2, b^\circ{}_2^2, b^\circ{}_3^2, b^\circ{}_4^2 ),\ \ b^\circ{}_\mu = \text{const} > 0\,, \tag{3.6.8}$$

thus setting the foundation for quantitative predictions of interior effects which are verifiable with contemporary experiments (see Vol. III).

## 3.7: ISOTOPIC UNIFICATION OF MINKOWSKI AND RIEMANNIAN SPACES

As indicated in Ch. I, isotopic techniques also have significant applications for *conventional* theories in vacuum. The best way to illustrate this possibility is by showing the new geometrical and physical insights permitted by the isotopies in gravitation. In turn, this can set the foundations for novel possible, such as an unambiguous operator form of conventional gravitation, or a novel approach to singularities.

Let us begin our study with the following now evident property:

***Corollary 3.3.1B:*** *The conventional, (3+1)-dimensional Riemannian spaces R(x,g,R) are isotope* $\hat{E}(x,\hat{\delta},\hat{R})$ *of the 4-dimensional Euclidean space* $E(x,\delta,\mathfrak{R})$ *of Class III characterized by the lifting of the Euclidean metric* $\delta$ = diag. (1, 1, 1, 1) *into the Riemannian metric g*

$$\delta = I_{4\times 4} \Rightarrow T\delta = g, \qquad (3.7.1)$$

*and by the corresponding lifting of the field*

$$R \Rightarrow \hat{R}, \quad \hat{1} = T^{-1} = g^{-1}. \qquad (3.7.2)$$

By recalling Corollary 3.3.1A, we lose any distinction at the abstract isotopic level between Euclidean,Minkowskian and Riemannian spaces of the same dimension. The following additional property also holds

**Corollary 3.2.1B:** *The conventional (3+1)-dimensional Riemannian spaces* $\mathfrak{R}(x,g,R)$ *can be reinterpreted as isotopes* $\hat{M}(x,\hat{\eta},\hat{R})$ *of the Minkowski space* $M(x,\eta,\mathfrak{R})$ *of Class I characterized by the lifting of the metric*

$$\eta = \text{diag.}\,(1, 1, 1, -1) \Rightarrow T(x)\,\eta = g(x), \qquad (3.7.3)$$

*and of the field*

$$R \Rightarrow \hat{R} \approx R\hat{1}, \quad \hat{1} = [\,T(x)\,]^{-1}. \qquad (3.7,4)$$

In fact, all possible Riemannian spaces must verify the isotopic decomposition of the metric

$$g(x) = T(x)\,\eta\ , \ T(x) > 0, \qquad (3.7.5)$$

where the positive-definiteness is evidently due to the locally Minkowskian character. The above reinterpretation of Riemannian spaces then follows.

A simple example is provided precisely by the Schwartzschild metric in spherical polar coordinates

$$ds^2 = (1-2M/r)^{-1}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + (1-2M/r)\,dt^2, \quad (3.7.6)$$

which exhibits a manifest isotopic structure with respect to the Minkowski space with characteristic b-functions

$$T = \text{diag.}\,\{\,(1 - 2M/r)^{-1},\ r^2,\ r^2 \sin^2\theta,\ (1 - 2M/r)\,\} \quad (3.7.7)$$

The above properties imply that *the transition from relativistic to gravitational formulations is an isotopy* [5]. This concept is at the foundations of the study we shall conduct in Ch. 5 of the *isosymmetries of conventional gravitational theories*, the isotopic formulation of gravitational singularities and other aspects.

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# 4: LIE-ISOTOPIC THEORY

## 4.1: STATEMENT OF THE PROBLEM

Lie's theory with the celebrated product

$$[A , B] = AB - BA, \qquad (4.1.1)$$

where AB is the conventional associative product, is the true structural foundation of quantum mechanics. In fact, most quantum mechanical laws, such as unitary time evolution or Heisenberg's equation, can be simply "read–off" Lie's theory via a mere interpretation of its generators as operators on a Hilbert space.

The isotopic generalization of Lie's theory under the name of *Lie–isotopic theory* was submitted by the author in memoir [1] of 1978 with basic product

$$[A \hat{,} B] = A * B - B * A = A T B - B T A = \qquad (4.1.2)$$

$$= AT(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots)B - BT(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots)A,$$

because it implies a step–by–step generalization of quantum mechanics with new dynamical equations, new interactions represented by the isotopic operator T, and new notions of symmetries. The existence of the new mechanics was confirmed in memoir [2] of the same year, and proposed for study under the name of *hadronic mechanics.*[20]

The isotopic content of memoir [1] was then developed in monographs [3,4] and in the initial papers [5–9] [21]. Additional structural advances in the Lie–isotopic theory were made in memoir [16–18] of 1989, which were then presented in the mathematical literature in ref.s [19–20] and developed in monographs [21,22].

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[20] It should be noted that the Lie–isotopic theory was submitted as a particular case of the yet more general Lie–admissible theory reviewed in Ch. 7.

[21] The more general Lie–admissible theory was developed in monographs [11,12] and initial papers [13–15].

**THE STRUCTURE OF LIE'S THEORY**

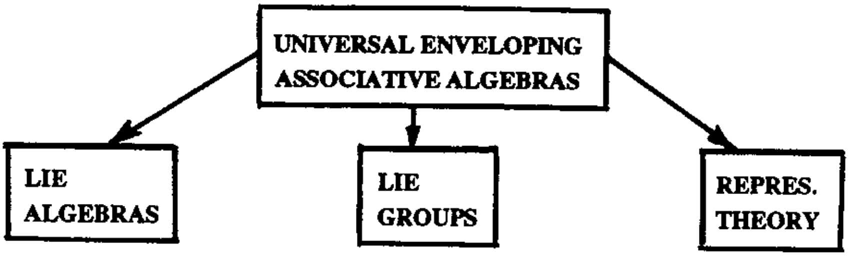



FIGURE 4.1.1: *Lie's theory* is today an articulated body of inter-related methods in algebras, geometries, functional analysis and other fields virtually encompassing all branches of mathematics [31–33]. Its most fundamental structure is the *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L [31] with conventional associative product AB among vector-fields A, B on a cotangent bundle or operators on a Hilbert space. In fact, the knowledge of $\xi$ permits the construction of: the Lie algebra L as the attached antisymmetric algebra $\xi \sim [\xi(L)]^-$; the corresponding connected Lie group G via exponentiations in $\xi(L)$; the representation theory; etc. In memoir [1] this author submitted the elements of the *Lie-isotopic theory* conceived as a step-by-step isotopic generalization of the above formulation of Lie theory, beginning with the isotopies of universal enveloping algebras, and then passing to the isotopies of Lie's algebras and groups, the isotopies of the representation theory, etc. The dominant motivation of the proposal was of purely physical character and consist in: a) achieving methods for the construction of *nonlinear-nonlocal-noncanonical symmetries for interior dynamical problems;* b) in such a way to preserve the abstract axioms of the contemporary *linear-local-canonical symmetries of exterior dynamical problems;* c) so as to achieve a unity of mathematical and physical thought admitting of both, exterior and interior problems merely expressed in different realizations.

Physical contributions on the Lie-isotopic theory by various authors are numerous. An independent review of contributions up to 1990 of primarily physical character is the monograph by Aringazin, Jannussis, Lopez, Nishioka and Veljanoski [30]. An update to include subsequent contributions is presented in Vol.s II and III. Besides the presentations at the meetings listed in Sect. 1.4, it may be significant to indicate the following physical contributions at meetings during the summer of 1993:

> *International Workshop on Symmetry Methods in Physics,* JINR, Dubna, Russia, July 1993, with a presentation of: nonlinear-nonlocal-noncanonical isosymmetries [40]; their application to the apparent possibility of building unstable hadrons via chemical synthesis of lighter hadrons and therefore controlling their artificial disintegration [41]; axiomatic isotopic reformulation of

the so-called q-deformations of Lie symmetry [42]; isotopic formulation of creation-annihilation operators [43]; and others;

> *International Workshop on Secrets of Quantum Logic and Intuition,* JINR, Dubna, July 1993, with a systematic study of the isotopies of the Poincaré symmetry and their applications to interior dynamical problems [44];

> *Third International Wigner Symposium,* Oxford University, England, September 1993, with a presentation of isosymmetries and Klimyk's rule for their isorepresentations (see Sect. 4.7); the isotopic reconstruction of the exact Lorentz's symmetry under Q-operator deformations; applications to reconstruction of the exact isospin symmetry in nuclear physics and other topics [45];

> *International Symposium Deuteron 1993,* JINR, Dubna, September 1993, with the presentation of the apparently first exact representation of the magnetic moment of the deuteron and of other few-body nuclei via the isotopic representation of nucleons as nonspherical-deformable charge distributions [46];

> *XVI International Conference in High Energy Physics,* IHEP, Protvino, Russia, September 1993, with a presentation of the isotopies of quark theories to achieve an exact confinement as well as convergent isoperturbative series [47], and the isotopies of Riemann for interior gravitational problems [48];

> *VI Trilateral Seminar in High Temperature Superconductivity,* JINR, Dubna, September 1993, with the application of the isotopic techniques for a quantitative representation of the attractive interactions among the electron pairing in superconductivity [49];

> *International Conference on the Frontiers of Physics,* Olympia, Greece, September 1993, with a quantitative representation of the anomalies in redshift and blueshift of quasars via the isotopies of conventional geometries for interior physical media [50], an axiomatic theory of complex time to represent open systems [51]; and other topics.

By comparison, pure mathematical studies on the Lie-isotopic theory (as referred to in Fig. 4.1.1) have been conspicuously absent until recently[22]. In fact, to the author's best knowledge, the first contribution in a mathematical Journal mentioning the words "Lie-isotopic algebras" is the review by (the physicists) Aringazin *et al.* [23] of 1990, some twelve years following their original proposal [1] in a physics journal. The only additional studies on Lie-isotopic theory appeared in the mathematical literature prior to the summer of 1993 are memoirs [19,20].

This situation is now changing rapidly. In fact, comprehensive mathematical studies in the Lie-isotopic theory are today available by the mathematicians Sourlas and Tsagas in monograph [24] and papers [25]. Other comprehensive studies, this time with emphasis on nonassociative algebras, are presented by the mathematicians Lôhmus, Paal and Sorgsepp in monograph [39]. Studies in the isorepresentation theory have been conducted by Lopez [26], as well

[22] This is not the case for mathematical studies on Lie-admissible algebras which, as we shall see in Ch. 7, have been quite numerous.

as by the mathematician Klimyk and this author [52]. Important mathematical advances have been reached by Kadeisvili in papers [27,28] on the structure of the Lie–isotopic algebras and groups, and in monograph [29], the latter with emphasis on the geometric profile. A study on isonumbers and isofields was presented by Kamiya [53] at the *International Workshop on Symmetry Methods in Physics*, JINR, Dubna, July 1993. Additional articles by pure mathematicians are forthcoming.

The difficulties in a first inspection (and appraisal) of the Lie–isotopic theory are, again, of mathematical nature. They are due to the understandable expectation that the current formulation of Lie's theory (see, e.g., ref.s [31–33] and literature quoted therein) being abstract, encompasses all possible realizations, thus including the isotopic formulation.

It is important to understand beginning with these introductory words that the Lie and Lie–isotopic theories are structurally inequivalent for the following reasons:

1) The map interconnecting Lie product (4.1.1) and its Lie–isotopic generalization (4.1.2) is *nonunitary*,

$$U\,U^{\dagger} = \hat{1} \neq I \tag{4.1.3a}$$

$$U\,[A, B]\,U^{\dagger} = U\,(A\,B - B\,A)\,U^{\dagger} = A'\,T\,B' - B'\,T\,A' = [A' \hat{,} B'], \tag{4.1.3b}$$

$$A' = U\,A\,U^{\dagger}\;,\; B' = U\,B\,U^{\dagger}, \tag{4.1.3c}$$

with isotopic element T given precisely by the inverse of $\hat{1}$ as needed for a correct isotopic formulation,

$$T = (U\,U^{\dagger})^{-1} = \hat{1}^{-1}; \tag{4.1.4}$$

2) Lie's theory is linear–local–canonical in its structure, while the Lie–isotopic theory has a nonlinear–nonlocal–noncanonical structure (when projected in the original carrier space, see Sect. 4.2) as a necessary condition to be directly applicable to interior dynamical problems. This implies a generalization under isotopies of basic symmetries of contemporary physics, such as rotations, Lorentz transformations, etc. into the most general possible nonlinear–nonlocal–noncanonical forms;

3) The isotopies alter conventional weights and, in general, the spectra of eigenvalues of the conventional Lie theory. Let X be a diagonal generator of a Lie algebra with spectrum of eigenvalues $S^{\circ}$ with respect to a basis $|\,b>$. Then, under isotopies the *same* generator X admits a *different* spectrum s, according to the lifting

$$X\,|b\rangle = S^\circ|b\rangle \;\Rightarrow\; X*|\hat{b}\rangle = X\,T\,|\hat{b}\rangle = S\,|\hat{b}\rangle\,,\; S \neq S^\circ\,; \quad (4.1.5)$$

4) The isotopies map Cartan's tensor and other structural elements of Lie's theory into suitable integral forms;
5) The topology of the current formulation of Lie's theory is notoriously local-differential, while that of the covering Lie isotopic theory is integro-differential (Fig. 1.1.4);

and other reasons.

In this chapter, which has been written for physicists, we outline only those aspects of the Lie-isotopic theory that are essential for the physical applications of Volumes II and III. Unless otherwise indicated, the presentation is specifically intended for the *Lie-isotopic theory of Class I* (that with isounits $\hat{1}$ which are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite, see Sect.s 1.4 and 2.3). An outline of the *Lie-isotopic theory of Class II* (with negative-definite isounits) is also presented because it is important for our subsequent study of antiparticles. We shall also study a few aspects of the *Lie-isotopic theory of Class III* because it unifies those of Classes I and II. The *Lie-isotopic theories of Classes IV* (singular isounits) *and V* (generic, e.g., discrete, isounits) are vastly unknown at this writing and will be discussed only briefly.

During the course of our analysis we shall assume that: all Lie algebras are finite dimensional; all Lie algebras basis and corresponding parameters are ordered; and all fields have characteristic zero (Def. 2.3.1). Mathematically inclined readers are suggested to consult the above quoted mathematical literature, e.g., ref.s [24,27,39].

A clear understanding is that the Lie-isotopic theory is still at its first infancy, particularly when compared to the current status of Lie's theory with vast mathematical and physical contributions by a large number of mathematicians and physicists for over one century (see ref.s [31–33] and related literature).

A technical knowledge of the conventional Lie theory is an evident pre-requisite for the understanding of this chapter. A prior reading of Appendix 4.A on basic notions of algebras and their isotopies is recommendable.

## 4.2: ISOTRANSFORMATIONS AND THEIR ISODUALS

An important notion requiring a clear understanding is that the Lie-isotopic theory is nonlinear, nonlocal-integral and noncanonical-nonhamiltonian only when projected in the original carrier space because, when treated in its appropriate isospace, it verifies the isotopic axioms of linearity, locality and canonicity and, for this reason, it is called an *isolinear, isolocal and isocanonical.*

Let S(x,R) be a conventional, real vector space with local coordinates x over the reals R(n,+,×), and let

$$x' = A(w)\,x\,, \quad w \in F\,, \quad x'^{\,t} = x'\ A^t(w) \tag{4.2.1}$$

be a conventional *right or left, linear, local and canonical transformation* on S(x,R), where t denotes transpose.

The isotopic lifting $S(x,R) \Rightarrow \hat{S}(x,\hat{R})$ studied in the preceding chapter requires a corresponding necessary isotopy of the transformation theory. In fact, it is instructive for the interested reader to verify that the application of transformations (4.2.1) to the isospace $\hat{S}(x,\hat{R})$ implies the loss of linearity, transitivity and other basic properties.

For these and other reasons, the author submitted in the original proposals [1,2] the isotopy of the transformation theory, today known as *isotransformation theory* [24,29,30], which is characterized by the *isotransformations*

$$x' = \hat{A}(\hat{w}) * x = \hat{A}(\hat{w})\,T\,x, \quad x'^t = x^t * \hat{A}^t(\hat{w}) = x^t\,T\,\hat{A}^t(\hat{w})\,, \tag{4.2.2a}$$

$$T = T^\dagger = \text{fixed}, \quad x \in \hat{S}(x,\hat{R}), \quad \hat{w} \in \hat{R}(\hat{n},+,*), \quad \hat{1} = T^{-1}. \tag{4.2.2b}$$

where the isotopic element T is here assumed to be of Kadeisvili's Class I

We can say that conventional transformations (4.2.1) are characterized by the *right modular associative action* Ax of A on $S(x,\hat{R})$. The isotransformations are then characterized by the *right isomodular associative action* action A*x of A on $\hat{S}(x,\hat{R})$. In fact, the preservation of the associativity is established by the properties

$$A * B * C * x = A * (B * C * x) = (A * B * C) * x\,, \text{etc.} \tag{4.2.3}$$

while the preservation of the modular character under isotopies is discussed in Sect. 4.7.

The most dominant aspect in the transition from transformations (4.2.1) to isotransformations (4.2.2) is that, while the former are linear, local and canonical, the latter are the most general known transformations that are *nonlinear* in the coordinates as well as other quantities and their derivatives of arbitrary order, *nonlocal–integral* in all these quantities, and *noncanonical*. In fact, from the unrestricted nature of the isotopic element T, the projection of isotransformation (4.2.2) in the original space S(x, R) reads

$$x' = \hat{A}(\hat{w})\, T(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, n, .....)\, x . \tag{4.2.4}$$

In turn, the above features are crucial for the achievement of nonlinear, nonlocal and noncanonical generalizations of conventional space-time symmetries as necessary for the interior problem.

But transformations (4.2.1) and their covering (4.2.2) coincide at the abstract level where we have no distinction between the modular action "Ax" and its isotopic form "A*x". We therefore have the following

**Proposition 4.2.1** [22]: *Isotransformations (4.2.2) are "isolinear" because they verify the conditions of linearity in isospaces,*

$$\hat{A} * (\hat{a} * x + \hat{b} * y) = \hat{a} * (\hat{A} * x) + \hat{b} * (\hat{A} * y), \tag{4.2.5a}$$

$$\forall\, x, y \in \hat{S}(x, \hat{R}), \quad \hat{a}, \hat{b} \in \hat{R} . \tag{4.2.5b}$$

*while coinciding with linear transformations at the abstract level. Isotransformations (4.2.2) are also "isolocal" because they are defined at the local point x in isospace thus coinciding with conventional local transformations at the abstract level. Finally, isotransformations (4.2.2) are "isocanonical", in the sense that they are derivable from a first-order principle in isospace (see Ch. 5 and Vol. II) thus coinciding with conventional canonical theories at the abstract level.*

The understanding of the above notions of isolinearity, isolocality and isocanonicity is crucial for an understanding of the Lie-isotopic theory and, thus, of hadronic mechanics.

In particular, the abstract identity of the conventional and isotopic transformations as compared to their structural differences is the very essence of isotopies.

The origin of these advances lies in the basic notion of isonumbers because it is "hidden" in their multiplication. In fact, if one assumes the traditional expression "two multiplied by two equals four", one implicitly assumes, for mathematical consistence, that the multiplicative unit is the trivial value 1, and the multiplication is the trivial expression "2×2". Then, one must consequently assume, also for mathematical consistency, that the operation of multiplication in transformation (4.2.1), i.e., the expression "Ax", is the same multiplication as that of the numbers, i.e.,

$$A\, x \equiv A \times x , \quad \times \in R(n,+,\times). \tag{4.2.6}$$

because the carrier space $\dot{S}(x, R)$ and its inner automorphisms are defined on R.

We have shown in Ch. 2 that the assumption "2 × 2 = 4" is un-necessarily restrictive because the basic unit can be an integral quantity $\hat{1}$, and the product of numbers can be consequently generalized into an axiom-preserving form. This implies that the isotopic expression "A*x" is defined via the same multiplication of the base isofield

$$A * x \;=\; A\,T\,x \;\;, \quad * \in \hat{R}(\hat{n},+,*). \tag{4.2.7}$$

The preservation of the abstract identity between transformations (4.2.6) and their isotopic coverings (4.2.7) is due to the fact that *abstract theories cannot distinguish between different values of the unit when positive-definite*. Thus, any isounit $\hat{1}$ will automatically coincide with the number $1$ at the abstract level,

$$1_{\text{abstract level}} \equiv \hat{1}\,. \tag{4.2.8}$$

The same occurrence then holds for the transformations

$$A \times x \;\equiv_{\text{abstract level}}\; A * x\,. \tag{4.2.9}$$

To put it differently, conventional transformations on $S(x,R)$ and isotopic transformations $\hat{S}(x,\hat{R})$ are geometrically equivalent. Their rather profound physical differences occur only when the isotransformations on $\hat{S}(x,\hat{R})$ are projected in the original space $S(x, R)$.

The notion of isolocality is also expressed by the underlying integro-differential topology of isotopic theory (Ch. I), in which the quantity $x$ of Eq.s (4.2.2) represents the trajectory of the center of mass, thus remaining fully local. The integral terms are corrective terms embedded in the unit to represent the contribution to the trajectory due to the interior conditions (extended character with motion within a physical medium).

Finally, the isocanonical character will be best understood in Vol. II when studying the physical foundations. At this stage we can say that a situation fully analogous to the linear and local profiles emerges also from the viewpoint of the Hamiltonian theory. In fact, the systems considered are nonhamiltonian by conception and, thus, nonderivable from a canonical variational principle in an ordinary space. However, the same systems are indeed derivable from a first-order variational principle in isospace.

Along these lines, the basic objectives of hadronic mechanics, that is, of the "isotopies of quantum mechanics", are a "nonlinear-nonlocal-noncanonical treatment of the strong interactions" although in an "identical isolinear, isolocal and isocanonical form".

The following additional property is mathematically trivial, but it carriers important physical implications.

**Proposition 4.2.2** [1,2]: *Given a nonlinear, nonlocal and noncanonical* *transformation on a vector space* S(x,R)

$$x' = X(w, x,...)\, x\,, \quad x \in S(x,R), \quad w \in R(n.+,\times), \tag{4.2.10}$$

*on a vector space* S(x,R), *there always exist an infinite variety of isotopies of the base field,* $R(n,+,\times) \Rightarrow \hat{R}(\hat{n},+,*)$, *and corresponding isotopies of the space* $S(x,R) \Rightarrow \hat{S}(x,\hat{R})$ *under which the transformation can be "identically" rewritten in an isolinear, isolocal and isocanonical form*

$$x' = X(x, ...)\, x \;\; \equiv \;\; \hat{A}(\hat{w}) * x\,, \quad T = \hat{A}^{-1}X\,. \tag{4.2.11}$$

The understanding of the above additional property is also important for an understanding of hadronic mechanics. In fact, the first role of isotopic techniques is that of generalizing conventional linear, local and canonical theories into less trivial nonlinear, nonlocal and noncanonical generalizations. The subsequent role is then that of *turning conventionally nonlinear, nonlocal and noncanonical theories into "identical" isolinear, isolocal and isocanonical forms,* with evident simplification of their treatment.

**Definition 4.2.2** [8,9]: *The "isodual isotransformations" of Class II are given by the image of isotransformations (4.2.2) under isoduality, i.e., are defined on the isodual isospace* $\hat{S}^d(x, \hat{R}^d)$,

$$x' = \hat{A}^d\,(\hat{w}^d) *^d x = -\hat{A}^d\,(\hat{w}^d) * x\,, \quad x \in \hat{S}^d(x,\hat{R}^d), \;\; \hat{w}^d \in \hat{R}^d(\hat{n},+,*^d) \tag{4.2.12a}$$

$$x^{t\,\prime} = x *^d \hat{A}^{t\,d}\,(\hat{w}^d) = -x * \hat{A}^{t\,d}\,(\hat{w}^d) \tag{4.2.12b}$$

*where* $\hat{A}^t$ *and* $\hat{A}^{t\,d}$ *will be identified later on in this chapter.*

Isodual isotransformations characterize the isodual Lie theory which, in turn, characterizes the isodual symmetries for our treatment of antiparticles, as we shall see.

The formulation of the Lie–isotopic theory presented in this chapter is in reality the *Lie–isotopic transformation theory,* namely, it is specifically developed for isotransformations as needed in hadronic mechanics. At any rate, the formulation of the theory via topological Lie–isotopic groups is still lacking at this writing.

## 4.3: ISOENVELOPES AND THEIR ISODUALS

As well known (see, e.g., ref.s [31–33]), the contemporary Lie theory is constructed with respect to a conventional unit, e.g., the N–dimensional unit matrix $I = \mathrm{diag.}(1, 1, ..., 1)$. The central idea of the Lie–isotopic theory [1,2] is the construction of the theory with respect to the most general possible isounit $\hat{1}$ with isolinear, isolocal and isocanonical dependence in all possible local variables and quantities. The lifting of the unit $I \Rightarrow \hat{1}$ therefore implies a corresponding compatible lifting of all branches of the conventional Lie theory (Sect. 1).

From the very outset one can therefore see the reachness of the Lie–isotopic theory as compared to the conventional theory because, as it is the case for isofields and isospaces, we have Kadeisvili's classification [28]

**Lie isotopic theory** properly speaking (Class I);
**Isodual Lie-isotopic theory** (Class II);
**Indefinite Lie-isotopic theory** (Class III);
**Singular Lie-isotopic theory** (Class IV);
**General Lie-isotopic theory** (Class V).

which applies to each of the branches of the generalized theory, thus resulting in isoenveloping algebras of Classes I–V, Lie–isotopic algebras of Classes I–V, Lie–isotopic groups of Classes I–V, isorepresentations of Classes I–V, etc.[23], each of which can be of isocharacteristic zero or p (Sect. 2.3).

Moreover, the isotopies imply the possibility of introducing fundamentally novel notions, such as *"Lie's theory on a singular unit"*, or formulating the *"Lie–isotopic theory of discrete groups over continuously varying units"*, or, viceversa, studying the *"Lie–isotopic theory of continuous groups over discrete units"*, etc.

The isotopies of enveloping algebras will be formulated in this section for Class III over a field of characteristic zero to unify the formulations of Classes I and II. As we shall see, this permits the unification of compact and noncompact groups of the same dimension of Cartan's classification into one single isotope.

To begin, let $\xi = \xi(L)$ be a *universal enveloping associative algebra* of an N–dimensional Lie algebra L (see, e.g., ref. [31] and Fig. 4.3.1) with generic elements A, B, C,... , trivial associative product AB (say, of matrices) and unit matrix in N–dimension I. Their isotopes $\hat{\xi}$ where first introduced in the original proposal [1] (see Fig. 4.3.1 for their definition) under the name of *universal enveloping isoassociative envelopes*, also called *isoenvelopes* for short. They coincide with $\xi$ as vector spaces but are equipped with the isoproduct $A*B$ so as to admit $\hat{1}$ as the correct (right and left) unit

$$\hat{\xi} \;:\; A * B = ATB, \quad T \;\; \text{fixed}, \tag{4.3.1a}$$

$$\hat{1} * A = A * \hat{1} \equiv A \qquad \forall A \in \hat{\xi}, \quad \hat{1} = T^{-1}. \tag{4.3.1b}$$

[23] The reader should keep in mind that, as originally presented in memoir [1], all these formulations are still particular cases of the more general Lie–admissible theory (Ch. 7).

Let the (ordered) basis of L be given by $\{X_k\}$, k = 1, 2, ..., N, over a field $F(a,+,\times)$, and let the infinite-dimensional basis of $\xi(L)$ be given by the *Poincaré-Birkhoff-Witt theorem* [31]

$$I\,,\;\; X_k \qquad X_i\,X_j \quad (i \leq j), \qquad X_i\,X_j\,X_k \quad (i \leq j \leq k),\;\ldots\ldots \tag{4.3.2}$$

where one recognizes the familiar *standard monomials.*

A fundamental property from which most of the Lie-isotopic theory and hadronic mechanics can be derived is the following

**Theorem 4.3.1- Isotopic generalization of the Poincare´-Birkhoff-Witt Theorem** [1]: *The cosets of* $\hat{1}$ *and the standard, isotopically mapped monomials form an infinite-dimensional basis of the universal enveloping isoassociative algebra* $\hat{\xi}(L)$ *of a Lie algebra* L *of Class III*

$$\hat{1}\,,\;\; X_k \qquad X_i * X_j \quad (i \leq j), \qquad X_i * X_j * X_k \quad (i \leq j \leq k),\;\ldots\ldots \tag{4.3.3}$$

A detailed proof can be found in ref. [4], pp. 154–163, or ref. [24], pp. 74–93, and it is not repeated here for brevity (although its knowledge is assumed for more advanced treatments).

Algebraically, the above theorem essentially expresses the property that the isotopic character of the lifting of the basic product $AB \Rightarrow A*B$, i.e.,

$$AB : (A\,B)\,C = A\,(B\,C) \;\Rightarrow\; A*B : (A*B)*C \equiv A*(B*C)\,, \tag{4.3.4}$$

implies the existence of consistent isotopies of the basis (4.3.2).

In turn, the existence of such isobasis has fundamental mathematical and physical implications. Recall that the conventional exponentiation is defined precisely via a power series expansions in $\xi$

$$e_{\xi}^{iwX} = I + (iwX)\,/\,1! + (iwX) \times (iwX)\,/\,2! + \ldots\ldots, \quad w \in F(a,+,\times)\,. \tag{4.3.5}$$

The above exponentiation is then inapplicable under isotopies because the quantity $I$ is no longer the basic unit of the theory, the conventional product $\times$ has no mathematical or physical meaning, etc.

In turn, this implies that all quantum mechanical quantities depending on the conventional exponentiation, such as time evolution, unitary groups, Dirac's delta distributions, Fourier transforms, Gaussian, etc. have no mathematical or physical meaning under isotopies and must be suitably lifted.

Isobasis (4.3.3) then permits the following

**Corollary 4.3.1.A:** *The "isoexponentiation" of an element* X *in* $\hat{\xi}$ *via* *isobasis (4.3.3) over an isofield $\hat{F}(\hat{a},+,*)$ is given by*

$$e_\xi^{\,i\,\hat{w}\,*\,X} \equiv e_\xi^{\,i\,w\,X} = \hat{1} + (iwX)/1! + (iwX) * (iwX)/2! + \ldots\ldots =$$

$$= \hat{1}\,\{e^{iwTX}\} = \{e^{iXTw}\}\,\hat{1}, \qquad \hat{w} \in \hat{F}(\hat{a},+,*). \tag{4.3.6}$$

We see in this way the nontriviality of the isotopies of Lie's theory, which is clearly expressed by the appearance of the nonlinear, nonlocal and noncanonical isotopic element T directly in the exponent of isoexponentiations (4.3.6). This is sufficient to see the emergence of the isotopic generalizations indicated earlier.

Note that isoexponentiation (4.3.6) is defined in terms of *conventional* generators X, because of the preservation of the basis under isotopies, and *isotopic* parameters $\hat{w} = w\hat{1}$ because the acting field is the isotopic one $\hat{F}(\hat{w},+,*)$. However, on practical grounds, we can express the isoexponentiation also with respect to the conventional parameters w X because of the identities (Ch. 2)

$$\hat{w} * X \equiv w\,X. \tag{4.3.7}$$

One should keep in mind the *uniqueness of isoexponentiation* (4.3.6). It originates from a crucial requirement of the Poincaré–Birkhoff–Witt theorem, the existence of a well defined left and right unit [31] which, in turn, implies the uniqueness of the isobasis (4.3.3). This property can then be compared with the lack of uniqueness of the exponentiations in other theories. As an example we shall study in Vol. II, the so-called q-deformations, which do not possess a unique exponentiation because they do not possess a unit [442].

By recalling the results of the preceding analysis on isodual fields and isodual spaces (particularly Proposition 3.2.1), we can see that the *isodual isoenvelopes* $\xi^d$ [8,9] are characterized by: the *isodual basis* and the *isodual parameters*

$$X^d{}_k = -X_k, \quad \hat{w}^d = w\,\hat{1}^d = -\hat{w}. \tag{4.3.8}$$

**Corollary 4.3.1B:** *The "isodual isoexponentiation" is the isodual image of isoexponentiation (4.3.6) on the isodual isofield $\hat{F}^d(\hat{w}^d,+,*^d)$*

$$e_{\xi^d}^{\,i^d\,\hat{w}^d\,*^d\,X^d} = -\{e^{\,i\,X\,T\,w}\}\,\hat{1} \tag{4.3.9}$$

Note that the preservation of the sign in the exponent is only apparent, i.e., when projected in an isofield, because, when properly written in the isodual isofield, one can use the expression

$$e_{\xi^d}^{-i^d \hat{w}^d *^d X} \equiv -\{e^{-iXTw^d}\}^\dagger \quad (4.3.10)$$

Isodual isoexponentiations play an important role for the construction of the isodual isosymmetries for antiparticles.

It is easy to see that Theorem 4.3.1 holds for envelopes of Class III, as originally formulated [1], thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, Theorem 4.3.1 was conceived to unify with one single Lie algebra basis $X_k$, but arbitrary isotopies in the envelope $\hat{\xi}(L)$, nonisomorphic compact and noncompact algebras of the same dimension N.

To clarify this aspect, recall [31] that a conventional envelope $\xi(L)$ represents only *one* algebra (up to local isomorphism),

$$L \approx [\xi(L)]^- . \quad (4.3.11)$$

On the contrary, one isoenvelope $\hat{\xi}(L)$ of Class III represents a family of generally nonisomorphic Lie algebras $\hat{L}$ as the attached antisymmetric algebras

$$\hat{L} \approx [\hat{\xi}(L)]^- . \quad (4.3.12)$$

Theorem 4.3.1 therefore permits a unified formulation of all Lie-isotopic algebras $\hat{L}$ of the same dimensions. This implies, the reduction of compact and noncompact structures of the same dimension to only one isotopic structure, and, for each given structure, the reduction of all possible linear and nonlinear, local and nonlocal, canonical and noncanonical realizations to one primitive algebraic notion, the isoenvelope $\hat{\xi}(L)$ (see Fig. 4.3.1 below for more details).

The above unification was illustrated in the original proposal [1] with an example that is still valid today. Consider the conventional Lie algebra so(3) of the rotational group on the Euclidean space E(r,δ,R) with unit I = diag. (1, 1, 1). The adjoint representation of so(3) is given by the familiar expressions

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.3.13)$$

The universal enveloping associative algebra $\xi(so(3))$ is then characterized by the unique infinite-dimensional basis from the conventional Poincaré-Birkhoff-Witt theorem [31]

$$I, \quad J_k, \quad J_i J_j \ (i \leq j), \quad J_i J_j J_k \quad (j_i \leq j \leq k), \dots \quad (4.3.14)$$

and characterizes only one algebra as the attached antisymmetric algebra

$$[\,\xi\,(so(3))\,]^- \approx so(3). \tag{4.3.15}$$

The isotopies $\hat{\xi}(so(3))$ of the envelope $\xi(so(3))$ of Class III are characterized by the the lifting of the basic carrier space E(r,δ,R) into the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ with isometric, isotopic element and isounit

$$\hat{\delta} = T\delta\,,\quad T = \mathrm{diag.}\,(g_{11}, g_{22}, g_{33})\,,\quad \hat{1} = \mathrm{diag.}\,(g_{11}^{-1}, g_{22}^{-1}, g_{33}^{-1})\,, \tag{4.3.16}$$

where the characteristic quantities $g_{kk}$ are real-valued, non-null but arbitrary functions of the local coordinates $g_{kk}(t, r, \dot{r}, \ddot{r}, ...)$ which, as such, can be either positive or negative. From Theorem 4.3.1, the isoenvelope $\hat{\xi}(so(3))$ is then characterized by the original generators (4.3.13), although expressed now in terms of the isoassociative product $J_i{*}J_j = J_iTJ_j$ and isounit $\hat{1}$ with unique infinite-dimensional basis from Theorem 4.3.1

$$\hat{1} = T^{-1},\quad J_k\,,\quad J_i\,T\,J_j\ (i \leq j)\,,\quad J_i\,T\,J_j\,T\,J_k\quad (\,j_i \leq j \leq k\,)\,, .... \tag{4.3.17}$$

It is now easy to see that the algebra characterized by the attached antisymmetric part of $\hat{\xi}(so(3))$ *is not unique*, evidently because it depends on the explicit values of the characteristic quantities $g_{kk}$. It was shown in ref.s [1,9] that the isoenvelope $\hat{\xi}(so(3))$ unifies: all possible compact and noncompact three-dimensional Lie algebra of Cartan classification, the algebras so(3) and so(2.1); all their infinitely possible isotopes $\hat{so}(3)$ and $\hat{so}(2.1)$; the compact and noncompact isodual algebras $so^d(3)$ and $so^d(3)$; as well as all their infinitely possible isodual isotopes $\hat{so}^d(3)$ and $\hat{so}^d(2.1)$, according to the classification

$$[\,\hat{\xi}\,(so\,(3)\,]^- : \quad \begin{array}{l} so(3) \text{ for } T = \mathrm{diag.}\,(1,\,1,\,1); \\ so(2.1) \text{ for } T = \mathrm{diag.}\,(1,\,-1,\,1); \\ \hat{so}(3) \text{ for sign. } T = (+,\,+,\,+); \\ \hat{so}(2.1) \text{ for sign. } T = (+,\,-,\,+); \\ so^d(3) \text{ for } T = (-1,\,-1,\,-1); \\ so^d(2.1) \text{ for } T = \mathrm{diag.}\,(-1,\,+1,\,-1); \\ \hat{so}^d(3) \text{ for sign. } T = (-,\,-,\,-); \\ \hat{so}^d(2.1) \text{ for sign. } T = (-,\,+,\,-). \end{array} \tag{4.3.18}$$

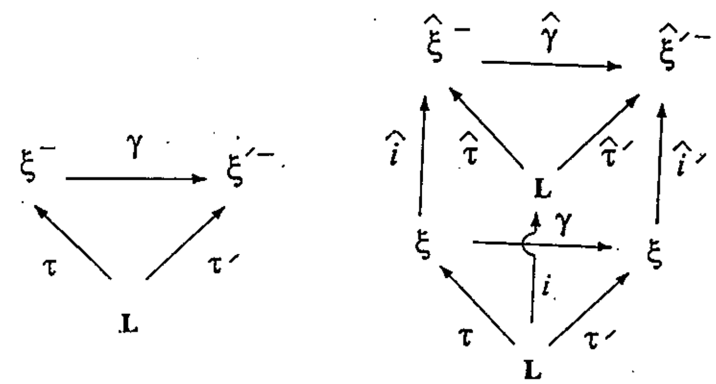
The explicit form of the Lie-isotopic algebras will be studied in the next section; an illustration of the isoexponentiation will be provided in Sect. 4.5; and a first example of physical applications will be given in Sect. 4.7. The isotopes and isoduals of so(3) will then be studied in detail in Vol. II their applications in Vol. III. The unification of all six-dimensional simple algebras is studied in Ch. II.8.

Whenever needed for clarity, isoenvelopes will be denoted with the symbol

$\hat{\xi}_T$ identifying the selected isotopic element T.

As concluding remarks, note that the lifting $\xi \Rightarrow \hat{\xi}$ is *necessary* under the isotopy of the unit because, in general, $\hat{1}A \neq A\hat{1} \neq A$.

## UNIVERSAL ISOASSOCIATIVE ENVELOPING ALGEBRAS



(a) (b)

FIGURE 4.3.1: The *universal enveloping associative algebra* $\xi(L)$ of a Lie algebra L [31] is the set $(\xi, \tau)$ where $\xi$ is an associative algebra and $\tau$ is a homomorphism of L into the antisymmetric algebra $\xi^-$ attached to $\xi$ such that: if $\xi'$ is another associative algebra and $\tau'$ is another homomorphism of L into $\xi'^-$ a unique isomorphism $\gamma$ between $\xi$ and $\xi'$ exists in such a way that the diagram (a) above is commutative. The above definition evidently expresses the *uniqueness* of the Lie algebra L (up to local isomorphisms) characterized by its universal envelopes $\xi(L)$.

With reference to diagram (b) above, the *universal enveloping isoassociative algebra* $\hat{\xi}(\hat{L})$ of a Lie algebra L was introduced [1] as the set $\{(\xi, \tau), i, \hat{\xi}, \hat{\tau}\}$ where: $(\xi, \tau)$ is a conventional envelope of L; i is an isotopic mapping $L \Rightarrow iL = \hat{L} \neq L$; $\hat{\xi}$ is an associative algebra generally nonisomorphic to $\xi$; $\hat{\tau}$ is a homomorphism of $\hat{L}$ into $\hat{\xi}^-$; such that: if $\hat{\xi}'$ is another associative algebra and $\hat{\tau}'$ another homomorphism of $\hat{L}$ into $\hat{\xi}'^-$, there exists a unique isomorphism $\hat{\gamma}$ of $\hat{\xi}$ into $\hat{\xi}'$ with $\hat{\tau}' = \hat{\gamma}\hat{\tau}$, and two unique isotopies $\hat{i}\xi = \hat{\xi}$ and $\hat{i}'\xi' = \hat{\xi}'$.

A primary objective of the isotopic definition is the achievement of *the lack of uniqueness of the Lie algebra characterized by the isoenvelope* or, equivalently, the characterization of a family of generally nonisomorphic Lie algebras via the use of only one basis. The illustration of the above notions for the case of the rotational algebra so(3) studied in the text is straighforward and can be expressed via the diagrams (c) and (d) below

$$\begin{tikzcd} \xi^- \arrow[rr, "\gamma"] & & \xi'^- \\ & SO(3) \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] & \end{tikzcd}$$

( c )

$$\begin{tikzcd} \hat{\xi}^- \arrow[rr, "\hat{\gamma}"] & & \hat{\xi}'^- \\ & SO(2.1) \arrow[ul, "\hat{\tau}"] \arrow[ur, "\hat{\tau}'"'] & \\ \xi \arrow[uu, "\hat{i}"] \arrow[rr, "\gamma"] & & \xi \arrow[uu, "\hat{i}'"'] \\ & SO(3) \arrow[ul, "\tau"] \arrow[ur, "\tau'"'] \arrow[uu, "i"] & \end{tikzcd}$$

( d )

where the isotopy is given by $I = \text{diag.}(1, 1, 1) \Rightarrow \hat{1} = \text{diag.}(1, -1, 1)$. The above definition then provides all infinitely possible isotopes and isodual isotopes.

The above notion of isoenvelope represents the essential mathematical structure of hadronic mechanics, namely, the preservation of the conventional basis, i.e., the set of observables, and the generalization of the operations on them via an infinite number of isotopies so as to admit a new class of interactions structurally beyond the possibilities of quantum mechanics.

The isoenvelopes are denoted $\hat{\xi}(L)$ and *not* $\hat{\xi}(\hat{L})$ to stress the preservation of the original basis of L under isotopies (Proposition 3.2.1), as well as to emphasize the existence of an infinite family of isoenvelopes for *each* original Lie algebra L.

The isotopy $\xi \Rightarrow \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators $X_k$ and the parameters $w_k$ are not changed by assumption and, thus, one is prohibited from using transformation theory under isotopies.

When transformations are admitted, the connection between Lie and Lie–isotopic theories is via nonunitary transformations (Sect. 4.1). We therefore have the following

**Proposition 4.3.1** [2]: *A conventional envelope $\xi$ and its isotopic image $\hat{\xi}$ are not unitarily equivalent.*

Despite the above lack of unitary equivalence, a given Lie algebra L and its isotope $\hat{L}$ of Class I are indeed isomorphic as we shall see in the next section.

## 4.4: LIE-ISOTOPIC ALGEBRAS AND THEIR ISODUALS

We are now equipped to introduce the fundamental notion of hadronic mechanics according to the following

**Definition 4.4.1** [1]: *A (finite-dimensional) isospace* $\hat{L}$ *over an isofield* $\hat{F}(\hat{a},+,*)$ *of isoreal numbers* $\hat{R}(\hat{n},+,*)$, *isocomplex numbers* $\hat{C}(\hat{c},+,*)$ *or isoquaternions* $\hat{Q}(\hat{q},+,*)$ *with isotopic element* $T$ *and isounit* $\hat{1} = T^{-1}$ *is called a "Lie-isotopic algebra" over* $\hat{F}$ *when there is a composition* $[A \hat{,} B]$ *in* $\hat{L}$, *called "isocommutator", which verifies the following "isolinear and isodifferential rules" for all* $\hat{a}, \hat{b} \in \hat{F}$ *and* $A, B, C \in \hat{L}$

$$[\hat{a} * A + \hat{b} * B \hat{,} C] = \hat{a} * [A \hat{,} C] + \hat{b} * [B \hat{,} C], \tag{4.4.1a}$$

$$[A * B \hat{,} C] = A * [B \hat{,} C] + [A \hat{,} C] * B. \tag{4.4.1b}$$

*and the "Lie-isotopic axioms", given by the following antisymmetry and Jacobi law*

$$[A \hat{,} B] = - [B \hat{,} A], \tag{4.4.2a}$$

$$[A \hat{,} [B \hat{,} C]] + [B \hat{,} [C \hat{,} A]] + [C \hat{,} [A \hat{,} B]] = 0. \tag{4.4.2b}$$

Note that the use of isoreals, isocomplexes and isoquaternions preserves the associative character of the underlying envelope. The use instead of isooctonions $\hat{O}(\hat{o},+,*)$ (Sect. 2.8) would imply the loss of such an associative character and, for this reason, isooctonions have been excluded as possible isofields in Definition 2.3.1 in a way fully parallel to conventional lines in number theory.

This point deserves an elaboration because important for the construction of any generalization of quantum mechanics, whether of isotopic type or not.

On algebraic grounds we can say that the lack of associativity of the octonions and isooctonions is not a reason, per se, for their exclusion, because there are nonassociative algebras U such that the attached antisymmetric algebra $U^-$ is Lie. In fact, as we shall see in Ch. 7, this is precisely the definition of Lie-admissible algebras. The reason for the exclusion at this time is that the notion of universal enveloping algebra has been essentially developed for associative algebras [31–33]. For the case of nonassociative ones, the envelope is known only for very restricted algebras of the so-called *flexible Lie-admissible algebras* (see in this respect ref. [12]).

The physical reasons for excluding isooctonions are however deeper then the above. They are related to the fact that associative envelopes of the type herein considered admit a consistent unit which is at the foundation of physical applications such as the measurement theory. On the contrary, nonassociative envelopes generally do not admit a unit[24], thus prohibiting the very formulation of the measurement theory.

Moreover, in Volume II we shall review *"Obuko's no-go theorem"* which prohibits the use of a nonassociative envelope for a consistent generalization of quantum mechanics, e.g., because of the loss of equivalence between the Heisenberg-type and the Schrödinger-type representations.

We reach in this way the following:

**Fundamental condition on Lie-isotopic theory 4.4.1** [2]: *All studies on the Lie-isotopic theory and hadronic mechanics will be restricted throughout our analysis to formulations based on an isoassociative character of the enveloping algebra with a well defined left and right isounit.*[25]

In the original proposal [1] this author proved the existence of consistent *isotopic generalization of the celebrated Lie's First, Second and Third Theorems.* For brevity, we refer the interested reader to ref. [4], pp. 163–184 or to the ref. [24], Ch. II. We here quote the *Isotopic First and Second Theorems* because useful in applications for the speedy construction of one realizations of Lie-isotopic algebras (see later on for more complex realizations).

**Theorem 4.4.1 - Lie-isotopic Second Theorem** [1]: *Let* $X = \{X_k\}$, $k = 1, 2, \ldots, N$, *be the (ordered set of) generators in adjoint representations of a Lie algebra* L *with commutation rules*

$$L: \ [X_i, X_j] = X_i X_j - X_j X_j = C_{ij}^{\ k} X_k , \tag{4.4.3)}$$

*where* $C_{ij}^{\ k}$ *are the "structure constants". Then, one realization of the Lie-isotopic images* $\hat{L}$ *of* L *is characterized by the same generators* X *with isocommutation rules*

$$\hat{L}: \ [X_i \hat{,} X_j] = X_i * X_j - X_j * X_i = X_i T X_j - X_j T X_i =$$

$$= X_i T(x, \dot{x}, ..) X_j - X_j T(x, \dot{x}, ...) X_i = \hat{C}_{ij}^{\ k}(t, x, \dot{x}, ...) * X_k =$$

---

[24] Recall that the enveloping algebra $\xi$ is associative while the attached Lie algebra $\xi^-$ is nonassociative. Thus, the fundamental unit I of the conventional Lie's theory is the unit of the envelope and not of the attached Lie algebra. In fact, the product [A, B], per se, admits no consistent unit because it would require an element E such that [E, A = [A, E] ≡ A, ∀ A ∈ L. Exactly the same situation occurs under isotopies.

[25] As we shall see in Ch. 7, this fundamental condition will persist also for the more general Lie-admissible formulations for which the underlying envelope must remain still isoassociative, and the units must still exist, although they are differentiated for the right and left multiplications.

$$= \hat{C}_{ij}{}^{k}(x, \dot{x}, \dots)\, X_k\,, \tag{4.4.4}$$

*where the* $\hat{C}_{ij}{}^{k}$ *are the "structure functions" in the isofield.*

**Theorem 4.4.2 - Lie-isotopic Third Theorem** [loc. cit.]: *The structure functions* $\hat{C}_{ij}{}^{k}$ *of a Lie-isotopic algebra* $\hat{L}$ *verify the conditions*

$$\hat{C}_{ij}{}^{k} = - \hat{C}_{ji}{}^{k}\,, \tag{4.4.5}$$

*and the property (when commuting with the generators* [26]*)*

$$\hat{C}_{ij}{}^{p} * \hat{C}_{pk}{}^{q} + \hat{C}_{jk}{}^{p} * \hat{C}_{pi}{}^{q} + \hat{C}_{ki}{}^{p} * \hat{C}_{pj}{}^{q} = 0\,. \tag{4.4.6}$$

We learn in this way that the structure "constants" of Lie's theory acquire a dependence on local variables similar to that of the isotopic element T, thus becoming structure "functions'.

It is important to illustrate the above theorems with an example. Consider the generators of the su(2) Lie algebra in their adjoint representation, which are given by the celebrated *Pauli's matrices* and related commutation rules

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4.7a}$$

$$[\,\sigma_n, \sigma_m\,] = \sigma_n\,\sigma_m - \sigma_m\,\sigma_n = 2\,i\,\epsilon_{nmk}\,\sigma_k\,, \tag{4.4.7b}$$

Theorem 4.4.1 states that the *same generators* $\sigma_k$ can characterize *one realization* of the Lie-isotopic $\hat{su}(2)$ algebra via the lifting of the structure constants into suitable functions.

This property is readily verified by introducing a Class III isotopic element assumed diagonal for simplicity, and then identifying the structure functions under which the algebra is closed, with explicit solution

$$[\,\sigma_n \hat{,} \sigma_m\,] = \sigma_n\,T\,\sigma_m - \sigma_m\,T\,\sigma_n = 2\,i\,\hat{\epsilon}_{nmk}\,T\,\sigma_k\,, \tag{4.4.8a}$$

$$T = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}, \; g_{kk} \neq 0, \;\; \Delta = \det T = g_{11}\,g_{22}\,, \tag{4.4.8b}$$

$$\hat{1} = \begin{pmatrix} g_{11}{}^{-1} & 0 \\ 0 & g_{22}{}^{-1} \end{pmatrix} = \Delta^{-1} \begin{pmatrix} g_{22} & 0 \\ 0 & g_{11} \end{pmatrix}, \tag{4.4.8c}$$

[26] If not, more general properties are easily derivable from Jacobi's law.

$$\hat{\epsilon}_{ijk} = \epsilon_{ijk}\begin{pmatrix} g_{22}/g_{11} & 0 \\ 0 & g_{11}/g_{22} \end{pmatrix}. \tag{4.4.8d}$$

The verification of Theorem 4.4.2 is trivial in this particular case because of the factorization of the old structure constants $\epsilon_{ijk}$ which multiplies the same matrix for all elements.

Note that the original structure "constants" $C_{ij}{}^{k}$ are elements of a field $F(a,+,\times)$ and, as such, are ordinary numbers. On the contrary, the structure "functions" $\hat{C}_{ij}{}^{k}$ are now elements of the isofield $\hat{F}(\hat{a},+,*)$ and, as such, are matrices. As such, they should be called more properly *structure isofunctions*, where the prefix "iso" stands precisely to represent their matrix character. Nevertheless, the term "structure functions" is now widely used in the literature in the field and it will be kept in these books for simplicity.

Note finally that Theorem 4.4.1 provide only *one method* for the speedy construction of an isotope $\hat{L}$ of a given Lie algebra L.In fact, another way of constructing Class I isotopes $\hat{L}$ of a given Lie algebra L is by generalizing the generators $X_k$ and keeping instead the old structure constants. This alternative approach will be used in a number of applications because it evidently ensures the local isomorphism $\hat{L} \approx L$ *ab initio*, while lifting conventional symmetries into the desired nonlinear–nonlocal–noncanonical form.

Theorems 4.4.1 and 4.4.2 were however conceived for specific physical needs. Recall that the generators of a Lie algebra represent physical quantities, such as linear momentum, angular momentum, energy, etc. As such, these quantities cannot be changed under isotopies, thus explaining the preservation of the original basis in Theorem 4.4.1. An additional motivation is that, among all possible realizations, the method of Theorem 4.4.1 results to be most effective in the computation of the symmetries of nonlinear–nonlocal–noncanonical systems, as we shall see in Sect. 4.6.

It is easy to prove the following:

**Theorem 4.4.3 [2]:** *The isotopies* $L \Rightarrow \hat{L}$ *of an N–dimensional Lie algebra* L *preserve the original dimensionality.*

In fact, the basis $e_k$ , $k = 1, 2, ...., N$ of a vector space and, thus, of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let then the commutation rules of L be given by

$$[\, e_i \,, e_j \,] = C_{ij}{}^{k}\, e_k. \tag{4.4.9}$$

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i \hat{,} \hat{e}_j] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \hat{C}_{ij}{}^k(t, x, \dot{x}, \ldots)\hat{e}_k . \tag{4.4.10}$$

One can then see in this way the necessity of lifting the structure "constants" into structure "functions", as correctly predicted by the Lie-isotopic Second Theorem. A number of examples will be provided during the course of our analysis.

We now review a few basic notions of Lie-isotopic algebras $\hat{L}$ which can be derived via an easy isotopy of the corresponding conventional notions (as available, e.g., in ref.s [31–33]). Lie-isotopic algebras $\hat{L}$ are said to be:

a) *isoreal (isocomplex)* when $\hat{F} = \hat{R}$ ($\hat{F} = \hat{C}$);

2) *isoabelian* when $[A \hat{,} B] \equiv 0$, $\forall$ A, B $\in \hat{L}$;

3) A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when

$$[\hat{L}_0 \hat{,} L_0] \subset \hat{L}_0 ; \tag{4.4.11}$$

4) An *isoideal* occurs when

$$[\hat{L} \hat{,} \hat{L}_0] \subset \hat{L}_0; \tag{4.4.12}$$

5) The *isocenter* of a Lie-isotopic algebra is the maximal isoideal $L_0$ which verifies the property

$$[\hat{L} \hat{,} \hat{L}_0] = 0. \tag{4.4.13}$$

**Definition 4.4.2** [27]: *The "general isolinear and isocomplex Lie-isotopic algebras", here denoted with* $\hat{GL}(n,\hat{C})$, *are the vector isospaces of all* $n \times n$ *complex matrices over* $\hat{C}(\hat{c},+,\hat{*})$, *and are evidently closed under isocommutators. The "isocenter" of* $\hat{GL}(n,\hat{C})$ *is then given by* $\hat{c} * \hat{1}$, $\forall\, \hat{c} \in \hat{C}$. *The subset of all complex* $n \times n$ *matrices with null trace is also closed under isocommutators, it is called the "special, isolinear, isocomplex, Lie-isotopic algebra", and denoted with* $\hat{SL}(n,\hat{C})$. *The subset of all antisymmetric* $n \times n$ *real matrices* X, $X^t = -X$, *is also closed under isocommutators, is called the "isoorthogonal algebra", and is denoted with* $\hat{O}(n)$.

By proceeding along similar lines, one can classify all classical, non-exceptional, Lie-isotopic algebras into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ according to the general rules [27]

$$\begin{aligned}
&\textbf{Class } \hat{A}_{n-1} = \hat{SL}(n, \hat{C});\\
&\textbf{Class } \hat{B}_n = \hat{O}(2n+1, \hat{C});\\
&\textbf{Class } \hat{C}_n = \hat{SP}(n, \hat{C}); \text{ and}\\
&\textbf{Class } \hat{D}_n = \hat{O}(2n, \hat{C}).
\end{aligned}$$

plus the *isoexceptional algebras* here ignored for brevity.

Each one of the above algebras then needs its own classification (evidently absent in the conventional case), depending on whether $\hat{1}$ is positive–definite (Class I), negative definite (Class II), indefinite (Class III), singular (IV) and general (Class V), as well as whether of isocharacteristic zero or p, thus illustrating the richness of the isotopic theory indicated above.

The notions of *homomorphism, automorphism and isomorphism* of two Lie–isotopic algebras $\hat{L}$ and $\hat{L}'$ are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct* and *semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

By following Kadeisvili [27] we now introduce an *isoderivation* $\hat{D}$ of a Lie–isotopic algebra $\hat{L}$ as an isolinear map of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A \hat{,} B]) = [\hat{D}(A) \hat{,} B] + [A \hat{,} \hat{D}(B)] \qquad \forall\, A, B \in \hat{L} \,. \tag{4.4.14}$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a}*\hat{D}_1 + \hat{b}*\hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie–isotopic algebra as in the conventional case.

The isolinear map $a\hat{d}(\hat{L})$ of $\hat{L}$ into itself defined by

$$a\hat{d}\, A(B) = [A \hat{,} B], \qquad \forall\; A, B \in \hat{L}, \tag{4.4.15}$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the Jacobi identity (4.4.2b). The set of all $a\hat{d}(A)$ is therefore an isolinear Lie–isotopic algebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Consider the algebras

$$\hat{L}^{(o)} = \hat{L}, \quad \hat{L}^{(1)} = [\hat{L}^{(o)} \hat{,} \hat{L}^{(o)}], \qquad \hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}], \quad \text{etc.,} \tag{4.4.16}$$

which are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$.

Consider also the sequence

$$L_{(o)} = L, \hat{L}_{(1)} = [\hat{L}_{(o)} \hat{,} \hat{L}], \hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}], \text{etc.} \tag{4.4. 17}$$

Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield

$$\hat{Tr}\, A = (\, Tr\, A)\hat{1} \in \hat{F}, \tag{4.4.18}$$

where TrA is the conventional trace. Then

$$\hat{Tr}\,(A * B) = (\hat{Tr}\, A) * (\hat{Tr}\, B), \qquad \hat{Tr}\,(\, B * A * B^{-\hat{1}}\,) = \hat{Tr}\, A\,. \tag{4.4.19}$$

Thus, $\hat{Tr}\, A$ preserves the axioms of $Tr\ A$, by therefore being a correct isotopy. Then, the isoscalar product

$$(\, A \hat{,} B\,) = \hat{Tr}\,[\,(\, A\hat{d}X) * (A\hat{d}\, B)\,] \tag{4.4.20}$$

is called the *isokilling form* as first studied by Kadeisvili [27]. It is easy to see that $(\, A \hat{,} B\,)$ is symmetric, bilinear, and verifies the property

$$(A\hat{d}\, X\,(Y)\hat{,} Z\,) + (\, Y \hat{,} Ad\, X\,(Z) = 0, \tag{4.4.21}$$

thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let $e_k$ , $k = 1, 2, \ldots, N$, be the basis of L with one-to-one invertible map $e_k \rightarrow \hat{e}_k$ into the basis $\hat{e}_k$ of $\hat{L}$. Generic elements in $\hat{L}$ can then be written in terms of local coordinates x, y, z,

$$A = x^i\,\hat{e}_i\,, B = y^j\,\hat{e}_j\,,\ C = z^k\,\hat{e}_k \ = \ [\, A \hat{,} B\,] = x^i\, y^j\,[\,\hat{e}_i \hat{,} \hat{e}_j\,] =$$

$$= \ x^i\, x^j\, \tilde{C}_{ij}{}^k\, \hat{e}_k\,. \tag{4.4.22}$$

Thus,

$$[\, A\hat{d}\ A\,(B)\,]^k = [\, A \hat{,} B\,]^k = \tilde{C}_{ij}{}^k\, x^i\, x^j\,. \tag{4.4.23}$$

By following again Kadeisvili [27], we now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of a Lie-isotopic algebra $\hat{L}$ via the definition $(\, A \hat{,} B\,) = \tilde{g}_{ij}\, x^i\, y^j$ yielding

$$\tilde{g}_{ij}(t, x, \dot{x}, \ddot{x}, \ldots) = \tilde{C}_{ip}{}^k\, \tilde{C}_{jk}{}^p\,. \tag{4.4.24}$$

Note that the isocartan tensor has the general dependence of the isometric tensor of the preceding chapter, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables $x, \dot{x}, \ddot{x}, \ldots$ .

The isocartan tensor also clarifies another important point of the preceding analysis, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic models of interior dynamical problems, and that their restriction to the nonlinear dependence on the coordinates x only, as needed for the exterior gravitational problem, would be manifestly un–necessary.

The isotopies of the structure theory of Lie algebras then follow, including the notion of *simplicity, semisimplicity,* etc. (see the monograph [24]) Here we limit ourselves to recall the following

**Definition 4.4.3** [27]: *A Lie–isotopic algebra $\hat{L}$ is called "compact" ("noncompact") when the isocartan form is positive– (negative–) definite.*

Numerous additional, more refined definitions of compactness and noncompactness are possible via the isotopies of the corresponding conventional definitions [31–33]. The above definition is however sufficient for our needs.

We now study a few implications of the isotopic lifting of Lie's theory.

**Theorem 4.4.4** [1]: *The isotopes of Class III $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the reader interested in becoming an expert in isotopic theories. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

**Definition 4.4.4** [8,8]: *Let $\hat{L}$ be a Lie–isotopic algebra with generators $X_k$ and isounit $\hat{1} = T^{-1} > 0$. The "isodual Lie–isotopic algebras" $\hat{L}^d$ is the isoalgebra with isodual generators $X_k = -X_k$ conventional structure functions over the isodual isofield $\hat{F}^d(\hat{a}^d,+,\hat{*}^d)$ with "isodual isocommutators"*

$$[X_i \hat{,} X_j]^d = -[X^d_i \hat{,} X^d_j] = -[X_i \hat{,} X_j] = \hat{C}_{ij}{}^k X^d_k = -\hat{C}_{ij}{}^k X_k. \quad (4.4.25)$$

*When the original algebra is a Lie algebra L the "isodual Lie algebra" is given by the structure $L^d$ over the isodual field $F^d(a^d,+,\times^d)$ with "isodual commutators"*

$$[X_i , X_j]^d = X_i I^d X_j - X_j I^d X_i = -[X_i , X_j] = -C_{ij}{}^k X_k . \quad (4.4.26)$$

$\hat{L}$ and $\hat{L}^d$ are then anti–isomorphic. Note that the isoalgebras of Class III contain all algebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie–isotopic theory can be naturally formulated for Class III, as implicitly done in ref. [1].

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit I $\Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product AB − BA into the isotopic form ATB − BTA.

The following property is mathematically trivial, yet carries important physical applications.

**Theorem 4.4.5** [1,5]: *All infinitely possible, isotopes $\hat{L}$ of Class I of a (finite-dimensional) Lie algebra* L *are locally isomorphic to* L, *and all infinitely possible isodual isotopes $\hat{L}^d$ of Class II are anti-isomorphic to* L.

The simplest possible proof is via the redefinition of the basis $X_k \Rightarrow X'_k = X_k\hat{1}$, under which isotopic algebras $\hat{L}$ acquire the same structure constants of L,

$$[X_i \hat{,} X_j] \;\Rightarrow\; [X'_i \hat{,} X'_j] \;=\; [X_i, X_j]\hat{1} \;=\; C_{ij}{}^k X'_k . \qquad (4.4.27)$$

We should however indicate that, even though the above reduction is possible, in general we have $\hat{C}_{ij}{}^k \neq C_{ij}{}^k \hat{1}$, as it is the case of example (4.4.8), thus rendering inapplicable the realization $X' = X\,\hat{1}$. Also the realization $X'_k = X_k\hat{1}$ does not yield the desired nonlinear-nonlocal-nonhamiltonian isosymmetries as we shall see in Sect. 4.6.

Despite the local isomorphism $L \approx \hat{L}$, the lifting $L \Rightarrow \hat{L}$ is not mathematically trivial because these two algebras are not unitarily equivalent. The physical relevance of the isotopies originates precisely from their local isomorphism, because it permits the construction of nonlinear, nonlocal and noncanonical isotopes of the rotational $S\hat{O}(3)$, Galilean $\hat{G}(3.1)$, Lorentz $\hat{O}(3.1)$, Poincaré $\hat{P}(3.1)$, $S\hat{U}(3)$ and other space-time and internal symmetries which are locally isomorphic to the original algebras.

Theorem 4.4.5 therefore represents the property which has permitted the achievement of methods for the nonlinear-nonlocal-noncanonical interior problems by preserving the analytic, algebraic and geometric axioms of the conventional, linear-local-canonical methods of the exterior problems [4].

For additional technical studies of the Lie-isotopic algebras we refer the reader to the forthcoming book [24].

We now illustrate the results of this sections with the isotopies and isodualities of the rotational algebra so(3) with generators in their adjoint form (4.3.13). For this purpose, the isounit and isotopic element of Class III, Eq. (4.3.16), can be realized in the form

$$\hat{1} \;=\; \mathrm{diag.}\,(\pm b_1^{-2}, \pm b_2^{-2}, \pm b_3^{-2}), \quad b_k(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0\,, \qquad (4.4.28a)$$

$$\delta \;=\; \dot{T} \;=\; \mathrm{diag.}\,(\pm b_1^2, \pm b_2^2, \pm b_3^2), \qquad (4.4.28b)$$

The Isotopic Second Theorem 4.4.1 then yields

$$[J_i \hat{,} J_j] = J_i\, T\, J_j - J_j\, T\, J_i = \hat{C}_{ij}{}^k(t, r, \dot{r}, \ddot{r}, \ldots)\, T\, J_k \qquad (4.4.29)$$

where the J's are the conventional adjoint generators (4.3.13) and the $\hat{C}$'s are the structure functions.

It is easy to see that all possible isoalgebras (4.4.8) are those of classification (4.3.18), and are given by [1,9]:

1) so(3) for $\hat{1} \equiv I = \text{diag.}(1, 1, 1)$ with commutation rules

$$[J_1, J_2] = J_3, \quad [J_2, J_3] = J_1, \quad [J_3, J_1] = J_2; \qquad (4.4.30)$$

2) so(2.1) for $T = \text{diag.}(1, -1, 1)$ with rules

$$[J_1 \hat{,} J_2] = J_3, \quad [J_2 \hat{,} J_3] = -J_1, \quad [J_3 \hat{,} J_1] = J_2; \qquad (4.4.31)$$

3) An infinite family of isotopes $\hat{so}(3)$ isomorphic to so(3) for $T = \text{diag.}(b_1^2, b_2^2, b_3^2)$ with rules

$$[J_1 \hat{,} J_2] = b_3^2\, J_3, \quad [J_2 \hat{,} J_3] = b_1^2\, J_1, \quad [J_3 \hat{,} J_1] = b_2^2\, J_2; \qquad (4.4.32)$$

4) An infinite family of isotopes $\hat{so}(2.1)$ isomorphic to so(2.1) for $T = \text{diag.}(b_1^2, -b_2^2, b_3^2)$ and rules

$$[J_1 \hat{,} J_2] = b_3^2\, J_3, \quad [J_2 \hat{,} J_3] = -b_1^2\, J_1, \quad [J_3 \hat{,} J_1] = b_2^2\, J_2; \qquad (4.4.33)$$

5) The isodual $so^d(3)$ of so(3) for $\hat{1} = \text{diag.}(-1, -1, -1)$ and rules

$$[J_1 \hat{,} J_2] = -J_3, \quad [J_2 \hat{,} J_3] = -J_1, \quad [J_3 \hat{,} J_1] = -J_2; \qquad (4.4.34)$$

6) The isodual $so^d(2.1)$ of so(2.1) for $\hat{1} = \text{diag.}(-1, 1, -1)$ and rules

$$[J_1 \hat{,} J_2] = -J_3, \quad [J_2 \hat{,} J_3] = J_1, \quad [J_3 \hat{,} J_1] = -J_2; \qquad (4.4.35)$$

7) The infinite family of isotopes $\hat{so}^d(3) \approx so^d(3)$ for $T = \text{diag.}(-b_1^2, -b_2^2, -b_3^2)$ and rules

$$[J_1 \hat{,} J_2] = -b_3^2\, J_3, \quad [J_2 \hat{,} J_3] = -b_1^2\, J_1, \quad [J_3 \hat{,} J_1] = -b_2^2\, J_2; \qquad (4.4.36)$$

8) The infinite family of isotopes $\hat{so}^d(2.1) \approx SO^d(2.1)$ for $T = \text{diag.}(-b_1^2, b_2^2, -b_3^2)$ and rules

$$[J_1 \hat{,} J_2] = -b_3^2\, J_3, \quad [J_2 \hat{,} J_3] = b_1^2\, J_1, \quad [J_3 \hat{,} J_1] = -b_2^2\, J_2; \qquad (4.4.37)$$

The reader can readily verify the above indicated local isomorphisms via the redefinition of the basis

$$J'_1 = b_1^{-1} b_3^{-1} J_1, \quad J'_2 = b_1^{-1} b_3^{-1} J_2, \quad J'_3 = b_1^{-1} b_2^{-1} J_3, \qquad (4.4.38)$$

in which case the b-terms in the r.h.s. of the commutation rules disappear and one recovers conventional structure constants of so(3) and so(2.1) under isotopies (see Ch. II–6 for details).

It is also significant that exactly the same classification exists for the isotopies of so(3) in classical mechanics, in which case the isoproduct is given by an isotopy of the conventional Poisson brackets (see ref. [22] for details). This latter occurrence is important to understand that the conventional quantization of the classical rotational symmetry carries over in its entirety to the isotopic and isodual coverings (Vol. II).

It is instructive for the interested reader to verify with the above examples various other notions introduced in this section, such as the isocartan's tensor, the isokilling form, etc. We shall have plenty of opportunities to study additional examples of Lie–isotopic algebras in Vol.s II and III.

As final comments, we discourage the reader from applying conventional notions of Lie's theory to the covering Lie–isotopic theory without their specific isotopic reformulation. This is due to the lack of general preservation of structural properties of the original Lie algebras, such as compactness, irreducibility, etc.

The reader should also be aware of the physical importance of preserving under isotopies the original generators $X_k$ (i.e., the original basis). In fact, the generators represent physical quantities, such as total energy, linear momentum, angular momentum, etc. which, as such, cannot be changed by isotopies or other techniques. Similarly, the parameters represent physics measurable quantities such as angles of rotation, velocities, etc. This also illustrates the preservation under isotopies of the conventional parameters $w \in F$ merely lifted into the form $\hat{w} = w\hat{1} \in \hat{F}$.

The implications of these results for hadronic mechanics are clear. As we shall see better in the next Volume, *the conventional total conservation laws of hadronic mechanics can be simply read-off the generators of the Lie–isotopic symmetries.*

## 4.5: LIE-ISOTOPIC GROUPS AND THEIR ISODUALS

As indicated earlier, the isotopies of a topological space are still lacking at this writing and so are the isotopies of topological Lie groups. Only the isotopies of Lie's transformation groups are available at this writing as per the original proposals [1,2].

**Definition 4.5.1** [1]: *A "right Lie-isotopic transformation group"* $\hat{G}$, *or "isotransformation group" for short, on an isospace* $\hat{S}(x,\hat{F})$ *over an isofield* $\hat{F}(\hat{a},+,*)$ *(of isoreal numbers* $\hat{R}$ *or isocomplex numbers* $\hat{C}$ *or isoquaternions* $\hat{Q}$*) is a group which maps each element* $x \in \hat{S}(x,\hat{F})$ *into a new element* $x' \in \hat{S}(x,\hat{F})$ *via the isotransformations*

$$x' = \hat{U} * x = \hat{U}\, T\, x, \qquad T \textit{ fixed}, \tag{4.5.1}$$

*such that:*
*1) The map* $(\hat{U}, x) \rightarrow \hat{U} * x$ *of* $\hat{G}\times\hat{S}(x,\hat{F})$ *onto* $\hat{S}(x,\hat{F})$ *is differentiable;*
*2)* $\hat{1} * \hat{U} = \hat{U} * \hat{1} = \hat{U}$, $\forall\, \hat{U} \in \hat{G}$; and
*3)* $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x$, $\forall\, x \in \hat{S}(x,\hat{F})$ *and* $\hat{U}_1, \hat{U}_2 \in \hat{G}$.
*A "left Lie-isotopic isotransformation group" is defined accordingly.*

The notions of *connected or simply connected transformation groups* (see, e.g., ref.s [31–33]) carry over to the Lie-isotopic groups in their entirety. We consider hereon the connected Lie-isotopic transformation groups (see Sect. 4.6 for the discrete parts).

Right or left Lie-isotopic groups are characterized by the following *isogroup laws* first introduced in ref. [1]

$$\hat{U}(\hat{0}) = \hat{1}, \tag{4.5.2a}$$

$$\hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \tag{4.5.2b}$$

$$\hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{1}, \qquad \hat{w} \in \hat{F}, \tag{4.5.2c}$$

Their most direct realization is that via the isoexponentiation (4.3.6) under which they can be reduced to ordinary transformations for computational convenience, i.e.,

$$x' = \hat{U} * x = \{ e_{\xi}^{\,i\,\hat{w} * X} \} * x = \{ e^{i\,X\,T\,w} \}\, x, \tag{4.5.3}$$

where X represents the infinitesimal generators of G and w the parameters, with the understanding that only the isotransformations are admitted on rigorous mathematical grounds.

The fundamental feature of the above transformations is their nonlinear, nonlocal and noncanonical character, as implied by the isotopic element $T(t, x, \dot{x}, \ddot{x}, ...)$ in the exponent.

Evidently, Eq.s (4.5.3) hold for some open neighborhood N of the isoorigin of $\hat{L}$ and which, in this way, characterizes some open neighborhood of the isounit of $\hat{G}$ (see in this respect [27,28]).

Still another important property permitting the isocomposition of Lie-isotopic groups is given by the following

**Theorem 4.5.1 - isotopic Baker-Campbell-Hausdorff theorem** [1,4]: *The conventional group composition laws admit a consistent isotopic lifting, resulting in the following "isotopic composition law"*

$$\hat{U}_1 * \hat{U}_2 = ( e_{\xi}^{X_1} ) * ( e_{\xi}^{X_2} ) = \hat{U}_3 = e_{\xi}^{iX_3}, \qquad (4.5.4a)$$

$$X_3 = X_1 + X_2 + [X_1 \hat{,} X_2]/2 + [(X_1 - X_2)\hat{,}[X_1 \hat{,} X_2]]/12 + \ldots \qquad (4.5.4b)$$

By following Kadeisvili [27], we now study the connection between Lie-isotopic groups and algebras. Let $\hat{L}$ be a (finite-dimensional) Lie-isotopic algebra with (ordered) basis $X_k$, $k = 1, 2, \ldots, N$. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(w) = \prod^{*}{}_{k=1,2,\ldots,N}\, e_{\xi}^{iX_k * \hat{w}_k}\,. \qquad (4.5.6)$$

which characterizes some open neighborhood M of the isounit $\hat{1}$ of $\hat{G}$.

The map

$$\Phi_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{1}}, \qquad (4.5.7)$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto itself. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering expression (4.5.7) in the neighborhood of the isounit $\hat{1}$, in which case we have

$$\hat{U}'_2 = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{1}} \cong \hat{U}_2 + w_1 w_2 [X_2 \hat{,} X_1] + O^{(2)}. \qquad (4.5.8)$$

By recalling the differentiability property of $\hat{G}$, we also have the following isotopy of the conventional expression in one dimension[27]

$$(1/i)\frac{d}{dw}\hat{U}\,|_{\hat{w}=0} = (1/i)\frac{d}{dw} e_{\xi}^{iwX}\,|_{w=0} = X * e_{\xi}^{iwX}\,|_{w=0} = X\,, \qquad (4.5.9)$$

---

[27] We should indicate that the conventional derivative d/dw needs a suitable isotopic formulation $\hat{\partial}/\hat{\partial}\hat{w}$ presented in Ch. I.6. The results, however, will be the same as those of Eq.s (4.5.9).

Thus, to every inner isoautomorphism of $\hat{G}$ there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form [27]

$$(\hat{L})_i^{\ j} = \hat{C}_{ki}^{\ \ j}\, w^k . \qquad (4.5.10)$$

The Lie-isotopic group $\hat{G}_a$ of all inner isoautomorphism of $\hat{G}$ is called the *isoadjoint group.* It is possible to prove that the Lie-isotopic algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$.

We mentioned before that the direct sum of Lie-isotopic algebras is the conventional operation because the addition is not lifted in our studies. The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be a Lie-isotopic group and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o{}_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o{}_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o{}_a$ of $\hat{G}$ and $\hat{G}^o{}_a$ is the Lie-isotopic group of all ordered pairs $(\hat{g}, \hat{\Lambda})$ with group isomultiplication

$$(\hat{g}, \hat{\Lambda}) * (g', \Lambda') = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}') . \qquad (4.5.11)$$

with total isounit given by

$$\hat{1}_{tot} = (\hat{1}, \hat{1}_{\hat{\Lambda}}), \qquad (4.5.12)$$

and inverse

$$(\hat{g}, \hat{\Lambda})^{-1} = (\hat{\Lambda}^{-1}(\hat{g}^{-1}), \hat{\Lambda}^{-1}) . \qquad (4.5.13)$$

As we shall see in Vol. II, the above notions play an important role in the isotopies of the inhomogeneous space-time symmetries, such as Galilei's and Poincaré's symmetries.

Let $\hat{G}_1$ and $\hat{G}_2$ be two Lie-isotopic groups with respective isounits $\hat{1}_1$ and $\hat{1}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the Lie-isotopic group of all ordered pairs $g = (\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1$, $\hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$g * g' = (\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2), \qquad (4.5.14)$$

total isounit

$$\hat{1}_{tot} = (\hat{1}_1, \hat{1}_2), \qquad (4.5.15)$$

and inverse

$$g^{-1} = (\hat{g}_1^{-1}, \hat{g}_2^{-1}) . \qquad (4.5.16)$$

**Definition 4.5.2** [8,9]: *Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators $X_k$, k = 1, 2, ..., N. The "isodual image" $\hat{G}^d$ of $\hat{G}$ is the N-dimensional isogroup with infinitesimal generators $X^d{}_k = -X_k$, isodual isounit $\hat{1}^d = -\hat{1}$ and isodual parameters $w^d = -w$ over the isodual isofield $\hat{F}^d(\hat{a}^d,+,*^d)$ with "isodual isotransformation" in a suitable neighborhood of $\hat{1}^d$*

$$x' = \hat{U}^d(\hat{w}^d) *^d x = \{ e_{\xi^d}^{\,i^d X^d *^d \hat{w}^d} \} *^d x = -\{ e_{\xi}^{\,i X T w} \} x . \qquad (4.5.17)$$

*In particular, the above antiautomorphic conjugation can also be defined for conventional Lie groups, yielding the "isodual Lie group" $G^d$ which is defined over the isodual field $F^d(a^d,+,\times^d)$ with generic "isodual transformations"*

$$x' = U^d(w^d) \times^d x = \{ e_{\xi^d}^{\,i X^d \times^d w^d} \} \times^d x = -\{ e_{\xi}^{\,i X w} \} x . \qquad (4.5.18)$$

In summary, any Lie group admits the following four realizations relevant for our analysis:

**Lie groups** G of conventional type;
**Lie-isotopic groups** $\hat{G}$;
**Isodual Lie groups** $G^d$; and
**Isodual Lie-isotopic groups** $\hat{G}^d$.

Realization G ($G^d$) is useful for the characterization of particles (antiparticles) in vacuum within the context of the exterior problem, while realization $\hat{G}$ ($\hat{G}^d$) is useful for the characterization of particles (antiparticles) in physical media within the context of the interior dynamical problem.

It is hoped the reader can see from the above foundations that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-isotopic formulation.

We now illustrate the primary results of this section with the isotopies and isodualities of the *rotational group* SO(3). Let $S\hat{O}(3)$ be the lifting of SO(3) of Class III on isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ with isometric and isounit (4.3.16). Let $\theta_k \in R(n,+,\times)$ be the conventional *Euler's angles* and $\hat{\theta}_k = \theta_k\hat{1} \in \hat{R}(\hat{n},+,*)$ their isotopes. Then, a generic isotransformation on $\hat{E}(r,\hat{\delta},\hat{R})$ can be written

$$r' = \hat{\mathfrak{R}}(\hat{\theta}) * r = \hat{\mathfrak{R}}(\theta)\, r, \qquad \hat{\mathfrak{R}} = \mathfrak{R}\,\hat{1} . \qquad (4.5.19)$$

We then have the realization of isoexponentials (4.5.6)

$$\mathfrak{R}(\theta) = \{ e_\xi^{\,J_1 * \hat\theta_1} \} * \{ e_\xi^{\,J_2 * \hat\theta_2} \} * \{ e_\xi^{\,J_3 * \hat\theta_3} \} =$$

$$= \{ e_\xi^{\,J_1\, T\, \theta_1} \} \{ e_\xi^{\,J_2\, T\, \theta_2} \} \{ e_\xi^{\,J_3\, T\, \theta_3} \} \hat{1} . \qquad (4.5.20)$$

where the J's are the (skew-symmetric) generators (4.3.13) of the adjoint representation of so(3), the θ's are the conventional Euler's angles, and T is the isotopic element (4.3.16) in realization (4.4.28b).

It is an instructive exercise to verity for structure (4.5.20): the validity of laws (4.5.2) ensuring its Lie-isotopic group structure; the validity of Theorem 4.5.1 ensuring its finite-dimensional (actually three-dimensional) character; Corollary 4.3.1A ensuring the correct isoexponentiation from the Lie-isotopic algebras to the corresponding Lie-isotopic groups; rule (4.5.9) on the inverse transition from isogroups (4.5.20) to the corresponding isoalgebras; and others.

It is finally instructive to verify the following classification of all possible isogroups (4.5.20)

$$\mathfrak{R}(\theta) : \quad \begin{array}{l} SO(3) \text{ for } T = \text{diag.}\,(1, 1, 1); \\ SO(2.1) \text{ for } T = \text{diag.}\,(1, -1, 1); \\ S\hat{O}(3) \text{ for sign. } T = (+, +, +); \\ S\hat{O}(2.1) \text{ for sign. } T = (+, -, +); \\ SO^d(3) \text{ for } T = (-1, -1, -1); \\ SO^d(2.1) \text{ for } T = \text{diag.}\,(-1, +1, -1); \\ S\hat{O}^d(3) \text{ for sign. } T = (-, -, -); \\ S\hat{O}^d(2.1) \text{ for sign. } T = (-, +, -). \end{array} \qquad (4.5.21)$$

thus illustrating the compatibility of the above classification with the corresponding one at the isoalgebra level, Eq.s (4.4.30)-(4.4.37), and the original one at the level of isoenvelope, Eq.s (4.3.18).

An example of isotopic rotation $\mathfrak{R}(\theta)$ will be presented in the next section. We shall have ample opportunities to study in Vol.s II and III additional Lie-isotopic groups and related isotransformations.

As we shall see better in Vol. II, the fundamental time evolution of hadronic mechanics [2] was conceived as a one-parameter Lie-isotopic group of inner isoautomorphisms, which can be written for an arbitrary operator Q in terms of a Hermitean Hamiltonian H on a Hilbert space $\mathcal{H}$ in the finite form

$$Q(t) = \{ e_\xi^{\,-itH} \} * Q(0) * \{ e_\xi^{\,itH} \} = \{ e^{-itTH} \}\, Q(0)\, \{ e^{itTH} \}, \quad H = H^\dagger , \qquad (4.5.23)$$

with corresponding infinitesimal form derived via rule (4.5.8)

$$i \frac{dQ}{dt} = [Q \hat{,} H] = Q * H - H * Q = Q T H - H T Q. \tag{4.5.24}$$

This confirms that the isotopies of Lie's theory are indeed at the structural foundations of hadronic mechanics.

The singular Lie-isotopic theory of Class IV is unexplored. It is hoped that experts in the field will indeed study its structure because it characterizes degenerate space-time symmetries for gravitational singularities (Sect. 3.7). The Lie-isotopic theory of Class V is equally unexplored at this writing.

## 4.6: THE FUNDAMENTAL THEOREM ON ISOSYMMETRIES

In this section we shall apply the isotopic methods for the construction of the *isosymmetries*, i.e., the symmetries of the isoseparation in an isospace $\hat{S}(x,\hat{g},\hat{F})$

$$(x - y)^{\hat{2}} = (x - y)^{\mu} \hat{g}_{\mu\nu}(t, x, \dot{x}, \ddot{x}, \psi, \psi^{\dagger}, \partial\psi, \partial\psi^{\dagger}, \mu, \tau, n, \ldots\ldots)(x - y)^{\nu}, \tag{4.6.1a}$$

$$\det \hat{g} \neq 0, \quad \hat{g} = \hat{g}^{\dagger}, \tag{4.6.1b}$$

The objective is the form-invariant characterization of the most general known interior dynamical problems which are:

1) nonlinear in the coordinates x (as available, e.g., in conventional gravitation) as well as nonlinear in the velocities (e.g., to represent the drag forces of missiles in atmosphere which are nowadays proportional to the tenth power of the velocity $\dot{x}^{10}$ and more), and in the accelerations (as requested by certain particular interior dynamical motions studied in Vol. II);
2) nonlocal-integral on some or all these variables to represent the extended character of the particles moving within physical media;
3) noncanonical as a necessary condition for interior dynamics, i.e., violation of the conditions of variational selfadjointness for the existence of a Hamiltonian [3];
4) Inhomogeneous, to represent experimental evidence on interior physical medium (e.g., local variation of the density $\mu$); and
5) Anisotropic, also to represent experimental evidence on interior media (e.g., as occurring under an intrinsic angular momentum of the media themselves).

The above invariance problem was solved by this author in ref.s [6–9] in

1982–1983 via: paper [8] on a general theorem on isosymmetries reviewed below; paper [9] on the first construction of the isotopies $\hat{O}(3)$ of the rotational symmetry O(3); and in paper [7] on the construction of the isotopies $\hat{O}(3.1)$ of the Lorentz symmetry O(3.1). The inclusion of the isotranslations to reach the isotopies $\hat{P}(3.1)$ of the Poincaré symmetry P(3.1) was done in memoir [16].[28]

The studies essentially permitted the formulation and proof of the following

**Theorem 4.6.1 - Fundamental Theorem on isosymmetries** [8]: *Let* G *be an N–dimensional Lie symmetry group of an m–dimensional metric or pseudo–metric space* S(x,g,F) *over a field* F(a,+,×) *of characteristic zero,*

$$G: \quad x' = A(w)\,x, \qquad x^{\dagger\prime} = x\,A^{\dagger}(w), \tag{4.6.2a}$$

$$(x'-y')^{\dagger}\,A^{\dagger}(w)\,g\,A(w)\,(x-y) \equiv (x-y)^{\dagger}\,g\,(x-y), \tag{4.6.2b}$$

$$A^{\dagger}\,g\,A = A\,g\,A^{\dagger} = g, \quad \mathrm{Det}\,A = \pm 1\,. \tag{4.6.2c}$$

*Then, the infinitely possible isotopies* $\hat{G}$ *of* G *characterized by the same generators and parameters of* G *and new isounits* $\hat{1}$ *(isotopic elements* T*) of Class III automatically leave invariant the isocomposition on the isospaces* $\hat{S}(x,\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{1} = T^{-1}$,

$$\hat{G}: \; x' = \hat{A}(\hat{w}) * x = \tilde{A}(\hat{w})\,x\,, \quad x^{\dagger\prime} = x^{\dagger} * \hat{A}^{\dagger}(\hat{w}) = x^{\dagger}\,\tilde{A}^{\dagger}(\hat{w})\,, \tag{4.6.3a}$$

$$(x'-y')^{\dagger} * \hat{A}^{\dagger}(\hat{w})\,\hat{g}\,\hat{A}(\hat{w}) * (x-y) \equiv (x-y)^{\dagger}\,\tilde{A}\,g\,\tilde{A}\,(x-y) = (x-y)^{\dagger}\,\hat{g}\,(x-y)\,, \tag{4.6.3b}$$

$$\hat{A}^{\dagger}\,\hat{g}\,\hat{A} = \hat{A}\,\hat{g}\,\hat{A}^{\dagger} = \hat{1}\,\hat{g}\,\hat{1}\,, \quad \textit{or} \tag{4.6.3c}$$

$$\tilde{A}^{\dagger}\,\hat{g}\,\tilde{A} = \tilde{A}\,\hat{g}\,\tilde{A}^{\dagger} = \hat{g}\,, \qquad \mathrm{Det}\,(\hat{A}\,T) = \mathrm{Det}\,\tilde{A} = \pm 1\,. \tag{4.6.3d}$$

As now familiar, the original symmetry G is generally linear–local–canonical, while the isosymmetries $\hat{G}$ are generally nonlinear, nonlocal and noncanonical when projected in the original space, owing to the arbitrary functional dependence of the isometric $\hat{g} = T(t, x, \dot{x}, \ddot{x}, ...)g$, although they are isolinear, isolocal and isocanonical in their proper isospace (Sect. 4.2).

The objective of this section is reduced in this way to the explicit construction of the isotopies of any given symmetry G via the original generators, the original parameters and the generalized metric $\hat{g}$. The invariance of the isoseparation is then guaranteed by Theorem 4.6.1.

[28] It should be noted that papers [8,9] were written prior to paper [7], but they ended up to be published in 1985, some two years after the appearance of the latter (1983) because of unreasonable editorial processing by several journals reported in details in page 26 of ref. [7].

Note that *the trivial isotopy* $X_k \Rightarrow X'_k = X_k \hat{1}$ *is excluded in Theorem 4.6.1, because it does not provide the invariance of the generalized metric.* This is due to the fact that the isotransformations characterized by the isoexponentiation of $X'_k$ coincide with the conventional ones

$$x' = \{ e_\xi^{\,i X' w} \} * x = \{ e^{\,i X \hat{1} T w} \} x = \{ e^{\,i X w} \} x, \qquad (4.6.4)$$

by losing in this way the crucial appearance of the isotopic element T in the exponent. This occurrence indicates the needs of using the Lie–isotopic theory in its entirety, and illustrates once more the reason for the preservation of the original basis under isotopies.

Let us now illustrate Theorem 4.6.1 with an explicit example. Consider the rotational symmetry G = SO(3) of the separation in Euclidean space E(r,δ,R)

$$G = SO(3):\ r' = \mathfrak{R}(\theta)\, r, \quad \mathfrak{R}(\theta)\, \mathfrak{R}^\dagger(\theta) = \mathfrak{R}^\dagger(\theta)\, \mathfrak{R}(\theta) = I,\ \text{Det.}\, \mathfrak{R} = +1, \qquad (4.6.5a)$$

$$r^2 = x^i\, \delta_{ij}\, x^j = = x^1 x^1 + x^2 x^2 + x^3 x^3 = \text{inv.}, \qquad (4.6.5b)$$

Consider now the most general possible deformation of the above invariant of Class III which, as such, can always be diagonalized into the form

$$r^2 = x^i\, \hat{\delta}_{ij}\, x^j = = x^1 g_{11} x^1 + x^2 g_{22} x^2 + x^3 g_{33} x^3 =$$

$$= \pm\, x^1 b_1^2 x^1 \pm x^2 b_2^2 x^2 \pm x^3 b_3^2 x^3 = \text{inv.}, \qquad (4.6.6a)$$

$$\hat{\delta} = T\delta = T = \text{diag.}(g_{11}, g_{22}, g_{33}) = \text{diag.}(\pm b_1^2, \pm b_2^2, \pm b_3^2), \qquad (4.6.6b)$$

$$b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) \neq 0. \qquad (4.6.6c)$$

Ref. [9] computed via Theorem 4.6.1 the symmetries of all infinitely possible deformations (4.6.6b). They are given by the isotopes (4.5.20) (see Ch. II.6 for details)

$$\hat{G} = S\hat{O}(3):\ r' = \hat{\mathfrak{R}}(\theta) * r, \qquad (4.6.7a)$$

$$\hat{\mathfrak{R}}(\theta) * \hat{\mathfrak{R}}^\dagger(\theta) = \hat{\mathfrak{R}}^\dagger(\theta) * \hat{\mathfrak{R}}(\theta) = \hat{1}, \qquad \text{Det.}(\hat{\mathfrak{R}}\, T) = +1, \qquad (4.6.7b)$$

$$\hat{\mathfrak{R}}(\theta) = \{ e_\xi^{\,J_1 * \hat{\theta}_1} \} * \{ e_\xi^{\,J_2 * \hat{\theta}_2} \} * \{ e_\xi^{\,J_3 * \hat{\theta}_3} \} =$$

$$= \{ e_\xi^{\,J_1 T \theta_1} \} \{ e_\xi^{\,J_2 T \theta_2} \} \{ e_\xi^{\,J_3 T \theta_3} \} \hat{1}. \qquad (4.6.7c)$$

where all quantities are known: the generators $J_k$ are in their adjoint representation (4.3.13), the parameters $\theta_k$ are the conventional Euler's angles, and the isotopic element T is that of deformation (4.6.6b).

The isosymmetry transformations can also be computed in the needed explicit form, because the convergence of isoexponentials (4.6.7c) is ensured by the original convergence plus the conditions for Class III (isotopic elements that are sufficient smooth, bounded, nowhere degenerate and Hermitean).

As an example, ref. [9] computed the following *isorotation* around the third axis:

$$\hat{\mathfrak{R}}(\theta_3) = \begin{pmatrix} \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{1} . \quad (4.6.8)$$

It is instructive to verify that isotransformations (4.6.7a) with realization (4.6.8) do indeed leave invariant generalized separation (4.6.6a). The following comments are now in order:

1) The $\hat{SO}(3)$ invariance of generalized separation (4.6.6a) is ensured by the original invariance SO(3) of the sphere (4.6.5b) for all infinitely possible deformations of the admitted Class III (Theorem 4.6.1);
2) The original SO(3) transformations (the ordinary rotations in Euclidean space) are linear, local and canonical, as well known. On the contrary, the covering $\hat{SO}(3)$ transformations (the isorotations (4.6.8)) are nonlinear, nonlocal and noncanonical owing to the arbitrary functional dependence of the $g_{kk}$–terms, although they are isolinear, isolocal and isocanonical in the sense of Sect. 4.2;
3) Owing to the general character of invariant (4.6.6a), Riemannian generalizations of the original Euclidean space are a particular case of isosymmetry (4.6.8) for $g_{kk} = g_{kk}(r)$, with the understanding that isosymmetries (4.6.8) are considerably more general than Riemann owing to their additional unrestricted dependence in the velocities, accelerations, etc.;
4) Isotransformations are already computed in the needed explicit form and there is no need of additional calculations. As an example, consider the lifting of the Euclidean metric $\delta$ into a Riemannian three–dimensional metric g(r), e.g., the space component of the Schwartzchild line element. Then the explicit symmetry of the latter is merely provided by plotting the $g_{kk}$ values in (4.6.8);
5) The classification of all possible isosymmetries (4.6.8) recovers again classification (4.3.18) at the level of the isoenvelopes, classification (4.4.30)–(4.4.37) at the level of Lie–isotopic algebras, and classification (4.5.21) at the level of Lie–isotopic groups, according to the following

invariances:[29]

$$
\begin{array}{ll}
SO(3): & x^1 x^1 + x^2 x^2 + x^3 x^3 = \text{inv.}, \\
SO(2.1): & x^1 x^1 - x^2 x^2 + x^3 x^3 = \text{inv.}, \\
S\hat{O}(3): & x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 = \text{inv.}, \\
S\hat{O}(2.1): & x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 + x^3 b_3{}^2 x^3 = \text{inv.}, \\
SO^d(3): & -x^1 x^1 - x^2 x^2 - x^3 x^3 = \text{inv.}, \\
SO^d(2.1): & -x^1 x^1 + x^2 x^2 - x^3 x^3 = \text{inv.}, \\
S\hat{O}^d(3): & -x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = \text{inv.}, \\
S\hat{O}^d(2.1): & -x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = \text{inv.},
\end{array}
\tag{4.6.9}
$$

In summary, the application of Theorem 4.6.1 to the deformations of the Euclidean space provides the invariance not only of all possible ellipsoidical deformations of the sphere, but also of all possible hyperbolic deformations, thus admitting as particular cases the conventional rotational and Lorentz symmetries, all their infinitely possible isotopes, their isoduals and all the infinitely possible isodual isotopes.

As we shall see in Ch. II.8, one of the first applications of the Lie–isotopic theory is the construction of the general invariant of conventional exterior gravitation and the proof that it is locally isomorphic to the conventional Poincaré symmetry of the special relativity.

In Ch. II–8 we shall show that, starting from the familiar invariance of the separation in Minkowski space

$$
P(3.1): \qquad x^\mu \eta_{\mu\nu} x^\nu = \text{inv.}, \qquad \eta \in M(x,\eta,R), \tag{4.6.10}
$$

Theorem 4.6.1 permits the construction of the general invariance for all possible Riemannian separations

$$
\hat{P}(3.1): \quad x^\mu g_{\mu\nu}(x) x^\nu = \text{inv.}, \quad g \in \Re(x,g,R), \tag{4.6.11}
$$

via the decomposition $g(x) = T(x)$ and the construction of the isosymmetries $\hat{P}(3.1)$ with respect to the generalized isounit $\hat{1} = [T(x)]^{-1}$. The invariance of the Riemannian separation (4.6.11) is then ensured by Theorem 4.6.1[30].

[29] Note that for hyperbolic invariants the trigonometric functions of (4.6.8) become hyperbolic functions, exactly as they should be).

[30] The attentive reader may have noted that isorotations (4.6.8) do already contain the solution for (2+1) dimensional Riemannian spaces. We are therefore referring here to the extension of the results to the full (3+1)-dimensional Riemannian space.

The isotopic unification of the Minkowski and Riemannian spaces of Sect. 3.3 will be carried over, in this way, to the unification of symmetries of the special and general relativities as a foundation for their isotopies for interior problems.

The relevance of Theorem 4.6.1 is further illustrated by the fact that all isosymmetries of hadronic mechanics studied in Vol.s II and III are particular applications of Theorem 4.6.1.

The understanding is that signature changing deformations, e.g., (+, +, +) → (+, −, +), cannot be reached in actual experiments and, therefore, they have a mere mathematical significance. This is the reason that practical applications of the isotopies are restricted to Class I which ensures the preservation of the original signature.

In summary, Theorem 4.6.1 is "directly universal" for all infinitely possible isotopies $g \Rightarrow \hat{g} = Tg$ of Class III. The "direct universality" of hadronic mechanics for the treatment of nonlinear–nonlocal–nonhamiltonian systems is then consequential, as we shall see.

## 4.7: ISOREPRESENTATION THEORY

Recall that the representation theory of Lie algebras has profound physical implications because it characterizes the contemporary notion of *particles* for the exterior problem in vacuum

A primary objective of the representation theory of the covering Lie-isotopic algebras is that of characterizing a generalized notion of particle for interior problems within physical media called *isoparticles* [22]. The still more general representation theory of Lie-admissible algebras will then characterize a yet more general notion of particles called *genoparticles* [loc. cit.] which are studied in Ch. I.7. The corresponding antiparticles are characterized via the corresponding isodual methods.

In this section we shall identify the mathematical difference between the notions of particles and isoparticles via a study of the isorepresentations of Lie-isotopic algebras of Class I or II over an isofield $\hat{F}(\hat{\alpha},+,*)$ of isocharacteristic zero. The mathematical differences with the more general notion of genoparticles will be studied in Ch. I.7.

The physical differences between the notions of particles, isoparticles and genoparticles will be studied in Vol. II after the identification of the isotopies and genotopies of the Galilei and Poincaré symmetries.

The representation theories of isoalgebras of Classes IV and V are unknown at this writing.

Consider a vector space U with elements a, b, c, ... and abstract composition (product) "ab" over a field $F(\alpha,+,\times)$. We shall say that U constitutes an *algebra* when it verifies the scalar and distributive laws (Definition 2.4.1). The algebra U is said to be *associative (nonassociative)* when ab is associative (nonassociative).

The *right and left multiplications* in U (see, e.g., ref. [34]) are given by the following linear transformations of U onto itself as a vector space

$$R_x : a \Rightarrow a\,x, \quad \text{or} \quad a\,R_x = a_x, \tag{4.7.1a}$$

$$L_x : a \Rightarrow x\,a, \quad \text{or} \quad L_x a = x\,a, \tag{4.7.1b}$$

for all a, x ∈ U, and verify the following general properties

$$(a\,\alpha)\,R_x = (a\,\alpha)\,x = a\,(\alpha\,x), \quad \text{or} \quad \alpha\,R_x = R_{\alpha x}, \tag{4.7.2a}$$

$$a\,R_{(x+y)} = a\,(x + y) = a\,R_x + a\,R_y = a(R_x + R_y),$$

$$\text{or}\ R_{(x+y)} = R_x + R_y, \tag{4.7.2b}$$

with evident similar properties for the left multiplications $L_x$.

When the algebra is associative, we have the additional properties

$$a\,(x\,y) = (a\,x)\,y, \quad \text{i.e.,} \tag{4.7.3a}$$

$$a\,R_{xy} = a\,R_x\,R_y, \ \text{or}\ R_{xy} = R_x\,R_y, \tag{4.7.3b}$$

$$(x\,y)\,a = x\,(y\,a), \ \text{i.e.} \tag{4.7.3c}$$

$$L_{xy}\,a = L_x\,L_y\,a, \quad \text{or}\ L_{xy} = L_x\,L_y. \tag{4.7.3d}$$

The above properties imply that the mapping $a \Rightarrow R_a$ ($a \Rightarrow L_a$) is a homomorphism (antihomomorphism) of A into the associative algebra V(A) of all linear transformations in A. Thus, they provide a *right representation* $a \Rightarrow R_a$ or a *left representation* $a \Rightarrow L_a$, respectively, of A, also called left or right $\mathrm{Hom}^A{}_F(V_p)$, for p = Right or Left.

If the algebra A contains the identity I, we have a *one–to–one (or faithful) representation* because $R_a = R_b$ implies $IR_a = IR_b$ which can hold iff a = b. When the space V is the algebra A itself, we have the so–called *adjoint representation* also called *fundamental or regular representation.*

In the case of nonassociative algebras, the mapping $a \Rightarrow R_a$ is no longer a homomorphism, and this illustrates the reason for the study of the representation theory of Lie algebras via that of the underlying universal enveloping associative algebra, as done in the mathematical literature, e.g., ref. [31] (but generally not in the physical literature).

Consider now an isoassociative algebra $\hat{A}$ over an isofield $\hat{F}(\hat{\alpha},+,*)$ with isounit $\hat{1}$ and isoassociative product a∗b. Introduce the *right and left* *isomultiplications*

$$\hat{R}_x: a \Rightarrow a * x, \quad \text{or} \quad a * \hat{R}_x = a * x, \tag{4.7.4a}$$

$$\hat{L}_x : a \Rightarrow x * a, \quad \text{or} \quad a * \hat{L}_x = x * a, \tag{4.7.4b}$$

for all $a \in \hat{A}$. It is then easy to see that properties (4.7.2) and (4.7.3) are lifted into the forms

$$\alpha * \hat{R}_x = \hat{R}_{\alpha * x}, \quad \hat{R}_{(x+y)} = \hat{R}_x + \hat{R}_y, \tag{4.7.5a}$$

$$\hat{R}_{x * y} = \hat{R}_x * \hat{R}_y, \quad \hat{1} * \hat{R}_a = \hat{1} * \hat{R}_b \quad \Rightarrow \quad a = b, \tag{4.7.5b}$$

with similar properties for the left isomultiplications.

It is easy to see that the mapping $a \Rightarrow \hat{R}_a$ characterizes a *right, faithful, isorepresentation* of $\hat{A}$ in the isoassociative algebra $\hat{V}(\hat{A})$ of isolinear transformations on $\hat{A}$ denoted $\mathrm{Hom}^{\hat{A}}{}_{\hat{F}}(\hat{V}_R)$, with similar results holding for the left isorepresentations.

The nontriviality of the isotopy is made clear by the following

**Lemma 4.7.1** [4]: *Isorepresentations of isoassociative algebras $\hat{A}$ over an isofield $\hat{F}(\hat{a},+,*)$ are isolinear and isolocal in $\hat{V}$ but generally nonlinear and nonlocal when projected in* V.

Thus, *the transition from Lie algebras to Lie-isotopic algebras generally implies the transition from linear-local-canonical to nonlinear-nonlocal-noncanonical representations*. Recall that the contemporary notion of point-like particle in vacuum is essentially a manifestation of the linear-local-canonical character of the theory. The more general isoparticles will then result to be a manifestation of the covering nonlinear-nonlocal-noncanonical character of the isorepresentation theory.

A *module* of an algebra U over a field F, also called *U-module* [34] is a linear vector space V over $F(\alpha,+,\times)$ together with a mapping $U \times V \Rightarrow V$ denoted with the symbol $(a, v) \Rightarrow av$ which verifies the distributive and scalar rules

$$a\,(v + t) = a\,v + a\,t, \qquad (a + b)\,v = a\,v + b\,v, \tag{4.7.6a}$$

$$\alpha\,(a, v) = (\alpha\, a, v) = (a, \alpha\, v), \tag{4.7.6b}$$

as well as all the axioms of U, for all $a, b \in U$, $v, t \in V$, and $\alpha \in F$.

The mappings $a \Rightarrow R_v = av$ and $a \Rightarrow L_v = va$ show that the space V is a left and right U-module.

The notion of one-sided, left or right *isomodule* of an isoalgebra $\hat{U}$ over an isofield $\hat{F}$ was introduced in ref. [35] and it is given by a straighforward isotopy of the preceding structure. The more general notion of two-sided, left and right *isobimodule* was also introduced in ref. [35] as reviewed in Ch. I.7.

The isomodules are sufficient for the representation theory of Lie-isotopic algebras, but the more general isobimodules are necessary for the more general Lie-admissible algebras. The above structures permit a first characterization of the notion of particles of hadronic mechanics as follows:

**1) Conventional particles,** those characterized by linear-local-canonical representations of Lie symmetries on a one sided module;

**2) Isoparticles,** those characterized by nonlinear-nonlocal-noncanonical representations of Lie-isotopic symmetries on one-sided isomodules;

**3) Genoparticles,** those characterized by nonlinear-nonlocal-noncanonical representations of Lie admissible algebras on two-sided isobimodules (CH. I.7).

As we shall see in Vol. II and III, the above characterizations yield notions of particles for physical conditions of increasing complexities, such as for a particle when members of an atomic structure (conventional particle), when member of a hadronic structure (isoparticle) and when in the core of a collapsing star (genoparticle).

By keeping in mind the results of preceding sections on isodual isoalgebras, we are now equipped to study the isorepresentations of Lie-isotopic algebras via an isotopy of the conventional representation theory in terms of enveloping associative algebras.

**Definition 4.7.1**: *Let $\hat{\xi}$ be the universal enveloping isoassociative algebra of a Lie-isotopic algebra $\hat{L} \approx \hat{\xi}^-$ of Class I. Then, the one-sided, right or left, "isorepresentations"* $\mathrm{Hom}^{\hat{\xi}}{}_{\hat{F}}(\hat{V}_p)$, $p = R$ *or* $L$, *of $\hat{\xi}$ on a corresponding, one sided isomodule $\hat{V}$ over an isofield $\hat{F}(\alpha,+,*)$ are characterized by*

$$\hat{R}_{a * b} = \hat{R}_a * \hat{R}_b , \tag{4.7.7a}$$

$$R_{\hat{e}} = \hat{1} . \tag{4.7.7b}$$

*The "isodual isorepresentations" of $\hat{L}^d$ on $\hat{V}^d$ over $\hat{F}^d(\hat{a}^d,+,*^d)$ are the isodual images of* $\mathrm{Hom}^{\hat{\xi}}{}_{\hat{F}}(\hat{V}_p)$ *characterized by*

$$\hat{R}(\hat{w})_a \rightarrow \hat{R}^d(\hat{w}^d)_a = -\hat{R}(-\hat{w})_a . \tag{4.7.8a}$$

$$\hat{R}_{\hat{e}} = \hat{1} \rightarrow \hat{R}^d{}_{\hat{e}^d} = \hat{1}^d = -\hat{1} . \tag{4.7.8b}$$

Conditions (4.7.7b) and (4.7.8b) ensure the invertibility of the elements, in the sense that

$$\hat{R}_{a * a^{-1}} = \hat{R}_a * \hat{R}_{a^{-1}} = \hat{R}_{\hat{e}} = \hat{1} . \tag{4.7.9a}$$

$$\hat{R}_{a^{-1}} = ( \hat{R}_a )^{-\hat{1}} . \tag{4.7.9a}$$

It should be indicated that isorepresentations (4.7.7) exhaust all isolinear and isolocal cases, but they are not expected to be unique. In fact, an additional class of nonlinear representations emerges in the conventional case [31–33] and a similar occurrence is expected in the isotopic case.

The matrix form of isorepresentations are also given by a simple isotopy of conventional matrix forms [31–33]. Let $\hat{e}_k$ , $k = 1, 2, ..., N$ be an isobasis of $\hat{A}$ which is *isoorthonormal*, i.e.,

$$( \hat{e}_i \hat{,} \hat{e}_j ) = \hat{\delta}_{ij} = \hat{1}\, \delta_{ij} \tag{4.7.10}$$

where $( \hat{,} )$ is the isoscalar product on the isomodule and $\hat{\delta}_{ij}$ is called the *isokronecker delta*. The desired matrix form of the isorepresentation is then given by

$$\hat{R}_a * \hat{e}_k = \hat{D}_{ik}(a) * \hat{e}_i , \tag{4.7.11a}$$

i.e.,

$$\hat{D}_{ij}(a) := ( \hat{R}_a * \hat{e}_j \hat{,} \hat{e}_i ) . \tag{4.7.11a}$$

Note that the matrix for of the product $\hat{R}_{a * b}$ is given by

$$\hat{D}_{ij} ( a * b ) = \textstyle\sum_{r,s} \hat{D}_{ir}(a)\, T_{rs}\, \hat{D}_{sj}(b) \tag{4.7.12}$$

From the above properties it is easy to see the following

**Lemma 4.7.2:** *The dimension of the representation of a Lie algebra does not change under isotopies.*

We now study the "degrees of freedom" of isorepresentations. Let $a \rightarrow R_a$ be an isorepresentation of an algebra $\hat{A}$ over $\hat{F}(\hat{\alpha},+,*)$ on an isomodule $\hat{V}$. Let

$$\hat{S}: \ \hat{V} \rightarrow \hat{V}' , \tag{4.7.13}$$

be a (bounded, sufficiently smooth and regular) isomap of $\hat{V}$ into $\hat{V}'$. Then the isomap

$$\hat{\Gamma} : \ a \ \rightarrow \ \hat{R}'_a = \hat{S} * \hat{R}_a * \hat{S}^{-\hat{1}} , \tag{4.7.14}$$

characterizes the image of the isorepresentation $\hat{R}_a$ in $\hat{V}'$ because

$$\hat{R}'_{a*b} = \hat{S} * \hat{R}_a * \hat{S} * \hat{S}^{-1} * \hat{R}_b * \hat{S}^{-1} = \hat{R}'_a * \hat{R}'_b \ , \quad \hat{R}'_{\hat{e}} = \hat{1} \, . \tag{4.7.15}$$

**Definition 4.7.2:** *Recall that a representation* $R_a$ *on a module* V *and* $R'_a$ *on another module* V′ *over* $F(a,+,\times)$ *are said to be "equivalent" when there is an invertible map* $S\colon V \to V'$ *such that*

$$R_a = S R_a S^{-1}, \tag{4.7.22}$$

*and they are said to be "unitarily equivalent" for the particular case*

$$R_a = S R_a S^{\dagger}, \quad S S^{\dagger} = S^{\dagger} S = I, \quad S^{-I} = S^{\dagger} . \tag{4.7.16}$$

*A isorepresentations* $\hat{R}_a$ *on an isomodule* $\hat{V}$ *and* $\hat{R}'_a$ *on* $\hat{V}'$ *over an isofield* $\hat{F}(\hat{a},+,*)$ *are said to be "isoequivalent" when there exist an sufficiently smooth invertible isomap* $\hat{S}\colon \hat{V} \to \hat{V}'$ *such that for all elements* $a \in \hat{A}$

$$\hat{R}'_a \equiv \hat{S} * \hat{R}_a * \hat{S}^{-1} , \tag{4.7.17}$$

*and they are said to be "isounitarily equivalent" for the particular case*

$$\hat{R}'_a = \hat{S} * \hat{R}_{a6} * \hat{S}^{\dagger} , \quad \hat{S} * \hat{S}^{\dagger} = \hat{S}^{\dagger} * \hat{S} = \hat{1} \, , \hat{S}^{\dagger} = \hat{S}^{-1} . \tag{4.7.18}$$

It is then easy to prove the following

**Lemma 4.7.3** : *Let* D *be a (finite-dimensional) representation of a Lie algebra* L *and* $\hat{D}$ *the corresponding isorepresentation of the Class I isotope* $\hat{L}$ *of* L, *in which case* $\hat{L}$ *is isomorphic to* L, $\hat{L} \approx L$, *and the dimensions of* $\hat{D}$ *and D are the same. Then* $\hat{D}$ *and* D *are not unitarily equivalent or equivalent.*

As in the conventional case (see, e.g., [33]), the notion of isoequivalence of isorepresentations is reflexive, symmetric and transitive. In fact, every isorepresentation is isoequivalent to itself; if an isorepresentation $\hat{R}_a$ is isoequivalent to $\hat{R}'_a$, then $\hat{R}'_a$ is isoequivalent to $\hat{R}_a$; etc. Thus, the set of all isorepresentations can be divided into *isoequivalence classes.*

In the conventional Lie theory only one matrix representation per each equivalence class is considered [33]. This is due to the fact that *the matrices of two equivalent representations can be made to coincide with a suitable selection of the basis.* In fact, the basis $e_i$ for V and $e'_i = Se_i$ for V′ yield the same matrix representations,

$$D(a)\,e_k = \sum_i D_{ik}\,e_i \;\rightarrow\; D'(a)\,e'_k = \sum_i D'_{ik}\,e'_i = S \sum D_{ik}\,e_i\,, \qquad (4.7.\,19)$$

Under isotopy we evidently have the corresponding image, in the sense that we can indeed select the isobasis $\hat{e}_i$ on $\hat{V}$ and $\hat{e}'_i = \hat{S} * \hat{e}_i$ on $\hat{V}'$, thus reaching the similar results

$$\hat{D}(a) * \hat{e}_k = \sum_{r,s} \hat{D}_{rk}\,T_{rs}\,e_s \;\rightarrow$$

$$\rightarrow\; \hat{D}'(a) * \hat{e}'_k = \sum_{r,s} \hat{D}'_{rk}\,T_{rs}\,\hat{e}'_s = \hat{S} * \sum_{r,s} \hat{D}_{rk}\,T_{rs}\,e_s\,. \qquad (4.7.\,20)$$

However, isoequivalent but different isorepresentations play an important role in the Lie–isotopic theory, particularly for physical applications, as illustrated below in this section.

Recall that for a conventional N–dimensional Lie algebra L with generators $X_i$, the structure constants $C_{ij}{}^k$ characterize the adjoint representation of $X_i$ with matrix elements

$$(X_i)_j{}^k = -\,C_{ij}{}^k\,. \qquad (4.7.\,21)$$

The repetition of the conventional proof via the use of the Isotopic Second and Third Theorem (Sect. 4.4) then leads to the following

**Lemma 4.7.4**: *Let $\hat{L}$ be a Lie–isotopic algebra with generators $X_i$ and structure functions $\hat{C}_{ij}{}^k(t, x, \dot{x}, \ddot{x}, \ldots) = C_{ij}{}^k \hat{1}$, Eq.s (4.4.7). Then, up to isoequivalence, one class of "isoadjoint isorepresentations' of $\hat{L}$ is characterized by the matrix elements*

$$(X_i)_j{}^k = -\,\hat{C}_{ij}{}^k(t, x, \dot{x}, \ddot{x}, \ldots)\,. \qquad (4.7.\,22)$$

Additional types of adjoint representations will be identified shortly.

Note the constancy of the elements of the adjoint representation in the conventional case, as compared to an arbitrary functional dependence of the corresponding elements under isotopy.

Consider an isolinear space $\hat{\mathcal{H}}$ equipped with an isoscalar product $(x \hat{,} y)$. As we shall see in Ch. I.6, an operator X of an isoenvelope $\hat{\xi}$ is called *isohermitean* when

$$(X^{\hat{\dagger}} * x \hat{,} y) \equiv (x \hat{,} X * y)\,. \qquad (4.7.\,23)$$

Consider now an isobasis $\hat{e}_i$ which is isonormalized with respect to the product $(\ldots \hat{,} \ldots)$, i.e., satisfying Eq.s (4.7.15).

**Definition 4.7.3:** *Let $\hat{D}$ be an isorepresentation of a Lie–isotopic group $\hat{G}$ with respect to basis (4.7.15) on an isolinear space $\hat{\mathcal{H}}$. Then the "isohermitean conjugate" $\hat{D}^{\dagger}$ of* D *is given by*

$$\hat{D}_{ij}^{\hat{\dagger}}(a) = \overline{\hat{D}_{ji}}(a^{-1}), \quad a \in \hat{G}. \tag{4.7.24}$$

*where the upper bar denotes complex conjugate. The isorepresentation is called "isounitary" when it coincides with its isohermitean conjugate,*

$$\hat{D} \equiv \hat{D}^{\dagger}. \tag{4.7.25}$$

An inspection of the structure of the isorepresentations leads to the following classification.

**Definition 4.7.4** [54]: *Let* $X_k$, k = 1, 2, .., n, *represent a maximal commuting set of a Lie algebra* L *(such as* $J^2$ *and* $J_3$ *for the* so(3) *algebra) and let* $S_k{}^{\circ}$ $\in$ R(n,+,×) *be its spectrum of eigenvalues with respect to a given basis* $| \, b >$,

$$X_k \, | \, b > = S_k{}^{\circ} \, | \, b >, \tag{4.7.26}$$

*(such as* L(L + 1) *and* M = L, L – 1, ..., –L, L = 0, 1, 2, ... *for* so(3)) *characterizing a set of representations* D *of* L. *Let* $\hat{L}$, $\hat{X}_k$ *and* $| \, \hat{b} >$ *be the corresponding isotopes of Class I, and let* $S_k(T) \in \hat{R}(\hat{n},+,*)$ *be the corresponding new spectrum of eigenvalues,*

$$X_k * | \, \hat{b} > = X_k \, T \, | \, \hat{b} > = \hat{S}_k(T) * | \, \hat{b} > = S_k(T) \, | \, \hat{b} >, \tag{4.7.27}$$

*Then, the isorepresentation* $\hat{D}$ *of* $\hat{L}$ *is said to be*
*A) "regular" when the isotopic spectrum* $S_k(T)$ *is entirely factorizable into the form*

$$S_k(T) = S^{\circ}{}_k \, f_k(\Delta), \quad \text{no sum}, \quad \Delta = \det T; \tag{4.7.28}$$

*where* $f_k(\Delta)$ *are smooth functions of* $\Delta$ *such that* $f_k(1) = 1$, k = 1, 2, ..., n;
*B) "irregular" when the above factorization does not exist for at least one element of the spectrum* $S_k(T)$; and
*C) "standard" when the isotopic and conventional spectra coincide,* $S_k \equiv S^{\circ}{}_k$, k = 1, 2, ..., n, *but the two representations* $\hat{D}$ *and* D *are not equivalent.*

We learn in this way that the spectrum of eigenvalues of a Lie representation can be preserved under a particular type of isotopy called standard, but the structure of the representation is generalized. This property is manifestly relevant for physical applications because, as we shall illustrate below and study in details in Vol. II and III, isotopic techniques permit the preservation of conventional quantum numbers under new functional degrees of freedom in the representations.

In turn, the latter permit physical applications which are prohibited in conventional theories, such as the exact representation of the still unknown total magnetic moments for few-body nuclei via the representation of the deformability of the charge distribution of protons and neutrons when members of a nuclear structure with consequential alteration of their intrinsic magnetic moments.

One explicit form of the regular and standard isoadjoint isorepresentations can be easily constructed from the corresponding representations via a rule here called *Klimyk's rule* [52] (although the rule does not apply for irregular and other isorepresentations). Let $D_k$ denote the adjoint representation of a given Lie algebra L, and introduce the matrix P such that

$$P = K \hat{1}, \quad P\,T = K^{-1}\, I, \qquad K \in F(\alpha,+,\times). \tag{4.7.29}$$

It is then easy to see that the expression

$$\hat{D}_k = D_k\, P, \qquad PT = K\, I, \tag{4.7.30}$$

characterizes the regular isoadjoint isorepresentation of the isotope $\hat{L}$ (up to isoequivalence). In fact, rule (4.7.35) reduces the isocommutator to the ordinary one according to the rule

$$[\hat{D}_i \hat{,} \hat{D}_j] = \hat{D}_i\, T\, \hat{D}_j - \hat{D}_j\, T\, \hat{D}_i = D_i\,(P\,T)\,D_j\, P - D_j\,(P\,T)\,D_i\, P =$$

$$= K\,(D_i\, D_j - D_j\, D_i)\,P = K\, C_{ij}{}^k\, D_k\, P = \hat{C}_{ij}{}^k * \hat{D}_k, \tag{4.7.31a}$$

$$\hat{C}_{ij}{}^k = K\, C_{ij}{}^k\, \hat{1}. \tag{4.7.31b}$$

The isoeigenvalues of the diagonal elements, say $\hat{D}_\alpha$, are also reduced to conventional eigenvalues multiplied by an isotopic factor,

$$\hat{D}_\alpha * |\hat{b}> = D_\alpha\,(P\,T)\,|\hat{b}> = K\, D_\alpha\,|\hat{b}> = K\, S_\alpha^\circ\,|\hat{b}>, \tag{4.7.32}$$

and the same holds for the isocasimir invariants, e.g.,

$$\hat{D}^2 * |\hat{b}> = \textstyle\sum_k D_k\,(P\,T)\,D_k\,(P\,T)\,|\hat{b}> = K^2\, D^2\,|\hat{b}> = K^2\, S\,|\hat{b}>, \tag{4.7.33}$$

thus confirming the "regular" character of the isorepresentation $\hat{D}$.

The "standard" adjoint isorepresentations are evidently the particular case of the regular when

$$\hat{D}_k = D_k P, \qquad P = \hat{1}, \quad PT = I, \quad K = I. \tag{4.7.34}$$

**Lemma 4.7.5 - Klimyk's rule** [52] : *Let* D *be the adjoint representation of a Lie algebra* L *and let* $\hat{L}$ *be a Class I isotope of L. Then, the regular adjoint isorepresentation of* $\hat{L}$ *is given by rule (3.7.30) and the standard adjoint isorepresentation of* $\hat{L}$ *is given by rule (4.7.34).*

It should be stressed that rules (4.6.30) and (4.7.34) *are not* equivalence transformations, i.e., *there exist no matrix* U *such that*

$$\hat{D}_k = D_k P = U D_k U^{-1}, \tag{4.7.35}$$

for all k = 1, 2, ..., N. Thus, adjoint representations and isorepresentations are not equivalent. Also, there exists no known rule for the construction of the irregular isorepresentations from conventional ones.

it is understood that the above differences between representations and isorepresentations characterize the desired mathematical differences between particles and isoparticles.

It is an instructive exercise for the interested reader to work out the definitions of *isoreducibility* and *isoirreducibility, isotensor product* and other known aspects of the conventional Lie's theory. For additional mathematical studies we refer the interested reader to ref. [24].

We now illustrate the results of this section with specific examples. Consider the adjoint representation of the su(2) Lie algebra on the complex Euclidean, two-dimensional, space E(z,δ,C). It is given by the celebrated *Pauli's matrices* we encountered in the representation of quaternions, Eq.s (2.7.6), and in the illustration of Theorem 4.4.1,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.7.36}$$

which satisfy the associative rules

$$\sigma_n \sigma_m = 2 i \epsilon_{nmk} \sigma_k + \delta_{nm}, \quad n, m = 1, 2, 3, \tag{4.7.37}$$

where $\epsilon_{nmk}$ is the conventional totally antisymmetric tensor of rank three, and the Lie rules

$$[\sigma_n, \sigma_m] = \sigma_n \sigma_m - \sigma_m \sigma_n = 2 i \epsilon_{nmk} \sigma_k, \tag{4.7.38}$$

with Casimir invariant $\sigma^2 = \sum_{k=1,2,3} \sigma_k{}^2$, maximal commuting set $X = \{\sigma_k, \sigma^2\}$, and eigenvalues on a two–dimensional orthogonal basis $|\,b>$

$$\sigma_3\,|\,b> \;=\; \pm\,|\,b>, \quad \sum_{k=1,2,3}\sigma_k{}^2\,|\,b> \;=\; 3\,|\,b>, \tag{4.7.39}$$

The isotopies of Pauli's matrices were outlined at the 1993 *International Third Wigner Symposium* at Oxford University [45] and then studied in detail in ref. [54]. They are reviewed in detail in Ch. II.6 where we also construct the irreducible isorepresentations of the Lie–isotopic algebra $\hat{su}(2)$. In this section it is sufficient to indicate that the isotopy here considered begins with the lifting of $E(z,\delta,R)$ into the complex isoeuclidean space

$$\hat{E}(z,\hat{\delta},\hat{R}):\;\; \hat{\delta} \;=\; T\,\delta, \quad T = \text{diag.}\,(\,g_{11}, g_{22}\,), \quad \hat{1} \;=\; \text{diag.}\,(\,g_{11}{}^{-1}, g_{22}{}^{-1}\,)\,. \tag{4.7.40}$$

The Lie–isotopic group $\hat{SU}(2)$ is then the invariance of the generalized expression

$$z^{\dagger}\,\hat{\delta}\,z \;=\; \bar{z}_1\,g_{11}\,z_1 + \bar{z}_2\,g_{22}\,z_2\,, \tag{4.7.41a}$$

$$z' \;=\; \hat{U} * z, \quad \hat{U} * \hat{U}^{\dagger} \;=\; \hat{U}^{\dagger} \hat{*}\, \hat{U} = \hat{1}, \quad \det\,(\hat{U}\,T) \;=\; 1, \tag{4.742b).}$$

$$\hat{U} \;=\; e_{\xi}{}^{\,i\,\hat{\sigma}_k * \hat{\theta}_k} = \{\,e^{\,i\,\hat{\sigma}_k\,T\,\theta_k}\,\}\,\hat{1}\,, \quad \text{tr.}\,(\hat{\sigma}\,T\,) = 0\,, \tag{4.7.43c)}$$

with Lie–isotopic algebra for the isoadjoint isorepresentation

$$[\,\hat{\sigma}_n \,\hat{,}\, \hat{\sigma}_m\,] \;=\; \hat{\sigma}_n\,T\,\hat{\sigma}_m - \hat{\sigma}_m\,T\,\hat{\sigma}_n \;=\; 2\,i\,\hat{C}_{nmk}(t, z, \bar{z}, ...)\,T\,\hat{\sigma}_k\,, \tag{4.7.44}$$

where the $\hat{C}$'s are the structure functions of $\hat{su}(2)$ as identified below. The following adjoint isorepresentations of $\hat{su}(2)$ were then constructed in ref.s [45,54]:

**A) Regular isopauli matrices**, they are given from rule (4.7.35) by

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 \;=\; \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -\,i\,g_{11} \\ +\,i\,g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 \;=\; \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -\,g_{11} \end{pmatrix} \tag{4.7.45a)}$$

$$P = \Delta^{-\frac{1}{2}}\,\text{dig.}\,(\,g_{22}, g_{111}) = \Delta^{\frac{1}{2}}\,T^{-1}, \;\; K \;=\; \Delta^{\frac{1}{2}}\,, \;\; \Delta = \det Q \;=\; g_{11}g_{22} > 0, \tag{4.7.45b)}$$

with the isoassociative rules

$$\hat{\sigma}_i\;T\,\hat{\sigma}_j = \;2\,\Delta^{\frac{1}{2}}\,i\,\epsilon_{ijk}\,\hat{\sigma}_k \;+\Delta^{\frac{1}{2}}\,\hat{1}\delta_{ij}, \tag{4.7.46}$$

and, consequential isocommutator rules

$$[\hat{\sigma}_n \hat{,} \hat{\sigma}_m] = \hat{\sigma}_n T \hat{\sigma}_m - \hat{\sigma}_m T \hat{\sigma}_n = 2 \Delta^{\frac{1}{2}} i \epsilon_{nmk} \hat{\sigma}_k , \qquad (4.7.47)$$

Note the identity in this case of the structure functions with the conventional structure constants of su(2) up to the multiplicative term $\Delta^{\frac{1}{2}}$, thus confirming the local isomorphism $\hat{su}(2) \approx su(2)$. The isoeigenvalues are generalized and are given by

$$\hat{\sigma}_3 * | \hat{b}_i^2 > = \pm \Delta^{\frac{1}{2}} | \hat{b}_i^2 >, \qquad (4.7.48a)$$

$$\hat{\sigma}^2 * | \hat{b}_i^2 > = \sum_k \hat{\sigma}_k * \hat{\sigma}_k * | \hat{b} > = 3 \Delta | \hat{b}_i^2 >, i = 1, 2 \qquad (4.7.48b)$$

thus confirming the "regular" character of the isotopy here considered (i.e., the factorizability of the isotopic contribution in the spectrum of eigenvalue). The isonormalized isobasis is then given by a trivial extension of the conventional basis, $| \hat{b} > = T^{-\frac{1}{2}} | b >$.

It is instructive to verify that isorepresentation (4.7.45a) is indeed derivable from Klimyk's rule.

**B) Irregular isopauli matrices,** they must be constructed via the full use of the isorepresentation theory resulting in expressions of the type

$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \ \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2 , \ \hat{\sigma}_3' = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat{1} \sigma_3, \quad (4.7.49)$$

with isocommutation rules

$$[ \hat{\sigma}'_1 \hat{,} \hat{\sigma}'_2] = 2 i \hat{\sigma}'_3, \quad [ \hat{\sigma}'_2 \hat{,} \hat{\sigma}'_3 ] = 2 i \Delta \hat{\sigma}'_1 , [ \hat{\sigma}'_1 \hat{,} \hat{\sigma}_1' ] = 2 i \Delta \hat{\sigma}'_2, \quad (4.7.50)$$

which evidently do not alter the local isomorphism $\hat{SU}_Q(2) \approx SU(2)$. The new isoeigenvalue equations are given by

$$\hat{\sigma}'_3 * | \hat{b}_i^2 > = \pm \Delta | \hat{b}_i^2 >, \quad \hat{\sigma}''^{\hat{2}} * | \hat{b}_i^2 > = \Delta ( \Delta + 2) | \hat{b}_i^2 > , \qquad (4.7.51)$$

which confirm the "irregular" character under consideration (i.e., lack of factorizability of the isotopic contributions in all elements of the spectrum). As one can see, isorepresentations (4.7.52) are far from trivial because they imply the lifting of the notion of spin ½ into a local quantity

$$s = \tfrac{1}{2} \quad \rightarrow \quad \hat{s} = \tfrac{1}{2} \Delta(t, z, \bar{z}, \psi, \psi^\dagger, .....) . \qquad (4.7.52)$$

as expected for a particle in hyperdense interior conditions (e.g., a proton in the core of a collapsing star).

It is instructive to verify that isorepresentation (4.5.49) *is not* derivable from Klimyk's rule.

**C) Standard isopauli matrices,** which occur when $K = 1$, resulting in the expressions

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & g_{22}^{-1} \\ g_{11}^{-1} & 0 \end{pmatrix}, \hat{\sigma}''_2 = \begin{pmatrix} 0 & -ig_{22}^{-1} \\ ig_{11}^{-1} & 0 \end{pmatrix}, \hat{\sigma}''_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \quad (4.7.53)$$

possessing isocommutation rules with conventional structure constants,

$$[\hat{\sigma}''_n \hat{,} \hat{\sigma}''_m] = \hat{\sigma}''_n T \hat{\sigma}''_m - \hat{\sigma}''_m T \hat{\sigma}''_n = 2 i \epsilon_{nmk} \hat{\sigma}''_k, \quad (4.7.57)$$

and admitting conventional eigenvalues

$$\hat{\sigma}''_3 * |\hat{b}> = \pm |\hat{b}, \quad \hat{\sigma}''^{\hat{2}} * |\hat{b}> = 3 |\hat{b}>. \quad (4.7.54)$$

Yet, isorepresentations (4.7.56) exhibit the "hidden variables" $g_{kk}$ in their very structure. Also, the above matrices are not unitarily equivalent to the conventional Pauli's matrices, thus establishing the "standard" character of the isorepresentation.

It is instructive to verify that isorepresentations (4.7.53) is indeed derivable from Klimyk's rule.

This illustrates Definition 4.7.4. We now study the degrees of freedom of the above isorepresentations. Those of the standard isorepresentations are trivially expressed by the arbitrariness of the factor K. The degrees of freedom of the other isorepresentations are less trivial.

**D) Isoequivalent irregular isopauli matrices,** which are illustrated by

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-\frac{1}{2}} \\ g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-\frac{1}{2}} \\ ig_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \quad (4.7.55)$$

with isocommutation rules

$$[\hat{\sigma}_1 \hat{,} \hat{\sigma}_2] = 2 i \Delta \hat{\sigma}_3, \ [\hat{\sigma}_2 \hat{,} \hat{\sigma}_3] = 2 i \hat{\sigma}_1, \ [\hat{\sigma}_3 \hat{,} \hat{\sigma}_1] = 2 i \hat{\sigma}_2, \quad (4.7.56)$$

and isoeigenvalues

$$\hat{\sigma}_3'' * |\hat{b}_i^2> = \pm |\hat{b}_i^2, \quad \hat{\sigma}^{\hat{2}} * |\hat{b}_i^2> = (1 + 2\Delta) |\hat{b}_i^2>, \quad (4.7.57)$$

where, as one can see, the eigenvalue of the third component is conventional, but that of the magnitude is generalized with a nonfactorizable isotopic contribution, thus confirming the "irregular" character of the isorepresentation. Again, the above isorepresentation is not derivable from Klimyk's rule.

**E) Isoequivalent standard isopauli matrices,** which are given by particularizations of the standard and irregular isorepresentations for the case

$$\Delta = g_{11}\, g_{22} = 1 \tag{4.7.58}$$

which holds under the identification

$$g_{11} = g_{22}^{-1} := \lambda \neq 0, \tag{4.7.59}$$

where $\lambda$ is a real value and nowhere null but arbitrary functions of local quantities, resulting in expressions of the type

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{4.7.60a}$$

$$\hat{\sigma}_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \quad \hat{\sigma}_3' = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{4.7.60b}$$

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ \lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}''_2 = \begin{pmatrix} 0 & -i\lambda^{\frac{1}{2}} \\ i\lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}''_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{4.7.60c}$$

They also satisfy isocommutation rules with conventional structure constants and possess conventional eigenvalues, yet remain inequivalent to the conventional Pauli's matrices as one can verify.

The physical nontriviality of the isotopies of Pauli's matrices, as well as of the isorepresentation theory at large, will be studied in the applications of Vol. III. In essence, the appearance of the "hidden parameter" (actually the "hidden function") $\lambda$ under conventional values of spin $s = \frac{1}{2}$ has numerous *novel* applications, that is, applications not possible with quantum mechanics, such as: the reconstruction of the exact isospin symmetry in nuclear physics under weak and electromagnetic interactions via equal proton and neutron masses in isospace represented by $\lambda^2$; the representation of total magnetic moments of few-body nuclei via a deformation of that of the individual nucleons conjectured since the early stages of nuclear physics but not treated via quantum mechanics; the characterization of a generalized notion of quark called *isoquark* which is indistinguishable with cónventional quarks (because the quantum number are the same), yet possessing an exact confinement because of the incoherence between the interior and exterior Hilbert spaces; and others.

In summary, the above example indicates that the SU(2)–spin symmetry, admits an isotopic image $S\hat{U}(2)$ which is isomorphic to the original symmetry, $S\hat{U}(2) \approx SU(2)$, because of the axiom-preserving character of the isotopies. Yet the isotopic $S\hat{U}(2)$ algebra and its isorepresentations are not unitarily equivalent to the original ones, and the spectra of eigenvalues are generally altered, thus illustrating the nontriviality of the isotopies.

## APPENDIX 4.A: ABSTRACT ALGEBRAS AND ISOALGEBRAS

The hadronic generalization of quantum mechanics was born thanks, specifically, to studies in abstract algebras [1,2]. A few rudimentary notions in that field appear therefore recommendable as an introduction to the content of this chapter. In particular, the notions (essentially derived from Sect. II.5, ref. [21]) are important to understand later on in Ch. 7 and in Vol. II the emergence, apparently for the first time in physics, of *Jordan algebras* as attached algebras to the more general Lie-admissible algebras (Ch. 7).

Let us recall from Sect. 2.4 that a (finite-dimensional) *linear algebra* U, or *algebra* for short (see, e.g., ref. [34]) is a linear vector space V over a field $F(\alpha,+,\times)$ (hereon assumed to be of characteristics zero (Sect. I.2.3)) equipped with a multiplication ab verifying the following axioms

$$\alpha(a\,b) = (\alpha a)b = a(\alpha b),\ (a\,b)\beta = a(b\beta) = (a\beta)b, \tag{4.A.1a}$$

$$a(b+c) = a\,b + a\,c,\ \ (a+b)c = a\,c + b\,c, \tag{4.A.1b}$$

called *right and left scalar and distributive laws,* respectively, which must hold for all elements a, b, c $\in$ U, and $\alpha, \beta \in$ F.

The reader should keep in mind that *the above axioms must be verified by all products to characterize an algebra as commonly understood* [34]. In particular, the distributive law is the basic axiom which prevented the lifting of the operation of addition as shown in Sect. I.2.3.

Among the existing large number of algebras [34], an understanding of hadronic mechanics requires a knowledge of the following primary algebras:

1) **Associative algebras** A, characterized by the additional axiom (besides laws (4.A.1))

$$a(b\,c) = (a\,b)c \tag{4.A.2}$$

for all a, b, c $\in$ A, called the *associative law*. Algebras violating the above law are called *nonassociative*. All the following algebras are nonassociative:

2) **Lie algebras** L which are characterized by the additional axioms

$$a\,b + b\,a \;=\; 0, \tag{4.A.3a}$$

$$a\,(b\,c) + b\,(c\,a) + c\,(a\,b) \;=\; 0. \tag{4.A.3b}$$

A familiar realization of the Lie product is given by

$$[a, b]_A = a\,b - b\,a, \tag{4.A.4}$$

with the classical counterpart being given by the familiar Poisson brackets among functions A, B in cotangent bundle (phase space) $T^*E(r,\delta,\mathfrak{R})$

$$[A,B]_{\text{Poisson}} = \frac{\partial A}{\partial r^k}\,\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\,\frac{\partial A}{\partial p_k}. \tag{4.A.5}$$

3) **Commutative Jordan algebras** J, characterized by the additional axioms

$$a\,b - b\,a \;=\; 0, \tag{4.A.6a}$$

$$(a\,b)\,a^2 \;=\; a\,(b\,a^2), \tag{4.A.6b}$$

A realization of the special commutative Jordan product is given by

$$\{a, b\}_A = a\,b + b\,a. \tag{4.CA7}$$

where a b is associative.

The *noncommutative Jordan algebras* are algebras U which verify Jordan's axiom (4A.6b) but not (4.A.6a).

Intriguingly, no realization of the commutative Jordan product in classical mechanics is known at this writing. As an example, the brackets

$$\{A, B\} = \frac{\partial A}{\partial r^k}\,\frac{\partial B}{\partial p_k} + \frac{\partial B}{\partial r^k}\,\frac{\partial A}{\partial p_k}, \tag{4.A.8a}$$

evidently verify axiom (4.A.6a), but violate axiom (4.A.6b).

4) **General Lie-admissible algebras** U [1,2,38] which are characterized by a product ab verifying laws (4.A.1), which is such that the attached product $[a, b]_U = a\,b - b\,a$ is Lie. This implies, besides (4.A.1), the unique axiom

$$(a, b, c) + (b, c, a) + (c, a, b) \; = \; (c, b, a) + (b, a, c) + (a, c, b), \tag{4.A.9}$$

where

$$(a, b, c) = a\,(\,b\,c\,) \; - \; (\,a\,b\,)\,c, \tag{4.A.10}$$

is called the *associator*.

Note that *Lie algebras are a particular case of the Lie-admissible algebras.* In fact, given an algebra L with product $ab = [a, b]_A$, the attached algebra $L^-$ has the product

$$[a, b]_U \; = \; 2\,[a, b]_A\,, \tag{4.A.11}$$

and, thus, L is Lie-admissible.

Therefore, the classification of the Lie Lie-admissible algebras contains all possible Lie algebras. Also, Lie algebras enter in the Lie-admissible algebras in a two-fold way: first, in their classification and, second, as the attached antisymmetric algebras. Finally, *associative algebras are trivially Lie-admissible.*

The first realization of general Lie-admissible algebras U in classical mechanics was identified by the author in memoir [1] and, in its simplest possible form, it is given by the following product for functions A(r,p) and B(r,p) in $T^*E(r,\delta,\Re)$

$$U: \quad (A, B) \; = \; \frac{\partial A}{\partial r^k}\,\frac{\partial B}{\partial p_k}\,, \tag{4.A.12}$$

namely, *the general, nonassociative Lie-admissible algebras are at the foundations of the structure of the conventional Poisson brackets* (4.A.5) which can be written

$$[A, B]_{Poisson} = \; [A, B]_U \; = (A, B) \; - (B, A), \tag{4.A.13}$$

The first operator realization of the general Lie-admissible algebras was also given by the author in the subsequent memoir [2] Sect. 4.14, and can be written

$$U: \quad (a, b)_A \; = \; a\,r\,b \; - \; b\,s\,a, \tag{4.A.14}$$

$$r, s \quad \text{fixed} \in A, \quad r \neq s, \qquad r, s \neq 0$$

where ar, rb, etc., are associative. In fact, the antisymmetric product attached to U is a particular form of a Lie algebra (see below)

5) **Flexible Lie-admissible algebras** U [1,2,38], which are characterized by

the axioms in addition to (5.1)

$$(a, b, a) = 0, \tag{4.A.15a}$$

$$(a, b, c) + (b, c, a) + (c, a, b) = 0, \tag{4.A.15b}$$

where condition (4.C.14a), called the *flexibility law* [38].is a simple generalization of the anticommutative law, as well as a weaker form of associativity. A first realization of the flexible Lie–admissible product was identified by this author back in 1967 [55]

$$(a, b)_U = \lambda\, a\, b - \mu\, b\, a, \quad \lambda, \mu \in F \tag{4.A.16}$$

where the products $\lambda a$, $ab$, etc. are associative. It is instructive to verify that the algebras characterized by the above product is a realization of the noncommutative Jordan algebras.

As we shall see in App. I.7.A, *a certain class of the so-called q-deformations* [56] *are a particular case of product (4.A.16) and, as such, they are flexible, Lie–admissible and Jordan–admissible, as well as noncommutative Jordan algebras.*

No classical realization of flexible Lie–admissible algebras has been identified until now, to our best knowledge. As an example, the brackets on $T^*E(r,\delta,\mathfrak{R})$

$$(A, B) = \lambda \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \mu \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} \tag{4.A.17}$$

are Lie–admissible, but violate the flexibility law.

6) **General Jordan-admissible algebras** U [1,2,38], which are characterized by a product ab verifying laws (4.C.1), such that the attached symmetric product $\{a, b\}_U = a\, b + b\, a$ is Jordan, i.e., verifies the axiom

$$(a^2, b, a) + (a, b, a^2) + (b, a^2, a) + (a, a^2, b) = 0. \tag{4.A.18}$$

Again, *associative and Jordan algebras are trivially Jordan- admissible.* Also, Jordan algebras enter in the Jordan–admissible algebras in a two-fold way, in the classification of the latter, as well as the attached symmetric algebras.

It is important for the operator formulation of the isotopies of Vol. II to point out the following important

**Proposition 4.A.1** [2]: *The general Lie–admissible product (4.A.14) is, jointly, Lie–admissible and Jordan–admissible .*

But Lie–admissible product (4.A.11) characterizes the brackets of the most possible general time evolution of hadronic mechanics. The Jordan algebra therefore enter as the attached form $U^+$ of the Lie–admissible algebras U of hadronic operators.

By comparison, there is no "Jordan content" in quantum mechanics, because the algebra of its time evolution is a Lie algebra L, whose attached symmetric part is identically null, $L^+ \equiv 0$.

As we shall see in Vol. II, the emergence of a nontrivial "Jordan content" has far reaching physical implications, such as the capability of constructing a quark theory with an "exact confinement", i,.e., with a transition probability for free quarks which is explicitly computed and rigorously proved to be identically null under any possible physical condition.

Intriguingly, this emergence of a "Jordan content" at the operator level has no known counterpart in classical mechanics. In fact, the classical Lie–admissible product (4.A.12) is only Lie–admissible and not jointly Jordan–admissible.

7) **Flexible Jordan-admissible algebras** U [1,2,38), which, in addition to axioms (4.A.1), are characterized by the axioms

$$a(ba) = (ab)a, \tag{4.A.19a}$$

$$a^2(ba) + a^2(ab) = (a^2 b)a + (a^2 a)b. \tag{4.A.19b}$$

The flexible Lie–admissible product (4.A.16) is also a flexible Jordan–admissible product, but the classical product (4.C.17) is only Lie–admissible, and not flexible Lie–admissible or Jordan–admissible.

8) **Alternative algebras** U, which are algebras characterized by the additional axioms encountered in Sect 2.2,

$$(a, a, b) = 0 \text{ and } (a, b, b) = 0, \ \forall\ a, b, c \in U \tag{4.A.20}$$

called right and left alternative laws. A realization of alternative algebras is given by the octonions (Sect. I.2.8).

9) **Power associative algebras** U, characterized by the additional law

$$a^n a^m = a^{n+m}, \quad \forall\ a \in U,\ n, m \text{ integers} \tag{4.A.21}$$

which constitutes the axiomatization of an important physical notion. In fact, algebras currently used used in physics are power associative.

For additional algebras we refer the reader to ref.s [34,38] and quoted literature.

We now pass to the study of the isotopies of the above notions.

**Definition 4.A.1** [1]: *An "isoalgebra", or simply an "isotope"* $\hat{U}$ *of an algebra* U *with elements* a,b,c,... *and product* ab *over a field* F, *is the same vector space* U *but defined over the isofield* $\hat{F}$, *equipped with a new product* a*b, *called "isotopic product", which is such to verify all original axioms of* U.

Thus, by definition, the isotopic lifting of an algebra does not alter the type of algebra considered.

It is important for these studies to review the isotopies of the primary algebras listed above.

Given an associative algebra A with product ab over a field F, its simplest possible *isotope* $\hat{A}$, called *associative-isotopic* or *isoassociative algebra* [1] is given by

$$\hat{A}_1 : \; a * b = \alpha\, a\, b, \qquad \alpha \in F \text{, fixed and} \neq 0, \tag{4.A.22}$$

and called a *scalar isotopy.* The preservation of the original associativity is trivial in this case. This is evidently the case of the q-deformations [56].

A second less trivial isotopy is the fundamental one of the Lie-isotopic theory, and it is characterized by the basic product of this chapter [2]

$$\hat{A}_2 : \quad a * b = a\, T\, b, \tag{4.A.23}$$

where T is an nonsingular (invertible) and Hermitean elements not necessarily belonging to the original algebra A.

The third known isotopy of A is given by [2]

$$\hat{A}_3 : \qquad a * b = w\, a\, w\, b\, w, \qquad w^2 = w\, w \; = \; w \; \neq 0, \tag{4.A.24}$$

Additional isotopies are given by the combinations of the preceding ones, such as

$$\hat{A}_4 : \qquad a * b = w\, a\, w\, T\, w\, b\, w, \quad w^2 = w\, w = w \; \neq 0 \tag{4.A.25a}$$

$$\hat{A}_5 : \; a * b = \alpha\, w\, a\, w\, T\, w\, b\, w, \;\; \alpha \in F, \quad w^2 = w, \quad a, w, T \neq 0. \tag{4.A.25b}$$

It is believed that the above isotopies exhaust all possible isotopies of an associative algebra over a field of characteristic zero, although this property has not been rigorously proved to this writing.

We now pass to the study of the *isotopes* $\hat{L}$ of a Lie algebra L with product ab over a field F, which are the same vector space L but equipped with a *Lie-isotopic product* [1] a⊙b over the isofield $\hat{F}$ which verifies the left and right scalar and distributive laws (4.A.1), and the axioms

$$a \odot b + b \odot a = 0, \tag{4.A.26a}$$

$$a \odot (b \odot c) + b \odot (c \odot a) + c \odot (a \odot b) = 0, \tag{4.A.26b}$$

Namely, the abstract axioms of the Lie algebras remain the same by assumption.

The simplest possible *realization of the Lie-isotopic product* is that attached to isotopes $\hat{A}_1$,

$$\mathfrak{L}_1 : [a, b]_{\hat{A}_1} = a \odot b - b \odot a = \alpha (a b - b a) = \alpha [a, b]_A, \quad \alpha \in F, \alpha \neq 0, \tag{4.A.27}$$

and it is also called a *scalar isotopy.* It is generally the first lifting of Lie algebras one can encounter in the operator formulation of the theory.

The second independent realization of the Lie-isotopic algebras is that characterized by the isotope $\hat{A}_2$ which is that of primary use in hadronic mechanics [1,2]

$$\mathfrak{L}_2 : \quad [a, b]_{\hat{A}_2} = a \odot b - b \odot a = a T b - b T a, \tag{4.A.28}$$

The third, independent isotopy is that attached to $\hat{A}_3$ [2]

$$\mathfrak{L}_3 : \quad [a,b]_{\hat{A}_3} = w a w b w - w b w a w, \quad w^2 = w w \neq 0. \tag{4.A.29}$$

A fourth isotope is that attached to $\hat{A}_4$, i.e.,

$$\mathfrak{L}_4 : \quad [a, b]_{\hat{A}_4} = w a w T w b w - w b w T w a w, \quad w^2 = w, \quad w, T \neq 0. \tag{4.A.30}$$

A fifth and final (abstract) isotope is that characterized by $\hat{A}_5$, i.e.

$$\mathfrak{L}_5 : [a, b]_{\hat{A}_5} = \alpha [a, b]_{\hat{A}_4}. \tag{4.A.31}$$

Again, it is believed that the above five isotopes exhaust all possible abstract Lie algebra isotopies (over a field of characteristics zero), although this property has not been proved to date on rigorous grounds.

Note that the Lie algebra attached to the general Lie-admissible product (4.A.12) are not conventional, but isotopic. In fact, we can write

$$[a, b]_U = (a, b)_A - (b, a)_A = a r b - b s a - b r a + a s b = \tag{4.A.32a}$$

$$= a T b - b T a = a * b - b * a, \tag{4.A.32b}$$

$$r \neq s, \quad r, s, T \neq 0, \quad T = r + s \neq 0.$$

As a matter of fact, the author first encountered the Lie–isotopic algebras by studying precisely the Lie content of the more general Lie–admissible algebras [1]

The following property can be easily proved from properties of type (5.30).

**Lemma 4.A.1** [1]: *An abstract Lie–isotopic algebra* $\mathfrak{L}$ *attached to a general, nonassociative, Lie–admissible algebra* U, $\mathfrak{L} \approx U^-$, *can always be isomorphically rewritten as the algebra attached to an isoassociative algebra* $\hat{A}$, $\mathfrak{L} \approx \hat{A}^-$, *and vice–versa, i.e.*

$$\mathfrak{L} \approx U^- \approx \hat{A}^-. \tag{4.A.33}$$

The above property has the important consequence that *the construction of the abstract Lie–isotopic theory does not necessarily require a nonassociative enveloping algebra because it can always be done via the use of an isoassociative envelope.* In turn, this focuses again the importance of knowing all possible isotopes of an associative algebra, e.g., from the viewpoint of the representation theory.

The most general possible, classical, *local–differential* [31] realization of Lie–isotopic algebras via functions A(a) and B(a) in $T^*E(r,\delta,\mathfrak{R})$ with local chart

$$a = (a^\mu) = (r, p) = (r^i, p_i), \quad i = 1, 2, \ldots, n, \; \mu = 1, 2, \ldots, 2n, \tag{4.A.34}$$

is provided by the *Birkhoffian brackets* [1,4] also called generalized Poisson brackets

$$[A , B]_{\text{Birkhoff}} = [A, B]_U = \frac{\partial A}{\partial a^\mu} \; \Omega^{\mu\nu}(a) \; \frac{\partial B}{\partial a^\nu}, \tag{4.A.35}$$

where $\Omega^{\mu\nu}$, called the *Lie–isotopic tensor* , is the contravariant form of (the exact, symplectic, *Birkhoff's tensor*

$$\Omega^{\mu\nu} = (\|\Omega_{\alpha\beta}\|^{-1})^{\mu\nu}, \tag{4.A.36a}$$

$$\Omega_{\mu\nu} = \frac{\partial R_\nu(a)}{\partial a^\mu} - \frac{\partial R_\mu(a)}{\partial a^\nu}, \tag{4.A.36b}$$

where the R's are the so–called *Birkhoff's functions.* The symplectic character of the covariant tensor ensures the Lie–isotopic character of brackets (see the geometric treatment of the next chapter).

[31] The nonlocal–integral realizations will be presented in the next chapter after studying the underlying nonlocal–integral geometries.

Recall that, unlike the conventional, abstract, Lie brackets (4.A.4), the conventional Poisson brackets (4.A.5) characterize a Lie algebra attached to a *nonassociative* Lie-admissible algebra U, Eq.s (4.A.13). It is then evident that the covering Birkhoff's brackets (4.A.35) are also attached to a nonassociative Lie-admissible algebra, although of a more general type (see ref. [4] for brevity).

For other classical Lie-isotopic brackets, such as *Dirac's generalized brackets* for systems with subsidiary constraints see the locally quoted references.

Note the lack of identification of the underlying generalized unit in Birkhoff's brackets (4.A.35). This is precisely the aspect which has requested the isotopies of conventional geometries of the next chapter.

Realizations of the abstract isotopes $\hat{U}$ of the Lie-admissible algebras can be easily constructed via the above techniques. For instance, an isotope of the general Lie-admissible product (4.A.14) is given by

$$\hat{U}: \quad (a \hat{,} b) = w\, a\, w\, r\, w\, b\, w \; - \; w\, b\, w\, s\, w\, a\, w, \qquad w^2 = w, \quad w, r, s \neq 0, \; r \neq s. \tag{4.A.37}$$

An isotope of the classical realization (4.A.11) is then given by

$$\hat{U}: \qquad (A \hat{,} B) = \frac{\partial A}{\partial a^{\mu}} \; S^{\mu\nu}(t,a) \frac{\partial B}{\partial a^{\nu}}, \tag{4.A.38}$$

where the tensor $S^{\mu\nu}$, called the *Lie-admissible tensor*, is restricted by the conditions of admitting Birkhoff's tensor as the attached antisymmetric tensor, i.e.,

$$S^{\mu\nu} - S^{\nu\mu} \equiv \Omega^{\mu\nu}, \tag{4.A.39}$$

Brackets (4.A.38] constitutes the basic product of the classical Lie-admissible studies of ref.s [10–14].

Historical notes on the origin of the isotopies are provided in ref. [3]. The broader genotopies will; be studied in Ch. I.7.

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# 5: ISOGEOMETRIES

## 5.1: STATEMENT OF THE PROBLEM

Contemporary theoretical physics is an articulated body of deeply interrelated and mutually compatible formulations including algebras, geometries, mechanics, and other fields. The isotopic generalization of contemporary algebras, by no means, can alone provide a new scientific horizon, unless complemented by interrelated and mutually compatible formulations for all remaining aspects.

The additional mandatory studies considered in this chapter are therefore the isotopies of contemporary geometries, such as the *Euclidean, Minkowskian, symplectic, affine, Riemannian and other geometries* (for conventional formulations see, e.g., ref.s [1–4] and literature quoted therein).[32]

As we shall see, these studies are intriguing indeed because they identify the following three types of new geometries of primary interest for hadronic mechanics:

**A) Isotopies of flat geometries,** such as the **isoeuclidean** and **isominkowskian geometries.** As well known, the Euclidean and Minkowskian geometries provide a geometrization of the homogeneity and isotropy of empty space. As such, they are *exactly valid* for the *nonrelativistic and relativistic exterior problems in vacuum,* respectively.

Our central problem here is the identification of covering geometries which permit a direct, classical geometrization of the inhomogenuity and anisotropy of physical media for *nonrelativistic and relativistic interior dynamical problems.*

An important illustration, particularly for applications, is the identification of the image of the *light cone* for interior conditions, in which case the speed of light is *locally varying* thus implying the loss of the very "cone".

[32] In this chapter we shall mainly refer to the monograph by Lovelock and Rund [4] of which we shall preserve the notations and symbol for clarity in the comparison of the results.

Even though evidently not unique, the isotopies of flat geometries are particularly suited to achieve: 1) the desired, direct, interior geometrization; 2) the representation of the most general possible nonlinear–nonlocal–nonhamiltonian interior systems; while 3) preserving the original axioms, and therefore admitting the original geometries as particular cases.

Moreover, the isotopies of the Euclidean and Minkowskian geometries acquire the most general known dependence not only on local coordinates x, but also on the velocities $\dot{x}$, accelerations $\ddot{x}$, and other variables. Despite that, the isotopies here considered preserve the original axioms of flatness, thus resulting in fundamentally novel geometries in which, for instance, the notions of angles and trigonometric functions can still be defined, although in a predictable generalized way.

As we shall see in Vol.s II and III, the applications and experimental verifications of these new geometries are considerable, and include nuclear physics, particle physics, astrophysics, superconductivity and unexpected other fields, such as conchology.

**B) Isotopies of the symplectic geometry,** known as the **isosymplectic geometry.** As well known, the symplectic geometry provides the geometrization of Lie algebras and, as such, is strictly local–differential, thus being inapplicable for the geometrization of nonlocal–integral systems.

Our primary objective here is therefore the identification of a covering of the symplectic geometry which is the geometric counterpart of the Lie–isotopic theory in its most general possible nonlinear–nonlocal–nonhamiltonian formulation.

The technical problem we shall address was indicated in Sect. 1.4, and can be treated now in more details. In essence, the Lie–isotopic algebras in their abstract formulation as presented in the preceding chapter showed since the original formulation [5] their natural capability to admit the most general possible nonlinear–nonlocal–nonhamiltonian systems owing to the arbitrariness of the isotopic element T in the isoproduct

$$[A \hat{,} B] = A\, T(t, x, \dot{x}, \ddot{x}, \ldots)\, B - B\, T(t, x, \dot{x}, \ddot{x}, \ldots)\, A\,. \qquad (5.1.1)$$

The geometry of the first studies [5] was the *conventional* symplectic geometry, although realized in its most general possible exact version, with noncanonical, symplectic, exact two–form on a 2n–dimensional manifold M(x,R) over the reals R(n,+,×) (see App. 5,A for an outline and monograph [6] for a detailed treatment)

$$\Theta = d\,\Phi = d\,[\, R_i(x)\, dx^i\, ] = \Omega_{ij}(x)\ dx^i \wedge dx^j\,. \qquad (5.1.2)$$

The covariant symplectic tensor

$$\Omega_{ij} = \partial_i R_j - \partial_j R_i, \ \partial_i = \partial / \partial x^i, i, j = 1, 2, ..., 2n. \tag{5.1.3}$$

and corresponding contravariant form

$$\Omega^{ij} = (|\Omega_{pq}|^{-1})^{ij}, \tag{5.1.4}$$

are manifestly noncanonical and therefore result to be a direct realization of the *Isotopic Lie's Theorems* (Sect. 4.5). In fact, the generalized brackets

$$[A \hat{,} B] = \frac{\partial A}{\partial x^i} \Omega^{ij}(x) \frac{\partial B}{\partial x^j}, \tag{5.1.5}$$

are Lie-isotopic, as ensured by the *Poincaré lemma* (see Sect. 5.2 and App. 5.A)[33]

$$d\Theta = d(d\Phi) \equiv 0. \tag{5.1.6}$$

This permitted a step-by-step generalization of classical Hamiltonian mechanics into a new discipline submitted in ref. [5] under the name of *Birkhoffian mechanics*, and subsequently elaborated in monograph [6].

However, brackets (5.1.5) are strictly local-differential, thus preventing a treatment of nonlocal-integral systems. In fact, the theorem of "direct universality" of Birkhoffian mechanics and of the related conventional symplectic geometry (ref. [6], p. 54 and ff., and Theorem 5.A.1 of App. 1.5.A) was specifically formulated for all possible nonlinear and nonhamiltonian systems, under the conditions that they are local-differential and verify the needed regularity and smoothness conditions.

Comparison of brackets (5.1.1) and (5.1.5) clearly reveals important structural differences. As well known [4], the symplectic tensor $\Omega_{ij}$, and, consequently, the Lie-isotopic tensor $\Omega^{ij}$, can only have a dependence on the local coordinates, $\Omega^{ij}(x)$, while the isotopic element T can have an arbitrary functional dependence, $T(t, x, \dot{x}, \ddot{x}, ...)$.

The above disparity between algebras and geometries persisted for a decade. Its solution required the author to conduct, again, a step-by-step generalization, this time, of Birkhoffian mechanics into the so-called *Birkhoff-isotopic (or isobirkhoffian) mechanics* and of underlying geometry into the so-called *symplectic-isotopic (isosymplectic) geometry*, as a necessary condition to achieve a complete equivalence between isoalgebras, isogeometries and isomechanics.

As we shall see, the solution was provided by the full implementation of

[33] The Lie-isotopic algebras were originally formulated precisely on these grounds, that is, by showing that the transition from Lie's theorems to their isotopic coverings imply the transition from the Poisson to generalized Lie brackets.

the same methods that had originated the Lie–isotopic theory: the systematic lifting of the entire structure of the symplectic geometry, including fields, vector spaces, exterior calculus, and the like.

Note that, as indicated earlier, the isoeuclidean and isominkowskian isogeometries *are not unique* for the description of interior problems. However, the isosymplectic geometry is the *only* geometric counterpart of the Lie–isotopic theory.

**C) Isotopies of curved geometries** such as the **isoriemannian geometry.** In the study of interior gravitational problems the need for a generalization of the Riemannian geometry of nonlinear and nonlocal type in the velocities and other variables becomes compelling.

This is due to the evidence that interior problems, such as a collapsing star, are not an aggregate of ideal points, but are instead composed of extended wavepackets/wavelengths/charge–distributions of hadrons in condition of total mutual penetration as well as of compression in large numbers into a small region of space. The emergence of the most general possible nonlinear and nonlocal, thus nonlagrangian structure under these conditions must be admitted in order not to exit the boundary of Science.

By looking at this occurrence in retrospect, we can say that the originators of current gravitational theories were fully aware of the distinction between the exterior and interior gravitational problem, the exact validity of the Riemannian geometry for the former problem, and its approximate character for the latter problem.

In fact, the geometry conceived by B. Riemann in 1868 [7] can be readily proved to be *exactly valid* as well as "directly universal" for the arena of its original conception, subsequently applied by Einstein [8] to dimensionless test bodies moving in empty space, and identified as the exterior gravitational problem.

Jointly, one should note the scientific honesty of authors of the early part of this century on the limitations of the Riemannian geometry for the interior problem. As an example, Schwartzschild wrote *two* articles [9] in which the distinction between exterior and interior problems is stated beginning from the title. In particular, Schwartzchild's first article is dedicated to the *exterior* gravitational problem with emphasis on the *exact* character of his celebrated solution, while the second (little known) article is dedicated to the *interior* gravitational problem with emphasis on the *approximate* character of the solution).

The insufficiency of the Riemannian geometry for interior problems is at times also called the *Cartan legacy*, because expressing Cartan's indication of the inability of the Riemannian geometry to recover under PPN and other limiting procedures *all* generally nonconservative Newtonian systems of our physical reality (such as missiles in atmosphere with drag forces of the type $-\gamma\dot{x}^{10}$).

The distinction between the exterior and interior gravitational problems was kept in the early treatises in gravitations (see, e.g., the title of Ch. VI of treatise [10] – with Einstein's preface –, or the titles of Sect.s 11.6, p. 439 and 11.7, p. 444, of treatise [11]).

Unfortunately, the emphasis on this distinction was progressively lost with the passing of time, up to the current trend of eliminating any distinction between exterior and interior gravitation. This is done via the (often tacit) abstraction of the interior problem as an ideal collection of dimensionless elementary particles which, as such, recover the exterior conditions in vacuum.

The scientific reality is that the interior, nonconservative and irreversible physical events, as majestically shown by the direct observation of Jupiter's structure, simply cannot be reduced to an ideal collection of dimensionless elementary particles in stable orbits, because of the *No-reductions Theorems* indicated in Ch. 1.

At any rate, we do not possess today an unambiguous operator formulation of gravity which is an evident pre-requisite for the reduction. Thus, interior gravitational problems must first be represented *classically* as they actually are in the physical reality, that is, with nonconservative irreversible interior effects. Their reduction to particle descriptions can only be studied thereafter, provided that they are capable of representing visual evidence of the interior problem, such as Jupiter's vortices with *continuously varying angular momenta*.

Also, the insistence on applying for the interior problem physical theories so clearly conceived for the exterior problem [7–11] leads to excessive approximations, such as the acceptance the "perpetual motion" within a physical environment, as necessary from the locally Lorentz character of Einsteinian theories with the consequential, *necessary*, local conservation of the angular momentum.

In conclusion, the generalization of the Riemannian geometry of nonlinear and nonlocal type in the velocities and other variables is compelling because of the *lack* of exact applicability of the conventional geometry in the interior problem on numerous independent grounds of analytic, geometric and topological character.

It should be stressed that, by no means, the isoriemannian geometry is the only geometry applicable for nonlocal interior problems, because numerous other (e.g., integral) geometries are equally conceivable and their study is encouraged here.

The isotopies of Riemann have been preferred over other possible geometries because they permit the achievement of the needed nonlinear-nonlocal-nonlagrangian generalization for the *interior* problem while preserving the geometric axioms of the *exterior* problem. In turn, this permits the unification of exterior and interior problems achieved in these volumes this time at the gravitational level on curved spaces.

We reach in this ways one of most important geometric notions of our analysis, that of *isogeodesic*, which represents trajectories within physical media while preserving the original trajectory in vacuum when represented in isospace. The notion of isogeodesic is also fundamental for an understanding of the Lie-isotopic theory, e.g., the preservation of exact space-time symmetries in interior conditions, such as the rotational and Lorentz symmetries for deformed spheres and Minkowski metrics, respectively.

Also, the isotopies of Riemann permit a clear separation between the original local-differential exterior structure, and the nonlinear-nonlocal interior effects. In turn, this separation is essential from an experimental viewpoint to separate the interior contributions from the conventional gravitational structures.

Needless to say, the Riemannian geometry does indeed remain *approximately valid* for interior gravitational problems, with consequential approximate validity of all interior studies, such as gravitational collapse, "black holes", "big bang", etc. The issue addressed in this chapter is a quantitative treatment suitable for experimental verifications of the (generally small) *corrections* to these studies expected from nonlinear-nonlocal-nonlagrangian interior effects.

One additional aspect requires a few introductory words. As studied in preceding chapters, the isotopies naturally imply a new antiautomorphic conjugation characterized by the now familiar isoduality

$$\hat{1} \;\Rightarrow\; \hat{1}^d = -\hat{1}\,, \qquad (5.1.7)$$

with corresponding conjugation of the isoreal numbers into their isoduals (Sect. 2.2) and of all remaining aspects. The isoduality therefore applies also to all conventional geometries, whether flat or curved.

Recall that our current description of the universe is based on a positive-definite norm, and therefore has a positive energy with a motion forward in time,

$$E > 0,\ t > 0,\ |E| > 0,\ |t| > 0,\ \text{etc.,} \qquad E,\ t \in R(n,+,\times)\,. \qquad (5.1.8)$$

while the isodual spaces are characterized by negative-definite norm on isodual isofields, thus having *negative energies, evolving backward in time, etc.*

$$E^d < 0, t^d < 0,\ |E^d|^d < 0, |t^d|^d < 0,\ \ E^d,\ t^d \in R^d(n^d,+,\times^d)\,. \qquad (5.1.9)$$

The above occurrences have permitted the identification of an new universe, called *isodual universe*, constituted by antiparticles, which coexists with our own universe and possesses rather intriguing features, e.g., an interconnection with our own universe because of the finite transition probability between the positive- and negative-energy solutions of conventional field equations.

It should be recalled that the concepts of negative time and negative energies for antiparticles are rather old, and actually date back to the early stages of particle physics. What is new is their systematic treatment via a body of formulations specifically conceived for that purpose, such as isodual numbers, isodual algebras, isodual geometries, etc.

The concepts of negative energies and time were generally abandoned in favor of rather artificial and still unsettled constructions because of their unphysical implications, i.e., predictions contrary to evidence when treated with conventional methods. A further novelty of our studies is that the quantitative treatment of negative energies and times via isodual methods leads to a fully physical behaviour.

As a further introductory aspect, the reader should be aware of the rather broad character of the new geometries. It is sufficient to note that Kadeisvili's classification of the isounits $\hat{1}$ into five classes carries over to all isogeometries studied in this chapter thus resulting in isotopies of conventional geometries of Classes I, II, III, IV and V.

The reader should be aware that the isogeometries imply new perspectives, with new geometric concepts, such as: *light cone for the interior of physical media; curvatures over singular units; connections over discrete units; geodesic motion for structures with negative-definite norm; reformulation of exterior gravitation in the isominkowski space*, and others.

The isotopies of the Euclidean and Minkowskian geometries were first proposed in papers [12–14] of 1983–1985. The isotopies of the symplectic geometry first appeared in print in memoir [15] of 1988, while the isotopies of the affine and Riemannian geometries first appeared in memoir [16] also of 1988. The isogeometries were then further studied in papers [17,18] of 1990 and in monographs [19,20] of 1991. The most recent contributions by this author on isogeometries are papers [21–23,41].

Contributions by independent authors are as follows. Aringazin [24] first proved the "direct universality" of the isominkowskian geometry for all possible deformations of the Minkowski metric. Lopez [25] studied certain implications of the interior isoriemannian geometry for the exterior problem. Kadeisvili [26] wrote the first comprehensive review with emphasis on the isoriemannian geometry. Sourlas and Tsagas have just completed monograph [27] with emphasis on the isosymplectic geometry and papers [40] on the first formulation of isomanifolds and related topology.

These are all contributions, specifically, on isogeometry available at this writing, to our best knowledge.[34] It is evident that the literature *indirectly* related to the isogeometries is vast indeed. We are here referring to several forms

[34] The paucity of contributions in the field is also due to truly unreasonable editorial obstructions for papers on isogeometries submitted by various authors to a number of journals, which have prevented several papers in the field from seeing the light.

of noneuclidean geometries and their possible relativistic extension which are based on the *conventional unit.* We regret our inability to review these geometries for brevity, except for marginal comments.

## 5.2: ISOEUCLIDEAN GEOMETRY

The *isoeuclidean geometry* [12–14,41] is the geometry of the isoeuclidean spaces on the isoreal field expressible in the diagonal form

$$\hat{E}(r,\hat{\delta},\hat{R}):\quad \hat{\delta} = T\delta,\ \delta = \text{diag.}(1,1,1),\ \hat{\delta} = \hat{\delta}^{\dagger},\quad \hat{1} = T^{-1}, \tag{5.2.1a}$$

$$T = T(t, r, \dot{r}, \ddot{r}, \ldots) = \text{diag.}(g_{11}, g_{22}, g_{33}) = T^{\dagger} > 0, \tag{5.2.1.b}$$

$$r^{\hat{2}} = (r^i\,\hat{\delta}_{ij}\,r^j)\hat{1} = (x\,g_{11}\,x + y\,g_{22}\,y + z\,g_{33}\,z)\hat{1} \in \hat{R}(\hat{n},+,*). \tag{5.2.1c}$$

or in nondiagonal realizations of the isounit. Its general form is of Class V (Sect. 3.4), although it will be studied here for the simpler Classes I, II and III.

The primary properties of the isoeuclidean geometry are the following:

**Property I:** *Reconstruction of angles and trigonometric functions in a space whose metrics depends on local coordinates and other quantities* (see Fig. 5.2.1);

**Property II:** *Unification of all ellipsoidical (for Class I) and hyperbolic (for class III) deformations of the sphere (See Fig. 5.2.2);*

**Property III:** *Preservation of the original spherical shape in isospace, called "isosphere", for all ellipsoidical (Class I) and hyperbolic (Class III) deformations of the sphere in our space (Fig. 5.2.3).*

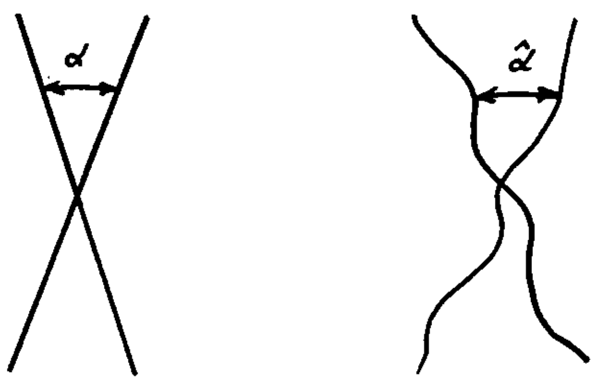
Let us begin with a study of Property I. Recall that in the transition from the Euclidean space $E(r,\delta,R)$ to the Riemannian space in the same dimension $\Re(r,g,R)$ there is the loss of angles and trigonometric functions evidently because of the loss of straight lines due to the curvature of the space, as expressed by the Riemannian metric $g = g(r)$.

In the transition from the Euclidean space $E(r,\delta,R)$ to the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ we acquire the most general possible curvature, this time dependent also in the velocities, accelerations and other quantities, as expressed by the isometric $\hat{\delta} = \hat{\delta}(t, r, \dot{r}, \ddot{r}, \ldots)$. Nevertheless, the lifting $E(r,\delta,R) \rightarrow \hat{E}(r,\hat{\delta},\hat{R})$ is an *isotopy,* that is, a generalization which preserves the original axioms.

Since the original space is flat, its isotopic image is then *isoflat,* that is, the curvature emerges only when $\hat{E}(r,\hat{\delta},\hat{R})$ is projected in $E(r,\delta,R)$ because, within the isospace itself, there is no curvature.

The novel geometric property of isoflatness then permits the reconstruction in $\hat{E}(r,\hat{\delta},\hat{R})$ of angles and trigonometric functions which is precluded in Riemannian spaces.

**RECONSTRUCTION OF ANGLES AND TRIGONOMETRIC FUNCTIONS IN ISOSPACE**



(A) (B)

FIGURE 5.2.1. Diagram (A) depicts the origin of the notion of angle in the conventional Euclidean plane from two straight intersecting lines, which can be analytically expressed via the familiar expression

$$\cos\theta = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{\frac{1}{2}} (x_2 x_2 + y_2 y_2)^{\frac{1}{2}}} . \tag{1}$$

In the transition to the Isoeuclidean plane, the above expression is lifted into the form

$$\text{isocos}\,\hat{\alpha} = \frac{x_1 g_{11} x_2 + y_1 g_{22} y_2}{(x_1 g_{11} x_1 + y_1 g_{22} y_1)^{\frac{1}{2}} (x_2 g_{11} x_2 + y_2 g_{22} y_2)^{\frac{1}{2}}} \tag{2}$$

which evidently does not characterize $\cos\theta$ any more, and it is assumed as the definition of the isocos $\hat{\theta}$, with $\hat{\theta} = \theta(g_{11}g_{22})^{\frac{1}{2}}$ derived from the underlying Lie-isotopic $S\hat{O}(2)$ symmetry. Note that in expression (2) we have no isounits because they cancel out in the quotient. Thus, isocos $\hat{\theta}$ is an ordinary scalar and not an isoscalar. Note also that expression (2) holds for an arbitrary functional dependence of the argument $\hat{\theta} = \hat{\theta}(t, r, \dot{r}, \ddot{r}, \ldots)$. The isotopic reconstruction of the other aspects of trigonometry is done in Appendix 6.A. The geometric origin of the reconstruction of the angle and of trigonometric functions is the fact that straight lines are lost in the isoeuclidean plane, although they remain *isostraight*, that is, they preserve the original axioms. More explicitly, the loss of the straight character is only apparent, when projected in the original plane as done in Fig. (B) above. However, when considered in the isoeuclidean plane, the images of straight lines remain perfectly straight (see also Fig. 5.2.3) and later on in this section the representation of vectors.

Let us consider first the isoeuclidean geometry in two dimension with isotopic element and isounit

$$T = \mathrm{diag.}(g_{11}, g_{22}), \quad \hat{1} = \mathrm{diag.}(g_{11}^{-1}, g_{22}^{-1}). \tag{5.2.2}$$

Let $\theta$ be the angle among two intersecting straight lines in $E(r,\delta,R)$. Then the corresponding angle $\hat{\theta}$ in $\hat{E}(r,\hat{\delta},\hat{R})$, called *isoangle*, is given by

$$\hat{\theta} = \theta\,(g_{11}\,g_{22})^{\frac{1}{2}}. \tag{5.2.3}$$

This result is established by the basic invariance of the space, the Lie-isotopic $\hat{O}(2)$ symmetry, or from an inspection of the arguments isorotations (4.6.8), as we shall have ample opportunity to verify later on.

A study of the interpretation of rule (5.2.3), particularly from the elaboration of experimental data (see Vol. III), as indicated that the isoangle $\hat{\theta}$ can also be interpreted as *the original angle prior to the deformation.* The angle $\theta$ in rule (5.2.3) is then *the angle of deformation measured in our space.* This has the nontrivial implication that, say, *when deforming a circle into an ellipsoid, the directional angle of a point is evidently altered in the Euclidean plane into the angle $\theta$, but the isoeuclidean plane reconstructs the original directional angle $\hat{\theta}$.* This notion will appear clearer later on in this section (Fig. 5.2.3) when we show that the original circle itself is preserved in its entirety in isospace.

The familiar $\sin\theta$ and $\cos\theta$ are lifted into the *isotrigonometric functions*

$$\sin\theta \quad \rightarrow \quad \mathrm{isosin}\,\hat{\theta} = g_{22}^{-\frac{1}{2}} \sin[\,\theta\,(g_{11}\,g_{22})^{\frac{1}{2}}\,], \tag{5.2.4a}$$

$$\cos\theta \quad \rightarrow \quad \mathrm{isocos}\,\hat{\theta} = g_{11}^{-\frac{1}{2}} \cos[\,\theta\,(g_{11}\,g_{22})^{\frac{1}{2}}\,]. \tag{5.2.4b}$$

In fact, the familiar property $\sin^2\theta + \cos^2\theta = 1$ is now lifted into the isotopic form

$$g_{11}\,\mathrm{isocos}^2\,\hat{\theta} + g_{22}\,\mathrm{isosin}^2\,\hat{\theta} = 1. \tag{5.2.5}$$

Note the association of the element $g_{11}$ with the isocosin$\theta$, rather than the isosin$\theta$, as necessary when $g_{11}$ is the isotopic element of the x-axis. Note also the rather complex functional dependence on the isoangles $\hat{\theta} = \hat{\theta}(t, r, \dot{r}, \ddot{r}, ...)$.

This is sufficient to indicate the existence of a consistent and intriguing isotopy of conventional trigonometry, which is studied in more detail in Appendix 6.A.

Property II can be enlarged to the unification of all possible quadrics (i.e., surfaces of the second order) in three-dimension via the formulation of the isogeometry for Class III. In fact, the symmetry of the isoeuclidean geometry, the Lie-isotopic group $\hat{SO}(3)$ of Class III is the invariance of all possible quadrics, as shown in Eq.s (4.6.9), and as inherent in the classification of the isometrics

$$\hat{\delta} = \mathrm{diag.}\,(g_{11}, g_{22}, g_{33}) = \mathrm{diag.}\,(\pm b_1^2, \pm b_2^2, \pm b_3^2)\,. \qquad (5.2.6)$$

As a result, the isoeuclidean geometry of Class III treats in a unified way all the following quadrics and related Class I or II symmetries[35]:

**1) The sphere**

$$SO(3): \qquad x^1 x^1 + x^2 + x^2 + x^3 x^3 = \text{inv.}, \qquad (5.2.7)$$

**2) The elliptic paraboloids** (paraboloids with one sheet)

$$SO(2.1): \qquad x^1 x^1 - x^2 x^2 + x^3 x^3 = \text{inv.}, \qquad (5.2.8)$$

**3) All deformations of the sphere** (prolate or oblate spheroidal ellipsoids)

$$\hat{SO}(3): \qquad x^1 b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = \text{inv.}, \qquad (5.2.9)$$

**4) All deformations of the elliptic paraboloids**

$$\hat{SO}(2.1): \qquad x^1 b_1^2 x^1 - x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = \text{inv.}, \qquad (5.2.10)$$

**5) The isodual sphere**

$$SO^d(3): \qquad -x^1 x^1 - x^2 x^2 - x^3 x^3 = \text{inv.}, \qquad (5.2.11)$$

**6) The hyperbolic paraboloid** (paraboloid with two sheets here reinterpreted as the isodual of the elliptic paraboloid)

$$SO^d(2.1): \qquad -x^1 x^1 + x^2 x^2 - x^3 x^3 = \text{inv.}, \qquad (5.2.12)$$

---

[35] Note that the classification also includes the *real cones* when the hyperbolic invariants are equal to zero, the *imaginary cones* when the elliptic invariants are equal to zero, and other structures. For a comprehensive list of all possible invariants one may consult ref. [1], p. 221–222.

**7) All isodual ellipsoids**

$$\hat{SO}^d(3): \qquad - x^1 b_1{}^2 x^1 - x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = \text{inv.}, \quad (5.2.13)$$

**8) All deformations of the hyperbolic paraboloid**

$$\hat{SO}^d(2.1): \qquad - x^1 b_1{}^2 x^1 + x^2 b_2{}^2 x^2 - x^3 b_3{}^2 x^3 = \text{inv.}, \quad (5.2.14)$$

The unifying power of the isoeuclidean geometry of Class IV is then even broader, beacuse of the inclusion of the various cones. Finally, the isogeometry of Class V introduces *novel* structures, such as the sphere defined with respect to a distribution or a step function as the basic unit.

It should be indicated that all *physical* applications known at this time are restricted to the isoeuclidean geometry of Class I, which unifies the sphere and all its ellipsoidical deformations (5.2.9), and to the isodual isoeuclidean geometry of Class II (see below), which unifies the isodual sphere (5.2.11) and all its ellipsoidical deformations (5.2.13). This is due to the current lack on knowledge of a physical deformations capable of altering, say, ellipsoids into hyperboloids, or viceversa.

Nevertheless, it is intriguing to note that quatitative representations of *biological* systems such as sea shells appear to require isoeuclidean geometries of at least Class III (see later on Fig. 5.2.5).

Note that the isoeuclidean geometry implies a revision of the very notion of distance, called *isodistance*, whose square is isoinvariant (5.2.1c), with remarkable cosmological implications studied in Vol. II.

We should finally note that the full formulation of the isoeuclidean geometry is that of Class V which incorporates all preceding classes, thus including Class IV with singular isounits. In fact, at the level of Class III, the isogeometry essentially contains all quadrics via four disjoint classes with signatures (+, +, +), (+, +, –), (–, –, –), and (–, –, +). The point is that the interconnection of these classes requires the transition from positive to negative values of the elements of the isometrics, thus requiring the transition through the null values of the elements $g_{kk}$ which characterizes precisely singular isounits and, thus, an isogeometry of Class IV.

We now study Property III. It essentially deals with the fact that *all quadrics (A)–(D) of Fig. 5.2.2 have the shape depicted only when projected in our Euclidean space, because when properly represented in isoeuclidean space they all are perfect circles.*

This intriguing property should not be surprising for the reader now familiar with isotopic liftings. In fact, as it was the case for straight lines, *the isotopies of a sphere must remain a sphere as a necessary condition for the achievement of the isotopies themselves.* The unification of the sphere with all its infinitely possible ellipsoidical deformations then follows, with evidently broader unifications for higher classes.

One can now understand why distances which are very large in our empirical perception of the universe in Euclidean space can become rather small in isospace. In fact, very large distances, say, in a hyperboloid are turned into relatively much shorter distances on the isosphere.

## ISOTOPIC UNIFICATION OF QUADRICS

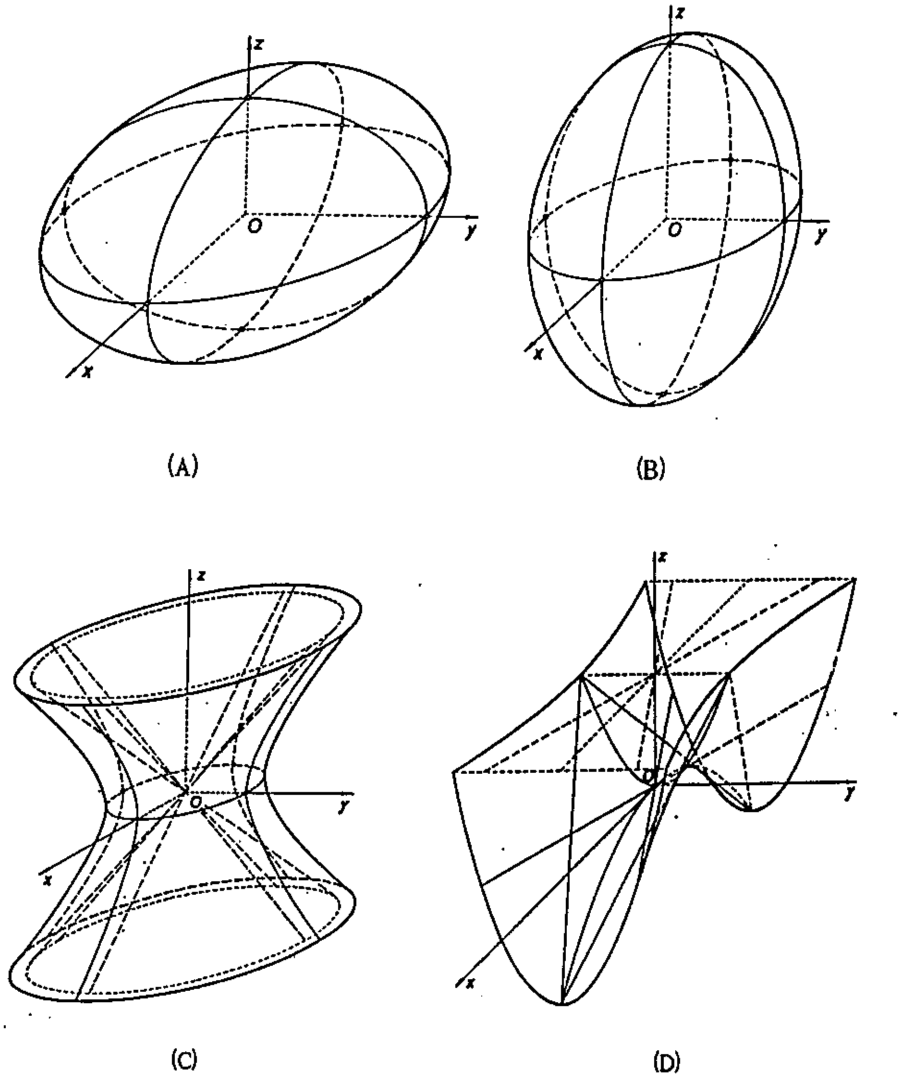



FIGURE 5.2.2: A schematic view of the prolate ellipsoid (A), oblate ellipsoid (B), one sheet hyperboloid (C) and two sheet hyperboloid (D), plus the related real and imaginary cones here omitted for brevity (see ref. [28] for complete classification). All these quadrics are unified into one single geometric notion in isospace of Class

III, the isoinvariant (5.2.1c) in classification (5.2.6). Moreover, each of the above quadrics admits in isoeuclidean geometry an isodual occurring when the isounits and isonorm become negative-definite. These isoduals are also unified with all preceding ones in one, single abstract, geometric entity in isospace.

The geometric origin for the unification of all quadrics into one single geometric entity is quite simple. *In conventional Euclidean geometry we perform the transition from the sphere to the ellipsoids*

$$\delta = \text{diag.}(1, 1, 1) \quad \rightarrow \quad \hat{\delta} = \text{diag.}(b_1^2, b_2^2, b_3^2), \tag{5.2.15}$$

*by preserving the conventional unit, thus implying the differentiation of the two quadrics. In isospace we perform the transition from the sphere to the ellipsoids, while jointly lifting the unit by exactly the inverse amount of the deformation*

$$I = \text{diag.}(1, 1, 1) \quad \rightarrow \quad \hat{1} = \text{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}), \tag{5.2.16}$$

*The preservation of the perfectly spherical shape in isospace is then intrinsic in the very structure of the isotopy.*

We consider now the representation of vectors and their operations in the isoeuclidean geometry. Recall that the basis of a vector space is not changed under isotopy (up to possible renormalization factors). Let $e_k$, k = 1, 2, 3, be the unit vectors on E(r,δ,R) directed along the x, y, z axes, and let $\hat{e}_k$ be the corresponding isobasis in $\hat{E}(r,\hat{\delta},\hat{R})$. Then, a vector V can be expressed in isospace as in the conventional case

$$V = x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3. \tag{5.2.17}$$

This is another way of expressing the fact that the vector V is straight in $\hat{E}(r,\hat{\delta},\hat{R})$, although its projection in E(r,δ,R) is curved. As usual under isotopies, *the operations on vectors are changed.* In fact, the scalar product $V_1 \cdot V_2$ of two vectors $V_1 = (x_1, y_1, z_1)$ and $V_2 = (x_2, y_2, z_2)$ is now lifted into the expression called *isoscalar product*

$$V_1 \odot V_2 = (x_1\, g_{11}\, x_2 + y_1\, g_{22}\, y_2\; z_1\, g_{11}\, z_2)\,\hat{1} \in \hat{R}(\hat{n},+,*). \tag{5.2.18}$$

Note that, as expected, the isoscalar product preserves the original axioms, i.e.

$$V_1 \odot V_2 = V_2 \odot V_1, \quad V_1 \odot (V_2 + V_3) = V_1 \odot V_2 + V_1 \odot V_3. \tag{5.2.19}$$

Moreover, the *isonorm* on $\hat{E}(r,\hat{\delta},\hat{R})$ is expressible in terms of the isoscalar product via the rule

$$\uparrow V \uparrow \; = \; (V \odot V)^{\frac{1}{2}} \hat{1} \in \hat{R}(\hat{n},+,*). \tag{5.2.20}$$

Thus, the isocosinus of the angle for two intersecting vectors (Fig. 5.2.1) can be

**THE ISOSPHERE IN ISOEUCLIDEAN SPACE**

$x\, g_{11}\, x \;+\; y\, g_{22}\, y \;=\; 1,\; \hat{1} \;=\; \mathrm{diag.}\,(g_{11}{}^{-1}, g_{22}{}^{-1}),\; g_{kk} \neq 0$

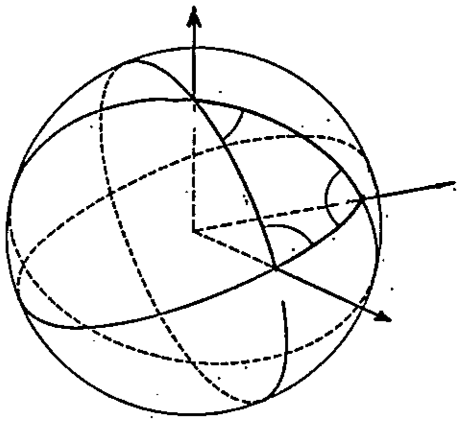


FIGURE 5.2.3: While the Euclidean "space" is unique, there exist infinitely many isoeuclidean "spaces", evidently because of the infinitely many possible isounits $\hat{1}$. This allows the unification of all infinitely possible quadrics into the isosphere depicted in the figure. The mechanism is, again, so simple to appear trivial. Consider the sphere with unit radius and, thus, with all identical semiaxes

$$a \;=\; b \;=\; c \;=\; 1. \tag{1}$$

Consider now its deformation into ellipsoids with semiaxes in our Euclidean space

$$a \;=\; b_1{}^2,\;\; b \;=\; b_2^2,\;\; c \;=\; b_3{}^2. \tag{2}$$

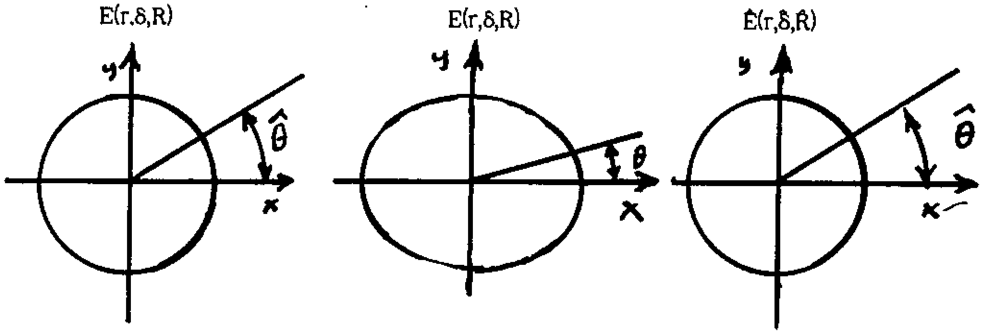
But the basic unit in isospace is changed into form (5.2.16). This implies that the semiaxes in isospace preserve the original numerical value 1,

$$\hat{a} \;=\; b_1{}^2\, b_1{}^{-2} \;=\; 1, \quad \hat{b} \;=\; b_2{}^2 b_2{}^{-2} \;=\; 1\,,\;\; \hat{c} \;=\; b_3{}^2\, b_3{}^{-2} \;=\; 1\,. \tag{3}$$

This illustrates the preservation under isotopy of the original axioms (1) of sphericity, and the unification of all possible quadrics and all their isoduals into the isosphere of this figure. Equivalently, the preservation of perfectly spherical character for isosurfaces of Class I can be derived via the isotrigonometric functions under which we have

$$x\,g_{11}\,x + y\,g_{22}\,y = \cos^2\hat\theta + \sin^2\hat\theta = 1, \quad \hat\theta = \theta\,(g_{11}\,g_{22})^{\frac{1}{2}}. \quad (4)$$

A similar result can be derived for the isosurfaces of Class II via the use of the isohyperbolic functions of the next section. The reader can now see the property before on the deformation of the circle




the angle $\hat\theta$ in rule (5.2.3) is the angle prior to the deformation; the angle $\theta$ is the angle of the deformation as measured in the Euclidean plane; and the isotopy, that is, the values $g_{11}$ and $g_{22}$, are such to reconstruct the original angle $\hat\theta = \theta(g_{11}g_{22})^{1/2}$ in isospace.

written as the isotopy of the conventional case

$$\text{isocos}\,\hat\theta = \frac{V_1 \odot V_2}{\uparrow V_1 \uparrow * \uparrow V_2 \uparrow}. \qquad (5.2.21)$$

Also, one can introduce the *directional isocosinuses* of a vector

$$\text{isocos}\,\hat\alpha = V_1 / \uparrow V \uparrow, \quad \text{isocos}\,\hat\beta = V_2 / \uparrow V \uparrow, \quad \text{isocos}\,\hat\gamma = V_3 / \uparrow V \uparrow. \quad (5.2.22)$$

Then, we have again the correct lifting of the corresponding conventional identity

$$(\,g_{11}\,\text{isocos}^2\,\hat\alpha + g_{22}\,\text{isocos}^2\,\hat\beta + g_{33}\,\text{isocos}^2\,\hat\gamma\,)\,\hat 1 = 1\,. \qquad (5.2.23)$$

Consider now two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Then the *isodistance* among them is the quantity

$$\hat D^{\hat 2} = [\,(x_1 - x_2)g_{11}(x_1 - x_2) + (y_1 - y_2)g_{22}(y_1 - y_2) + (z_1 - z_2)g_{33}(z_1 - z_2)\,]\,I \quad (5.2.24)$$

it is evidently unique (for each given isounit) and permits a study of the isotopy of the original Euclid axioms (Fig. 5.2.4)

Similarly the vectorial product $V_1 \wedge V_2$ is lifted in the expression called *isovectorial product*

$$V_3 = V_1 \hat{\wedge} V_2, \quad V_{3k} = \epsilon_{kij}(g_{ij}^{\frac{1}{2}} x_{i1})(g_{jj}^{\frac{1}{2}} x_{2j}), \; i, j, k = 1, 2, 3. \quad (5.2.25)$$

which satisfies the basic axioms of a vector product

$$V_1 \hat{\wedge} V_2 = - V_2 \hat{\wedge} V_1, \quad V_1 \hat{\wedge} (V_2 + V_3) = V_1 \hat{\wedge} V_2 + V_1 \hat{\wedge} V_3. \quad (5.2.26)$$

**ISOTOPY OF EUCLID'S FIFTH AXIOM**

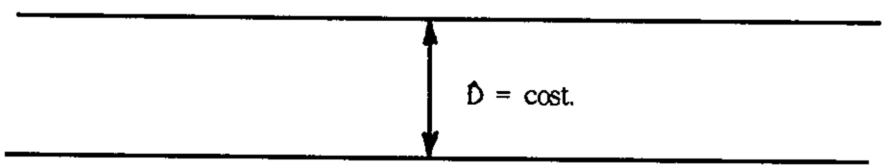



FIGURE 5.2.4. A schematic view of Euclid's celebrated Fifth Axiom on parallel lines, which can be reformulated in isoeuclidean spaces, thus leading to the notion of *isoparallel lines.*

It is instructive for the interested reader to verify the preservation of *Lagrange's identity* under isotopies among four vectors A, B, C, D in $\hat{E}(r,\hat{\delta},\hat{R})$

$$(A \hat{\wedge} B) \odot (C \hat{\wedge} \hat{D}) = (A \odot C) * (B \odot D) - (B \odot C) * (A \odot D). \quad (5.2.27)$$

Other properties can be easily derived by the interested reader via similar procedures.

A few comments are now in order on other geometries, particularly the so-called *non-Euclidean geometries* (see, e.g., ref. [28] and quoted literature). As well known, Euclid's Fifth Axiom lead to a historical controversy that lasted for two millennia, until solved by Lobacevskii in a rather unpredictable way, via the introduction of a new, non-Euclidean geometry today appropriately called *Lobacevskii geometry* (see [loc. cit.]).

As well known, Lobacevskii geometry is also based on certain liftings of Euclidean expressions, although defined on the conventional unit. Thus, the Lobacevskii and isoeuclidean geometries are structurally different.

Nevertheless, it is also important to understand that *the Lobacevskii geometry is a particular case of the projection of the isoeuclidean geometry in the Euclidean plane.* To see this point consider the following celebrated transformations

$$x' = \frac{x + a}{1 + a x}, \qquad y' = \frac{y(1 - a^2)^{\frac{1}{2}}}{1 + ax}, \qquad |a| < 1, \qquad (5.2.28)$$

which have the peculiar property of carrying straight lines into straight lines and circles into circles (see ref. [28] for details) *while keeping the unit the same.* Now, the isoeuclidean space $E(r,\delta,R)$ in two dimensions can be equivalently reinterpreted as an ordinary Euclidean plane $E(\bar{r},\delta,R)$ in the new coordinates

$$\bar{x} = g_{11}^{\frac{1}{2}} x, \quad \bar{y} = g_{22}^{\frac{1}{2}} y, \qquad (5.2.29)$$

under which we have the identity

$$\bar{x}\,\bar{x} + \bar{y}\,\bar{y} = x\, g_{11}\, x + y\, g_{22}\, y. \qquad (5.2.30)$$

It is then evident that Lobacevskii transformations (5.2.28) are contained as a particular case of the much larger class of isotransformations (5.2.29).

The connection between Lobacevskii and isoeuclidean geometries can therefore be expressed by saying that:

A) *the Lobacevskii geometry identifies "one" particular lifting of the Euclidean geometry preserving straight lines and circles under the conventional value of the unit;* while

B) *the isoeuclidean geometry identifies "an infinite class" of liftings of the Euclidean geometry which preserve straight lines and circles under a joint lifting of the unit.*

Note finally that the Lobacevskii geometry itself can be subjected to an isotopic lifting which has not been studied here for brevity.[36]

Numerous other noneuclidean geometries exist in the literature (besides the Minkowskian, symplectic, affine and Riemannian geometries studied later on in this section). One particularly intriguing geometry is the so-called *nondesarguesian geometry* studied by Shoeber [29], which has a significant connection with the studies of these volumes because it is also capable of representing variationally nonselfadjoint (that is, nonhamiltonian) systems.

This latter geometry too is different from the isoeuclidean one, again, because it is based on the conventional unit. However, the underlying mapping between the Euclidean and nondesarguesian geometry is also contained as a particular case of the infinite transformations (5.2.29) of the isoeuclidean geometry.

---

[36] Note that the *isolobacevkii geometry* is no longer contained as a particular case of the isoeuclidean geometry because the original axioms of the two geometries are different.

These comments are significant to focus the attention on an additional reason for our selection of the isoeuclidean geometry over other possible choices, its "direct universality" of incorporating "all" infinitely possible maps of the Euclidean geometry (including singular maps for Class IV and discrete maps for Class V).

In summary, the isoeuclidean geometry appears to be unique because encompassing all possible noneuclidean geometries when projected in the conventional space, yet remaining *Euclidean* in isospace.

Vol.s II and III contain numerous physical applications of the isoeuclidean geometry. Its primary function is to provide a geometry directly applicable to interior dynamical problems, that is, applicable to the most general possible nonlinear–nonlocal–nonhamiltonian systems studied in these volumes.

This physical objective is achieved via *the geometrization of physical media,* that is, via the characterization of the deviations in the geometric axioms of empty space caused by the presence of a physical medium. The geometrization is done via the restriction of the isogeometry to be of Class I, in which case the isometric is restricted to the positive–definite form

$$\hat{\delta} = T\delta \equiv T = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2)\,, \qquad (5.2.31)$$

where the b's, called *the characteristic functions of the medium considered,* have an unrestricted functional dependence of the type

$$b_k = b_k(t, r, \dot{r}, \ddot{r}., \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots) > 0,\ k = 1, 2, 3, \qquad (5.2.32)$$

including a dependence on basic physical characteristics of the medium to be geometrize, such as local density $\mu$, local temperature $\tau$, local index of refraction n, etc.

The above characterization is evidently not unique and can be done via other methods. However, *to be consistent with physical reality, such a characterization must be done with any appropriate methods "other than adding a potential to a Lagrangian or a Hamiltonian".* The is due to the intrinsically nonpotential character of the effects here considered. This basic condition is so compelling that the possible treatment of interior effects via a potential would imply trajectories not related to those of the physical reality.

The isoeuclidean geometry has been preferred over other possibilities because it verifies the above nonlagrangian–nonhamiltonian representation of interior effects, while preserving the same geometric axioms of empty space, thus permitting the geometric unity in the treatment of both exterior and interior problems achieved in these volumes.

The mechanism for the representation of interior problems is so simple to appear trivial. It is based on the now familiar lifting of the product of the conventional Euclidean geometry.

Consider an extended free particle in empty space, which is evidently represented via the kinetic energy alone

$$L = \tfrac{1}{2} m \dot{r} \cdot \dot{r} \in E(r, \delta, R) \qquad (5.2.33)$$

where r represents the trajectory of the center of mass.

Suppose now that the particle at a given value of time penetrates within a physical medium, thus experiencing nonpotential forces. *The transition from the exterior to the interior problem is merely expressed by the transition from the Euclidean geometry to its isoeuclidean covering of Class I.*

In turn, the transition is represented by writing the original Lagrangian in isospace, thus reaching the following *isolagrangian*

$$\hat{L} = \tfrac{1}{2} m \dot{r} \odot \dot{r} \in \hat{E}(r, \hat{\delta}, \hat{R}) . \qquad (5.2.34)$$

The geometric aspect important for this section is that the two Lagrangians L and $\hat{L}$ coincide at the abstract level for all Class I isospaces.

Numerous classical examples are now available (see ref.s [6,20]). the simplest one is the particle with linear velocity–damping

$$\ddot{x} + \gamma \dot{x} = 0, \quad m = 1, \quad \gamma > 0 , \qquad (5.2.35)$$

which is merely represented via the particular realization of the isotopic element and isounit

$$T = e^{\gamma t}, \quad \hat{1} = e^{-\gamma t}, \quad \gamma > 0 . \qquad (5.2.36)$$

as the reader is encouraged to verify (see ref. [6], p. 101). The isorepresentation can be enlarged into the form

$$T = \mathrm{diag.}\,(b_1^2, b_2^2, b_3^2)\, e^{\gamma t} , \qquad (5.2.37)$$

exhibiting a feature completely absent in Euclidean geometry, *a direct representation of the actual nonspherical shape of the particle considered* here assumed to be an ellipsoid with semiaxes $b_k^2$. The understanding is that the isoeuclidean geometry can also be realized via *nondiagonal isotopic elements*, as requested by the case at hand.

Note that the representation of shape is completely absent in Newton's equation of motion (5.2.35) and it is a sole feature of the isoeuclidean geometry we shall study and apply in detail in Vol.s II and III. In fact, after computing the equation of motion, the "shape factor" cancels out.

But perfectly rigid objects do not exist in the physical reality. The isoeuclidean geometry then permits a *direct representation of all infinitely* *possible deformations of the original shape,* which can be easily achieved via a dependence of the characteristics b–quantities in the local pressure, velocity, etc.

Note that $T > 0$ and $\hat{1} > 0$ as verified for all known cases of particles in interior conditions (while for antiparticles we have $T = -e^{\gamma t} < 0$ resulting in the same equation of motion).

In summary, the isoeuclidean geometry has the following primary applications in physics: A) geometrization of the physical medium considered; B) representation of the resistive effects on the motion of extended particles; and C) representation of the actual, extended, deformable shape of particles.

The *nonrelativistic description* of these volumes will be based on the following four geometries

**I) Euclidean geometry,** i.e., the conventional geometry on the Euclidean space over the reals[37]

$$E(r,\delta,R), \quad \delta = \mathrm{diag.}\,(1,1,1), \quad R = R(n,+,\times), \quad I = \delta^{-1} = \mathrm{diag.}\,(1,1,1)\,, \qquad (5.2.38a)$$

$$r^2 = r^i\,\delta_{ij}\,r^j = x\,x + y\,y + z\,z = \mathrm{inv.} \qquad (5.2.38b)$$

which will be used for the description of *p(articles in exterior nonrelativistic conditions.*

**II) Isoeuclidean geometry,** which, unless otherwise stated, will be referred to Class I, i.e., the isogeometry defined on the following diagonal realization of the isospace and related isofields

$$\hat{E}(r,\hat{\delta},\hat{R}): \ \hat{\delta} = T\delta, \ \ \delta = \mathrm{diag.}\,(1,1,1)\,, \ \ \hat{R} = \hat{R}(\hat{n},+,*), \ \ \hat{n} = n\hat{1}, \ \hat{1} = T^{-1}, \qquad (5.2.39a)$$

$$T = T(t,r,\dot{r},\ddot{r},\ldots) = \mathrm{diag.}\,(b_1^{\,2}, b_2^{\,2}, b_3^{\,2}) > 0, \ \ b_k > 0, \qquad (5.2.39.b)$$

$$\hat{r}^2 = (r^i\,\hat{\delta}_{ij}\,r^j)\,\hat{1} = (x\,b_1^{\,2}\,x + y\,b_2^{\,2}\,y + z\,b_2^{\,2}\,z)\,\hat{1} \in \hat{R}(\hat{n},+,*)\,. \qquad (5.2.39c)$$

or to nondiagonal realizations which preserve the positive–definiteness of the isounit. Such isogeometry will be solely used for the description of *particles in interior nonrelativistic conditions.*

**III) Isodual euclidean geometry,** the isodual of the *conventional* geometry over the isodual space and isodual field

$$E^d(r,\delta^d,R^d), \ \ \delta^d = -\mathrm{diag.}\,(1,1,1), \ \ R^d = R^d(n^d,+,\times^d), \ \ n^d = n\,I^d\,, \qquad (5.2.40a)$$

---

[37] Note the difference between the unit 1 of the field $R(n,+,\times)$ and the unit $I = \mathrm{diag.}\,(1,1,1)$ of the geometry. Note also that the geometry and underlying field can be trivially reformulated with respect to the unit I.

$$I^d = \delta^{d-1} = -\,\mathrm{diag.}\,(1, 1, 1)\,, \qquad (5.2.40a)$$

$$r^2 = (-\,r^i\,\delta_{ij}\,r^j\,)\,\hat{1}^d = (-\,x\,x\; -\; y\,y\; -\; z\,z\,)\,I^d \in R^d \qquad (5.2408b)$$

which will be used for the description of *antiparticles in exterior nonrelativistic conditions* and

**IV) Isodual isoeuclidean geometry,** defined on an isodual isospace of Class II over an isodual isofield of the same class in the diagonal realization

$$\hat{E}^d(r,\hat{\delta}^d,\hat{R}^d){:}\quad \hat{\delta}^d = T^d\delta,\ \delta = \mathrm{diag.}\,(1, 1, 1),\, T^d = -\,T\,,\hat{1}^d = -\hat{1} = -T^{-1},\ (5.2.41a)$$

$$T^d = T^d(t, r, \dot{r}, \ddot{r}, ...) = -\,\mathrm{diag.}\,(\,b_1{}^2, b_2{}^2, b_3{}^2\,) = T^\dagger,\ b_k > 0, \qquad (5.2.41.b)$$

$$r^{\hat{2}\,d} = (\,-\,r^i\,\hat{\delta}_{ij}\,r^j\,)\,\hat{1}^d = (\,-\,x\,b_1{}^2\,x - y\,b_2{}^2\,y - z\,b_2{}^2\,z\,)\,\hat{1}^d \in \hat{R}^d(\hat{n}^d,+,*^d)\,.\ (5.2.41c)$$

or in a nondiagonal realization preserving the negative–definite character of the isounit. This isodual isogeometry will be solely used for the description of *antiparticles in nonrelativistic interior conditions.*

The restriction to Class I for particles (or, separately, to Class II for antiparticles) is *necessary* because the presence of matter cannot modify the topology of empty space from a compact to a noncompact form and viceversa, to our best knowledge at this time.[38]

In Vol. II and III we study examples and applications of these isogeometries in nuclear physics, high energy physics and other fields. One application in the field of conchology is particularly significant to deserve an outline here, not only because it is unexpected and thus intriguing, but also because it permit the illustration of the fallacy of our geometric perception of Nature.

A mathematical representation of the growth of sea shells has been achieved by Illert [30]. The main result is that *sea shells should crack if their growth occurs via the strict application of the Euclidean geometry,* as established by computer visualizations. The issue addressed here is therefore the identification of the appropriate geometry permitting a consistent representation of their growth.

As well known, sea shells grow by discrete increments $\Delta\xi$, thus requiring discrete methods. Their analytic representation in $E(r,\delta,R)$ has a "kinetic term" $K = \frac{1}{2}(\Delta\xi/\Delta t)\cdot(\Delta\xi/\Delta t)$ and a "potential" term similar to that of the harmonic oscillator, $V = \frac{1}{2}\Delta\xi\cdot\Delta\xi$. The emerging Lagrangian in $E(r,\delta,R)$ is therefore of the type

$$L = \tfrac{1}{2}\,(\Delta\xi/\Delta t)\cdot(\Delta\xi/\Delta t) + \tfrac{1}{2}\ \Delta\xi\cdot\Delta\xi\,. \qquad (5.2.42)$$

[38] The knowledge of physical events capable of changing the Euclidean signature (+, +, +) into the hyperbolic form (+, +, –) would imply the theoretical possibility mentioned earlier of transforming very large distances in Euclidean space into very small distances in isoeuclidean space of Class III.

Illert's studies [loc. cit.] show that:

1) Euclidean models of type (5.2.42) are insufficient to represent the actual growth of sea shells, as illustrated by the disparity between reality and computer modelling.

2) The problem of growth of sea shells is analytically similar to the interior dynamical problem, evidently because growth is a "nonconservative" process; and

3) The growth of sea shells can be quantitatively represented via noneuclidean Lagrangians of the type

$$L = \tfrac{1}{2}\, e^{\Omega(\phi)} (\Delta\xi/\Delta t) \cdot (\Delta\xi/\Delta t) + \tfrac{1}{2}\, e^{\Omega(\phi)} \Delta\xi \cdot \Delta\xi\,, \qquad (5.2.43)$$

where $\Omega(\phi)$ is a function, varying from shell to shell, of the characteristic angle $\phi$ of growth of each shell (see ref. [30] for details).

It is then evident from the above results that *sea shells do not evolve in Euclidean space, while they do admit a quantitative interpretation as evolving in isoeuclidean space with isorepresentation and fundamental isounit*

$$\hat{L} = \tfrac{1}{2} (\Delta\xi/\Delta t) \odot (\Delta\xi/\Delta t) + \tfrac{1}{2}\, \Delta\xi \odot \Delta\xi\,, \quad \hat{1} = e^{-\Theta(\phi)}. \qquad (5.2.44)$$

Again, sea shells *appear* to evolve in our Euclidean space because of the peculiar nature of the isogeometries of preserving the shapes of our space, but they evolve in a more complex geometry which is representable via the isotopy of the product $AB \rightarrow ATB$ *and* the joint isotopy of the unit $1 \rightarrow \hat{1} = T^{-1}$. Note that the three-dimensional character of the geometry remains completely unaffected in the transition to a higher geometry.

Sea shells also constitute an illustration of the need for the broadest possible isoeuclidean geometry, that of Class V. In fact, the interpretation of their growth at bifurcations clearly shows the need for an isogeometry of Class III (Fig. 5.2.5). Moreover, their structure is discrete, thus requiring, in general, isounits of Kadeisvili's Class V which include all other classes as particular cases.

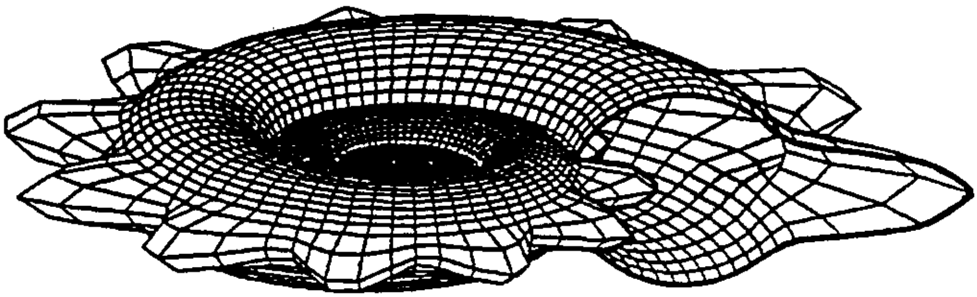
This is per se intriguing inasmuch as we have just indicated the *necessary* condition for *physical* events of restricting the isoeuclidean geometry, separately, to Class I for particles and to Class II for antiparticles. It therefore appears that *biological events* have a structurally more general geometry of at least Class III encompassing both motions forward and backward in time.

The application of the isoeuclidean geometry to the growth of sea shells, even though evidently not unique, is instructive in suggesting an act of scientific humility: expressing doubts prior to claiming final achievement of knowledge via a perception of Nature based on our manifestly limited three Eustachian tubes.

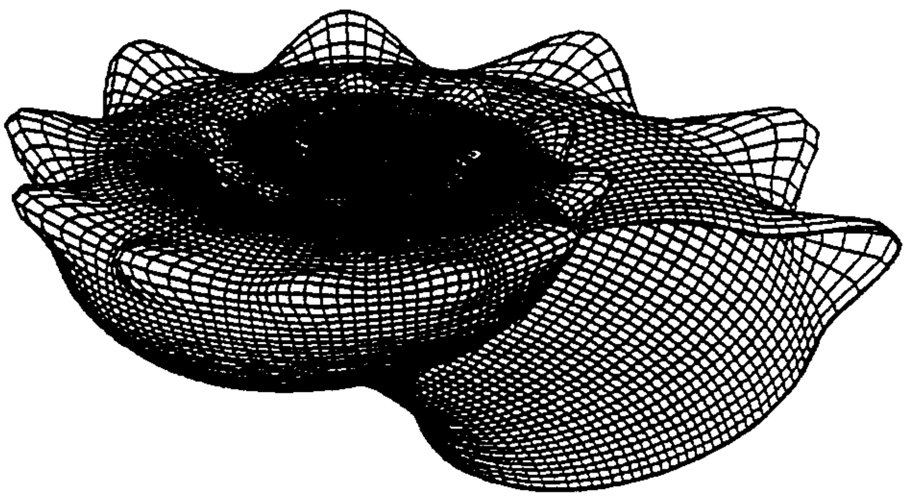
In fact, relatively "simple" biological entities such as sea shells, even though appearing to our perception as belonging to a three-dimensional Euclidean world, evolve in reality according to a structurally more general geometry.

This is an clear indication that, rather than having achieved "final" geometrical knowledge, the complexity of the geometry of biological entities, such as a DNA molecule, is simply beyond the grasp of human comprehension at this time.

**ISOEUCLIDEAN EVOLUTION OF SEA SHELLS**



(A)



(B)

FIGURE 5.2.5: The computer visualization of two relatively "simple" sea shells from Illert [30], p. 64, the *Phanerotinus Spiralis* (A) and the *Angaria Delphinius* (B). The

computer visualization clearly shows that the shells would crack during their growth if the Euclidean geometry is strictly implemented. However, the same computer modeling shows that growth is normal in isoeuclidean space with isorepresentation (5.2.44). In the transition to more complex sea shells, e.g., those with bifurcations, the need for noneuclidean geometries appear more compelling. In fact, a quantitative interpretation of growth at the bifurcations in Euclidean space would require *a discontinuous inversion of time* (see ref. [30], p. 98 and ff.). As we shall see in Vol. II, the isoeuclidean geometry of Class III permits instead a direct representation of the bifurcations without discontinuities [42].

## 5.3: ISOMINKOWSKIAN GEOMETRY

The *isominkowskian geometry* [12,21–23,41] is the geometry of the isominkowski spaces of Class V over isoreal fields of the same class (Sect. 3.5)

$$\hat{M}(x,\hat{\eta},\hat{R}): \quad \eta = \mathrm{diag.}\,(1, 1, 1, -1)\,, \quad \hat{\eta} = T(s, x, \dot{x}, \ddot{x}, \ldots..)\,\eta, \quad \hat{1} = T^{-1}, \tag{5.3.1a}$$

$$x^{\hat{2}} = [x^{\mu}\,\hat{\eta}_{\mu\nu}(s, x, \dot{x}, \ddot{x},\ldots)\,x^{\nu}\,]\,\hat{1} \in \hat{R}(\hat{n},+,*), \quad x = \{x^{\mu}\} = (r, x^4), \; x^4 = c_o t, \tag{5.3.1b}$$

$$ds^{\hat{2}} = (-\,dx^{\mu}\,\hat{\eta}_{\mu\nu}\,dx^{\nu}\,)\,\hat{1}, \tag{5.3.1c}$$

where $c_o$ is the *speed of light in vacuum* and s is called the *isotime* (t being the ordinary time), which, for Class I can be assumed in the diagonal realization

$$\hat{1} = \mathrm{diag.}\,(\,b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}\,) > 0, \tag{5.3.2a}$$

$$x^2 := (\,x^1\,b_1{}^2\,x^1 + x^2\,b_2{}^2\,x^2 + x^3\,b_2{}^2\,x^3 - x^4\,b_4{}^2\,x^4\,)\,\hat{1}\,. \tag{5.3.2b}$$

or in a nondiagonal form preserving the positive-definiteness of the isounit (see later on in this section for an example).

The isominkowski geometry is therefore characterized by four functions $b_\mu$ which: A) are called *relativistic characteristic functions of the medium considered*; B) have a generally nonlinear and nonlocal dependence on the isotime s, coordinates x, wavefunctions $\psi$, $\psi^\dagger$ their derivatives of arbitrary order, $\dot{x}$, $\ddot{x}$, $\partial\psi$, $\partial\psi^\dagger$...., as well as the physical characteristics of the medium considered, such as the local density $\mu$, the local temperature $\tau$, the local index of refraction n, etc.; and C) are assumed to be positive-definite for reason indicated below

$$b_\mu = b_\mu(s, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \mu, \tau, n, \ldots\ldots..) > 0\,, \quad \mu = 1, 2, 3, 4\,. \tag{5.3.3}$$

The above general functional dependence is needed for the local study of interior dynamical problems; that is, the trajectory of an extended relativistic particle within a physical medium at one given interior point x.

If one studies global effects of physical media, such as the average speed of light throughout the medium, the characteristic functions can be averaged into the constants

$$b^{\circ}_{\mu} = < \hat{b}_{\mu}(s, x, \dot{x}, \ddot{x}, ...) > , \qquad (5.3.4)$$

where $< .... >$ represents an average appropriate for the problem at hand.

As we shall see in Ch. II.8, the above average preserves a quantitative representation of interior effects, while permitting the recovering of conventional inertial systems for an outside observer.

Note that *the space–component of the isominkowskian geometry is the isoeuclidean geometry in its entirety.*

The first property to keep in mind is that *the Minkowskian and isominkowskian geometries coincide, by construction, at the abstract, realization–free level.* This is due to the positive–definiteness of the b–quantities or, equivalently to the preservation under isotopies of the signature (+, +, +, –).

This is the fundamental property for which the isominkowskian geometry was built in the first place [12]. The evident objective is the preservation of the axioms of the special relativity in the transition from the exterior to the interior problem, as studied in detail in Vol.s II and III. The subtle implication is that *criticisms on the isominkowskian geometry may in the final analysis result to be criticisms on Einstein's axioms themselves.*

The primary physical application of the isominkowskian geometry is the relativistic geometrization of physical media (see Fig. 5.3.1 for comments).

A first point which should be stressed is that, *by no means, is the isominkowskian geometry the only possible one for the geometrization here considered.* In fact, the use of other geometries is conceivable, and their study is encouraged, because one of the beauties of mathematical and physical inquiries is their polyhedric character.

A second point stressed in Ch. I.3 is that *studies on the propagation of classical electromagnetic waves via operator approaches in first and second quantizations should be strictly precluded because they would suppress the very characteristics to be represented.*

In essence, when first exposed to the propagation of light in our inhomogeneous and anisotropic atmosphere, a very natural mental attitude is the study of the propagation via old methods, e.g., via scattering of photons on the atoms of our atmosphere.

This is the approach which should be avoided on both theoretical and experimental grounds. Theoretically, the event depicted in Fig. 5.3.1 is *purely classical,* thus requiring a purely classical description, rather than the use of photons in second quantization. *After* the achievement of a geometric representation of the inhomogenuity and anisotropy of physical media at the classical level, studies based on first and second quantization should be considered.

## ISOMINKOWSKIAN GEOMETRIZATION OF PHYSICAL MEDIA

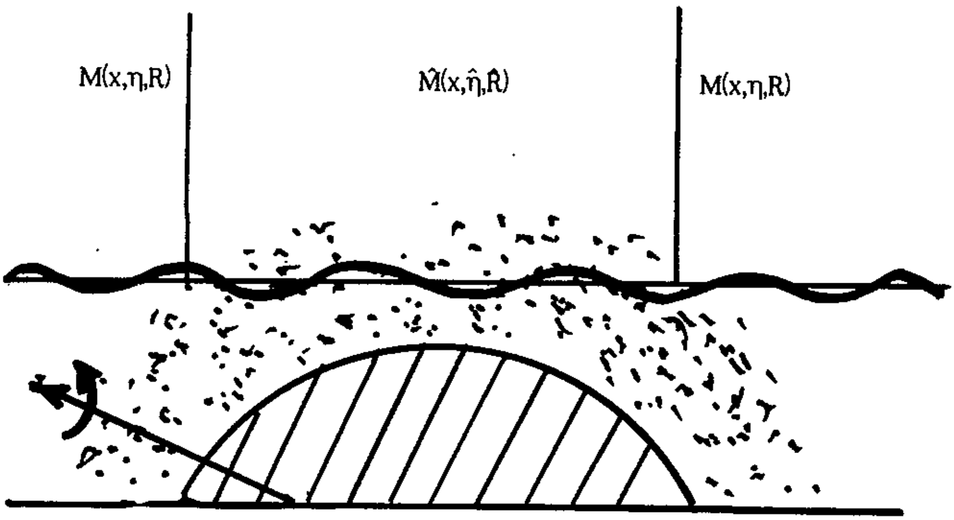



FIGURE 5.3.1. A schematic view of a primary physical application of the isominkowskian geometry: *the quantitative treatment in a form suitable for experimental verifications of the dynamical effects caused by the inhomogenuity and anisotropy of physical media in the propagation of electromagnetic waves and particles.* Recall that the Minkowskian geometry is a geometrization of the homogeneity and isotropy of empty space. All predictions based on the Minkowskian geometry, such as Doppler's effects, dilation of time, etc., are therefore crucially dependent on the homogeneity and isotropy of empty space. Consider now an electromagnetic wave originating from a distant star which travels, first, in empty space (in which case the Minkowskian representation is exactly valid), then travels throughout our atmosphere, and finally returns to travel in empty space. Now, our atmosphere is manifestly inhomogeneous, and anisotropic, as discussed earlier in this volume. The physical issue requiring experimental verifications (which is studied in detail in Vol. III) is whether the inhomogenuity and anisotropy imply measurable deviations from the conventional Minkowskian predictions. Specifically, the experimental issue is whether Doppler's effect, time dilation, etc. have the same numerical values for events within inhomogeneous and anisotropic media or deviations are experimentally measurable. As we shall see in Vol. III a rather considerable body of experimental evidence supports the latter expectation, although in a predictable preliminary way. The mathematical issue considered here is therefore the achievement of a

geometric representation, specifically, of the inhomogenuity and anisotropy of our atmosphere. The isominkowskian geometry appears to be particularly suited to: A) provide a direct geometric treatment of physical media, B) in a form suitable for experimental verifications, while C) preserving the basic Einsteinian axioms at the abstract level.

At any rate, the reduction of the event of Fig. 5.3.1 to photons scattering on atoms guarantees the elimination of the inhomogenuity and anisotropy to be represented.

But the strongest support against the preservation of old knowledge for the novel physical conditions of Fig. 5.3.1 comes from experimental data. In fact, as we shall see in Vol. III, physical media imply *shifts toward both, the red or the blue depending on their characteristics.* Assuming that adequate manipulations permit the interpretation of shift toward the red via scattering of photons on atoms, the same theory cannot evidently represent the opposite shift, precisely because lacking the characteristics to be represented.

A first intuitive understanding of the isominkowskian geometrization of physical media can be reached by writing the isoseparation in the equivalent form

$$x^2 = x^1 b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_2^2 x^3 - x^4 b_4^2 x^4 =$$
$$= x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_2^{-2} x^3 - x^4 n_4^{-2} x^4 )\hat{1}, \qquad (5.3.5)$$

(where we have ignored the multiplicative factor $\hat{1}$ for simplicity), namely, by expressing the characteristic functions in the equivalent form $b_\mu = 1/n_\mu$. Now, the fourth term,

$$b_4 = 1 / n_4 , \qquad (5.3.6)$$

is already known, and represents the local index of refraction within a given medium, yielding the local speed of light

$$c = c_o b_4 = c_o / n_4 = c(x, \mu, \tau, ...) . \qquad (5.3.7)$$

One can therefore see the above distinction between the characteristic *functions* $b_\mu$ and the characteristic *constants* $b°_\mu$. In fact, the quantity $n_4$ is the local index of refraction at one given point in space-time (characterizing the speed of light at one point of our atmosphere in Fig. 5.3.1), while $n°_4$ is the average index of refraction (characterizing the average speed of light throughout our entire atmosphere).

A first meaning of the isominkowskian geometry is therefore that of providing a *relativistic generalization of the familiar index of refraction* $n_4$ *to* *all space–time components* $n_\mu$. At any rate, such an extension is requested by the very notion of space–time covariance, or it can be derived from the use of the conventional Lorentz transformations.

At a deeper level, recall that only a small portion of physical media is transparent to light. A second meaning of the isominkowskian geometrization is therefore that of *extending the index of refraction to all physical media, whether transparent or not to light.* In the latter case the quantity $b_4 = 1/n_4$ acquires a purely geometric meaning similar, say, to the component $g_{44}$ the Riemannian metric.

As we shall see in Vol. III, experimental evidence indicates quite clearly that *the space characteristic functions* $b_k$, $k = 1, 2, 3$, *have a velocity and other dependence, while the fourth characteristic quantity* $b_4$ *generally provides a geometrization of its density.*

The isominkowskian representation of the inhomogenuity and anisotropy of physical media is now evident. In fact, the former can be represented, e.g., via a dependence of the characteristic functions on the local density, while the latter can be represented, e.g., via a differentiation of the space–time quantities, $b_k \neq b_4$.

As a first example, a direct representation of water is given by the simplest possible isotopy, called *relativistic scalar isotopy* (see Ch. II.8 for details)

$$\hat{x}^2 = \frac{1}{n^\circ} x^2, \quad n^\circ_\mu = n^\circ, \ \mu = 1, 2, 3, 4. \tag{5.3.8}$$

where $n^\circ$ is a known numerical quantity and $x^2$ is evidently the conventional Minkowskian invariant. In fact, water is a homogeneous and isotropic medium whose characteristics are then represented by isoinvariant (5.3.8).

A second example is our inhomogeneous and anisotropic atmosphere which requires the full isoinvariant (5.3.5) for its representations. The numerical values of the $b^\circ$–constants will be computed in Vol. III from astrophysical data. Needless to say, the deviations of the $b^\circ$–quantities from the value 1 are very small for our atmosphere, yet they produce measurable effects, as we shall see.

Intriguingly, isoinvariant (5.3.8) and related isospecial relativity permit a direct representation of relativistic kinematics in water, such as: the *decrease* of the speed of light according to law (5.3.7); the propagation of electrons faster than the local speed of light (Cherenkov's effect); the correct relativistic addition of speeds in water; and others (see Ch. II.8).

The extension of the results to inhomogeneous and anisotropic media is then consequential, and equally consequential are deviations from the Minkowskian prediction in vacuum.

The main features of the isominkowskian geometry are intriguing because they are contrary to our ordinary perception of geometric structures in the physical reality. The space component has been studied in the preceding section and it is hereon assumed. We can therefore restrict ourselves to a study of the particular case in the 3–4 dimension with *isolight cone*

$$x^2 = x^3\, b_3^2\, x^3 - x^4\, b_4^2\, x^4 = 0\,,\ \hat{1} = \mathrm{diag.}\,(\, b_3^{-2}, b_4^{-2})\,, \qquad (5.3.9)$$

The above structure clearly represents a *locally varying deformation of the light cone.* This feature is evidently *necessary* for physical consistency because the constancy of the speed of light is a philosophical abstraction, while the speed of light in the physical reality is indeed a locally varying quantity depending on the characteristics of the medium at hand. A locally varying light cone as represented by Eq.s (5.3.9) is therefore established by physical evidence beyond credible doubts.

The intriguing point is that deformation (5.3.9) appears only in the projection of the isominkowskian description in the original Minkowski space, because at the level of the isospace itself there is no deformation.

Moreover, the preservation of the original perfect cone under isotopies is such to include the preservation of its characteristic angle; that is, the preservation of the speed of light $c_o$ at the abstract isotopic level.

The proof is trivial for the isolight cone in water. In fact, isoinvariant (5.3.9) for infinitesimal values $\Delta x$ and $\Delta t$ reads

$$\frac{\Delta x}{\Delta t} = \frac{b_4}{b_3}\, c_o \equiv c_o\,, \qquad (5.3.10)$$

(because $b_3 \equiv b_4$ in water).

The understanding of the isominkowskian geometry requires the knowledge that cone (5.3.10) is purely geometric because the speed of light in water is not $c_o$ but $c = c_o/n^\circ$. The actual light cone is therefore that characterized by c and not $c_o$.

It is easy to prove that the above results hold for arbitrary media, that is, they also hold for a locally varying speed of light within inhomogeneous and anisotropic media. In fact the general invariant for infinitesimal $\Delta x$ and $\Delta t$ is given by

$$\frac{\Delta x}{\Delta t} = \frac{b_4}{b_3}\, c_o\,. \qquad (5.3.11)$$

but in the 3–4 isoplane we have

$$\Delta x = D\, b_4 \sin\hat{\alpha}, \qquad \Delta t = D\, b_3 \cos\hat{\alpha}\,, \qquad (5.3.12)$$

where D is the isoipothenuse. This implies that, even in physical media, the isolight cone remains a perfect cone and its angle remains solely characterized by $c_o$.

A deeper knowledge of the (3–4)–isoplane can be gained via its Lie–isotopic isosymmetry $\hat{SO}(1.1)$, and related isotopy of the hyperbolic functions. These isotopies have already been studied in Ch. 3.6 as the realization of the isotope $\hat{SO}(2)$ with signature (+, –).

Also, the isominkowskian geometry in (1+1) dimension is a particular case of the two–dimensional isoeuclidean geometry of Class III, merely realized for the values $g_{11} > 0$, $g_{22} < 0$[39].

**ISOLIGHT CONE**

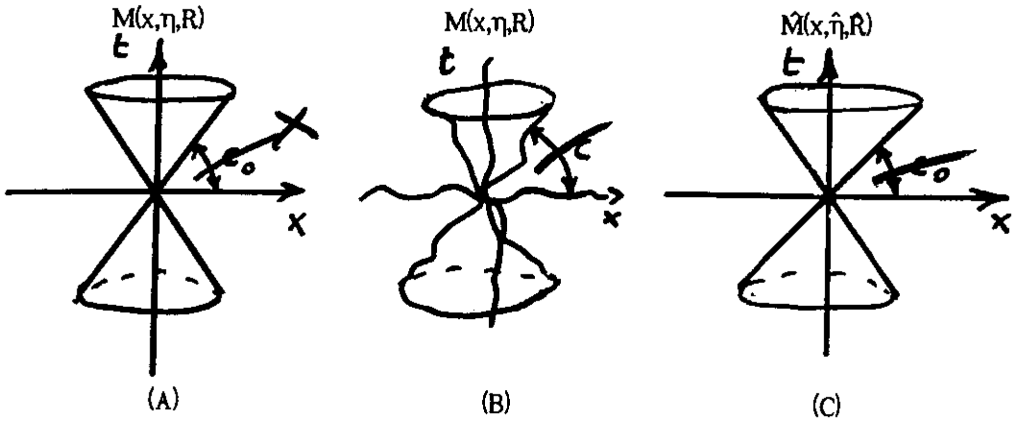



FIGURE 5.3.2. The three light cones of the isominkowskian geometry. Cone (A) is the conventional one in Minkowski space with $c_0 = 1$ and $\alpha = 45°$. "Cone" (B) is the physical one in our space–time for a locally varying speed of light propagating within a generic medium. Cone (C) the *isolight cone*, that is, an isotopy of the original perfect cone. As such, it is also a perfect cone provided that it is computed in isospace. We learn in this way that the isolight cone essentially maps the locally varying "cone" (B) into the perfect cone (C). The axiom–preserving character of the isotopy is so strict to preserve the original numerical value $c_0$, i.e., the original angle $\alpha = 45°$. An understanding of these geometric occurrences is essential for the understanding of the isotopies of the special relativity studied in Ch. II.8. In fact, as expected, the isospecial and the special relativities coincide at the abstract level to such an extent, as to admit the same light cone with the same speed of light $c_0$. Yet the physical predictions of the two relativities are profoundly different, as indicated by the inapplicability of the linear–local–canonical Lorentz transformations in favor of suitable nonlinear–nonlocal–noncanonical coverings,

[39] Evidently the (3+1)–dimensional isominkowskian geometry is a particular case of the four–dimensional isoeuclidean geometry of Class V with signature (+, +, +, –). This illustrates once again that all distinctions are lost at the appropriate isotopic level between Euclidean and Minkowskian formulations (Ch. 3). Equivalently, the occurrence is a mere realization of the capability of the Lie–isotopic algebra to unify in one single isotope O(2) and O(1.1) or (O(4) and O(3.1) (Sect. 4.5).

of the need to abandon the locally varying speed of light as the "universal invariant" for a geometrically more appropriate notion.

Consider a *hyperbolic angle*, i.e., a speed v. Then, Eq.s (4.6.8) for $g_{22} < 0$ provide the *isohyperbolic angle*

$$\hat{v} = v\, b_3\, b_4 \tag{5.3.13}$$

The isotopies of hyperbolic functions are then readily given by isotrigonometric functions (5.2.4) for $g_{22} < 0$, resulting in the *isohyperbolic functions*

$$\text{isosinh}\, \hat{v} = b_4^{-1} \sinh(v\, b_3\, b_4), \quad \text{isocosh}\, \hat{v} = b_3^{-1} \cosh(v\, b_3\, b_4), \tag{5.3.14}$$

which do verify the expected property

$$b_3^2\, \text{isocosh}^2\, \hat{v} - b_4^2\, \text{isosinh}^2\, \hat{v} = 1\,. \tag{5.3.15}$$

Note again the association of $b_3$ with isocosh $\hat{v}$, rather than with isosinh $\hat{v}$

The isominkowskian geometry is studied in more detail in Vol. II after constructing its isosymmetry, the isotopic Poincaré symmetry. We shall then study the axiom-preserving lifting of the basic postulates of the special relativity which is inherent in the geometry, and review Aringazin's proof of its "direct universality" for all possible deformations of the Minkowski metric. Experimental verifications are studied in Vol. III. The gravitational content of the geometry is studied in App. 5.B. During these studies we shall use the following four isogeometries:

**I) Minkowskian geometry,** which is the conventional one for the description of *particles in exterior relativistic conditions;*

**II) Isominkowskian geometry,** as described above for the description of *particles in interior relativistic conditions;*

**III) Isodual Minkowskian geometry,** which is the isodual image of the conventional one over the isodual space and related isodual field

$$M^d(x,\eta^d,R^d): \qquad \eta^d = -\eta, \qquad I^d = -I, \tag{5.3.16a}$$

$$x^{2\,d} = [x^\mu\, \eta^d_{\mu\nu}\, x^\nu\,]\, I^d = -[\,-x^\mu\, \eta^d_{\mu\nu}\, x^\nu\,]\, I \equiv x^2 \tag{5.3.16b}$$

$$ds^{2\,d} = (\,+\,dx^\mu\, \eta^d_{\mu\nu}\, dx^\nu\,)\, I^d \equiv ds^2 \in R^d(n^d, +, x^d)\,, \tag{5.3.16c}$$

which is used for the description of antiparticles in *exterior relativistic conditions;* and

**IV) Isodual isominkowski geometry,** which is the geometry of the isodual isominkowski space of Class II over an isodual isoreal field of the same class (Sect. 3.5)

$$\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d): \ \hat{\eta}^d = T^d(s, x, \dot{x}, \ddot{x}, \mu, .....)\,\eta = -\hat{\eta}, \ \hat{1}^d = (T^d)^{-1} = -\hat{1}, \qquad (5.3.17a)$$

$$x^{\hat{2}\,d} = [x^\mu\,\hat{\eta}^d{}_{\mu\nu}(s, x, \dot{x}, \ddot{x},...)\,x^\nu\,]\hat{1}^d = -[-x^\mu\,\hat{\eta}^d{}_{\mu\nu}\,x^\nu\,]\hat{1} \equiv x^{\hat{2}} \qquad (5.3.17b)$$

$$ds^{\hat{2}\,d} = (+\,dx^\mu\,\hat{\eta}^d{}_{\mu\nu}\,dx^\nu\,)\hat{1}^d \equiv ds^{\hat{2}} \in \hat{R}^d(\hat{n}^d,+,*^d), \qquad (5.3.17c)$$

which can be assumed in the diagonal form

$$\hat{1}^d = -\,\mathrm{diag.}\,(\,b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}\,) > 0, \qquad (5.3.18a)$$

$$x^{2\,d} = (-\,x^1\,b_1{}^2\,x^1 \ - \ x^2\,b_2{}^2\,x^2 \ - \ x^3\,b_2{}^2\,x^3 \ + \ x^4\,b_4{}^2\,x^4\,)\hat{1}^d \in \hat{R}^d. \qquad (5.3.18b)$$

or in any nondiagonal realization preserving the negative–definite character of the isodual isounit. This latter geometry will be used for the description of *antiparticles in relativistic interior conditions.*

As one can see, the isodual invariant coincides with the original isoinvariant (5.3.1b). However, the isominkowskian geometry and its isodual are physically different. In fact, the latter is defined on an isofield with negative–definite norm, thus implying negative–definite energies, motion backward in time, etc.

Note that the characteristic b–quantities remain the same for both geometries. The isolight cones also apply to the isodual geometries, although the references axe and the local variables are inverted in sign. As we shall see in Vol. II, the positive– and negative–energy solutions of conventional field equations imply that they can be defined on the tensorial product $M(x,\eta,R)\times M^d(x,\eta^d,R^d)$. The isotopies of field equations will then be defined on $\hat{M}(x,\hat{\eta},\hat{R})\times\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$.

We have considered until now isominkowskian geometries with a diagonal isotopic element T. The reader should be aware of the existence of rather intriguing applications for isogeometries with *nondiagonal isotopic elements and isounits.* One of the most significant cases was proposed by Dirac [31] in two of his last (and little known) papers dealing with a generalization of his own equation. The ensuing "Dirac's generalization of Dirac's equation" recently resulted to possess an essential isotopic structure, evidently without Dirac's awareness,[40] as we shall study in detail in Vol. II.

In this chapter we would like to identify only the rather intriguing isominkowskian geometry of Dirac's papers [31]. In essence, Dirac studied a deformation of the Minkowski space characterized by the nondiagonal element

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad (5.3.19)$$

[40] Dirac's papers [31] are of 1971–1972, while the isotopies were formulated in 1978 [5].

with properties

$$\det T = 1, \qquad 1 = T^{-1} = T^{t}, \tag{5.3.20}$$

where t denotes transposed, which therefore qualify it as a fully acceptable isotopic element of Class I.

The isogeometry characterized by isotopic element (5.3.19) is intriguing indeed. Its most salient property is that *the isometric is nondegenerate,* $\det \hat{\eta} = -1$, *but the isoinvariant is degenerate,*

$$x^2 = x^{\mu}\,\hat{\eta}_{\mu\nu}\,x^{\nu} = x^1\ x^3\ x^3\ x^4 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} =$$

$$= x^1 x^3 - x^2 x^4 - x^3 x^1 - x^4 x^2 = -2\,x^2 x^4, \tag{5.3.21}$$

namely, the isoseparation is contracted under Dirac's isotopy from four to two dimensions. In turn, this contraction has truly remarkable implications, such as the lifting of the original spin s = ½ to spin s = 0, as originally derived by Dirac [31] and as confirmed by isotopic methods (seee Vol. II).

It is instructive for the interested reader to see that the same dimensional contraction occurs for other realizations, such as for $\eta = (+1, -1, -1, -1)$ and related ordering of the components $x = \{x^4, x^1, x^2, x^3\}$. As a result, the dimensional contraction (1, 2, 3, 4) → (2,4) is intrinsic in the isogeometry here considered, and so are its rather peculiar properties, such as the contraction of the three-dimensions (1, 2, 3) down to the line along the y-axis.

We shall have ample opportunities in Vol.s II and III to study the above isogeometry, the related "Dirac's generalization of Dirac's equation", and its novel physical implications.

## 5.4: ISOSYMPLECTIC GEOMETRY

**5.4A: Statement of the problem.** In this section we shall study a generalization of the symplectic geometry[41] which is nonlinear (in all possible variables *and* their derivatives of arbitrary order), nonlocal-integral (also in all variables and their derivatives) and nonlagrangian-nonhamiltonian, yet preserving the original symplectic axioms.

The new geometry was introduced in memoir [15] under the name of

[41] An outline of the basic notions of the conventional symplectic geometry is presented in App. 5.A. For technical presentations, one may consult ref.s [3,4]. A comprehensive bibliography is contained in ref. [6].

*symplectic-isotopic geometry,* or *isosymplectic geometry* for short, and plays an important role in hadronic mechanics, e.g., for the isotopies of symplectic quantization, for interior gravitational problems, and others.

In particular, the isosymplectic geometry is the geometric counterpart of the Lie-isotopic theory of the preceding chapter. The identification of the corresponding generalized analytic structures will be presented in Vol. II, including explicit examples of "direct representations" of nonlinear-nonlocal-noncanonical systems.

In this section we shall merely review the main lines of the new geometry, and refer the reader to ref.s [15–20,25,26] for more details.

All quantities considered are assumed to verify the needed continuity conditions, e.g., of being of Class $\mathcal{C}^{\infty}$, which shall be hereon omitted for brevity. Similarly, all neighborhoods of given points are assumed to be star-shaped, or have a similar topology also ignored hereon for brevity.

Let M(R) be an n-dimensional manifold over the reals R(n.+,×) and let T*M(R) be its cotangent bundle. We shall denote with $T^*M_1(R)$ the manifold M(R) equipped with the *canonical one-form* θ [3,4]

$$\theta \;:\; T^*M_1(R) \Rightarrow T^*(T^*M_1(R)), \quad \theta \in \Lambda_1(T^*M_1(R)). \tag{5.4.1}$$

The *fundamental (canonical) symplectic form* is then given by

$$\omega = d\theta, \tag{5.4.2}$$

which is nowhere degenerated, exact and closed (see App. 5.A). The manifold M(R), when equipped with the symplectic two-form ω becomes an (exact) *symplectic manifold* $T^*M_2(R)$ in canonical realization. The *symplectic geometry* is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

The isotopies permit a dual nonlinear, nonlocal and noncanonical generalization of all that, one of isotopic type (Class I) and the other of isodual character (Class II).

In the original presentations we used the notation by Lovelock and Rund [4] in order to facilitate the identification of the differences between conventional and isotopic geometries, and we shall adopt the same approach here. Latin indices i, j, p, q, etc. will be used for a generic manifold, while Greek indices μ, ν, etc. will be used for specific physical applications.

**5.4B: Isodifferential calculus and its isodual.** The first visible implication of the isotopies for the symplectic geometry is that the basic differential calculus becomes inapplicable. This implies that the very notion of one-form θ or two-form ω are inapplicable and must be suitably generalized.

Consider an n-dimensional isomanifold $\hat{M}(\hat{R})$ (see ref. [26] for a technical definition) with local chart x over the isoreals $\hat{R}$, and let $T^*\hat{M}(\hat{R})$ be its *"isocotangent bundle"*, that is, the bundle of isoforms as more appropriately defined below. Introduce one of the infinitely possible, symmetric, nonsingular and real-valued isounits of Class I of the same dimension of M(R),

$$\hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, ...) = (\hat{1}^i_{\ j}) = (\hat{1}_j^{\ i}) = (\hat{1}^j_{\ i}) = (\hat{1}_i^{\ j}) = T^{-1}. \tag{5.4.3}$$

For mathematical consistency (e.g., to preserve isolinearity, see Sect. 4.2), conventional linear transformations on T*M(R), $x' = Ax$, or $x'^i = A^i_{\ j}\, x^j$, must be generalized on $T^*\hat{M}(\hat{R})$ into the *isotransformations*

$$x' = A * x, \qquad \text{or} \qquad x' = A^i_{\ r}\, T^r_{\ s}\, x^s. \tag{5.4.4}$$

In the conventional case, the differentials dx and dx′ of the two coordinate systems are related by the familiar expression $dx' = Adx$, or $dx'^i = A^i_{\ j} dx^j$, with consequential known properties, e.g., for coordinates transformations [4,6].

However, the same differentials dx and dx′ are inapplicable in the isocotangent bundle $T^*\hat{M}(\hat{R})$. The author therefore introduced the generalized notion of *isodifferentials* $\hat{d}x$ and $\hat{d}\bar{x}$ [15] which hold when interconnected by the isotopic laws

$$\hat{d}\bar{x} = A * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = A^i_{\ r}\, T^r_{\ s}\, \hat{d}x^s, \tag{5.4.5}$$

with the particular realization, say, for the case of the isotransformations $x \Rightarrow \bar{x}(x)$

$$\hat{d}\bar{x} = \frac{\partial \bar{x}}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^r}\, T^r_{\ s}\, \hat{d}x^s. \tag{5.4.6}$$

As we shall see in the next chapter, two possibilities are significant for the characterization of the isodifferentials, $\hat{d}x = dx$ or $\hat{d}x = Tdx$. They are connected with the corresponding isointegrals $\hat{\int}\hat{d}x = \hat{x} \in \hat{R}$ or $\hat{\int}\hat{d}x = x \in R$. In this chapter we shall addume the former for simplicity and leave the latter to the reader.

Let $\phi(x)$ be a scalar function on $T^*\hat{M}(\hat{R})$. Then its isodifferential is given by

$$\hat{d}\phi = \frac{\partial \phi}{\partial x} * \hat{d}x, \quad \text{or} \quad \hat{d}\phi(x) = \frac{\partial \phi}{\partial x^r}\, T^r_{\ s}\, \hat{d}x^s. \tag{5.4.7}$$

where the partial derivative is the conventional one.

Similarly, a contravariant *isovector-field* $X = (X^i)$ on $T^*\hat{M}(\hat{R})$ is an ordinary vector-field although defined on an isospace. Then its isodifferential is given by

$$\hat{\partial} X = \frac{\partial X}{\partial x} * \hat{\partial} x, \quad \text{or} \quad \hat{\partial} X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{\partial} x^s, \tag{5.4.8}$$

Thus, an isovector-field on $T^*\hat{M}(\hat{\mathfrak{R}})$ transforms according to the isotopic laws

$$\overline{X}(\overline{x}) = \frac{\partial \overline{X}}{\partial x} * X(x), \quad \text{or} \quad \overline{X}^i(\overline{x}) = \frac{\partial \overline{X}^i}{\partial x^r} T^r{}_s(x)\, X^s(x). \tag{5.4.9}$$

Note that, while for conventional transformations $dx' = Adx$ on $T^*M(x,R)$ we have $\partial x'/\partial x = A$, and thus we now have for isotransformations

$$\frac{\partial \overline{x}^i}{\partial x^j} = A^i{}_r T^r{}_j + A^i{}_r \frac{\partial T^r{}_s}{\partial x^j} x^s. \tag{5.4.10}$$

By using the above results and the usual chain rule for partial differentiation, one easily gets

$$\frac{\partial \overline{X}^j}{\partial \overline{x}^k} = \frac{\partial^2 \overline{x}^j}{\partial x^s \partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} T^i{}_r X^r + \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} Tir \frac{\partial X^r}{\partial x^s} + \frac{\partial \overline{x}^j}{\partial x^i} \frac{\partial x^s}{\partial \overline{x}^k} \frac{\partial T^i{}_r}{\partial x^s} X^r. \tag{5.4.11}$$

Thus, in addition to the isotopy of the conventional two terms of this expression (see ref. [4], Eq.s (3.5), p. 67), we obtain an additional third term. Note that the quantity $\partial \overline{X}_j/\partial \overline{x}^k$ is not a mixed tensor of rank (1.1), exactly as it happens in the conventional case.

From the preceding results one can then compute the isodifferential of a contravariant isovector-field

$$\hat{\partial} \overline{X}^j = \frac{\partial \overline{X}^j}{\partial \overline{x}^k} T^k{}_r \hat{\partial} \overline{x}^r =$$

$$= \frac{\partial T^i{}_r}{\partial x^s \partial x^i} T^i{}_r X^r \hat{\partial} x^s + \frac{\partial^2 \overline{x}^j}{\partial x^i} T^i{}_r \frac{\partial \overline{x}^j}{\partial x^s} \hat{\partial} x^s + \frac{\partial X^r}{\partial x^i} \frac{\partial \overline{x}^j}{\partial x^s} X^r \hat{\partial} x^s. \tag{5.4.12}$$

A *contravariant isotensor* $X^{ij}$ of rank two on $\hat{M}(\hat{R})$ is evidently characterized by the transformation laws

$$\overline{\overline{X}}^{(2)}(\overline{x}) = \frac{\partial \overline{x}}{\partial x} * \frac{\partial \overline{x}}{\partial x} * X^{(2)}(x), \quad \overline{X}^{ij}(\overline{x}) = \frac{\partial \overline{x}^i}{\partial x^r} T^r{}_p \frac{\partial \overline{x}^j}{\partial x^s} T^s{}_q X^{pq}(x), \tag{5.4.13}$$

Similar extensions to higher orders, as well as to contravariant isotensors of rank (0.s) and to generic tensors of rank (r.s) are left as an exercise for the interested reader.

All preceding expressions (5.4.4)–(5.4.13) have been written in both, the abstract form and their realization in local coordinates, to illustrate that the notion of isotransformations and isodifferentials do constitute isotopies, in the sense that all distinctions between conventional and isotopic notions cease to exist at the abstract, realization-free level.

For the identification of the *isodual isodifferential calculus* [15,19] recall that, under isodualities we have

$$\hat{d}\bar{x}^d = A^d *^d \hat{d}x^d = A^d * \hat{d}x . \qquad (5.4.14)$$

The rest of the isocalculus can then be easily derived.

To have a guide in the use and meaning of isoduality, the reader should keep in mind that its primary classical function at this level is the characterization of the map from positive to negative energies. But energies are represented by the Hamiltonian H. A good guide for isodualities is therefore the map

$$\text{Energy } H > 0 \quad \overset{\text{isoduality}}{\Rightarrow} \quad \text{Energy } H^d = -H < 0 . \qquad (5.4.15)$$

As it has been the case for all preceding aspects studied so far, we have four distinguishable formulations:

A) the **conventional differential calculus** over the ordinary reals $R(n,+,\times)$ with basic transformation law $dx' = A \times dx$ which is and will remain the fundamental calculus for the exterior problem of ordinary matter in vacuum;

B) the **isodual differential calculus** over the isodual reals $R^d(n^d,+,\times^d)$ with law $dx' = A \times^d dx = A^d dx$ which is assumed the basic calculus for antimatter also in vacuum;

C) *the* **isodifferential calculus** over the isoreals $\hat{R}(\hat{n},+,*)$ with law $\hat{d}\bar{x} = A * \hat{d}x$ which is assumed as the basic calculus for the interior dynamical problem of matter; and

D) *the* **isodual isodifferential calculus** over the isodual isoreals $\hat{R}^d(\hat{n}^d,+,*^d)$ which is assumed as the basic calculus for the interior problem of antimatter.

The reader should keep in mind that all the above formulations can be unified by the abstract isotopic treatment of Class III, although in these volumes we shall study the individual formulations for clarity.

**5.4C: Isoforms and their isoduals.** The isotopies of the symplectic geometry of Class I were constructed [15] via the use of the isodifferential calculus, which permits the introduction of the following *one–isoform*

$$\hat{\Phi}_1 = A \hat{*} \hat{\partial}x = A_i T^i{}_j \hat{\partial}x^j. \tag{5.4.16}$$

and the study of the algebraic operations on them. The *isocotangent bundle* $T^*\hat{M}_1(\hat{R})$ is then the bundle of all possible one–isoforms. The sum of two one–isoforms $\hat{\Phi}_1{}^1 = A\hat{*}\hat{\partial}x$ and $\hat{\Phi}_1{}^2 = B\hat{*}\hat{\partial}x$ is the conventional expression

$$\hat{\Phi}_1{}^1 + \hat{\Phi}_1{}^2 = (A + B) \hat{*} \hat{\partial}x. \tag{5.4.17}$$

The isoproduct of one–isoform $\hat{\Phi}_1 = A\hat{*}\hat{\partial}x$ with an isonumber $\hat{n} \in \hat{R}$ is the conventional product,

$$\hat{n} \hat{*} \hat{\Phi}_1 \equiv n \hat{\Phi}_1. \tag{5.4.18}$$

For the product of two or more one–isoforms $\hat{\Phi}_1{}^k = A^k\hat{*}\hat{\partial}x$, $k = 1, 2, 3, \ldots$ we introduce the *isoexterior*, or *isowedge product* denoted with the symbol $\hat{\wedge}$, which verifies the same axioms of the conventional exterior product, that is, the distributive laws

$$(\hat{\Phi}_1{}^1 + \hat{\Phi}_1{}^2) \hat{\wedge} \hat{\Phi}_1{}^3 = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^3 + \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^3, \tag{5.4.19a}$$

$$\hat{\Phi}_1{}^1 \hat{\wedge} (\hat{\Phi}_1{}^2 + \hat{\Phi}_1{}^3) = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 + \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^3, \tag{5.4.19b}$$

and the antisymmetry law

$$\hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 = - \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^1. \tag{5.4.20}$$

although it is defined on an isomanifold.

The isoproduct of two one–isoforms $\hat{\Phi}_1{}^1 = A\hat{*}\hat{\partial}x$ and $\hat{\Phi}_1{}^2 = B\hat{*}\hat{\partial}x$ is the *two–isoform*

$$\begin{aligned} \hat{\Phi}_2 &= \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 = A_i T^i{}_r B_j T^j{}_s \hat{\partial}x^r \hat{\wedge} \hat{\partial}x^s = \\ &= \tfrac{1}{2} (A_i T^i{}_r B_j T^j{}_s - A_i T^i{}_s B_j T^j{}_r) \hat{\partial}x^r \hat{\wedge} \hat{\partial}x^s = \\ &= \tfrac{1}{2} A_i B_j (T^i{}_r T^j{}_s - T^i{}_s T^j{}_r) \hat{\partial}x^r \hat{\wedge} \hat{\partial}x^s, \end{aligned} \tag{5.4.21}$$

which characterizes the isocotangent bundle $T^*\hat{M}_2(x,\hat{R})$. Note the clear deviations from the conventional exterior calculus (compare with ref. [4], p. 132).

The isoexterior product of three one–isoforms yields the *three–isoform*

$$\hat{\Phi}_3 = \hat{\Phi}_1{}^1 \hat{\wedge} \hat{\Phi}_1{}^2 \hat{\wedge} \hat{\Phi}_1{}^3 =$$

$$= A^1{}_{i_1} A^2{}_{i_2} A^3{}_{i_3} \delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} \hat{\partial} x^{k_1} \wedge \hat{\partial} x^{k_2} \hat{\wedge} \hat{\partial} x^{k_3}, \quad (5.4.22)$$

where [4,6]

$$\delta^{i_1 i_2}{}_{j_1 j_2} = \det.\begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} \end{pmatrix}, \delta^{i_1 i_2 i_3}{}_{j_1 j_2 j_3} = \det.\begin{pmatrix} \delta^{i_1}{}_{j_1} & \delta^{i_1}{}_{j_2} & \delta^{i_1}{}_{j_3} \\ \delta^{i_2}{}_{j_1} & \delta^{i_2}{}_{j_2} & \delta^{i_2}{}_{j_3} \\ \delta^{i_3}{}_{j_1} & \delta^{i_3}{}_{j_2} & \delta^{i_3}{}_{j_3} \end{pmatrix}, \quad (5.4.23)$$

with a consequential extension to *p–isoforms*

$$\Phi_p = A_{i_1 i_2 \ldots i_p} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \times\times\times T^{i_p}{}_{j_p} \hat{\partial} x^{j_1} \hat{\wedge} \hat{\partial} x^{j_2} \hat{\wedge} \ldots \hat{\wedge} \hat{\partial} x^{j_p}, \quad (5.4.24)$$

characterizing the corresponding isocotangent bundle $T^*\hat{M}_p(\hat{R})$.

Given n one–isoforms $\Phi_1{}^k = A^k * \hat{\partial} x$, k = 1, 2,... n, they are said to be *isolinearly dependent* when $\Phi_1{}^1 \hat{\wedge} \ldots\ldots \hat{\wedge} \Phi_1{}^n \equiv 0$. Note that given n one–isoforms linearly dependent on $M(x,\Re)$, they can be isolinearly independent, evidently because of the functional dependence of the isotopic product.

In an n–dimensional isomanifold $\hat{M}(\hat{R})$ there exist a maximum of n linearly independent one–isoforms as in the conventional case, with isobasis $\hat{\partial} x^1, \ldots, \hat{\partial} x^n$. The isobasis of $T^*\hat{M}_2(\hat{R})$ are then given by the ordered set $\hat{\partial} x^i \hat{\wedge} \hat{\partial} x^j$, $i < j$. A similar situation occurs for p–isoforms and related isomanifolds $T^*\hat{M}_p(\hat{R})$.

As an incidental note we point out without treatment the *Grassmann–isotopic algebra* $\hat{G}$, or *isograssmann algebra*, which is given by the direct sum [15]

$$\hat{G} = \sum_{k = 0,1,2,\ldots,n} T^*\hat{M}_k(\hat{R}). \quad (5.4.25)$$

The necessary and sufficient conditions for a two–isoform to be identically null are

$$\delta^{i_1 i_2}{}_{j_1 j_2} A^1{}_{k_1} A^2{}_{k_2} T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} = A^1{}_{k_1} A^2{}_{k_2} (T^{k_1}{}_{i_1} T^{k_2}{}_{i_2} - T^{k_1}{}_{i_2} T^{k_2}{}_{i_1}) \equiv 0. \quad (5.4.26)$$

A similar situation occurs for p–isoforms.

The reader should keep in mind the nontriviality of the above liftings. As an example, the linear, local and canonical one– and two–forms are lifted into the respective structures

$$\hat{\Phi}_1 = A_i T^i{}_j(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots) \hat{\partial} x^j, \quad (5.4.27a)$$

$$\hat{\Phi_2} = A_i\, T^i{}_r(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots B_j\, T^j{}_s(x, \dot{x}, \ddot{x}, \mu, \tau, n, \ldots)\, \hat{d}x^r \,\hat{\wedge}\, \hat{d}x^s \tag{5.4.27b}$$

which are nonlinear (in the local coordinates x as well as their derivatives), nonlocal–integral (in all variables) and noncanonical when projected in the original manifold M(R). What is remarkable is that the above forms are *isolinear, isolocal and isocanonical* (Sect. 4.2) on $\hat{M}(\hat{R})$, that is, they coincide with the corresponding conventional forms at the abstract level despite the indicated differences. Perhaps, this abstract unity is the reason why the isosymplectic geometry has been discovered only recently.

It is evident that all the above quantities admit a new image under isoduality. To begin, the basic manifold M(R) is now mapped into the isodual manifold $M^d(R^d)$ with the isobasis $dx^{1d}, \ldots, dx^{nd}$ over $R^d$. We therefore have the *isodual one–forms*

$$\theta^d = A^d \times^d d\, x^d , \tag{5.4.28}$$

and the isodual operations in them. We then have th *isodual one–isoforms*

$$\hat{\theta}^d = A *^d \hat{d}x = - \hat{\theta} , \tag{5.4.29}$$

and isodual isooperations on them.

**5.4D: Isotopies and isodualities of the Poincare´ lemma.** The Poincaré lemma (see. e.g., ref.s [3,4,6) has a particular mathematical and physical meaning inasmuch as it establishes that the symplectic geometry is the geometry underlying Lie's theory. For the case of two–forms on a 2n–dimensional manifold

$$d\omega = d\,[\,\omega_{ij}\, dx^i \wedge dx^j\,] = d(d\theta) \equiv 0, \tag{5.4.30}$$

the Poincaré lemma provides the integrability conditions for the brackets characterized by the contravariant tensor $\omega^{ij} = |\,(\,\omega_{rs}\,)^{-1}\,|^{ij}$

$$[\,A\,,B\,] = \frac{\partial A}{\partial x^i}\quad \omega^{ij}\quad \frac{\partial A}{\partial x^j}\,, \tag{5.4.31}$$

to be Lie. Thus, the rather complex integrability conditions for brackets (5.4.31) to be Lie (see, e.g., the detailed study in ref. [6]) are reduced to the simple and elegant geometric property $d\omega = d(d\theta) \equiv 0$.

A central objective of memoir [15] was to show that a similar situation occurs under isotopy, namely; that the isosymplectic geometry is the correct integro–differential geometry underlying the Lie–isotopic algebras. This property was established by showing that the Poincaré lemma is a true geometric axiom because it persists under isotopies.

To review this result, let us first study the isodifferential calculus of p-isoforms. Let $\hat{\Phi}_1 = A \hat{*} \hat{d}x$ be a one-isoform. The *isoexterior derivative* $\hat{d}\hat{\Phi}_1$ of $\hat{\Phi}_1$ (also called *isoexterior differential*) is the two-isoform [loc. cit.]

$$\hat{\Phi}_2 = \hat{d}\hat{\Phi}_1 = \frac{\partial(A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2} \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} = \qquad (5.4.32)$$

$$= \Big( \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} =$$

$$= \tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \Big( \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big) \hat{d}x^{k_1} \hat{\wedge} \hat{d}x^{k_2}$$

from which one can see that $\hat{d}\hat{\Phi}_1$ is no longer the curl of the vector field $A_{i_1}$, but the more general *isocurl* encountered in Sect. 5.2.

The isoexterior derivative of a two-isoform (5.2.32) is given by the three-isoform

$$\hat{\Phi}_3 = \hat{d}\hat{\Phi}_2 = \Big( \frac{\partial A_{i_1 i_2}}{\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + A_{i_1 i_2} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} +$$

$$+ A_{i_1 i_2} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \Big) \hat{d}x^{j_1} \hat{\wedge} \hat{d}x^{j_2} \wedge \hat{d}x^{j_3}. \qquad (5.4.33)$$

It is easy to see that the isoexterior derivative of the isoexterior product of a p-isoform $\Phi_p$ and a q-isoform $\hat{\Phi}_q$ is given by

$$\hat{d}(\hat{\Phi}_p \hat{\wedge} \hat{\Phi}_q) = (\hat{d}\hat{\Phi}_p) \hat{\wedge} \hat{\Phi}_q + (-1)^p \hat{\Phi}_p \hat{\wedge} (\hat{d}\hat{\Phi}_q). \qquad (5.4.34)$$

A p-isoform $\hat{\Phi}_p$ is then said to be *isoexact* when there exists a (p−1)-isoform $\hat{\Phi}_{p-1}$ such that $\hat{\Phi}_p = \hat{d}\hat{\Phi}_{p-1}$, and *isoclosed* when $\hat{d}\hat{\Phi}_p \equiv 0$. We are thus equipped to formulate the following important

**Lemma 5.4.1 - Isotopic Poincare´ lemma** [15,19]: *The Poincaré Lemma admits an infinite number of isotopic liftings of Class I, i.e., given an exact p-form* $\Phi_p = d\Phi_{p-1}$, *there exists an infinite number of isotopies*

$$\Phi_{p-1} \Rightarrow \hat{\Phi}_{p-1}, \quad \Phi_p = d(\Phi_{p-1}) \Rightarrow \hat{\Phi}p = \hat{d}(\hat{\Phi}_{p-1}), \qquad (5.4.35)$$

*for each of which the isoexterior derivative of the isoexact p–isoforms are identically null,*

$$\hat{d}\,(\hat{d}\,\Phi_{p-1}) \equiv 0. \qquad (5.4.36)$$

The proof is an instructive exercise for the reader interested in acquiring a knowledge of the isotopic techniques. We merely note that $\hat{d}\Phi_1 \equiv 0$, iff

$$\tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2}\,\Big(\frac{\partial A_{i_1}}{\partial x^{i_2}}\,T^{i_1}{}_{j_1}\,T^{i_2}{}_{j_2} + A_{i_1}\frac{\partial T^{i_1}{}_{j}}{dx^{i_2}}\,T^{i_2}{}_{j_2}\Big) \equiv 0, \qquad (5.4.37)$$

namely, the isoclosure of a one–isoform does not imply that the conventional curl of the vector A is null, but that the isocurl is null.

Similarly, given a exact two–isoform $\Phi_2 = \hat{d}\Phi_1$ , the property $\hat{d}\Phi_2 \equiv 0$ holds iff

$$\delta^{j_1 j_2 j_3}{}_{k_1 k_2 k_3}\Big(\frac{\partial^2 A_{i_1}}{\partial x^{i_2}\,\partial x^{i_3}}\,T^{i_1}{}_{j_1}\,T^{i_2}{}_{j_2}\,T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}}\,\frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}}\,T^{i_2}{}_{j_2}\,T^{i_3}{}_{j_3} +$$

$$+ \frac{\partial A_{i_1}}{\partial x^{i_2}}\,T^{i_1}{}_{j_1}\,\frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}}\,T^{i_3}{}_{j_3}\Big) \equiv 0. \qquad (5.4.38)$$

Thus, the abstract axioms $\hat{d}\Phi_2 = \hat{d}(\hat{d}\Phi_1) \equiv 0$, $\hat{d}\Phi_3 = \hat{d}(\hat{d}\Phi_2) \equiv 0$, etc., admit the conventional linear–local–canonical realization based on an ordinary manifold, as well as an infinite number of additional, nonlinear–nonlocal–noncanonical realizations for each given original form, via covering isomanifolds. The latter realizations are geometrically equivalent among themselves, but physically inequivalent owing to the generally different isotopic elements or isounits.

The *isodualities of the Poincaré lemma* can now be easily formulated. We first have the isodual Poincaré lemma which is characterized by the isodual calculus, and then the isodual isopoincaré lemma based on the isodual isodifferential calculus.

**5.4E- Isosymplectic geometry and its isodual.** Let us review the interplay between exact symplectic two–forms and Lie–isotopic algebras. Recall (see ref. [6] for details) that the most general possible, *local–differential* and conventional two–form on an even, 2n–dimensional manifold $T^*M_2(R)$ with covariant geometric tensor $\Omega_{i_1 i_2}$

$$\Phi_2 = \tfrac{1}{2}\Omega_{i_1 i_2}(x)\,dx^{i_1} \wedge dx^{i_2}, \qquad (5.4.39)$$

characterizes, in its corresponding contravariant version, the brackets among functions A(x) and B(x) on $T^*M_2(R)$

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{i_1}} \Omega^{i_1 i_2} \frac{\partial B}{\partial x^{i_2}}, \qquad \Omega^{i_1 i_2} = (|\Omega_{j_1 j_2}|^{-1})^{i_1 i_2}. \tag{5.4.40}$$

Now, the integrability conditions for two-form (5.4.39) to be an exact symplectic two-form are given by [loc. cit]

$$\Omega_{i_1 i_2} + \Omega_{i_2 i_1} \equiv 0, \qquad \frac{\partial \Omega_{i_1 i_2}}{\partial x^{i_3}} + \frac{\partial \Omega_{i_2 i_3}}{\partial x^{i_1}} + \frac{\partial \Omega_{i_3 i_1}}{\partial x^{i_2}} \equiv 0, \tag{5.4.41}$$

which general solution in terms of 2n functions $R_i(x)$

$$\Phi_2 = d\,[\,R_i(x)\,dx^i\,], \qquad \Omega_{j_1 j_2} = \frac{\partial R_{j_1}}{\partial x^{j_2}} - \frac{\partial R_{j_2}}{\partial x^{j_1}}, \tag{5.4.42}$$

characterizing the *Birkhoffian generalization of Hamiltonian mechanics* [5,6]. The above conditions are equivalent to the integrability conditions

$$\Omega^{i_1 i_2} + \Omega^{i_2 i_1} \equiv 0, \tag{5.4.43a}$$

$$\Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_2 k} \frac{\partial \Omega^{i_3 i_1}}{\partial x^k} + \Omega^{i_3 k} \frac{\partial \Omega^{i_1 i_2}}{\partial x^k} \equiv 0, \tag{5.4.43b}$$

for generalized brackets (5.4.40) to be Lie-isotopic, i.e., to verify the Lie algebra axioms in their most general possible, classical, regular realization on $T^*M_2(R)$

$$[A \hat{,} B] + [B \hat{,} A] \equiv 0, \quad [[A \hat{,} B] \hat{,} C] + [[B \hat{,} C] \hat{,} A] + [[C \hat{,} A] \hat{,} B] \equiv 0. \tag{5.4.44}$$

Thus, the exact character of the general two-form $\Phi_2 = d\Phi_1$ implies its closure $d\Phi_2 \equiv 0$ (Poincaré Lemma), which, in turn, guarantees that the underlying brackets are Lie-isotopic, with the canonical case being a trivial particular case (see the analytic, algebraic, and geometric proofs in ref. [6], Sect. 4.1.5).

Lemma 5.4.1 establishes that the above general but local-differential interplay between algebra and geometry persists under the most general possible nonlocal-integral isotopies. We therefore have the following:

**Definition 5.4.1** [15,19]: *"The "exact isosymplectic manifolds" of Class I are 2n-dimensional isomanifolds* $T^*\hat{M}_2(x,\hat{R})$ *over the isofields* $\hat{R}(\hat{n},+,*)$ *with isounits* $\hat{1}_2$ *equipped with an isoexact and nowhere degenerate two-isoform*

$$\Phi_2 = \tfrac{1}{2}\,\Omega_{j_1 j_2}(x,\dot{x},\ddot{x},...)\,\partial x^{j_1} \hat{\wedge}\, \partial x^{j_2} = \partial\Phi_1 = \frac{\partial(A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2}\, \partial x^{j_1} \hat{\wedge}\, \partial x^{j_2} =$$

$$= \Big( \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \partial x^{j_1} \hat{\wedge}\, \partial x^{j_2} =$$

$$= \tfrac{1}{2}\delta^{j_1 j_2}{}_{k_1 k_2} \Big( \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \Big)\, \partial x^{k_1} \hat{\wedge}\, \partial x^{k_2} , = \tag{5.4.45}$$

*which is such to admit the factorization*

$$\Phi_{2_2} = \Omega_{i_1 k}(x) \times T_2{}^{k}{}_{i_2}(s, x, \dot{x}m, \ddot{x}, ...)\, \partial x^{i_1}\, \partial x^{i_2}, \qquad \hat{T}_2 > 0, \tag{5.4.46}$$

*where* $T_2$ *is the isotopic element of the underlying isofield, i.e., it is such that* $\hat{1}_2 = T_2^{-1}$ *and*

$$\Omega_{i_1 i_2} = \frac{\partial A_{i_2}}{\partial x^{i_1}} - \frac{\partial A_{i_1}}{\partial x^{i_2}} , \tag{5.4.47}$$

*is Birkhoff's tensor [6], i.e., the most general possible local, exact symplectic tensor. The corresponding Lie-isotopic theory is then characterized by the brackets*

$$[A \hat{,} B] = \frac{\partial A}{\partial x^{i_1}} \hat{1}_2{}^{i_1}{}_{k}(s, x, \dot{x}, \ddot{x}, ...)\, \Omega^{k i_2}(x) \frac{\partial B}{\partial x^{i_2}}, \tag{5.4.48a}$$

$$\hat{1}_2 = \hat{T}_2^{-1}, \quad (\hat{\Omega}^{i_1 i_2}) = (\|\Omega_{k_1 k_2}\|^{-1}), \tag{5.4.48b}$$

*where* $\hat{1}_2 = \hat{T}_2^{-1}$ *is the isounit of the universal enveloping isoassociative algebra. The "isosymplectic geometry" is the geometry of the isosymplectic manifolds.*

*The "exact isodual isosymplectic manifolds" are defined by the isodual exact two-isoforms*

$$\Phi^d{}_2 = \Omega_{i_1 k}(x) \times T^d{}_2{}^{k}{}_{i_2}(s, x, \dot{x}, \ddot{x}, ...)\, \partial x^{i_1} \hat{\wedge}\, \partial x^{i_2}, \quad \hat{T}^d{}_2 < 0, \tag{5.4.49}$$

*which are now defined on the "isodual isocotangent bundle"* $T^*\hat{M}^d(\hat{R}^d)$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d, +, *^d)$ *with isodual isounit* $\hat{1}^d{}_2 = (T^d{}_2)^{-1} = -\hat{1}_2$.

Note the complete lack of restriction in the functional dependence of the isotopic element $T_2$ which is at the foundation of the "direct universality" of the isogeometry for all possible nonlinear, nonlocal and noncanonical systems (the "direct universality" for the nonlinear and noncanonical but local systems was proved in ref. [6]).

Note also that factorization (5.4.46) is possible for all two–isoforms (5.4.45). In the above definition one can either pre–assign an isounit $\hat{1}_2$ and then select the two–isoforms (5.4.46) verifying the condition $\hat{1}_2 = T_2^{-1}$, or pre–assign the two–isoform (5.4.46) and then select the isounit $\hat{1}_2$ accordingly. In this way, *all two–isoforms whose antisymmetric tensor $\Omega_{ij}$ is symplectic can always be interpreted as characterizing an isosymplectic manifold.* As a matter of fact, this is an illustration of the existence of the infinite variety of isotopies $\hat{R}(\hat{n},+,*)$ of the field of real numbers $R(n,+,\times)$.

The isosymplectic geometry focuses the attention on a subtle aspect which is absent in the conventional formulation of the geometry [3,4]: *the relationship between the two–forms and the underlying unit.* For the conventional Hamiltonian (or Birkhoffian) case, the underlying unit is the unit of the enveloping associative algebra of the related Lie algebra. As such it is the 2n–dimensional unit

$$I = (I^i_{\ j}) = (I_i^{\ j}) = \mathrm{Diag.}\,(1, 1, ..., 1) \quad (2n\ \mathrm{dim})\,. \tag{5.4.50}$$

which is trivially symmetric. The most general possible symplectic two–form is then characterized in a local chart by *two* tensors, the totally antisymmetric symplectic tensor $\Omega_{ij}$ and the totally symmetric one $I^{-1} = \mathrm{diag.}\,(1, 1, ..., 1) \equiv I$

$$\Omega = d\Theta = \Omega_{i_1 k}(x)\, I^k_{\ i_2}\, dx^{i_1} \wedge dx^{i_2}\,, \tag{5.4.51}$$

In the transition to an arbitrary isounit $\hat{1}_2$ the symplectic tensor $\Omega_{ij}$ is preserved, but the totally symmetric tensor $I = (I^i_{\ j})$ is lifted for mathematical consistency into the isotopic form $T_2 = (T_2{}^i_{\ j}) = \hat{1}_2{}^{-1}$ or, equivalently, the totally symmetric tensor in the factorization (5.4.46) must be interpreted as the isotopic element of the related enveloping associative algebra.

The geometrical and physical implications of the above isotopies and isodualities are intriguing, and it is hoped that they will received a much needed attention by geometers. As an example, it has been assumed until now in differential geometry that the only possible degeneracy is that in the symplectic tensor, e.g.,

$$\mathrm{Det}\,\Omega(x) \equiv 0. \tag{5.4.52}$$

in which case one evidently lose the symplectic character of the geometry and the possibility to characterize a corresponding Lie algebra owing to the impossibility to perform the transition to the contravariant tensor $\Omega^{ij}$.

The isotopies imply the existence of a second "hidden" degeneracy, that of the isotopic element

$$\mathrm{Det}\ T_2(x) \ \equiv\ 0. \qquad (5.4.53)$$

which the symplectic tensor is nondegenerate, det $\Omega \neq 0$, which characterizes the *isosymplectic geometry of Class IV.* This latter form of isogeometry represents gravitational collapse into a singularity at x and, as such, need suitable study.

Note that the primitive notion here is that of *isonumbers with a singular unit.* The degeneracy of the geometry is only consequential.

The generalized analytic equations characterized by the isosymplectic geometry will be identified in Vol. II, jointly with explicit examples.

**6.EF: Isodual representation of negative energies.** As well known in particle physics, antiparticles are characterized by negative–energy solutions of field equations. In this section (which is evidently purely classical), we can only study the geometric characterization of the negative–energies via isodualities.

First, we simply note that a conventional Hamiltonian representing the kinetic energy over R(n,+,×), naturally becomes negative–definite when mapped into the isodual field $R^d(n^d+,\times^d)$

$$H_R = \dot{r} \times \dot{r}\ \ 2m > 0 \ \Rightarrow\ H^d = H_{R^d} = \dot{r} \times^d \dot{r}\ /\ 2m \ =\ -\dot{r} \times \dot{r}\ /\ 2m \ =\ -H < 0 \qquad (5.4.54)$$

and evidently the same holds for the isoduality of the Lagrangian

$$L_{\Re} = \tfrac{1}{2}\, m\, \dot{r} \times \dot{r} > 0 \ \Rightarrow\ L^d = L_{\Re^d} = \tfrac{1}{2}\, m\, \dot{r} \times^d \dot{r} \ = -\tfrac{1}{2}\, m\, \dot{r} \times \dot{r} \ = -L < 0\ . \qquad (5.4.55)$$

Jointly, the equation of motion reads

$$m^d \times^d \ddot{r}^d \ \equiv\ -\, m\, \ddot{r} \ =\ 0, \qquad (5.4.56)$$

with a similar result for arbitrary equations of motion (see Ch. II.1).

More generally, the *isodual Legendre transform* is given by

$$L = p \times \dot{r} - H \ \backslash \ \Rightarrow\ \ L^d = p \times^d \dot{r} - H^d = -p \times r + H\,. \qquad (5.4.57)$$

The construction of the *isodual isolegendre transform* is an instructive exercise for the interested reader.

All these features of antiparticles in vacuum are directly represented by the isodual symplectic geometry. In fact, the integrand of the conventional action is precisely the one–form of the symplectic geometry

$$A = \int_{[\,,\,]} dt\, L(t, r, \dot{r}) = \int_{[\,,\,]} R_k(t, r) \times dr_k = \int_{[\,,\,]} \Theta \,. \qquad (5.4.58)$$

The property identified earlier of the change of sign of a one-form under isoduality then constitutes precisely the desired geometrization of the negative-energy solutions of field equations.

## 5.5: ISOAFFINE GEOMETRY

**5.5A: Isoaffine spaces and their isoduals.** As an intermediate step prior to the isotopies of Riemann, we shall now review the isotopies of the affine geometry introduced in in memoir [16] under the name of *affine-isotopic geometry*, or *isoaffine geometry* for short, then studied in ref.s [17–21] and reviewed in ref. [25]. This author is aware of no additional studies in the new geometry at this writing.

The central technical objective is the achievement of a generalization of basic notions such as connection, curvature, etc., which is of *integral* type, as well as dependent on the *velocities and accelerations* in a nonlinear and nonlocal way, while jointly *preserving the original axioms of the geometry.*

The literature in the conventional affine geometry is predictably vast, although Scrödinger's presentation [2] remains valid to this day. In this section we shall continue to follow the treatise by Lovelock and Rund [4] of which we preserve the notation unchanged for clarity in the comparison of the results.

Let M(x,R) be an n-dimensional *affine space* here referred as a differentiable manifold with local coordinates $x = (x^i)$, $i = 1, 2, ..., n$, over the reals $R(n,+,\times)$. We shall denote: the conventional scalars on M(x,R) with $\phi(x)$; contravariant and covariant vectors with $X^j(x)$ and $X_j(x)$, respectively; and mixed tensors of rank (r.s) with the notation $X^{(r.s)} = X^{j_1 j_2 \dots j_r}{}_{k_1 k_2 \dots k_s}(x)$. Unless otherwise stated, all tensors considered on $M(x,\mathfrak{R})$ will be assumed to be local-differential and to verify all needed continuity conditions.

**Definition 5.5.1** [16,19]: *The infinitely possible isotopic liftings* $\hat{M}(x,\hat{R})$ *of Class I of an n-dimensional affine space* M(x,R) *over the reals* $R(n,+,\times)$, *called "isoaffine spaces", are characterized by the same local coordinates* x *and the same local-differential tensors* $X^{(r.s)}$ *of* M(x,R) *but now defined over the isoreals* $\hat{R}(\hat{n},+,*)$ *for all infinitely possible n-dimensional isounits* $\hat{1}$ *of Class I. "The "isoaffine geometry" of Class I is then the geometry of vectorfields on* $\hat{M}(x\hat{R})$. *The "isodual isoaffine spaces" of Class II* $\hat{M}^d(x, \hat{R}^d)$ *are the original spaces* $\hat{M}(x,\hat{R})$ *defined over the isodual isoreals* $\hat{R}^d(\hat{n}^d,+,*^d)$. *The 'isodual isoaffine geometry" is then the geometry of isodual vectorfields on* $\hat{M}^d(x,\hat{R}^d)$.

Recall in the conventional case that, given two contravariant vectors $x_1$ and $x_2$ on M(x,R), their difference $\Delta x = x_1 - x_2$ is a contravariant vector iff the transformation is *linear* (as well as local) [4]. Similarly, $\Delta x$ is a contravariant vector on $\hat{M}(x,\hat{R})$ iff the transformation is *isolinear* (as well as isolocal).

The first difference between affine and isoaffine spaces can be seen by noting that coordinate differences which are not contravariant in the conventional geometry can be turned into a contravariant form via a suitable selection of the isotopic element.

The *left and right modular isotransformations* on $\hat{M}$(x,R) are defined by

$$\bar{x}^t = x^t * A^t = x^t \, T \, A^t, \qquad \bar{x} = A * X = A \, T \, x, \tag{5.5.1}$$

where t denotes conventional transpose. The *inverse, right-modular isotransformations* are given by

$$x = A^{-\hat{1}} * \bar{x} = A^{-\hat{1}} \, \bar{T} \, x, \tag{5.5.2}$$

where $A^{-\hat{1}}$ is the *isoinverse*, i.e., it verifies the isotopic rules $A^{-\hat{1}} * A = A * A^{-\hat{1}} = \hat{1}$ and $\bar{T} = T(\bar{x}, \dot{\bar{x}}, ...) \equiv T(x, \dot{x}, ...)$. Note the preservation of the isotopic element for the left and inverse isotransformations which is ensured by the assumed Hermiticity of the element T for Classes I and II herein considered, and it is at the very foundations of the Lie–isotopic theory. Isoaffine spaces $\hat{M}(x,\hat{R})$ are then *isomodules* for the isorepresentations of Lie–isotopic algebras, while the isodual spaces $\hat{M}^d(x, \hat{R}^d)$ are correct isodual isomodules for the isorepresentations of the isodual algebras.

**5.5B: Isocovariant differentials and their isoduals.** Recall that the conventional differentials dx′ and dx interconnected by the linear and local transformations $dx' = A \times dx$ cannot be defined under isotopies and must be lifted into the *isodifferentials* $\hat{d}\bar{x}$ and $\hat{d}x$ interconnected by the isotopic rules (5.5.5).

The *isodifferential of a scalar* $\phi(x)$ on $\hat{M}(x,\hat{R})$ is then given by law (5.5.7); the *isodifferential of a contravariant isovector* $X = (X^i(x))$ on $\hat{M}(x,\hat{R})$ is given by rule (5.5.8); the *isotransformation laws* of the contravariant isovector is rule (5.5.9); and the *isotransformations of a contravariant isotensor* $X^{ij}$ of rank two on $\hat{M}(x,\hat{R})$ is given by Eq.s (5.5.13).

By using these results, the *isodifferential of a contravariant isovectorfield* on $\hat{M}(x,\hat{R})$ is given by

$$\hat{d}\,\bar{X}^j = \frac{\partial \bar{X}^j}{\partial \bar{x}^k} T^k{}_r \, \hat{d}\bar{x}^r = \tag{5.5.3}$$

$$= \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} T^i{}_r X^r \hat{\partial} x^s + \frac{\partial \bar{x}^j}{\partial x^i} T^i{}_r \frac{\partial X^r}{\partial x^s} \hat{\partial} x^s + \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r \hat{\partial} x^s$$

By using the above quantities, one can introduce the *isocovariant (or isoabsolute) differential* [10]

$$\hat{D} X^j = \hat{\partial} X^j + \hat{P}^j(x, X, \hat{\partial} x), \tag{5.5.4}$$

under the condition that it preserves the original axioms (see ref. [4], p. 68), i.e.,

1) $\hat{D}(X^j + Y^j) = \hat{D} X^j + \hat{D} Y^j$, which can hold iff $\hat{P}^j$ is isolinear in $X^r$;

2) $\hat{D} X^j$ is isolinear in $\hat{\partial} x^s$; and

3) $\hat{D} X^j$ transforms as a contravariant isovector.

By again using Lovelock–Rund's symbols with a "hat" to denote isotopy, we can write

$$\hat{D} X^j = \hat{\partial} X^j + \hat{\Gamma}_h{}^j{}_k T^h{}_r X^r T^k{}_s \hat{\partial} x^s, \tag{5.5.5}$$

where the $\hat{\Gamma}$'s are called the component of an *isoaffine connection.*

By lifting the conventional procedure, one can readily see that the necessary and sufficient conditions for the $n^3$ quantities $\hat{\Gamma}_m{}^s{}_n$ to be the coefficient of an isoaffine connection are given by

$$\hat{\Gamma}_m{}^j{}_p T^m{}_r \frac{\partial \bar{x}^r}{\partial x^s} T^s{}_t X^t T^p{}_q \frac{\partial \bar{x}^q}{\partial x^w} T^w{}_z \hat{\partial} x^z =$$

$$= \frac{\partial \bar{x}^j}{\partial x^r} T^r{}_s \hat{\Gamma}_m{}^s{}_n T^m{}_p X^p T^n{}_q \hat{\partial} x^q - \frac{\partial^2 \bar{x}^j}{\partial x^s \partial x^i} T^i{}_r X^r \hat{\partial} x^s +$$

$$+ \frac{\partial x^j}{\partial x^i} T^h{}_r \frac{\partial X^r}{\partial x^s} (T^s{}_t \hat{\partial} x^t - \hat{\partial} x^s) - \frac{\partial \bar{x}^j}{\partial x^i} \frac{\partial T^i{}_r}{\partial x^s} X^r \hat{\partial} x^s. \tag{5.5.6}$$

As in the conventional case, the $\hat{\Gamma}$'s do not constitute a tensor of rank (1.2). The extra terms in conditions (5.5.6), therefore, do not affect the consistency of the isoaffine geometry, but constitute the *desired generalization.*

The extension of the above notions to the *isocontravariant derivatives* is evidently given by

$$\hat{D} X_j = \hat{\partial} X_j - \hat{\Gamma}_j{}^s{}_n T_s{}^r X_r T^n{}_p \hat{\partial} x^p. \tag{5.5.7}$$

As a result, *the isocovariant derivative of a scalar coincides with the isodifferential*, as in the conventional case, i.e., $\hat{D}\phi = \hat{D}(X^i X_i) = \hat{\partial}\phi$.

The isoaffine connection is *symmetric* if $\hat{\Gamma}_m{}^s{}_n = \hat{\Gamma}_n{}^s{}_m$. Also, the isotopic image of a symmetric connection is symmetric in isospace. However, the following property can be easily proved (but carries important consequences).

**Proposition 5.5.1** [16,19]: *The isotopic image* $\hat{\Gamma}_h{}^j{}_k$ *of a symmetric affine connection* $\Gamma_h{}^j{}_k$ *is not necessarily symmetry when projected in* M(x,R).

The isotopic liftings of all remaining properties of covariant derivatives, as well as the extension to the isocovariant differential of tensors, will be left for brevity to the interested reader.

It is important to verify that the isocovariant (isoabsolute) differential preserves the basic axioms of the conventional differential because this is a necessary condition for consistency of the isotopies. In fact, we have the following axioms which coincide at the abstract level with the conventional ones (ref. [4], p. 74),

**Axiom 1:** *The isocovariant differential of a constant is identically null; that of a scalar coincides with the isodifferential; and that of a tensor of rank (r.s) is a tensor of the same rank.*
**Axiom 2:** *The isocovariant differential of the sum of two tensors of the same rank is the sum of the isoabsolute differentials of the individual tensors. And*
**Axiom 3:** *The isocovariant differential of the product of two tensors of the same rank verifies the conventional chain rule of differentiation.*

By following again the conventional formulation, and as a natural generalization of the isocovariant differential, we introduced the *isocovariant derivative* of a contravariant vector field $X^p$ [loc. cit.]

$$X^j{}_{\uparrow k} := \frac{\partial X^j}{\partial x^k} + \hat{\Gamma}_h{}^j{}_k \, T^h{}_r \, X^r, \tag{5.5.8}$$

under which the isocovariant differential can be written

$$\hat{D}X^j = X^j{}_{\uparrow k} \, T^k{}_s \, \hat{d}x^s. \tag{5.5.9}$$

It is an instructive exercise for the interested reader to prove that the isocovariant derivatives (5.5.9) constitute the components of a (1.1) isotensor. It is also easy to verify that the isocovariant derivatives preserve the axioms of the conventional covariant derivatives (ref. [4], p. 77):

**Axiom 1':** *The isocovariant derivative of a constant is identically null; that of a scalar is equal to the conventional partial derivative; and that of*

*an isotensor of rank (r. s) is an isotensor of rank (r. s+1).*
**Axiom 2':** *The isocovariant derivative of the sum of two tensors of the same rank is the sum of the isocovariant derivatives of the individual tensors. And*
**Axiom 3':** *The isocovariant derivative of the product of two isotensors of the same rank is that of the usual chain rule of partial derivatives.*

It is easy to see that all the preceding notions admit a consistent and significant image under isoduality. Intriguingly, the isodifferential of a vectorfield does not change under isoduality $\hat{\partial}^d X^d = \hat{\partial} X$. Similarly, we have the following isodual isocovariant differentials (see the 2-nd edition of Vol. I, ref. [47])

$$\hat{D}^d X^{dj} = \hat{\partial}^d X^{dj} + \Gamma^{d\ j}_{\ hk} T^{dh}{}_r X^{dr} T^{dk}{}_s \hat{\partial} x^s = -\hat{D}X^j, \qquad (5.5.10a)$$

$$\hat{D}^d X_j = \hat{\partial}^d X_j - \Gamma^{d\ s}_{\ jn} T^{d\ r}_{\ s} X_r T^{dn}{}_p \hat{\partial} x^p = \hat{D}X_j. \qquad (5.5.10b)$$

We therefore have the following important

**Proposition 5.5.2** [loc. cit.]: *The isoaffine connection changes sign under isoduality,*

$$\hat{\Gamma}^{d\ s}_{\ jn} = -\hat{\Gamma}^{\ s}_{jn} \qquad (5.5.11)$$

The preservation of all basic axioms, although in their isodual form, is then consequential.

Axioms 1, 2, 3 and 1', 2', 3' imply the most important result of this section, which can be expressed via the following

**Theorem 5.5.1** [16,19]: *All infinitely possible nonlinear, nonlocal and noncanonical isoaffine geometries of Class I coincide with the conventional affine geometry at the abstract, coordinate-free level, while all the infinitely possible isodual isoaffine geometries of Class II coincide with the isodual affine geometry at the abstract level.*

**5.5C: Isocurvature, isotorsion and their isoduals.** We now pass to the study of a central notion of the isoaffine geometry, the generalized curvature, called *isocurvature,* and generalized torsion, called *isotorsion,* which are inherent in the isoaffine geometry prior to any introduction of an isometric (to be done in the next section).

For this purpose, let us study the lack of commutativity of the isocovariant derivatives on isoaffine spaces $\hat{M}(x,\hat{R})$ with respect to an arbitrary, not necessarily symmetric, isoconnection $\hat{\Gamma}_h{}^j{}_k$. Via a simple isotopy of the corresponding equations (see ref. [4], pp. 82–83), and by noting that

$$X^j{}_{\hat{\uparrow}h\hat{\uparrow}k} = \frac{\partial}{\partial x^k}(X^j{}_{\hat{\uparrow}h}) + \hat{\Gamma}^2{}_p{}^j{}_k\, T^p{}_q (X^q{}_{\hat{\uparrow}h}) - \hat{\Gamma}^2{}_h{}^p{}_k\, T_p{}^q (X^j{}_{\hat{\uparrow}q}), \tag{5.5.12}$$

one gets the expression

$$X^j{}_{\hat{\uparrow}h\hat{\uparrow}k} - X^j{}_{\hat{\uparrow}k\hat{\uparrow}h} = \Big(\frac{\partial \hat{\Gamma}^2{}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}^2{}_l{}^j{}_k}{\partial x^h} +$$

$$+ (\hat{\Gamma}^2{}_m{}^l{}_k\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_h - \hat{\Gamma}^2{}_m{}^l{}_h\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_k)\, T^l{}_s X^s - (\hat{\Gamma}^2{}_h{}^l{}_k - \hat{\Gamma}^2{}_k{}^l{}_h) T_l{}^r X^j{}_{\hat{\uparrow}r} -$$

$$- (\hat{\Gamma}^2{}_l{}^j{}_h \frac{\partial T^l{}_r}{\partial x^k} - \hat{\Gamma}^2{}_l{}^j{}_k \frac{\partial T^l{}_r}{\partial x^h}) X^r, \tag{5.5.13}$$

**Definition 5.5.2** [16,19]: *The "isocurvature" of a vector field $X^r$ on an n-dimensional isoaffine space $\hat{M}(x,\hat{R})$ is given by the isotensor of rank (1.3)*

$$\hat{R}_l{}^j{}_{hk} = \frac{\partial \hat{\Gamma}_l{}^j{}_h}{\partial x^k} - \frac{\partial \hat{\Gamma}_l{}^j{}_k}{\partial x^h} + \hat{\Gamma}_m{}^j{}_k\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_h - \hat{\Gamma}_m{}^j{}_h\, T^m{}_r\, \hat{\Gamma}^2{}_l{}^r{}_k +$$

$$+ \hat{\Gamma}_r{}^j{}_h \frac{\partial T^r{}_s}{\partial x^k} \hat{1}^s{}_l - \hat{\Gamma}_r{}^j{}_k \frac{\partial T^r{}_s}{\partial x^h} \hat{1}^s{}_l\,; \tag{5.5.14}$$

*while the "isotorsion" is given by the isotensor*

$$\hat{\tau}_h{}^l{}_k = \hat{\Gamma}_h{}^l{}_k - \hat{\Gamma}_k{}^l{}_h\,; \tag{5.5.15}$$

*The "isodual isocurvature" and "isodual isotorsion" are the opposite of the corresponding isotopic quantities.*

Expression (5.5.12) can then be written

$$X^j{}_{\hat{\uparrow}h\hat{\uparrow}k} - X^j{}_{\hat{\uparrow}k\hat{\uparrow}h} = \hat{R}_l{}^j{}_{hk}\, T^l{}_s X^s - \hat{\tau}_h{}^l{}_k\, T^s{}_l\, X^j{}_{\hat{\uparrow}s}. \tag{5.5.16}$$

Comparison with the corresponding conventional expression (Eq.s (6.9), p. 83, ref. [4]) is instructive to understand the *modification of the curvature as well as of the torsion caused by the isotopic geometrization of interior physical media.* As we shall see, this modification is the desired feature to avoid excessive approximations of physical reality, such as the admission of the perpetual motion within a physical environment which is inherent in all rotationally invariant (torsionless) theories.

The extension of the results to a (0.2)–rank tensor is tedious but trivial, yielding the expression

$$X^{jl}{}_{\uparrow h \uparrow k} - X^{jl}{}_{\uparrow k \uparrow h} = \hat{R}_r{}^j{}_{hk} T^r{}_s X^{sl} + \hat{R}_r{}^l{}_{hk} T^r{}_s X^{js} - \hat{\tau}_h{}^r{}_k T_r{}^s X^{jl}{}_{\uparrow s}, \qquad (5.5.17)$$

Similarly, for contravariant isovectors and isotensors one obtains the expressions

$$X_{j\uparrow h \uparrow k} - X_{j\uparrow k \uparrow h} = -\hat{R}_j{}^r{}_{hk} T_r{}^s X_s - \hat{\tau}_h{}^r{}_k T_r{}^s X_{j\uparrow s}, \qquad (5.5.18a)$$

$$X_{jl\uparrow h \uparrow k} - X_{jl\uparrow k \uparrow h} = -\hat{R}_j{}^r{}_{hk} T_r{}^s X_{sl} - \hat{R}_l{}^r{}_{hk} T_r{}^s X_{js} - \hat{\tau}_h{}^r{}_k T^s{}_k X_{jl\uparrow s}. \qquad (5.5.18b)$$

called the *isoricci identities*.

The following first property is an easy derivation of definition (5.5.14).

**Property 1:**

$$\hat{R}_l{}^j{}_{hk} = -\hat{R}_l{}^j{}_{kh}. \qquad (5.5.19)$$

The second property requires some algebra, which can be derived via a simple isotopy of the conventional derivation (ref. [26], pp. 91–92).

**Property 2:**

$$\begin{aligned} \hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{lh} + \hat{R}_k{}^j{}_{lh} &= \hat{\tau}_l{}^j{}_{h\uparrow k} + \hat{\tau}_h{}^j{}_{kl\uparrow l} + \hat{\tau}_k{}^j{}_{l\uparrow h} + \\ &+ \hat{\tau}_l{}^j{}_r T^r{}_s \hat{\tau}_h{}^s{}_k + \hat{\tau}_h{}^j{}_r T^r{}_s \hat{\tau}_k{}^s{}_l + \\ + \hat{\tau}_k{}^j{}_r T^r{}_s \hat{\tau}_l{}^s{}_h &+ \hat{\tau}^j{}_{rh} \frac{\partial T^r{}_s}{\partial x^k} 1^s{}_l + \hat{\tau}^j{}_{rk} \frac{\partial T^r{}_s}{\partial x^l} 1^s{}_h + \hat{\tau}^j{}_{rl} \frac{\partial T^r{}_s}{\partial x^h} 1^s{}_k. \end{aligned} \qquad (5.5.20)$$

where, again, the reader should note the isotopies of the conventional terms, plus two new terms.

Note that, for a symmetric isoconnection, the isotorsion is null and the above property reduces to the familiar form

$$\hat{R}_l{}^j{}_{hk} + \hat{R}_h{}^j{}_{kl} + K_k{}^j{}_{lh} = 0. \qquad (5.5.21)$$

The third property identified in ref.s [16,19] also requires some tedious but simple algebra given by an isotopy of the conventional derivation (ref. [4], pp. 92–93), which results in

**Property 3:**

$$(\hat{R}_{j\,h\hat{|}p}^{\ l} + \hat{R}_{j\,kp\hat{|}h}^{\ l} + \hat{R}_{j\,ph\hat{|}k}^{\ l})\,Y_l =$$
$$= (\hat{S}_{h\ k}^{\ r}\,T_r^{\ s}\,\hat{R}_{j\ sp}^{\ l} + \hat{S}_{k\ p}^{\ r}\,T_r^{\ s}\,\hat{R}_{j\ sh}^{\ l} + \hat{S}_{k\ h}^{\ r}\,T_r^{\ s}\,\hat{R}_{j\ sk}^{\ l})\,Y_l +$$
$$+ (\hat{R}_{j\ hk}^{\ r}\,T_{r\hat{|}p}^{\ l} + \hat{R}_{j\ kp}^{\ r}\,T_{r\hat{|}h}^{\ l} + \hat{R}_{j\ ph}^{\ r}\,T_{r\hat{|}k}^{\ l})\,Y_l +$$
$$+ (\hat{S}_{h\ k}^{\ r}\,T_{r\hat{|}p}^{\ l} + \hat{S}_{k\ p}^{\ r}\,T_{r\hat{|}h}^{\ l} + \hat{S}_{p\ h}^{\ r}\,T_{r\hat{|}k}^{\ l})\,TY_{j\hat{|}l}, \tag{5.5.22}$$

called the *isobianchi identity*, and which can be written in a number of equivalent forms here left to the interested reader (see an alternative expression in the next section).

Again, as it was the case for property (5.5.20), the isobianchi identity for the case of a symmetric isoconnection reduces to

$$\hat{R}_{j\,hk\hat{|}p}^{\ l} + \hat{R}_{j\,kp\hat{|}k}^{\ l} + \hat{R}_{j\,ph\hat{|}k}^{\ l} \equiv 0. \tag{5.5.23}$$

The corresponding properties for isodual quantities can be easily derived. This completes the identification of the primary properties of an isocurvature tensor prior to the introduction of the isometric.

## 5.6: ISORIEMANNIAN GEOMETRY

**5.6A: Statement of the problem.** The *isoriemannian geometry* of Class V is the most general possible geometry on a curved manifold possessing:

A) a *nonlinear, nonlocal and nonlagrangian structure* in the local coordinates and their derivatives of arbitrary order;

B) "directly universality" for all possible interior gravitational problems; and

C) admitting the conventional Riemannian geometry and exterior gravitation as a particular case when the isounits $\hat{1}$ recovers the conventional unit $I = \text{diag.}\,(1, 1, 1, 1)$ (physically, when motion returns to be in vacuum).

In this section we shall solely study some of the mathematical properties of the new geometry for the specific case of Class I, with only basic elements for the case of Class II. All physical applications are deferred to Ch. II.8, while experimental verifications are be studied in Vol. III.

The new geometry was proposed in memoir [16], developed in more details in ref. [18,19] and applied to the generalization of Einstein's gravitation for the interior problem in ref.s [20,21]. The only additional contributions on the new *geometry* on record at this time (Spring 1993) are Lopez's [24] application to the exterior problem and Kadeisvili's review [25].

Additional contributions in the field are those by Gasperini [32–34] who was the first to study the isotopies of Einstein's gravitation and to introduce the notion of *locally isopoincaré theory.* However, Gasperini formulated his studies on a *conventional* Riemannian geometry, while the primary emphasis of this section is on the *generalization* of the Riemannian geometry. A review of Gasperini studies is available in ref. [35].

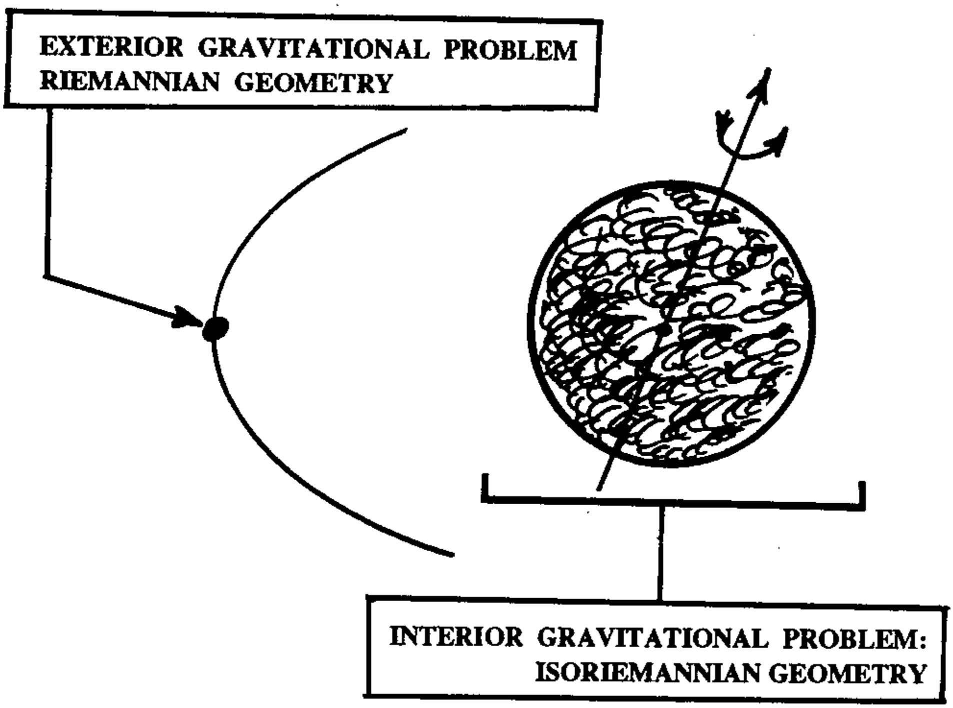



FIGURE 5.6.1: A schematic view of the dual geometric treatment of gravitation characterized by: A) the conventional Riemannian geometry on spaces $\Re(x,g,R)$ assumed as exact for the exterior motion of dimensionless test body in vacuum; B) the covering isoriemannian geometry on isospaces $\hat{\Re}(x,\hat{g},\hat{R})$ for the interior structural problem; C) under the general condition that the latter recovers the former identically in vacuum, e.g., for null density $\mu$, $\hat{\Re}(x,\hat{g},\hat{R})_{\mu=0} \equiv \Re(x,g,R)$. As we shall see in Ch. I.7, despite the considerable enlargement of the scientific horizon, the use of the isoriemannian geometry alone is still insufficient for the interior problem because it is time-reversible, thus particularly suited for the "global" treatment of the structure as a whole with conserved total quantities. The complementary approach of Ch. I.7 will then be the irreversible treatment of one interior test body, while considering the rest of the system as external.

Also, Gasperini formulated his locally isopoincaré studies everywhere in space-time, thus reaching predictable restrictions from available exterior experiments. On the contrary, in the studies herein considered, all generalized geometrical and physical theories are specifically formulated for the *interior* problem only under Condition C) of recovering identically the conventional formulations in the exterior problem.

In this way all available experiments in gravitation have no bearing on the interior isotopic treatment by construction. As we shall see in Vol. II, the test of the isoriemannian geometry for the interior problem requires *novel* experiments, that is, experiments which cannot be even formulated, let alone quantitatively treated with the conventional Riemannian geometry.

The central technical objective [16] is the achievement of an axiom-preserving generalization of the Riemannian geometry with an isometric $\hat{g}$ which, besides being sufficiently smooth, bounded, real valued and symmetric, possesses the most general possible dependence on all needed quantities

$$\hat{g}_{ij}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...) \equiv \hat{g}_{ji}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...). \qquad (5.6.1)$$

as a pre-requisite to achieve the desired "direct universality" for the interior gravitational problem.

**5.6B: Isoriemannian spaces and their isoduals.** To begin, let us perform the transition from the n-dimensional isoaffine spaces $\hat{M}(x,\hat{R})$ of the preceding section, to the corresponding isospaces $\hat{M}(x,\hat{g},\hat{R})$ equipped with the symmetric isotensor (5.6.1) on $\hat{M}(x,\hat{R})$, called *isometric*.

Similarly, we perform the transition from the isodual isoaffine spaces $\hat{M}^d(x,\hat{R}^d)$ to the corresponding spaces $\hat{M}^d(x,\hat{g}^d,\hat{R}^d)$ equipped with the isodual symmetric isotensor $\hat{g}^d = (\hat{g}^d{}_{ij})$.

**Definition 5.6.1** [16,19]: *The "isotopic liftings" of Class I $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ of a conventional Riemannian space $\mathfrak{R}(x,g,R)$ in n-dimension, called "isoriemannian spaces", are the isoaffine spaces $\hat{M}(x,\hat{R})$ in the same dimension equipped with an isometric*

$$\hat{g} = \hat{g}(s, x, \dot{x}, \mu, \tau, n, ...) = T(s, x, \dot{x}, \mu, \tau, n, ...)\, g(x)\,, \quad \hat{g} \in \hat{R},\ g \in R \qquad (5.6.2)$$

*where* T *is the isotopic element of the underlying isofield* $\hat{R}(\hat{n},+,*)$, $\hat{n} = n\hat{1}$, $\hat{1} = T^{-1}$, *which characterizes a symmetric isoaffine connection, called "isochrisoffell symbols of the first kind"*

$$\hat{\Gamma}^1{}_{hlk} = \tfrac{1}{2}\left(\frac{\partial\hat{g}_{kl}}{\partial x^h} + \frac{\partial\hat{g}_{lh}}{\partial x^k} - \frac{\partial\hat{g}_{hk}}{\partial x^l}\right) = \hat{\Gamma}^1{}_{klh} \qquad (5.6.3)$$

*as well as the "isochristoffel symbols of the second kind"*

$$\hat{\Gamma}^2{}_h{}^i{}_k = \hat{g}^{ij}\,\hat{\Gamma}^1{}_{hjk} = \hat{\Gamma}^2{}_k{}^i{}_h \qquad (5.6.4)$$

*where the capability for an isometric of raising and lowering the indices is understood (as in any affine space), and* $\hat{g}^{ij} = |\,(\hat{g}_{rs})^{-1}|^{ij}$. *The* *"isoriemannian geometry" is the geometry of isospaces* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$.

*The "isodual isoriemannian isospaces" are then given by the isodual map of isospaces* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$

$$\hat{\mathfrak{R}}^d(x, \hat{g}^d, \hat{R}^d), \quad \hat{g}^d = T^d g = -\hat{g}, \quad \hat{R}^d \sim \hat{R}\hat{1}^d, \hat{1}^d = (T^d)^{-1} = -\hat{1}, \qquad (5.6.5)$$

*with "isodual isochristoffel symbols"*

$$\Gamma^{d}{}_{1\,hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}^d{}_{kl}}{\partial x^h} + \frac{\partial \hat{g}^d{}_{lh}}{\partial x^k} - \frac{\partial \hat{g}^d{}_{hk}}{\partial x^l}\right) = -\Gamma^{1}{}_{hlk} \qquad (5.6.6a)$$

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$$\Gamma^{d\,2\,i}{}_{hk} = -\Gamma^{2\,i}{}_{hk} \qquad (5.6.6b)$$

In essence, the above definition is centered on the requirement that the alteration (also called *"mutation"* [loc. cit.] $g(x) \Rightarrow T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)\, g(x) = \hat{g}$ of the original Riemannian metric g is characterized by the isotopic element T of the base field and, thus of the base multiplication. The joint liftings $g \Rightarrow \hat{g} = Tg$ and $R(n,+,\times) \Rightarrow \hat{R}(\hat{n},+,*)$, $\hat{n} = n\hat{1}$, $\hat{1} = T^{-1}$, leave the functional dependence of the isometric totally unrestricted, thus verifying the fundamental pre-requisite for "direct universality".

The above new structures imply that the transformation theory of the conventional Riemannian space must be lifted into the isotopic form of the preceding sections. In turn, this ensures that the isoriemannian geometry is isolinear, isolocal and isolagrangian (Sect. 4.2) on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, although generally nonlinear, nonlocal and nonlagrangian when projected on $\mathfrak{R}(x,g,R)$.

On physical grounds, the isotopies $\mathfrak{R}(x,g,R) \Rightarrow \hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ imply that we have performed the transition from the exterior to the interior gravitational problem. Throughout our analysis the reader should keep in mind that the isotopic elements T (or isounit $\hat{1}$) assume their conventional unit value I = diag. (1, 1, 1, 1) everywhere in the exterior of the minimal surface $S^\circ$ encompassing all matter of the interior problem, i.e., for null density $\mu$, in which case $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})_{\mu=0} \equiv \mathfrak{R}(x,g,R)$.

Note that *each given Riemannian geometry can be subjected to an infinite number of isotopic liftings* which are expected to represent the infinite number of possible, different, interior physical media for each given total gravitational mass. This is the reason for the use the plural in "isotopies".

As indicated in Definition 5.6.1, the introduction of a metric on an affine space implies the capability of raising and lowering the indices. The same property evidently persists under isotopies. Given a contravariant isovector $X^i$ on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, one can define its covariant form via the familiar rule

$$X_i = \hat{g}_{ij} X^j. \qquad (5.6.7)$$

Similar conventional rules apply for the lowering of the indices of all other quantities.

It is easy to see that the inverse $g^{-1}$ is a bona-fide contravariant isotensor of rank (2.0). Given a covariant isovector $X_j$ on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, its contravariant form is then defined by

$$X^i = \hat{g}^{ij} X_j. \tag{5.6.8}$$

Rules (5.6.7) and (5.6.8) can then be used to raise or lower the indices of an arbitrary isotensor of rank (r. s).

The first important property of the isoriemann geometry can be derived by writing from Eq.s (5.6.3)

$$\frac{\partial \hat{g}_{hl}}{\partial x^k} = \hat{\Gamma}_{hlk} + \hat{\Gamma}_{lhk}, \quad \hat{g}_{hl\uparrow k} = \frac{\partial g_{hl}}{\partial x^k} - \hat{\Gamma}_{hlk} - \hat{\Gamma}_{lhk}, \tag{5.6.9}$$

for which,

$$\hat{g}_{hl\uparrow k} \equiv 0, \qquad \hat{g}^{hl}{}_{\uparrow k} \equiv 0, \tag{5.6.10}$$

with similar results for the isodual isometrics. We reach in this way the following

**Lemma 5.6.1 – Isoricci lemma** [16,19]: *All isotopic liftings of Class I and II of the Riemannian geometry preserve the vanishing character of the covariant derivative of the isometrics.*

In different terms, the familiar property of the Riemannian geometry $g_{ij\,|\,k} = 0$ is a true geometric axiom because it is invariant under all infinitely possible isotopies. As shown below, this property is not shared by all gravitational quantities, such as Einstein's tensor.

The isotransformation law of the isometric $\hat{g}$ is given by expression of type (5.4.13). By repeating the conventional procedure (ref. [4], pp. 78–70) under isotopy, one obtains the following expression for the *isochristoffel symbol of the first kind*

$$\hat{\Gamma}_{hlk} = \tfrac{1}{2}\left(\frac{\partial \hat{g}_{kl}}{\partial x^h} + \frac{\partial \hat{g}_{lh}}{\partial x^k} - \frac{\partial \hat{g}_{hk}}{\partial x^l}\right) =$$

$$= \hat{g}_{jp}\, T^j{}_r \frac{\partial^2 x^r}{\partial x^h \partial x^k} T^p{}_s \frac{\partial x^s}{\partial x^l} + \frac{\partial \bar{g}_{jp}}{\partial \bar{x}^m} T^j{}_r T^p{}_s \left( \frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} \frac{\partial \bar{x}^m}{\partial x^l} + \right.$$

$$\left. + \frac{\partial \bar{x}^r}{\partial x^k} \frac{\partial \bar{x}^s}{\partial x^l} \frac{\partial \bar{x}^m}{\partial x^h} - \frac{\partial \bar{x}^r}{\partial x^l} \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^m}{\partial x^k} \right) + \tfrac{1}{2}\hat{g}_{jp}\, T^p{}_s \left[ \frac{\partial T^j{}_r}{\partial x^l} \left( \frac{\partial \bar{x}^r}{\partial x^h} \frac{\partial \bar{x}^s}{\partial x^k} + \frac{\partial \bar{x}^s}{\partial x^h} \frac{\partial \bar{x}^r}{\partial x^k} \right) + \right.$$

$$+ \frac{\partial T^{j}{}_{r}}{\partial x^{h}}\left(\frac{\partial \bar{x}^{r}}{\partial x^{k}}\frac{\partial \bar{x}^{s}}{\partial x^{l}} + \frac{\partial \bar{x}^{s}}{\partial x^{k}}\frac{\partial x^{r}}{\partial x^{l}}\right) - \frac{\partial T^{j}{}_{r}}{\partial x^{k}}\left(\frac{\partial \bar{x}^{r}}{\partial x^{l}}\frac{\partial \bar{x}^{s}}{\partial x^{h}} + \frac{\partial \bar{x}^{s}}{\partial x^{l}}\frac{\partial \bar{x}^{r}}{\partial x^{h}}\right), \qquad (5.6.11)$$

with a number of alternative formulations and simplifications, e.g., for diagonal isotopic elements T, which are left to the interested reader for brevity.

**5.6C: Basic identities.** In order to proceed with our review, we need the following

**Definition 5.6.2** [loc. cit.]: *Given an n-dimensional isoriemannian space $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ of Class I, the "isocurvature tensor" is given by*

$$\hat{R}_{l}{}^{j}{}_{h} = \frac{\partial \Gamma^{2}{}_{l}{}^{j}{}_{h}}{\partial x^{k}} - \frac{\partial \Gamma^{2}{}_{l}{}^{j}{}_{k}}{\partial x^{h}} + \Gamma^{2}{}_{m}{}^{j}{}_{k}\, T^{m}{}_{r}\, \Gamma^{2}{}_{l}{}^{r}{}_{h} - \Gamma^{2}{}_{m}{}^{j}{}_{h}\, T^{m}{}_{r}\, \Gamma^{2}{}_{l}{}^{r}{}_{k} +$$

$$+ \Gamma^{2}{}_{r}{}^{j}{}_{h} \frac{\partial T^{r}{}_{s}}{\partial x^{k}} \hat{1}^{s}{}_{l} - \Gamma^{2}{}_{r}{}^{j}{}_{k} \frac{\partial T^{r}{}_{s}}{\partial x^{h}} \hat{1}^{s}{}_{l}, \qquad (5.6.12)$$

*and can be rewritten*

$$\hat{R}_{l}{}^{j}{}_{hk} = \tfrac{1}{2}\, \hat{g}^{jp}\left(\frac{\partial \hat{g}_{ph}}{\partial x^{k}\,\partial x^{l}} - \frac{\partial \hat{g}_{pk}}{\partial x^{h}\,\partial x^{l}} + \frac{\partial \hat{g}_{lk}}{\partial x^{h}\,\partial x^{j}} - \frac{\partial \hat{g}_{lh}}{\partial x^{k}\,\partial x^{j}}\right) +$$

$$+ \hat{g}^{jp}\left(\Gamma^{1}{}_{prh}\, T^{r}{}_{s}\, \hat{\Gamma}^{2}{}_{l}{}^{s}{}_{h} - \hat{\Gamma}^{1}{}_{prk}\, T^{r}{}_{s}\, \Gamma^{2}{}_{l}{}^{s}{}_{k}\right) +$$

$$+ \Gamma^{2}{}_{r}{}^{j}{}_{h} \frac{\partial T^{r}{}_{s}}{\partial x^{k}} \hat{1}^{s}{}_{l} - \Gamma^{2}{}_{r}{}^{j}{}_{k} \frac{\partial T^{r}{}_{s}}{\partial x^{h}} \hat{1}^{s}{}_{l}; \qquad (5.6.13)$$

*the "isoricci tensor" is given by*

$$\hat{R}_{lh} = \hat{R}_{l}{}^{j}{}_{hj} = g^{ij}\hat{R}_{lihj}; \qquad (5.6.14)$$

*the "isoeinstein tensor" is given by*

$$\hat{G}_{i}{}^{j} = \hat{R}_{i}{}^{j} - \tfrac{1}{2}\delta_{i}{}^{j}\hat{R}; \qquad (5.6.15)$$

*and the "completed isoeinstein tensor" is given by*

$$\hat{S}_{i}{}^{j} = \hat{R}_{i}{}^{j} - \tfrac{1}{2}\delta_{i}{}^{j}\hat{R} - \tfrac{1}{2}\delta_{i}{}^{j}\hat{\Theta}; \qquad (5.6.16)$$

*where $\hat{R}$ is the "isocurvature isoscalar"*

$$\hat{R} = \hat{R}^{i}{}_{i} = \hat{g}^{ij}\hat{R}_{ij}, \tag{5.6.17}$$

*and $\hat{\Theta}$ is the "isotopic isoscalar"*

$$\begin{aligned}\hat{\Theta} &= \hat{g}^{jh}\hat{g}^{lk}(\hat{\Gamma}^{1}{}_{rjk}T^{r}{}_{s}\hat{\Gamma}^{2s}{}_{lh} - \hat{\Gamma}^{1}{}_{rjh}T^{r}{}_{s}\hat{\Gamma}^{2s}{}_{lk}) = \\ &= \hat{\Gamma}^{1}{}_{rjk}T^{r}{}_{s}\hat{\Gamma}^{2s}{}_{lh}(\hat{g}^{jh}\hat{g}^{lk} - \hat{g}^{jk}\hat{g}^{lh}).\end{aligned} \tag{5.6.18}$$

*Isodual quantities are defined accordingly.*

We are now equipped to review the isotopies of the various properties of the Riemannian geometry [10,12]. From definition (5.4.12) we readily obtain

**Property 1: Antisymmetry of the last two indices of the isocurvature tensor**

$$\hat{R}_{l}{}^{j}{}_{hk} = - \hat{R}_{l}{}^{j}{}_{kh}. \tag{5.6.19}$$

The specialization of properties (3.22) to the case at hand easily implies the following

**Property 2: Vanishing of the totally antisymmetric part of the isocurvature tensor**

$$\hat{R}_{l}{}^{j}{}_{hk} + \hat{R}_{h}{}^{j}{}_{kl} + \hat{R}_{k}{}^{j}{}_{lh} = 0, \tag{5.6.20}$$

or, equivalently,

$$\hat{R}_{lmhk} + \hat{R}_{hmkl} + \hat{R}_{kmlh} = 0. \tag{5.6.21}$$

The use of property (5.6.19) and Lemma 5.6.1 then yields

**Property 3: Antisymmetry in the first two indices of the isocurvature tensor**

$$\hat{R}_{jlhk} = \hat{R}_{ljhk}, \tag{5.6.22}$$

or, equivalently,

$$\hat{R}_{ljhk} = \hat{R}_{hklj}. \tag{5.6.23}$$

From Definition (5.6.12) and the use of Lemma 5.6.1, after tedious but simple calculations, we obtain the following:

**Property 3: Isobianchi identity**

$$\hat{R}_{1\,hk\uparrow p}^{\;j} + \hat{R}_{1\,ph\uparrow k}^{\;j} + \hat{R}_{1\,kp\uparrow h}^{\;j} = \hat{S}_{1\,hkp}^{\;j}, \tag{5.6.24}$$

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where

$$\hat{S}_{1\,hkp}^{\;j} = \hat{\Gamma}^{2\,j}_{\;r\,h}\,( T^r_{\;s\uparrow k}\,\hat{\Gamma}^{2\,s}_{\;1\,p} - T^r_{\;s\lceil p}\,\hat{\Gamma}^{2\,s}_{\;1\,k} ) + \hat{\Gamma}^{2\,j}_{\;r\,p}\,( T^r_{\;s|h}\,\hat{\Gamma}^{2\,s}_{\;1\,k} - T^r_{\;s|k}\,\hat{\Gamma}^{2\,s}_{\;1\,h} ) + \hat{\Gamma}^{2\,j}_{\;r\,k}\,( T^r_{\;s\lceil p}\,\hat{\Gamma}^{2\,s}_{\;1\,h} - T^r_{\;s|h}\,\hat{\Gamma}^{\;s}_{1\,p} ) +$$

$$+ \hat{\Gamma}^{2\,j}_{\;r\,h}\,( \hat{Q}_{k\;1\lceil p}^{\;r} - \hat{Q}_{p\;1\lceil k}^{\;r} ) + \hat{\Gamma}^{2\,j}_{\;r\,p}\,( \hat{Q}_{h\;1\lceil k}^{\;r} - \hat{Q}_{k\;1\lceil h}^{\;r} ) + \hat{\Gamma}^{2\,j}_{\;r\,k}\,( \hat{Q}_{p\;1\lceil h}^{\;r} - \hat{Q}_{h\;r\lceil p}^{\;r} ) \tag{5.4.25a}$$

$$\hat{Q}_{k\;1\uparrow p}^{\;r} = \left(\frac{\partial T^r_{\;s}}{\partial x^k}\,\hat{1}^s_{\;1}\right)_{\uparrow p}. \tag{5.6. 25b}$$

For isotopic liftings independent from the local coordinates (but dependent on the velocities and other variables, as it is often the case for the characteristic functions of interior physical media, isodifferential property (5.6.25) assumes the simpler form

$$\hat{R}_{1\,hk\uparrow p}^{\;j} + \hat{R}_{1\,ph\uparrow k}^{\;j} + \hat{R}_{1\,kp|h}^{\;j} \equiv 0. \tag{5.6.26}$$

The isobianchi identity can also be equivalently written in the general case

$$\hat{R}_{1jhk\uparrow p} + \hat{R}_{1jph\uparrow k} + \hat{R}_{1jkp|h} = \hat{S}_{1jhkp}, \tag{5.6.27}$$

where the $\hat{S}$–term is that defined by Eq.s (5.6.26), with the reduced form for the isotopies not dependent on the local coordinates (or constant)

$$\hat{R}_{1jhk\uparrow p} + \hat{R}_{1jph\uparrow k} + \hat{R}_{1jkp\uparrow h} \equiv 0. \tag{5.6.28}$$

We now consider the isotopic liftings of *Freud identity* which was originally identified by Freud [36] in 1939, reviewed in details by Pauli []37, and then forgotten for a long time by virtually all textbooks in gravitation. The identity was "rediscovered" by Yilmaz [38] who brought it to the attention of this author. The identity was then subjected to a mathematical study by Rund [39] (in perhaps his last paper). In memoir [19] published jointly with Rund's article [39], this author followed Rund's treatment, and reached the following property:

**Property 5: The isofreud identity**

$$\hat{U}^{k}{}_{j} + \hat{G}^{k}{}_{j} = \frac{\partial \hat{V}^{kl}{}_{j}}{\partial x^{l}}\,, \tag{5.6.29}$$

where

$$\hat{V}^{kl}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\,(\,\hat{g}^{rs}\,(\,\delta^{k}{}_{j}\,\hat{\Gamma}^{2\,l}{}_{r\,s} - \delta^{l}{}_{j}\,\hat{\Gamma}^{2\,k}{}_{r\,s}\,) +$$

$$+ (\delta^{l}{}_{j}\,\hat{g}^{kr} - \delta^{k}{}_{j}\,\hat{g}^{lr}\,)\,\hat{\Gamma}^{2\,s}{}_{r\,s} + \hat{g}^{lr}\,\hat{\Gamma}^{2\,k}{}_{j\,r} - \hat{g}^{kr}\,\hat{\Gamma}^{2\,l}{}_{j\,r}\,), \tag{5.6.30a}$$

$$\hat{U}^{k}{}_{j} = \tfrac{1}{2}\hat{\Delta}^{\frac{1}{2}}\,(\frac{\partial \hat{G}}{\partial \hat{g}^{lm}{}_{\uparrow k}}\,\hat{g}^{lm}{}_{\uparrow j} - \delta^{k}{}_{j}\,\hat{G})\,, \tag{5.6.30b}$$

$$\hat{G} = \hat{g}^{jk}\,(\,\hat{\Gamma}^{2\,p}{}_{j\,s}\,T^{q}{}_{p}\,\hat{\Gamma}^{2\,s}{}_{k\,q} - \hat{\Gamma}^{2\,p}{}_{j\,k}\,T^{q}{}_{p}\,\hat{\Gamma}^{2\,s}{}_{q\,s}\,)\,, \tag{5.6.30c}$$

$$\hat{G}^{k}{}_{j} = \hat{\Delta}^{\frac{1}{2}}\,\hat{G}^{k}{}_{j}, \quad \hat{\Delta}^{\frac{1}{2}} = \sqrt{\hat{g}}\,. \tag{5.6.30d}$$

Rund's [39] reached the important result that *the Freud identity holds for all symmetric and nonsingular metrics on a (conventional) Riemannian space of dimension higher than one.* The same property evidently persist under isotopies. Thus, *Property 5 is automatically satisfied for all symmetric and nonsingular isometrics on isoriemannian spaces of dimension higher than one.* Despite this inherent compatibility of the identity with the geometry, the Freud identity and its isotopic image have important consequences in gravitation, e.g., for the vexing problem of the *source* of the gravitational field in vacuum.

In fact, Yilmaz's [38] points out that the conventional Freud identity on a Riemannian space raises the fundamental question, apparently still open to debates at this writing, whether a sourceless gravitational theory in vacuum does or does not verify all axioms of the Riemannian geometry.

We are now in a position to identify some of the first consequences of the isoriemannian geometry. First, it is an instructive exercise for the reader interested in acquiring a technical knowledge of the isotopies of the Riemannian geometry to prove the following important property:

**Lemma 5.6.2** [19]: *Einstein's tensor* $G^{i}{}_{j} = R^{i}{}_{j} - \frac{1}{2}\delta^{i}{}_{j}R$ *does not preserve under isotopies the vanishing value of its covariant divergence (contracted Bianchi identity)*

$$G^{i}{}_{j|i} = R^{i}{}_{j|i} - \tfrac{1}{2}\,\delta^{i}{}_{j}\,R_{|i} \equiv 0, \tag{5.6.31}$$

*that is, the isoeinsteinian tensor (5.6.15) is such that*

$$\hat{G}^{i}_{j\uparrow i} = \hat{R}^{i}_{j\uparrow i} - \tfrac{1}{2}\delta^{i}_{j}\hat{R}_{\uparrow i} \neq 0, \tag{5.6.32}$$

*Therefore, Einstein's tensor does not possess an axiomatically complete structure.*

This unexpected occurrence has rather deep connections with the Freud identity, and implications for the identification of the correct theory of exterior gravitation in vacuum because it raises again the fundamental question, this time from an independent viewpoint, of the geometric consistency of a sourceless theory in vacuum.

It is interesting to note that *the Freud identity is a true geometric axiom of the Riemannian geometry in the sense that it persists under isotopies, while the contracted Bianchi identity is not, evidently because not preserved by isotopies.*

These occurrences shift the emphasis, from the historically predominant use of the contracted Bianchi identity, to the geometrically more rigorous Freud identity with predictable important implications for the entire theory of gravitation, both external and internal.

The following property can also be proved via tedious but simple calculations from isodifferential property (5.6.25).

**Lemma 5.6.3** [16,19]: *The completed isoeinstein tensor does possess an identically null isocovariant isodivergence, i.e.,*

$$\hat{S}^{i}_{j\uparrow j} = (\hat{R}^{i}_{j} - \tfrac{1}{2}\delta^{i}_{j}\hat{R} - \tfrac{1}{2}\delta^{i}_{j}\hat{\Theta})_{\uparrow i} \equiv 0. \tag{5.6.33}$$

*called the "completed and contracted isobianchi identity".*

**5.6.D: The fundamental theorem for interior isogravitation.** As now familiar, we have initially considered conventional gravitational theories on $\Re(x,g,R)$ which have *null torsion*, and have reached an infinite family of isotopies all of which also have a *null isotorsion* on $\hat{\Re}(x,\hat{g},\hat{R})$ because of the axiom-preserving character of the isotopies. In fact, the original symmetric connection $\Gamma^{2}{}_{h}{}^{s}{}_{k}$ has been lifted into an infinite family of isoconnections which are also symmetric

$$\tau_{h}{}^{s}{}_{k} = \Gamma^{2}{}_{h}{}^{s}{}_{k} - \Gamma^{2}{}_{k}{}^{s}{}_{h} \equiv 0 \;\Rightarrow\; \hat{\tau}_{h}{}^{s}{}_{k} = \hat{\Gamma}^{2}{}_{h}{}^{s}{}_{k} - \hat{\Gamma}^{2}{}_{k}{}^{s}{}_{h} \equiv 0. \tag{5.6.34}$$

However, *the null value of torsion occurs at the level of isospace $\hat{\Re}(x,\hat{g},\hat{R})$ which is not the physical space of the experimenter*, the latter remaining the conventional space-time in vacuum (see for details ref. [20], Ch. V).

The physical issue whether or not the isotopies of Einstein's gravitation for interior conditions have the non-null torsion required to avoid perpetual motion approximations, must therefore be inspected in the physical space and not in the geometrical isospace.

This can be done by projecting the isocovariant derivative of an isovector on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ in the ordinary space $\mathfrak{R}(x,g,R)$, i.e.,

$$X^i{}_{\uparrow k} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_h{}^i{}_k\, T^h{}_r\, X^r = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_r{}^i{}_k\, X^r, \tag{5.6.35a}$$

$$\hat{\Gamma}^2{}_r{}^i{}_k = \hat{\Gamma}^2{}_h{}^i{}_k\, T^h{}_r. \tag{5.6.35b}$$

It is then evident that, starting with a symmetric isoconnection $\hat{\Gamma}_h{}^i{}_k$ on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, the corresponding connection $\hat{\Gamma}_r{}^i{}_k$ on $\mathfrak{R}(x,g,R)$ is no longer necessarily symmetric, and we have the following

**Theorem 5.6.1** [18,19]: *The isotopic liftings $\Gamma^2{}_h{}^i{}_k \Rightarrow \hat{\Gamma}^2{}_h{}^i{}_k$ of a symmetric connection $\Gamma^2{}_h{}^i{}_k$ on a Riemannian space $\mathfrak{R}(x,g,R)$ into an infinite family of isotopic connections $\hat{\Gamma}^2{}_h{}^i{}_k$ on Isoriemannian spaces $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ of the same dimension, imply that the isospace always possesses a null isotorsion, but, when the isotopies are projected into the original space, a non–null torsion generally occurs.*

The above property was first reached by Gasperini in ref. [32–34] in the language of conventional differential forms on a conventional Riemannian space. The geometrization of the property into a symmetric isotorsion was achieved by the author in ref. [18].

Theorem 5.6.1 is physically fundamental inasmuch as it ensures the needed structural differences for a realistic, quantitative representation of interior trajectories. We are referring to a representation of the *differences* in the trajectory of a test body from motion in vacuum with stable orbit (and thus null torsion) to motion within a physical medium with an unstable trajectory (and, therefore, non–null torsion, but null isotorsion).

Theorem 5.6.1 is also fundamental for our achievement of a geometric unit between the exterior and interior problem which will be more evident later on in this section. In fact, the instability of the interior trajectories is achieved via the same geometric axiom (null torsion) of the exterior problem, although realized in its most general possible isotopic form.

Finally, Theorem 5.6.1 necessarily requires *two different, but compatible theories: one for the exterior gravitational problem with null torsion, and one for the interior gravitational problem with null isotorsion but non–null torsion* .

The most important result of the analysis of this section can be expressed via a repetition under isotopies of ref. [26], p. 313 and the Theorem of p. 321, with the addition of the isofreud identity plus the completed Einstein's tensor (5.6.16), lead to the following:

**Theorem 5.6.2 –Fundamental theorem for interior gravitation** [18,19]: *In a (3+1)–dimensional isoriemannian isospace of Class I, $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, the most general possible isolagrangian equations*

$$\hat{E}^{ij} = 0, \tag{5.6.36}$$

*verifying the properties:*
*1) symmetric condition*

$$\hat{E}^{ij} = \hat{E}^{ji}, \tag{5.6.37}$$

*2) the contracted isobianchi identity*

$$\hat{E}^{il}{}_{\uparrow l} = 0, \tag{5.6.38}$$

*and 3) the isofreud identity*

$$\hat{\Theta}^{k}{}_{j} + \hat{G}^{k}{}_{j} = \frac{\partial \hat{V}^{kl}}{\partial x^{l}}, \tag{5.6.39}$$

*are characterized by the isolagrangian principle* [42]

$$\begin{aligned} \delta\hat{A} &= \delta\int L(\hat{g}_{ij}, \hat{g}_{ij,k}, \hat{g}_{ij,kl}, \tau_{ij}, t_{ij})\, dx = \\ &= \delta\int \hat{\Delta}^{\frac{1}{2}} [\lambda(\hat{R} + \hat{\Theta}) + 2\Lambda + \rho(\hat{\tau} + \hat{t})]\, dx = 0, \end{aligned} \tag{5.6.40}$$

*where $\lambda$, $\Lambda$, and $\rho$ are constants, $\hat{t}$ is the isotopic generalization of stress–energy tensor, $\hat{\tau}$ is an isotopic source tensor, $\hat{R}$ the isotopic curvature scalar and $\hat{\Theta}$ the isotopic scalar. For the case $\lambda = \rho = 1$, $\Lambda = 0$ and appropriate units, the isolagrange equations are given by*

$$\hat{E}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} - \tfrac{1}{2}\hat{g}^{ij}\hat{\Theta} - \hat{\tau}^{ij} - \hat{t}^{ij} = 0, \tag{5.6.41}$$

*and can be written in terms of the completed isoeinstein tensor*

$$\hat{S}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} + \tfrac{1}{2}\hat{\Theta} = \hat{\tau}^{ij} + \hat{t}^{ij}, \tag{5.6.42}$$

[42] We are now in a position to clarify the meaning of "non–first–order–Lagrangians" in interior gravitation. As now well known, the Lagrangians emerging under isotopies, when projected in the original space, are of arbitrary order higher than the first, $L = L(s, x, \dot{x}, \ddot{x}, \ldots)$. However, the isolagrange equations remain of the second–order, evidently because they only depend on the second–order derivative of the isometric with respect to the local coordinates.

*or, equivalently, in terms of the isoeinstein tensor*

$$\hat{G}^{ij} = \hat{R}^{ij} - \tfrac{1}{2}\hat{g}^{ij}\hat{R} + \tfrac{1}{2}\hat{\theta} = \hat{T}^{ij} + \hat{t}^{ij}, \quad \hat{T}^{ij} = \hat{\tau}^{ij} - \tfrac{1}{2}\hat{\Theta}, \tag{5.6.43a}$$

The reformulation of the above theorem in terms of isointegrals (Sect. 6.7) is an intriguing exercise for the interested reader.

The physical implications of the above theorem will be studied in Vol. II. Here were merely note the *dual* revision of conventional equations, one caused by the isoscalar $\hat{\Theta}$ and the other by the Freud identify which implies the identifiocation

$$\hat{M}^{ij}{}_{\text{Einstein}} = \hat{\tau}^{ij} + \hat{t}^{ij}. \tag{5.6.44}$$

As we shall see in Vol. II, this turns the exterior "description" of the gravitational field in vacuum into an *interior theory on the origin of the gravitational field* wth numerous, rather intriguing and far reaching implications.

**5.6E: Description of antimatter via the isodual isoriemannian geometry.** We close this section with a brief study of the image of the isoriemannian geometry under isoduality, including the isodual definition of operations (such as fraction and derivatives) which can be expressed via the following

**Theorem 5.6.3** [20,21]: *The interior problem of antimatter verifies Theorem 5.6.2 under isoduality characterized by the following maps:*

| | |
|---|---|
| Basic unit | $\hat{1} \rightarrow \hat{1}^d = -\hat{1},$ |
| Isotopic element | $T \rightarrow T^d = -T,$ |
| Isometric | $\hat{g} = Tg \rightarrow \hat{g}^d = -\hat{g},$ |
| Isoconnection coefficients | $\hat{\Gamma}_{klh} \rightarrow \hat{\Gamma}^{d\,l}{}_{klh} = -\hat{\Gamma}^l{}_{klh},$ |
| Isocurvature tensor | $\hat{R}_{lijk} \rightarrow \hat{R}^d{}_{lijk} = -\hat{R}_{lijk},$ |
| Isoricci tensor | $\hat{R}_{\mu\nu} \rightarrow \hat{R}^d{}_{\mu\nu} = -\hat{R}_{\mu\nu},$ |
| Isoricci scalar | $\hat{R} \rightarrow \hat{R}^d = \hat{R},$ |
| Isoeinstein tensor | $\hat{G}_{\mu\nu} \rightarrow \hat{G}^d{}_{\mu\nu} = -\hat{G}_{\mu\nu},$ |
| Isotopic scalar | $\hat{\Theta} \rightarrow \hat{\Theta}^d = \hat{\Theta},$ |
| Compl. isoeinstein tensor | $\hat{S}_{\mu\nu} \rightarrow \hat{S}^d{}_{\mu\nu} = -\hat{S}_{\mu\nu},$ |
| Electromagnetic potentials | $\hat{A}_\mu \rightarrow \hat{A}^d{}_\mu = -\hat{A}_\mu,$ |
| Electromagnetic field | ${}_{\mu\nu} \rightarrow \hat{F}^d{}_{\mu\nu} = -\hat{F}_{\mu\nu},$ |
| Elm energy-mom. tensor | $\hat{T}_{\mu\nu} \rightarrow \hat{T}^d{}_{\mu\nu} = -\hat{T}_{\mu\nu},$ |
| Stress-energy tensor | $\hat{t}_{\mu\nu} \rightarrow \hat{T}^d{}_{\mu\nu} = -\hat{t}_{\mu\nu}.$ |

The proof of the above properties is simple but instructive. In particular, it can show the necessity of the use of the isodual spaces to reach negative energies. In fact, in conventional Minkowski $M(x,\eta,R)$ and Riemannian spaces $\Re(x,g,R)$ the electromagnetic potentials and fields do change sign for antiparticles, but the energy–momentum tensor remains the same. The latter changes sign only when computed in isodual Minkowski spaces $M^d(x,\eta^d,R^d)$ and isodual Riemannian spaces $\Re^d(x,g^d,R^d)$. These basic properties then persist when passing to the covering isospaces $\hat{M}(x,\hat{\eta},\hat{R})$, $\hat{\Re}(x,\hat{g},\hat{R})$ and their isoduals $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$, $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$.

The proof of Theorem 5.6.3 also shows that antimatter represented via the isodual isoriemannian geometry evolves "backward in time", as anticipated in Sect. 5.1, with intriguing epistemological conceptual and geometrical possibilities for advances, e.g., a theoretical conception of antigravity [43] studied in Vol. II.

## 5.7: ISOTOPIES OF PARALLEL TRANSPORT AND GEODESIC MOTION

A geometrically consistent generalization of the Riemannian geometry and of Einstein's gravitation cannot be reached without consistent isotopic coverings of conventional *parallel transport* and *geodesic motion* [4].

These generalized notions were introduced for the first time in memoir [16], expanded in ref.s [18,19], applied to interior gravitation in ref. [20,21] under the names of *isoparallel transport* and *isogeodesic motion* and reviewed in [25].

The new notions represent the maximal geometric achievements of the isotopies. They can be stated in figurative terms by saying that "physical media disappear under their isogeometrization". In fact, as we shall see, the trajectories of the iosoparallel transport and the isogeodesics coincide with the original trajectories in vacuum when represented in isospaces.

Their knowledge is particularly important for hadronic mechanics. Recall that the sections of the perfect sphere, i.e., the circles, are geodesics of the rotational symmetry O(3). Isogeodesics are then important to understand that the sections of the ellispoidically deformed charge distributions of hadrons, the ellipses, are *bona fide* geodesics of the isorotational symmetry $\hat{O}(3)$ in isospace.

Since the times of Galileo Galilei and his experiments at the Pisa tower (1609), we know that *the free fall of a body in Earth's gravitational field is geodesic only in the absence of the resistive forces due to our atmosphere.* It is therefore well know that the trajectory of a test particle within a physical medium is not geodesic, owing to the resistive forces. Our isogeodesic then permits an ultimate geometric unity of motion in vacuum and within physical media which is the true foundation of the isorelativities of Vol. II.

Moreover, it is also well known since Lagrange's and Hamilton's times (see the historical notes of ref. [20]) that the forces between the body and the medium are of nonpotential type and, thus, of a type outside the representational capabilities of the conventional, local-differential, Riemannian geometry. A fully similar situation occurs for parallel transport, thus implying the inapplicability of the geometry itself for interior conditions.

Isoparallel transport and isogeodesic motions are crucial for a technical understanding of the isotopic relativities and of their underlying form-invariant description of physical laws via isosymmetries, because they complete the abstract geometric unity between interior and exterior problems found at the preceding levels in vector spaces, algebras, groups, etc.. In fact, parallel transport and geodesic motion are reached in interior conditions via the same abstract axioms of the corresponding quantities in vacuum, only realized in their most general possible way.

To begin, let $\Re(x,g,R)$ be a conventional n-dimensional Riemannian space. Under sufficient smoothness and regularity conditions hereon assumed, a vector field $X^i$ on $\Re(x,g,R)$ is said to be *parallel along a curve* C if it satisfies the differential equation along C [4]

$$DX^i = X^i{}_{|s}\, dx^s = \left( \frac{\partial X^i}{\partial x^s} + \Gamma 2_r{}^i{}_s X^r \right) dx^s = 0, \qquad (5.7.1)$$

where $\Gamma 2_r{}^i{}_s$ is a symmetric connection. Then, by recalling the notions of isodifferential of Sect. 5.4, we have the following

transported by

**Definition 5.7.1** [16,19]: *An isovector field* $X^i$ *on an n-dimensional* differ
*isoriemannian space of Class I* $\hat{\Re}(x,\hat{g},\hat{R})$ *is said to be "isoparallel" ~~along~~ a curve* C *on* $\hat{\Re}(x,\hat{g},\hat{R})$, *iff it verified the isotopic equations along* C

$$\hat{D}X^i = X^i{}_{\hat{|}r}\, T^r{}_s(x, \dot{x},..)\, \hat{\partial} x^s = \left[ \frac{\partial X^i}{\partial x^s} + \hat{\Gamma} 2_r{}^i{}_s\, T^r{}_t(x, \dot{x},..)\, X^t \right] T^s{}_p(x, \dot{x},..) \,]\, \hat{\partial} x^p = 0 \qquad (5.7.2)$$

*where* $\hat{\Gamma}2_r{}^i{}_s$ *is the symmetric isoconnection and* $T = (T^r{}_s)$ *is the isotopic element of the underlying isofield* $\hat{R}(\hat{n},+,*)$.

The identity of axioms (5.7.1) and (5.7.2] at the abstract level is evident, again, because of the loss of all distinction between the right, modular, associative product, say Xx, and its isotopic generalization $X*x = XT(s, x, \dot{x}, \ddot{x}, ...)x$.

To understand the physical differences between the above two definitions, let us consider the independent (invariant) parameter s, such that the isovector field $\dot{x} = \hat{\partial} x/\hat{\partial} s$ is tangent to C, and let $X^i = X^i(s)$. Consider the curve C at a point P(1) for $s = s_1$ and let $X^i(1)$ be the corresponding value of the isovector field $X^i$ at P(1).

Consider now the transition from P(1) to P(2), i.e., from $s_1$ to $s_1 + \hat{\partial} s$. The corresponding transported value of the isovector field $X^i(2) = X^i(1) + \partial X^i$ is said to occur under an *isoparallel displacement* on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ in accordance with Definition 5.7.1, iff

$$\hat{\partial} X^i = \frac{\partial X^i}{\partial x^r} T^r{}_s \hat{\partial} x^s = - \hat{\Gamma}^2{}_{r\,s}{}^i T^r{}_p X^p T^s{}_q \hat{\partial} x^q. \qquad (5.7.3)$$

The iteration of the process up to a finite displacement is equivalent to the solution of the integro-differential equation

$$\frac{\hat{\partial} X^i}{\hat{\partial} s} = \frac{\partial X^i}{\partial x^r} T^r{}_s \frac{\hat{\partial} x^s}{\hat{\partial} s} = - \Gamma^2{}_{r\,s}{}^i T^r{}_p X^p T^s{}_q \frac{\hat{\partial} x^q}{\hat{\partial} s}, \qquad (5.7.4)$$

By integrating the above expression in the finite interval $(s_1, s_2)$, one reaches the following property (expressed in terms of isointegrals of Sect. 6.7)

**Lemma 5.7.1** [loc. cit.]: *The isoparallel transport of an isovectorfield* $X^i(s)$ *on an n–dimensional isoriemannian space* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *of Class I from the point* $s_1$ *to a point* $s_2$ *on a curve* C *verifies the isotopic laws*

$$\hat{X}^i(2) = \hat{X}^i(1) - \hat{\int}_1^2 \hat{\Gamma}^2{}_{r\,s}{}^i(x, \dot{x},..)\, T^r{}_p(x, \dot{x},..)\, X^p(x)\, T^s{}_q(x, \dot{x},..)\, \dot{x}^q\, \hat{\partial} s, \qquad (5.7.5)$$

*where*

$$\hat{X}^i(2) - \hat{X}^i(1) = \hat{\int}_1^2 \hat{\partial} X^i = \int_1^2 \frac{\partial X^i}{\partial x^p} T^p{}_q \frac{\hat{\partial} x^q}{\hat{\partial} s} \hat{\partial} s. \qquad (5.7.6)$$

The physical implications are pointed out by the fact that the isotransported isovector does not start at the value $X^i(1)$, but at the *modified value* $\hat{X}^i(1)$ characterized by Eq.s (5.7.5). Additional evident modifications are characterized by the isotopic connection $\hat{\Gamma}^2{}_{r\,s}^{\,i}$ and the two isotopic elements T of the r.h.s. of Eq.s (57.5).

These departures from the conventional case can be better understood in a *flat isospace*, via the following evident

**Corollary 5.7.1A** [Loc. cit.]: *In a flat isospace, such as the isominkowski space* $\hat{M}(x,\hat{\eta},\hat{R})$ *in (3.1)–space–time dimensions, or the isoeuclidean space* $\hat{E}(r,\hat{\delta},\hat{R})$ *in 3–dimension, the conventional notion of parallelism no longer holds, in favor of the following flat isoparallelism*

$$\hat{X}^i(2) - \hat{X}^i(1) = \int_1^2 \hat{\partial} X^i = \hat{\int}_1^2 \frac{\partial X^i}{\partial x^p} T^p{}_q \frac{\hat{\partial} x^q}{\hat{\partial} s} \hat{\partial} s. \qquad \hat{\Gamma}^2{}_{r\,s}{}^i \equiv 0, \qquad (5.7.7)$$

Consider, as an illustration, a straight line C in conventional Euclidean space $R_t \times E(r,\delta,R)$, with only two space-components. Then a vector $\bar{R}(1)$ at $s = t_1$ is transported in a parallel way to $\bar{R}(2)$ at $s = t_2$ by keeping unchanged the characteristic angles with the reference axis, i.e.,

$$\hat{R}^k(2) - \hat{R}^k(1) = \hat{\int}_1^2 \left(\frac{\partial R^k(r)}{\partial x^1} \hat{d}x^1 + \frac{\partial R^k(r)}{\partial x^2} \hat{d}x^2\right. \qquad (5.7.8)$$

Under isotopy, the situation is no longer that trivial. In fact, assume the simple diagonal isotopy

$$T = \text{diag.}\,(b_1{}^2(\hat{r}), b_2{}^2(\hat{r})) > 0. \qquad (57.9)$$

Then Eq.s (5.7.5) yield into the form

$$\hat{R}^k(2) - \hat{R}^k(1) = \hat{\int}_1^2 \left( \frac{\partial R^k(r)}{\partial r^1} b_1{}^2(\hat{r})\, \hat{d}r^1 + \frac{\partial R^k(r)}{\partial r^2} b_2{}^2(\hat{r})\, \hat{d}r^2\right. \qquad (5.7.10)$$

**ISOPARALLEL TRANSPORT**

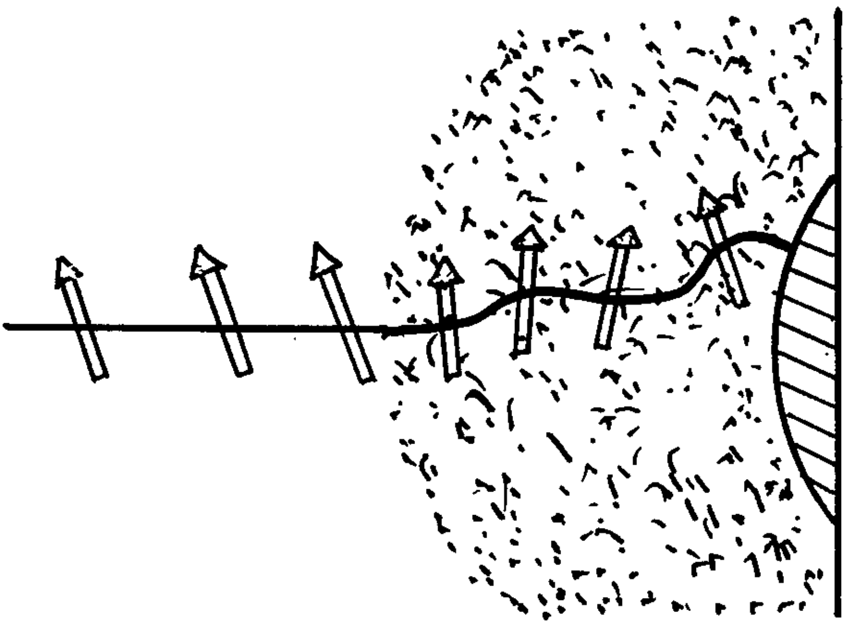

FIGURE 5.7.1: A schematic view of the isotopic representation of parallel transport in isoriemannian space. Consider, say, a rocket under parallel transport in empty

space, e.g., due to free fall toward Earth. When penetrating within physical media, the same object is, first, twisted depending on its shape, and then moves along an anomalous trajectory. The isoriemannian geometry permits the geometrization of the latter motion via the *isoparallel transport*. Its understanding requires the knowledge that the anomalous trajectory depicted in the figure occurs in our space, while in isospace the object continues with exactly the same original trajectory.

The *irreducibility* of the notion of isoparallel transport to the conventional notion can be illustrated even in the case of null curvature. In fact, consider for simplicity the isominkowski-space $M(x,\hat{\eta},\hat{R})$ with local coordinates $x = (x^\mu)$, $\mu = 1, 2, 3, 4$, with constant diagonal isotopy

$$\hat{\eta} = T\eta, \; T = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2, b_4{}^2) > 0. \tag{5.7.11}$$

and introduce the redefinitions $\hat{x}^\mu = b_\mu{}^2 x^\mu$ (no sum), $X^\mu(x(\hat{x})) = \hat{X}^\mu(\hat{x})$.
Then Eq.s (5.7.5) become

$$\int_1^2 [\frac{\partial X^\mu(x)}{\partial x^\alpha} b_\alpha{}^2 dx^\alpha \equiv \int_1^2 \frac{\partial \hat{X}^\mu(\hat{x})}{\partial \hat{x}^\alpha} b_\alpha{}^2 d\hat{x}^\alpha, \tag{5.7.12}$$

namely, the isotopy persists even under the simplest possible constant isotopy (5.7.11), thus confirming the achievement of a novel geometrical notion.

By submitting the conventional treatment (ref. [4], Sect. 3.7) to isotopies, one can identify the *integrability conditions for the existence of isoparallelism* result in the condition

$$\frac{\partial X^i}{\partial x^s \partial x^t} = - \frac{\partial \hat{\Gamma}^2{}_r{}^i{}_s}{\partial x^t} T^r{}_p X^p + \hat{\Gamma}_r{}^i{}_s T^r{}_q \hat{\Gamma}^2{}_m{}^q{}_t T^m{}_n X^n +$$

$$+ \hat{\Gamma}^2{}_r{}^i{}_s \frac{\partial T^r{}_p}{\partial x^t} X^p = - \frac{\partial^2 X^i}{\partial x^t \partial x^s} = \frac{\partial \hat{\Gamma}^2{}_r{}^i{}_t}{\partial x^s} T^r{}_p X^p +$$

$$+ \hat{\Gamma}^2{}_r{}^i{}_t T^r{}_p \hat{\Gamma}^2{}_m{}^p{}_s T^m{}_n X^n + \hat{\Gamma}^2{}_r{}^i{}_t \frac{\partial T^r{}_p}{\partial x^s} X^p \tag{5.7.13}$$

from which the following property holds.

**Lemma 5.7.2** [loc. cit.]: *Necessary and sufficient conditions for the existence of an isoparallel transport of an isovector* $X^i$ *on an n-dimensional isoriemannian isospace* $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ *are that all the following* *equations hold*

$$\hat{R}_{l\ hk}^{\ i} T^{l}_{s} X^{s} = 0, \tag{5.7.14}$$

*where* $\hat{R}_{r\ pq}^{\ i}$ *is the isocurvature , Eq.s (5.6.12).*

The re-emergence of the isocurvature tensor as part of the integrability conditions of isoparallel transport, can then be considered as a confirmation of the achievement of a novel geometrical notion.

We now pass to the *isogeodesics motion* . Let s be an invariant parameter and consider the tangent $\dot{x}^i = \hat{\partial} x^i / \hat{\partial} s$ of the curve C on an n-dimensional isoriemannian space $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$. Its absolute isodifferential is given by

$$\hat{D}\dot{x}^i = \hat{\partial}\dot{x}^i + \hat{\Gamma}^{2\ i}_{\ r s} T^{r}_{\ p} \dot{x}^{p} T^{s}_{\ q} \hat{\partial} x^{q}. \tag{5.7.15}$$

In accordance with Definition 5.6.3, $\hat{D}x^i$ remains isoparallel along C iff $\hat{D}\dot{x}^i = 0$. We can therefore introduce the following

**ISOGEODESIC MOTION**

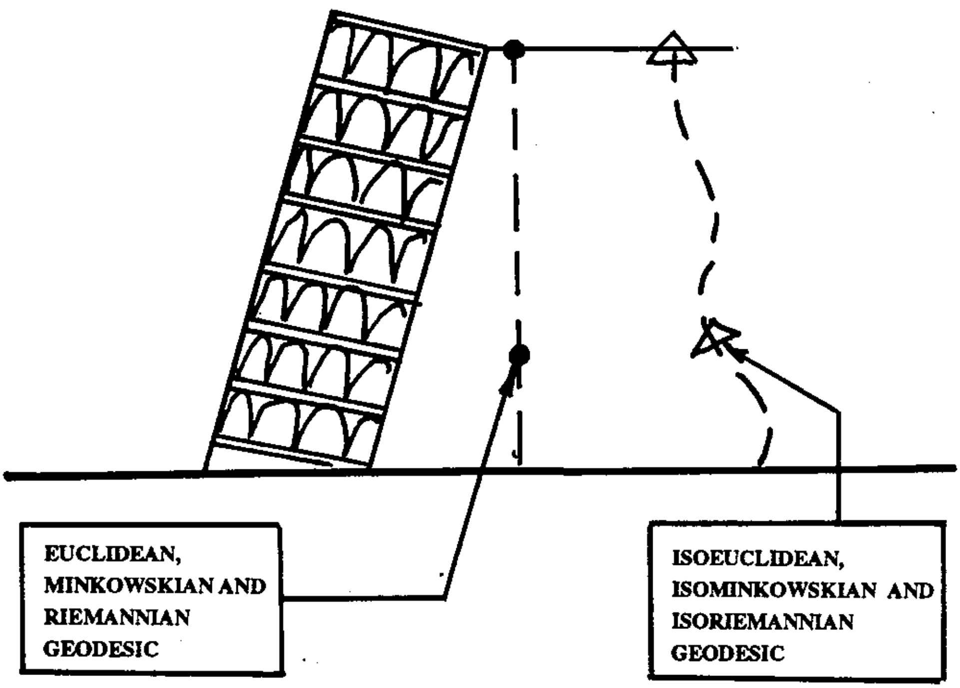


FIGURE 5.7.1: The birth of the notion of geodesic motion can be seen in Galilei's

historical conception of uniform motion in vacuum, i.e., via the celebrated Galilei's boosts

$$r'^k = r^k + t^\circ v^{\circ k}, \qquad p'_k = p_k + m v^\circ_k , \tag{1}$$

which can be formulated in terms of the contemporary modular action

$$T(v^\circ)\, r^k = r^k + t^\circ v^{\circ k}, \quad T(v^\circ)\, p_k = p_k + m v^\circ_k . \tag{2}$$

As well known, Galilei established the above law by ignoring the friction due to the air. Our studies essentially aim at the achievement of a geodesic characterization of the motion of free objects within physical media in such a way as to preserve the original axioms of the free motion in vacuum.

Stated in different terms, the understanding of the content of this chapter can be reduced to the understanding that the irregular trajectory of this figure describing the free fall of an objective under the resistive force due to the atmosphere does indeed verify the same geodesic axioms of Galilei's free fall in the absence of the atmosphere. In fact, in isoeuclidean space it is a straight isoline (Sect. 5.2), exactly as the trajectory in the absence of air, and a similar occurrence holds for curved spaces.

The fundamental tool is provided by the isospaces. In fact, we represent the transition from motion in vacuum to motion within a physical medium via the transition from conventional Euclidean, Minkowskian or Riemannian space to the corresponding isoeuclidean, isominkowskian and isoriemannian spaces, respectively. By recalling that the conventional spaces provide a geometrization of the vacuum (empty space), one can then confirm the isogeometrization of interior physical media of Sect.s 5.2 and 5.3.

This yields the most general possible, nonlinear, nonlocal and noncanonical generalization of laws (1) in $\hat{E}(r,\hat{\delta},\hat{\Re})$

$$r'^k = r^k + t^\circ v^\circ B^{k-2}(t,r,p,...), \qquad p'_k = p_k + m v^\circ B_k^{-2}(t,r,p,...), \tag{3}$$

and represented via the isotopic group action (see ref. [20] for a detailed classical treatment and Vol. II for the operator counterpart)

$$\hat{T}(v^\circ) * r^k = r^k + t^\circ v^{\circ k} B^{k-2}, \quad \hat{T}(v^\circ) * p_k = p_k + m v^\circ_k B_k^{-2}. \tag{4}$$

where the B's are certain nonlinear-nonlocal functions computable from the knowledge of the isounit.

The arbitrariness of the isounits, that is, of the B-function then illustrate the "direct universality" of the *isogalilean relativity* for the form-invariant description of interior trajectories. The preservation of the original Galilean axioms can also be seen by nothing that isoboosts (4) form an isogroup (Sect. 4.5), e.g., the composition of two successive Galilean boosts

$$T(v^\circ)\, T(v^{\circ\prime}) = T(v^\circ + v^{\circ\prime}), \tag{5}$$

is lifted into the isocomposition of two isoboosts

$$\hat{T}(v^\circ) * \hat{T}(v^{\circ\prime}) = \hat{T}(v^\circ + v^{\circ\prime}) . \qquad (6)$$

The abstract identity of the Galilean and isogalilean relativities then follows from the manifest abstract identity of group (5) with its isotopic covering (6), that is, *isogeodesics in isospace coincide with the original geodesics in vacuum.* The same result can be directly reached via principle (5.7.17) which shows that, jointly with the deformation $b_k{}^2$ along the k–axis, the unit along the same axis is deformed of the inverse amount $b_k{}^{-2}$.

**Definition 5.7.2** [loc. cit.]: *The "isogeodesics" of an n–dimensional isoriemannian manifold of Class I, $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, are the solutions of the differential equations*

$$\frac{\hat{d}^2 x^i}{\hat{d}s^2} + \hat{\Gamma}^{2}{}_{r}{}^{i}{}_{s}(x, \dot{x}, \ddot{x}..)\, T^r{}_p(x, \dot{x}, \ddot{x}..)\, \frac{\hat{d}x^p}{ds}\, T^s{}_q(x, \dot{x}, \ddot{x}.)\, \frac{\hat{d}x^q}{ds} = 0 . \qquad (5.7.16)$$

It is a simple but instructive exercise to prove the following

**Lemma 5.7.2** [loc. cit.]: *The isogeodesics of an n–dimensional isoriemannian space $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ are the curves verifying the principle*

$$\delta\int \hat{d}s = \delta\int [\, \hat{g}_{ij}(x, \dot{x}, \ddot{x}..)\, \hat{d}x^i\, \hat{d}x^j\, ]^{\frac{1}{2}} = 0. \qquad (5.7.17)$$

We discover in this way a new important role of the isometric essentially similar to the corresponding role of conventional metric in geodesic motion. Also, the appearance of the isometric in the variational principle characterizing isogeodesic motion is a confirmation of the achievement of a novel geometry.

## APPENDIX 5.A: ELEMENTS OF THE SYMPLECTIC GEOMETRY

In this appendix we shall outline the rudiments of the conventional symplectic geometry from ref.s [3,4,6] in its local–differential, canonical as well as Birkhoffian versions. The presentation will then result to be useful for reader not familiar with the field, not only for the nonlocal–integral extension of this chapter, but also for the isotopies of symplectic quantization of Vol. II.

As done in Sect. 5.4, all quantities are assumed to verify the needed continuity conditions, e.g., of being of Class $\mathcal{C}^\infty$, and all neighborhoods of given points are assumed to be star–shaped, or have a similar topology also ignored hereon for brevity.

Let M(R) be an n–dimensional manifold over the reals R(n,+,×) . A *tangent vector* $X_m$ at a point $m \in M(R)$ is a linear function defined in the neighborhood of m with values in R satisfying the rules

$$X_m(\alpha f + \beta g) = \alpha X_m(f) + \beta X_m(g), \qquad (5.A.1a)$$

$$X_m(fg) = f(m) X_m(g) + g(m( X_m(f), \qquad (5.A.1b)$$

for all $f, g \in C^{\infty}(M)$, $\alpha, \beta, \in R$.

The *tangent space* $T_mM$ at m is the vector space of all tangent vectors at m. The *tangent bundle* is the 2n–dimensional space TM = $U_m$ $T_mM$ equipped with a structure (see below). The cotangent bundle $T^*M$ is the dual of TM given by the space of all linear functional on TM also equipped with a structure.

Let $x = \{x^1, ..., x^n\}$ be a local chart in the neighborhood of m. Then it can be shown that the ordered set dx forms a basis of $T^*M$, while $\partial/\partial x$ forms a basis of TM. An element $\theta \in T^*M$ and $\chi \in TM$ can the be written in local coordinates

$$\theta = \theta_i(m)\, dx^i, \quad X = X^i(m)\, \partial / \partial x^i, \qquad (5.A.2)$$

θ is then called the *canonical form.* The cotangent bundle $T^*M$ equipped with θ is at times denoted $T^*M_1(R)$. The *fundamental (canonical) symplectic form* is then given by the two–form

$$\omega = d\theta, \qquad (5.A.3)$$

which is nowhere degenerated, exact and therefore closed; i.e., such that $d\omega \equiv 0$. The manifold $T^*M(R)$, when equipped with two–form ω becomes an (exact) *symplectic manifold* $T^*M_2(R)$ in canonical realization. The *symplectic geometry* is the geometry of symplectic manifolds as characterized by exterior forms, Lie's derivative, etc.

Let H be a function on $T^*M_2(R)$ called the *Hamiltonian.* A vector–field X on $T^*M_2(R)$ is called a *Hamiltonian vector–field* when it verifies the condition

$$X \rfloor \omega = -\, dH. \qquad (5.A.4)$$

The above equation provides a global, coordinate–free characterization of the conventional *Hamilton's equations* (those without external terms) for the case of *autonomous systems;* i.e., systems without an explicit dependent in the independent variable (time t).

Finally, we recall that the *Lie derivative* of a vector–field Y with respect to the vector field X on $T^*M_2(R)$ can be defined by

$$L_X Y = [X,Y], \tag{5.A.5}$$

where [X, Y] is the canonical commutator. The case of *nonautonomous systems* (those with an explicit dependence on time) requires the further extension to the *contact geometry* (see, e.g., ref. [3]). However, the Lie content is contained in the symplectic part of the geometry.

The Birkhoffian generalization of the above canonical geometry is straightforward, and was worked out in ref.s [5,6]. Introduce in the same cotangent bundle $T^*M_1(R)$ the most general possible one-form $\Theta$, called the *Birkhoffian* or *Pfaffian one-form* . The *Birkhoffian two-form* is then given by

$$\Omega = d\Theta\,, \tag{5.A6}$$

under the condition that it is nowhere degenerate. $\Omega$ is exact by construction and therefore closed, that is, symplectic. The manifold $T^*M(R)$, when equipped with the two-form $\Omega$, becomes an *exact, Birkhoffian, symplectic manifold* $T^*M_2(R)$.

Let B be another function on $T^*M_2(R)$ called the *Birkhoffian.* Then, a non-Hamiltonian vector-field $\hat{X}$ on $T^*M_2(R)$ is called a *Birkhoffian vector-field* when it verifies the property

$$\hat{X} \rfloor \Omega = - dB. \tag{5.A.7}$$

which provides a global, coordinate-free characterization of *Birkhoff's equations for autonomous systems.*

Similarly, we recall that the *Lie-isotopic derivative* of a vector-field $\hat{Y}$ with respect to a *nonhamiltonian* vector field $\hat{X}$ [5,6] can be written

$$\hat{L}_{\hat{X}}\hat{Y} = [\hat{X} \hat{,} \hat{Y}], \tag{5.A.8}$$

where the brackets are now Birkhoffian (see below for the explicit form).

The realization of the above global structures in local coordinates is straightforward. Interpret the space M(R) as an Euclidean space $E(r,\delta,R)$ with local coordinates $r = (r_i)$, i= i, 2, ..., n. Then, the cotangent bundle $T^*M$ becomes $T^*E(r,\delta,R)$ with local coordinates $(r,p) = (r_i,\ p_i)$, where $p = dr/dt$ represents the tangent vectors, and we ignore for simplicity of notation the distinction between contravariant and covariant indices in Euclidean spaces (but not in the cotangent bundle). The canonical one-form (5.A.2) then admits the local realization

$$\theta = p_i\, dr_i. \tag{5.A.9}$$

The Hamiltonian two-form (5.A.3) admits the realization

$$\omega = d\theta = dp_i \wedge dr_i, \tag{9.10}$$

from which one can easily verify that $d\omega \equiv 0$. A vector–field can then be written

$$X = A_i(r,p)\, \partial / \partial r_i + B_i(r,p)\, \partial / \partial p_i, \tag{5.A.a}$$

$$A_i\, dr_i + B_i\, dp_i = -dH, \tag{5.A.b}$$

which can hold iff Hamilton's equations are verified, i.e.,

$$\frac{dr_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = - \frac{\partial H}{dr_i}, \tag{5.A.12}$$

Finally, Lie's derivative (5.A.4) admits the simple realization

$$L_X Y = [X,Y] = \frac{\partial X}{\partial r_i}\frac{\partial Y}{\partial p_i} - \frac{\partial Y}{\partial r_i}\frac{\partial X}{\partial p_i}, \tag{5.A.13}$$

where one recognizes in the commutator the familiar Poisson brackets.

The realization of the Birkhoffian generalization of the above structures requires the introduction of the unified notation

$$a = (a^\mu) = (r, p) = (r_i, p_i), \quad \mu = 1, 2, ..., 2n, \quad i = 1, 2, ..., n, \tag{5.A.14}$$

where we preserve the distinction between contravariant and covariant indices. The canonical one–form can then be rewritten

$$\theta = R^\circ_\mu\, da^\mu \equiv p_i\, dr_i, \quad R^\circ = (p, 0), \tag{5.A.15}$$

and Hamiltonian two–form (5.A.10) becomes

$$\omega = d\theta = \tfrac{1}{2}\, \omega_{\mu\nu}\, da^\mu \wedge da^\nu \equiv dp_i \wedge dr_i, \tag{5.A.16}$$

where $\omega_{\mu\nu}$ is the *covariant, canonical, symplectic tensor* (5.A.15), i.e.,

$$(\omega_{\mu\nu}) = \left( \frac{\partial R^\circ_\nu}{\partial a^\mu} - \frac{\partial R^\circ_\mu}{\partial a^\nu} \right) = \begin{pmatrix} 0_{n\times n} & -I_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{pmatrix} \tag{5.A.17}$$

A vector–field can then be written

$$X = X_\mu(a)\,\partial/\partial a^\mu. \tag{5.A.18}$$

The conditions for a Hamiltonian vector-field become

$$\omega_{\mu\nu}\, X^\mu\, da^\mu = -dH, \tag{5.A.19}$$

and can hold iff

$$X = X_\mu \frac{\partial}{\partial a^\mu} = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu} \frac{\partial}{\partial a^\mu}, \tag{5.A.20}$$

where

$$\omega^{\mu\nu} = (|\omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.A.21}$$

namely, iff Hamilton's equations (5.A.12) hold, which in the unified notation can be written

$$\dot{a}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial a^\nu}. \tag{5.A.22}$$

Finally, Lie's derivative becomes

$$L_X Y = [X,Y] = \frac{\partial X}{\partial a^\mu}\, \omega^{\mu\nu}\, \frac{\partial Y}{\partial a^\nu}, \tag{5.A.23}$$

The transition to the Birkhoffian realization [6] is now straight-forward [5,6]. In fact, it merely requires the transition from the canonical quantities $R°(a) = (p, 0)$ to arbitrary quantities $R(a)$ on $T^*E_1(r,\delta,R)$ under which the Birkhoffian one-form (5.A.5) assumes the realization

$$\Theta = R_\mu(a)\, da^\mu, \tag{5.A.24}$$

while the Birkhoffian two-form (5.A.6) becomes

$$\Omega = d\Theta = \tfrac{1}{2}\, \Omega_{\mu\nu}(a)\, da^\mu \wedge da^\nu. \tag{5.A.25}$$

where $\Omega_{\mu\nu}$ is the *(covariant) symplectic Birkhoff's tensor*

$$\Omega_{\mu\nu}(a) = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu} \tag{5.A.26}$$

A Birkhoffian vector-field $\hat{X}$ can no longer be decomposed in the simple form (5.A.11), but can be written

$$\hat{X} = X^\mu \ \partial / \partial a_\mu . \tag{5.A.27}$$

The conditions for a vector-field $\hat{X}$ to be Birkhoffian, Eq.s (5.A.7), then become

$$\hat{X} \rfloor \Omega = \Omega_{\mu\nu} \ \hat{X}^\nu \, da^\mu = - dB, \tag{5.A.28}$$

and they hold iff

$$\hat{X} = \hat{X}^\mu \frac{\partial}{\partial a^\mu} = \Omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} \ \frac{\partial}{\partial a^\mu}, \tag{5.A.29}$$

where

$$\Omega^{\mu\nu} = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu}, \tag{5.A.30}$$

which can hold iff the *autonomous Birkhoff's equations* hold, i.e.,

$$\dot{a}^\mu = \hat{X}^\mu = \Omega^{\mu\nu}(a) \frac{\partial B(a)}{\partial a^\nu}. \tag{5.A.31}$$

Similarly, the Lie-isotopic derivative (5.A.8) assumes the realization

$$\hat{L}_{\hat{X}} \hat{Y} = [\hat{X} \,\hat{,}\, \hat{Y}] = \frac{\partial \hat{X}}{\partial a^\mu} \ \Omega^{\mu\nu}(a) \ \frac{\partial \hat{Y}}{\partial a^\nu}, \tag{5.A.32}$$

For additional aspects, the reader may consult reef. [6], the appendices of Ch. 4.

Note that an arbitrary vector-field $\hat{X}$ is not Hamiltonian in a given local chart. A central result of ref. [6] can be reformulated as follows

**Theorem 5.A.1 – Direct universality of the symplectic geometry for local nonhamiltonian Newtonian systems** [6]: *An arbitrary, local-differential, nonhamiltonian, analytic and regular vector-field $\hat{X}$ on a given chart on* $T^*M_2(r,R)$ *always admits in a star-shaped neighborhood of the local variables a direct representation as a Birkhoffian vector-field, i.e., a representation via Birkhoff's equations directly in the chart* *considered.*

The physical implications are as follows. When considering conservative-potential systems of the exterior dynamical problem (Ch. I.1), the vector-fields are evidently Hamiltonian in the frame of the experimenter. However, when considering the nonconservative systems of the interior dynamical problem, the vector-fields are generally nonhamiltonian in the frame of the experimenter.

Now, under sufficient topological conditions, the *Lie-Koening theorem* (see ref. [6] and quoted literature) ensures that a local-differential nonhamiltonian vector-field can always be transformed into a Hamiltonian form under a suitable change of coordinates.

However, since the original vector-field is nonhamiltonian by assumption, the transformations must necessarily be *noncanonical* and *nonlinear*, thus creating evident physical problems, e.g., conventional relativities become inapplicable because turned into *noninertial* formulations.

This creates the need of the "direct representation" of the physical systems considered; that is, their representation, first, in the frame of the experimenter, as per Theorem 5.A.1. Once this basic task is achieved, then the judicious use of the transformation theory may have some physical value.

Intriguingly, the identification of the Lie-Koening transformation $a \Rightarrow a'$ turning nonhamiltonian systems $\hat{X}(a)$ into Hamiltonian forms $\hat{X}(a(a')) = X(a')$, implies the Birkhoffian representation of Theorem 5.A.1 in the a-frame of the observer. In fact, Birkhoff's equations (5.A.31) in the a-frame can be characterized precisely via a *noncanonical* transformation $a' \Rightarrow a$ of Hamilton's equations (5.A.22) in the a'-frame, i.e.,

$$\omega_{\mu\nu}\dot{a}'^{\nu} - \frac{\partial H(a')}{\partial a^{\mu}} = \frac{\partial a^{\rho}}{\partial a'^{\mu}}\left[\Omega_{\rho\sigma}(a) - \frac{\partial B(a)}{\partial a^{\rho}}\right] = 0, \tag{5.A.33a}$$

$$H(a'(a)) = B(a), \tag{5.A.33b}$$

(see ref. [6], p.130 for details).

As an introduction to the covering isosymplectic geometry (Sect. 5.4), the above canonical and Birkhoffian forms can be expressed in a yet more general way. Consider again the original cotangent bundle $T^*M(R)$, and let

$$I^{\circ} = (I_{n\times n}) = \text{diag.}(1, 1, \ldots, 1) \equiv T^{\circ\,-1} \tag{5.A.34}$$

be its unit. Then, the canonical one form (5.A.2) can be identically written in terms of the factorization

$$\theta \equiv \hat{\theta}^{\circ} = \theta \times T^{\circ} : T^*\hat{M}_I{}^{\circ} \Rightarrow T^*(T^*\hat{M}_I{}^{\circ}), \tag{5.A.35}$$

$$\theta \equiv \hat{\theta}^{\circ} = \theta \times T^{\circ} \; : \; T^{*}\hat{M}_{1}{}^{\circ} \Rightarrow T^{*}(T^{*}\hat{M}_{1}{}^{\circ}), \tag{5.A.35}$$

while the canonical two–form (5.A.3) becomes

$$\omega \equiv \hat{\omega}^{\circ} = d\hat{\theta}^{\circ} = (d\theta) \times T^{\circ} + \theta \, dT^{\circ} \equiv \omega \times T^{\circ} \tag{5.A.36}$$

This implies that, in the realization $T^{*}E(r,\delta,R)$ of $T^{*}M(R)$ with local chart a = (r, p), we can write

$$\hat{\omega}^{\circ}{}_{\mu\nu} = T^{\circ}{}_{\mu}{}^{\alpha} \, \omega_{\alpha\nu}, \tag{5.A.37}$$

Then, its contravariant versionis exhibited in the Lie–tensor of the theory,

$$\hat{\omega}^{\circ\mu\nu} = \omega^{\mu\alpha} \, I^{\circ}{}_{\alpha}{}^{\nu}. \tag{5.A.38}$$

The transition to the isosymplectic geometry in Birkhoff–isotopic realization is then performed by assuming that the isotopic element and unit are no longer the trivial unit, but arbitrary integro–differential quantities.

In the latter generalization one central property persists: *the transition from the canonical to the Birkhoffian and Birkhoffian–isotopic formulations requires noncanonical transformations.* This is the geometric–analytic counterpart of the corresponding algebraic property. In fact, *the transition from the classical (operator) formulation of Lie's theory to its isotopic covering necessarily requires noncanonical transformations (nonunitary transformations)*

the above results imply that *quantum mechanics and its covering hadronic mechanics are inequivalent because not interconnected by a unitary transformations* (see Vol. II for details).

In closing we mention the so–called *multisymplectic generalization* of the content of this appendix, as presented in the recent monograph by Sardanashvili [42] and related jet manifolds which have intriguing possibilities for further isotopic formulation and application to interior dynamical problems.

## APPENDIX 5.B: GRAVITATION IN ISOMINKOWSKIAN SPACE

Isotopic techniques permit *novel* approaches to gravitation, i.e., approaches not permitted by conventional Riemannian methods. One of them is *the equivalent study of gravitation on a flat geometry.*

This approach is not a mere mathematical curiosity, but resolves a rather old problematic aspect of current gravitational theories: the absence of weight in relativistic theories. Consider a test body experiencing a gravitational field at a space–time point x in a Riemannian space $\Re(x,g,R)$. As well known [10,11,37], *gravitation is entirely represented by the curvature in current theories,* i.e., by the metric g(x) (for null total charge). In passing at the tangent Minkowski space $M(x,\eta,R)$ at the same point x, all gravitational effects disappear (equivalence principle), which is contrary to experimental evidence [21]. Weight is preserved in current theories in flat spaces, but only in the limit into the *Euclidean* space.

In the physical reality, weight is present irrespective of our treatment, whether nonrelativistic, relativistic or gravitational. Any consistent treatment of gravitation must therefore have a well defined Minkowskian counterpart.

The above problematic aspect of current theories is resolved by the isotopies because of the geometric equivalence between the Riemannian and isominkowskian spaces of Sect. 3.7,

$$\Re(x,g,R) \approx \hat{\Re}(x,g,\hat{R}) \equiv \hat{M}(x,\hat{\eta},\hat{R}) \sim M(x,\eta,R), \qquad (5.B.1a)$$

$$g(x) = T(x)\,\eta = \hat{\eta}, \qquad \hat{1} = [\,T(x)\,]^{-1}\,. \qquad (5.b.1b)$$

where $\hat{1}$ (T) is called the *gravitational isounit (isotopic element).*

In fact, all gravitational theories admit the decomposition $g = T\eta$ with $T > 0$ as a necessary condition to be locally Minkowskian. Then $\hat{1} > 0$ and the equivalence chain (5.B.1a) follows.

Current gravitational theories are formulated in a *curved space* with metric g(x) with respect to the *conventional unit* $I = \text{diag.}\,(1,\ 1,\ 1,\ 1)$. Isotopic theories permit the treatment of exactly the same metric $g(x) \equiv \hat{\eta}(x)$ although referred to the gravitational isounit $\hat{1}$ in the isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$.

Note that curvature is entirely contained in the isotopic element T(x) of decomposition $g(x) = T(x)\eta$. The isominkowskian treatment therefore implies the study of the curvature via $g \equiv T\eta$ at x, while assuming at the same point x an isounit which is the "inverse of the curvature", $\hat{1}(x) = [T(x)]^{-1}$. This is precisely the mechanism that renders the treatment of gravitation locally flat or, more technically in our terminology, *locally isoflat.*

It is an instructive exercise for the interested reader to reconstruct in $\hat{M}(x,\hat{\eta},\hat{R})$ all properties of the Riemannian geometry, including Ricci lemma, Einstein's tensor, field equations, etc. One can therefore see in this way that all the results on $\Re(x,g,R)$ equally hold on $\hat{M}(x,\hat{\eta},\hat{R})$.

Besides resolving the problematic aspect of the "disappearance of weight" at the tangent Minkowski space, isotopic methods permit a *novel* approach to gravitational singularities, which now become *the singularities of the isounit,*

$$T(x) \rightarrow 0\,, \qquad \hat{1}(x) \rightarrow \infty\,, \qquad (5.B.2)$$

or *the singularities of the isotopic element,*

$$T(x) \rightarrow \infty\,, \qquad \hat{1}(x) \rightarrow 0\,. \qquad (5.b.3)$$

As an example, the celebrated Schwartzchild's line element in spherical polar coordinates admits the isotopic factorization into

$$T = \text{diag.} \{ (1 - 2M/r)^{-1}, r^2, r^2 \sin^2\theta, (1 - 2M/r) \} . \tag{5.B.4}$$

where one should keep in mind our ordering (+, +, +, –). We then have the following

**Proposition 5.B.1** [21]: *The Schwartzchild's singularity at the horizon r = 2M is a zero of the isounit, while its singularity at the origin r = 0 is a zero of the isotopic element.*

The reader should be aware that the above novel perspectives on gravitational collapse are studied merely as a basis for the intended studies, their treatment via the interior nonlocal isoriemannian geometries. In fact, the equivalence chain (5.B.1a) can also be formulated at the fully isotopic level of Class I

$$\hat{R}(x,\hat{g},\hat{R}) \approx \hat{M}(x,\hat{\eta},\hat{R}), \tag{5.B.5a}$$

$$\hat{g} = T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)\, g(x) \equiv T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, ...)\, \eta. \tag{5.B.5b}$$

As a result, *gravitational singularities on the horizon are the zeros of the general isotopic element of the isoriemannian geometry*

$$T(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, .....) = 0 , \tag{5.B.6}$$

*while the singularities at the origin ate the zeros of the isounit*

$$\hat{1}(s, x, \dot{x}, \ddot{x}, \mu, \tau, n, .....) = 0 . \tag{5.B.7}$$

As a matter of fact, the latter reformulation is done precisely to study the contributions to singularities expected from nonlinear–nonlocal–nonlagrangian interior effects.

The broadening of the scientific horizon from Eq.s (5.B.2)–(5.B.3) to (5.B.6)–(5.B.7) )is evident, as we illustrate in more detail in Vol. II and III.

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# 6: FUNCTIONAL ISOANALYSIS

## 6.1: STATEMENT OF THE PROBLEM

The transition from Newtonian to quantum mechanics implies the preservation of the basic mathematical notions such ash numbers, angles, metric spaces, special functions, etc., and only the reformulation of observable on a Hilbert space.

The transition from quantum to hadronic mechanics is much deeper because it requires a suitable generalization of all basic mathematical notions of quantum mechanics, beginning with numbers, angles, metric spaces, special functions, etc., and then passing to a generalization of Hilbert spaces themselves.

The above occurrence can be expressed by the fact that *functional analysis remained unchanged in the transition from classical to quantum mechanics.* On the contrary, *the transition from quantum to hadronic mechanics requires a structural generalization of functional analysis into a new discipline called "functional isoanalysis".*

The need for an isotopic lifting of numbers, angles and trigonometric functions has been indicated earlier in this volume, jointly with that for the generalization of other ordinary functions, such as exponentiation, hyperbolic functions, logarithm, etc. The need for a lifting of special functions is then consequential, as studied in this chapter.

Let us identify here the need for lifting Hilbert spaces themselves. As recalled earlier, hadronic mechanics was originally submitted [1] in 1978 as an isoassociative enveloping algebra $\xi_T$ (Sect. 4.3)[43] of operators A, B, ... with isotopic product

$$\xi_T = A * B := A\,T\,B, \quad \hat{1} = T^{-1}, \qquad (6.1.1)$$

on a *conventional* Hilbert space $\mathcal{H}$ with elements $\psi$ , $\phi$ , ... with familiar inner

[43] For clarity due to the subsequent analysis, in this chapter we shall identify with a subscript the isotopic element of a given structure, such as in $\xi_T$.

product

$$\mathcal{H}: \ <\psi|\phi> \ = \ \int d^3r\, \psi^\dagger(r)\, \phi(r) \quad \in \ C(c,+,\times)\,. \tag{6.1.2}$$

which is indeed a mathematically correct formulation.

However, this original formulation had the physical problematic aspect that operators of the original envelope $\xi$ which are Hermitean on $\mathcal{H}$ did not remain necessarily Hermitean under lifting $\xi \rightarrow \hat{\xi}_T$. This is due to the fact that, as we shall see in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\mathcal{H}$ is given by

$$H^{\hat{\dagger}} \ = \ T\ H^\dagger\ T^{-1}\,. \tag{6.1.3}$$

where $H^\dagger$ is the conventional Hermiticity. Since the operators T and H do not necessarily commute, we have in general that $H \neq H^\dagger$.

This implied that observable of quantum mechanics, such as the total energy H, the linear momentum p, etc., do not necessarily remain observable under under isotopies $\xi \rightarrow \hat{\xi}_T$ over $\mathcal{H}$.

This clearly called for an appropriate generalization of the underlying Hilbert space $\mathcal{H}$ in such a way to preserve observability under isotopies. These studies were initiated by this author immediately after proposal [1], e.g., in ref. [2] of 1979. The resolution of the problem received a first rigorous treatment by Myung and Santilli in ref. [3] of 1983 via the introduction of the notion of *isotopic Hilbert space* $\hat{\mathcal{H}}_T$ or *isohilbert space* for short, which is essentially the image of $\mathcal{H}$ under the lifting of the composition

$$\begin{aligned}\hat{\mathcal{H}}_T: \ <\psi\hat{|}\phi> \ &:= \ <\hat{\psi}|T|\hat{\phi}>\hat{1} \ = \ <\hat{\psi}|*|\hat{\phi}>\hat{1} \ = \\ &= \ \hat{1}\int d^3r\, \psi^\dagger(r)\, T(t, r, \dot{r}, \ddot{r}, \ldots)\, \phi(r) \quad \in \ \hat{C}(\hat{c},+,*)\,.\end{aligned} \tag{6.1.4}$$

where, as one can see, the assumption of the positive-definiteness of the isounit (Class I) implies the preservation of the inner character of the composition and, thus, of the Hilbert character of the space $\hat{\mathcal{H}}_T$.

Isohilbert space (6.1.4) did indeed achieve the desired objective because, as we shall see better in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\hat{\mathcal{H}}_T$ coincides with the conventional Hermiticity,

$$H^{\hat{\dagger}} \ \equiv \ H^\dagger\,, \tag{6.1.5}$$

thus permitting the preservation of Hermiticity under isotopies.

The importance of this result should be indicated for readers not familiar with isotopic techniques. A central objective of hadronic mechanics is to complement conventional quantum mechanical descriptions of interacting particles at large mutual distances ( $\gg$ 1 fm), with additional internal, short range, nonlinear–nonlocal–nonhamiltonian interactions when in conditions of mutual penetration of their wavepackets at very short distances ( < 1 fm) (see Fig. I.1.1.1)

This implies that the operators such as the energy $H = K + V$ of the particle in exterior conditions does not change in the transition to the condition of total mutual immersion of the particles considered because the additional interactions have no potential by conception. Still in turn, this implies that *a necessary condition for the physical consistency of the isotopies is the preservation of the <observability> of the original energy* H, *i.e., the preservation of its Hermiticity.*

Thus, ref. [3] identified the fundamental carrier space of (the Lie–isotopic branch of) hadronic mechanics, the space $\hat{\mathcal{H}}_T$ , which fulfills the fundamental task of preserving the observability of conventional physical quantities. However, the lifting $\mathcal{H} \rightarrow \hat{\mathcal{H}}_T$ implies a structural revision of the conventional Hilbert space theory into five classes, as we shall see.

The subsequent studies by Mignani, Myung and Santilli [4] of 1983 indicated that formulation (6.1.4) is still restrictive because the enveloping isoassociative algebra (6.1.1) could be consistently formulated also in the different isospace

$$\hat{\mathcal{H}}_G : \ <\psi \hat{\uparrow} \phi> \ : \ <\hat{\psi}|G|\hat{\phi}>\hat{1} \ = <\hat{\psi}|\circ|\hat{\phi}>\hat{1} \ =$$

$$= \ \hat{1}\int d^3r\, \psi^\dagger(r)\, G(t, r, \dot{r}, \ddot{r}, ...)\, \phi(r) \quad \in \ \hat{C}(\hat{c},+,*)\,, \qquad (6.1.6)$$

where G is an operator *independent* of T. The lifting $\hat{\mathcal{H}}_T \rightarrow \hat{\mathcal{H}}_G$ implies again the general loss of Hermiticity because, as we shall see in details in this chapter, the condition of Hermiticity of an operator $H \in \hat{\xi}_T$ on $\hat{\mathcal{H}}_G$ is given by

$$H^{\hat{\dagger}} = G^{-1}\, T\, A^\dagger\, G\, T^{-1}\,, \qquad (6.1.7)$$

which includes as particular case condition (6.1.3).

Subsequent studies indicated that, despite the general loss of the original Hermiticity, the formulation of hadronic mechanics via isoenvelopes $\hat{\xi}_T$ on the isohilbert space $\hat{\mathcal{H}}_G$ with $T \neq G$ is important in certain specific cases in which the formulation on $\hat{\mathcal{H}}_T$ is not sufficient. In fact, the introduction of an isotopic element G in the Hilbert space different than T represents an additional "hidden degree of freedom" of the theory.

The motivations are linked to *the reconstruction of exact Lie symmetries at the isotopic level of hadronic mechanics when believed to be broken at the simpler quantum mechanical level.* The use of only one isotopic element T for both the envelope and the Hilbert space is sufficient for the reconstruction of the exact symmetry in a number of cases, such as the reconstruction of the exact rotational symmetry when believed to be broken by ellipsoidical deformations of the sphere [5–7], the reconstruction of the exact Lorentz symmetry when believed to be broken by signature–preserving deformations of the Minkowski metric [8], the reconstruction of the exact isospin symmetry in nuclear physics with equal proton and neutron masses in isospace, and others.

However, there exist cases in which one sole degree of freedom is insufficient, and two different isotopic elements T and G are needed. This is the case for the ongoing attempts (see the initial effort [9] studied in more detail in Vol. II) to *reconstruct parity at the isotopic level as an exact symmetry for "weak" interactions* via the embedding of all symmetry breaking terms in the isotopic elements.

The technical issue is the identification of which isotopic element should incorporate all symmetry violating terms. Recall from Sect. I.4.5 that in the lifting of continuous symmetries we have the appearance of the isotopic element T in the isoexponentiation. Thus, *the embedding of the symmetry breaking terms in the isotopic element* T *of the isoenvelope* $\hat{\xi}_T$ *and isofields* $\hat{F}_T$ *is generally sufficient for the reconstruction of exact "continuous" symmetries.*

The case for discrete transformation is different because they admit no isoexponentiation, and actually admit the reduction to the corresponding conventional transformations (Sect. I.4.7), e.g.,

$$\hat{\pi} * \psi(r) = \pi\,\psi(r) = \psi(-r), \qquad \hat{\pi} = \pi\,T^{-1}. \tag{6.1.8}$$

The general insuffiency of the isotopic element T is then evident. As a result, *the reconstruction of exact "discrete" symmetries generally requires the embedding of the symmetry breaking terms in both the isotopic element* T *of the isoenvelopes* $\hat{\xi}_T$ *and of the isofield* $\hat{F}_T$ *as well as in the isotopic element* G *of the isohilbert space* $\hat{\mathcal{H}}_G$.

In summary, the part of functional isoanalysis dealing with isohilbert spaces implies a rather broad lifting of conventional quantum mechanical formulations consisting of a double generalization, the first via the same isotopic element of the envelope and the second based on the differentiation between the isotopy of the envelope and that of the Hilbert space.

We now pass to a few comments on the lifting of the remaining aspects of functional analysis. Recall that the first step that lead to hadronic mechanics was the isotopy of the Poincaré–Birkhoff–Witt theorem resulting in a generalized notion of eponentiation (Sect. I.4.3).It was then known since the original proposal [1] that the isotopies of the enveloping operator algebra, $\xi \rightarrow \hat{\xi}_T$, imply a generalization of all familiar structures of quantum mechanics such as Dirac's δ–function, the Fourier transforms, Gauss distributions, etc.

A first formulation of the isotopic δ–function appeared in ref. [3], while its systematic study was presented in memoirs [10,11], jointly with the first formulations of the isotopies of Fourier series and transforms isotopies studied in this chapter.

The full implications of these studies for conventional functional analysis (see, e.g., ref.s [12–13] and quoted references) was however identified only recently by Kadeisvili [14] who understood that *the isotopies of fields* $F(\alpha,+,\times) \rightarrow \hat{F}_T(\hat{\alpha},+,*)$, *enveloping algebras* $\xi \rightarrow \hat{\xi}_T$ *and Hilbert spaces* $\mathcal{H} \rightarrow \hat{\mathcal{H}}_T$ *imply a nontrivial, nonlinear–nonlocal–noncanonical isotopic generalization of the totality of functional analysis, that is, not only of square integral, Banach, Hilbert and other spaces, but also of conventional special polynomials (such as the Legendre polynomial), special functions (such as Bessel and Legendre functions), transforms (such as Fourier and Laplace transform), etc.* In fact, the terms "functional isoanalysis" appeared for the first time in ref. [14].

The mathematical relevance of these isotopies will be evident during the analysis of this chapter. Their physical relevance can be best illustrated with the fact that, in the subsequent paper [15], Kadeisvili reinspected the isotopies of the Fourier transforms of ref. [10] and discovered that they imply a necessary generalization of Heisenberg's uncertainties precisely into the form submitted by this author [16] back in 1981

$$\Delta x \;\Delta k \;\geq\; \tfrac{1}{2} < \hat{1} >, \qquad (6.1.9)$$

where $\hat{1}$ is the isounit and < ..... > is a certain form of the expectation value to be studied in Vol. II.

In fact, Kadeisvili [15] showed that the Fourier isotransform, when applied to a Gaussian distribution, implies the map (in term of the isoexponentiation of Sect. I.4.3)

$$\psi(x) \approx e_\xi^{\,-x^2/2a} \equiv e_\xi^{\,-x^2\,T\;2a^2} \Rightarrow \phi(k) \approx e_\xi^{\,-k^2a^2/2} \equiv e_\xi^{\,-k^2\,T\,a^2/2} \qquad (6.1.10)$$

as a result of which we have the isotopic behaviour

$$\Delta x \approx a\,/\,T^{\frac{1}{2}}, \qquad \Delta k \approx 1\,/\,a\,T^{\frac{1}{2}}, \qquad (6.1.11)$$

yielding precisely isouncertainties (6.1.9).

We reach in this way the first illustration of the fact that *the isotopies imply such a generalization of the mathematical structure of quantum mechanics for the exterior problem in vacuum to result in fundamentally more general physical laws for the interior problem.*

For future need in Ch. I.7, note the *uniqueness* of the generalizations originating from the uniqueness of the exponentiation (Sect. I.4.3).

It should be indicated that a number of other generalizations of functional analysis have been initiated, most notably that of the so-called *q-special functions*, as outlined in App. 6.B. These generalizations are different than those needed for hadronic mechanics for numerous reasons, such as:

A) the q-special functions are deformations preserving the original unit while in hadronic mechanics, as now familiar, we have deformations under the joint lifting of the unit;

B) q–deformations are *q–number* deformations, while hadronic mechanics requires *Q–operator* deformations;

C) q–deformations are defined on an ordinary space, while the Q–operator deformations are defined on an isospace;

and others. Despite these differences, the interconnection between q–number special functions and the isotopic ones is intriguing and deserving attention. The relationship between q–deformations and hadronic mechanics is studied in App. I.7.A.

In this chapter we presenting the rudiments of functional isoanalysis with the understanding that this discipline too is at its first infancy and so much remains to be done. Also, a number of additional aspects, such as special isofunctions needed for specific applications, will be worked out in Vol.s II and III.

## 6.2: ISOHILBERT SPACES AND THEIR ISODUALS

It is significant for this chapter to recall that functional analysis (see, e.g., ref.s [12,13]) was born and developed primarily because of specific physical motivations, rather than abstract mathematical needs.

In fact, the French mathematician J. B. J. Fourier identified his celebrated series and transforms during his study on heat conduction; Freedholm worked on integral equations because of specific problems in classical electromagnetism; von Neumann conducted most of his studies on operator algebras because of specific physical needs; not to mention the fundamental physical role of Hilbert studies in quantum mechanics.

It is intriguing to note that, much along the same lines, the new discipline of *functional isoanalysis* , was also born out, specifically, of physical problems, given this time by the author's studies of nonlinear, nonlocal and noncanonical systems of the interior dynamical problem.

Conventional functional analysis can be seen as the discipline which is and will remain fundamental for the *exterior* dynamical problem of particles in vacuum (see Sect. 1.1), while functional isoanalysis is a covering discipline specifically conceived for the more general *interior* dynamical problem of extended particles moving within physical media.

Despite its rather vast current dimension, contemporary functional analysis remains based on conventional notions, such as conventional fields, conventional vector spaces, conventional operations, etc. It is then inevitable that the isotopic generalizations of these structural foundations imply the existence of a consequential, corresponding generalization of the entire theory.

It is also significant to note that functional isoanalysis was born and completely developed in physical publications until very recently. In fact, Kadeisvili papers [14,15] are the first papers appeared very recently in a mathematical Journal, to the author's best knowledge.

The foundations of functional isoanalysis are those reviewed in the preceding chapters, and consist of the isotopies of fields, vector spaces, transformation theory, algebras, groups, geometries, etc. This section is solely devoted to the isotopies of Hilbert spaces, while additional aspects will be studied in the following sections.

The first notion of isoanalysis is the isofield $\hat{F}(\hat{a},+,*)$ with isonumbers $\hat{a} = a\hat{1}$, conventional sum +, isoproduct $* = \times T \times$, and isounit $\hat{1} = T^{-1}$. For simplicity, we shall restrict $\hat{F}$ to have isocharacteristic zero and to represent the isofields of real isonumbers $\hat{R}(\hat{n},+,*)$ and of complex isonumbers $\hat{C}(\hat{c},+,*)$. More general formulations of isoanalysis on isoquaternions are left to the interested reader.[44]

The second fundamental notion is a generic, finite-dimensional vector isospace $\hat{S}(x,\hat{C})$ on the isofield $\hat{C}$. The abstract identity of $\hat{C}(\hat{c},+,*)$ and $C(c,+,\times)$ and that of $\hat{S}(x,\hat{C})$ and $S(x,C)$ should be kept in mind to anticipate that *functional isoanalysis coincides with the conventional formulation at the abstract level by construction* (although only for the case of isounits of Class I, see below).

Recall that conventional complex numbers c can be reinterpreted as being complex isonumbers under the isotopy of the multiplication. Along similar lines, a conventional function f(x) on S(x,C) can be reinterpreted as being a function on $\hat{S}(x,\hat{C})$. In fact, it is not the value of the function f(x) which identifies the distinction between S(x,C) and $\hat{S}(x,\hat{C})$, but rather the operations on it.

Finally, the reader should recall that the isotopies automatically generalize a linear, local and canonical theory into an axiom-preserving, nonlinear, nonlocal and noncanonical form because of the arbitrary functional dependence of the isounit $\hat{1} = \hat{1}(t, x, \dot{x}, \ddot{x}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \dots)$.

The first isotopic operation among functions on $\hat{S}(x,\hat{C})$ is the *isoscalar product* (or *isoproduct* for short) of two functions $f_1(x)$ and $f_2(x)$, which is given by

$$f_1(x) * f_2(x) \ := \ f_1(x)\, G(x, \dots)\, f_2(x) \ \in \ \hat{S}(x,\hat{C}), \qquad (6.2.1)$$

where the isotopic element G is fixed, and different than T.

The *isoinner product* of two functions $f_1(x)$ and $f_2(x)$ on $\hat{S}(x,\hat{C})$ is the composition with elements in $\hat{C}$ introduced in ref. [3][45]

43 It should be recalled that, on strict mathematical grounds, even the formulation on isocomplex numbers is inessential owing to the unification of all numbers and isonumbers in the abstract field of isoreals (Sect. 2.7).

45 It should be indicated that, as shown in Sect. 6.7, the measure dx is lifted into the form $\hat{d}x = d(\hat{1}x)$. However, for $\hat{1}$ independent of x, we have $\int \hat{d}x * \bar{f}_1(x) * f_2(x) = \int dx\, \bar{f}_1(x) * f_2(x)$.

$$(f_1 \hat{,} f_2) \ := \int_a^b dx\, \bar{f}_1(x)\, G\, f_2(x) = \int_a^b dx\, \bar{f}_1(x) \odot f_2(x) \ \in \hat{C}(c,+,*), \quad (6.2.2)$$

where $\bar{f}$ denotes ordinary complex conjugation and $\hat{C}(c,+,*)$ is the isofield of Proposition 2.3.1 (that without the lifting of the numbers in which case the isounit must necessarily be an element of the original field).

The above foundations then imply the lifting of the conventional quantity $|f(x)|$ into the *isoabsolute value* $\uparrow f(x) \uparrow$ characterized by

$$\uparrow f(x) \uparrow^{\hat{2}} = (\overline{f(x)}\, G\, f(x))\, \hat{1} = (\overline{f(x)} \odot f(x))\, \hat{1}, \quad (6.2.3)$$

and given by

$$\uparrow f(x) \uparrow = (\bar{f}\, G\, f)^{\frac{1}{2}}\ \hat{1} = (\bar{f} \odot f)^{\frac{1}{2}} \times \hat{1}. \quad (6.2.4)$$

where $\hat{1}^{\frac{1}{2}}$ is a conventional square root. The *isonorm* $|\uparrow f(x) \uparrow|$ of a function $f(x)$ is then defined by the element of the isoreals

$$|\uparrow f(x) \uparrow|^{\hat{2}} := (f \hat{,} f)\hat{1} = \hat{1} \int_a^b dx\, \bar{f}(x)\, G\, f(x) \quad \in \hat{R}(n,+,*), \quad (6.2.5)$$

and given by

$$|\uparrow f(x) \uparrow| = (f \hat{,} f)^{\hat{\frac{1}{2}}} = (f_1, f_2)^{\frac{1}{2}}\ \hat{1}\ . \quad (6.2.6)$$

It should be indicated from the outset that the above definitions are not unique, owing to the degrees of freedom of the isotopies. In fact, one can consider the maps

$$f \ \rightarrow\ \hat{f} = f\,\hat{1} \in \hat{S}(\hat{x},\hat{C}), \quad c \rightarrow \hat{c} = c\,\hat{1} \in \hat{C}(\hat{c},+,*), \quad (6.2.7)$$

in which case we have the map of the isoproduct

$$f_1\, G\, f_2 \quad \rightarrow \quad \hat{f}_1\, G\, \hat{f}_2 = f_1\, \hat{1}\, G\, f_2\, \hat{1} = f_1\, f_2\, \hat{1}, \quad (6.2.8)$$

with corresponding definitions for isoabsolute value

$$\uparrow \hat{f}(x) \uparrow \ := (\ \bar{f}\ f\, \hat{1}\ )^{\frac{1}{2}}\hat{1}, \quad (6.2.9)$$

isoinner product

$$(\hat{f} \hat{,} \hat{f}) \ := \hat{1} \int_a^b dx\, \bar{f}(x)\, f(x)\, \hat{1}(x, ..) \ \in \hat{R}(\hat{n},+,*). \quad (6.2.10)$$

and isonorm

$$|\hat{\uparrow} f(x) \hat{\uparrow}| := (\hat{\uparrow} \hat{;} \hat{\uparrow})^{\hat{1}} = (\hat{\uparrow}_1, \hat{\uparrow}_2)^{\hat{1}} \hat{1} . \qquad (6.2.11)$$

The transition from the preceding formulation in terms of ordinary numbers and functions to the latter one was introduced by the author in ref. [10] for the particular case of $T \equiv G$ under the name of *reciprocity transformation* because based on the replacement

$$T \rightarrow \hat{1}, \qquad \hat{1} \rightarrow T^{-1}, \qquad (6.2.12)$$

the case $T \neq G$ being a simple generalization. The formulation on isocomplex numbers $\hat{C}(\hat{c},+,*)$ is that primarily used in physics because it implies that the isotopic eigenvalues are the conventional ones (see below in this section), although both formulations emerge rather naturally, e.g., in the lifting of Dirac delta-function (see Sect. 6.4).

Needless to say, maps (6.2.7) are, by far, nonunique and a number of additional maps implying nontrivial alterations of the isoproduct are possible. Nevertheless the above two alternatives are sufficiently to identify the foundations of isoanalysis.

From these rudimentary notions it is sufficient to see the need to use again Kadeisvili classification:

**Primary classification:** based on the characteristics of the isounit (Sect. 1.5):
**Class I: Functional isoanalysis** properly speaking;
**Class II: Isodual functional isoanalysis**;
**Class III: Indefinite functional isoanalysis**;
**Class IV: Singular functional isoanalysis**;
**Class V: General functional isoanalysis**.

**Secondary classification:** based on the assumed realization of isofields and isovector spaces
**Subclass A: characterized by** $\hat{F}(a,+,*)$ **and** $\hat{S}(x,\hat{F})$, i.e., isofields whose elements are ordinary numbers and with ordinary functions $f(x)$ on $\hat{S}(x,\hat{F})$.
**Subclass B: characterized by** $\hat{F}(\hat{a},+,*)$ **and** $\hat{S}(\hat{x},\hat{F})$, i.e., isofields with elements elements $\hat{a} = a\,\hat{1}$ and with isofunctions $\hat{f}(x) = f(x)\,\hat{1}$ on $\hat{S}(x,\hat{F})$.

By no means the above classification is complete. In fact, a further structural generalization is that suggested by the more general, one-sided, Lie-admissible formulations of the next chapter. Nevertheless, the above classification is sufficient to identify the new discipline and initiate its systematic study.

A first purpose of the above classification is to separate the axiom-preserving liftings from the more general ones. As an example, an "inner" product remains inner for Classes I , but not necessarily for Class III.

The mathematician can now see the novel concepts implied by isoanalysis, such as [10]: negative-definite composition (Class II); functional analysis based on a singular isounit (Class IV); isohilbert space whose unit is a lattice, or a distribution (Class V), etc.

Note that *the isoinner product is invariant under isoduality,*

$$(f_1 , f_2)^d := 1^d \int_a^b dx\, \bar{f}_1(x)\, 1^d\, f_2(x) \equiv \int_a^b dx\, \bar{f}_1(x)\, f_2(x) . \qquad (6.2.13)$$

However, one should recall that positive numbers are negative when referred to isodual fields, evidently because their unit is negative-definite. This point is clarified below when studying the isodual isohilbert spaces.

From now on, unless otherwise stated, we shall study in this section only the isoanalysis of Class IA, and IB, and their isoduals IIA and IIB. The study of the remaining classes must be deferred for brevity to the individual researcher.

Let us consider first Class IA. The problem of *isocontinuity* , that is, continuity on an isospace, was first studied by Kadeisvili in ref. [14] via the *isocontinuity of a function* f(x) *at a point* $x \in \hat{S}(x,\hat{F})$ , which occurs when $|\uparrow f(x) \uparrow| \rightarrow 0$ implies $\uparrow f(x + \epsilon) - f(x) \uparrow \rightarrow 0$.

Note that all conventionally continuous functions are also isocontinuous for Class IA, although the viceversa is not necessarily true under relaxed properties of the isounits. As a matter of fact, *functions that are conventionally discontinuous can be turned into isocontinuous forms via suitable selection of the isounit.*

The *isoschwartz inequality* , introduced in ref. [3] for the case T = G, is given by the simple isotopy of the conventional expression

$$\uparrow (f_1 \hat{,} f_2) \uparrow \;\; \hat{\leq} \;\; \uparrow f_1 \uparrow * \uparrow f_2 \uparrow \;, \qquad (6.2.14)$$

and its validity (again, for Class I) can be easily proved.

A function f(x) on $\hat{S}(x,\hat{C})$ is said to be *isosquare integrable* [14] in the interval [a, b] when the integral

$$\int_a^b dx\, | f(x) |^{\hat{2}} = 1 \int_a^b dx\, \bar{f}(x)\, G\, f(x) , \qquad (6.2.15)$$

exists and is finite. The set of all isosquare integrable functions in [a, b] will be denoted with $\hat{\mathcal{L}}^{(2)}[a,b]$. One can now begin to see some of the novel applications of isoanalysis. In fact, *a function which is not square integrable in a given interval, can be turned into an isosquare integrable form via a suitable selection of the isotopic element* with evident computational advantages (see below for an example).

A sequence $f_1$, $f_2$, ..... is said to be *strongly isoconvergent to f* when

$$\mathrm{Lim}_{k\to\infty} |\hat{\uparrow} f_k - f \hat{\uparrow}| = 0 \qquad (6.2.16)$$

with a similar definition holding for series. Again, for Class IA, strong convergence implies the strong isoconvergence, which is a trivial occurrence.

A nontrivial property is that the opposite is not necessarily true, namely, *a sequence (or, more generally, a series) which is strongly isoconvergent is not necessarily conventionally convergent*. This property has fundamental physical relevance that motivated this authors and several independent researchers to study hadronic mechanics,

In fact, as well known, electromagnetic interactions do have a convergent perturbative theory due to the low value of the coupling constant, which permits several numerical calculations suitable for experimental tests. On the contrary, strong interactions do not have such a convergent perturbative theory in their current formulation within the context of ordinary functional analysis, with evident consequential limitations of the theory.

As we studied in detail in Vol. II, the fundamental physical point here is that the covering functional isoanalysis offers real possibilities for the construction of a *convergent isoperturbation theory for strong interactions.*

The *isocauchy condition* is the isotopic property verified by every strong isoconvergence

$$|\hat{\uparrow} f_m - f_n \hat{\uparrow}| < \delta \qquad (6.2.17)$$

with $\delta > 0$ real arbitrary and for all m and n greater than a suitably chosen $N(\delta)$.

It is easy to see that, again for Class IA, when the isoinner product is isocontinuous, the isonorm is isocontinuous. The extension of the preceding results to Class IB is evident and will be tacitly implied hereon.

We now present the following notion introduced in ref.s [3,4,10]

**Definition 6.2.1:** *An "isohilbert space" $\mathcal{H}_{IB,G}$ of Class I.B and isotopic element G is an isospace over the isofield $\hat{C}(\hat{c},+,*)$ characterized by the following axioms:*

A.1: *$\mathcal{H}_{IB,G}$ is an isolinear and isolocal space (Sect. 4.2), i.e., for given elements $\hat{\psi}_1$, $\hat{\psi}_2$ of $\mathcal{H}_{IB,G}$, complex numbers $\hat{c}_1$, $\hat{c}_2 \in \hat{C}$ and operator $\hat{O}$ acting on $\mathcal{H}_{IB,G}$, we have*

$$\hat{O} * (\hat{c}_1 * \hat{\psi}_1 + \hat{c}_2 * \hat{\psi}_2) = \hat{c}_1 * \hat{O} * \hat{\psi}_1 + \hat{c}_2 * \hat{O} * \hat{\psi}_2 ; \qquad (6.2.18)$$

*where the isotopic product is given by $* = \times T \times$, and the isounit is $\hat{1} = T^{-1}$;*

A.2: *$\mathcal{H}_{IB,G}$ is equipped with an isoinner product defined for every pair of* *elements* $\hat{\psi}_1, \hat{\psi}_2 \in \hat{\mathcal{H}}_{IB,G}$ *by*

$$(\hat{\psi} \hat{,} \hat{\psi}) \ := \ \hat{1}\int_a^b dx\, \hat{\psi}\dagger(x)\, G(x, \dot{x}, ...)\, \hat{\psi}(x) \ \in \ \hat{R}(\hat{n},+,*)\,, \qquad (6.2.19a)$$

$$(\hat{\psi}_1 \hat{,} \hat{\psi}_2) \ = \ \overline{(\hat{\psi}_2 \hat{,} \hat{\psi}_1)} \in \hat{C}(\hat{c},+,*)\,, \qquad (6.2.19b)$$

$$(\hat{c} * \hat{\psi}_1 \hat{,} \hat{\psi}_2) = \bar{c} * (\hat{\psi}_1 \hat{,} \hat{\psi}_2)\,, (\hat{\psi}_1 \hat{,} \hat{c} * \hat{\psi}_2) = (\hat{\psi}_1 \hat{,} \hat{\psi}_2) * \hat{c}, \quad (6.2.19b)$$

$$(\hat{\psi}_1 + \hat{\psi}_2 \hat{,} \hat{\psi}) = (\hat{\psi}_1 \hat{,} \hat{\psi}) + (\hat{\psi}_2 \hat{,} \hat{\psi})\,, \qquad (6.2.19c)$$

$$\hat{\psi}_k \ \in \hat{\mathcal{H}}_{IB,G}\,, \ \hat{c} = c\hat{1} \in \hat{C}(c,+,*), \ G \neq T,$$

A.3: *The isonorm* $||\hat{1}(x)||$ *is always positive definite, or null for* $\hat{1} = 0$, *and verifies the isoschwartz inequality (6.3.14), thus implying that both isotopic elements are of Class I (sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive definite),*

$$T > 0, \qquad G > 0; \qquad (6.2.20)$$

A.4: $\hat{\mathcal{H}}_{IB,G}$ *is countable, i.e., there exists a countable set of elements* $\hat{e}_1$, $\hat{e}_2$, ..., $\hat{e}_n$ *approximating every element* $\hat{\psi} \in \hat{\mathcal{H}}_{IB,G}$,

$$\hat{\psi} = \sum_{k=1,..,n} \hat{c}_k * \hat{e}_k \in \hat{\mathcal{H}}_{IB,G}\,, \ \hat{c} \in \hat{C}\,, \qquad (6.2.21)$$

*with arbitrary accuracy, i.e.,*

$$||\hat{\psi} - \sum_{k=1,...,n} \hat{c}_k * \hat{e}_k \hat{1}|| < \delta \qquad (6.2.22))$$

*for arbitrary* $\delta > 0$ *and sufficiently large* n. *The elements* $\hat{\psi}$, $\hat{\phi}$, *etc. of an isohilbert space are called "isostates".*
*A.5:* $\hat{\mathcal{H}}_{IB,G}$ *is conventionally complete* [12,13].

The reason for the formulation of isohilbert spaces for Class IB is now evident. In fact, for Class IA, we have in general $G = G(t, x, \dot{x}, \psi, \bar{\psi}, ...)$, as a result of which, in general,

$$(c * \psi_1 \hat{,} \psi_2) \neq \bar{c} * (\psi_1 \hat{,} \psi_2), \quad (\psi_1 \hat{,} c * \psi_2) \neq (\psi_1 \hat{,} \psi_2) * c\,. \qquad (6.2.23)$$

As a result, we have the following

**Proposition 6.2.1** [14]: *Isohilbert spaces of Class IB are Hilbert, but those of Class IA are generally not.*

However, in most physical applications, we have the single isotopic element $T = G$ which can be assumed to be independent of $x$ and $\psi$. In this latter case isohilbert spaces of Class IA do verify all axioms of Definition 6.2.1, including the axioms

$$(c * \psi_1 \hat{,} \psi_2) = \bar{c} * (\psi_1 \hat{,} \psi_2), \quad (\psi_1 \hat{,} c * \psi_2) = (\psi_1 \hat{,} \psi_2) * c . \qquad (6.2.24)$$

by therefore being Hilbert.

**Definition 6.2.2** [10]: *Two elements* $\hat{\psi}_1$ *and* $\hat{\psi}_2$ *of an isohilbert space* $\hat{\mathcal{H}}_{IB,G}$ *over the isofield* $\hat{C}$ *are said to be "isoorthogonal "when*

$$(\hat{\psi}_1 \hat{,} \hat{\psi}_2) \equiv 0 ; \qquad (6.2.25)$$

*an element* $\hat{\psi}$ *is said to be "isonormalized" when*

$$(\hat{\psi} \hat{,} \hat{\psi}) \equiv \hat{1} ; \qquad (6.2.26)$$

*and a basis* $\hat{e}_1 , \ldots, \hat{e}_n$ *is said to be "isoorthonormal " when it verifies the rules*

$$(\hat{e}_i \hat{,} \hat{e}_j) = \hat{\delta}_{ij} = \hat{1}\, \delta_{ij} \qquad (6.2.27)$$

*The corresponding expression for spaces of Class IA are given by*

$$(\psi_1 \hat{,} \psi_2) \equiv 0, \qquad (\psi \hat{,} \psi) = 1 , \qquad (e_i \hat{,} e_j) = \delta_{ij} . \qquad (6.2.28)$$

**Definition 6.2.3** [14]: *An isobanach space* $\hat{B}_{IB}$ *of class IB is an isospace over an isofield* $\hat{C}(\hat{c},+,*)$ *characterized by the following axioms :*

A.1: $\hat{B}_{IB}$ *is an isolinear space ;*

A.2: *For every element* $\hat{f} \in \hat{B}_{IB}$ *there is an isonorm* $|\hat{\uparrow} f \hat{\uparrow}|$ *with values in* $\hat{R}(\hat{n},+,*)$ *verifying the properties*

$$|\hat{\uparrow}\hat{c} * \hat{f}\hat{\uparrow}| = \hat{\uparrow}\hat{c}\hat{\uparrow} * |\hat{\uparrow}\hat{f}\hat{\uparrow}|, \quad |\hat{\uparrow}\hat{f}_1 + \hat{f}_2\hat{\uparrow}| \leq |\hat{\uparrow}\hat{f}_1| | + ||\hat{f}_2|| \qquad (6.2.29)$$

$|\hat{\uparrow}\hat{f}\hat{\uparrow}|$ *is positive-definite, or null for* $\hat{f} = 0$*; and*

A.3: $\hat{B}_{IB}$ *is (conventionally) complete as for the isohilbert space.*

Again, one can see that *an isobanach space of Class IB is Banach, but one of Class IA is not necessarily so, unless the isounit is independent from the local coordinates* .

The classification given above for functional isoanalysis evidently applies also to square integrable, Hilbert, Banach and other spaces, resulting in isospaces of Class IA, IB, IIA, IIB, IIIA, IIIB, etc.

To study the isodual image of isohilbert spaces it is best to use Dirac's notation via bras and kets. Recall that the elements of a conventional Hilbert space $\mathcal{H}$ are the *states* $|\psi>$ with familiar inner product and normalization

$$<\psi|\phi> = \int d^3r\, \psi^\dagger(r)\, \phi(r) \;\; \epsilon \;\; C(c,+,\times), \qquad <\psi|\psi> = 1. \qquad (6.2.30)$$

The *dual Hilbert space* $\tilde{\mathcal{H}}$ is then the space with *dual states* $<\psi|$ equipped with the same composition (6.2.30) over C(c,+,×). As well known, $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are not independent, but interconnected with the conjugation

$$<\psi| \;=\; (|\psi>)^\dagger. \qquad (6.3.31)$$

In the above formulation, the isohilbert space $\hat{\mathcal{H}}$ is an isolinear space of *isostates* $|\hat{\psi}>$ (with $\hat{\psi}$ genmerally different from $\psi$) equipped with the isoinner product and isonormalization

$$<\hat{\psi}\uparrow\hat{\phi}> \;=\hat{1}\int d^3r\, \hat{\psi}^\dagger(r)\, T(r, ...)\, \phi(r) \;\; \epsilon \;\; \hat{C}(\hat{c},+,*), \qquad (6.3.32a)$$

$$<\hat{\psi}\uparrow\hat{\psi}> = \hat{1}. \qquad (6.3.32b)$$

The *isodual isohilbert space* $\hat{\mathcal{H}}^d$ can then be defined As the isolinear space with *isodual isostates* $<\hat{\psi}|^d$ equipped with the same composition (6.3.32a) *but now referred over the isodual isofield* $\hat{C}^d(\hat{c}^d,+,*^d)$, with calls for an isonormalizationnbwith respect to $\hat{1}^d = -\hat{1}$.

This implies that $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^d$ are interconnected by the conjugation

$$<\hat{\psi}|^d \;=\; -(|\hat{\psi}>)^\dagger, \qquad (6.3.33)$$

which is the extension to Hilbert spaces of the isodual conjugation for complex numbers $c \to c^d = -\bar{c}$.

The *isodual isoinner product* and *isodual isonormalization* can then be written

$$<\hat{\psi}\uparrow\hat{\phi}>^d \;=\; (<\hat{\psi}|^d)|\hat{\psi}> \;=-\hat{1}^d\int d^3r\, \hat{\psi}^\dagger(r)\, T^d(r, ...)\, \phi(r) \in \hat{C}(\hat{c},+,*), \quad (6.3.34a)$$

$$<\hat{\psi}\uparrow\hat{\psi}>^d = \hat{1}^d \;=\; -\hat{1}. \qquad (6.3.34b)$$

In summary, the following four spaces will have a primary relevance for our analysis:

**A) Conventional Hilbert spaces** $\mathcal{H}$, which are and will remain at the foundation of *particles in eterior conditions*;

**B) Isodual Hilbert spaces** $\mathcal{H}^d$, occurring for $\hat{1}^d = -1$, which are assumed as the basic spaces to represent *antiparticles in eterior conditions;*

**C) Isohilbert spaces** $\hat{\mathcal{H}}$ (generally assumed of Class I), which are the basis of the representation of *particles in interior conditions;* and

**D) Isodual isohilbert spaces** $\hat{\mathcal{H}}^d$(generally assued of Class II), which are assumed at the basis of *antiparticles in interior conditions.*

We leave to the interested reader for brevity the study of the isohilbert spaces of Classes III, IV and V, as well as the *isodual square integrable spaces* $\hat{\mathcal{L}}^{2d}[a, b]$ and the *isodual isobanach spaces* $\hat{B}^d$.

The fundamental character of the isotopy of the unit $I \Rightarrow \hat{1}$ is evident from the preceding structures. Note that the integral realizations of $\hat{1}$ mentioned above characterizes the particular type of integral topology of Fig. I.1.4.1. In this sense, functional isoanalysis constitutes an integral generalization of the conventional analysis.

Numerous examples of integral isounits will be given in Vol.s II and III. They essentially represent the overlapping of the wavepackets as a necessary condition to have an interior dynamical system, in such a way that, when the overlapping is null, the isounits $\hat{1}$ recover the conventional unit I. In this way, functional isoanalysis recovers the conventional functional analysis identically, by construction at the limit $\hat{1} \to I$.

Whenever needed for clarity, isospaces will be denoted with symbols of the type $\hat{\mathcal{L}}^{(2)}{}_{IAT}[a, b]$, $\hat{\mathcal{H}}_{IAT}$ , $\hat{B}_{IAT}$ , etc., identifying the class as well as the selected isotopic element.

All conventional operations and properties of linear–local operators on Hilbert and other spaces (such as determinant, trace, Hermiticity, unitarity, etc.) admit a consistent isotopic generalization studied in the next section.

At this point we indicate that the conventional eigenvalue equation $H\psi = E\psi$ on $\mathcal{H}$ is lifted on $\hat{\mathcal{H}}_{IBT}$ into the *isoeigenvalues equations* [1,2,3]

$$H * \hat{\psi} = \hat{E} * \hat{\psi} \equiv E\,\hat{\psi} \ , \quad \hat{E} = E\,\hat{1} \in \hat{C}(\hat{c},+,*), \ E \in C(c,+,\times) . \qquad (6.2.35)$$

This illustrates the reasons indicated earlier for the preference in physical calculations of formulations of Class IB. In fact, the identity $\hat{E}*\hat{\psi} \equiv E\hat{\psi}$ implies that the "numbers" of the theory are the conventional values E, rather than the isovalues $\hat{E} = E\hat{1}$ even when $\hat{1}$ is an operator.

We can now indicate the nontriviality of the isotopies of Hilbert spaces. To begin, *the lifting* $\mathcal{H} \to \hat{\mathcal{H}}_{IB}$ *implies the alteration of the eigenvalues of an operator,* as clearly illustrated by Eq.s (6.2.35). Moreover, *Hilbert and isohilbert spaces are not unitarily equivalent,* that is, there exist no (conventionally) unitary transformation mapping $\mathcal{H}$ into $\hat{\mathcal{H}}_{IB}$. However, $\mathcal{H}$ and $\hat{\mathcal{H}}_{IB}$ are indeed interconnected by a *conventionally nonunitary* transformation [1]. In fact, the maps

$$|\psi\rangle \to |\psi'\rangle = U|\psi\rangle, \quad \langle\psi| \to \langle\psi'| = \langle\psi|U^\dagger, \qquad U\,U^\dagger = \hat{1} \neq I\,, \quad (6.2.36)$$

implies the map of the inner product into the isotopic form

$$<\psi|\psi> \rightarrow <\psi'|T|\psi'>, \quad T = (U\,U^{\dagger})^{-1} \equiv T^{\dagger}. \qquad (6.2.37)$$

while, jointly, the unit is mapped into the isounit $1 \rightarrow \hat{1} = T^{-1} = UU^{\dagger}$.

The physical inequivalence of the Hilbert and isohilbert formulations is then established. Note that the isotopic element T emerging from mapping (6.2.37) is Hermitean, as it should be for Class IA or IB.

The remarkable properties of the isotopies is that, despite these physical and structural differences, *Hilbert and isohilbert spaces coincide at the abstract level.* In fact, for the particular case in which T = G = cost. or independent from the intergration variables for Class IB,[46] the isoinner product has been constructed in such a way to coincide with the conventional product, $<\hat{\psi}|\hat{\psi}> = \hat{1} <\hat{\psi}|T|\hat{\psi}> = \hat{1}T<\hat{\psi}|\hat{\psi}> \equiv <\hat{\psi}|\hat{\psi}>$. Nevertheless, eigenvalues and isoeigenvalues remain different even for a constant isounit $\hat{1} \neq 1$.

As a result, *functional analysis and its isotopic covering also coincide at the abstract level by construction.*

An example of functions which are not square integrable but are isosquare integrable is given by

$$f(x) = 1 / \sqrt{x}\,, \qquad (6.2.38)$$

which is known not to be square integrable in the interval [0, 1]. In fact, function (6.2.38) becomes isoquare integrable in the same interval for the isotopic element $T(x) = x^{1/6}$. A significance of the isospaces is therefore given by the fact that if a functional space does not constitute a conventional $\mathcal{L}^{(2)}$, Hilbert or Banach space, there may exist an isotopic element T such that the same sets does indeed form an $\hat{\mathcal{L}}^{(2)}$, isohilbert or isobanach space.

In any case, functional isoanalysis establishes that statements such as "a given function f(x) is or is not square integrable" need, for necessary mathematical consistency, the joint identification of the unit of the underlying space.

A simple example of a set of functions isoorthonormal on $\hat{\mathcal{H}}_{IA,T}$ is given by

$$f_n(x) = (2\pi)^{-1/2}\, e^{inTx}, \quad n = 0, \pm 1, \pm 2, \ldots \qquad (6.2.39)$$

for $x \in [-\pi/T, +\pi/T]$ and T independent of x (but dependent on $\dot{x}$ and other variables). In fact, we can write

$$(f_n \hat{,} f_m) = (1/2\pi)\int dx\, e^{-inTx}\, T\, e^{imTx} =$$

[46]Note that for Class IA, $\hat{1}$ is an element of the original field, that is, a constant.

$$= (1/2\pi)\int dx\, e^{-inz}\, e^{+imz}\, dz = \delta_{nm} . \qquad (6.2.40)$$

It also important to have an idea of the physical applications of functional isoanalysis in general, and of hadronic mechanics in particular. An example is the representation of unstable hadrons as "cold fusion" of lither hadrons which implies the inverse possibility of stimulating their decay artificially, with an apparent new technology. These possibilities are strictly precluded for the conventional functional analysis, and require instead the covering isoanalysis for their quantitative treatment.

A number of other important applications of isoanalysis also exist with a simpler structure which, as such, can be outlined here. We mention in this respect the possibility indicated earlier of achieving *a convergent perturbation theory of the strong interactions.* In fact, we have the following

**Theorem 6.2.1** [10]: *Given a perturbative series which is conventionally divergent on a Hilbert space $\mathcal{H}$, there always exist an isotopy under which the series becomes isoconvergent on isohilbert space $\hat{\mathcal{H}}_T$.*

The proof is so simple to be trivial. Consider, e.g., a divergent canonical expansion of an operator $A(k)$, $k \in R(n,+,\times)$, on $\mathcal{H}$ in terms of a Hermitean Hamiltonian $H = H^\dagger$ for a large value of k

$$A(k) = A(0) + k\,[\,A\,,H]\,/\,1! + k^2[\,[\,A\,,H\,],H\,]\,2! + \ldots\ldots \rightarrow \infty\,, \quad k \gg 1\,, \qquad (6.2.41)$$

where $[A, H] = AH - HA$ is the Lie product. Theorem 6.2.1 then establishes that there *always* exists an isotopy of the unit $I \Rightarrow \hat{1} = T^{-1}$ and a reinterpretation of A(k) and H on $\hat{\mathcal{H}}_T$ under which the series becomes isoconvergent

$$A(k) = A(0) + k\,[A\hat{,}H]\,/\,1! + k^2\,[[A\hat{,}H\,],H\,]\,2! + \ldots\ldots \rightarrow K < \infty\,, \quad k \gg 1 \qquad (6.2.42)$$

where $[A\hat{,}H] = ATH - HTA$ is the Lie-isotopic product. In fact, a solution exists even for a constant isotopic element T when sufficiently smaller than k, e.g.,

$$T = k^{-n}, \qquad (6.2.43)$$

with n a sufficiently large positive integer.

Yet another important application of functional isoanalysis in physics occurs when the conventional Hilbert space $\mathcal{H}$ and its isotopic image $\hat{\mathcal{H}}_{IB}$ are *incoherent*, in the sense that the transition probability among states belong to $\mathcal{H}$ and $\hat{\mathcal{H}}_{IB}$ is identically null.

This mathematically simple property implies the possibility of resolving a vexing problem of contemporary particle physics, the lack of exact confinement of quarks beginning at the discrete nonrelativistic level. In fact, as preliminarily studied in ref. [11], and studied in detail in Vol. III, quarks possess an *exact confinement* wheh treated via hadronic mechanics, i.e., when belonging to $\hat{\mathcal{H}}_{IB}$, because they have an identically null probability of escaping to the exterior world represented by the conventional space $\mathcal{H}$ *even for collision with infinite energies or in the absence of a potential barrier.* In addition, as we shall see, the isotopy $\mathcal{H} \rightarrow \hat{\mathcal{H}}_{IB}$ preserves all axiomatic properties and quantum numbers (for the case of standard isorepresentations) of SU(3), while permitting convergent isoseries.

Intriguingly, it appears that the lack of exact confinement is essentially due to the insistence of current quark theories of using conventional, rather than isotopic, functional analysis.

Further *novel* applications of isoanalysis (that is, applications which cannot be formulated with the conventional analysis, let alone treated quantitatively) are possible via isotopies of the remaining classes. For instance, the singular isoanalysis of Class IV is given by the isotopic element characterizing the space component in spherical coordinates $(r, \theta, \phi)$ of Schwartzschild's metric for the exterior gravitational problem (Sect. 5.4)

$$T = \mathrm{diag}\,(r / (r - 2M),\, r^2,\, r^2 \sin\theta)\,. \tag{6.2.44}$$

The singular character of the isoanalysis at the limit when the astrophysical bodies collapse into a singularity with $T = 0$ is evident.

To close this section with a few comments of historical character, let us recall that the appearance of the isotopic element G in composition (6.2.2) has considerable connections with the known *weight function* of the conventional functional analysis [12,13]. As a matter of fact, the techniques known for the latter are extendable to the former.

The extension of Hilbert spaces $\mathcal{H}$ to the form $\mathcal{H}_T$ with a weight function T and composition on ordinary fields C

$$(f_1 \hat{,} f_2) = \int_a^b dx\, \overline{f_1}(x)\, T(x)\, f_2(x) \in C\,, \tag{6.2.51}$$

is known since the first part of this century in both mathematical and physical literature [12,13]

The novelty of the isotopies here studied is the introduction of the isotopic function G *jointly* with the lifting of the underlying fields $F \rightarrow \hat{F}$. The nontriviality of the latter as compared to the former is easily illustrated by the fact that the basic unit remains unchanged for the former although it is generalized for the latter, or by the fact that the latter has a generally nonlocal-integral topology as compared to the local-differential topology of the former, or by the fact that the isohilbert spaces $\hat{\mathcal{H}}_{IB.G}$ coincide with the conventional ones $\mathcal{H}$ at the abstract level, which is not generally the case for structures (6.2.51). In turn this is an additional illustration of the remarkable implications of the isotopies of the unit.

## 6.3: OPERATIONS ON ISOHILBERT SPACES

A property of functional analysis with fundamental physical implications is the linearity of the operations on a Hilbert space $\mathcal{H}$, from which superposition principle, causality, measurement theory, and other physical laws follow (see, e.g., ref.s [17,18] and literature quoted therein).

A property of functional isoanalysis of equally fundamental physical character is the capability of representing the most general known nonlinear, nonlocal and nonhamiltonian interactions via an operator theory on isohilbert spaces which preserves the original axioms of linearity, thus permitting the achievement of consistent generalizations of conventional physical laws.

To state it differently, the isotopic methods disprove the rather widespread belief that a nonlinear and nonlocal formulation of the strong interactions implies the loss of superposition principle, causality, measurement theory, and all that.

These physical aspects will be studied in Vol. II. In this section we shall study the elements of the *isooperator theory* , i.e., the theory of operators on isohilbert spaces.

The understanding of this section requires a knowledge of the preceding parts, with particular reference to the notions of: isolinear and isolocal transformations (Sect. 4.2); isomodules (.App. 4.A); isoexponentiations (Sect. 4.3); etc.

**Proposition 6.3.1** [10]: *Lei* $\xi_T$ *be an isoassociative enveloping algebra of operators* A, B, C, *with isoproduct* $A*B = ATB$ *and isounit* $\hat{1} = T^{-1}$ *acting on an isohilbert space* $\hat{\mathcal{H}}_{IB,G}$ *over an isofield* $\hat{F}(\hat{\alpha},+,*)$ *of isoreal or isocomplex numbers. Then,* $\xi_T$ *is isolinear and isolocal on* $\hat{\mathcal{H}}_{IB,G}$ *, i.e., it verifies the properties*

$$A*(\hat{\alpha}*\psi + \hat{\beta}*\phi) = \hat{\alpha}*(A*\psi) + \hat{\beta}*(A*\phi), \qquad (6.3.1a)$$

$$(\hat{\alpha}*A + \hat{\beta}*B)*\psi = \hat{\alpha}*(A*\psi) + \hat{\beta}*(B*\psi), \qquad (6.3.1b)$$

$$(A*B)*\psi = A*(B*\psi), \qquad (6.3.1c)$$

$$\hat{1}*\psi = \psi, \quad \forall\ A, B, \in \xi_T,\ \psi \in \hat{\mathcal{H}}_{IB,G}, \quad \hat{\alpha}, \hat{\beta} \in \hat{F}, \qquad (6.3.1d)$$

*and* $\hat{\mathcal{H}}_{IB,G}$ *is a one–sided, left or right isomodule of* $\xi_T$.

We now study of the isotopies of conventional operations of quantum mechanics. Let $A$ be an isolinear and isolocal operator on $\xi_T$ over $\hat{F}$. Then the n-th *isopower* $A^{\hat{n}}$ of $A$ is given by

$$\hat{A}^{\hat{n}} = A * A * ... * A \ \text{(n times)}. \tag{6.3.2}$$

In particular, the *isoquare* of an operator $A$ on $\mathcal{H}_{IB,G}$ is $A^{\hat{2}} = A*A$. Thus, conventional powers, such as that of the linear momentum "$p^2$" = pp, or of the angular momentum "$J^2$" = JJ, etc. have no mathematical or physical meaning for hadronic mechanics and their use actually leads to a host of easily demonstrable (but often undetected) inconsistencies.

The *isoinverse* $A^{\hat{-1}}$ of $A \in \xi_T$ on $\hat{F}(\hat{\alpha},+,*)$ is defined by the conditions

$$A * A^{\hat{-1}} = A^{\hat{-1}} * A = \hat{1}, \tag{6.3.3}$$

and given by

$$A^{\hat{-1}} = \hat{1}\, A^{-1}\, \hat{1}, \tag{6.3.4}$$

where $A^{-1}$ is the conventional quantum mechanical inverse.

By ignoring hereon the isolinearity and isolocality for simplicity, we then have the following

**Definition 6.3.1** [3,4,10]: *Let $\mathcal{H}_{IB,G}$ be an isohilbert space with isoinner product (6.2.19) over an isofield $\hat{F}_T$. Then, the "isohermitean conjugate" $A^{\hat{\dagger}}$ of an operator $A \in \xi_T$ on $\mathcal{H}_G$ is defined by*

$$[\,(\psi * \hat{A}^{\hat{\dagger}}\,]\hat{,}\psi) = (\psi\hat{,}[\,A * \psi\,)\,] \tag{6.3.5}$$

**Proposition 6.3.2** [4,10]: *Necessary and sufficient condition for an operator $A \in \xi_T$ on $\mathcal{H}_{IB,G}$ to be isohermitean is that* [47]

$$A^{\hat{\dagger}} = G^{-1}\, T\, A^{\dagger}\, G\, T^{-1}, \tag{6.3.6}$$

The following properties can also be readily proved

$$(\,\hat{\alpha} * A + \hat{\beta} * B\,)^{\hat{\dagger}} = \bar{\hat{\alpha}} * A^{\hat{\dagger}} + \bar{\hat{\beta}} * B^{\hat{\dagger}}, \tag{6.3.7a}$$

$$(\,A * B\,)^{\hat{\dagger}} = B^{\hat{\dagger}} * A^{\hat{\dagger}}, \tag{6.3.7b}$$

$$(\,A^{\hat{\dagger}}\,)^{\hat{\dagger}} \equiv A, \quad \forall\ \hat{\alpha},\hat{\beta} \in \hat{F},\ A \in \xi_T. \tag{6.3.7c}$$

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[47] Note that the isotopic elements T and G are inverted in ref. [4] as compared to their use in this presentation, which is the notation now widely adopted in the literature.

where the upper bar denotes complex conjugation.

An example of isohermitean operator on $\mathcal{H}_G$ is given by

$$A = G^{-1}|\psi><\psi|T^{-1} = A^{\dagger}. \tag{6.3.8}$$

**Definition 6.3.2** [3,4,10]: *Let $\mathcal{H}_{IB,G}$, $\hat{F}_T$ and $\hat{\xi}_T$ be as in Definition 6.3.1. Then, an operator $\hat{U} \in \hat{\xi}_T$ on $\mathcal{H}_{IB,G}$ is called "isounitary" if it verifies the condition*

$$(\psi\hat{,}\hat{U}^{\dagger} * \hat{U} * \psi) \equiv (\psi\hat{,}\psi), \tag{6.3.9}$$

*i.e.,*

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{1}, \quad \text{or} \quad \hat{U}^{\dagger} = \hat{U}^{-1}. \tag{6.3.10}$$

**Proposition 6.3.3** [loc. cit.]: *Let $\hat{U}$ be a isounitary operator and $A$ an isohermitean operator on $\mathcal{H}_{IB}$. Then, the transformation*

$$A' = \hat{U} * A * \hat{U}^{-1} \tag{6.3.11}$$

*is isohermitean.*

It is an instructive exercise for the interested reader to prove the following property (see ref. [3] for a detailed treatment)

**Proposition 6.2.4** [loc. cit.]: *Isounitary operators $\hat{U} \in \hat{\xi}_T$ on $\mathcal{H}_{IB,G}$ over $\hat{F}_T$ always admit the following realization via the isoexponentiation of an isohermitean operator $X = X^{\dagger} \in \hat{\xi}_T$, $\hat{w}$ $\hat{F}_T$,*

$$\hat{U}(w) = e_{\hat{\xi}}^{\,i\,\hat{w} * X} \equiv e_{\hat{\xi}}^{\,i\,w\,X} = \hat{1}\,\{e^{-wTX}\} = \{e^{iXTw}\}\,\hat{1}, \tag{6.3.12}$$

As we shall seee in Vol. II, the above property has fundamental character for the Lie-isotopic theory of Ch. 4 because it permits the realization of continuous Lie-isotopic transformation groups via isounitary operator on a isohilbert space with isocomposition rules (Sect. 4.5)

$$\hat{U}(\hat{0}) = \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{1}, \tag{6.3.13a}$$

$$\hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \tag{6.3.13b}$$

In particular, the time evolution in hadronic mechanics is characterized precisely by a isounitary transformation admitting of the above isoexponentiation and, thus, forming a Lie-isotopic group.

A further important property is given by (see ref. [3], p. 1304 for proof)

**Proposition 6.3.5 :** *Any isolinear, isolocal and isohermitean operator* $A = A^\dagger$ *is bounded.*

We now pass to a study of the isotopies of eigenvalues equations.

**Definition 6.3.3** [loc. cit.]: *Let* H *be a (not necessarily isohermitean) operator on an isohilbert space* $\mathcal{H}_{IB,G}$. *Then a generally isocomplex number* $\hat{c} \in \hat{C}_T$ *is called an "isoeigenvalue" of* H *if there exist an isostate* $\psi \in \mathcal{H}_G$ *such that*

$$H * \psi = \hat{c} * \psi \equiv c\,\psi . \qquad (6.3.14)$$

We therefore confirm that *the isoeigenvalues* $\hat{c}$ *of an operator* H *on* $\mathcal{H}_{IB,G}$ *coincide with the conventional eigenvalues c of the operator* $\hat{H} = HT$. Thus, the "numbers" predicted by hadronic mechanics for measurements are conventional numbers.

The following property is important for the applications of hadronic mechanics.

**Proposition 6.3.6** [3]: *A set of isocomplex numbers* $\hat{c} = c\,\hat{1}$ *are the isoeigenvalues of an operator* $H \in \xi_T$ *on* $\mathcal{H}_{IB,G}$ *iff they are the solution of the so-called "isocharacteristic equation" of* H

$$\mathrm{Det}\,(HT - c) = 0 . \qquad (6.3.15)$$

A number of conventional properties of the eigenvalue theory (see, e.g., ref.s [17,18]) persist under isotopies, thus implying that they are indeed genuine axioms of quantum mechanics. This is the case for the following important property (see ref. [3], p. 1310, or ref. [4], p. 1922).

**Proposition 6.3.7** [loc. cit.]: *All isoeigenvalues of isohermitean operators* $H \in \xi_T$ *on* $\mathcal{H}_{IB,G}$ *are real.*

The above property establishes that the reality (observability) of the eigenvalues of Hermitean operators is a true axiom of quantum mechanics because it persists under isotopies. Another important property which also persists under isotopy is expressed by the following

**Proposition 6.3.8** [loc. cit.]: *The isoeigenvalues of isohermitean operators are invariant under isounitary transformations.*

However, there are a number of properties of quantum mechanics which are not invariant under isotopy and, as such, they cannot be considered as true axioms of the theory. The first is the rather popular belief that Hermitean operators admit a unique set of eigenvalues which is disproved by the following :

**Proposition 6.3.9** [10]: *A Hermitean operator does not admit a unique set of real eigenvalues, but admits instead an infinite number of different sets of eigenvalues, each of which is real, depending on the assumed basic unit.*

Let H be conventionally Hermitean and consider for this purpose the conventional eigenvalues $H\psi = E_o\psi$ . Consider now an isotopy of the preceding equations under which H remains Hermitean (as anticipated in Sect. 6.1, this is always the case when T = G). Then, we have *different* isoeigenvalues for the *same* operator H, i.e., the isotopies imply for the eigenvalues equations the lifting

$$H\psi = E_o \rightarrow H * \hat{\psi} = E_T \hat{\psi}, \quad E_T \neq E_o, \tag{6.3.16}$$

which is inherent in the basic isotopy of these volumes, Eq.s (1.1.1), i.e.

$$I \rightarrow \hat{1}. \tag{6.3.17}$$

But an infinite number of different isotopic elements T are possible. This proves that a Hermitean operator H can have an infinite number of different sets of eigenvalues $E_T$ depending on the selected isounit $\hat{1}$ or isotopic element T.

As we shall see in Vol. II, *expression (6.3.16) permits an explicit realization and operator generalization of the so-called "hidden variables"*. We shall also see that *Bell's inequality, von Neumann's theorem and other properties are not preserved under isotopies and, as such, they are not true axioms of quantum mechanics.* As we shall see, these and other intriguing occurrences permit an isotopic completion of quantum mechanics into hadronic mechanics which is intriguingly close to the historical argument of Einstein, Rosen, Podolsky and others.

**Definition 6.3.4** [3,4,100]: *Let* A *be an operator on a finite-dimensional isohilbert space* $\hat{\mathcal{H}}_{IB,G}$, *and let* $\hat{c}_1, \hat{c}_2, ..., \hat{c}_n$ *be its isoeigenvalues. Then the "isotrace"* $\hat{T}\hat{r}A$ *of* A *is given by*

$$\hat{T}\hat{r}A = \hat{c}_1 + \hat{c}_2 + ...... + \hat{c}_n. \tag{6.3.18}$$

*The "isodeterminant"* $\hat{D}et\, A$ *of a matrix* A *is the isoscalar defined by*

$$\hat{D}et\, A = [\,Det\,(AT)\,]\hat{1} \in \hat{C}_T. \tag{6.3.19}$$

A further instructive exercise for the interested reader is to prove the following properties:

**Proposition 6.3.10** [loc. cit.]: *Isolinear and isolocal operators* A, B, C $\in \hat{\xi}$ *on a finite-dimensional isohilbert space verify the following properties*

$$\hat{Tr}\ A = (Tr\ A)\hat{1}, \tag{6.3.20a}$$

$$\hat{Tr}\ (A \times B) = (\hat{Tr} A) * (\hat{Tr} B), \tag{6.3.20b}$$

$$\hat{Tr}\ (B * A * B^{-\hat{1}}) = \hat{Tr}\ A, \tag{6.3.20c}$$

$$\hat{Det}\ (A * B) = (\hat{Det}\ A) * (\hat{Det}\ B), \tag{6.3.20d}$$

$$\hat{Det}\ (A^{-\hat{1}}) = (\hat{Det}\ A)^{-\hat{1}}, \tag{6.3.20e}$$

$$\hat{Det}\ \{ e_{\hat{\xi}}{}^{A} \} = e_{\hat{\xi}}{}^{\hat{Tr}\ A}, \tag{6.3.20f}$$

**Definition 6.3.5** [3]: *Let* A *be an isolinear and isolocal operator on* $\hat{\mathcal{H}}_{IB,G}$. *Then the "isospectrum"* $\hat{Sp}A$ *of* A *is defined as the set of isocomplex numbers* $\hat{c} = c\,\hat{1}$ *which are such that the quantity* $(A - \hat{c})$ *is not invertible in* $\hat{\xi}_T$, *and admits the realization in term of the conventional spectrum* SpA *of* A

$$\hat{Sp}\ A = (\ Sp\ AT\ )\hat{1} \in \hat{C}_T . \tag{6.3.21}$$

We now pass to the study of the isotopies of another important notion of conventional quantum mechanics, that of projection operators.

**Definition 6.3.6** [4]: *Two "isosubspaces"* $\hat{\mathcal{H}}^1{}_{IB,G}$ *and* $\hat{\mathcal{H}}^2{}_{IB,G}$ *of* $\hat{\mathcal{H}}_{IB,G}$ *are said to be "isoorthogonal" when all their isostates are isoorthogonal (Definition 6.2.2). For any given subspace* $\hat{\mathcal{H}}^o{}_{IB,G}$ *of* $\hat{\mathcal{H}}_{IB,G}$ *the isoorthogonal complement* $\hat{\mathcal{H}}^c{}_{IB,G}$ *is the isoorthogonal subset for which we have the direct sum decomposition*

$$\hat{\mathcal{H}}_{IB,G} = \hat{\mathcal{H}}^o{}_{IB,G} + \hat{\mathcal{H}}^c{}_{IB,G} . \tag{6.3.22}$$

One can then study the isotopies of similar properties of conventional quantum mechanics [17,18].

**Definition 6.3.7** [4,10]: *An operator* $\hat{P}$ *on* $\hat{\mathcal{H}}$ *is called "isoidempotent" when it verifies the property*

$$\hat{P}^2 = \hat{P} * \hat{P} \equiv \hat{P}, \tag{6.3.23}$$

*An isoidempotent operator* $\hat{P}$ *is an "isoprojection" of* $\hat{\mathcal{H}}_{IB,G}$ *onto* $\hat{\mathcal{H}}^o{}_{IB,G}$ *when it verifies the properties*

$$\hat{P} * \psi = \psi_o, \tag{6.3.24}$$

*for all* $\psi \in \hat{\mathcal{H}}_{IB,G}$, *with* $\psi_o \in \hat{\mathcal{H}}_{IB,G}{}^o$.

The following property is important for the applications of hadronic mechanics (see ref. [3,4] for its proof).

**Proposition 6.3.11:** *An isolinear and isolocal operator* $P \in \hat{\xi}_T$ *acting on a finite-dimensional isohilbert space* $\hat{\mathcal{H}}_{IB,G}$ *is an isoprojection operator iff it is isohermitean and isoidempotent.*

The explicit realization of isoprojection operators is given by the following

**Proposition 6.3.12** [3,4,10]: *Let* $\hat{\mathcal{H}}_{IB,G}{}^o$ *be a closed subspace of a (finite-dimensional) isohilbert space* $\hat{\mathcal{H}}_{IB,G}$, *and let* $\psi_o{}^k$ *be the isoorthogonal basis of* $\hat{\mathcal{H}}_{IB,G}{}^o$. *Then, an operator* $\hat{P}$ *is an isoprojection operator of* $\hat{\mathcal{H}}_{IB,G}$ *onto* $\hat{\mathcal{H}}^o{}_{IB,G}$ *if it has the explicit realization*

$$\hat{P} = \sum^k |\psi_o{}^k> < \psi_o{}^k| \; G\, T^{-1}, \tag{6.3.25}$$

**Corollary 6.3.12A:** *Under realization (6.3.25) the isoprojection operator of* $\hat{\mathcal{H}}^o{}_{IB,G}$ *onto the complement* $\hat{\mathcal{H}}^c{}_{IB,G}$ *is given by*

$$\hat{P}^c = \hat{1} - \hat{P}. \tag{6.3.26}$$

This completes the notions of isooperator algebras on isohilbert spaces that are minimally sufficient for the initiation of physical applications of Vol. II. Additional, more detailed aspects will be studied when needed. The reader interested in acquiring a technical knowledge of isotopic methods is however suggested to work out a systematic study of the isotopies of conventional operator algebras [17,18].

We consider now the isodual isohilbert spaces of Class II B first studied in ref. [10]. For this purpose let us recall the isodual image $\mathcal{H}^d$ of the conventional Hilbert space $\mathcal{H}$, called *isodual Hilbert space,* which must be defined for consistency over the isodual field $C^d(c^d,+,\times^d)$ and with isodual states given by

$$\psi^d = \psi^\dagger 1^d = -\psi^\dagger . \qquad (6.3.27)$$

Its most salient property is that the isodual norm, i.e., the image under duality of quantity (6.2.6) is now negative-definite

$$|\uparrow\psi^d\uparrow|^d < 0 . \qquad (6.3.28)$$

The isodual inner product is then given by

$$\mathcal{H}^d : (\psi\hat{,}\psi)^d = 1^d \int d^3r\, \psi^d(r) \times^d \psi(r) \equiv -(\psi, \psi) . \qquad (6.3.29)$$

As now familiar, this property is important for a study of antiparticles via isoduality.

The *isodual isohilbert spaces* are then isospaces defined over $\hat{F}^d(\hat{a}^d,+,\hat{*}^d) = \hat{R}^d$ or $\hat{C}^d$ with isodual isostates $\hat{\psi}^d = \psi\dagger\hat{1}^d = -\hat{\psi}^\dagger$

**Definition 6.3.8** [10]: *An operator* H *is said to be "isodual isohermitean" on* $\mathcal{H}^d_{IIB,G}$ *when it verifies the condition*

$$H^{\hat{\dagger}d} = T^{-1} G A^\dagger T G^{-1} , \qquad (6.3.30)$$

*The isodual isoprojection operators on* $\mathcal{H}^d_{IIB,G}$ *are then given by*

$$\hat{P}^d = T G^{-1} \sum^k |\psi_o{}^k> < \psi_o{}^k| . \qquad (6.3.31)$$

By comparing the above definition with Proposition 6.3.3, we have the following intriguing property of hadronic mechanics in its general formulation under consideration here with $T \neq G$.

**Proposition 6.3.13** [10]:*An operator* H *which is isohermitean on* $\mathcal{H}_{IB,G}$ *is not necessarily isohermitean in its isodual* $\mathcal{H}^d_{IIB,G}$.[48]

In summary we have four primary mathematical structures at the foundation of the Lie-isotopic branch of hadronic mechanics:

**A) Linear operator theory** for the representation of particles in exterior conditions;

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[48] This property, however, is dependent on the assumed notion of duality, that based on a *conventional* conjugation. It is evident that a more general notion of duality is possible in such a way to preserve the operation of isohermiticity, but this approach has other undesirable implications (e.g., for normalizations) and it has not been adopted until now in the applications of hadronic mechanics.

**B) Isodual operator theory** for the representation of antiparticles in exterior conditions;

**C) Isolinear operator theory** for the representation of particles in interior conditions; and

**D) Isodual isolinear operator theory** for the representation of antiparticles in interior conditions.

As we shall see in Vol.s II and III, each of the above parts of functional isoanalysis will have significant applications.

## 6.4: ISODELTA FUNCTIONS

As indicated in Sect. 6.1, the isotopies imply a generalization not only of main structural foundations of functional isoanalysis, as outlined in Sect.s 6.2 and 6.3, but also of all conventional distributions, special functions and transforms.

This is a topic of such a dimension to require a separate volume for its detailed treatment. In this section we shall merely illustrate these generalizations for the case of Dirac's delta function. In the remaining sections we shall then provide examples of isotopic generalizations of special series and transforms.

As well known (see, e.g., ref. [19] and quoted bibliography), the conventional *Dirac delta function* is not a function, but a distribution representing a rather delicate limit procedure in a conventional functional space, such as the Hilbert space $\mathcal{H}$, with a mathematically well defined meaning only when it appears under an integral.

When the singularity is at the point $x = 0$, the $\delta$–function can be defined in terms of a well behaved function $f(x)$ on a one–dimensional space $S(x,R)$ over the reals $R$ by [loc. cit.]

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx \;=\; f(0), \qquad \int_{-\infty}^{+\infty} \delta(x)\,dx \;=\; 1 \tag{6.4.1}$$

This essentially means that $\delta(x) = 0$ everywhere except at $x = 0$ where it is singular. Nevertheless, what is mathematically and physically significant is the behaviour near that point, which permits explicit realizations, such as the familiar integral form

$$\delta(x) = (1\,/\,2\pi)\int_{-\infty}^{+\infty} e^{\,ixy}\,dy\,. \tag{6.4.2}$$

If the singularity is at a point $x \neq 0$, then we can write [loc. cit.]

$$f(x) \;=\; \int_{-\infty}^{+\infty} f(x')\,\delta(x'-x)\,dx'\,. \tag{6.4.3}$$

Finally, the δ–function verifies the basic properties

$$\delta(x) = \delta(-x) \quad , \quad \delta(x - x') = \int_{-\infty}^{+\infty} dz\, \delta(x - z)\, \delta(z - x') . \qquad (6.4.4)$$

The delta function is evidently inapplicable when dealing with functional isospaces, such as the isohilbert spaces $\hat{\mathcal{H}}$. In particular, exponentials of the type appearing in the integrand of Eq. (6.4.2) are no longer defined in isospaces, and must be replaced by the isoexponentials.

These occurrences rendered mandatory the studies of the isotopies of the delta function. Their origin can therefore be traced back to the isotopies by this author of the Poincaré–Birkhoff–Witt theorem reviewed in Sect. 4.3. The existence of a consistent isotopic generalization of Dirac's delta function was indicated in ref. [1], first studied in detail in ref. [3], subjected to systematic studies and classification in ref. [10], and finally applied to a number of cases reviewed in Vol.s II and III. Inspection of the recent treatment by Kadeisvili [15] is also recommended.

In particular, six mathematically and physically distinguishable isotopies of the Dirac delta function are identified in ref. [10] under the name of *isodelta functions*. Their outline is recommendable as an application of functional isoanalysis, and as a pre–requisite for the isotopies of the Fourier transforms studied in the subsequent sections.

Consider a one–dimensional isospace of Class I, denoted $\hat{S}_I(x,\hat{R})$ with (conventional) real coordinates x over the isofield of real numbers $\hat{R}(n,+,*)$ with conventional elements n and sum +, but isotopic multiplication $n_1 * n_2 := n_1 T n_2$, where T is the isotopic element and $\hat{1} = T^{-1}$ is the multiplicative isounit of Class I.

Let f(x) be an ordinary function defined on $\hat{S}_{IA}(x,\hat{R})$ which verifies the conditions of strong isocontinuity of Sect. 6.2 in all possible subintervals of $[-\infty, +\infty]$. Recall that the isotopic element T of Class I is a strongly isocontinuous, bounded, real valued, and positive–definite function of the coordinate x as well as its derivatives with respect to an independent variable of arbitrary order and any other needed quantity, $T = T(x, \dot{x}, \ddot{x}, ...)$.

Then, the *isodelta function of the first kind*, denoted $\hat{\delta}_1$, can be defined in terms of the expression

$$\int_{-\infty}^{+\infty} f(x) * \hat{\delta}_1(x)\, dx = f(0), \qquad (6.4.5)$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} T(x, \dot{x}, \ddot{x}, ...)\, \hat{\delta}_1(x)\, dx = 1 . \qquad (6.4.6)$$

The isotopic image of (6.4.3) is then given by

$$f(Tx) = \int_{-\infty} f(Tx') * \hat{\delta}_1(x' - x)\, dx' . \qquad (6.4.7)$$

namely, it is not possible any longer to map the dependence on x to the dependence at x', but rather the dependence on Tx to Tx'. This confirms the very peculiar nonlocality of the topology underlying the isotopies discussed earlier.

In fact, the isotopic element T can have an integral dependence on the interval $x \in [a, b]$ centered at x. In this case the singularity of the Dirac $\delta$ at x can be spread over the interval [a, b] by the isodelta function for a suitable selection of T.

In several cases of physical interest, T can be assumed as having an explicit dependence only on the derivatives $T = T(\dot{x}, \ddot{x}, ...)$, with consequential identity $Tdx \equiv d(Tx)$. In this case, the projection of the $\hat{\delta}_1$–function into the original functional space S(x,R) implies the equivalence

$$\hat{\delta}_1(x) \approx \delta(Tx). \qquad (6.4.8)$$

It is easy to see that, under the assumption of T being independent from x (which is the case for Class IA), the $\hat{\delta}_1$–function admits the integral representation

$$\hat{\delta}_1(x) = (1/2\pi)\int_{-\infty}^{+\infty} T\, e_{\xi}^{\;ixy}\, dy = (1/2\pi)\int_{-\infty}^{+\infty} e^{\,ixTy}\, dy, \qquad (6.4.9)$$

(where we have used the fundamental Theorem 6.3.1 on isoexponentiation), and verifies the properties

$$\hat{\delta}_1(x) = \hat{\delta}_1(-x) \quad , \quad \hat{\delta}_1(x - x') = \int_{-\infty}^{+\infty} dz\, \hat{\delta}_1(x - z) * \hat{\delta}_1(z - x') . \qquad (6.4.10)$$

For the case of an isospace of Class IB, $\hat{S}_{IB}(\hat{x},\hat{R})$, with isofunctions $\hat{f}(x) = f(x)\hat{1}$, a different isotopic expression emerged in ref. [10], called *isodelta function of the second kind*, and denoted $\hat{\delta}_2$ , which is characterized by the property

$$\int_{-\infty}^{+\infty} \hat{f}(x) * \hat{\delta}_2(x)\, dx = \int_{-\infty}^{+\infty} f(x)\, \hat{\delta}_2(x)\, dx = \hat{f}(0) = f(0)\hat{1}, \qquad (6.4.12)$$

In this case the $\hat{\delta}_2$–function must necessarily be an isofunction, i.e., admitting a structure of the type $\hat{\delta}_2(x) = \tilde{\delta}_2(x)\hat{1}(x, \dot{x}, \ddot{x}, ...)$. Then, for $\hat{T} = \hat{1}$, we have

$$\int_{-\infty}^{+\infty} \hat{\delta}_2(x)\, dx = \int_{-\infty}^{+\infty} \tilde{\delta}_2(x)\, \hat{1}(x, \dot{x}, ...)\, dx = \hat{1} \qquad (6.4.13)$$

and the isotopic image of (6.4.3) is given by

$$\hat{f}(x) = \int_{-\infty}^{+\infty} \hat{f}(x') * \hat{\delta}_2(x' - x)\, dx' . \qquad (6.4.14)$$

One can see that the projection of the $\hat{\delta}_2$–function in the original functional space S(x,R) implies the equivalence (again for isounits independent of the integration variable)

$$\hat{\delta}_2(x) \approx \delta(x)\,\hat{1}(\dot{x}, \ddot{x}, \ldots)\,. \qquad (6.4.15)$$

It is easy to see that, under the same assumptions, the $\hat{\delta}_2$–function admits the integral representation [6]

$$\hat{\delta}_2(x) = 1 / 2\pi \int_{-\infty}^{+\infty} T\, e_{\xi}^{\;ixy}\, dy = \hat{1} / 2\pi \int_{-\infty}^{+\infty} e_{\xi}^{\;ixTz}\, d(Tz) \qquad (6.4.16)$$

and verifies the properties

$$\hat{\delta}_2(x) = \hat{\delta}_2(-x) \quad , \quad \hat{\delta}_2(x - x') = \int_{-\infty}^{+\infty} dz\; \hat{\delta}_2(x - z) * \hat{\delta}_2(z - x')\,. \qquad (6.4.17)$$

It is an intriguing exercise for the reader interested in learning the isotopic techniques to prove that *the first and second kind isodelta functions can be interconnected by the reciprocity transformation* $T \to \hat{1}$.

To present the *isodelta function of the third kind* $\hat{\delta}_3$, let us recall [10] that the separation on a generic, n–dimensional isospace $\hat{S}(x,\hat{g},\hat{R})$, $\hat{g} = Tg$, $\hat{R} \approx R\,\hat{1}$, $\hat{1} = T^{-1}$ (see Sect. 3.2 for details), can be formally written as that of a fictitious conventional space in the same dimension $S(\tilde{x},g,R)$, according to the simple rule

$$x^{\hat{2}} = x^t\,\hat{g}\,x \;\equiv\; \tilde{x}^t\,\tilde{x} = \tilde{x}^2, \qquad \tilde{x} = T^{\frac{1}{2}}\,x\,. \qquad (6.4.18)$$

This implies that a number of problems in isospaces can be worked out in this fictitious conventional space in the $\tilde{x}$–variables, and the results then re–expressed in the x–variables.

The $\hat{\delta}_3$–function emerged precisely from reduction of this type. It can be defined via the conditions [10]

$$\int_{-\infty}^{+\infty} f(\tilde{x})\; \hat{\delta}_3(\tilde{x})\; d\tilde{x} = \int_{-\infty}^{+\infty} f(T^{\frac{1}{2}}\, x)\; \hat{\delta}_3(T^{\frac{1}{2}}x)\; d(T^{\frac{1}{2}}\, x) = f(0), \quad T^{\frac{1}{2}} = T^{\frac{1}{2}}(x, \dot{x}, \ldots) \qquad (6.4.19)$$

from which we obtain for f = 1

$$\int_{-\infty}^{+\infty} \hat{\delta}_3(T^{\frac{1}{2}}x)\, d(T^{\frac{1}{2}}x) = 1\,. \qquad (6.4.20)$$

with realization in terms of the conventional δ–function

$$\hat{\delta}_3(x) \approx \delta(\tilde{x}) = \delta(T^{\frac{1}{2}}x)\,. \qquad (6.4.21)$$

It should be stressed that, while the isodelta functions of the first and second kind are bona–fide isotopies of the conventional expression, this is not the case for $\hat{\delta}_3$ which is merely a pragmatic tool for simplifying calculations, rather than a mathematically rigorous structure.

The above expressions have been presented for the case of one-dimensional coordinates x. The extension to three-dimensions is trivial, and given by isotopic products of the type

$$\hat{\delta}_I(r) = \hat{\delta}_I(x) * \hat{\delta}_I(y) * \hat{\delta}_I(z) . \tag{6.4.22}$$

Consider now the isodual isospace $\hat{S}_{IIA}{}^d(x^d,\hat{R}^d)$ over the isodual isoreals $\hat{R}^d$, with isotopic element $T^d = -T$ and isounit $\hat{1}^d = -\hat{1}$. The isodelta functions on isodual isospaces can then be defined accordingly, by reaching three additional quantities $\hat{\delta}^d{}_I$ , $\hat{\delta}^d{}_2$ and $\hat{\delta}^d{}_3$ called *isodual isodelta functions.* The following property then holds.

**Proposition 6.4.1** [10]: *The isodual isodelta functions of the first and second kind change their overall sign under isoduality.*

In fact, by recalling that $x^d = -x$, $y^d = -y$, $i^d = -i$, we have the *isodual isodelta function of the first kind*

$$\hat{\delta}^d{}_I(x^d) = (1/2\pi)\int_{-\infty}^{+\infty} T^d\, e_{\xi^d}{}^{i^d x^d y^d}\, dy^d = -(1/2\pi)\int_{-\infty}^{+\infty} T\, e_{\xi}{}^{ixy}\, dy\,, \tag{6.4.23}$$

with a similar expression for the second kind. Note that $\hat{\delta}_3$ has no isoselfdual structure, evidently because it is not an isotopic structure.

The properties of the isodelta functions for all the remaining Classes III, IV and V are vastly unknown at this writing. Additional generalizations of the delta functions are expected in the one-sided Lie-admissible formulations of the next chapter.

Note that, while the Dirac delta function is unique, there exist infinitely possible isodelta functions for each of the above six kinds, evidently because of the infinitely possible isounits or isotopic elements. The reader may have noted the intriguing character of the general case of isodelta functions (6.4.5) and (6,4,12) for $T = T(x,...)$, which are hoped to receive an attention in the literature much needed for physical advances.

As well known, the locality of quantum mechanics is precisely expressible via the Dirac delta function. The nonlocality of the isotopies of quantum mechanics is then expressed by the isodelta functions. In turn, such nonlocality is necessary for a quantitative treatment of the extended character of hadrons with consequential nonlocal components in the strong interactions due to mutual overlapping of the wavepackets and charge distributions of the particles.

While the Dirac delta is a *bona fide* distribution, the isodelta functions are not necessarily so because the original singularity at x can be spread over an interval of which x is the center. Nevertheless, in specific cases, such as when T = cost., the isotopic $\hat{\delta}$-functions are distributions similar to $\delta(x)$.

## 6.5: ISOSERIES

We indicate in Sect. 6.1 that all conventional series of functional analysis admit significant isotopic generalizations. In this section we shall illustrate this occurrence via the isotopies of the *Fourier series* (see, e.g., ref. [20,21]). The interested reader can then compute the isotopies of other series.

As well known a sufficiently smooth function $f(\theta)$ on a conventional one-dimensional space $S(\theta,R)$ over the reals R, which is periodic in $[0, 2\pi]$, admits the representation in term of the Fourier series [loc. cit.]

$$f(\theta) = \sum_{n=-\infty,+\infty} a_k\, e^{in\theta}, \tag{6.5.1}$$

where

$$a_n = (1/2\pi)\int_0^{2\pi} e^{-in\theta}\, f(\theta)\, d\theta. \tag{6.5.2}$$

If the function is periodic in the interval $[0,L]$, we have instead

$$f(x) = \sum_{n=-\infty,+\infty} b_n\, e^{i2\pi nx/L}, \tag{6.5.3}$$

in which case

$$b_n = (1/L)\int_0^{L} e^{-i2\pi nx/L}\, f(x)\, dx. \tag{6.5.4}$$

When the underlying functional space is lifted into a functional isospace of Class IA, the above Fourier series are no longer applicable, again, because of the loss of basic definitions, such as that of exponential. For this reason this author [10] studied the isotopies of the Fourier series, resulting in a generalization called *isofourier series* [10] which can be defined for a function $f(\theta)$ also periodic in $[0, 2\pi]$ on an isospace of Class IA, $\hat{S}(\theta,\hat{R})$ over the isoreals $R_{IA}(n,+,\hat{*})$, via the expression

$$f(\theta) = \sum_{n=-\infty,+\infty} A_n * e_{\xi}^{in\theta} = \sum_{n=-\infty,+\infty} A_n\, e^{inT\theta}, \tag{6.5.5}$$

where we have again used the properties of isoexponentiation of Sect. 4.3. Then, for T independent of $\theta$, by using the isoorthogonality of the isoexponentials, Eq.s (6.2.34), we have

$$A_n = (1/2\pi)\int_0^{2\pi} \{ T\, e_{\xi}^{-in\theta} \} * f(\theta)\, d\theta. \tag{6.5.6}$$

If the function is periodic in the interval $[0, L]$, we have instead

$$f(x) = \sum_{n=-\infty,+\infty} B_n * e_{\xi}^{i2\pi nx/L} = \sum_{n=-\infty,+\infty} B_n\, e^{i2\pi nxT/L}, \tag{6.5.7}$$

in which case

$$B_n = 1/L \int_0 \{T\, e_\xi^{\,i2\pi nx/L}\} * f(x)\, dx \,. \tag{6.5.8}$$

When one deals with functional isospaces of Class IB, the preceding results are essentially multiplied by the isounit $\hat{1}$. The extension of the results to the isodual isospaces of Classes IIA and IIB is equally simple, and will be implied hereon.

An important application of the above isoseries occurs in the transition from Cartesian to polar coordinates in isospaces. This transition is linked to a central property of isorelativities, their capability to represent particles with their actual, generally nonspherical shape, jointly with all their infinitely possible deformations. In turn, this formulation originates from the isotopies of the rotational symmetry [19,20] (see also review [21]).

Consider a two-dimensional Euclidean space

$$E(r,\delta,\mathfrak{R}) : r = (x,\, y),\ \ \delta = \mathrm{diag.}\,(1,\, 1),\ \ r^2 = xx + yy \in R \,. \tag{6.5.9}$$

The transition to polar coordinates is provided by the familiar expressions

$$x = r \cos\theta \,, \qquad y = r \sin\theta \,, \tag{6.5.10}$$

A sufficiently smooth function g(x,y) on E(r,δ,R) can then be represented in the unit circle r = 1 via the expansion

$$g(x,y) = \mathrm{Lim}_{N\to\infty} \sum\nolimits_{n,m=0,\dots,N} a_{nm}\, x^n\, y^m =$$

$$= g(\cos\theta,\, \sin\theta) = f(\theta) = \sum\nolimits_{n,m=0,\infty} a_{nm} \cos^n\theta \sin^m\theta \tag{6.5.11}$$

The use of the expressions

$$e^{i\theta} = \cos\theta + i\sin\theta,\quad \cos\theta = (e^{i\theta} + e^{-i\theta})/2,\quad \sin\theta = (e^{i\theta} - e^{-i\theta})/2i \tag{6.5.12}$$

then implies the well known Fourier series [20,21]

$$f(\theta) = \sum\nolimits_{n=-\infty,+\infty} a_n e^{in\theta} = c_0/2 + \sum\nolimits_{n=1,\infty} [\, c_n \cos n\theta + d_n \sin n\theta\,),$$

$$c_n = (1/\pi) \int_{-\pi}^{+\pi} d\theta\, f(\theta) \cos n\theta \,, \quad n = 0,\, 1,\, 2,\, \dots$$

$$d_n = (1/\pi) \int_{-\pi}^{+\pi} d\theta\, f(\theta) \sin n\theta \,, \quad n = 1,\, 2,\, \dots \tag{6.5.13}$$

On physical grounds, the central geometric object is the perfect and rigid circle, as requested by the fact that *the rotational symmetry is a symmetry for rigid bodies* .

Conceptual, mathematical and physical advances are permitted by the transition to the covering isoeuclidean space of Class IA (Sect. 3.3)

$$\hat{E}(r,\hat{\delta},\hat{R}) : r = (x, y), \quad \hat{\delta} = T\delta = \mathrm{diag.}\,(b_1{}^2, b_2{}^2), \quad \hat{1} = T^{-1}, \qquad (6.5.14)$$

whose separation

$$r^{\hat{2}} = x\, b_1{}^2 x + y\, b_2{}^2 y = \mathrm{inv.} \qquad (6.5.15)$$

represents all possible signature preserving deformations of the circle, i.e, an infinite family of possible ellipses with semiaxes $a = b_1{}^{-2}$ and $b = b_2{}^{-2}$. In this case, the invariance is provided by the isotopic covering $\hat{O}(2)$ of the rotational symmetry O(2) [6,7] we shall study in detail in Vol. II.

What is important for this section is that the conventional transformations to polar coordinates, Eq.s (6.5.10), is no longer applicable in isoeuclidean space (6.5.14) and must be generalized into the *isopolar coordinates* of App. 6.A [10]

$$x = r\, b_1{}^{-1}\, \mathrm{isocos}\, \Delta^{\frac{1}{2}}\, \theta, \quad y = r\, b_2{}^{-1}\, \mathrm{isosin}\, \Delta^{\frac{1}{2}}\, \theta. \qquad (6.5.16)$$

where isocos $\hat{\alpha}$ and isosin $\hat{\alpha}$ are the *isotrigonometric functions* and $\Delta = \det T = b_1{}^2 b_2{}^2$.

We can therefore study the isotopy of expansion (6.5.11) for the unit case

$$r^{\hat{2}} = x\, b_1{}^2 x + y\, b_2{}^2 y = 1, \qquad (6.5.17)$$

in the form

$$g(x,y) = \mathrm{Lim}_{N\to\infty} \sum\nolimits_{n,m=0,\dots N} A_{nm} * x^{\hat{n}} * y^{\hat{m}} =$$

$$= g(b_1{}^{-1}\mathrm{isocos}\,(\Delta^{\frac{1}{2}}\,\theta),\, b_2{}^{-1}\mathrm{isosin}\,(\Delta^{\frac{1}{2}}\,\theta)) = f(\theta) =$$

$$= \sum\nolimits_{n,m=0,\infty} A_{nm} * (b_1{}^{-1}\mathrm{isocos}\,(\Delta^{\frac{1}{2}}\,\theta))^n\, (b_2{}^{-1}\,\mathrm{isosin}\,(\Delta^{\frac{1}{2}}\,\theta))^m. \qquad (6.5.18)$$

A physical significance of isoseries is in the expansion of an intensity in isospace, i.e., in expressions of the type

$$(1/L)\int_0^L |f(x)|^{\hat{2}}\, dx = (1/L)\int_0^L \overline{f(x)} * f(x)\, dx =$$

$$= (1/L)\int_0^L dx\, \{\sum\nolimits_n \bar{B}_n\, e^{-i2\pi n x T/L}\} * \{\sum\nolimits_m B_m\, e^{i2\pi m x T/L}\} =$$

$$= \sum\nolimits_{n=-\infty,+\infty} \bar{B}_n * B_n = \sum\nolimits_{n=-\infty,+\infty} |B_n|^{\hat{2}}. \qquad (6.5.19)$$

In the simple case here considered, the original intensity is reduced to the sum of the individual isocontributions $\overline{B_n}*B_n$ without interference terms $\overline{B}_n*B_m$ .

However, a rather complex interference pattern occurs for the case of T explicitly dependence on the integration variable, or merely when T is a nondiagonal matrix [10]. This is a representation of the nonlocal character of the isotopic wave-theory, namely, the representation of wavepackets of particles in conditions of mutual penetration.

Note that additional generalizations of isoseries are possible for liftings of the addition, but their would imply the loss of the distributive law (Sect. 2.3).

Note also that the same function $f(\theta)$ can be expanded in Fourier series (6.5.1), when dealing with ordinary functional spaces, as well as in the isoseries (6.5.18), when dealing with isospaces. The selection of which series holds is therefore relinquished again to the basic multiplicative unit.

## 6.6: ISOTRANSFORMS

We also indicated in Sect. 6.1 that conventional transforms of functional analysis admit nontrivial isotopic generalizations. In this section we shall illustrate this occurrence via the isotopies of the Fourier and Laplace transforms (see, e.g., ref.s [20,21] for their conventional forms). The reader can then work out any needed additional isotransform with the same techniques.

Six different isotopies of the Fourier transforms were identified by this author in ref. [10] in correspondence with the six different types of isodelta function of Sect. 6.4. They apply for correospondently different mathematical and physical conditions, and can be presented as follows.

Consider a one-dimensional functional isospace $\hat{S}_{IA,T}(x,\hat{R})$ over the isoreals $\hat{R}$, with the isotopic element T and isounits $\hat{1} = T^{-1}$, and the Fourier isoseries in the interval [–L, L] for strongly isocontinuous functions f(x) on $\hat{S}_{IA,T}(x,\hat{R})$ with 2L periodicity

$$f(x) = (2L)^{-\frac{1}{2}} \sum_{n=-\infty,+\infty} g_n * e_{\xi}^{in\pi X/L} . \qquad (6.6.1a)$$

$$g_n = (1L)^{-\frac{1}{2}} \int_{-L}^{+L} f(x) * e_{\xi}^{-in\pi x/L} dx . \qquad (6.6.1b)$$

As in the conventional case [20,21], set $(\pi/L)^{\frac{1}{2}} x = y$ and $(n\pi L)^{\frac{1}{2}} = k_n$ , so that $(n\pi/L)\, x = k_n y$, $\Delta k_n = k_{n+1} - k_n = (\pi/L)^{\frac{1}{2}}$ , and $(2L)^{-\frac{1}{2}} = \Delta k_n (2\pi)^{-\frac{1}{2}}$. Then Eq.s (6.6.1) become

$$f(y) = (2\pi)^{-\frac{1}{2}} \sum_{k_n=-\infty,+\infty} g_{k_n} * e_{\xi}^{ik_n y} \Delta k_n . \qquad (6.6.2a)$$

$$g_{k_n} = (2\pi)^{-\frac{1}{2}} \int_{-(\pi L)^{\frac{1}{2}}}^{(\pi L)^{\frac{1}{2}}} f(y) * e_{\xi}^{-ik_n y}\, dy \,. \tag{6.6.2b}$$

At the limit $L \to \infty$, we have the *Fourier isotransforms of the first kind* [10]

$$f_1(x) = (1/2\pi)\int_{-\infty}^{+\infty} g_1(k) * e_{\xi}^{ikx}\, dk =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} g_1(k)\, e^{ikTx}\, dk \,, \tag{6.6.3a}$$

$$g_1(k) = (1/2\pi)\int_{-\infty}^{+\infty} f_1(x) * e_{\xi}^{-ikx}\, dx =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} f_1(x)\, e^{-ikTx}\, dx \,. \tag{6.6.3b}$$

The reason why the above isotransforms are called of the first kind is that they are linked to the isodelta function of the same kind, Eq.s (6,4,9), as illustrated by the following

**Theorem 6.6.1** [10]: *The isotopic Fourier integral theorem reads*

$$f_1(x) = (1/2\pi)\int_{-\infty}^{+\infty} dk\, e_{\xi}^{ikx} * \{\int_{-\infty}^{+\infty} f_1(x') * e_{\xi}^{ikx'}\, dx'\} =$$

$$= \int_{-\infty}^{+\infty} dx'\;\; f_1(x') * \{\, (1/2\pi)\int_{-\infty}^{+\infty} e_{\xi}^{i(x - x')\, k}\, dk =$$

$$= \int_{-\infty}^{+\infty} dx'\;\; f_1(x') * \delta_1(x - x') \tag{6.6.4}$$

In particular, it is easy to prove the following isotopy of the corresponding conventional property

$$\int_{-\infty}^{+\infty} dx\, |\, f_1(x)\,|^{\hat{2}} = \int_{-\infty}^{+\infty} dx\, \overline{f_1(x)}\, T\, f_1(x) \equiv$$

$$\equiv \int_{-\infty}^{+\infty} dk\, |\, g_1(k)\,|^{\hat{2}} = \int_{-\infty}^{+\infty} dk\, \overline{g_1}(k)\, T\, g_1(k) \,. \tag{6.6.5}$$

The entire theory of Fourier transforms can therefore be subjected to step–by–step isotopic liftings. Studies along these general lines have been initiated by Kadeisvili [15] and their continuation is left to the interested reader.

The *Fourier isotransforms of the second kind* are defined on isospaces $\hat{S}_{IB,T}(x,\hat{R})$, that is, for isofunctions $\hat{f}(x) = f(x)\hat{I}$, are given by

$$\hat{f}_2(x) = (1/2\pi)\int_{-\infty}^{+\infty} \hat{g}_2(k) * e_{\xi}^{ikx}\, dk \,, \tag{6.6.6a}$$

$$\hat{g}_2(k) = (1/2\pi)\int_{-\infty}^{+\infty} \hat{f}_2(x) * e_{\xi}^{-ikx}\, dx \tag{6.6.6b}$$

and can be written for isounits independent on the integration variable

$$\hat{f}_2(x) = (\hat{1}/2\pi)\int_{-\infty}^{+\infty} g_2(k)\, e_\xi^{ikTx}\, dk\,, \qquad (6.6.7a)$$

$$\hat{g}_2(k) = (\hat{1}/2\pi)\int_{-\infty}^{+\infty} f_2(x)\, e_\xi^{-ikTx}\, dx\,. \qquad (6.6.7b)$$

Note that, again, the isotransforms of first and second kind are interconnected by the reciprocity transforms $T \rightarrow \hat{1}$.

The *Fourier isotransforms of the third kind* are defined on an ordinary space $S(\tilde{x},C)$ with local coordinates $\tilde{x} = T^{\frac{1}{2}}\, x$, and can be written [10]

$$f_3(x) = (1/2\pi)\int_{-\infty}^{+\infty} g_3(k)\, e_\xi^{ikT^{\frac{1}{2}}x}\, d(T^{\frac{1}{2}}k)\,, \qquad (6.6.8a)$$

$$g_3(k) = (1/2\pi)\int_{-\infty}^{+\infty} f_3(x)\, e_\xi^{-ikT^{\frac{1}{2}}x}\, d(T^{\frac{1}{2}}x)\,. \qquad (6.6.8b)$$

The remaining three cases are of isodual character. The *isodual isotransform of the first kind* on isospaces $\hat{S}_{IIA,T}(x,\hat{R})$ are given by [10]

$$f^d{}_1(x^d) = (1/2\pi)\int_{-\infty}^{+\infty} g^d{}_1(k^d)\, e^{\,i^d k^d T^d x^d}\, dk^d =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} g_1(k)\, e^{\,ikTx}\, dk \qquad (6.6.9a)$$

$$g^d{}_1(k^d) = (1/2\pi)\int_{-\infty}^{+\infty} f^d{}_1(x^d)\, e^{-i^d k^d T^d x^d}\, dx^d =$$

$$= (1/2\pi)\int_{-\infty}^{+\infty} f_1(x)\, e^{-ikTx}\, dx \qquad (6.6.9b)$$

from which we have the following simply but significant

**Proposition 6.6.1** [10]: *The Fourier isotransforms of the first and second kind are isoselfdual.*

The isotransforms of the third kind are not isoselfdual, as it is the case for the corresponding isodelta function, because they are not genuine isotopies. As a matter of fact, isoduality turns the structure of $\delta_3$ into a *Laplace isotransform* of the next section.

The reader has noted the simplicity of the isotransforms for isounits independent from the local coordinates, $\hat{1} = \hat{1}(\dot{x}, \ddot{x}, ...)$ which will be used in the great majority of physical applications of Vol.s II and III. However, their general expression, e.g., for isounits of gravitational type $\hat{1} = \hat{1}(x, ...)$, is nontrivial and substantially unexplored at this writing.

The extension of the above analysis to more than one dimension is trivial and shall be tacitly implied. The formulation and properties of the Fourier isotransforms for Classes III, IV and V are also unknown at this writing.

Note that despite their abstract equivalence, isotransforms and conventional transforms are inequivalent, as directly shown by the appearance of the isotopic element T in the exponent of the isotransforms or by the fact that ordinary transforms are linear and local, while the isotransforms are isolinear and isolocal.

Note that the Fourier transform is unique for a given function f(x). On the contrary, the same function can be subjected to an infinite variety of Fourier isotransforms, evidently depending on the infinitely possible isounits. This degree of freedom is necessary for physical consistency. In fact, empty space (the vacuum) is unique, and represented by the trivial unit I = diag. (1, 1, 1). A unique transform is then fully consistent. On the contrary, there exist infinitely possible physical media due to infinitely possible densities, pressures, temperatures, etc., which are represented by the infinitely possible isounits. A corresponding infinite class of isotransforms is then necessary.

The Fourier isotransforms can evidently be applied to a large variety of nonlinear, nonlocal and nonhamiltonian problems. Their relevance was elegantly established by Kadeisvili [15] by proving that *the isotopies of the Fourier transform imply a necessary generalization of Heisenberg's uncertainty relations for particle in vacuum* (exterior dynamical problem of Sect. 1.1)

$$\Delta x \, \Delta k \approx 1 , \tag{6.6.10}$$

into the *isouncertainties* for particles moving within physical media (interior problem)

$$\Delta x \, \Delta k \approx < \hat{1} > , \tag{6.6.11}$$

first proposed by this author in ref. [16] and then re-examined in ref.s [10,11] (see Vol. II for detailed studies).

The proof of conventional uncertainties via Fourier transforms within the context of functional analysis is well known (see, e.g., ref. [18], p. 47–49), although it is worth reviewing for comparative purposes. Consider an ordinary Hilbert space $\mathcal{H}$ with states $\psi(x)$ depending on a variable $x$ belonging to an ordinary one–dimensional space S(x,C), which are normalized to one.

*Gauss distribution* within the above context can be written

$$\psi(x) = n \, e^{-x^2 / 2 a^2} , \qquad n = a^{-1/2} \pi^{-1/4} \in \Re \tag{6.6.13}$$

The conventional Fourier transform of the above expression is given by

$$\phi(k) = (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \psi(x) \, e^{-ikx} \, dx = n' \, e^{-k^2 a^2 / 2} ,$$

$$n' = an = a^{1/2} \pi^{-1/4} \in \Re . \tag{.6.6.13}$$

Now, the width of distribution (6.6.13) is of the order of $\Delta x \approx a$, while the width of transform (6.6.14) is of the order of $\Delta k \approx 1/a$. The conventional Heisenberg's uncertainties then follow,

$$\Delta x \approx a\,, \quad \Delta k \approx 1/a\,, \quad \Delta x\, \Delta k \approx 1\,. \tag{6.6.14}$$

We now reinspect the above formulation under isotopies within the context of functional isoanalysis. Consider an isohilbert space $\hat{\mathcal{H}}_{IA,T}$ with states $\psi(x)$, where x is the local coordinates on an isospace $\hat{S}_{IA,T}(x,\hat{C})$ on the isofield $\hat{C}(c,+,*)$ with isounit $\hat{1}$ and isotopic element $T$ which are then independent of x, and suppose that $\psi$ is isonormalized (Definition 6.2.2)

$$(\psi\,\hat{,}\,\psi) \;:=\; \hat{1}\int_{-\infty}^{+\infty} dx\, \overline{\psi(x)} * \psi(x) \;=\; \hat{1}\,. \tag{6.6.15}$$

The conventional Gauss distribution cannot any longer be consistently defined under isotopies, e.g., because of the lack of meaning of the conventional exponentiation (Sect. 4.3). Its image in $\hat{\mathcal{H}}_{IA.T}$ is instead given by the *Gauss isodistribution* [10]

$$\psi(x) \;=\; N * e_{\xi}^{-x^2/2a^2} \;=\; N\, e^{-x^2 T/2a^2}\,, \tag{6.6.16}$$

where

$$N = T^{-1/2} a^{-1/2}\, \pi^{-1/4} \tag{6.6.17}$$

The conventional Fourier transform has no mathematical or physical meaning in isohilbert spaces, and must be replaced by the Fourier isotransform of the first kind which yields after simple algebra

$$\Phi(k) \;=\; (1/2\pi)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \psi(x) * e_{\xi}^{-ikx}\, dx \;=$$

$$= N' e^{-k^2 T a^2/2}\,, \qquad N' \;=\; aN \;=\; T^{-1/2} a^{1/2}\, \pi^{-1/4}\,. \tag{6.6.18}$$

Now, the width of isodistribution (6.6.16) is given by $\Delta x \approx a/T^{\frac{1}{2}}$, while the width of its isotransforms (6.6.18) is $\Delta k \approx 1/(aT^{\frac{1}{2}})$, and this establishes isouncertainties (6.6.11),

$$\Delta x \approx a/T^{\frac{1}{2}}\,, \quad \Delta k \approx 1/(aT^{\frac{1}{2}})\,, \quad \Delta x\, \Delta k \approx \hat{1}\,. \tag{6.6.19}$$

The implications of the above findings are manifestly far reaching. In fact, they confirm the existence and consistency of a step-by-step isotopic generalization of quantum mechanics into hadronic mechanics, which has been conceived and worked out for physical conditions of particles (those of the interior dynamical problem) fundamentally different than those of conception, applicability and experimental verification of quantum mechanics (those of the exterior dynamical problem).

Heisenberg's uncertainties are mathematically and physically valid in the arena of their conception and experimental verification, e.g., for an electron moving in an atomic orbit in vacuum. The isouncertainties have instead been conceived for the same electron when moving within hyperdense physical media, such as the core of a collapsing star. In this latter case, the isotopy $1 \Rightarrow \hat{1}$ is expected to provide a quantitative treatment suitable for experimental verifications of *integral* corrections to Heisenberg's uncertainties due to: the total immersion of the wavepacket of the electron within those of the surrounding particles; the inhomogenuity and anisotropy of the medium; and other *physical differences* with respect to motion in vacuum (see Vol. II for details).

Note that isouncertainties (6.6.11) depend on the preservation of the isogaussian character under Fourier isotransforms. In turn, this is dependent on the basic isoexponentiation of the Lie–isotopic theory. In fact, starting from the isoexponential

$$e_{\xi}^{-a^2 / 2 a^2} = \hat{1}\, e^{-r^2 T / 2 a^2}, \tag{6.6.20}$$

we also end up with the isoexponential

$$e_{\xi}^{-k^2 a^2 / 2} = \hat{1}\, e^{-k^2 T a^2 / 2}, \tag{6.6.21}$$

In turn, the preservation of this isoexponential character is precisely the mechanism that alters Heisenberg's uncertainties via the isotopy $1 \Rightarrow \hat{1}$.

The isotopic techniques used in this section for the Fourier transforms are easily extendable to other transforms. We mention as an example, the *Laplace isotransform* also studied, apparently for the first time, in ref. [10]

$$\hat{F}(z) = \int_0 dx\, \hat{f}(x) * e_{\xi}^{-zx}, \quad z = \text{cost.} + iy, \tag{6.6.22}$$

The same techniques are then applicable to the *isotopies of Hankel, Mellin, Hilbert and other transforms* [see ref.s [20,21] for their conventional forms).

## 6.7: ISOFUNCTIONS AND THEIR OPERATIONS

We shall now study the isotopies of a few representative elementary functions and the primary operations on them.

**Definition 6.7.1:** *Let* f(x) *be an ordinary function verifying the needed regularity and continuity conditions on a given closed interval of the real variable* $x \in R(n,+,\times)$. *Then the "isotopic image"* $\hat{f}(\hat{x})$ *of* f(x), *is a function of the corresponding closed isointerval of the isoreal number* $\hat{x} = x\hat{1} \in \hat{R}(\hat{n},+,*)$ *generally given by the rule*

$$\hat{f}(\hat{x}) = \hat{1}\, f(x) . \tag{6.7.1}$$

We have already encountered several *elementary isofunctions* during our analysis, such as the *isopower*

$$\hat{f}(\hat{x}) = \hat{x}^{\hat{n}} = \hat{x} * \hat{x} * ..... * \hat{x}\ (\text{n-times}) = \hat{1}\,(x^n) . \tag{6.7.2}$$

A most fundamental isofunction is the *isoexponentiation* of Sect. I.4.3. When written in terms of an isonumber $\hat{x}$, it also follows rule (6.7.1),

$$\hat{e}^{\hat{x}} := e_{\xi}{}^{\hat{x}} = \hat{1}\,\{e^{T\hat{x}}\} = \hat{1}\,\{e^{T\hat{1}x}\} = \hat{1}\, e^{x} , \tag{6.7.3}$$

where $e^x$ is the ordinary exponentiation.

The *isologarithm* of an isonumber $\hat{a} \in \hat{R}(\hat{a},+,*)$ on isobasis $\hat{e} = e\hat{1}$ can be then defined as the quantity $\hat{\log}_{\hat{e}}\, \hat{a}$ such that

$$\hat{e}^{\hat{\log}_{\hat{e}} \hat{a}} = \hat{a} , \tag{6.7.4}$$

with evident(and unique) solution

$$\hat{\log}_{\hat{e}}\, \hat{a} = \hat{1}\, \log_e a . \tag{6.7.5}$$

where $\log_e a$ is the ordinary logarithm on basis e of the ordinary number a.

It is easy to se that the above definition of the isologarithm characterizes a correct isotopy because it preserves all the conventional properties of log a, such as (we ignore in the following the subscripts $\hat{e}$ and e for simplicity)

$$\hat{\log}\, \hat{e} = \hat{1}, \quad \hat{\log}\, \hat{1} = 0, \tag{6.7.5a}$$

$$\hat{\log}\, \hat{a} * \hat{b} = \hat{\log}\, \hat{a} + \hat{\log}\, \hat{b} , \quad \hat{\log}\, \hat{a}\, \hat{/}\, \hat{b} = \hat{\log}\, \hat{a} - \hat{\log}\, \hat{b} , \tag{6.7.6b}$$

$$\hat{\log}\, \hat{a}^{-\hat{1}} = -\,\hat{\log}\, \hat{a} , \quad \hat{b} * \hat{\log}\, \hat{a} = \hat{\log}\, \hat{a}^{\hat{b}} , \text{ etc.} \tag{6.7.6c}$$

A similar situation occurs for the isotopy of most, but not all functions. In fact, two exceptions are given by the isotopy of the trigonometric and hyperbolic functions, which were preliminarily identified in Ch. I.5, and are studied in more detail in App. I.6.A. 6C

The isotopies of derivatives and integrals are intriguing because of the *variety* of the emerging novel notions. In fact, by again assuming T independent of x for simplicity, we can introduce the *three* different isodifferentials $\hat{d}_1 x = dx$, $\hat{d}_2 x = d(Tx) = T dx$ and $\hat{d}_3 = d(Ix) = \hat{1} dx$. We then have the *isoderivatives of the first, second and third kind*

$$\frac{\hat{d}_1}{\hat{d}_2 x}\hat{f}(\hat{x}) = \frac{\hat{d}_2}{\hat{d}_2 x} f(x) = \hat{1}\frac{d}{dx} f(x), \quad \frac{\hat{d}_3}{\hat{d}_3 x}\hat{f}(\hat{x}) = \frac{d}{dx} f(x) \qquad (6.7.7)$$

as the reader can derive via more rigorously via *isolimts* here omitted for brevity.Similarly, we have the three indefinite *isointegrals of the first, second and third kind* $\hat{\int}_1 = \int$, $\hat{\int}_2 = \hat{1}\int$ and $\hat{\int}_3 = T\int$ which verify the axions $\hat{\int}_k \hat{d}_k x = x$, $k = 1, 2, 3$. *Definite isointegrals* can be defined accordingly, e.g., for $\sigma = [a, b]$ being a closed isointerval of x,

$$\hat{\int}_{\hat{k}_\sigma} \hat{f}(\hat{x}) * \hat{d}_k x = \int_{\hat{\sigma}} f(x)\, dx, \quad k = 1, 2, 3. \qquad (6.7.8)$$

A virtually endless number of isotopic liftings of conventional treatments (see, e.g., ref. [22]) can be introduced, but its study is here left to the interested mathematician. We finally mention that the *isospecial functions*, such as the isotopies of Legendre, Bessel and other special functions, will be studied in Vol. II as needed for specific applications.

## APPENDIX 6.A: ISOTRIGONOMETRIC, ISOHYPERBOLIC FUNCTIONS AND ISOGAUSS PLANE

In Sect. 2.5 we pointed out the inapplicability of the conventional trigonometry and related Gauss plane for the characterization of the isocomplex numbers. In Sect. 5.2 we then showed the inapplicability of trigonometry. In this appendic we study its isotopic generalization, called *isotrigonometry* first studied in [10].

Consider a conventional two-dimensional Euclidean space $E(r,\delta,R)$, $\delta = \text{diag.}(1, 1)$ over the reals $R(n,+,\times)$. The fundamental notion of this space, the familiar distance among two points

$$D^2 = (x_1 - x_2)(x_1 - x_2) + (y_1 - y_2)(y_1 - y_2) \in R(n,+,\times), \qquad (6.A.1)$$

then represents the celebrated *Pythagorean theorem* expressing the hypothenuse D of a right triangle with sides A and B, $D^2 = A^2 + B^2$.

A property of the space $E(r,\delta,R)$ permitting its reinterpretation as the Gauss plane and other applications is the trigonometric notion of "$\cos\alpha$" where $\alpha$ is the angle between two intersecting straight vectors which, for the case when they originate from the origin and go to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{\frac{1}{2}} (x_2 x_2 + y_2 y_2)^{\frac{1}{2}}} . \tag{6.A.2}$$

From the above definition one can derive the entire conventional trigonometry. For instance, by assuming that the points are on a circle of unit radius $D^2 = 1$, we have for $P_1(x_1, y_1)$ and $P_2(1, 0)$ $\cos\alpha = x_1$, for $P_2(0, 1)$ we have $\sin\alpha = y_1$, with consequential familiar property of the Gauss plane $\sin^2\alpha + \cos^2\alpha = 1$, etc

All the above properties lose mathematical and physical significance under isotopy for numerous independent reasons, such as the loss of the central notion, that of distance, Eq. (6.A.1), the generally curved character of the lines, etc. The use of conventional geometries, those defined over ordinary fields, is also inapplicable because they are based on the conventional unit 1, while under isotopies we must necessarily redefine the unit.

These and other reasons have rendered mandatory the generalization of conventional trigonometry under isotopy. Besides numerous predictable applications, the problem has central relevance for the generalization of the eigenfunction of the rotational symmetry, the Legendre functions and spherical harmonics (see Vol. II).

Consider the isotopic lifting $E(r,\delta,R)$ into the now familiar *two-dimensional isoeuclidean space* over the isoreals $\hat{R}(n,+,*)$ (Sect. 3.3) here assumed for simplicity in the diagonalized form

$$\hat{E}(r,\hat{\delta},\hat{R}) : r = (x, y),\ \hat{\delta} = T\delta = \mathrm{diag.}\,(b_1^{\,2}, b_2^{\,2}),\ \ \hat{R} = \hat{R}(n,+,*) , \tag{6.A.3a}$$

$$\hat{1} = T^{-1} = \mathrm{diag.}\,(b_1^{\,-2}, b_2^{\,-2}), \quad b_k = b_k(t, r, \dot{r}, \ddot{r}, ....) > 0,\ k = 1, 2 . \tag{6.A.3b}$$

Consider now two points $P_1(x_1, y_1)$ , $P_2(x_2, y_2) \in \hat{E}(r,\hat{\delta},\hat{R})$. Then the conventional distance (6.A.1) is necessarily (and uniquely) generalized into the *isodistance*

$$D^{\hat{2}} = (x_1 - x_2)\, b_1^{\,2} (x_1 - x_2) + (y_1 - y_2)\, b_2^{\,2} (y_1 - y_2) \ \in \hat{R} . \tag{6.A.4}$$

It evidently characterizes an infinite family of isotopies of the Pythagorean theorem, called *isopythagorean theorem*, according to which the image of expression (6.A.2) under isotopies is given by

$$D = (A\, b_1^{\,2}(t, r, \dot{r}, ...)\, A + B\, b_2^{\,2}(t, r, \dot{r}, ...)\, B)^{\frac{1}{2}}\, \hat{1}^{\frac{1}{2}} , \tag{6.A.5}$$

Suppose now that the two points $P_1$ and $P_2$ also represent *isovectors* originating from the origin of $\hat{E}(r,\hat{\delta},\hat{R})$. Let us denote with $\hat{\alpha}$ the *isoangle* between these two isovectors to be identified below. We can then introduce the notion of *isocosinus* of the isoangle $\hat{\alpha}$ via the definition[49]

[49] The attentive reader may have noted the conventional square in the denominator. However, the isofactor in the denominator cancels out with that of the numerator,

$$\text{isocos}\,\hat{\alpha} \;=\; \frac{x_1\, b_1{}^2\, x_2 \;+\; y_1\, b_2{}^2\, y_2}{(\, x_1\, b_1{}^2 x_1 \;+\; y_1 b_2{}^2\, y_1\,)(\, x_2\, b_1{}^2\, x_2 \;+\; y_2\, b_2{}^2\, y_2\,)} \tag{6.A.6}$$

We now assume again the two points to be on the ellipse

$$D^2 = \quad x\, b_1{}^2\, x \;+\; y\, b_2{}^2\, y \;=\; 1\,, \tag{6.A.7}$$

and for the values $P_1\,(x_1, y_1)$, $P_2\,(b_1{}^{-1}, 0)$, the isocosinus becomes

$$x_1 = b_1{}^{-1} \cos\hat{\alpha} \;=\; \text{isocos}\,\hat{\alpha}\;. \tag{6.A.8}$$

We can then define the isosin $\hat{\alpha}$ by assuming $P_2\,(\,0\,,\, b_2{}^{-2}\,)$ for which

$$y_2 \;=\; b_2{}^{-1} \sin\hat{\alpha} \;=\; \text{isosin}\,\hat{\alpha}\;. \tag{6.A.9}$$

The following property then holds

X

$$b_2^2\;\text{isocos}^2\,\hat{\alpha} \;+\; \overset{b_1^2}{\text{isosin}^2}\,\hat{\alpha} \;=\; 1 \tag{6.A.10}$$

where we have ordinary squares rather than isosquares.

The introduction of the isotopies of conventional polar coordinates can help in understanding better the above formalism, as well as in identifying the explicit form of the isoangle $\hat{\alpha}$ from the theory of isorotations in a plane [7],

$$\hat{\alpha} \;=\; b_1\, b_2\, \alpha \;=\; \hat{\alpha}(t, r, \dot{r}, \ldots)\,. \tag{6.A.11}$$

By recalling Eq.s (6.A.8) and (6.A.9), the *isopolar coordinates* in the isogauss plane with value (6.A.11) are then given by

$$x \;=\; \text{isocos}\,\hat{\alpha} \;=\; b_1{}^{-1} \cos\,(\, b_1\, b_2\, \alpha\,)\,, \tag{6.A. 12a}$$

$$y \;=\; \text{isosin}\,\hat{\alpha} \;=\; b_2{}^{-1}\, \sin\,(\, b_1\, b_2\, \alpha\,)\,. \tag{6.A.12b}$$

whose verification of property (6.A.11) is evident

$$x\, b_1{}^2\, x \;+\; y\, b_2{}^2\, y \;=\; \overset{b_1^2}{\text{isosin}^2}\,\hat{\alpha} \;+\; \overset{b_2^2}{\text{isocos}^2}\,\hat{\alpha} \;=\; 1\,. \tag{6.A.13}$$

We reach in this way the *isotopy of the trigonometric functions*

$$\cos\alpha \quad\rightarrow\quad \text{isocos}\;(\, b_1\, b_2\, \alpha\,)\,, \tag{6.A.14a}$$

$$\sin\alpha \quad\rightarrow\quad \text{isosin}\,(\, b_1\, b_2\, \alpha\,)\,. \tag{6.A.14b}$$

The construction of the isotrigonometry in all the necessary details can be conducted accordingly.

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resulting in expression (6.A.6).

**ISOTOPY O OF THE PYTHAGOREAN THEOREM**

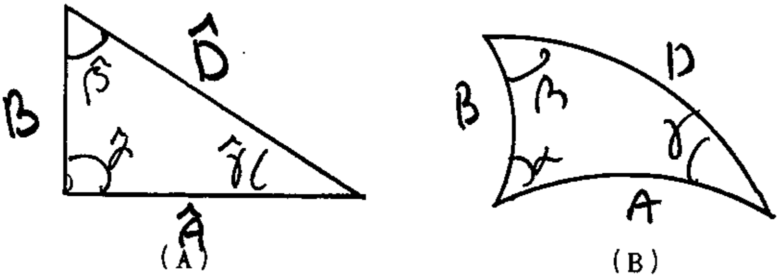


FIGURE 6.A.1: A schematic view of the isotopic image of the Pythagorean Theorem. Its understanding requires the knowledge that *a right triangle (A) remains a right triangle in isoeuclidean space, called "right isotriangle".* Diagram (B) is only *the projection of the right isotriangle in the conventional Euclidean space* (see Sect. 5.2) for more details. The considerations below are for the latter. To begin, the conventional sides are lifted into curves. Then, the original angle $\alpha$ between two generic sides is lifted into the isoangle $\hat{\alpha} = b_1 b_2 \alpha$. Despite these generalizations, a relationship between the *isohypothenuse* and the *isosides* still exists, and it is given by

$$D^2 = A\, b_1{}^2 A + B\, b_2{}^2 B . \tag{1}$$

As a matter of fact, this is precisely the central geometric meaning of isoeuclidean spaces. It may be intriguing and instructive for the interested reader to note that all conventional trigonometry on a plane then admits a consistent and nontrivial isotopy. As an example, the isoside $A$ is given by

$$A = D\, b_1{}^{-1} \cos \hat{\alpha} = D \text{ isocos } \hat{\alpha} , \tag{2}$$

while the isoside $B$ is given by

$$B = D\, b_2{}^{-1} \sin \hat{\alpha} = D \text{ isosin } \hat{\alpha} , \tag{3}$$

which are again consistent with basic theorem (1), and can independently be derived via the use of the isotopy of trigonometric functions (6.A.18). Note that the trigonometric functions are deformed both in intensity and in their angle as shown below, which properties render them particularly intriguing, e.g., for the study of deformation of potential wells in nuclear physics.

The rest of the properties of triangles are lifted accordingly. As an example, the original $\alpha, \beta, \gamma, \alpha + \beta + \gamma = 180°$, are deformed under isotopies into the new values $\alpha, \beta, \gamma$ such that $\alpha + \beta + \gamma \neq 180°$. However, the peculiarity of the isoeuclidean geometry is such that the deformations $\hat{\alpha} \rightarrow$

$\alpha$, etc. are compensated in such a way that $b_1 b_2 \alpha = \hat{\alpha}$. The sum of the angle of (any) isotriangle is then

$$b_1\, b_2 ( \alpha + \beta + \gamma ) = \hat{\alpha} + \hat{\beta} + \hat{\gamma} = 180°.$$

All other properties can then be derived accordingly. To understand the novelty one should note that the above generalization of the Pythagorean theorem does not exist in conventional geometries, those constructed with respect to conventional units, such as the Riemannian geometry. As we shall see in Vol. II, the applications of these novel ideas are novel and intriguing indeed.

We here limit ourself to mention only a few properties. For instance, *the isotrigonometric functions are also periodic of period* $2\pi$,

$$\text{isocos}\ ( \hat{\alpha} + 2\pi ) \equiv \text{isocos}\ \hat{\alpha}, \quad \text{isosin}\ ( \hat{\alpha} + 2\pi ) \equiv \text{isosin}\ \hat{\alpha}. \tag{6.A.15}$$

The following symmetry properties then follow as in the conventional case

$$\text{isocos}\ -\hat{\alpha} \equiv \text{isocos}\ \hat{\alpha}, \quad \text{isosin}\ -\hat{\alpha} = -\ \text{isosin}\ \hat{\alpha}. \tag{6.A.16}$$

Similarly, the theorems of isoaddition become

$$\text{isosin}\ ( \hat{\alpha} \pm \hat{\beta} ) = b_1^{-1} ( \text{isosin}\ \hat{\alpha}\ \text{isocos}\ \hat{\beta} \pm \text{isocos}\ \hat{\alpha}\ \text{isosin}\ \hat{\beta} ), \tag{6.A.17a}$$

$$\text{isocos}\ ( \hat{\alpha} + \hat{\beta} ) = b_1^{2} ( b_2^{-2}\ \text{isocos}\ \hat{\alpha}\ \text{isocos}\ \hat{\beta} \pm b_1^{-2}\ \text{isosin}\ \hat{\alpha}\ \text{isosin}\ \hat{\beta} ) \tag{6.A.18b}$$

$$\text{isosin}\ \hat{\alpha} + \text{isosin}\ \hat{\beta} = 2\, b_1^{-1}\ \text{isosin}\ \tfrac{1}{2} ( \hat{\alpha} + \hat{\beta} )\ \text{isocos}\ \tfrac{1}{2} ( \hat{\alpha} - \hat{\beta} ), \tag{6.A.19c}$$

$$\text{isocos}\ \hat{\alpha} + \text{isocos}\ \hat{\beta} = 2\, b_1^{-1}\ \text{isocos}\ \tfrac{1}{2} ( \hat{\alpha} + \hat{\beta} )\ \text{isocos}\ \tfrac{1}{2} ( \hat{\alpha} - \hat{\beta} ), \tag{6.A.19d}$$

etc.

The connection between isotrigonometric functions and isoexponentiation calls for the preferred isounit

$$\hat{1} = b_1^{-1}\, b_2^{-1} = ( \det \hat{1} )^{\frac{1}{2}}, \quad \hat{T} = b_1\, b_2 = ( \det T )^{\frac{1}{2}} \tag{6.A.20}$$

under which we have the alternative definition of isotrigonometric functions

$$\hat{e}^{\ i\,\hat{\alpha}} = \hat{1}\, e^{\ i\, b_1\, b_2\, \alpha} = b_2^{-1}\ \text{isocos}\ \hat{\alpha} + i\, b_1^{-1}\ \text{isosin}\ \hat{\alpha}, \tag{6.A.21}$$

where the isoenvelope is now defined for product $a * b = a \hat{T} b$ and isounit $\hat{1}$. We can then also introduce the *isohyperbolic functions*

$$\text{isocosh}\ \hat{\alpha} = b_1^{-1} \cosh ( b_1\, b_2\, \alpha ), \quad \text{isosinh}\ \hat{\alpha} = b_2^{-1} \sinh ( b_1\, b_2\, \alpha ), \tag{6.A.22}$$

with basic property

$$b_1^2 \text{ isocosh}^2 \hat{\alpha} - b_2^2 \text{ isosinh}^2 \hat{\alpha} = 1, \qquad (6.A.23)$$

and their derivation via the isoeponentiation

$$\hat{e}^{\hat{\alpha}} = \hat{1}\, e^{b_1 b_2 \alpha} = b_2^{-1} \text{ isocosh } \hat{\alpha} + b_2^{-1} \text{ isosinh } \hat{\alpha}\,. \qquad (6.A.24)$$

It may be intriguing and instructive for the interested reader to work out additional properties of isotrigonometric and isohyperbolic functions.

We close this appendix with the following introguing propeerty of the isogauss theory

**Lemma 6.B.1:** *All possible algebraic or transcendental functions* $f(x, y) = 0$ *in the Gauss plane can be unified into the isocircle in the isogauss plane of Class III*

In fact, for any $f(x,y) = 0$ there always exist elements $\hat{\delta}_{ij}$ such that $f(x,y) \equiv r^i \hat{\delta}_{ij} r^j - 1 = 0$. The above property can be illustrated via a certain geometric complementarity between tyebcircle $xx + yy = 1$ and the tractrix

$$y\, e^{1 - (1 - y^2)^{\frac{1}{2}}} + (1 - y2)^{\frac{1}{2}} = 1 \qquad (6.A.25)$$

recently studied by Hecht [27] with intriguing gravitational implications studied in Vol. II (see Fig. 6.B.1) It is evident that tractrix (6.B.1) is unified into the isocircle by the diagonal isometric with elements $\hat{\delta}_{11} = yx^{-2}\exp\{1 - (1 - y^2)^{\frac{1}{2}}\}$, $\hat{\delta}_{22} = y^{-1}(1 - y^2)^{\frac{1}{2}}$.

## APPENDIX 6.B: OTHER GENERALIZATIONS OF FUNCTIONAL ANALYSIS

A considerable number of generalizations of functional analysis of non-isotopic type exist in the literature, some of which dating back to the past century. They are all *independent* from the isotopic generalization because derived from different assumptions. As such, they all have their own value. Regrettably, we cannot review them here for brevity, and must limit ourselves to indicate those most significant for our studies.

The generalization of functional analysis based on the so-called *q-deformations* (see, e.g., ref.s [23]) is particularly relevant for hadronic mechanics, and includes q-number-generalizations of ordinary and special functions, the operations defined on them, etc.

The differences between the isotopic and q-functional analysis are numerous, such as the central dependence of the former is on th lifting of th unit and the preservation of the conventional unit for the latter, the validity of the former for arbitrary integro-differential operators T, and that of the latter for q-numbers, etc. (see App. I.7.A). Nevertheless, a knowledge of the q-functional analysis is unquestionably useful for the construction of the isospecial functions, as we shall see in Vol. II.

**GEOMETRIC COMPLEMENTARITY BETWEEN CIRCLE AND TRACTRIX**

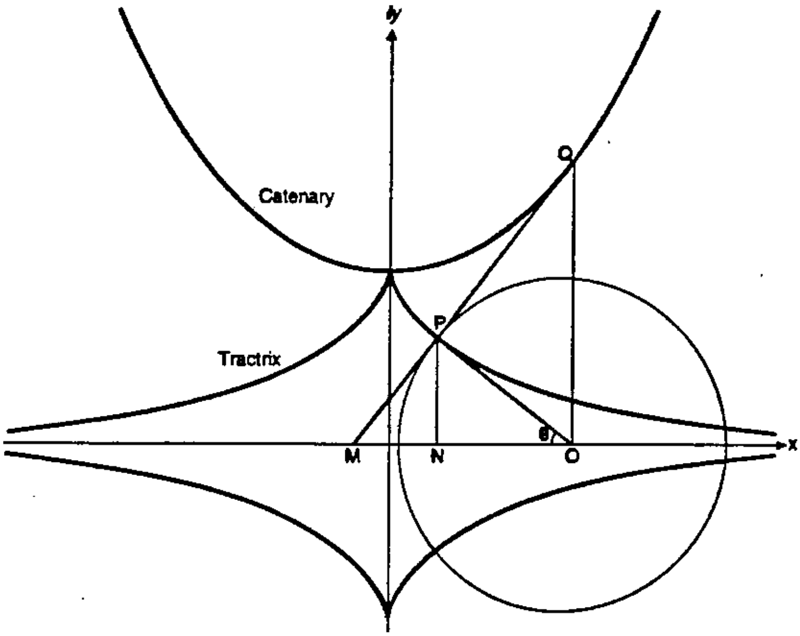


FIGURE 6.A.2: A reproduction of of one of the figures in Hecht [27] showing a geometric complementiarity between the *circle* and the *tractrix* (which is the lower curve, the upper one being the *catenary* of which the tractrix is the involute). By recalling that the circle is the curve with constant positive curvature, $K = +1/r^2 = \text{cost.}$, Hecht [loc. cit.] has shown that *the tractrix is the curve with the constant negative curvature*, $K = -1/r^2$, where r is now the length of the equitangent. The tractix is therefore significant for theoretical studies on antigravity (see Vol. II). The interpretation o thef y axis as bei g purely imaginary, has then permitted Hecht to reach *a complementary formulation of transcendental functions via the trigonometric ones*, e.g.,

$$\sin^2\theta - \cos^2\theta = 1, \; 1 - \text{tag}^2\theta = \sec^2\theta, \text{cotg}^2\theta - 1 = \text{cosec}^2\theta \quad \text{(a)}$$

which is an intriguing realization of the isogauss plane.

Among the great variety of q-number theories, a significant realization was conceived by Dirac (see the review [24], p. 320 ff.) whose study is also recommended here. Unfortunately, the differences between Dirac's q–numbers and the others q–numbers are so great to be misleading.[50] For this reason we shall refer to them via the alternative name of *queer numbers* suggested by Dirac himself.

Yet another generalization, this time, of the conventional differential calculus is the so–called *Helmholtz's calculus* (see, ref. [25]). This generalization too is significant for these volumes because it leads to an inevitable generalization of conventional relativities although different and independent from the isotopic one.

Additional special forms of differential calculus exist in the literature, depending on the needs at hand. We indicate, for instance, the *small derivative calculus* developed by Gonzalez–Diaz and Jannussis [26], which is specifically conceived for small distances and exhibits rather intriguing properties.

By no means the above indications exhaust all existing generalizations of conventional functional analysis. Additional novel possibilities can be found in the monograph by Lôhmus, Paal and Sorgsepp [28]. A further approach is presented in the monograph by Vougiouklis [29] via the *algebraic hyperstructures* (also called *multivalued algebras*) and the so–called $H_V$–*structures*, in which associativity, distributivity and commutativity are replaced by their weak forms. The latter approach also implies the chain of generalized *hyperfields, hyperspaces, hyperalgebras, hypergroups*, etc. with intriguing possibilities for isotopic reformulation and application to interior dynamical problems.

Nevertheless, none of these generalizations require a lifting of the basic unit, thus illustrating the uniqueness as well as independence of the isoanalysis of this section.

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# 7: LIE-ADMISSIBLE THEORY

## 7.1: STATEMENT OF THE PROBLEM

A central assumption of hadronic mechanics is that one operator alone, the Lagrangian or the Hamiltonian, is *insufficient* to represent physical reality, which needs instead (3N + 1)–quantities, the Lagrangian or Hamiltonian, plus the 3N diagonal elements of the isounit $\hat{1}$.

By no means is the above assumption new, because its origins go back to Lagrange [1], Hamilton [2], Jacobi [3] and other founders of analytic dynamics. The novelty is merely in the *realization* of the 3N additional quantities via the elements of the isounit.

The equations originally conceived by Lagrange and Hamilton *are not* those available in the contemporary mathematical and physical literature, but equations *with external terms*. In fact, the *true Lagrange's equations* for a system of N particles in three–dimensional Euclidean space are given by [1]

$$\frac{d}{dt}\frac{\partial L(t, r, \dot{r})}{\partial \dot{r}^{ka}} - \frac{\partial L(t, r, \dot{r})}{\partial r^{ka}} = F_{ka}(t, r, \dot{r}), \qquad (7.1.1a)$$

$$L = K(\dot{r}) - V(t, r, \dot{r}), \quad k = 1, 2, 3 \,(= x, y, z), \quad a = 1, 2, ..., N, \qquad (7.1.1b)$$

the *true Hamilton's equations* are [2]

$$\dot{r}^{ka} = \frac{\partial H(t, r, p)}{\partial p_{ka}}, \quad \dot{p}_{ka} = -\frac{\partial H(t, r, p)}{\partial r^{ka}} + F_{ka}(t, r, p), \qquad (7.1.2a)$$

$$H = K(p) + V(t, r, p), \qquad (7.1.2b)$$

and the *true Jacobi theorem* [3] is also that with external terms.

As one can see, Eq.s (7.1.1) and (7.1.2) require precisely (3N + 1)–quantities for the representation of physical reality, a Lagrangian or a Hamiltonian, plus the 3N external forces $F_{ka}$.

By comparison, the analytic equations of the contemporary literature are the so–called *"truncated Lagrange's and Hamilton's equations"*; i.e., those *without* external terms. As a result of a scientific process still ignored by contemporary historians, the external terms were progressively removed from the analytic equations sometime by the end of the past century, to acquire the form almost universally used nowadays.

The origin of this "truncation" appears to be the birth of Lie's theory [4] in the second part of the past century. In fact, the brackets of the true Hamilton's equations, not only violate the Lie algebras axioms, but actually violate all conditions to constitute an algebra, whether Lie or not (see below). The achievement of a classical realization of Lie algebras therefore required the elimination of the external terms. The historical successes of the truncated Hamilton's equations in the description, first, of planetary systems (see, e.g. ref. [5]) and then of the atomic structure (see, e.g., ref. [6]) provided a major drive toward the current elimination of the external terms.

However, by no means, has this scientific process suppressed the vision of Lagrange and Hamilton. In fact, the "truncated analytic equations" can directly represent[51] only conservative systems and a restricted class of other systems. By comparison, the "true analytic equations" are directly universal for all possible systems of the physical reality, whether conservative or not. In fact, the Lagrangian and Hamiltonian represent all conservative forces, while all remaining forces are directly represented with the external terms.

The primary difference is that, while the truncated equations represent *closed–isolated systems* with conserved total energy, the true equations represent instead *open–nonconservative systems* with the following *time rate of variation of the energy* [52]

$$d H / dt = \sum_{ka} v_{ka} F_{ka} \neq 0, \tag{7.1.3}$$

due to interactions with systems interpreted as external.

This author conducted his graduate studied at the Universitá di Torino, Italy, where Lagrange did most of his work, thus having the opportunity of studying Lagrange's original papers and and comparing them with contemporary literature. The latter is essentially based on the trend to reduce all physical systems, whether classical or quantum mechanical, to a form representable by the truncated analytic equations. By contract, Lagrange and Hamilton were fully aware that the quantities today called "Lagrangian" and "Hamiltonian" cannot

[51] We are here referring to a representation with a direct physical meaning of all algorithms at hand, whereby r represents the actual frame of the observer, H represents the actual total energy K + V, p represents the actual linear momentum $m\dot{r}$, etc.

[52] We here adopt the definition of *nonconservation* of ref.s [8,9] in which the energy can monotonically *increase or decrease* in time, while *dissipation* is referred to the case when the energy solely *decreases* in time.

represent the totality of the physical reality, but only a very small part of it and, for this reason, they introduced the external terms to their equations.

Subsequent rigorous studies on the integrability conditions for the existence of a Lagrangian or a Hamiltonian, *Helmholtz's conditions of variational selfadjointness* [7,8], proved Lagrange's and Hamilton's vision in its entirety. In fact, the broadest possible class of Newtonian systems (those of the *interior dynamical problem*) result in being nonrepresentable with a Lagrangian or a Hamiltonian in the coordinates of the observer (These are the so-called *essentially nonselfadjoint systems* [8]).

We mentioned earlier the contemporary trend of using the Lie-Koening theorem (see ref. [9] and quoted literature) to turn nonlagrangian-nonhamiltonian systems into equivalent forms which are representable with the truncated analytic equations. Yet, Lagrange and Hamilton's vision remains broader then the contemporary one in this respect too for the reasons indicated earlier (lack of general applicability of the Lie-Koening theorem, e.g., for integral or discontinuous systems, lack of realization of the transformed frame in the laboratory, loss of conventional relativities because of the highly noninertial character of the transformed frame, etc.).

But even ignoring all this, and assuming that some artificial construction permits the construction of a Lagrangian or a Hamiltonian for the truncated equations, the physical significance of these quantities is unclear, controversial and manifestly misleading, particularly in the operator treatment of nonconservative forces.[53]

Because of the above occurrences, this author spent his research life studying the true, historical, Lagrange and Hamilton equations *with* external terms, according to the following two main lines:

**Isotopies** [7] These are the methods outlined in the preceding chapters possessing a *Lie-isotopic structure*, which can now be reinspected from a different viewpoint. In fact, these methods were conceived to preserve the basic

[53] The literature in particle physics is full of models in which the physical structure of the Hamiltonian

$$H = K + V = \text{Kin. Energy} + \text{Pot. Energy}. \qquad (1)$$

is generalized into canonical expressions of the type

$$H = p^2/2m + V(r), \qquad p = \alpha\, e^{\beta r t}, \quad \alpha, \beta \in R. \qquad (2)$$

Yet H is continued to be interpreted as "the total energy" while in reality H is a pure mathematical quantity (a first integral). The real total energy $E = K + V$ is nonconserved because of interactions not properly identified as being external, i.e. $[E, H] \neq 0$. The "physical conclusions" of these models are unsettled at best. This is another illustration of *the paramount importance of solely using "direct representations" (as identified in the preceding footnote) whenever studying nonpotential forces.*

axioms of the truncated analytic equations, yet requiring (3N+1)–quantities for the representation of physical reality and permitting the same direct representation of the true analytic equations with external terms.

**Genotopies** [7]. These are the more general methods outlined in this chapter with the covering *Lie–admissible structure.*

A central property represented by Eq.s (7.1.1) and (7.1.2) is that conventional closed–conservative systems are a particular case of the more general open–nonconservative ones. In fact, the conventional conservation law of the energy is a particular case of the more general laws (7.1.3) on the *time–rate–of–variation of the energy.* As a result, we expect the existence of covering methods for the treatment of open nonconservative systems which admit the conventional Lie and Lie–isotopic methods as particular cases.

*A central problem for a quantitative study of open nonconservative systems is therefore the identification of a covering of both, Lie and Lie–isotopic theories which permits a direct representation of the time–rate–of–variation of physical quantities; that is, a representation in which all quantities* H, r, p, r∧p, *etc., have a direct physical meaning, change their value in time and admit conservation laws as particular cases.*

Recall that the conservation of the energy for Lie and Lie–isotopic formulations,

$$dH/dt = [H, H] = H\,H - H\,H \equiv 0, \qquad (7.1.3a)$$

$$dH/dt = [H\hat{,}H] = H\,T\,H - H\,T\,H \equiv 0, \qquad (7.1.3b)$$

are a consequence of the anticommutativity of the products [A, B] and $[B\hat{,}A]$. Thus, the above requirements can be expressed by the following conditions, originally submitted in ref.s [12–14] and then studied in detail in ref.s [7,10,11]:

**Condition 7.1.1:** The brackets, say A⊙B, of the analytic equations characterizing interactions under *external* forces must not be anticommutative, $A \odot B \neq - B \odot A$, as a necessary condition to represent the time–rate–of–variation of the energy and of other physical quantities

$$i\, dH / dt = H \odot H = f(t) \neq 0; \qquad (7.1.4)$$

**Condition 7.1.2:** The new brackets A⊙B must recover the isotopic $[A\hat{,}H]$ or conventional Lie product [A, H] when all external forces are null

$$\lim_{\text{Ext. Forces} = 0} A \odot B \equiv [A\hat{,}B] \text{ or } [A, B]; \qquad (7.1.5)$$

**Condition 7.1.3:** The new brackets $A \odot B$ must, first, define a consistent algebra, and, second, that algebra must be a covering of the Lie and Lie-isotopic algebras, therefore admitting the latter in their classification.

As originally identified in ref.s [12-14], and confirmed in the subsequent studies [7,10,11], the algebras which verify all the above conditions are the *Lie-admissible algebras* preliminarily presented in App. I.4.A.

When joint with studies on the isotopic formulations, this occurrence permits the identification of the following chain of covering formulations:

$$\left[\begin{array}{c}\text{LIE}\\ \text{FORMULATIONS:}\\ \text{Closed-isolated}\\ \text{local-diff.}\\ \text{Hamiltonian}\\ \text{systems}\end{array}\right] \subset \left[\begin{array}{c}\text{LIE-ISOTOPIC}\\ \text{FORMULATIONS:}\\ \text{Closed-isolated}\\ \text{nonlocal-integral}\\ \text{nonhamiltonian}\\ \text{systems}\end{array}\right] \subset \left[\begin{array}{c}\text{LIE-ADMISSIBLE}\\ \text{FORMULATIONS:}\\ \text{Open-nonconserv.}\\ \text{nonlocal-integral}\\ \text{nonhamiltonian}\\ \text{systems}\end{array}\right]$$

In fact, the Lie-isotopic formulations were introduced in ref. [7] precisely as a particular form of the more general Lie-admissible structures because the antisymmetric algebras generally attached to the Lie-admissible ones are *not* Lie, but Lie-isotopic.

In this chapter we shall outline the rudiments of the Lie-admissible formulations with the understanding that they are considerably less developed than the corresponding Lie-isotopic methods, and so much remains to be done.

The mathematical relevance of the Lie-admissible theory is self-evident from their covering character over the conventional Lie and Lie-isotopic theories. Their physical relevance can be understood only after a knowledge of the problematic aspects of current formulations of nonconservative systems outlined in Sect. 7.2 below. The Lie-admissible formulations result to be as rather unique for a number of applications, including nonconservative systems, q-deformations, nonlinear and nonlocal theories, and others.

However, the primary mathematical and physical relevance of the Lie-admissible theory for which it was conceived [7-11] rests in the capability of providing an *axiomatic formulation on the origin of irreversibility*.

The scientific scene prior to the advent of hadronic mechanics is well known. On one side, experimental evidence establishes that macroscopic structures generally are irreversible at the Newtonian, statistical, thermodynamical and other levels. On the other side, only *one* theory for the macroscopic world, the reversible quantum mechanics, was then available. All past efforts in irreversibility have therefore been centered in attempting a *reconciliation of the macroscopic evidence of irreversibility with the only available microscopic theory* (see, e.g., the recent account [15] and quoted literature).

The advent of hadronic mechanics has altered this scientific scenario because of the structural irreversibility of its Lie-admissible branch. In fact, quantum mechanics emerges as being exact for the *exterior particle problem in vacuum* (such as the atomic structure) which is reversible also at the classical level (such as the planetary structure). Hadronic mechanics then emerges as the applicable theory for the *interior particle problem* which is irreversible at the particle level (such as the structure of a neutron star) and remains irreversible at its classical counterpart (such as the structure of Jupiter).

At any rate, the current studies on irreversibility cannot resolve the paradox caused by the *No-Reduction Theorems* of Ch. I.1, according to which an irreversible interior system, such as a satellite during re-entry in a monotonically decaying orbit, simply cannot be decomposed in any consistent way into a collection of elementary particles all in stable-reversible orbits, while such paradox is indeed resolved by hadronic mechanics (see Fig. 7.1.1).

The epistemological origins are the limitations of quantum mechanics (Sect. I.1.2) caused by its local-differential structure which does not permit an exact description of the nonlocal-integral conditions of interior problems. The advent of a structurally irreversible mechanics specifically built for interior problems evidently alter the scenario.

**ORIGIN OF IRREVERSIBILITY**

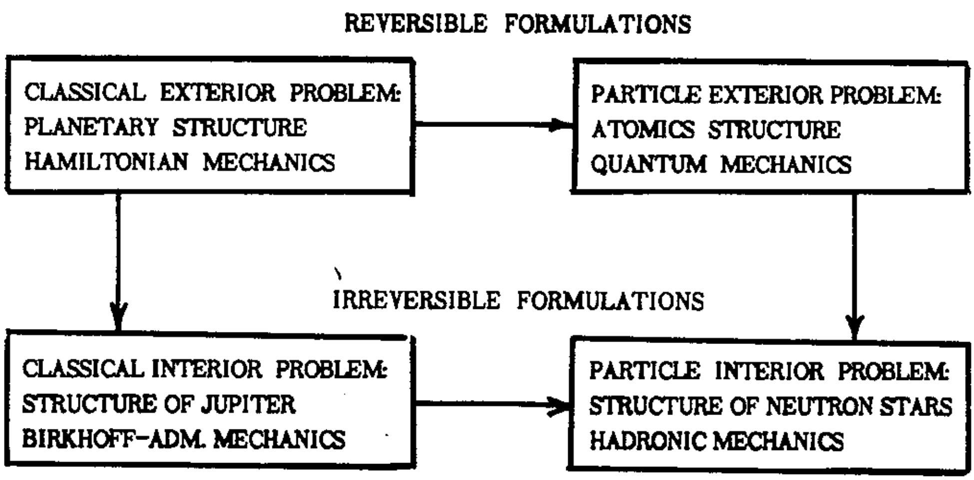


FIGURE 7.1.1. A schematic view of the scenario on irreversibility after the advent of hadronic mechanics. As well known, *exterior dynamical problems* of point particles in

vacuum are reversible at both the classical and particle levels, and so are the corresponding mechanics, the Hamiltonian and quantum mechanics. Irreversibility emerges in nature for *interior dynamical problems.* Once this fundamental distinction is understood, the scenario in irreversibility is completely altered. One begins with the need of a covering mechanics at the *purely classical level* because Hamiltonian mechanics cannot represent all interior Newtonian systems in the frame of the experimenter [8,9]. These studies have resulted in the construction of a generalization of Hamiltonian mechanics submitted in ref.s [10,11] under the name of *Birkhoff–admissible mechanics* which is structurally irreversible and therefore directly compatible with irreversibility at the subsequent statistical and thermodynamical levels. In these volumes we shall study the Lie–admissible branch of hadronic mechanics as the particle counterpart of the Birkhoff–admissible mechanics. Irreversibility then emerges as originating at the ultimate elementary level of *interior* particle problems, and then merely persists at the macroscopic level.

## 7.2: PROBLEMATIC ASPECTS OF CONTEMPORARY FORMULATIONS OF OPEN SYSTEMS

The best to way initiate the study of the Lie–admissible formulations is to see (and admit) rather serious problematic aspects in the contemporary formulation of open–nonconservative systems *beginning at the purely classical level,* which then persist at different levels.

They can be identified by inspecting the brackets of the time evolution at the various levels off description, such as in:

**Classical mechanics,** where nonconservative systems of N particles (labeled with a = 1, 2, ..., N) in Euclidean space with local coordinates $x^k$ (k = 1, 2, 3)represented via external forces $F_{ka}$, result in the following dynamical evolution of a quantity A(r,p)

$$dA / dt \overset{\text{def}}{=} A \times H = [A , H] + \frac{\partial A}{\partial p_{ka}} F_{ka} , \tag{7.2.1}$$

with

$$[A, H] = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} \tag{7.2.2}$$

being the conventional Poisson brackets;

**Quantum mechanics,** where nonconservative systems are generally represented by nonhermitean Hamiltonians of the type

$$H = H_0 + i V \neq H^\dagger , \tag{7.2.3}$$

as rather popular in nuclear physics, resulting in the dynamical equations

$$i\, dA / dt = A \times H = A H - H^{\dagger} A \qquad (7.2.4)$$

**Statistical mechanics,** where collisions and other effects are expressed also with external terms, classically and quantum mechanically, resulting in the following dynamical evolution of the density matrix $\rho$

$$i\, d\rho / dt = \rho \times H = [\rho, H] + C \qquad (7.2.5)$$

with

$$[\rho, H] = \rho H - H \rho \qquad (7.2.6)$$

being the classical or quantum, canonical brackets.

Note that all the above formulations correctly describe the time-rate-of-variation of the energy,

$$i\, dH / dt = H \times H = f(t) \neq 0 \qquad (7.2.7)$$

Therefore, the brackets A×H do indeed describe an open nonconservative system, by verifying Condition 7.1.1. The admission of the conventional Lie brackets as a particular case is trivial, and brackets A×H also verify Condition 7.1.2. The central point is that the above formulations violate the crucial Condition 7.1.3.

**Proposition 7.2.1** [7,10,11]: *The brackets of conventional formulations of open nonconservative systems, Eq.s (7.2.1), (7.2.4) and (7.2.5), do not constitute an "algebra" as commonly understood in mathematics (see Sect. 2.4 and App. 4.A) because they verify the right scalar and distributive laws,*

$$\alpha \times (A \times B) = (\alpha \times A) \times B = A \times (\alpha \times B), \qquad (7.2.8a)$$

$$(A + B) \times C = A \times B + B \times C, \qquad (7.2.8b)$$

*but violate the left distributive and scalar laws, i.e., for any scalar* $\alpha \neq 0$, $\alpha \in F$, *and elements* A, B, C, *we have*

$$(A \times B) \times \alpha \neq A \times (B \times \alpha) \neq (A \times \alpha) \times B, \qquad (7.2.9a)$$

$$A \times (B + C) \neq A \times B + A \times C, \qquad (7.2.9b)$$

In different terms, *in the transition from the conventional Lie formulations characterized by brackets* [A, H] *to the above classical, quantum or statistical brackets* A×H, *we have not only the loss of all Lie algebras, but in actuality we have the loss of all possible consistent algebraic structures.*

Additional mathematical properties are the following.

**Proposition 7.2.2** [loc. cit.]: *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent enveloping algebra.*

This can be seen in a number of ways, the most effective one being the fact that *Eq.s (7.2.5) cannot be exponentiated as for conventional Lie equations,* because they do not admit a consistent infinite basis (no Poincaré–Birkhoff–Witt theorem–see Sect. I.4.3).

**Proposition 7.2.3** [loc. cit.]: *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent unit.*

This can also be seen in a number of ways, e.g., from the lack of a consistent envelope needed to define the unit of the theory.

Rather than being mere mathematical curiosities, the physical implications of the above occurrences are rather serious, and can be summarized as follows (for a detailed study see ref. [11,16]):

**Problematic aspect 7.2.1:** *Eq.s (7.2.1), (7.2.4) and (7.2.5) do not admit a consistent measurement theory.* The fundamental notion of all measurements theories, whether classical, or quantum mechanical or statistical, is the unit with respect to which the measurements are referred to[54]. No formulation without a unit can therefore have a measurement theory usable for contemporary experiments. Note that one may indeed conduct measures. However, the insidious aspect is that they have no rigorous relationship to the theory at hand. The lack of existence of the unit for the equations considered can be established on numerous independent counts, e.g., from the lack of the envelope itself in which the unit is defined. The physical implications for plasma physics and other fields are self–evident.

**Problematic aspect 7.2.2:** *The angular momentum, spin, and other*

[54] The issue is technically deeper. In fact, a Hilbert space can certainly be defined over a field which, as such, possesses the unit 1, even for nonconservative systems. The point is that the enveloping operator algebra of the theory here considered has no unit, which implies that no operator can be "measured" in a consistent way. To understand the occurrence one should think at a quantum mechanical measurement in which Planck's constant cannot be defined.

*physical quantities characterized by Lie>s theory cannot be consistently defined under the generalized brackets A×H of the equations considered.* As well known, the angular momentum and spin are centrally dependent on the exact O(3) and SU(2) theory, respectively. Then, the same quantities are manifestly meaningless, mathematically and physically, for Eq.s (7.2.1), (7.2.4) and (7.2.5), trivially, because they have lost not only the entire Lie's theory, but the very notion of algebra. This is another occurrence which should not be taken lightly. As an example, the use of the terms "protons and neutrons with spin ½" has no mathematical or physical meaning when referred to Eq.s (7.2.4) in nuclear physics or Eq.s (7.2.5) in plasma physics.

**Problematic aspect 7.2.3:** *Loss of the conventional notion of particle.* Eq.s (7.2.4) have been generally used in nuclear physics over the past decades to describe nonconservative processes of nucleons. However, the quantum mechanical notion of protons and neutrons can be rigorously proved to be inapplicable to these equations and, if applied, to imply a host of inconsistencies. They are technically due to the loss of all means to characterize the conventional notion of particle.

**Problematic aspect 7.2.4:** *Loss of the rotational, Lorentz and other fundamental space–time symmetries.* This is evidently due to the lack of a consistent exponentiation and other technical reasons. Stated explicitly, *the open–nonconservative systems generally represented in the contemporary literature imply the inapplicability of Galilei's, Einstein's special and Einstein's general relativities.*

**Problematic aspect 7.2.5:** *Eq.s (7.2.1), (7.2.4) and (7.2.5) cannot consistently represent irreversibility.* As well known, Lie's theories for Hermitean Hamiltonians verify the *Theorem of Detailed Balancing* (se, e.g., ref. [17]) and, as such, they do consistently represent reversibility from first principles (Fig. 7.1.1). Such a theorem becomes manifestly inapplicable under nonunitary transformations as those underlying Eq.s (7.2.4), but no consistent generalization of the theorem of detailed balancing exists for brackets A×H, to our best knowledge. Thus, the equations considered cannot consistently represent irreversibility (see Vol. II for a Lie–admissible, irreversible generalization of the Theorem of Detailed Balancing [17]).

For additional problematic aspects the interested reader may consult ref.s [10,11,16].

It is hoped the reader can see *the need for a fundamental structural revision in the treatment of open nonconservative systems in their classical, particle and statistical formulation,* because any attempt at reconciling these systems with old knowledge will inevitably lead to inconsistencies.

## 7.3: HISTORICAL NOTES

The notion of *Lie-admissible algebra* was introduced by A. A. Albert in paper [18] of 1948. A generally nonassociative algebra U with elements a, b, c, ... and (abstract) product ab over a field $F(\alpha,+,\times)$[55] is called Lie-admissible when the attached algebra $U^-$, which is the same vector space as U (that is, the elements of U and $U^-$ coincide) but equipped with the product $[a, b]_U = ab - ba$, is Lie.

Since the attached product $[a, b]_U$ is antisymmetric, the sole condition for a product ab to be Lie-admissible is that the attached product $[a,b]_U$ verifies the *Jacobi identity*, i.e., the following axiom, called *axiom of general Lie-admissibility*, is identically verified

$$(a, b, c) + (b, c, a) + (c, a, b) - (c, b, a) - (b, a, c) - (a, c, b) = 0, \qquad (7.3.1)$$

where

$$(a, b, c) = (a\,b)\,c - a\,(b\,c) \qquad (7.3.2)$$

is called the *associator* (see also App. 4.A), and represents the departure of the algebra from an associative one.

Albert [18] identified only one nontrivial subcase of Lie-admissible algebras called *flexible Lie-admissible algebras* and characterized by the axioms

$$(a, b, a) = 0, \qquad (7.3.3a)$$

$$(a, b, c) + (b, c, a) + (c, a, b) = 0. \qquad (7.3.3b)$$

where condition (7.3.3a), called the *flexibility law*, is a simple generalization of the anticommutative law. No additional study, e.g., of the structure theory, was conducted by Albert in his original paper [18].

In the subsequent two decades, only two additional brief notes appeared by mathematicians in Lie-admissible algebras, one in 1957 and one in 1962 (see the general bibliography [19]), but without any detailed mathematical study.

The Lie-admissible algebras made their first appearance in classical mechanics in paper [12-14] 1967-68 via their identification in the fundamental brackets of the time evolution of Hamilton's equations with external terms, when properly written (see bellow). The algebras were then studied in more details in ref.s [7,10,11]

By introducing the unified notation $a = (a^\mu) = (r_{ka}, p_{ka})$, $\mu = 1, 2, ..., 6N$, the main result of ref.s [7,13-15] can express via the re-formulation of brackets

[55] Assumed throughout this presentation as of characteristic zero.

A⊙B of Eq.s (7.1.5)

$$dA / dt = [A,H] + \frac{\partial A}{\partial p_{ka}} F_{ka} \equiv (A, H) := \frac{\partial A}{\partial a^{\mu}} S^{\mu\nu} \frac{\partial B}{\partial a^{\nu}}, \tag{7.3.4a}$$

$$S^{\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu} = \omega^{\mu\alpha} 1^{>}{}_{\alpha}{}^{\nu}, \quad s^{\mu\nu} = \text{diag.}\{ 0, F/(\partial H / \partial p) \}, \tag{7.3.4b}$$

$$1^{>}{}_{\alpha}{}^{\nu} = I_{\alpha}{}^{\nu} + s_{\alpha}{}^{\nu}, \tag{7.3.4c}$$

where $\omega^{\mu\nu}$ is Lie's tensor characterizing the Poisson brackets, and $1^{>}$ is a quantity to be identified shortly. It is then easy to verify the existence of: the consistent exponentiation of Eq.s (7.3.4a) into the finite form

$$A(t) = e^{t\, S^{\mu\nu} (\partial_{\nu} H) (\partial_{\mu})} A(0); \tag{7.3.5}$$

the direct representation of the time-rate-of-variation of the energy

$$\dot{H} = H - e^{t\, S^{\mu\nu} (\partial_{\nu} H) (\partial_{\mu})} H = v_{ka} F^{NSA}{}_{ka}; \tag{7.3.6}$$

and underlying equations of motion in explicit form

$$\dot{r}_{ka} = \frac{\partial H}{\partial p_{ka}} = p_{ka} / m_{a}, \tag{7.3.7a}$$

$$\dot{p}_{ka} = - \frac{\partial H}{\partial r_{ka}} + s_{ka} \frac{\partial H}{\partial p_{ka}} = F^{SA}{}_{ka} + F^{NSA}{}_{Ext \cdot ka}, \tag{7.3.7b}$$

where SA stands for the *conditions of variational selfadjointness* and NSA stands for their violation [7,8].

The verification of the right and left scalar and distributive laws by brackets (A, B) is evident.[56] Equally evident is their Lie-admissibility because their attached antisymmetric brackets are Lie,

$$(A, B) - (B, A) \equiv 2 [A, B]. \tag{7.3.8}$$

Thus, Lie-admissible equations (7.3.7) resolve all problematic aspects of Eq.s (7.1.2).

Very few additional papers appeared in the decade following ref.s [12–14]

[56] Note that the addition of a second term in the equation for $\dot{r}_{ka}$ would imply the loss, in general, of the physical meaning of the linear momentum $p_{ka} = m_{l}\dot{r}_{ka}$.

(see the "genealogical tree" on Lie-admissible algebras, ref. [7], p. 304 and quoted literature in pp. 414–415). However, following paper [7] of 1978, the study of Lie-admissible algebras increased considerably, also as a result of a series of *Workshops on Lie-admissible Formulations* organized by this author (see the general bibliography [19]).

The Lie-admissible algebras made their first appearance in operator mechanics ref. [20], p. 746, of 1978 as the central structural algebras of hadronic mechanics via the basic dynamical equations

$$i\, dA / dt = (A \hat{,} B) = A\, R\, B - B\, S\, A, \tag{7.3.9a}$$

$$R, S, R + S \neq 0, \quad R \neq S^{\dagger}, \tag{7.3.9b}$$

with exponentiated form (ref. [20], Sect. 4.18, p. 779 ff.)

$$A(t) = e^{+ i H S t}\, A(0)\, e^{- i t R H}, \tag{7.3.10}$$

and time-rate-of-variation of the energy operator

$$i\, dH / dt = (H \hat{,} H) = H\, (R - S)\, H\, . \tag{7.3.11}$$

It is evident that the product (A, B) characterizes a general Lie-admissible algebra because the attached algebra is Lie-isotopic (rather than Lie)

$$(A \hat{,} B) - (B \hat{,} A) = [A \hat{,} B] = A\, T\, B - B\, T\, A\,, \quad T = R + S\,. \tag{7.3.12}$$

The algebra characterized by the following brackets

$$(A, B) = p\, AB - q\, BA, \tag{7.3.13}$$

with p and q non-null scalars (or functions, was introduced by the author [12] back in 1967[57] as a realization of flexible Lie-admissible and Jordan-admissible algebras (App. I.4.A).

Subsequently, the Lie-admissible algebras made their first appearance in statistical mechanics in paper [21] of 1979 via the master equation for the density matrix

$$i\, d\rho / dt \equiv [\rho\, , H] + C \equiv (\rho, H) = \rho\, R\, H - H\, S\, \rho, \tag{7.3.14}$$

---

[57] The sole realization of the Lie-admissible product introduced by Albert [18] is (a, b) = $\lambda ab - (1 - \lambda) ba$ which does not include the so-called *q-deformations* (App. I.7.A) as a particular case, while the latter are indeed a particular case of the Lie-admissible algebras with product (7.3.13).

which admits conventional equations of type (7.2.5) as a particular case with the identifications

$$\rho H - H\rho + C \equiv \rho R H - H S \rho, \; R = 1, \; S = 1 + H^{-1}\rho^{-1} C, \qquad (7.2.15)$$

although Eq. (7.3.15) are transparently more general than (7.2.5).

Since that time (1979), Lie-admissible algebras have been submitted to considerable, mathematical and physical studies by numerous authors. A comprehensive bibliography until 1984 can be found in ref. [19]. More recent accounts can be found in Vol.s II and III.

Monograph [11] presents the Lie-admissible theory in classical realization. In this chapter we shall outline the foundations of the *Lie-admissible theory* in its operator realization. Applications will be studied in the subsequent volumes.

This is a line of study conducted by this author [7,10,11] which is considerably different than the studies generally listed in bibliography [19]. In fact, the latter were conducted within the context of abstract nonassociative algebras, while the former refer, specifically, to a step-by-step generalization of enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The understanding is that all studies in Lie-admissibility, whether explicitly or implicitly oriented for the generalization of Lie's theory, are relevant for these volumes because they deal with the mathematical structure of hadronic mechanics.

The inspection of classical studies [11] is recommended for the reader interested in acquiring a technical knowledge of the field, because all the basic concepts of the the Lie-admissible formulations already exist at the classical level, where they find theirs clearest realization.[58]

## 7.4: GENONUMBERS

The technical understanding of the Lie-admissible formulations requires the knowledge that they are based on a generalized theory of numbers beyond that of isonumbers.

Let $F(\alpha,+,\times)$ be a conventional field (Sect. 2.3) with multiplication $\alpha\beta := \alpha\times\beta$. In Ch. I.2 we have reviewed a generalization of this basic operation into the isotopic form $\alpha * \beta = \alpha T \beta$. Both products $\alpha\beta$ and $\alpha*\beta$ are based on the assumption that they apply irrespective of whether $\alpha$ multiplies $\beta$ from the left, or $\beta$

[58] Particularly important is the classical realization of the *Lie-admissible symmetries* which provide a structural generalization of Noether's theorem whereby the Lie-admissible symmetries characterize the time-rate-of-variation of physical quantities, thus admitting as a particular case Lie symmetries and conservation laws. Rather oddly, these covering notions have remained virtually ignored in the physical literature.

multiplies $\alpha$ from the right. We can therefore introduce the following:

**Definition 7.4.1 - Ordering of the multiplication** [22]: *The multiplication of two numbers $\alpha$ and $\beta$ is ordered to the right, and denoted $\alpha>\beta$, when $\alpha$ multiplies $\beta$ to the right, while it is ordered to the left, and denoted $\alpha<\beta$ when $\beta$ multiplies $\alpha$ from the left.*

Note that the above ordering is compatible with other properties and axioms of number theory. As an example, if the original field F is commutative, it remains commutative after the above ordering, that is, if $\alpha\beta = \beta\alpha$, then $\alpha>\beta = \beta>\alpha$ and $\alpha<\beta = \beta<\alpha$. The same occurrence holds for other properties, such as associativity while the verification of the left and right distributive laws is evident. Thus, the entire Definition 2.3.1 can therefore be reformulated under ordering by characterizing fully acceptable fields.

The point at the foundations of the Lie-admissible theory is that *the multiplications of the same numbers in different orderings are generally different*, $\alpha>\beta \neq \beta<\alpha$. In turn, this implies the possibility of introducing *two ordered isounits*, called *genounits*, one per each ordering

$$\hat{1}^{>}: \ \hat{1}^{>} > \alpha = \alpha > \hat{1}^{>} \equiv \alpha , \tag{7.4.1a}$$

$$^{<}\hat{1}: \ ^{<}\hat{1} < \alpha = \alpha < {}^{<}\hat{1} \equiv \alpha , \tag{7.4.1b}$$

The above features permit a dual generalization of Definition 2.3.1, one for ordering to the right, yielding the *right genofield*

$$\hat{F}^{>}(\hat{\alpha}^{>},+,\hat{*}^{>}), \qquad \hat{\alpha}^{>} = \alpha\,\hat{1}^{>}, \tag{7.4.2}$$

whose elements $\hat{\alpha}^{>}$ are called *right genonumbers*, and one to the left, yielding the *left genofield*

$$^{<}\hat{F}(^{<}\hat{\alpha},+,^{<}\hat{*}), \qquad ^{<}\hat{\alpha} = {}^{<}\hat{1}\,\alpha , \tag{7.4.3}$$

whose elements $^{<}\hat{a}$ are called *left genonumbers*. The above two different genofields are often denoted with the unified symbol $^{<}\hat{F}^{>}(^{<}\hat{\alpha}^{>},+,^{<}\hat{*}^{>})$, with the understanding that the orderings can solely be used individually and not jointly.

The realization of the genoproducts used in these volumes is given by the following two different isotopic multiplications, one to the right and one to the left,

$$\alpha > \beta := \alpha\, R\, \beta , \tag{7.4.4a}$$

$$\alpha < \beta := \alpha\, S\, \beta , \tag{7.4.4b}$$

where R ≠ S, with realization of the genounits

$$\hat{1}^{>} = R^{-1}, \qquad \hat{1}^{>} > \alpha \equiv \alpha, \tag{7.4.5a}$$

$$^{<}\hat{1} = S^{-1}. \qquad \alpha < {}^{<}\hat{1} \equiv \alpha. \tag{7.4.5b}$$

The entire theory of isonumbers of Ch. 2, including isoreal, isocomplex, isoquaternions and isooctonions numbers, then admits a generalization into the *theory of genonumbers* first introduced in ref. [22].

Note the need for a prior isotopy $\alpha\beta \rightarrow \alpha T\beta$ in order to construct genotopies (7.4.4). In fact, no ordering is evidently meaningful for the conventional multiplication $\alpha\beta = \alpha 1 \beta$.

So far we have presented in this section the right and left genomultiplications and related isounits as disjoint, in which case the isounits can indeed be Hermitean and real-valued, thus admitting of Kadeisvili classification into Classes I, II, III, IV, V.

Nevertheless, the realizations used in physics are those when the right and left genounits are inter-related by a conjugation, such as the Hermitean conjugation

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}, \tag{7.4.6}$$

In this case Kadeisvili's classification still holds, but must be referred to the Hermitean parts of the genounits. More specifically, we shall decompose the genounits into the products

$$\hat{1}^{>} = \hat{1}\,P, \qquad {}^{<}\hat{1} = Q\,\hat{1}, \qquad \hat{1} = \hat{1}^{\dagger}, \qquad P^{\dagger} = Q, \tag{7.4.7}$$

where $\hat{1}$ is the maximal Hermitean part. We can then classify the theory of genonumbers into Kadeisvili's Classes I, II, III, IV and V now referred to the maximal Hermitean part of the genounits.

As it will be soon evident, *under the above interconnection, the product ordered to the right can be interpreted as characterizing motion forward in time, while that ordered to the left can represent motion backward in time.* In different term, the ordering of Definition 7.4.1 can represent *Eddington's "arrows of time"*, and we have the following:

**Lemma 7.4.1** [22]: *An axiomatization of irreversibility in number theory is given by: A) the ordering of the multiplications to the right and to the left, representing motion forward and backward in time, respectively; B) the differentiation of these two multiplications; and C) the assumption of an* *interconnecting map representing time-reversal ~~invariance~~*

As we shall see in the rest of this chapter and in Vol.s II and III, the theory of genonumbers that with interconnecting map is the true foundation of the Lie-admissible branch of hadronic mechanics,

Note that the simpler theory of isonumbers is a subcase of that of genonumbers under the simple condition

$$R = S = R^{\dagger}, \tag{7.4.8}$$

This illustrates that the origin of the reversibility of the Lie and Lie-isotopic theories can be seen in their respective theories of numbers and isonumbers and, more specifically, from the fact that their multiplications to the right and to the left are identical, $\alpha > \beta \equiv \alpha < \beta$.

We close this section with a few mathematical comments. Realization (7.4.4) is evidently not unique. In fact, other realizations of ordered multiplications are given by

$$\alpha > \beta = W \alpha W R W \beta W, \quad W^2 = W, \tag{7.4.9a}$$

$$\alpha < \beta = Z \alpha Z S Z \beta Z, \quad Z^2 = Z, \tag{7.4.9b}$$

where $R \neq S$ and $W \neq Z$. The latter realizations are not used in physics to our best knowledge at this writing, because they do not verify the Fundamental Condition 4.4.1 of admitting unique, left and right units.

Conjugation (7.4.8) is used in physics, but in mathematics one can introduce any other conjugation, such as that characterized by isoduality

$$1^{>} = ({}^{<}1)^{d} = -{}^{<}1, \tag{7.4.10}$$

or have no conjugation at all.

Finally, note that the notion of isoduality applies also to genofields, yielding the *isodual genofields* $^{<}\hat{F}^{>d}(^{<}\hat{a}^{>d}, +, ^{<}*^{>d})$.

In the preceding chapters we have indicated the truly remarkable, novel mathematical developments permitted by the theory of isonumbers. The yet broader theory of genonumbers permit additional mathematical developments that are simply inconceivable with conventional theories.

As an illustration, the Lie product AB −BA originates from two envelopes, one for the multiplication to the right with product BA, and one to the left with product AB, as we shall see, even though these two multiplications are evidently identical. Then, the theory of genonumbers permits *the reinterpretation of Lie algebras as commutative Jordan algebras defined on two genofields interconnected by isoduality*, i.e.

$$A\,B\ -\ B\,A\ \equiv\ A\,1\,B\ +\ B\,1^{d}\,A\,. \tag{7.4.11}$$

The remark is important to indicate that *Jordan legacy (i.e., a possible quantum mechanical content of Jordan algebras) is still open.*

We finally note that the ordering of the multiplication can also be extended to the addition, although it must necessarily be lifted to be nontrivial. This further generalization is not used in physics because it violates the distributive law as studied in Sect. I.2.3. Nevertheless, the extension is significant to point out that the most general notion of "numbers" introduced by this author [22], the *theory of genonumbers and their isoduals.* It can be expressed by the unified symbol $^{<}\hat{F}^{>}(^{<}\hat{\alpha}^{>},\ ^{<}\hat{+}^{>},^{<}\hat{\times}^{>})$, representing: three separate generalizations of the numbers $\alpha \to \hat{\alpha} \to \hat{\alpha}^{>} \to {}^{<}\hat{\alpha}$; characterized by three separate generalizations of the operations $+ \to \hat{+} \to \hat{+}^{>} \to {}^{<}\hat{+}$ and $\times \to \hat{\times} \to \hat{\times}^{>} \to {}^{<}\hat{\times}$; with three separate generalizations of the additive units $0 \to \hat{0} \to \hat{0}^{>} \to {}^{<}\hat{0}$, and multiplicative units $1 \to \hat{1} \to \hat{1}^{>} \to {}^{<}\hat{1}$, plus the image of all these structures under isoduality.

Such genonumbers can be not only of dimension 1 *(genoreals)*, 2 *(genocomplex)*, 4 *(genoquaternions)* and 8 *(genooctonions)*, but also have dimension 3, 5, 6, 7 (called *"hidden numbers"* because they hidden in the operations as for the case of isonumbers (see App. I.2.A and ref. [22] for brevity).

## 7.5: GENOSPACES

The entire theory of isospaces of Ch. I.3 admits a consistent and significant genotopic covering. Let S(x,g,R) be a conventional metric or pseudo-metric space and $\hat{S}(x,\hat{g},\hat{R})$ its family of isotopes. Then, the following *left and right genospaces* hold

$$\hat{S}^{>}(x,\hat{g}^{>},\hat{R}^{>}) \,\Big|\, \hat{g}^{>}\ =\ g\,R\,,\ x^{\hat{2}>} = x^{t}\,\hat{g}^{>}\,x,\quad \hat{1}^{>}\ =\ R^{-1}\,, \tag{7.5.1a}$$

$$^{<}\hat{S}(x,{}^{<}\hat{g},{}^{<}\hat{R}) \,\Big|\, {}^{<}\hat{G}\ =\ S\,g\,,\quad x^{<2} = x\,{}^{<}\hat{g}\,x^{t}\,,\quad {}^{<}\hat{1}\ =\ S^{-1}\,. \tag{7.5.1b}$$

$$\hat{1}^{>}\ =\ (\,{}^{<}\hat{1}\,)^{\dagger}\,, \tag{7.5.1c}$$

†

A most visible difference between genospaces and isospaces is therefore that the invariant in the former is unique, while in the latter we have two different invariants, one for the multiplication to the right and one to the left.

When the two multiplications are interconnected by conjugation (7.5.1c), we have *two different genospaces one for motion forward in time, and one for motion backward in time.*

The most significant genospaces, denoted with a unified notation $^{<}\hat{S}^{>}(x, {}^{<}\hat{g}^{>}, {}^{<}\hat{R}^{>})$, are given by:

**I) genoeuclidean spaces** $^{<}\hat{E}^{>}(x, {}^{<}\hat{\delta}^{>}, {}^{<}\hat{R}^{>})$ and their isoduals;

**II) genominkowskian spaces** $^{<}\hat{M}^{>}(x, {}^{<}\hat{\eta}^{>}, {}^{<}\hat{R}^{>})$ and their isoduals;

**III) genoriemannian spaces** $^{<}\hat{\Re}^{>}(x, {}^{<}\hat{g}^{>}, {}^{<}\hat{R}^{>})$ and their isoduals,

where $\hat{\delta}$, $\hat{\eta}$, $\hat{g}$ are the isometric of the corresponding isospaces of Ch. I.3.

It should be noted that conventional spaces, such as the Euclidean space $E(r,\delta,R)$, admit a nontrivial isodual images $E^d(r,\delta^d,R^d)$. However, their genoimages $^{<}E^{>}(r, {}^{<}\delta^{>}, {}^{<}R^{>})$ without a joint isotopy are trivial, evidently because $^{<}\delta^{>} \equiv \delta$. This occurrence is similar to that of the preceding section whereby ordinary fields $F(a,+,\times)$ admit nontrivial isoduals $F^d(a^d, \times d)$ without isotopies, but trivial genotopes, $^{<}F^{>} \equiv F$, because $a>b \equiv b<a$ for ordinary fields.

The lack of a significant "arrow of time" in the conventional numbers and spaces is the axiomatic origin of their *reversibility*. By comparison, the presence of a structural "arrow of time" in the theory of genonumbers and genospaces renders them particularly suited to represent *irreversibility*.

The use of conventional transformation theory for genospaces also violates linearity, transitivity and other basic laws. For this reason it must be lifted into the *right and left genotrasformations*

$$x' = \hat{U}^{>} > x \,/\!:= \hat{U}^{>} R\, x\,, \qquad (7.5.2a)$$

$$x' = x < {}^{<}\hat{U} \,/\!= x\, S\, {}^{<}\hat{U}\,. \qquad (7.4.2b)$$

The above transformations are one–sided isolinear, isolocal and isocanonical as it occurs for the isotransformations. This illustrates again that the ordering of the multiplication does indeed preserve all basic axioms. The remaining aspects of isospaces (Ch. I.3) and their transformation theory therefore admit a consistent and intriguing generalization into left and right theories.

## 7.6: LIE-ADMISSIBLE THEORY

Recall that the conventional unit I is at the foundation of Lie's theory, and the same occurrence holds for the Lie–isotopic theory.

The distinction of the multiplication to the right from that to the left with corresponding different genounits implies an evident generalization of the entire Lie and Lie–isotopic theories whose study has only been initiated at this writing [11]. We here indicate the existence of two genoassociative enveloping algebras $\hat{\xi}^{>}$ and $^{<}\hat{\xi}$ with the same elements A, B, C, ... denoted with the joint symbol $^{<}\hat{\xi}^{>}$, buy different genoproducts and genounits

$$\xi^{>} : A > B := A R B, \quad 1^{>} = R^{-1}, \tag{7.6.1a}$$

$$^{<}\xi : A < B := A S B, \quad ^{<}1 = S^{-1}. \tag{7.6.1b}$$

defined over corresponding genofields $^{<}F^{>}(^{<}\hat{a}^{>},+,^{<}*^{>})$.

It is easy to see that the isotopic Poincaré–Birkhoff–Witt theorem (Sect. 4.3) can be consistently generalized for each direction of the multiplication, yielding an infinite-dimensional base for each genoassociative envelope.[59]

This allows the introduction of the unique, fundamental notions of *genoexponentiation*

$$U^{>} = e_{\xi^{>}}^{\,iXw} = \{ e^{\,i X R w} \} 1^{>}, \tag{7.6.2a}$$

$$^{<}U = e_{^{<}\xi}^{\,i w X} = {}^{<}1 \{ e^{\,i w S X} \}, \tag{7.6.2b}$$

which, in turn, permit the formulation of the *Lie-admissible group* first introduced in refg. [7] (see also ref.s (10,11]), which is given by the left and right genotransformations of a generic quantity $Q \in {}^{<}\xi^{>}$

$$Q(w) = U^{>} > Q(0) < {}^{<}U = \{ e_{\xi^{>}}^{\,iXw} \} > Q(0) < \{ e_{^{<}\xi}^{\,i w X} \} =$$
$$= \{ e^{\,i X R w} \} Q(0) \{ e^{\,i w S X} \}, \tag{7.6.2b}$$

Its most fundamental feature is of admitting a non-Lie/non-Lie-isotopic but Lie-admissible algebra in the neighborhood of the genoidenties

$$i \frac{dQ}{dw} = (Q \hat{,} X) = Q < X - X > Q, \tag{7.6.3}$$

thus confirming the existence of a Lie-admissible generalization of Lie's theory at all various levels (enveloping algebras, Lie algebras, Lie groups, etc.). Structure (7.6.3) also confirms that the l.h.s. of the product $(Q \hat{,} X)$ is characterized by the backward genoenvelope, while the r.h.s. is characterized by the forward genoenvelope, as anticipated earlier.

An important point for the correct interpretation and use of the theory is that *the envelopes underlying the Lie-admissible formulations remain associative*, thus verifying Fundamental Condition I.4.4.1. In different terms, structure (7.6.3) is a generalization of the corresponding Lie and Lie-isotopic

[59] This is possible because, again, the genoalgebras admit well defined right and left units. By comparison, q-deformation have no such unit and, therefore, do not admit a unique basis for their exponentiation (App. I.7.A).

structures

$$i \frac{dQ}{dw} = [Q, H] = QH - HQ, \tag{7.6.4a}$$

$$i \frac{dQ}{dw} = [Q, H] = QTH - HTQ, \tag{7.6.4b}$$

where, as now familiar, the brackets $[\ ,\ ]$ and $[\ \hat{,}\ ]$ are nonassociative, but their envelopes with respective product QX and QTX are indeed associative.

Exactly the same occurrence holds for the more general Lie-admissible formulations. In fact, the brackets $(\ \hat{,}\ )$ are evidently nonassociative, but the underlying envelopes with products $Q{>}H$ and $H{<}Q$ are isoassociative.

In Vol. II we shall study the basic laws of the Lie-admissible representation of interior systems. In particular, we shall identify the *Lie-admissible symmetries* and show that they characterize *time-rate-of-variation of physical quantities*, by providing in this way an operator counterpart of the corresponding classical notions [11], and by reaching an intriguing covering of the corresponding notions for Lie and Lie-isotopic theories.

The most important application of the Lie-admissible theory is the characterization of the most general known notion of particle, called *genoparticle*, as studied in more details in Vol.s II and III. At this moment we simply list the notions of particles used in hadronic mechanics:

**Conventional particles,** which is characterized by Lie symmetries in a stable-reversible orbit, such as an electron of an atomic structure;

**Isoparticles** which is characterized by the Lie-isotopic symmetries also on stable orbits, such as the constituents of few-body nuclei and hadrons; and

**Genoparticles,** which is characterized by Lie-admissible symmetries on the most general known nonconservative, unstable and irreversible orbit, such as an electron in the core of a star undergoing gravitational collapse

plus all their isoduals.

The best way to understand the conceptual, mathematical and physical advances permitted by the Lie-admissible theory is by inspecting the underlying representations called *genorepresentations.*

In Sect. I.4.7. we have studied the isorepresentation theory which is based on the notion of module implying only one action, e.g., that to the right. By comparison genorepresentations of Lie-admissible algebras require a *two-sided isobimodule* called *genomodule.*

Consider an algebra U over a field $F(\alpha,+,\times)$. Let V be a vector space over F and introduce the direct sum

$$S = U \oplus V \tag{7.6.5}$$

in such a way that S is an algebra verifying the same axioms of U while V is a two sided ideal of S.

This can be done as follows [23]:

1) retain the product of U;
2) introduce a left and a right composition av and va, for all elements $a \in U$ and $v \in V$ which verify all axioms of U (including the right and left scalar and distributive laws); and
3) to complete the requirement that V is an ideal of S, assume vt = tv = 0 for all elements of V.

When all the above properties are verified, V is called a *two-sided, left and right module,* or a *bimodule* of U, and the algebra S is called a *split null extension* of U [loc. cit.].

Bimodules clearly provide a generalized, left and right representation theory of all algebras, whether associative or nonassociative. It is important to understand why bimodules are not needed for the representation theory of conventional Lie algebras (i.e., for the conventional notion of particle) as well as of Lie-isotopic algebras (i.e., for the generalized notion of isoparticle), but they become essential for the covering Lie-admissible algebras (i.e., for the most general possible notion of genoparticle).

A *bimodule* V *of a Lie algebra* L or *Lie-bimodule* [24] is characterized by left and right compositions av and va, $a \in L$, $v \in V$, verifying the properties

$$a\,v = -\,v\,a, \tag{7.6.5a}$$

$$v(a\,b) = (v\,a)\,b - (v\,b)\,a, \tag{7.6.5b}$$

which can be identically expressed via the left and right multiplications

$$L_a = -R_a, \tag{7.6.7a}$$

$$R_{ab} = R_a R_b - R_b R_a, \tag{7.6.7b}$$

The mappings $a \Rightarrow R_a$ and $a \Rightarrow L_a$ then provide *a left and right representation,* or a *birepresentation,* of the Lie algebra L over the bimodule V as a $\mathrm{Hom}^L{}_F(V_R,V_L)$.

However, owing to property (7.6.6a), the left representation is trivially equivalent to the right representation, $R_a = -L_a$.[60] This is the reason why only one-sided representations of Lie algebras are significant in quantum mechanics.

The notions of isomodules and isobimodules were introduced for the first

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[60] Note again the intriguing possibility of reinterpreting the left representation as an isodual of the right here left to the interested reader.

time in ref. [24] of 1979, although they do not appear to have been studied thereafter in the mathematical or physical literature. In essence, a *Lie-isobimodule* is an isovector space $\hat{V}$ over an isofield $\hat{F}(\alpha,+,*)$ with left and right isocompositions $a * v$ and $v * a$ verifying the distributive and scalar laws, and the rules

$$a * v = - v * a, \tag{7.6.8a}$$

$$v * (a * b) = (v * a) * b - (v * b) * a, \tag{7.6.8b}$$

or, equivalently in terms of isomultiplications

$$\hat{R}_a = - \hat{L}_a, \tag{7.6.9a}$$

$$\hat{R}_{a * b} = \hat{R}_a * \hat{R}_b - \hat{R}_b * \hat{R}_a, \tag{7.6.9b}$$

An *isobirepresentation* of a Lie-isotopic algebra $\hat{L}$ is then given by $\mathrm{Hom}^{\hat{L}}{}_{\hat{F}}(\hat{V}_R, \hat{V}_L)$.

However, the left and right isorepresentations are again equivalent because of the property $\hat{R}_a = - \hat{L}_a$. Thus, *only one-sided isomodules and one-sided isorepresentations are needed for the Lie-isotopic branch of hadronic mechanics*, and this explains the reason fro our silence on them in Ch. I.4.

Note also that the above equivalence between the right and left isomodular actions is an axiomatic representation of reversibility. This implies that isoparticles as characterized by one-sided isorepresentations are on stable-reversible orbits.

The *two-sided isorepresentations*, or *genorepresentations*, become *necessary* when studying Lie-admissible algebras evidently because of the loss of the antisymmetric (or symmetric) character of the product. As a result, the representation theory of the Lie-admissible algebras is much richer than those of the Lie and Lie-isotopic algebras.

A *Lie-admissible bimodule* $\hat{V}$, or *genomodule* for short, is a vector isospace over a genofield $^{<}\hat{F}^{>}$ equipped with two, inequivalent, right and left compositions $a>v$ and $v<a$ such that the attached composition $a \odot v = a<v - v>a$ verifies the axioms

$$a \odot v = - v \odot a, \tag{7.6.10a}$$

$$v \odot (a \odot b) = (v \odot a) \odot b - (v > b) \odot a. \tag{7.6.10b}$$

Thus, genomodules are characterized by their attached composition $a>b - v<a$, rather then each individual actions $a>v$ and $v<a$. They can be equivalently expressed via the right and left isomultiplications

$$\hat{R}_{a>v} - \hat{v}_{>a} + \hat{L}_{a>b} - {}_{b<a} = [(\hat{r}_a - \hat{L}_a), (\hat{R}_b - \hat{L}_b)]. \qquad (7.6.11)$$

A *genorepresentation of a Lie-admissible algebras* U over the genofields $^{<}\hat{F}^{>}$ is therefore given by the $\mathrm{Hom}^{U}{}_{<\hat{F}>}(^{<}\hat{V}_{\hat{L}}, \hat{V}^{>}{}_{\hat{R}})$.

The physical meaning of the Lie-admissible theory is identified by the following:

**Lemma 7.6.1** [11]: *An axiomatization of irreversibility at the algebraic-group theoretical level is provided by the differentiation of enveloping associative algebras of Lie's theory into two genotopic forms of the Lie-admissible theory* $\hat{\xi}^{>}$ *and* $^{<}\hat{\xi}$ *and related genorepresentations, characterizing motion forward and backward in time, respectively, with a corresponding interconnecting conjugation, and related forward and backward genounits* $\hat{1}^{>}$, $^{<}\hat{1}$, *for corresponding right and left actions.*

The axiomatic nature of the above characterization is expressed by the fact that irreversibility is intrinsic in the theory, i.e., it holds also for time-reversible Hamiltonians, as we shall see better in Vol. II. By comparison, both Lie and Lie-isotopic theories are structurally reversible.

The implications of the above axiomatization of irreversibility are far reaching. In fact, as we shall study in detail in Vol.s II and III, the lifting of the Poincaré symmetry into its Lie-admissible covering (first proposed at the classical level in ref. [11]) characterizes the most complex known notion of particle with locally varying intrinsic characteristics, as expected to represent the most complex known physical conditions in Nature, such as for a neutron in the core of a neutron star.

## 7.7: GENOGEOMETRIES

As stressed throughout our studies, physical theories in general, and relativities in particular, are a symbiotic expression of deeply interconnected and mutually compatible analytic, algebraic and geometric formulations.

In the preceding sections we have presented the analytic and algebraic structures of the Lie-admissible theory. It is therefore important to show that, exactly as it occurs for the Lie and Lie-isotopic theories, the Lie-admissible theory also admits a fully defined geometric counterpart.

This problem was studied in ref.s [7,11] and resulted in the submission of new geometries, more general than the isogeometries of Ch. I.5, called *genogeometries*, according to the following main lines.

**7.7.A: Genoeuclidean and genominkowskian geometries.** They are the geometries of the genospaces ${}^<\hat{E}^>(r, {}^<\hat{\delta}^>, {}^<\hat{R}^>)$ and ${}^<\hat{M}^>(r, {}^<\hat{\delta}^>, {}^<\hat{R}^>)$, respectively, and are essentially given by the isoeuclidean and isominkowskian geometries of Sect. I.5.2 and I.5.3, in different realizations for each "arrow of time".

The most important difference between the iso- and genogeometries is therefore that the metric of the former is unique for both directions of time, while the metric of the latter is differentiated depending on the assumed direction of time, $\hat{\delta}^> \neq {}^<\hat{\delta}$ and $\hat{\eta}^> \neq {}^<\hat{\eta}$.

However, the base unit is lifted in correspondence to each of these generalized metrics according to the rules

$$g \Rightarrow \hat{g}^> = g\,T^>, \quad I \Rightarrow \hat{1}^> = (T^>)^{-1}, \tag{7.7.1a}$$

$$g \Rightarrow {}^<\hat{g} = {}^<T\,g, \qquad I \Rightarrow {}^<\hat{1} = ({}^<T)^{-1}, \tag{7.7.1b}$$

as a result of which all the peculiar properties of the isogeometries are preserved for forward and, separately, backward genogeometries.

This implies the existence of two different deformations of the sphere, the light cone, etc., for interior dynamical problems, one per each direction of time, each of which is mapped into the perfect sphere and the perfect cone* in genospace.

The extension of the remaining properties of isogeometries into the genotopic form is an instructive exercise for the interested reader, and it will be assumed hereon.

**7.7.B: Genosymplectic geometry.** Recall that the symplectic geometry is the geometry underlying Lie's theory, while the isosymplectic geometry (Sect. I.5.5) is that underlying the Lie-isotopic theory. In ref.s [7,11] this author showed that the yet more general Lie-admissible theory also admits a fully defined underlying geometry, evidently of a generalized nature submitted under the name of *symplectic-admissible geometry*, or *genogeometry* for short.

Recall from App. I.5.A Birkhoff's brackets in $T^*E(r,\delta,\Re)$ and related exact symplectic two-form in the now familiar unified notation $a = (a^\mu) = (r, p)$, $\mu = 1, 2, \ldots, 2n$,

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu}\,\Omega^{\mu\nu}(a)\,\frac{\partial B}{\partial a^\nu}, \tag{7.7.2a}$$

$$\Omega = \tfrac{1}{2}\,\Omega_{\mu\nu}(a)\,da^\mu \wedge da^\nu, \tag{7.7.2b}$$

where the algebraic-contravariant and geometric-covariant tensors are interconnected by the familiar rule

$$\Omega^{\mu\nu} = (\| \Omega_{\alpha\beta} \|^{-1})^{\mu\nu}. \tag{7.7.3}$$

In the transition to the Birkhoff-isotopic brackets on isospaces $T^*\hat{E}_2(r,\hat{\delta},\hat{R})$ with isounit $\hat{1}_2$ (Sct. I.5.4),

$$[A \hat{,} B] = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\alpha}(a) \hat{1}_{2\alpha}{}^{\nu}(t, a, \dot{a}, ....)) \frac{\partial B}{\partial a^\nu}, \tag{7.7.4}$$

we have the transition to the isosymplectic geometry characterized by the isoexact two-isoform

$$\hat{\Omega} = \tfrac{1}{2} T_\mu{}^\alpha(t, a, \dot{a}, ...) \Omega_{\alpha\nu}(a) \hat{d}a^\mu \hat{\wedge} \hat{d}a^\nu , \tag{7.7.5}$$

where, again, the algebraic and geometric tensors are interconnected by the rule

$$\Omega^{\mu\alpha} \hat{1}_{2\alpha}{}^{\nu} = [ (T_{2\alpha}{}^{\rho} \Omega_{\rho\beta})^{-1} ]^{\mu\nu}. \tag{7.7.6}$$

The problem of the geometry underlying the *Birkhoff-admissible brackets* [7,11]

$$(A \hat{,} B) = \frac{\partial A}{\partial a^\mu} <\hat{S}>^{\mu\nu}(t,a) \frac{\partial B}{\partial a^\nu} , \tag{7.7.7a}$$

$$\hat{S}^{\mu\nu} = \Omega^{\mu\alpha} <\hat{1}>_\alpha{}^\nu, \tag{7.7.7b}$$

$$\Omega^{\mu\nu} = - \Omega^{\nu\mu}, \tag{7.7.7c}$$

$$<\hat{1}>_\alpha{}^\nu = <\hat{1}>_\nu{}^\alpha, \tag{7.A.7d}$$

was resolved via the introduction of a geometry more general than the symplectic and the isosymplectic ones.

The first point to realize is that *the symplectic geometry and related exterior calculus, whether in their conventional or isotopic formulations, are intrinsically unable to characterize the Lie-admissible algebras.*

This is due to the fact that the calculus of exterior forms is essentially *antisymmetric* in the indices, and so remains under isotopies by assumption, while the Lie-admissible tensors $<\hat{S}>^{\mu\nu}$ are not antisymmetric, and the same occurs for the covariant counterpart

$$<\hat{S}>_{\mu\nu}(t, a) = (\| <\hat{S}>^{\alpha\beta} \|^{-1})_{\mu\nu} \neq \pm <\hat{S}>_{\nu\mu} \tag{7.7.8}$$

In fact, the construction of a conventional exterior two-form with the above tensor implies the reduction

$$< \hat{S}^{>}_{\mu\nu}\, da^{\mu} \wedge da^{\nu} \equiv \tfrac{1}{2}\, \Omega_{\mu\nu}\, da^{\mu} \wedge da^{\nu}, \tag{7.7.9}$$

namely, the symplectic geometry automatically eliminates the symmetric component of the S–tensor, thus characterizing only its Lie content.

The main idea of the symplectic–admissible geometry is that of generalizing the conventional exterior calculus, say, of two differentials

$$da^{\mu} \wedge da^{\nu} = - da^{\nu} \wedge da^{\mu}, \tag{7.7.10}$$

into a more general calculus, called *exterior–admissible calculus*, or *genoexterior calculus*, which is defined over the genofield of real numbers $^{<}\hat{R}^{>}(^{<}\hat{n}^{>},+,^{<}*^{>})$ based on a product, say $\odot$, which is neither totally symmetric nor totally antisymmetric, but such that its antisymmetric component is the conventional exterior one [7,11],

$$da^{\mu} \odot da^{\nu} = da^{\mu} \wedge da^{\nu} + da^{\nu} \times da^{\nu}, \tag{7.7.11a}$$

$$da^{\mu} \wedge da^{\nu} = - da^{\nu} \wedge da^{\mu}, \tag{7.7.11b}$$

$$da^{\mu} \times da^{\nu} = da^{\nu} \times da^{\mu}, \tag{7.7.11c}$$

The isocotangent bundle is then further generalized into the *genocotangent bundle* $T^{*<}\hat{E}^{>}(r,^{<}\hat{\delta}^{>},^{<}\hat{R}^{>})$upon selection of one given ordering in the multiplication.

This allows the introduction of the *exterior–admissible forms* or *genoforms*, via the sequence

$$^{<}\hat{S}^{>}_{0} = {}^{<}\phi^{>}(a), \tag{7.7.12a}$$

$$^{<}\hat{S}^{>}_{1} = {}^{<}\hat{S}^{>}_{\mu}\, da^{\mu}, \tag{7.7.12b}$$

$$^{<}\hat{S}^{>}_{2} = {}^{<}\hat{S}^{>}_{\mu\nu}\, da^{\mu} \odot da^{\nu}, \tag{7.7.12c}$$

..................

The *exact exterior–admissible forms* or *exact genoforms*, are then given by

$$^{<}\hat{S}^{>}_{1} = d\,^{<}\hat{S}^{>}_{0} = \frac{\partial\,^{<}\phi^{>}}{\partial a^{\mu}}\, da^{\mu}, \tag{7.7.13a}$$

$$^{<}\hat{S}^{>2} = d\,^{<}\hat{S}^{>}_{1} = \frac{\partial\,^{<}A^{>}_{\nu}}{\partial a^{\mu}}\, da^{\mu} \odot da^{\nu}, \tag{7.7.13b}$$

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The calculus of exterior–admissible forms can indeed characterize the Lie–admissible algebras, because it characterizes not only the antisymmetric component of the Lie–admissible brackets, but also their symmetric part, via the two–forms

$$\langle\hat{S}\rangle_2 = \langle\hat{S}\rangle_{\mu\nu}(t,a)\, da^\mu \odot da^\nu =$$

$$= \hat{\Omega}_{\mu\nu}(a)\, da^\mu \wedge da^\nu + \langle\hat{T}\rangle_{\mu\nu}(t, a)\, da^\mu \times da^\nu, \qquad (7.7.14)$$

Structures (7.7.14) are *symplectic–admissible two–forms* because their antisymmetric component is symplectic, in a way fully parallel to the property whereby the antisymmetric part of the Lie–admissible algebras is Lie. Structure (7.7.14) are also called *genosymplectic two–forms*, when emphasis is needed on the loss of the original antisymmetric axiom. The spaces $T^*\langle\hat{E}\rangle(r,\langle\hat{\delta}\rangle,\langle\hat{R}\rangle)$, again selected either for the multiplication to the right or to the left, when equipped with two–form (7.7.14) are called *symplectic–admissible manifolds* or *genosymplectic manifolds*, and the related geometry is called *symplectic–admissible geometry* or *genosymplectic geometry.*

As incidental comments, note that the dependence on time appears only in the symmetric part, as needed for consistency in the symplectic component. Also, under inversion (7.7.8), we generally have

$$(\hat{\Omega}_{\mu\nu}) \neq (\Omega^{\alpha\beta})^{-1}, \quad (\langle\hat{T}\rangle_{\mu\nu}) \neq (\langle T\rangle^{\alpha\beta})^{-1}, \qquad (7.7.15)$$

which is a rather intriguing feature of the generalized geometry here considered, whereby the symplectic content of a contravariant tensor is more general than the symplectic counterpart of the covariant tensor. (see ref. [11] for details)

The most salient departure from the exterior calculus in its conventional or isotopic formulation is that *the Poincare' Lemma no longer holds for the genosymplectic geometry,* i.e., for exact symplectic–admissible two–forms we have

$$\langle\hat{S}\rangle_2 = d\langle\hat{S}\rangle_1, \qquad (7.7.16a)$$

$$d\langle\hat{S}\rangle_2 = d(d\langle\hat{S}\rangle_1) \neq 0. \qquad (7.7.16b)$$

In actuality, within the contest of the exterior–admissible calculus, the Poincare' Lemma is generalized into a rather intriguing geometric structure which evidently admits the conventional Lemma as a particular case when all symmetric components are null.

The geometric understanding of the Lie-isotopic algebras requires the understanding that *the validity of the Poincaré Lemma within the context of the isosymplectic geometry is a necessary condition for the representation of the conservation of the total energy under nonhamiltonian internal forces,* as studied in the main sections of this volume.

By the same token, the geometric understanding of the more general Lie-admissible formulations requires the understanding that *the lack of validity of the Poincaré Lemma within the context of the symplectic-admissible geometry is a necessary condition for the representation of the nonconservation of the energy of an interior dynamical system.*

**7.7.C: Genoriemannian geometry.** Despite impressive and historical advances in gravitation during this century, gravitation is still at its first infancy, particularly when compared to the problems yet to be addressed, let alone solved.

In Ch. I.5 we identified the need of an integral generalization of the Riemannian geometry for a more adequate representation of interior gravitational problems, such as gravitational collapse, "black holes", "big bang", etc., and submitted a generalization of the Riemannian geometry of the desired integral type called isoriemannian geometry.

The point to be stressed here is that physics is a discipline that will never admit final theories. No matter how advanced the isoriemannian geometry is over the Riemannian one, it is not expected to be "the" final geometry. Instead, the isoriemannian geometry is "one" geometry specifically conceived for one purpose, the treatment of closed-isolated interior systems with total conservation laws under a generalized interior structure.

Another fundamental physical problem in gravitation which has not even been addressed so far, let alone solved, is the dichotomy expressed by experimental evidence in the observation, say, of Jupiter, according to which the center-of-mass of the celestial body is time-reversal invariant, while its interior dynamics is manifestly irreversible. It is evident that the conventional Riemannian geometry is insufficient to represent the interior irreversibility in the needed axiomatic form.

It is at this point that the dual Lie-isotopic and Lie-admissible formulations become useful. In fact, as indicated earlier, the Lie-isotopic formulations are structurally reversible while the Lie-admissible formulations are intrinsically irreversible. The dual representation of reversible center-of-mass-trajectories *versus* irreversible interior dynamics, is then permitted by the complementarity of the Lie-isotopic and Lie-admissible formulations because of their inter-relation discussed in this chapter (see also Fig. 7.1.1).

Note the necessity of the Lie-isotopic formulations for this complementarity. In fact, reversible, conventionally Lie formulations for the global-exterior description are not compatible with irreversible, Lie-admissible, interior descriptions because their attached Lie algebra is not Lie but Lie-isotopic.

It may therefore be of some value to indicate a conceivable generalization of the Riemannian geometry, under the name of *Riemannian-admissible geometry* or *genoriemannian geometry*, originally submitted in ref. [11] which provides an irreversible description of interior gravitation in a way compatible with and complementary to the reversible description of the isoriemannian geometry of Sect. I.5.6. The understanding is that, unlike the isoriemannian geometry, the genoriemannian extension is vastly unexplored at this writing.

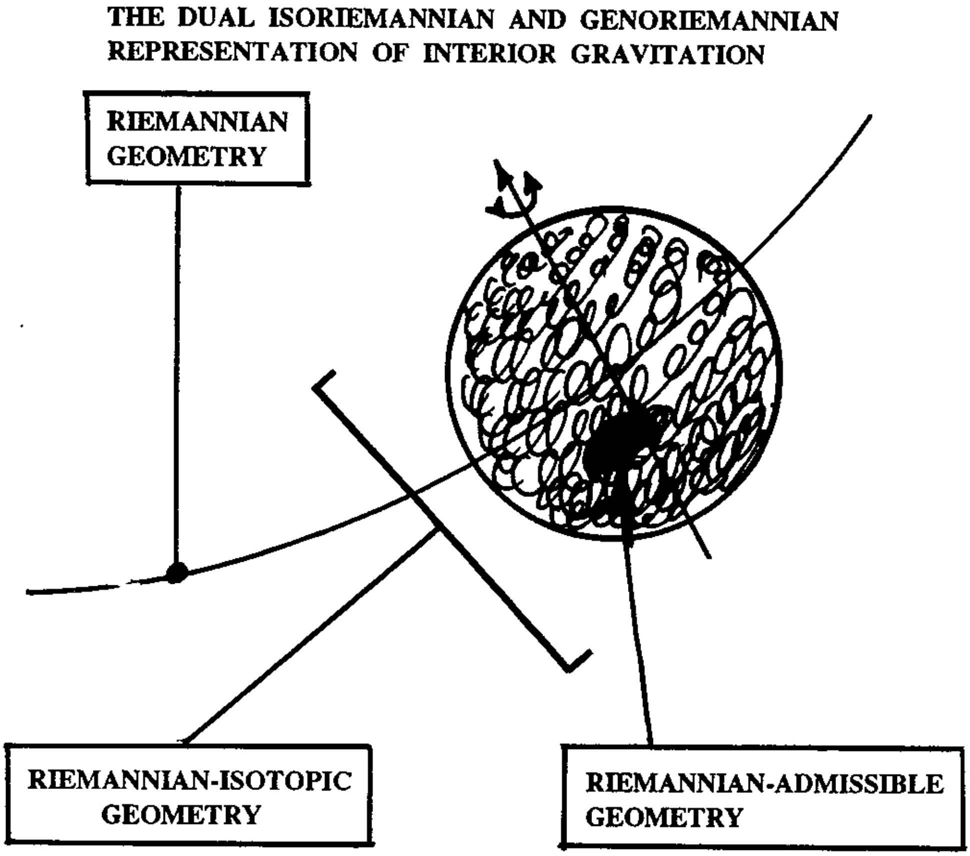


FIGURE 7.7.1: A schematic view of the geometric treatment of gravitation studied in these volumes. The Riemannian geometry is local-differential, thus being exact for the exterior problem of point-like test bodies in vacuum, but only approximate for the interior one. Moreover, a fundamental condition of the geometry is the *symmetric* character of the metric, $g = g^t$, thus implying its *reversible* character, with consequential inability to represent the experimental evidence of the interior irreversibility, say, of Jupiter. The Riemannian-isotopic geometry does solve the first problem, by permitting a direct representation of internal nonlocal-nonlagrangian effects in a way conform with total conservation laws. However, the isometric $\hat{g} = Tg$ of this latter geometry is also *symmetric*, $T = T^t$, thus being also

structurally *reversible*. A *necessary* condition for the construction of an interior geometry for the direct representation of irreversibility is the use of a *nonsymmetric* metric. The construction of a generalization of the Riemannian geometry with a nonsymmetric metric via the use of conventional methods (those over a conventional field) presents simply unsurmontable difficulties for current mathematical knowledge, which explains its absence at this writing. However, the same objective can be achieved via isotopic techniques in such a simple way to appear elementary. The main idea is based on two different nonsymmetric liftings of the isometric, one for motion forward in time $\hat{g} \Rightarrow \hat{g}^{>} = \hat{g}T^{>}$ and one backward in time $\hat{g} \Rightarrow {}^{<}T\hat{g}$, with $T^{>}$ and ${}^{<}T$ being different nonsymmetric real-valued tensors (evidently of the same dimension of $\hat{g}$) interconnected by the conjugation $T^{>} = ({}^{<}T)^{t}$. The definition of the forward isometric $\hat{g}^{>}$ over the genofield $\hat{R}^{>}(\hat{n}^{>},+,*^{>})$, and of the backward isometric ${}^{<}\hat{g}$ over ${}^{<}\hat{R}({}^{<}\hat{n},+,{}^{<}*)$ for the conjugation ${}^{<}T = (T^{>})^{t}$ or over its isodual for ${}^{<}T = (T^{>})^{d}$ then removes all technical difficulties indicated earlier because it implies the deformation $\hat{g} \Rightarrow T^{>}g$ while jointly deforming the unit of the amount inverse of the deformation, $I \Rightarrow \hat{1}^{>} = (T^{>})^{-1}$, and the same occurs for motion backward in time. Recall that deformed spheres, cones etc. are perfect spheres, cones, etc. at the level of the isogeometry (Ch. I.5). Along similar lines, the understanding of the genogeometry requires the knowledge that its nonsymmetric character appears only when the genogeometry is projected in the conventional Riemannian space because at the abstract level conventional, isotopic and genotopic geometries coincide.

The notion of genospace of Sect. I.7.3 can be specialized to that of *genoaffiine manifolds* as the manifolds ${}^{<}\hat{M}^{>}(x,{}^{<}\hat{R}^{>})$ which possess the same dimension, local coordinates and continuity properties of a conventional affine manifold M(x,R), but are defined over an isofield ${}^{<}\hat{R}^{>}$ with two different isounits $\hat{1}^{>}$ and ${}^{<}\hat{1}$ for the modular-isotopic action to the right and to the left, respectively,

$$x'^{>} = A{>}x = AT^{>}x, \quad \hat{1}^{>} = (T^{>})^{-1}, \tag{7.7.17a}$$

$${}^{<}x' = x{<}A = x\,{}^{<}TA, \quad {}^{<}\hat{1} = ({}^{<}T)^{-1}, \tag{7.A.17.1b}$$

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}. \tag{7.7.17.1c}$$

A *"Riemannian-admissible manifold" or genoriemannian manifold* can then be thought as an isoriemannian manifold (Definition I.5.6.1) with inequivalent isomodular actions to the right (forward in time) and to the left (backward in time), here denoted ${}^{<}\hat{R}^{>}(x,{}^{<}\hat{g}^{>},{}^{<}\hat{R}^{>})$; namely, a manifold characterized by the "genometrics for motions forward and backward in time"

$$g^{>} = T^{>}(s, x, \dot{x}, \ddot{x},...)\, g(x), \tag{7.7.18a}$$

$$^{<}g = {}^{<}T(s, x, \dot{x}, \ddot{x},...)\, g(x), \tag{7.7.18b}$$

where the two motions (multiplications) are interconnected by a suitable conjugation, e.g.,

$$T^{>} = ({}^{<}T)^{t} \quad \text{or} \quad ({}^{<}T)^{d}, \tag{7.7.19}$$

and equipped with two nonequivalent isoaffine connections, one for the modular–isotopic action to the right and the other to the left, the *Christoffel–admissible symbols of the first kind*

$$\Gamma^{>1}_{\ hlk} = \tfrac{1}{2}\left(\frac{\partial g^{>}_{kl}}{\partial x^{h}} + \frac{\partial g^{>}_{lh}}{\partial x^{k}} - \frac{\partial g^{>}_{hk}}{\partial x^{l}}\right) \neq \Gamma^{>1}_{\ klh}, \tag{7.7.20a}$$

$$^{<}\Gamma^{1}_{\ hlk} = \tfrac{1}{2}\left(\frac{\partial\, {}^{<}g_{kl}}{\partial x^{h}} + \frac{\partial\, {}^{<}g_{lh}}{\partial x^{k}} - \frac{\partial\, {}^{<}g_{hk}}{\partial x^{l}}\right) \neq {}^{<}\Gamma^{1}_{\ klh}, \tag{7.7.20b}$$

with corresponding *Christoffel–admissible symbols of the second kind*

$$\Gamma^{>2\ i}_{\ \ h\ k} = g^{>ij}\, \Gamma^{>1}_{\ hjk} = \Gamma^{>2\ i}_{\ \ k\ h}, \tag{7.7.21a}$$

$$^{<}\Gamma^{2\ i}_{\ \ h\ k} = {}^{<}g^{ij}\ {}^{<}\Gamma^{1}_{\ hjk} = {}^{<}\Gamma^{2\ i}_{\ \ k\ h}. \tag{7.7.21b}$$

The capability of a genometric to raise and lower the indices is understood (as in any affine space), and

$$g^{>ij} = |(g^{>}_{rs})^{-1}|^{ij}. \tag{7.7.22a}$$

$$^{<}g^{ij} = |({}^{<}g_{rs})^{-1}|^{ij}. \tag{7.7.22b}$$

The *Riemannian–admissible geometry* or *genoriemannian geometry* for short, is the geometry of genospaces $^{<}\hat{R}^{>}(x, {}^{<}\hat{g}^{>}, {}^{<}\hat{R}^{>})$.

Its explicit construction can be done via the appropriate generalization of the isoriemannian geometry, with particular reference to the isoconnections which, besides being different for the right and left modular–isotopic action, can now be *nonsymmetric* depending on the assumed characteristics of the genotopic elements $T^{>}$ and $^{<}T$.

What is important is the mechanism of the lifting, which consists of a deformation of the original metric while jointly lifting the unit by the inverse of the deformation, Eq.s (7.7.1). The consistency of the new geometry is then consequential (see Fig. 7.7.1).

The above results permits the following

**Lemma 7.7.1** [11]: *An axiomatization of irreversibility in interior gravitation is provided by inequivalent deformations of modular actions, metrics and connections to the right (forward in time) and to the left (backward in time) under a joint lifting of the unit per each direction of time characterized by the inverses of the deformations.*

As we shall see in Vol.s II and III, the above geometrization does indeed permit the representation of open-nonconservative-irreversible interior trajectories in Jupiter, such as a representation of interior vortices with monotonically varying angular momenta, although in a way compatible with the reversibility of the closed-isolated systems.

The interconnection of Lemmas 7.5.1, 7.6.1 and 7.7.1 should be kept in mind.

It is hoped that geometers in the field will be intrigued by the Riemannian-admissible geometry and develop it in the necessary technical details needed for quantitative studies of irreversible interior gravitation.

## 7.8: FUNCTIONAL GENOANALYSIS

In Sect. I.7.2 we have pointed out a number of problematic aspects of the current representation of nonconservative systems, such as their representation via the addition of a fictitious "imaginary potential" to the Hamiltonian, $H = K + iV$. This evidently implies a trajectory different from the physical one because the forces originating the nonconservation are generally of *nonpotential* type.

This approach to open nonconservative systems has yet another fundamental problematic aspects, and it is given, on one side, by the evident lack of Hermiticity of the Hamiltonian with consequential loss of observability, while, on the other side, the loss of energy is indeed observed and physically measured.

Hadronic mechanics has been conceived and constructed to resolve these evident problematic aspects. In fact, the nonpotential forces responsible for the nonconservation are not represented with a potential but with other means. Moreover and most importantly, the nonconserved Hamiltonian remains fully Hermitean and, thus observable, during the nonconservative process.

As we shall see in Vol. II, these are not mere mathematical curiosities, because they have direct experimental consequence. As an example, the achievement of the Hermiticity of the Hamiltonian during its time-rate-of-variation requires a suitable, corresponding revision of the data elaboration (such as a structural alteration of the expectation values) resulting in different numerical predictions and interpretations for the same event, as we shall see.

The preservation of the Hermiticity/observability of a Hamiltonian when nonconserved is achieved by a further generalization of the functional isoanalysis of the preceding chapter, this time, of genotopic character.

Recall that the Lie-isotopic theory admits a formulation via operators on a *conventional* Hilbert space $\mathcal{H}$, as originally proposed in ref. [20]. However, in so doing the observability is lost even for *conservative* processes. The observability can however be preserved if one lifts the Hilbert space via the same isotopic element T of the enveloping algebra.

A fully similar situation occurs for the more general Lie-admissible theory. In fact, it can be well defined on both a conventional Hilbert space $\mathcal{H}$ and its isotope $\hat{\mathcal{H}}_T$. However, a Hamiltonian is generally nonhermitean in both.

We reach in this way the *left and right genohilbert spaces*

$$\hat{\mathcal{H}}^{>} : \quad (\psi \hat{,} \Phi)^{>} = \hat{1}^{>} \int d^3r \, \psi^{\dagger}(r) > \phi(r) \in \hat{R}^{>} , \tag{7.8.1a}$$

$$^{<}\hat{\mathcal{H}} : \quad ^{<}(\psi \hat{,} \phi) = {}^{<}\hat{1} \int d^3r \, \phi(r) < \psi^{\dagger}(r) \in {}^{<}\hat{R} . \tag{7.8.1b}$$

It is easy to prove that a Hamiltonian which is conventionally Hermitean, remains Hermitean under the above genotopies, thus being observable, even though it is nonconserved,

$$i \, dH / dt = H ( R - S ) H \neq 0 . \tag{7.8.2}$$

A simple example can be instructive here. Consider the free quantum mechanical particle with Hamiltonian $H_o = \frac{1}{2} p_o^2$, $m = 1$, which is evidently Hermitean over $\mathcal{H}$. Suppose now that this particle at a given instant of time $t_o$ enters within a resistive medium, thus losing energy to the medium itself. Assume the simplest possible decay, the linearly damped one

$$H = e^{-\gamma t} H_o = e^{-\gamma t} \frac{1}{2} p_o^2 . \tag{7.8.3}$$

As we shall see in Vol. II, the Lie-admissible branch of hadronic mechanics permits an *axiomatic representation* of the above system; that is, a formulation derivable from first principles which is invariant under its own time evolution.

At this point we are merely interested in illustrating the basic dynamical equations, the underlying genohilbert spaces, and the Hermiticity-observability of the Hamiltonian.

It is easy to see that the desired Lie-admissible representation of system (7.8.3) is characterized by the realizations of the R-S quantities

$$R = -\tfrac{1}{2} i \gamma H_o^{-1}, \quad S = + i \tfrac{1}{2} \gamma H_o^{-1}, \quad R = S^{\dagger}, \tag{7.8.4}$$

The Lie-admissible group of the time evolution of a quantity Q is then given by

$$\begin{aligned} Q(t) &= \{ e_{\xi>}^{\,i H (t_o - t)} \} > Q(t_o) < \{ e_{<\xi}^{\,i (t - t_o H} \} = \\ &= e^{\,i H_o S (t_o - t)} Q(t_o)\, e^{-i (t_o - t) R H} . \end{aligned} \tag{7.8.5}$$

with infinitesimal Lie-admissible equation

$$i\, dQ / dt = Q < H_o - H_o > Q \tag{7.8.6}$$

which becomes for the energy

$$i\, dH_o / dt = -i \gamma H_o, \tag{7.8.7}$$

thus verifying law (7.8.4).

The underlying genohilbert spaces are then given by

$$\mathcal{H}^{>} : \quad (\psi \hat{,} \phi)^{>} = 1^{>} \int d^3r\, \psi^{\dagger}(r) > \phi(r) \in \hat{R}^{>},\ 1^{>} = i\, 2 \gamma^{-1} H_o \tag{7.8.8a}$$

$$^{<}\mathcal{H} : \quad ^{<}(\psi \hat{,} \phi) = {}^{<}1 \int d^3e \quad \phi(r) < \psi^{\dagger}(r) \in {}^{<}\hat{R},\ {}^{<}1 = -i\, 2 \gamma^{-1} H_o . \tag{7.8.8b}$$

The Hermiticity/observability of the Hamiltonian during the decaying process can be easily verified.

Note the formal identities for the case considered

$$(\psi \hat{,} \phi)^{>} \equiv {}^{<}(\psi \hat{,} \phi) \equiv (\psi , \phi) = \int d^3r\, \psi^{\dagger}(r)\, \phi(r) , \tag{7.8.9}$$

namely, the compositions of the genohilbert spaces coincide with the conventional one. The Hermiticity/observability under decay emerges from the definition of the same composition in an invariant form on a genofield, that is, the decay is represented by the *operations* on $H_o$ and not by the Hamiltonian. In turn, this simple example illustrates the truly fundamental character of the theory of isonumbers and genonumbers for hadronic mechanics. More complex nonconservative systems will be studied Vol. II and III along structurally the same lines.

All isotopic generalizations of trigonometry, Dirac's δ-function, Fourier series, Fourier transforms, etc. then admit a significant and intriguing genotopic extension which is hereon assumed.

## 7.9: FUNDAMENTAL EQUATIONS OF HADRONIC MECHANICS AND THEIR DIRECT UNIVERSALITY

We are finally in a position to identify the fundamental equations of the two branches of hadronic mechanics indicated in Sect. 1.5

**7.9.A: Lie-isotopic branch of hadronic mechanics.** This branch describes *closed-isolated, composite systems with conserved total energy and other physical quantities and nonlinear-nonlocal-nonpotential internal forces.*

The nonrelativistic characterization of systems via this branch requires two operators, the Hamiltonian H and one isotopic operator T. The mathematical structure of the branch is characterized by one single isotopic product for both the right and the left with one single space isounit

$$\hat{1} = \hat{1}^{\dagger} = T^{-1}, \quad \hbar = 1 \qquad (7.9.1)$$

and it is based on the following main structures:

A–1) Isofields of isoreal or isocomplex numbers $\hat{F}(\hat{\alpha},+,*)$,

A–2) Enveloping isoassociative operator algebras $\hat{\xi}_T$,

A–3) Isohilbert spaces $\hat{\mathcal{H}}_G$, $G = G^{\dagger} > 0$, $G \neq T$,

which characterize the fundamental dynamical equations in the infinitesimal form

$$i\,\hat{1}_t \frac{dQ}{dt} = [Q \hat{,} H] := Q * H - H * Q = Q\,T\,H - H\,T\,Q, \qquad (7.9.2)$$

where $\hat{1}_t = T_t^{-1} \neq \hat{1}$ is the time isounit and $[Q \hat{,} H]$ are the Lie-isotopic brackets, with finite form

$$\begin{aligned} Q(t) &= \hat{U} * Q(0) * \hat{U}^{\dagger} = (e_{\xi}^{\,iHt}) * Q(0) * (e_{\xi}^{\,-itH}) = \\ &= (e^{\,iHTt})\, Q(0)\, (e^{\,-itTH}), \end{aligned} \qquad (7.9.3)$$

yielding a Lie-isotopic group of isounitary transformations on $\hat{\mathcal{H}}_G$,

The corresponding, isoequivalent Schrödinger-type representation in isospaces $\hat{E}(t,\hat{R}_t)\times\hat{E}(r,\hat{\delta},\hat{R})$ [61] are given by

$$i\,\hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) = H * \hat{\psi}(t, r) = H\,T\,\hat{\psi}(t, r) \qquad (7.9.4a)$$

---

[61] We introduce here and in the following isounits not dependent explicitly in the local coordinates to avoid gravitational considerations at this time.

$$- i\, \hat{\psi}^{\dagger}(t, r) \frac{\partial}{\partial t} \hat{1}_t = \hat{\psi}^{\dagger}(t, r) * H = \hat{\psi}^{\dagger}(t, r)\, T H . \qquad (7.9.4b)$$

and fundamental isocommutation rules

$$[\, a^{\mu} \hat{,} a^{\nu} \,] = \begin{pmatrix} [\, r^i \hat{,} r^j \,] & [\, r^i \hat{,} p_j \,] \\ [\, p_i \hat{,} r^j \,] & [\, p_i , p_j \,] \end{pmatrix} = \begin{pmatrix} 0 & i\hat{1} \\ - i\hat{1} & 0 \end{pmatrix} \qquad (7.9.5)$$

The above Lie–isotopic branch is structurally reversible, can be therefore used for either direction of time, and is divided into Kadeisvili's Classes I, II, III, IV and V depending on the characteristics of the basic isounit $\hat{1}$ (see Vol. II for details and relativistic extensions).

**7.9.B: Lie-admissible branch of hadronic mechanics,** which characterizes *open–nonconservative systems with nonconserved energy and other physical quantities under the most general possible nonlinear–nonlocal–nonpotential external interactions.*

Physical systems are represented nonrelativistically in this branch by three operators, the Hamiltonian H and the genotopic elements R and S which are however interconnected by the conjugation $R^{\dagger} = S$. This second branch is characterized by two different space genounits, one for motion forward, and one for motion backward in time interconnected by Hermitean conjugation

$$\hat{1}^{>} = R^{-1} ,\ A > B : = A R B,\ \ {}^{<}\hat{1} = S^{-1} ,\ A < B := A S B ,\ \ \hat{1}^{>} = ( {}^{<}\hat{1} )^{\dagger} , \qquad (7.9.6)$$

and it is based on the following main structures:

B–1) Genofields of genoreal or genocomplex numbers ${}^{<}\hat{F}^{>}({}^{<}\hat{\alpha}^{>},+,{}^{<}\hat{*}^{>})$,

B–2) Enveloping genoassociative operator algebras ${}^{<}\hat{\xi}^{>}$ ,

A–3) Genohilbert spaces ${}^{<}\hat{\mathcal{H}}^{>}$ ,

which characterize the fundamental dynamical equations in the infinitesimal form

$$i\, {}^{<}\hat{1}^{>}_t \frac{dQ}{d^{<}t^{>}} = (Q \hat{,} H) : = Q < H - H > Q = Q R H - H S Q \qquad (7.9.7)$$

where $t^{>}$ (${}^{<}t$) represents forward (backward) time with corresponding genounits $\hat{1}^{>}_t$ (${}^{<}\hat{1}_t$) and $(Q \hat{,} H)$ are the Lie–admissible brackets, with finite form

$$Q(t) = \hat{U}^{>} > Q(0) < {}^{<}\hat{U}^{\dagger} = \{ e_{\hat{\xi}^{>}}^{\, i H t} \} > Q(0) < \{ e_{{}^{<}\hat{\xi}}^{\, - i t H} \} =$$

$$= \{ e^{\, i H R t} \}\, Q(0)\, \{ e^{\, - i t S H} \} , \qquad (7.9.8)$$

yielding a Lie–admissible group of genounitary transformations on ${}^{<}\hat{\mathcal{H}}^{>}$, The corresponding genoequivalent Schrödinger-type representation in genospaces $^{<}\hat{E}^{>}(t, ^{<}\hat{R}^{>}_{t}) \times {}^{<}\hat{E}^{>}(r, ^{<}\hat{\delta}^{>}, ^{<}\hat{R}^{>})$ is characterized by the equations

$$i\,\hat{1}^{>}_{t}\ \frac{\partial}{\partial t^{>}}\hat{\psi}^{>}(t, r) \;=\; H > \psi^{>}(t, r) \;=\; H\,R\,\hat{\psi}^{>}(t, r) \tag{7.9.9a}$$

$$-\ i\ {}^{<}\hat{\psi}(t, r)\,\frac{\partial}{\partial^{<} t}\,{}^{<}\hat{1}_{t} = {}^{<}\hat{\psi}(t, r) < H \;=\; {}^{<}\hat{\psi}(t, r)\,S\,H\,. \tag{7.9.9b}$$

with fundamental genocommutation rules

$$(\,a^{\mu}\,\hat{,}\,a^{\nu}\,) = \begin{pmatrix} (\,r^{i}\,\hat{,}\,r^{j}\,) & (\,r^{i}\,\hat{,}\,p_{j}\,) \\ (\,p_{i}\,\hat{,}\,r^{j}\,) & (\,p_{i}\,,\,p_{j}\,) \end{pmatrix} = i\,{}^{<}\hat{S}^{>\mu\nu} \tag{7.9.10}$$

where ${}^{<}\hat{S}^{>\mu\nu}$ is the operator image of the corresponding classical Lie-admissible tensor also originating from the fundamental genocommutation rules.

The Lie-admissible branch is intrinsically irreversible, must be used for each given direction of time, and is also divided into Kadeisvili's Classes I, II, III, IV and V referred to the maximal Hermitean part of the genounits $^{<}\hat{1}^{>}$ and $^{<}\hat{1}^{>}$.

The crucial Lie-isotopic and Lie-admissible equations for the linear momentum will be presented in Vol. II.

It is evident that the Lie-isotopic branch is a particular case of the Lie-admissible branch, and this illustrates the reason why hadronic mechanics was originally submitted [20] in terms of the fundamental Lie-admissible equations (7.9.7) and (7.9.8). A detailed study of the derivation, properties and basic axioms of the above equations is presented in Vol. II.

**7.7.C: Direct Universality of Hadronic Mechanics.** We shall now outline the "direct universality" of hadronic mechanics, that is, its capability to represent all possible linear and nonlinear, local and nonlocal, Hamiltonian and nonhamiltonian, continuous or discrete, and other systems (universality), directly in the frame of the observer (direct universality).

It should be stressed from the outset that this *does not* means that hadronic mechanics is the only applicable mechanics, because numerous other approaches are indeed possible for the elaboration of the same system (see the appendix).

The direct universality however implies the remarkable occurrence that, while other theories generally treat only *one* class of systems, hadronic mechanics can treat them all. The selection of one theory versus another does not evidently depend on personal taste, but rather on the intrinsic consistency of the theory at hand, as well as the experimental verification.

The identification whether a given theory is a particular case of hadronic mechanics implies:

1) The identification of possible departures from conventional quantum mechanical laws which are inherent in the theory considered;

2) The identification of corresponding generalized physical laws, as well as the physical conditions for their applicability, as a basis for experimental resolution; and

3) The availability of rigorous axiomatic methods for the quantitative treatment of the theory considered in a way demonstrably consistent with the basic assumption. As we shall see, this basic condition is lacking for a number of theories which, while possessing a generalized structure, elaborate data with conventional quantum mechanical assumptions, thus leading to insidious problematic aspects in their physical interpretation and applications.

In this final section, it may be recommendable to provide the primary guidelines for detailed study later on, as expressed by the following two theorems.

**Theorem 7.9.1 - Direct universality for systems with conserved energy:** *All possible linear or nonlinear, local-differential or nonlocal-integral, continuous or discrete operator, nonrelativistic or relativistic, equations representing a system with conserved total energy admit a direct representation via the Lie-isotopic branch of hadronic mechanics in the frame of the experimenter in one of the Classes I, II, III,. IV and V.*

The above theorem is transparently proved, e.g., by Eq.s (7.9.4) when written explicitly

$$i\, \hat{1}_t(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi, \ldots) \frac{\partial}{\partial t} \hat{\psi} = H\, T(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, \ldots \ldots)\, \hat{\psi}\,, \tag{7.9.11a}$$

$$i\, dH/dt \equiv 0. \tag{7.9.11b}$$

which provide a direct representation of any given operator equations with conserved total energy. A similar situation occurs at the relativistic level (Vol. II).

Note that the Lie-isotopic equations permit an *infinite number of different representations of the same system* evidently due to the availability of two operators H and T for the same equation. However, such an infinity is reduced to only one, up to isoequivalence, when H is restricted to represent the total energy of the system considered.

**Theorem 7.9.2 - Direct universality for systems with nonconserved energy:** *All possible linear or nonlinear, local-differential or nonlocal-*

*integral, continuous or discrete, nonrelativistic or relativistic, operator equations representing a system with nonconserved energy admit a direct representation via the Lie-admissible branch of hadronic mechanics in the frame of the experimenter in one of the Classes I, II, III,. IV and V.*

Again, the property is transparently exhibited by Eq.s (7.9.9) in their explicit form

$$i\, \hbar^{>}_t(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ..) \frac{\partial}{\partial t} \hat{\psi} = H\, R(t, r, p, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ..)\, \hat{\psi}\,, \quad (7.9.12a)$$

$$i\, dH / dt \neq 0. \quad (7.9.12b)$$

As an illustration, one, among the infinitely possible reformulation of Eq.s (7.2.4) in terms of Lie-admissible equations submitted since the original proposal of the hadronic mechanics [20] is given by

$$(A \,\hat{,}\, H) \equiv A\,RH - HSA \equiv AH - H^\dagger A = A \times H, \quad (7.9.13a)$$

$$R = 1, \qquad S = H^{-1} H^\dagger. \quad (7.9.13b)$$

We now close this section with the necessary conditions for the existence of a bona-fide generalized mechanics. When inspecting any generalized theory, the fundamental issue is whether conventional quantum mechanical laws and axioms are preserved or generalized. In turn, this issue sets the stage for the elaboration via conventional or generalized methods, thus resulting in different numbers predicted by the theories for the same system.

The above issue can be answered via the following:

**Basic criterion 7.9.1 - Identification of conventional vs generalized theories:** *Any theory whose fundamental commutation rule coincide with or are unitarily equivalent to the canonical commutation rules*

$$[a^\mu, a^\nu] = \begin{pmatrix} [r^i, r^j] & [r^i, p_j] \\ [p_i, r^j] & [p_i, p_j] \end{pmatrix} = \begin{pmatrix} 0 & i\,I \\ -i\,I & 0 \end{pmatrix}, \quad (7.9.14)$$

*is structurally equivalent to quantum mechanics, with corresponding cases occurring for relativistic and field theoretical theories. A necessary condition for the existence of a generalization of quantum mechanics is therefore the presence of generalized fundamental commutation rules* *which are not unitarily equivalent to those of quantum mechanics.*

As one can see, the situation is clear–cut, without possibilities of using generalized theories while preserving old physical laws: generalized fundamental canonical commutation rules demand the use of generalized physical laws and methods. A good example is given by generalized commutation rules of the type

$$[\, r, p] \; = \; r\, p \; - \; p\, r \; = \; i\, f(r,p) \qquad (7.9.15)$$

where f(r,p) is a function or even a number different than $\hbar = 1$ (see also App. I.7.A.). Then the theory is noncanonical and must be reformulated via the redefinition of the unit and of the commutators themselves into the isotopic form

$$[\, r \hat{,} p] \; = \; r\, T\, p \; - \; p\, T\, r \; = \; i\, \hat{1} \; = \; i\, f(r, p) \;, \quad T \; = \; [\, f(r, p)\,]^{-1}, \qquad (7.9.16)$$

which is now axiomatic, that is, derivable from first principles and invariant under its own time evolution. On the contrary, it is easy to prove that the "noncanonical" brackets (7.9.15) expressed in terms of the "conventional" Lie product rp – pr, do not preserve their form, and are in actuality mapped precisely into the isotopic form (7.9.16), as shown in Eq.s (4.1.3).

This is due to the fact that the only possible transformations capable of reducing the noncanonical value f(r, p) to 1 are nonunitary, even when the function f reduces to a constant.

Equivalently, we can say that noncanonical brackets (7.9.15) are based on a generalization of the unit precisely of the fundamental form (1.1.1). The reconstruction of the entire structure of quantum mechanics into that of the covering hadronic mechanics is then necessary for consistency of the formalism, as well as for its axiomatic, form–invariant character.

The physically relevant issue here is that *the quantum mechanical data elaboration of the theory based on isocommutation rules (7.9.15) are different from those based on rule (7.9.16).* The formulation of rule (7.9.15) via the conventional Lie product therefore gives only a misleading *impression* of having preserved quantum mechanics.

We should insist in this important point and indicate some of the problematic aspects of formulation (7.9.15), such as: the belief that conventional quantum mechanical energy, linear momentum, etc., remain Hermitean and thus observable under generalized commutation rules (7.9.15) at all times. It is easily proved that, under the nonunitary time evolution, the enveloping algebra becomes isotopic, while the Hilbert space remains unchanged, and this implies the general loss of Hermiticity, as familiar from Ch. I.6.

In short, the "fundamental canonical commutation rules" are truly "fundamental". Any structural deviation from them implies a necessary, consequential and compatible generalization of the structure of quantum mechanics. This is the case of the large variety of models of type (7.9.15) and other models (App. I.7.A).

After having understood (and, most importantly, admitted) the generalized character of a given theory, the next basic issue is the determination whether the total energy is conserved or not, so as to determine which methods to use as per Theorem 7.9.1 and 7.9.2.

**Basic criterion 7.9.2 - Conservation of the energy in generalized theories:** *A necessary condition for the total energy* H *of a generalized theory (as per Criterion 7.9.1) to be conserved is that the generalized fundamental commutation rules are isounitarily equivalent to the Lie-isotopic rules (7.9.5).*

Note that the above condition is necessary but not sufficient. In fact, the establishing that the total energy is conserved requires the additional conditions that: 1) H is the generator of the time evolution; 2) the canonical algorithm "p" represents the physical linear momentum, $p = m\dot{r}$; 3) H consists of the sum of two terms, $H = K + V$, the physical kinetic energy K and the physical potential energy V, etc. (for a detailed study of this aspect one may consult ref. [8,9].

Note also the necessary use of isounitary transformation. In fact, the use of unitary transformations would be futile, inasmuch as fully within conventional quantum mechanical settings.

**Basic criterion 7.9.3 - Nonconservation of the energy in generalized theories:** *A necessary condition for the operator* H *of a generalized theory (as per Criterion 7.9.1) to represent the nonconserved energy of the system is that the generalized fundamental commutation rules are isounitarily equivalent to the Lie-admissible rules (7.9.10).*

Stated in different terms, the identification of an essential Lie-admissible structure guarantees the nonconservation of the energy. It is evident that, by no means does this implies any violation of any basic law of physics. The Lie-admissible formulations merely identify the *external* character of the interactions represented via the R and S operators. The understanding is that when the nonconservative system is completed with these external interactions, one regains the conservation of the total energy in full.

The above outline should be sufficient for the identification, first, whether a given theory is a generalization of quantum mechanics or not and, second, whether a generalized theory has a Lie-isotopic or Lie-admissible structure according to Theorem 7.9.1 or 7.9.2. Once these basic identifications have been made, then the methods of Vol. II are applicable for an axiomatic, form-invariant characterization of the theory, the identification of their physical laws, and the correct elaboration of data for experimental verifications.

## APPENDIX 7.A: CONNECTION BETWEEN HADRONIC MECHANICS AND OTHER GENERALIZED THEORIES

As indicated since the Preface of this volume, hadronic mechanics has a direct connection with *all* generalizations of quantum mechanics attempted until now, with no exception known to this author. This is due to the universality theorems 7.9.1 and 7.9.2 which imply the inclusion of generalizations of nonlinear, nonlocal, discrete, algebraic, geometric, or any other type.

All existing generalized theories have been conceived and developed in a way *independent* from hadronic mechanics. Such independence is here confirmed as well as supported because of the polyhedric nature of mathematical and physical inquiries indicated earlier.

At the same time, another aspect of scientific inquiries is the need to study inter–relationship among different theories, because of the evident scientific gains reached in the comparison.

Along the latter lines, the primary contribution expected by the reformulation of a given generalized theory in terms of hadronic mechanics is of primary *physical character*, and deals with the identification of the axiomatic form invariant under time evolution, the applicable physical laws, and the applicable formalism for the data elaboration, so as to reach predictions with the necessary consistency needed for experimental consideration.[62]

The axiomatic formulation, the applicable basic laws and the methods for the data elaboration are studied in Vol. II, jointly with primary applications such as to the origin of irreversibility, gauge theories, and the like. In this appendix we merely illustrate the connection between hadronic mechanics and a few representative generalized theories.

**7.A.1: Hadronic mechanics and q-deformations.** Albert's paper [18] of 1948 studied the generalized product

$$a \times b = p\, a\, b - (1 - p)\, b\, a\,, \qquad (7.A.1)$$

where ab can be assumed for simplicity to be associative and p is an element of the field, as a realization of the *noncommutative Jordan algebras*[63] which were

[62] The reader should always keep in mind the numerous papers existing in the literature with *noncanonical* commutation rules, yet the elaboration of data via *conventional quantum mechanics*, whose predictions have no credibility warranting a consideration for experiments.

[63] Those are algebras with product $a \times b \neq b \times a$ verifying Jordan's axiom $(a \times b) \times (a \times a) = (a \times (b \times (a \times a))$ [18].

of particular interest in the mathematics of the time.

Besides being a realization of noncommutative Jordan algebras, the above product is Lie-admissible, Jordan-admissible and admits the commutative Jordan algebras as a particular case for $p = \frac{1}{2}$, but it does not admit Lie algebras for any (finite) value of p. For this reason, this author introduced [12] back in 1967 as part of his graduate studies in physics at the University of Torino, Italy, and apparently for the first time in both mathematical and physical literature, the generalized product

$$(a, b) = p\,a\,b - q\,b\,a, \tag{7.A.2}$$

where p and q are elements of the base field or functions, under the name of *(p,q)-mutations of associative algebras.* As one can see, product (7.A.2) is Lie-admissible, Jordan-admissible, admits both Lie algebras and commutative Jordan algebras as particular cases for finite values of p and q, and constitutes a realization of the noncommutative Jordan algebras (see ref. [12] for details).

The above initial studies were then expanded by the author [20] in 1978 into the Lie-admissible time evolution (7.9.7), i.e.,

$$i\,dA / dt = A\,P\,H - H\,Q\,A, \tag{7.A.3}$$

where P and Q are now unrestricted integro-differential operators, and in the fundamental Lie-admissible commutation rules (7.9.10), i.e.,

$$(a^\mu \hat{,} a^\nu) = \begin{pmatrix} (r^i \hat{,} r^j) & (r^i \hat{,} p_j) \\ (p_i \hat{,} r^j) & (p_i , p_j) \end{pmatrix} = i \langle S \rangle^{\mu\nu} \tag{7.A.4)}$$

Subsequent studies along Albert's notion of Lie-admissibility have been reported in this chapter.

Independently from the above, various authors studied in the early 80's a generalization of canonical commutation rules of the type

$$(r, p) = r\,p - q\,p\,r, \tag{7.A.5}$$

under the name *q-deformation*[64], and more recently referred in a highly improper way as *quantum groups*[65] (see the recent ref.s [25] and literature

[64] In his original proposal of 1967 [12], this author had intentionally used a term other than "deformation" (and suggested the term "mutation" because most of the so-called q-deformations *are not* "deformations" as conventionally understood in mathematics. Nevertheless, the terms "q-deformations" are now widely used, and they will be kept in this volume to avoid confusion.

contained therein). As one can see, product (7.A.5) is the particular case (0, q) of the (p, q)-mutations (7.A.2), but *it is not* a particular case of product (7.A.1). As such, product (7.A.5) is also Lie-admissible, Jordan-admissible, admits Lie and commutative Jordan algebras as particular case, and it is a realization of the noncommutative Jordan algebras.

The studies in the field have recently multiplied and extended to various parts of quantum mechanics, including the q-deformation of the Poincaré algebra (see, e.g., ref.s [26]). [66]

The "q-deformations" are an ideal example to illustrate the relationship between generalized theories and hadronic mechanics. In fact, their *mathematical* consistency is impeccable, their *independence* from hadronic mechanics is established, e.g., by comparing q-special functions and isospecial functions (Ch. I.6), and their *beauty* is undeniable as shown by the number of researchers attracted to the field.

However, the q-deformations are afflicted by a number of problematic aspects of a *physical nature* which cannot be ignored. To identify them, let us recall that the terms "q-deformations" are now refereed to a variety of generalized theories all generally defined at a fixed value of time, such as:

I) **Deformation of the enveloping associative algebra.** Let $\xi(L)$ be the universal enveloping associative algebra of a Lie algebra L (Sect. I.4.3) with elements A, B, ... and conventional associative product AB over a field $F(a,+,\times)$. This first type is characterized by the following generalization of the associative product AB[67]

$$A\,B \quad \Rightarrow \quad A * B = q\,A\,B\,, \qquad (7.A.6)$$

where q is an element of the base field (or a function), without the joint lifting of the basic field as adopted by isotopic theories (Ch.s

[65] The use of the terms "quantum groups" is discouraged, and will not be adopted in these volumes because excessively misleading. In fact, the terms were historically referred, first, to a structure forming a conventional *Lie group* and, second, to the realization of such group in quantum mechanics. The use of the same terms for the q-deformation is therefore misleading on at least two counts, first, because the q-deformations do not yield a group as conventionally understood, and, second, because their structure is incompatible with the very notion of quantum of energy.

[66] It should be noted that the first Lie-admissible, P-Q-operator deformation of the Poincaré symmetry was introduced by the author in ref. [11] via the notion of Lie-admissible isobimodules or genomodules.

[67] The reader should be aware that the form "qAB" of the product is correct only for q-numbers or functions and *not* for q-operators, in which case the product must be written "AqB", as done throughout this volume. In fact, if AB is an associative algebra, the product $A \times B = qAB$ with q a fixed operator violates the left scalar and distributive laws and, as such, it does not constitute any algebra of any kind.

I.1, and I.2);

**II) Deformation of the Lie product.** Let L be a Lie algebra in quantum mechanical realization on a Hilbert space $\mathcal{H}$ over a field F(a,+,×) with generators A, B, ...and fundamental commutation rules rp – pr = i ($\hbar$ = 1). This second type of q-deformation is based on the generalization of the canonical commutators

$$r\,p - p\,r \quad \Rightarrow \quad r\,p \;-\; q\,p\,r \;=\; i\,f(q, ...) \qquad (7.A.7)$$

which is evidently of type (7.A.5).

**III) Deformation of the structure constants.** let L be an n-dimensional Lie algebra with ordered basis $X_i$, envelope $\xi(L)$ and commutation rules $[X_i, X_j] = C_{ij}{}^k\, X_k$ over a field F(a,+,×). This third type of deformations is based on the preservation of the original product $X_iX_j$ of $\xi(L)$ and of the original Lie product $X_iX_j - X_jX_i$ of L, while deforming this time the structure constants

$$X_i\,X_j \;-\; X_j\,X_i \;=\; C \;=\; C_{ij}{}^k\,X_k \;\;\Rightarrow\;\; X_i\,X_j \;-\; X_j\,X_i \;=\; F_{ij}{}^k(q, ..)\,X_k\,, \quad (7.A.8)$$

where the quantities $F_{ij}{}^k$ are similar to the "structure functions" of the Lie-isotopic theory (this type includes deformations characterized by the *Hopf algebras* and numerous others);

plus additional deformations, such as those characterized by the combination of *deformed* commutators (7.A.7) and *conventional* Heisenberg equations for the time evolution,[68] the deformation of creation-annihilation operators of the above Types I, II, III, etc. (see ref.s [25,26] or brevity).

Again, all the above q-deformations have an impeccable mathematical consistency and un undeniable beauty. However, when considered for *physical applications* they require the necessary use of the dynamical time evolution, in which case a number of problematic aspects emerge as recently studied by Lopez [27], such as:

*I) General loss of the Hermiticity/observability of the Hamiltonian.* As now familiar from the studies presented in this volume, deformations of the Types I, II, III above generally imply a nonunitary time evolution, as necessary from the lack of canonicity of the commutation rules, and demonstrable, e.g., via quantization of the corresponding, classical, noncanonical theories. In turn, nonunitary time evolutions imply the lifting of the envelope into the isotopic

[68] This latter class evidently requires *two different envelopes*, a generalized one for the characterization of the generalized commutation rules, and a conventional one for the characterization of the conventional time evolution. Even though *mathematically correct*, this class multiplies, rather than reduces the *physical problematic aspects* discussed below.

form for all Types I, II, III,

$$\xi: \; A\,B \;\Rightarrow\; \hat{\xi}: \; A' * B' = A'\,T\,B', \quad A' = U\,A\,U^\dagger, B' = U\,B\,U^\dagger \,, \tag{7.A.9a}$$

$$U\,U^\dagger = \hat{1} \neq I, \quad T = (U\,U^\dagger)^{-1}, \quad \hat{1} = T^{-1}. \tag{7.A.9b}$$

Still in turn, this implies the loss of the Hermiticity/observability of the Hamiltonian and of other physical quantities because q–deformations are defined on a *conventional Hilbert space* $\mathcal{H}$, while the preservation of Hermiticity under lifting (7.A.4) demands the joint lifting of the base field $F \Rightarrow \hat{F}_T$ and of the Hilbert space $\mathcal{H} \Rightarrow \hat{\mathcal{H}}_T$ (Sect. I.6.3).[69]

*2) General loss of the measurement theory.* Most q–deformations are deformations of the basic associative product AB and/or of Planck's constant $\hbar$ = , and/or of the structure constants without a corresponding redefinition of the unit as done in the isotopic theories. Therefore, *q–deformations are theories without a left and right unit which remains invariant under the time evolution.*

This occurrence is transparent in lifting (7.A.6) which deforms the product AB $\Rightarrow A*B = qAB = ATB$ without jointly deforming the unit as done in the foundations of hadronic mechanics

$$I \;\Rightarrow\; \hat{1} = T^{-1} = q^{-1}. \tag{7.A.10}$$

The lack of basic unit can also be established for deformations of Types II and III, e.g., under time evolution with ensuing nonunitary structure, and unification of all envelopes into isotopic form (7.A.4). The loss of the unit then implies the evident loss of the measurement theory, owing to the necessary condition of the existence of a well defined, left and right unit for the very concept of measurement.[70]

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[69] It should be indicated for clarity that, when nonunitary time evolutions are admitted also for the Hilbert space, Hermiticity can be preserved. In fact, in this case the conventional inner product is lifted into the form

$$<\phi\,|\,\psi> = \int d^2r\, \phi^\dagger(r)\, \psi(r) = \int d^3r\, \phi'^{\,\dagger}\, T\, \psi', \;\; \phi' = U\phi, \psi' = U\psi, \;\; T = (U\,U^\dagger)^{-1},$$

which is precisely of the isotopic type. However, the correct preservation of Hermiticity requires the joint lifting of the base field into the isofield with isounit $\hat{1} = T^{-1}$, in which case the correct form of the isoinner product is given by

$$<\phi' \hat{|}\, \psi'> = \hat{1} \int d^3r\, \phi'^\dagger\, T\, \psi',$$

(and coincides with the original product for T independent of the integration variables), thus implying the entire structure of hadronic mechanics.

*3) General lack of uniqueness of Gaussian distributions and related physical laws.* One of the strengths of quantum mechanics is the *uniqueness* of its various formulations (such as the Gaussian) which evidently implies the known uniqueness of its physical predictions (such as the uniqueness of Heisenberg's uncertainties, see Sect. I.6.1). This uniqueness can be mathematically traced to the uniqueness of the basic unit of the theory, Planck's constant, as well as to the existence of a right and left unit of the universal enveloping operator algebra $\xi(L)$. The general lack of the basic unit then implies that *q-deformations do not possess a consistent formulation of the Poincaré-Birkhoff-Witt theorem which is applicable at all times.* In fact, a necessary condition for the very formulation of the theorem is the existence and uniqueness of a left and right unit.

This means *the lack of existence of a unique, infinite-dimensional basis for the envelopes of q-deformations* and, therefore, *the lack of existence of a unique form of exponentiation.* In fact, q-deformations are known for their variety of "exponentiations" .

The above occurrences add to the mathematical beauty of the theory, but have rather serious physical consequences, such as *the lack of uniqueness of a Gaussian distribution with consequential lack of uniqueness of the generalized uncertainties.* A similar situation occurs for other physical laws.

It should be stressed that the above occurrences *are not* referred to different physical laws for different q-deformations, which would be physically acceptable, but to different physical laws which can be introduced in each q-deformation.

*4) General loss of special functions under time evolution.* As recalled earlier, q-deformations are formulated at a fixed value of time, and so are their special functions (Ch. I.6). But under time evolution the q-*number* is replaced by the isotopic *operator* T. The inapplicability of the q-special functions under time evolution is then consequential.

From a *mathematical* viewpoint, this occurrence may be irrelevant. The physical implications are however rather serious, such as the impossibility of performing a partial wave-analysis and the like.

*5) General loss of Einstein's axioms.* As well known (but not fully identified in the literature), all q-deformations imply a structural departure from *all* basic axioms of the special (and general) relativity, as established by the noncanonicity of the commutation rules, or the nonunitary character of the time evolution, or the deformation of the structure constants of the Poincaré symmetry, etc.

Again, this occurrence can be mathematically intriguing, but it carrier rather

[70] We are here intentionally silent, as a test of technical knowledge of isotopic techniques studied earlier, on the need for an axiomatic, form-invariant theory to unify the unit of the base field with that of the envelope.

serious physical problems in the compliance with physical reality which must be addressed prior to any physical application.

*Hadronic mechanics offers realistic possibilities of resolving all the above problematic aspects while leaving the results of q-deformations fundamentally unaffected,* and this illustrates the relationship between hadronic mechanics and generalized theories.

In fact, hadronic mechanics does not require any change of the assumed structural lines of q-deformations (such as the explicit form of q, f(q,...) or $F_{ij}{}^k(q, ...)$), but only their reformulation in the axiomatically correct form which is invariant under the time evolution of the theory.

The hadronic reformulation of q-deformations is so simple as to appear trivial. For Type I it merely requires the joint lifting of the associative product *and* of the basic unit

$$AB \Rightarrow A{*}B = A\,q\,B, \qquad I \Rightarrow \hat{1} = q^{-1}, \tag{7.A.11}$$

with consequential reformulation of the theory with respect to isofield, isospaces, isotransformations, etc.

The reformulation for Type II was first studied by Jannussis and his collaborators [28] on conventional fields. That on genofields requires the selection of one "time arrow" and then the interpretation of the function f(q,...) in rules (7.A.6) as the genounits for that direction. Jointly, the q-deformation of the second term in the l.h.s. is not axiomatic and must be lifted into the inverse of the selected genounit, resulting in the reformulation

$$r\,p - q\;p\,r = i\,f(q,...) \;\Rightarrow\; \begin{cases} r < p - p > = r\,R\,p - p\,S\,r = \hat{1}^{>}, \\ \hat{1}^{>} = f(q, ...)\,/\,q, \quad S = q\,/\,f(q, ...), \quad R = f(q,...) \\ \text{or} \\ r < p - p > = r\,R\,p - p\,S\,r = {}^{<}\hat{1}\,, \\ {}^{<}\hat{1} = f(q, ...)\,/\,q, \quad S = f(q, ...), \quad R = q\,/\,f(q,...) \end{cases} \tag{7.A.12}$$

The entire theory must then be reformulated on genofields, genospaces, genotransformations, etc. of the selected direction of time.

The hadronic reformulation of q-deformations of Type III is more complex owing to their general character. The procedures has however been studied in detail in this volume and it is applicable to each case considered. The end-result is that, to achieve an axiomatic formulation for given deformed structure constants $F_{ij}{}^k(q, ...)$, one must identify an isotopic element T(q, ...) such that the original Lie-deformation is turned into a Lie-isotopic algebra with $F_{ij}{}^k$ as structure functions

$$X_i X_j - X_j X_i = F_{ij}^{k}(t, q, ...) X_k \Rightarrow X_i T X_j - X_j T X_i = F_{ij}^{k}(t, q, ...) X_k. \quad (7.A.13)$$

The axiomatic reformulation of other q–deformations can be done with one or the other methods studied in this volume.

The researcher in q–deformations is urged to prove the form–invariance of the above isotopic reformulations under the time evolution of the theory. Equivalently, to understand the relationship between q–deformations and hadronic mechanics, one should study the image of all commutators under nonunitary time evolutions, e.g.,

$$r\,p - q\,p\,r = i\,f(q,...) \Rightarrow r' R\,p' - p'(q\,R)\,r = i\,\hat{1}, \quad (7.A.14a)$$

$$\hat{1} = f(q, ...)\,U\,U^{\dagger}, \qquad R = \hat{1}^{-1}. \quad (7.A.14b)$$

As a result, starting from the (0,q)–number deformation (7.A.6) at a fixed value of timer, one reaches at arbitrary times the general (P,Q)–deformations, that is, the Lie–admissible equations at the foundations of hadronic mechanics. This shows the inevitability of the hadronic reformulation even when not desired.

It is equally instructive for the researcher in q–deformation to see that the above isotopic reformulations resolve *all* the problematic aspects indicated earlier. To begin, hadronic mechanics has been built from the beginning (Sect. 1.1.1) under the condition of possessing a generalized, but well defined left and right unit $\hat{1}$. As now familiar, this implies a corresponding compatible isotopy of the base fields and Hilbert space, thus ensuring the Hermiticity/observability of the Hamiltonian and other operators at all times (Sect. I.6.3).

The basic assumptions of hadronic mechanics are centered in fundamental condition I.4.4.1 that the enveloping algebra (of both the Lie–isotopic and Lie–admissible branches) must have a well defined left and right unit. This implies the existence of a generalized Poincaré–Birkhoff–Witt theorem (Sect. I.4.3). The applicability of the measurement theory is proved in Vol. II by showing that the correct isoexpectation values of the isounit $\hat{1}$ turns out to be the conventional Planck value,

$$\langle \hat{1} \rangle = \hbar = 1. \quad (7.A.15)$$

As a result, the measurement theory of hadronic mechanics is the *conventional* one, as necessary for physical consistency and applicability to actual experiments, evidently because measures are conducted in our classical frame and, as such, cannot be modified by theoretical deformations introduced in the microworld.

The Lie–isotopic theory is also based on the existence of a unique infinite-dimensional basis which implies the uniqueness of the exponentiation in hadronic mechanics with consequential uniqueness of the physical laws defined on them.

The applicability of the isospecial functions at all times is evident from the studies of Ch. I.6, because they are constructed for an arbitrary integrodifferential operator T admitting of $T^{-1}$ as the correct unit, rather than with respect to a q-number without a unit.

Finally, the most important objective of all the isotopic techniques is the preservation of Einstein's axioms under T-integral-operator-deformations and only their *realizations* in a nonlinear-nonlocal-noncanonical form as needed for interior problems. The important point stressed throughout our analysis is that both the exterior and interior problems are characterized by a unique set of algebraic-geometric-dynamical axioms.

This is stressed by the local isomorphism between Minkowski and isominkowski spaces, or the Poincaré and isopoincaré symmetries

$$M(x,\eta,R) \approx \hat{M}_Q(x,\hat{\eta},\hat{R}), \qquad P(3.1) \approx \hat{P}_Q(3.1), \tag{7.A.16}$$

which should be compared the corresponding lack of isomorphisms for conventional q-deformations

$$M(x,\eta,R() \not\approx M_q(x,\eta,R(, \qquad P(3.1) \not\approx P_q(3.1). \tag{7.A.17}$$

As a final note we should indicate that, even after reaching a fully axiomatic formulation of the Lie-admissible type (7.A.12), there is one additional problematic aspect requiring consideration. It deals with the relationship between the R- and S-operators which should be be restricted to verify the conjugation

$$R = S^{\dagger}, \tag{5.A.18}$$

in which case one has a direct applicability to all possible nonconservative systems. Physical applications for $R \neq R^{\dagger}$, even though evidently possible, are unknown at this writing, to our best knowledge.

Perhaps the best way to see the relationship between q-deformations and hadronic mechanics is to inspect Vol. II on the applications to specific physical problems and Vol. III on the experimental verification. It is at that stage where the researchers in q-deformation can see the inevitability of an axiomatic reformulation in order to reach a form acceptable for experimental verifications.

**7.A.2: Hadronic mechanics and nonlinear theories.** As well known, *nonlinear generalizations of Schrödinger's equations*, here referred to those nonlinear in the wavefunctions (only), have been proposed since the early stages of quantum mechanics, such as the nonlinear equation proposed by E. Fermi [29] back in 1927

$$i \frac{\partial}{\partial t} \psi_k = [ - \frac{1}{2m} \Delta + V(r) + \hat{V}(\psi\bar{\psi}) ] \psi_k , \tag{7.A.19}$$

Since that time, the generalizations have been studied by a considerable number of authors and constitute today a new segment of theoretical physics. These studies are evidently valuable because they focus the attention on one of the expected limitations of quantum mechanics for interior dynamical problems (Sect. I.1.2), which is precisely the linearity in the wavefunctions.

The issue addressed by hadronic mechanics in Vol. II is the identification of *methods* appropriate for the elaboration of nonlinear *equations*, that is, methods verifying all the necessary principles, including the superposition principle and the conventional measurement theory.

More recently, a method for the study of the above type of nonlinear equations was proposed by S. Weinberg [30] in 1989 which is essentially characterized by an *enveloping algebra* U with product

$$\mathrm{U}: \qquad A \,\hat{\times}\, B = \frac{\partial A}{\partial \psi_k} \, \frac{\partial B}{\partial \overline{\psi}_k} \,, \tag{7.A.20}$$

"Heisenberg-type" equation for a physical quantity Q

$$i\, dQ / dt = Q \,\hat{\times}\, H - H \,\hat{\times}\, A \,, \tag{7.A.21}$$

and "schrödinger's type" equation

$$i \frac{\partial}{\partial t} \psi_k = \frac{1}{2m} \Delta \psi_k + \frac{\partial H}{\partial \overline{\psi}_k} \,, \tag{7.A.22}$$

where H is certain functional of $\psi_k$ and $\overline{\psi}_k$, all equations being defined over a conventional Hilbert space $\mathcal{H}$ on a conventional field $F(a,+,\times)$.

Weinberg's nonlinear theory provides another illustration of the relationship between hadronic mechanics and generalized theories, this time, from a viewpoint different than that of the q-deformations.

In fact, the elegance of the theory and its independence from other methods are evident. Yet the theory is afflicted by a number of problematic aspects which are, again, of *physical* nature, as studied in detail by Jannussis, Mignani and Santilli [31].

The most dominant characteristic of Weinberg's nonlinear theory is that its envelope U is *a general, nonassociative, Lie-admissible algebra* (App. I.4.A and Sect. I.7.3). In fact, product (7.A.20) is *nonassociative* because

$$\mathrm{U}: \quad A \,\hat{\times}\, (B \,\hat{\times}\,) C) \neq (A \,\hat{\times}\, B) \,\hat{\times}\, C \,; \tag{7.A.23}$$

it is *Lie-admissible* because the attached antisymmetric product is Lie

$$Q \hat{\times} H - H \hat{\times} Q = \text{Lie}, \tag{7.A.24}$$

and it is a *general* Lie-admissible algebra in the sense that it is characterized by the general law (7.3.1) without verifying simpler versions of the same, such as that of flexibility.

The immediate consequence is that *Weinberg's nonlinear theory does not admit a unit* (unless reduced to the trivial case of only one dimension). As a result, the theory suffers of a number of problematic aspects somewhat similar to those of q-deformations, such as [31]:

1) lack of existence of the measurement theory evidently because of the lack of existence of the unit

2) lack of well defined Casimir invariants, evidently because of the lack of the center of the envelope;

3) lack o the Poincaré-Birkhoff-Witt theorem for the basis of U[71];

4) lack of a consistent exponentiation, because of the lack of the needed infinite-dimensional basis;[72]

5) lack of a consistent formulation of space-time symmetries in their finite (exponentiated) form uniquely derivable fro their Lie algebra;

6) lack of the general equivalence between the "Heisenberg-type" and the "Schrödinger-type" equations;[73]

[71] As recalled in Sect. I.4.3, the largest nonassociative envelope admitting ordered monomials and a formulation of the Poincaré-Birkhoff-Witt theorem is given by the *flexible Lie-admissible algebras* while extreme technical problems exist in the formulation of the theorem for general Lie-admissible algebras.

[72] Note that, by comparison, exponentiations do exist for q-deformations, although they are not unique.

[73] This is a typical area of study of Vol. II. We here mention the origin of the problematic aspect which is due, on one side, to the *nonassociative* character of the envelope of the "Heisenberg-type" equations (i.e., the nonassociativity of the product $A \hat{\times} B$,), and the *associative* character of the modular structure of of the "Schrödinger-type" equations (i.e., the associativity of the action $\Delta\psi_k$, under which no equivalence is evidently possible. At the same time, a nonassociative reformulation of the modular action of the "Schrödinger's type" equation such as $H \hat{\times} \psi_k$ to achieve structural equivalence with the envelope of the the "Heisenberg-type" equation is confronted with large technical problems, because it would require a nonassociative generalization of Schrödinger's theory, i.e., one for which

$$A \hat{\times} (B \hat{\times} \psi_k) \neq (A \hat{\times} B) \hat{\times} \psi_k .$$

In summary, the mathematical structures of the Heisenberg-type and Schrödinger-type equations are inequivalent in Weinberg's nonlinear theory, and the attempts at rendering them structurally equivalent are confronted with considerable technical problems which, at any rate, would leave the other problematic aspects completely unaffected. The above occurrence is rather synthetically expressed by the so-called *Okubo's No-Quantization Theorem* [32]

7) lack of a well defined notion of particles because of the lack of well defined physical characteristics, such as spin, which evidently require a well defined Lie algebra, with an envelope possessing a well defined center, with a unique exponentiation to a well defined group, etc.

Again, the above occurrences do not prevent the theory from being *mathematically definable*. In fact, the occurrences have been called "intrinsic features" of the theory. The point is that they simply cannot be ignored for *physical applications.*

As it was the case for the q–deformations, *hadronic mechanics permit an axiomatic reformulation of Weinberg's nonlinear theory which, while leaving the physical content completely unchanged, avoids problematic aspects 1)–7) above.*

To understand the occurrence one must distinguish between the nonlinear *"equations"* represented by the theory, and Weinberg's nonlinear *"theory"* per se. Then, all possible Weinberg's nonlinear "equations" are an evident particular case of the isoschrödinger's equation of hadronic mechanics owing to its direct universality (Theorem 7.9.1)

$$i \frac{\partial}{\partial t} \psi_k = H(t, r, p)\, T(t, r, p, \dot{p}, \psi, \overline{\psi}, \partial\psi, \partial\overline{\psi}, \ldots\ldots)\, \psi_k \,. \qquad (7.A.25)$$

As a matter of fact, while Weinberg's '"theory" admits only one particular class of "equations" nonlinear in the wavefunctions, isoschrödinger's equations are much broader because they admit: 1) all possible nonlinear equations in the wavefunctions; 2) all possible equations nonlinear in the derivative of the wavefunctions; as well as 3) all possible equations which are nonlocal in the wavefunctions and their derivatives of arbitrary order.

The resolution of the problematic aspects in the treatment of the same "equations" then follows from their isotopic representation (7.A.25).

As an incidental note, one should be aware of the differences in the intended physical applicability of Weinberg's nonlinear theory and hadronic mechanics. In fact, the former has been formulated for what we essentially refer to as the *exterior dynamical problem in vacuum;* while the latter has been formulated for the *interior dynamical problem within physical media.*

This point is important to stress that the limitations emerged from experiments on Weinberg's theory [33] (essentially dealing with atomic structures), have no bearing of any nature for hadronic mechanics, evidently because they are not applicable, say, to a proton in the core of a collapsing star. In fact, a primary nonlinearity of interior conditions is expected to be in the derivative of the wavefunctions which is absent in Weinberg's theory.

The different origins of the problematic aspects in q–deformations and in Weinberg's theory should be identified because instructive. All q–deformations possess a fully *associative* algebra, with consequential full capability to identify

studied in Vol. II.

its correct left and right unit. By comparison, Weinberg's nonlinear theory is based on a *nonassociative* envelope with consequential *impossibility* to define the right and left unit.

Additional critical inspection of Weinberg's nonlinear theory can be found in ref.s [34]. An intriguing *reformulation* of Weinberg's theory which avoid some of the problematic aspects of the original formulation has been proposed by Jordan [35]. The identification of the algebraic origin of these resolutions is useful to cast additional light on the issues here considered.

Jordan [loc. cit.] introduces the following *generalization* of envelope (7.A.20)

$$U^*: \qquad A * B = \frac{\partial A}{\partial w_{jk}} w_{lk} \frac{\partial B}{\partial w_{lj}}, \tag{7.A.26}$$

The commutator $[A, B]_{U*} = A*B - B*A$ is Lie and, therefore $U^*$ remains a general nonassociative Lie-admissible algebra as in Weinberg's case.

Jordan's reformulation does however allow the treatment of spin and other conventional quantum mechanical quantities. This is due to the fact that the space of functions A, B,... is restricted to those with the structure

$$A = w_{jk}\, a_{kj} \ , \quad B = w_{jk}\, b_{jk}\, , \tag{7.A.27}$$

where the terms in the r.h.s. are interpreted as matrix elements. The commutator $[A, B]_{U*}$ computed in the *nonassociative* envelope $U^*$ is then turned into an equivalent commutator turned into an *associative* envelope,

$$[A,B]_{U*} = a_{kj}\, w_{lk}\, b_{jl} - b_{kj}\, w_{lk}\, a_{jl} = [A, B]_{A*} \tag{7.A.28}$$

the correct formulation of the Poincaré–Birkhoff–Witt theorem, space-time symmetries, exponentiation, Gaussian distribution, etc. is then consequential.

In fact, structure (7.A.28) is a realization of the Lie-isotopic product with an isoassociative envelope and isotopic element $T = (w_{ij})$ precisely of the type at the foundation of hadronic mechanics. More specifically, Jordan's transformation of Weinberg's nonassociative envelope into an equivalent isoassociative form is precisely a realization of Lemma I.4.A.1.

Jordan's reformulation itself is not immune of problematic aspects which are this time similar to those of the q-deformations (lack of joint isotopy of fields and Hilbert spaces, etc.).

The important information originating from these occurrences is that (Fundamental Condition I.4.4.1), according to current knowledge, physically meaningful theories should be formulated with respect to an *associative* envelope with a well defined left and right *unit*, as it is the case for quantum mechanics and its hadronic covering.

There is little doubt that a next generation of theories will likely be based on *nonassociative* envelopes, precisely along Weinberg's lines [30]. The researchers interested in these latter lines should however be aware of the rather serious technical problems involved, both *mathematical* (e.g., the Poincaré–Birkhoff–Witt theorem) and *physical* (e.g., the equivalence of Heisenberg-type and Schrödinger-type equations for nonassociative moduli).

**7.A.3: Hadronic mechanics and nonlocal theories.** Deformations of quantum mechanics (Sect. 7.A.2) focus the attention on the relevance of *noncanonical* theories, while Weinberg's theory of the preceding section focuses the attention on the relevance of *nonlinear* theories. The next logical step along the lines of these volumes is to focus the attention on *nonlocal* theories.

We assume the reader is familiar with the variety of notions of nonlocality existing in the literature. Those particularly relevant for these volumes are the studies initiated by Russian physicists, such as Blochintsev [36] which have subsequently seen the most comprehensive development by Efimov and his associates (see monographs [37] and quoted literature).

Most significant for these volumes is the original motivation which stimulated the studies of nonlocal theories: *remove the divergencies which are inherent in the local in the local character of quantum field theories.*

Note that studies [36,37] deal with nonlocal formulations of *quantum field theory* while the studies of these volumes deal with nonlocal formulations of *quantum mechanics.* Despite that, the rather intriguing connections and possibilities for further advances are already identifiable.

Hadronic mechanics can be conceived as a generalization of quantum mechanics which can remove the singularity of Dirac's delta function *ab initio* precisely via a nonlocal formulation (Sect. I.6.6.4).

The field theoretical extension of the isodirac delta function has been preliminarily studied by Nishioka [38] and, as we shall see in Vol. II, it does indeed contains the necessary elements for the possible, future construction of a *nonlocal-isotopic field theory* which is also free of singularities *ab initio.*

Again, all results achieved in ref.s [36,.27] remain unchanged in their possible isotopic reformulation, which essentially provides mere alternative methods for their treatment.

One point appears to be certain: the conventional local-differential field theories have reached and surpassed the limits of their applicability. Irrespective of which theory will eventually result to be more viable, the need for nonlocal-integral theories is simply beyond credible doubts. We are not referring to ideal point-like particles moving in vacuum (exterior problem) in which the exact validity of local field theories is incontrovertible, but to extended wavepackets moving within those of other particles (interior problem).

At any rate, there exist physical systems simply beyond the descriptive capacities of local field theories, such as the *attractive* interaction of the *same* electrons of the Cooper pair in superconductivity, which can be quantitatively interpreted via a suitable nonlocal representation of the overlapping of the wavepackets of the electrons (Vol. III). Similar needs for nonlocal theories exist in nuclear, particle and statistical physics, theoretical biophysics and other disciplines.

**7.A.4: Hadronic mechanics and and discrete theories.** Another field of research that is currently gaining momentum is at times known under the name of *discrete theories.* This area too is quite vast, by encompassing the use of discrete groups, lattices, discrete calculus, etc. We here focus the attention on only one aspect, the *discrete–time theories* which is sufficient to illustrate all other discrete theories.

Discrete time theories can be traced back to Caldirola's studies [39] of 1956. More recent studies have been conducted by Wolf [40] and others (see Vol. II).

These studies focus the attention on *the possibility that time has a discrete structure at a sufficiently small scale*, a possibility clearly deserving the proper attention in the mathematical, theoretical and experimental communities.

It was shown by Jannussis and his collaborators [41] that Caldirola's equations do have a structure precisely of the Lie–admissible type

$$i\hbar \frac{\rho(t) - \rho(t-\tau)}{\tau} = H\, R\, \rho(t) - \rho(t)\, S\, H\,, \tag{7.A.29}$$

where $\tau$, called *Caldirola's chronon,* is a measure the duration the interaction among particles. The full applicability of hadronic mechanics along universality Theorem I.7.9.2 is then completed by noting that the difference in the l.h.s. is a realization of the *isoderivative with discrete isounits* (Sect. I.6.7).

Thus, discrete time theories constitute an intriguing particular case of hadronic mechanics of Class V. Note that this interpretation permits an intriguing connection with q–deformations which does not appear to have been sufficiently identified in the literature.

By recalling that the basic axioms of quantum mechanics are preserved under isotopies, and only realizes in a more general way, the above hadronic reformulation is intriguing indeed because it shows that *discrete–time theories are admitted by the abstract axioms of quantum mechanics itself.*

The above unexpected property will be proved in Vol. II via the the isoexpectation value of the isounit $< \hat{1} > = 1$, which applies also for discrete isounits $\hat{1}$. To put it differently, a discrete structure of time emerges as admitted by the quantum mechanical axioms, evidently in a more general realization, when dealing with the microcosm. Nevertheless, when the theory is reduced to numbers suitable for macroscopic experiments via the isoexpectation values, such discreteness disappears. In fact, the future resolution of the possible discrete structure of time requires experiments specifically conceived for that purpose, whose study has been initiated by Wolf [40].

The current formulation of the discrete–time theories is also afflicted by problematic aspects of *physical character* due to the fact that, on one side, they generalize the structure of quantum mechanics while, on the other side, they preserve conventional quantum mechanical formulations (conventional expectation values, conventional physical laws and principles, etc.) in the elaboration of the theories.

In effect, the transition from the continuous time of quantum mechanics to a time with a discrete structure implies a necessary *generalization of the underlying unit of time*, from the trivial unit 1 to a generalized unit of Kadeisvili's discrete Class V. In turn, this demands, for evident need of consistency, a step-by-step generalization of the entire quantum mechanics, including expectation values, physical laws and data elaboration needed for experiments.

**7.A.5: Hadronic mechanics and other approaches.** By no mean the preceding examples exhaust all possible connections between hadronic mechanics and ongoing research.

Among a number of additional aspects we shall study in Vol. II, it may be recommendable to indicate the following ones. Kadyshevsky and his associates [42] have constructed *a generalized quantum field theory with a fundamental length at small distances* which exhibits numerous intriguing connections to q-deformations, nonlocal field theories, etc. The re-inspection of the above theory with isotopic methods is significant because it can indicate that a fundamental length can be reconciled with the very axioms of quantum mechanics, evidently when realized in a sufficiently general way.

Another intriguing topic is the Lie-admissible re-interpretation of *conventional external electromagnetic interactions*, such as the studies by Studenikin, and others [43]. Even though these studies deal with purely quantum mechanical settings, their Lie-admissible reinterpretation may be intriguing and instructive for various reasons. After all, interactions with external fields imply the nonconservation of the energy or of some other physical quantity of the particle considered, thus implying the direct applicability of the Lie-admissible formulations.

Note that the reinterpretation identifies another hitherto unknown application of the q-deformations (the treatment of open systems due to external electromagnetic and other fields), when also treated with Lie-admissible techniques.

The implications of the reinterpretation are nontrivial. Recall that the electromagnetic interactions verify the Poincaré symmetry. Their reinterpretation as open systems and treatment via the Lie-admissible theory then permits the construction of the equivalent *Poincaré-admissible symmetry* (Sect. I.7.6). Once such genosymmetry has been established in the known grounds of electromagnetic interactions, its extension to more complex systems is then expected, such as to the characterization of a neutron in the core of a neutron star.

The *Bogoliubov method of group variables* [44] is yet another field, as studied, e.g., by Khrustalev and his associates [45], which is particularly intriguing for hadronic mechanics. As well known, the method essentially consists of using collective group variables which greatly simplify complicated models in field theory and gravitation. But the method also has a nonlocal structure, and exhibits a clear connection with the Lie–isotopic branch of hadronic mechanics.

While the independent studies of Bogoliubov methods are evidently encouraged, their reinterpretation in terms of hadronic mechanics is also recommendable because of the predictable additional knowledge one can gain in the process for both approaches.

New bound states of hadrons are recently emerging such as the so-called *di–baryons* (see ref.s [46] and quoted literature). Such systems have a particular importance for hadronic mechanics because one of its primary objective is the study of the apparent cold fusion of massive particles into heavier particles (see ref.s [47] and Vol. III).

The entire field of *hidden variables* (see, e.g., ref. [48]) has a direct connection with hadronic mechanics. In fact, the isoeigenvalue equations

$$H * \psi = H T \psi = \hat{E}_T * \psi = E_T \psi \tag{7.A.30}$$

is an explicit and concrete realization of the theory of hidden variables, which are actually turned into "hidden operators". This occurrence has rather deep implications studied in Vol. II, which lead to the reinterpretation of hadronic mechanics as a *completion of quantum mechanics* along the celebrated Einstein–Podolsky–Rosen argument [49].

Additional related studies of particular interest for hadronic mechanics are the novel studies on *hidden symmetries* initiated by Smorodinsky and Winternitz [50] and continued by Sissakian, Pogosyan and their associates [51]. These studies too are particularly significant for hadronic mechanics because they permit the identification of generalized bound states deeply linked to the hadronic bound states. In fact, the *isosymmetries of hadronic mechanics are hidden symmetries.*

Yet another topic of particular relevance is the *variational method to regain convergence in perturbative treatments* by Sissakian and his collaborators [52]. In fact, as indicated in Sect. I.6.2, one of the objectives of the isotopies of Hilbert spaces is precisely that of turning conventionally divergent series into isotopically convergent ones under the mere selection of isotopic elements such that $|T| \ll 1$. The above variational method can therefore be particularly useful for the isotopic achievement of convergent series.

The interested reader can find along similar lines the connection between hadronic mechanics and other topics, such as *Berry's phase, squeezed states,* and others.

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**Ruggero Maria Santilli** was born and educated in Italy where he received his Ph. D. in theoretical physics in 1967 from the University of Torino. In 1967 he moved with his family to the USA where he held academic positions in various institutions including the Center for Theoretical Physics of the University of Miami in Florida, the Department of Physics of Boston University, the Center for Theoretical Physics of the Massachusetts Institute of Technology, the Lyman Laboratory of Physics and the Department of Mathematics of Harvard University. He is currently President and Professor of Theoretical Physics at The Institute for Basic Research, which operated in Cambridge from 1983 to 1991 and then moved to Florida. Santilli has visited numerous academic institutions in various Countries. He is currently a Honorary Professor of Physics at the Academy of Sciences of the Ukraine, Kiev, and a Visiting Scientist at the Joint Institute for Nuclear Research in Dubna, Russia. Besides being a referee for various journals, Santilli is the founder and editor in chief of the *Hadronic Journal* (sixteen years of regular publication), the *Hadronic Journal Supplement* (nine years of regular publication) and *Algebras, Groups and Geometries* (eleven years of regular publication). Santilli has been the organizer of the five *International Workshops on Lie-admissible Formulations* (held at Harvard), the co-organizer of five *International Workshops on Hadronic Mechanics* (held in the USA, Italy and Greece) and of the *First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment* (held at the Université d'Orléans, France). He is the author of over one hundred and fifty articles published in numerous physics and mathematics journals; he has written nine research monographs published by Springer-Verlag (in the prestigious series of "Texts and Monographs in Physics"), the Academy of Sciences of Ukraine and other publishers; he has been the editor of over twenty conference proceedings; he is the originator of new branches in mathematics and physics, some of which are studied in these books; he has received research support from the U. S. Air Force, NASA and the Department of Energy; and he has been the recipient of various honors, including the *Gold Medals for Scientific Merits* from the Molise Province in Italy and the City of Orléans, France. Santilli has been nominated for the Nobel Prize in Physics by various senior scholars since 1985.

## ABOUT THE BOOKS

These are the first books written on the *Hadronic Mechanics*, which is an axiom-preserving generalization of quantum mechanics for the study of strong interactions with nonlinear, nonlocal, and nonpotential contributions due to the overlapping of wavepackets and charge distributions of hadrons. After being proposed by Santilli at Harvard University in 1978 under D.O.E. support, the new mechanics has been developed by numerous scholars, discussed at eleven international meetings and studied in numerous papers in various journals. The new mechanics is based on a generalization of the mathematical structure of quantum mechanics, called of *isotopic type* (in the sense of being axiom-preserving). All basic quantum mechanical laws are then reformulated in an isotopic form admitting of nonlinear-nonlocal-nonpotential interatcions in an axiomatic way which is invariant under time evolution. The unrestricted functional character of the isotopies renders the new mechanics "directly universal" for all interactions considered.

VOLUME I contains the nonlinear-nonlocal-nonpotential isotopies of numbers, fields, spaces, algebras, groups, symmetries, geometries and functional analysis.

VOLUME II presents a step-by-step isotopic generalization of quantum mechanical physical laws, Heisenberg's and Schrödinger's representations, Galilei's and Einstein's relativities, including the isotopies of perturbation theory with convergent expansions, and of the scattering theory for inelastic collisions with nonlocal-nonhamiltonian internal effects due to mutual penetrations.

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