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SOMMARIO

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NONLOCAL-INTEGRAL ISOTOPIES OF DIFFERENTIAL CALCULUS, MECHANICS AND GEOMETRIES

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We present a simple, axiom-preserving, isotopic generalization of the ordinary differential calculus, here called *isodifferential calculus*, which is based on the generalization of the basic unit with compatible generalizations of fields, vector spaces and manifolds. The new calculus is applied to the isotopic lifting of Newton's equations with a number of novel possibilities, such as: the representation of the actual nonspherical and deformable shape of particles (which is absent in Newtonian mechanics); the admission of nonlocal-integral forces (which is not possible for the topology of Newton's equations); and the capability to turn Newtonian systems which are non-Hamiltonian in the frame of the observer into a form in the same frame which is Hamiltonian in isospaces. We then show that the isodifferential calculus implies corresponding liftings of analytic and quantum mechanics, as well as their interconnecting map, with the same capabilities of the isotopic Newton's equations, e.g., representation of extended, nonspherical and deformable particles at the level of first quantization without any need of form factors. We finally apply the isodifferential calculus to the construction of novel isotopies of the symplectic and Riemannian geometries which result to be nonlinear in (coordinates and) velocities, integro-differential and non-first-order-Lagrangians, thus being significant for interior dynamical problems, such as the geometrization of locally varying speeds of light. We also indicate the existence of further generalizations, called genotopic and hyperstructural methods, and an antiautomorphic map for all of them called isoduality, which is particularly suited to represent antimatter. The paper is written by a physicist to stimulate mathematical studies on nonlinear, nonlocal and noncanonical systems which have recently emerged in particle physics, astrophysics, superconductivity and other disciplines.

1. Background notions on isotopies

1.1. Introduction. The basic notion of this paper, that of *isotopies*, is rather old. As Bruck [7] recalls, the notion can be traced back to the early stages of set theory where two Latin squares were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin square can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups, then to algebras and more recently to most of mathematics. As an illustration, the isotopies of Jordan algebras were studied by McCrimmon [21], those of Lie algebras by Santilli [28], and subsequently extended to fields, vector spaces, manifolds, groups, functional analysis, etc. A comprehensive literature on isotopies up to 1984 can be found in Tomber's bibliography [4] while subsequent references can be found in the recent monograph by Lõhmus, Paal and Sorgsepp [19].

In this paper we present the isotopies of differential calculus, here called *isodifferential calculus*, and identify the consequential isotopies of mechanics and geometries. The isocalculus is presented here for the first time, although it is implicit in other studies by this author [37,38], as we shall indicated later on. In this section we recall only those aspects of the isotopies which are essential for the understanding of this paper. Dynamical applications are indicated in Sect. 2 while geometric applications are presented in Sect. 3. Due to the emphasis on applications, our treatment is local, while abstract, realization-free profiles are left to the interested mathematicians.

It should be indicated that the isotopies studied in this paper are a particular case of the broader *genotopies* first introduced by Santilli in memoir [28]. In turn, the genotopies are a particular case of the still broader *hyperstructures with a multivalued unit* first introduced in the recent memoir [40] (see also [39] for the first study of hyperstructures with single-valued units and [46] for a general presentation on hyperstructures). All results of this paper can be extended to the broader genotopic and hyperstructural forms, although these extensions are merely indicated without treatment.

This paper is specifically devoted to the broadening of mathematical methods via the *generalization of the unit* and we cannot study for brevity numerous other generalizations existing in the literature. The author would be grateful for the indication of contributions directly or indirectly connected with the above problem for proper quotation in future works.

1.2. Isotopies of the unit. The fundamental *isotopies* from which all others can be uniquely derived are those of the *unit* [28], i.e., the liftings of the n-dimensional unit $I = \mathrm{diag.}(1, 1, 1, \dots)$ of conventional vector or metric spaces into real-valued and symmetric n×n matrices $\hat{I} = (\hat{I}^i_j) = \hat{I}^t$ whose elements $\hat{I}^i_j$ have an unrestricted functional dependence in the coordinates x, velocities $v = dx/dt$, accelerations $a = dv/dt$, local density μ, local temperature τ, and any needed other characteristics of the problem considered,

$$I \quad \rightarrow \quad \hat{I} = \hat{I}(x, v, a, \mu, \tau, \dots) = \hat{I}^t. \tag{1.1}$$

The above liftings were classified by Kadeisvili [15] into: **Class I** (generalized units that are nondegenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking); **Class II** (the same as Class I although $\hat{I}$ is negative-definite, characterizing the so-called *isodualities*); **Class III** (the union of Class I and II); **Class IV** (Class III plus the zeros of the generalized unit, $\hat{I} = 0$); and **Class V** (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this paper we shall study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. The isotopies of Classes IV and V are vastly unexplored at this writing.

The *genotopies* [28] occur when the generalized unit is no longer symmetric, $\hat{I} \neq \hat{I}^t$, thus resulting in two different quantities denoted as follows

$$^<\hat{I}(t, x, v, a, \mu, \tau, \dots),\ \hat{I}^>(t, x, v, a, \mu, \tau, \dots),\quad ^<\hat{I} = \hat{I},\ \hat{I}^> = \hat{I}^t,\ ^<\hat{I} = (\hat{I}^>)^t. \tag{1.2}$$

The *hyperstructures* (of the class here considered [40]) occur when each of the preceding structure is given by a finite and ordered set denoted as follows

$$\{^<\hat{I}\} = \{^<\hat{I}_1, {^<\hat{I}_2}, {^<\hat{I}_3}, \dots\},\ \ \{\hat{I}^>\} = \{\hat{I}^>_1, \hat{I}^>_2, \hat{I}^>_3, \dots\},\ \ \ \{^<\hat{I}\} = \{\hat{I}^>\}^t, \tag{1.3}$$

where the last operation is referred to each element of the ordered sets.

1.3. Isotopies of fields. The first significant application of the isotopies of the unit is that for the liftings of conventional numbers and fields, which was presented by Santilli at the meeting on *Differential Geometric Methods in*

Mathematical Physics held at the University of Clausthal, Germany, in 1980 (see the latest mathematical study [34] and the physical presentation in [37]).

Let $F = F(a,+,\times)$ be a field (hereon assumed to have characteristic zero) with elements a, b, ..., sum a + b, multiplication $ab = a\times b$, additive unit 0, multiplicative unit I, and familiar properties $a + 0 = 0 + a = a$, $a\times I = I\times a = a$, $\forall\, a \in F$, etc. We have in particular the field $R(n,+,\times)$ of *real numbers* n, the field $C(c,+,\times)$ of *complex numbers* c, and the field $Q(q,+,\times)$ of *quaternions* q.

Definition 1.1 [34]: *An "isofield" $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ is a ring with elements $\hat{a} = a\times\hat{1}$, called "isonumbers", where $a \in F$, and $\hat{1}$ is a Class III quantity generally outside F, equipped with two operations $(+, \hat{\times})$, where $\hat{+} \equiv +$ is the conventional sum of F with conventional additive unit $\hat{0} \equiv 0$, and $\hat{\times}$ is a new multiplication* [28]

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{T}\times\hat{b}, \tag{1.4}$$

called "isomultiplication", where $\hat{T}$ is nowhere singular such that $\hat{1} = \hat{T}^{-1}$ is the left and right unit of $\hat{F}$,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\,\hat{\times}\,\hat{1} \equiv \hat{a}, \ \forall\ \hat{a} \in \hat{F}, \tag{1.5}$$

in which case (only), $\hat{1}$ is called the "isounit" and $\hat{T}$ is called the "isotopic element". Under these assumptions $\hat{F}$ is a field, i.e., it satisfies all properties of F in their isotopic form:

1. *The set $\hat{F}$ is closed under addition, $\hat{a} + \hat{b} \in \hat{F}$, $\forall\ \hat{a}, \hat{b} \in \hat{F}$.*
2. *The addition is commutative, $\hat{a} + \hat{b} = \hat{b} + \hat{a}$, $\forall\, \hat{a}, \hat{b} \in \hat{F}$,*
3. *The addition is associative, $\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}$, $\forall\, \hat{a}, \hat{b}, \hat{c}, \in F$,*
4. *There is an element $\hat{0} \equiv 0$, the "additive unit", such that $\hat{a} + \hat{0} = \hat{0} + \hat{a} = \hat{a}$, $\forall\, \hat{a} \in \hat{F}$*
5. *For each element $\hat{a} \in \hat{F}$, there is an element $-\hat{a} \in \hat{F}$, called the "opposite of $\hat{a}$", which is such that $\hat{a} + (-\hat{a}) = \hat{0}$;*
6. *The set $\hat{F}$ is closed under isomultiplication, $\hat{a}\hat{\times}\hat{b} \in \hat{F}$, $\forall\, \hat{a}, \hat{b} \in \hat{F}$,*
7. *The multiplication is generally non-isocommutive, $\hat{a}\hat{\times}\hat{b} \neq \hat{b}\hat{\times}\hat{a}$, but isoassociative", $\hat{a}\hat{\times}(\hat{b}\hat{\times}\hat{c}) = (\hat{a}\hat{\times}\hat{b})\hat{\times}\hat{c}$, $\forall\, \hat{a}, \hat{b}, \hat{c} \in \hat{F}$;*
8. *The quantity $\hat{1}$ in the factorization $\hat{a} = a\times\hat{1}$ is the "multiplicative isounit" of $\hat{F}$ as per Eq.s (2.3)*
9. *For each element $\hat{a} \in \hat{F}$, there is an element $\hat{a}^{-\hat{1}} \in \hat{F}$, called the "isoinverse", which is such that $\hat{a}\hat{\times}(\hat{a}^{-\hat{1}}) = (\hat{a}^{-\hat{1}})\hat{\times}\hat{a} = \hat{1}$.*
10. *The set $\hat{F}$ is closed under joint isomultiplication and addition,*

$$\hat{a}\,\hat{\times}(\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b})\hat{\times}\,\hat{c} \in \hat{F}, \quad \forall\ \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \tag{1.6}$$

11. All elements $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ verify the right and left "isodistributive laws"

$$\hat{a}\,\hat{\times}(\hat{b}+\hat{c}) = \hat{a}\,\hat{\times}\,\hat{b} + \hat{a}\,\hat{\times}\,\hat{c}, \qquad (\hat{a}+\hat{b})\hat{\times}\,\hat{c} = \hat{a}\,\hat{\times}\,\hat{c} + \hat{b}\,\hat{\times}\,\hat{c}, \tag{1.7}$$

When there exists a least positive isointeger $\hat{p}$ such that the equation $\hat{p}\hat{\times}\hat{a} = \hat{0}$ admits solution for all elements $\hat{a} \in \hat{F}$, then $\hat{F}$ is said to have "isocharacteristic p". Otherwise, $\hat{F}$ is said to have "isocharacteristic zero". Unless otherwise stated, all isofields considered hereon shall be Class III isofields of isocharacteristic zero.

We then have the isofield $\hat{R}(\hat{n},+,\hat{\times})$ of *isoreal numbers* $\hat{n}$; the isofield $\hat{C}(\hat{c},+,\hat{\times})$ of *isocomplex isonumbers* $\hat{c}$; and the isofield $\hat{Q}(\hat{q},+,\hat{\times})$ of *isoquaternions* $\hat{q}$ (see [34] for the *isooctonions*). Since all infinitely possible $\hat{F}$ preserves by construction all axioms of F, they are called *isotopes* of F and the liftings $F \to \hat{F}$ are called *isotopies*.

All conventional operations dependent on the multiplication on F are generalized on $\hat{F}$ in a simple yet unique way, yielding *isopowers, isosquare roots, isoquotients*, etc.

$$\hat{a}^{\hat{2}} = \hat{a}\,\hat{\times}\,\hat{a} = (a\times a)\times\hat{1}, \ \hat{a}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}}\times\hat{1}^{\frac{1}{2}}, \quad \hat{n}\,/\,\hat{m} = (n\,/\,m)\times\hat{1}, \text{ etc.} \tag{1.8}$$

It is then easy to see that *the isounit verifies all axiomatic properties of the conventional unit*, e.g.,

$$\hat{1}^{\hat{n}} = \hat{1}\hat{\times}\hat{1}\,\hat{\times}...\hat{\times}\,\hat{1} \equiv \hat{1}, \quad \hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \quad \hat{1}\hat{/}\hat{1} \equiv \hat{1}, \quad \text{etc.} \tag{1.9}$$

Despite its simplicity, the liftings $F \to \hat{F}$ have significant implications in number theory itself. For instance, real numbers which are conventionally prime (under the *tacit* assumption of the unit I) are not necessarily prime with respect to a different unit [34]. This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit simple, yet intriguing and significant isotopies (for more details, see [37], App. 2.B).

It is important to understand that an isofield of Class III, $\hat{F}_{III}(\hat{a},+,\hat{\times})$, is the union of two disjoint isofields, one of Class I, $\hat{F}_I(\hat{a},+,\hat{\times})$, in which the isounit is positive-definite, and one of class II, $\hat{F}_{II}(\hat{a},+,\hat{\times})$, in which the isounit is negative-definite,

$$\hat{F}_{III}(\hat{a},+,\hat{\times}) = \{ \hat{F}_I(\hat{a},+,\hat{\times}), \hat{1} > 0 \} \cup \{ \hat{F}_{II}(\hat{a},+,\hat{\times}), \hat{1} < 0 \}, \quad (1.10)$$

with interconnecting map called *isoduality* [35]

$$\hat{1} > 0 \ \rightarrow \ \hat{1}^d = -\hat{1} < 0, \quad (1.11)$$

The Class II isofields are also written in the literature $\hat{F}^d(\hat{a}^d,+,\hat{\times}^d)$ and called *isodual isofields* with *isodual isonumbers* $\hat{a}^d = a \times \hat{1}^d = -\hat{a}$, isodual isoproduct $\hat{\times}^d = \times \hat{T}^d \times = -\hat{\times}$., etc. The reader should keep in mind that decompositions of the type (2.10) also apply to all subsequent isotopies of Class III, such as isospaces, isotopologies, isogeometries, etc. Whenever no ambiguity arises, we shall omit the subscript I, II or III.

The isoduality also applies to *ordinary* numbers, yielding the *isodual fields* $F^d(n^d,+,\times^d)$ with *isodual unit* $1^d = -1$, *isodual numbers* $n^d = -n$, $c^d = -\bar{c}$, and $q^d = -q^\dagger$ (where the upper-bar $^-$ and upper-symbol † represents complex and Hermitean conjugation, respectively), and isodual multiplication $n^d \times^d m^d = n^d \times (-1) \times m^d$, etc. One should keep in mind the difference between the ordinary *negative numbers*, which are referred to a *positive* unit +1, and the *isodual negative numbers*, which are referred to a *negative* unit −1. As such, the latter are novel numbers.

Note that the imaginary quantity i is *isoselfdual*, i.e., invariant under isoduality (because $i^d = -\bar{i} \equiv i$). Note that the set of imaginary numbers *is not* a field, but the isoreal isofield with $\hat{1} = i$ is indeed a field. Note finally that the sum is not lifted under isotopies otherwise there is the loss of the axioms of a field. For these and other aspects, one can inspect ref.s [34,37].

The *isonorm* of an isofield of Class III is defined by [34]

$$\hat{\uparrow} \hat{a} \hat{\uparrow} = |a| \times \hat{1}, \quad (1.12)$$

where $|a|$ is the conventional norm. It is therefore easy to see that *the isonorm of isofields of Class I is positive-definite, while that of Class II is negative definite.*

A most significant aspect of isoduality is that it is an *antiautomorphic map.* As such, it has permitted a novel representation of *antimatter* which begins at the *classical* level and then persists at the operator level where it results to be equivalent to charge conjugation [38]. In particular, all characteristics which are conventionally positive for matter, becomes negative-definite for antimatter, thus including energy, time, curvature, etc.

To understand the latter occurrence, one should keep in mind the

equivalence of *positive characteristics referred a positive unit* and *negative characteristics referred to a negative unit.* This mathematically elementary property has rather intriguing applications, such as the prediction of *antigravity for elementary antiparticles in the field of matter,* and others [38].

All isotopies of class I are hereon restricted to represent *matter,* and those of Class II to represent *antimatter.* For certain reasons related to bifurcations, the isotopies of Class III appear to be significant in theoretical biology [40].

Examples of isounits are presented in Sect.s 2 and 3. Note that all isounits considered in this paper are outside the original set F (for the simpler case when $\hat{1} \in F$ see [34]). Note also that all isounits (isodual isounit) used in this paper are restricted by the condition of admitting the conventional unit +1 (isodual unit −1) as a particular case.

The reader should be aware that the isonumbers are a particular case of the broader *genonumbers* [34] which occur when the isounit remains nowhere singular, but it is no longer symmetric (or, more generally, Hermitean). This requires *two* different generalized units $^<\hat{1}$ and $\hat{1}^>$, $^<\hat{1} = (\hat{1}^>)^\dagger$, as in Eq. (1.2). By introducing the *genonumbers* $^<\hat{a} = {}^<\hat{1} \times a$ and $\hat{a}^> = a \times \hat{1}^>$ with corresponding *genoproducts ordered to the left and to the right,*

$$^<\hat{a} < {}^<\hat{b} = {}^<\hat{a} \times {}^<\hat{T} \times {}^<\hat{b}, \qquad \hat{a}^> > \hat{b}^> = \hat{a}^> \times \hat{T}^> \times \hat{b}^>, \quad (1.13)$$

where $^<\hat{T}$ and $\hat{T}^>$ are nonsingular, it is easy to see that the quantities $^<\hat{1} = ({}^<\hat{T})^{-1}$ and $\hat{1}^> = (\hat{T}^>)^{-1}$ are the correct left and right units of the corresponding ordered products

$$^<\hat{1} < {}^<\hat{a} = {}^<\hat{a} < {}^<\hat{1} \equiv {}^<\hat{a}, \quad \hat{1}^> > \hat{a}^> = \hat{a}^> > \hat{1}^> \equiv \hat{a}^>, \quad (1.14)$$

in which case (only) $^<\hat{1}$ and $\hat{1}^>$ are called *left and right genounits* and $^<\hat{T}$, $\hat{T}^>$ are called *left and right genotopic elements.*

The important property is that all axioms of a field are satisfied under each ordered product, resulting in two new *genofields,* one *to the left* $^<\hat{F}(^<\hat{a},+,<)$ and the other *to the right* $\hat{F}^>(\hat{a}^>,+,>)$ [34]. Note that, when the original field is commutative, $a \times b = b \times a$, each of the two genofields is also commutative, $^<\hat{a} < {}^<\hat{b} = {}^<\hat{b} < {}^<\hat{a}$ and $\hat{a}^> > \hat{b}^> = \hat{b}^> > \hat{a}^>$. However, $\hat{a}^> > \hat{b}^> \neq {}^<\hat{a} < {}^<\hat{b}$. Note the *necessity* of the prior isotopy $a \hat{\times} b = \hat{a} \times \hat{T} \times \hat{b}$ for a meaningful genotopy. In fact, the ordering of the product can also be introduced in ordinary fields resulting in two separate fields, $F^>(a^>,+,>)$ and $^<F(^<a,+,<)$, one per each ordering, each of which satisfies all axioms of a field. However, in the latter case $a^> = a \times 1 \equiv {}^<a = a \times 1$. Then, $a < b \equiv a > b$ and the ordering is inessential, as tacitly assumed in the conventional number theory. Note that, unless the genounit is complex, genotopies and isotopies with one-dimensional (scalar) unit coincide.

The genonumbers are, in turn, particular cases of the *hypernumbers* as defined in [40]. In this case, each genounit is a finite and ordered set as in Eq.s (1.3), the genotopic elements become the *hyperelements to the left and to the right*

$$\{^<\hat{T}\} = \{^<\hat{T}_1, ^<\hat{T}_2, ^<\hat{T}_3, ...\}, \qquad \{\hat{T}^>\} = \{\hat{T}^>_1, \hat{T}^>_2, \hat{T}^>_3, ...\}, \tag{1.15}$$

with interconnecting relations $\{\hat{I}^>\} = \{\hat{T}^>\}^{-1}$ and $\{^<\hat{I}\} = \{^<\hat{T}\}^{-1}$ interpreted term by term. By defining the *hypernumbers to the left and to the right* the quantities $\{^<\hat{a}\} = \{^<\hat{I}\}\times a$ and $\{\hat{a}^>\} = a\times\{\hat{I}^>\}$, with corresponding *hypermultiplications ordered to the left and to the right*

$$\{^<\hat{a}\}\{<\}\{^<\hat{b}\} = \{^<\hat{a}\}\times\{^<\hat{T}\}\times\{^<\hat{b}\},$$

$$\{\hat{a}^>\}\{>\}\{\hat{b}^>\} = \{\hat{a}^>\}\times\{\hat{T}^>\}\times\{\hat{b}^>\}, \tag{1.16}$$

it is easy to see that the quantities $\{\hat{I}^>\}$ and $\{^<\hat{I}\}$ are correct left and right units for each ordered product

$$\{^<\hat{I}\}\{<\}\{^<\hat{a}\} = \{^<\hat{a}\}\{<\}\{^<\hat{I}\} \equiv \{^<\hat{a}\},$$

$$\{\hat{I}^>\}\{>\}\{\hat{a}^>\} = \{\hat{a}^>\}\{>\}\{\hat{I}^>\} \equiv \{\hat{a}^>\}. \tag{1.17}$$

in which case (only) they are called *hyperunits to the left and to the right*. It is then possible to prove that the sets $\{^<\hat{F}\}(\{^<\hat{a}\},+,\{<\})$ and $\{\hat{F}^>\}(\{\hat{a}^>\},+,\{>\})$ individually verify all axioms for a field and are called *hyperfields to the left and to the right*, respectively (again, in the sense of ref. [40]).

The isotopies establish that the abstract axioms of a field do not require that the basic unit must necessarily be the trivial number +1. The genotopies establish that the same abstract axioms do not require that the basic unit is necessarily symmetric. The hyperstructures finally establish that the same axioms admit a multivalued nonsymmetric unit.

In regard to applications, the *isonumbers* are used as in this paper as the basic numbers for the representation of reversible nonhamiltonian vector fields, i.e., vector field which do not admit a Hamiltonian in the coordinates considered and verify the time-reversal invariance, in which case no ordering of the product is

needed. The *genonumbers* are used for the characterization of *irreversible nonhamiltonian vector fields*, i.e., nonhamiltonian vector fields which violate the time-reversal invariance. The *hypernumbers* are used for the characterization of complex biological structures [40].

The reader is suggested to meditate an instant on the fact that the entire contemporary mathematical knowledge is based on the simplest possible unit +1 which has essentially remained unchanged for thousands of years since its inception during biblical times. One can therefore see the horizon of new possibilities permitted by the generalization of such a fundamental notion.

1.4: Isospaces. The second significant application of the isotopies is the lifting of the conventional vector and metric spaces, first presented in paper [31] of 1983 (see monographs [37] for detailed treatment). In this section we shall review the main lines of the isotopies of the Euclidean space.

Let $E(x,g,R)$ be an n-dimensional Euclidean space, with local chart $x = \{x^k\}$, $k = 1, 2, ..., n$, n-dimensional metric $\delta = (\delta_{ij}) = \text{diag.}(1, 1, ..., 1)$, and interval between two points $x, y \in E$,

$$(x-y)^2 = (x^i - y^i)\,\delta_{ij}\,(x^j - y^j) \in R(n,+,\times), \tag{1.18}$$

where the convention on the sum of repeated indices is assumed hereon. It is evident that the isotopic liftings of the field require, for consistency, corresponding liftings of the spaces, and we have the following:

Definition 1.2 [31,37]: *The "isoeuclidean spaces" $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ of Class III are n-dimensional metric spaces defined over an isoreal isofield $\hat{R}(\hat{n},+,\hat{\times})$ with an n×n-dimensional real-valued and symmetric isounit $\hat{1} = \hat{1}^t$ of the same class, equipped with the "isometric"*

$$\hat{\delta}(t, x, v, a, \mu, \tau, ...) = (\hat{\delta}_{ij}) = \hat{T}(t, x, v, a, \mu, \tau, ...)\times\delta = \hat{\delta}^t, \quad \hat{1} = \hat{T}^{-1} = \hat{1}^t, \tag{1.19}$$

local chart in contravariant and covariant forms

$$\hat{x} = \{\hat{x}^k\} = \{x^k\times\hat{1}\}, \quad \hat{x}_k = \hat{\delta}_{ki}\hat{x}^i = \hat{T}_k{}^r\,\delta_{ri}\,x^i\times\hat{1}, \quad x^k, x_k \in E; \tag{1.20}$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{E}$ also called (the square of the) "isoeuclidean distance"

$$(\hat{x}-\hat{y})^{\hat{2}} = [(\hat{x}-\hat{y}^i)\times\hat{\delta}_{ij}\times(\hat{x}^j-\hat{y}^j)]\times\hat{1} \in \hat{R}. \quad (1.21)$$

The "isoeuclidean geometry" is the geometry of the isoeuclidean spaces. The same apply for the definition of isominkowskian, isoriemannian, isofinslerian and other isospaces and of related geometries.

The primary property of the liftings $E(x,\delta,R) \rightarrow \hat{E}(\hat{x},\hat{\delta},\hat{R})$ is the preservation of the original geometric axioms, as necessary for isotopies. In actuality, $E(x,\delta,R)$ and $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ coincide at the abstract level for Class I isotopies by conception [31]. This is due to the construction of the isospaces via the deformation of the metric δ into the isometric $\hat{\delta} = \hat{T}\times\delta$ while jointly the original unit I is deformed in the amount which is the *inverse* of the deformation of δ, $\hat{1} = \hat{T}^{-1}$. This mechanism then ensures the preservation of all original geometric properties, as we shall see in Sect. 3.

Note that the isoeuclidean spaces of Class II are *antiautomorphic* to the original spaces, and this renders them particularly suited to represent antimatter [38].

Note also that in the conventional space $E(x,\delta,R)$ the unit of the field is the *number* +1, while the unit of the space is the *matrix* $I = \text{diag.}(1, 1, 1, ...)$. For $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ the isounits of the isofield and that of the isospaces coincide and are given by $\hat{1}$.

Note also that the isoseparation $\hat{x}^{\hat{2}}$, for consistency, must be an element of the isofield, that is, must have the structure of a number n multiplied by the isounit $\hat{1}$. This isoscalar character is expressed by the isomultiplication

$$\hat{x}^{\hat{2}} = \hat{x}^k \hat{\times} \hat{x}_k = (x^k\times\hat{1})\times\hat{T}\times(x_k\times\hat{1}) = (x^k\times x_k)\times\hat{1} = n\times\hat{1}. \quad (1.22)$$

But the contraction over the repeated index k is in isospace. We recover in this way isoseparation (1.20), as one can see.

Because of the above occurrences, whenever no confusion arises isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\hat{x}^k = x^k\hat{1}$. The symbols x, v, a, ... will be used for conventional spaces, while the symbols $\hat{x}$, $\hat{v}$, $\hat{a}$, ... will be used for isospaces. When writing $\hat{\delta}(x, v, a, ...)$ we refer to the *projection* of the isometric $\hat{\delta}$ in the original space.

Despite their simplicity, the liftings $E(x,\delta,R) \rightarrow \hat{E}(\hat{x},\hat{\delta},\hat{R})$ have significant implications. In fact, the functional dependence of the isounit $\hat{1}$ remains unrestricted under isotopies. Besides being well behaved, real valued and symmetric, the isometrics $\hat{\delta}$ therefore have the unrestricted functional dependence indicated earlier $\hat{\delta} = \hat{\delta}(x, v, a, \mu, \tau, ...)$. As a consequence, *the isoeuclidean metric contains as particular cases all possible metrics of the same dimension and signature.* Therefore, *the isoeuclidean geometry of Class I permits a unified treatment of all conventional geometries of the same dimension and signature, plus all their infinitely possible isotopies.* The isogeometry of Class III permits the unified treatment of all the latter geometries irrespective of their signature. The isogeometries of Classes IV and V are basically novel and unexplored at this writing (see [37] for details).

A most salient characteristics of the isoeuclidean spaces is that of altering the *units* of the conventional space. Recall that the units of the three-dimensional space $E(x,\delta,R)$ are *equal for all axes* and are given by the number +1, i.e., $I_k = +1$, k = 1, 2, 3 (= x, y, z). Consider now the corresponding three-dimensional isotope of Class I $\hat{E}(\hat{x},\hat{\delta},\hat{R})$. Since $\hat{1}$ is positive-definite, it can always be diagonalized into the form

$$\hat{1} = \text{diag.}(n_1^2, n_2^2, n_3^2), \quad \hat{1}_k = n_k^{-2}, \ k = 1, 2, 3, (= x, y, z), \ n_k \neq 0. \quad (1.23)$$

This means that, not only the original units are now lifted into arbitrary (non-null) values, but *the units of different axes generally have different values.* But the component of the metric are however lifted by the *inverse* amounts, $\hat{\delta}_{kk} = n_k^2$. This permits the preservation of the Euclidean character *in isospace over isofields.*

The above features permit a number of novel applications. We here mention the unification of all possible ellipsoids $x^2 = x_1^2/n_1^2 + x_2^2/n_2^2 + x_3^2/n_3^2 = r^2 \in R(n,+,\times)$ in $E(x,\delta,R)$ into the so-called *isosphere*

$$\hat{x}^{\hat{2}} = (x_1^2/n_1^2 + x_2^2/n_2^2 + x_3^2/n_3^2)\times\hat{1} = R^2\times\hat{1} \in \hat{R}(\hat{n},\hat{+},\hat{\times}), \quad (1.24)$$

which is the perfect sphere in isospace over isofields. In fact, the deformation of the semiaxes of the sphere $I_k = +1 \rightarrow n_k^2$ when the corresponding units are deformed of the inverse amounts, $I_k = +1 \rightarrow n_k^2$ preserve the original perfect sphericity. The use of Class III permits the unified treatment of all compact and noncompact conics, while the use of Class IV permits the inclusion of all cones.

Similarly, the main mechanism of the isotopy $E(x,\delta,R) \rightarrow \hat{E}(\hat{x},\hat{\delta},\hat{R})$ leaves invariant the product (length)×(unit). this permits the conception of a new mathematical propulsion, called *geometric propulsion* [37], in which a massive point-particle is moved from one point to another *without any application of a force, but via the alteration of the underlying geometry.* In particular, conventionally very large (very small() distances can be mathematically made as small (large) as desired. In fact, for Aver.(n_k) sufficiently large (small) the distance R in (1.24) can be made as small (large) as desired with respect to the original euclidean distance (see [37] for details and other applications).

We should also mention the existence of the *genoeuclidean spaces to the left and to the right* [37]

$$ {}^<\hat{E}({}^<\hat{x},{}^<\hat{\delta},{}^<\hat{R}), \quad \hat{E}^>(\hat{x}^>,\hat{\delta}^>,\hat{R}^>): \quad {}^<\hat{x} = {}^<\hat{1}\times x, \quad \hat{x}^> = x\times\hat{1}^>, $$

$$ {}^<\hat{\delta} = {}^<\hat{T}\times\delta, \quad \hat{\delta}^> = \delta\times\hat{T}^>, \quad {}^<\hat{1} = ({}^<\hat{T})^{-1}, \quad \hat{1}^> = (\hat{T}^>)^{-1} = ({}^<\hat{1})^t, \qquad (1.25) $$

respectively, which emerge when the isounit is no longer Hermitean, thus requiring the selection of an ordering in the product. Note that *both genospaces coincide with the conventional spaces at the abstract level,* because the deformation of the metric is "compensated" by an inverse deformation of the original units.

Note that, unless the genounit is complex, forward and backward genospaces with one-dimensional, real-valued genounits coincide among themselves and with isospaces. In fact, for real-values scalar units we have $\hat{1}^> = {}^<\hat{1} = \hat{1} = \hat{1}^t$. Genospaces are therefore recommended for use only when there is the need of generalized units of more than one dimension.

In turn, genospaces are expected to be particular cases of the much broader *hyperspaces to the left and to the right* [40]

$$ \{{}^<\hat{E}\}(\{{}^<\hat{x}\},\{{}^<\hat{\delta}\},\{{}^<\hat{R}\}), \quad \{\hat{E}^>\}(\{\hat{x}^>\},\{\hat{\delta}^>\},\{\hat{R}^>\}): \ \{{}^<\hat{x}\} = \{{}^<\hat{1}\}\times x, \ \{\hat{x}^>\} = x\times\{\hat{1}^>\}, $$

$$ \{{}^<\hat{\delta}\} = \{{}^<\hat{T}\}\times\delta, \ \{\hat{\delta}^>\} = \delta\times\{\hat{T}^>\}, \ \{{}^<\hat{1}\} = \{{}^<\hat{T}\}^{-1}, \ \{\hat{1}^>\} = \{\hat{T}^>\}^{1}, \qquad (1.26) $$

respectively, which occur for *multivalued hypercoordinates, hypermetrics and hyperunits* and are defined over hyperfields (Sect. 1.3). Note that, again, *both hyperspaces coincide with the conventional space at the abstract level* for the same reasons as occurring for genospaces and isospaces.

We should finally indicate that all the above isospaces, genospaces and hyperspaces admit an antiautomorphic image under isoduality $\hat{1} \to \hat{1}^d = -\hat{1}$, whose explicit study is left to the interested reader for brevity.

In essence, the isotopies establish that the abstract axioms of the Euclidean geometry do not require that the metric is necessarily restricted to the quantity δ = diag. (1, 1, 1, ...) = δ^t, but can be given by a matrix with the same dimension and topological properties, yet with unrestricted functional dependence, $\hat{\delta} = \hat{T}\times\delta = \hat{\delta}(t, x, v, a, ...) = \hat{\delta}^t$, thus rendering the Riemannian spaces *particular cases* of the isoeuclidean spaces. The genotopies establish that the same abstract axioms of the Euclidean geometry do not require that the metric must necessarily be symmetric, $\hat{\delta} = \hat{T}\delta \neq \hat{\delta}^t$, provided that the nonsymmetric component $\hat{T}$ is entirely embedded in the genounit, $\hat{1} = \hat{T}^{-1} \neq \hat{1}^t$. The hyperstructure establish that, in addition, the metric need not necessarily to be single valued, $\{\hat{\delta}\} = \{\hat{T}\}\times\delta$, provided that the multivalued nonsymmetric component $\{\hat{T}\}$ is entirely embedded in the hyperunit, $\{\hat{1}\} = \{\hat{T}\}^{-1} \neq \{\hat{1}\}^t$.

The above features permit a number of intriguing applications. Isospaces have no ordering in product and are therefore irreversible (for isounits not explicitly dependent on time). As such, they are particularly suited to represent reversible nonhamiltonian systems, as we shall see shortly. Genospaces do instead require an ordering of the product which is associated with motion in a given direction of time. As such, they are *structurally irreversible,* that is, irreversible irrespective of the reversibility of the Lagrangian and isounit. Genospaces (with the genounits of more than one dimension) are therefore particularly suited to permit a novel geometrization of irreversibility which is reduced to primitive geometric axioms. Hyperspaces have the additional multivalued character, resulting to be particular suited for the representation of biological systems which are notoriously irreversible.

In turn, these characteristics have intriguing implications, such as the reduction of macroscopic irreversibility to that at the ultimate level of nature, that of elementary particles in interior conditions, e.g., for a proton in irreversible conditions in the core of a star [37,38]

Also, the abstract identity of generalized and conventional spaces permits the compatibility of the broader geometries expected in biological structures with our empirical Euclidean perception of the same, which compatibility would be otherwise prohibited.

1.5: Isodifferential calculus. We are finally equipped to introduce the *isotopies of the ordinary differential calculus,* or *isodifferential calculus* for short, hereon referred to the image of the conventional calculus under the isotopies of the unit of Class I. The new calculus was presented for the first time at the *International Workshop on Differential Geometry and Lie algebras* held at the Department of Mathematics of Aristotle University, Thessaloniki, Greece, in December 1994, but appears in print in this paper apparently for the first time.

The isodifferential calculus is here introduced on flat isospaces. Topological aspects are considered in Sect.s 1.6 and 1.7. The formulation in curved isospaces in considered in Sect. 3.

Let $E(x,\delta,R)$ be the ordinary n-dimensional Euclidean space with local coordinates $x = \{x^k\}$, k = 1, 2, ..., n, and metric δ = diag. (1, 1, 1, ...) over the reals

$R(n,+,\times)$. Let $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ denote the isotopic images with local coordinates $\hat{x} = \{\hat{x}^k\}$, isometric $\hat{\delta} = \hat{T}\times\delta$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$ with real-valued and symmetric isounit of Class III, $\hat{1} = (\hat{1}_i^j) = \hat{T}^{-1} = (T_i^j)^{-1} = \hat{1}^t$ whose elements have a smooth but otherwise arbitrary dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{1} = \hat{1}(\hat{x}, \hat{v}, \hat{a}, ...)$. For simplicity we shall ignore the isounit in the definition $\hat{x} = x\times\hat{1}$ because it cancels out ij the isomultiplication with any quantity Q, $\hat{x}\hat{\times}Q = x\times\hat{T}^{-1}\times\hat{T}\times Q \equiv x\times Q$.

Definition 1.3: *The "first-order isodifferentials" of Class I of the contravariant and covariant coordinates $\hat{x}^k$ and $\hat{x}_k$, on an isoeuclidean space $\hat{E}$ of the same class are given by*

$$\hat{\partial}\hat{x}^k = \hat{1}^k{}_i(x,...)\,dx^i, \quad \hat{\partial}\hat{x}_k = \hat{T}_k{}^i(x,...)\,dx_i, \qquad (1.27)$$

where the expressions $\hat{\partial}\hat{x}^k$ and $\hat{\partial}\hat{x}_k$ are defined on $\hat{E}$ while the corresponding expressions $\hat{1}^k{}_i dx^i$ and $\hat{T}_k{}^i dx_i$ are the projections on the conventional Euclidean space E. Let $\hat{f}(\hat{x})$ be a sufficiently smooth isofunction on a closed domain $\hat{D}(\hat{x}^k)$ of contravariant coordinates $\hat{x}^k$ on $\hat{E}$. Then the "isoderivative" at a point $\hat{q}^k \in \hat{D}(\hat{x}^k)$ is given by

$$\hat{f}'(\hat{q}^k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}\Big|_{\hat{x}^k=\hat{q}^k} = \hat{T}_k{}^i\frac{\partial f(x)}{\partial x^i}\Big|_{\hat{x}^k=\hat{q}} = \text{Lim}_{\hat{\partial}\hat{x}^k\to\hat{0}^k}\frac{\hat{f}(\hat{q}^k+\hat{\partial}\hat{x}^k) - \hat{f}(\hat{q}^k)}{\hat{\partial}\hat{x}^k} \qquad (1.28)$$

where $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$ is computed on $\hat{E}$ and $\hat{T}_k{}^i\partial f(x)/\partial x^i$ is the projection in E. The "isoderivative" of a smooth isofunction $\hat{f}(\hat{x})$ of the covariant variable $\hat{x}_k$ at the point $\hat{q}_k \in \hat{D}(\hat{x}_k)$ is given by

$$\hat{f}'(\hat{q}_k) = \frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k}\Big|_{\hat{x}_k=\hat{q}_k} = \hat{1}^k{}_i\frac{\partial f(x)}{\partial x_i}\Big|_{\hat{x}_k=\hat{q}_k} = \text{Lim}_{\hat{\partial}\hat{x}_k\to\hat{0}_k}\frac{\hat{f}(\hat{q}_k+\hat{\partial}\hat{x}_k) - \hat{f}(\hat{q}_k)}{\hat{\partial}\hat{x}_k} \qquad (1.29)$$

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various properties of the conventional differential calculus. We here mention for brevity only the following isotopic properties. The *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k(\hat{x}_k)$ on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ are defined according to the respective rules

$$\hat{\partial}\hat{f}(\hat{x})_{contrav.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{\partial}\hat{x}^k = \hat{T}_k{}^i\frac{\partial f}{\partial x^i}\hat{1}^k{}_j dx^j = \frac{\partial f}{\partial x^k}dx^k = \frac{\partial f}{\partial x^i}\hat{T}_i{}^j dx^j,$$

$$\hat{\partial}\hat{f}(x)_{covar.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\hat{\partial}\hat{x}_k = \hat{1}^k{}_i\frac{\partial f}{\partial x_i}\hat{T}_k{}^j dx_j = \frac{\partial f}{\partial x_k}dx_k = \frac{\partial f}{\partial x_j}\hat{1}^i{}_j dx_i, \qquad (1.30)$$

where the last expression originate from the fact that the contraction is in isospace, thus with respect to the isometric. An iteration of the notion of isoderivative leads to the *second-order isoderivatives*

$$\frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^{k\,2}} = \hat{T}_k{}^i\hat{T}_k{}^j\frac{\partial^2 f(x)}{\partial x^i\partial x^j}, \quad \frac{\hat{\partial}^2\hat{f}(\hat{x})}{\hat{\partial}\hat{x}_k{}^2} = \hat{1}^k{}_i\hat{1}^k{}_j\frac{\partial^2 f(x)}{\partial x_i\partial x_j}, \qquad (1.31)$$

(where there is no sum on k)and similarly for isoderivatives of higher order. The *isolaplacian* on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is given by

$$\hat{\Delta} = \hat{\partial}_k\hat{\partial}^k = \partial^i\hat{\delta}_{ij}\partial^j = \hat{1}^i_k\,\partial^k\,\delta_{ij}\,\partial^j, \qquad \hat{\partial}_k = \partial/\partial\hat{x}^k,\ \partial_k = \partial/\partial x^k, \quad (1.32)$$

and results to be different than the corresponding expression on a Riemannian space $\Re(x,g,R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2}\,\partial_i\,\hat{\delta}^{1/2}\,\hat{\delta}^{ij}\,\partial_j$, *even though the isoeuclidean metric* $\hat{\delta}(x, v, a, ...)$ *is more general than the Riemannian metric* $g(x)$. The following properties then follow,

$$\hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}^j = \delta^i{}_j, \quad \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}_j = \delta_i{}^j, \quad \hat{\partial}\hat{x}_i/\hat{\partial}\hat{x}^j = \hat{T}_i{}^j, \quad \hat{\partial}\hat{x}^i/\hat{\partial}\hat{x}_j = \hat{1}^i{}_j. \qquad (1.33)$$

For completeness we mention the (indefinite) *isointegration* which is the inverse of the isodifferential, e.g.,

$$\hat{\int}\hat{\partial}\hat{x} = \int\hat{T}\hat{1}\,dx = \int dx = x, \qquad (1.34)$$

namely, $\hat{\int} = \int\hat{T}$. Definite isointegrals are formulated accordingly. Due to its simplicity we shall tacitly assume the isotopies of integration hereon.

The above basic notions are sufficient for our limited needs in this paper. The isotopies of additional properties and theorems of the differential calculus (see, e.g., [47]) is left to the interested mathematician. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$.

An important property of the above calculus is that the isodifferentials and isoderivatives preserve the basic isounit $\hat{1}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, ..., on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isofield $\hat{R}(\hat{n},+,\hat{\times})$ with isounit $\hat{1}$ are mapped via isoderivatives into a set of isofunctions $\hat{f}'(\hat{x})$, $\hat{g}'(\hat{x})$, ..., defined over a *different* isofield because of the alteration of the isounit. Physically, the condition is also *necessary* because the unit is a prerequisite for measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d(\hat{1}^k{}_i x^i) = [(\partial_i \hat{1}^k{}_r) x^r + \hat{1}^k{}_i]\, dx^i = \hat{W}^k{}_i dx^i, \tag{1.35}$$

would imply the alteration of the isounit, $\hat{1} \rightarrow \hat{W} \neq \hat{1}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives of independent variables, say, isoderivatives on coordinates and time, the above rule does not apply and we have

$$\hat{\partial}_t \hat{\partial}_k \hat{f}(t, \hat{x}) = \hat{\partial}_t [\hat{\partial}_k \hat{f}(t, \hat{x})] = \hat{\partial}_t [T_k{}^i(t, x, \ldots) \partial_i f(t, x)]. \tag{1.36}$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

Note that the ordinary differential calculus is local-differential on E. The isodifferential calculus is instead local-differential on $\hat{E}$ but, when projected on E, it becomes *integro-differential* because it incorporates integral terms in the isounit.

The isodifferential calculus admit a simple extension to the *genodifferential calculus*, here introduced apparently for the first time, which can be constructed in terms of the following *genodifferentials and genoderivatives to the left and to the right*

$$^{<}\hat{d}^{<}\hat{x} = {}^{<}\hat{1}^k{}_i d^{<}\hat{x}^i, \quad {}^{<}\hat{\partial} / \partial^{<}\hat{x}^k = {}^{<}\hat{T}_k{}^i \partial / \partial^{<}\hat{x}^i,$$

$$\hat{d}^{>}\hat{x}^{>k} = \hat{1}^{>k}{}_i d\hat{x}^{>i}, \quad \hat{\partial}^{>} / \partial\hat{x}^{>k} = \hat{T}^{>}{}_k{}^i \partial / \partial\hat{x}^{>i}. \tag{1.37}$$

In turn, the genodifferential calculus is itself a particular case of the *hyperdifferential calculus*, also introduced here apparently for the first time, which can be constructed via the following *hyperdifferentials and hyperderivatives to the right*

$$\{\hat{d}^{>}\}(\hat{x}^{>k}) = \{\hat{1}^{>}{}_1{}^k{}_i d(\hat{x}^{>i}), \hat{1}^{>}{}_2{}^k{}_i d(\hat{x}^{>i}), \hat{1}^{>}{}_3{}^k{}_i d(\hat{x}^{>i}), \ldots\},$$

$$\{\hat{\partial}^{>} / \partial\}(\hat{x}^{>k}) = \{\hat{T}^{>}{}_{1k}{}^i \partial / \partial(\hat{x}^{>i}), \hat{T}^{>}{}_{2k}{}^i \partial / \partial(\hat{x}^{>i}), \hat{T}^{>}{}_k{}^i \partial / \partial(\hat{x}^{>i}), \ldots\}, \tag{1.38}$$

with corresponding conjugate expressions for the hypercalculus to the left, where one should note the multiplicity of the multivaluedness.

The preceding iso-, geno- and hyper-differential calculi admit antiautomorphic images via the isodual map which reverses the sign of all generalized units. For instance, the *isodual isodifferential calculus* is characterized by the following *isodual differentials and isodual isoderivatives*

$$\hat{d}^d\hat{x}^{kd} = \hat{1}^k{}_i{}^d d\hat{x}^{id} \equiv \hat{d}x^k, \quad \hat{\partial}^d / \partial\hat{x}^{kd} \quad \hat{T}_k{}^{id} \partial / \partial\hat{x}^{id} \equiv \hat{T}_k{}^i \partial / \partial\hat{x}^i, \tag{1.39}$$

namely (by keep ignoring the multiplicative isounit), *the isodifferential calculus is isoselfdual* because invariant under isoduality. This elementary mathematical property is important to establish that the physical laws applying for matter also apply for antimatter (see next section).

1.7: Kadeisvili's isocontinuity. The notion of *isocontinuity* on an isospace was first studied by Kadeisvili [15] and resulted to be easily reducible to that of conventional continuity for Class III isotopies because the *isomodulus* $\hat{|}\hat{f}(\hat{x})\hat{|}$ of a function $\hat{f}(\hat{x})$ on the isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isofield $\hat{R}(\hat{n},+,\hat{\times})$ is given by the conventional modulus $|\hat{f}(\hat{x})|$ multiplied by the a well behaved isounit $\hat{1}$,

$$\hat{|}\hat{f}(\hat{x})\hat{|} = |\hat{f}(\hat{x})| \times \hat{1}. \tag{1.40}$$

Definition 1.4 [15]: *An infinite sequence of isofunctions of Class I $\hat{f}_1, \hat{f}_2, \ldots$ is said to be "strongly isoconvergent" to the isofunction $\hat{f}$ of the same class, when*

$$\mathrm{Lim}_{k\rightarrow\infty} \hat{|}\hat{f}_k - \hat{f}\hat{|} = \hat{0}, \tag{1.41}$$

while the "isocauchy condition" can then be expressed by

$$\hat{|}\hat{f}_m - \hat{f}_n\hat{|} < \hat{\delta} = \delta \times \hat{1}, \tag{1.42}$$

where δ is real and m *and* n *are greater than a suitably chosen* N(δ).

The isotopies of other notions of continuity, limits, series, etc. can be easily constructed [15,37].

Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent. However, *a series which is strongly isoconvergent is not necessarily* *strongly convergent.* As a result, *a series which is conventionally divergent can be turned into a convergent form under a suitable isotopy* ([37], Sect. 6.5). This mathematically trivial property has rather important applications, e.g., for the reconstruction of *convergent* perturbative series for *strong* interactions, which are conventionally divergent.

Similarly, the reader may be interested in knowing that, given a function which is not square-integrable in a given interval, there always exists an isotopy which turns the function into a square-integrable form [37]. The novelty is due to the fact that the underlying mechanism *is not* that of a weight function, but that of *altering the underlying field.*

The *isodual isocontinuity* is the image of the preceding continuity under isoduality. No study on continuity for genotopic and hyperstructural methods has been conducted until know, to our best knowledge.

1.7. Tsagas-Sourlas isotopology. The notion of n-dimensional *isomanifold* was first studied by Tsagas and Sourlas [44,45]. The main lines can be summarized as follows. All isounits of Class III can always be diagonalized into the form

$$\hat{1} = \text{diag.}(B_1, B_2, \ldots, B_n), \quad B_k(x, \ldots) \neq 0, \, k = 1, 2, \ldots, n, \tag{1.43}$$

Consider then n isoreal isofields $\hat{R}_k(\hat{n},\hat{+},\hat{\times})$ each characterized by the isounit $\hat{1}_k = B_k$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_n . \tag{1.44}$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^n \approx R^n$, where R^n is the Cartesian product of n conventional fields $R(n,+,\times)$. But the total unit of $\hat{R}^n$ is expression (1.43). Therefore, one can introduce a topology on $\hat{R}^n$ via the simple isotopy of the conventional topology on R^n,

$$\hat{\tau} = \{\emptyset, \hat{R}^n, \hat{R}_l\}, \tag{1.45}$$

where $\hat{R}_l$ represents the subset of $\hat{R}^n$ defined by

$$\hat{R}_l = \{\hat{P} = (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \,/\, \hat{n}_l < \hat{a}_l, \hat{a}_2, \ldots, \hat{a}_n < \hat{m}_l, \hat{n}_l, \hat{m}_l, a_k \in \hat{R}\}. \tag{1.46}$$

As one can see, the above topology coincides everywhere with the

conventional topology τ of R^n *except at the isounit* $\hat{1}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{1}$ which can incorporate integral terms. The above structure is called the *Tsagas-Sourlas isotopology* or an *integro-differential topology.*

Definition 1.5 [44,45]: *A "topological isospace" $\hat{\tau}(\hat{R}^n)$ of Class I is the isospace $\hat{R}^n$ of the same class equipped with the isotopology $\hat{\tau}$ of Eq. (1.43). An "Isocartesian isomanifold" $\hat{M}(\hat{R}^N)$ of Class I is the topological isospace $\hat{\tau}(R^n)$ of the same class equipped with a vector structure, an affine structure and the mapping*

$$\hat{1}: \hat{R}^n \to \hat{R}^n, \quad \hat{1}: \hat{a} \to \hat{1}(\hat{a}) = \hat{a}, \quad \forall\, \hat{a} \in \hat{R}. \tag{1.47}$$

An "isoeuclidean isomanifold" of Class I $\hat{M}(\hat{E}(\hat{x},\hat{\delta},\hat{R}))$ occurs when the n-dimensional isospace $\hat{E}$ of the same class is realized as the Cartesian product

$$\hat{E}(\hat{x},\hat{\delta},\hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ldots \times \hat{R}_n , \tag{1.48}$$

and equipped with the isotopology $\hat{\tau}$ with isounit (1.43) *of the same class. The "isodual topological isospaces", "isodual isocartesian isomanifolds" and "isodual isoeuclidean isomanifolds" are the images of their corresponding structures under isoduality.*

For all additional aspects of isomanifolds and related topological properties we refer the interested reader to Tsagas and Sourlas [44,45]. It should be noted that their study is referred to $M(\hat{R}^n)$, and that the structure $\hat{M}(\hat{R}^n)$ is introduced here for the first time.

1.8. Isotopies of functional analysis. The isotopies imply simple, yet nontrivial generalizations of *all* conventional mathematical structures of functional analysis, with no exception known to this author. This study was initiated by Kadeisvili in ref. [15] under the name of *functional isoanalysis.* The latter discipline begins with the isotopies of continuity outlined in Sect. 1.6, and includes the isotopies of: conventional square-integrable, Banach and other spaces; conventional and special functions and transforms; etc. (see [37] for brevity).

The reader should be alerted that the use of *conventional* functions, such as the trigonometric or hyperbolic functions, within the context of *isotopic* spaces leads to a number of inconsistencies which may remain undetected by non-experts in the field.

1.9. Isotopies of Lie's theory. The conventional formulation of Lie's theory can only characterize classical and operator systems which are *linear, local-differential and potential-Hamiltonian,* as it is well known, thus possessing clear limitations.

The isotopies outlined in this section were introduced by Santilli [28] for the specific purpose of constructing a broader, yet axiom-preserving formulation of the various branches of Lie's theory, which are capable of characterizing *nonlinear, nonlocal-integral and nonpotential-nonhamiltonian systems.* The latter formulation was originally introduced as *Lie-isotopic theory,* studied by the author in various works (see [37] for a review up to 1995), and it is today known as the *Lie-Santilli isotheory* [3,14,19,43]. The latter is characterized by nonlinear, nonlocal and noncanonical isotopies of all aspects of Lie's theory (universal enveloping associative algebras, Lie algebras, Lie groups, representation theory. etc.), which are however such to reconstruct linearity, locality and canonicity on isospaces over isofields [loc. cit.].

We here limit ourselves to recall from [37] that *a (finite-dimensional) isospace* $\hat{L}$ *over an isofield* $\hat{F}(\hat{a},+,\hat{\times})$ *of isoreal numbers* $\hat{R}(\hat{n},+,\hat{\times})$*, isocomplex numbers* $\hat{C}(\hat{c},+,\hat{\times})$ *or isoquaternions* $\hat{Q}(\hat{q},+,\hat{\times})$ *with isotopic element* $\hat{T}$ *and isounit* $\hat{1} = \hat{T}^{-1}$ *of Class I is called a "Lie-Santilli isoalgebra" over* $\hat{F}$ *when there is a composition* $[\hat{A} \hat{,} \hat{B}]$ *in* $\hat{L}$*, called "isocommutator", which satisfies the following "isolinear and isodifferential rules" for all* $\hat{a}, \hat{b} \in \hat{F}$ *and* $\hat{A}, \hat{B}, \hat{C} \in L$

$$[\hat{a} \hat{\times} \hat{A} + \hat{b} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{a} \hat{\times} [\hat{A} \hat{,} \hat{C}] + \hat{b} \hat{\times} [\hat{B} \hat{,} \hat{C}],$$

$$[\hat{A} \hat{\times} \hat{B} \hat{,} \hat{C}] = \hat{A} \hat{\times} [\hat{B} \hat{,} \hat{C}] + [\hat{A} \hat{,} \hat{C}] \hat{\times} \hat{B}. \tag{1.49}$$

and the "Lie-Santilli isoaxioms",

$$[\hat{A} \hat{,} \hat{B}] = -[\hat{B} \hat{,} \hat{A}],$$

$$[\hat{A} \hat{,} [\hat{B} \hat{,} \hat{C}]] + [\hat{B} \hat{,} [\hat{C} \hat{,} \hat{A}]] + [\hat{C} \hat{,} [\hat{A} \hat{,} \hat{B}]] = 0. \tag{1.50}$$

In this paper we shall identify novel *realizations* of the Lie-Santilli isotheory which are permitted by the isodifferential calculus of this section. The formulation of the isotheory itself via the isodifferential calculus is presented in the adjoint paper by Kadeisvili [16]. Its knowledge is hereon assumed as an integral part of this paper, because all subsequent analytic and geometric studies are aimed

at achieving compatibility with the Lie-Santilli isotheory. The ultimate objective is that of lifting conventional space-time and internal symmetries for linear, local and canonical systems into the corresponding *isosymmetries* for more general systems.

The *isodual Lie-Santilli isotheory* is the image of the isotheory under isoduality [37,16]. The *Lie-Santilli genotheory and hypertheory* and their isoduals have been preliminarily addressed in [37,40] as a step-by-step liftings of the various aspects of Lie's theory (rather than within the context of nonassociative algebras), but they remain vastly unexplored at this writing.

2. Isotopic liftings of Newtonian, analytic and quantum mechanics.

2.1. Introduction. The contemporary formulations of Newtonian, analytic and quantum mechanics have been constructed for the characterization of systems which are local-differential and canonical-Hamiltonian. These systems were originally introduced by Lagrange [17], Hamilton [12] and other founders of analytic dynamics as characterizing the so-called *exterior dynamical problem,* namely, a finite number of isolated particles which can be well approximated as point-like when moving in vacuum under action-at-a-distance, potential interactions.

This class of systems is characterized by equations of motion which are *local-differential and variationally selfadjoint* (SA), as studied in details in monograph [29]. A typical classical examples is given by the planetary system, while operator examples are given by the atomic systems, as well as the electromagnetic and weak interactions ar large for which conventional formulations are *exactly* valid.

The founders of analytic dynamics also introduced the broader *interior dynamical problem,* namely, a finite number of extended, nonspherical and deformable particles moving within a physical medium, thus experiencing conventional potential interactions, plus resistive interactions due to the medium which are not derivable from a potential or, more generally, from a Hamiltonian.

Contrary to a rather popular belief, an inspection of the original contributions [17,12] reveals that Lagrange and Hamilton were fully aware that the functions today called "Lagrangian" or "Hamiltonian" *cannot* representing the entire physical reality in the coordinates of the observer. For this reason, they introduced their celebrated equations *with external terms.*

The contemporary literature on analytic dynamics is almost entirely restricted to Lagrange's and Hamilton's equations *without external terms* (also called the "truncated" analytic equations), thus representing the inertial coordinates of the observer only a rather limited class of systems, essentially the exterior systems in vacuum.

On the contrary, the original formulation by Lagrange and Hamilton [loc. cit.] permitted the representation of all possible systems in the coordinates of the observer, by representing potential forces with their celebrated functions and nonpotential forces with the external terms.

The latter systems are today characterized by equations of motion which are *arbitrarily nonlinear in coordinates and velocities, integro-differential and variationally nonselfadjoint* (NSA). Thus the equations include ordinary differential terms for the center-of-mass trajectory x(t) plus integral terms of surface or volume type representing the correction to the preceding characterization due to the *size and shape* of the bodies. Moreover, the equations are variationally nonselfadjoint in the sense of violating the integrability conditions for the existence of a Lagrangian or a Hamiltonian [29].

In the transition from analytic equations without external terms to those with external terms the following major methodological problem soon emerges. The brackets of Hamilton's equation *without* external terms characterize a Lie algebra, as well known. On the contrary, the brackets of Hamilton's equations *with* external terms violate the conditions to characterize any algebra (the scalar and distributive laws), let alone those of a Lie algebra [28].

Therefore, external systems can be treated with the large variety of methodological tools of contemporary physics, while *none* of them is applicable to the interior systems when represented with analytic equations with external terms.

In this section we shall study the above problem from an analytic profile, while the corresponding geometric profile is studied in Sect. 3.

The isotopies have permitted a first resolution of the problem herein considered, because they permit the applicability of the *conventional* methods of linear, local and Hamiltonian systems also to the more general nonlinear, nonlocal and nonhamiltonian systems, when properly expressed in isospaces over isofields [37,38].

The main idea is that the isounits introduced on mathematical grounds in the preceding section assume the physical role of replacing the external terms of Lagrange and Hamilton. In different terms, all conventional potential forces are represented as usual via the conventional Lagrangian or Hamiltonian while the nonlocal-nonpotential forces are represented via the isounit. This approach first

permits the characterization of an algebra by the brackets of the time evolution and, second, the algebra results to be a Lie-Santilli isoalgebra. The connection between the isounit and the external terms is so deep that both requires the same number of independent functions.

The above analytic isotopies were constructed via the use of the *isotopic degrees of freedom of the product* [28,32,33,37,38]. In this paper we present, apparently for the first time, the same isotopies although constructed with the isodifferential calculus of the preceding section. A comparison of the two formulations soon reveals that the latter approach is preferable over the former because it permits a much more impressive identity of conventional and isotopic methods at the abstract level, thus permitting a unified treatment of exterior and interior problem in which both, the conventional and the isotopic methods, are mere *realizations* of the same axioms.

Studies of this type can be best initiated with the isotopies of the fundamental equations of all contemporary physical theories, Newton's equations, and then pass to their analytic and operator counterparts.

However, as indicated in Sect. 1, the isotopies are structurally reversible, that is, they are reversible for reversible Lagrangians and isounits, while interior systems are structurally irreversible.

The representation of irreversibility can be best done via the *genotopies of Newtonian analytic and quantum mechanics*, which also permit the regaining of the conditions for the brackets of the time evolution to characterize an algebra, although that algebra results to be of the broader *Lie-admissible algebra* (see [28] for details). The latter genotopies and the still broader hyperformulations are merely indicated without treatment for brevity.

Recall from Sect. 1.3 that just the change of the sign of the charge is basically insufficient for the representation of antimatter, because the transition from matter to antimatter requires an antiautomorphic map as it is the case for charge conjugation. A great asymmetry then emerges in contemporary mathematical and physical treatment of matter and antimatter, in the sense that matter is treated in a prioritarian way with a large body of methods, only some of which are applicable to antimatter, those on a Hilbert spaces.

This unbalance is resolved by the isotopies because we can introduce, apparently for the first time in this paper, antiautomorphic images under isoduality (change of sign of the unit) of *all* basic equations of Newtonian, analytic and quantum mechanics. The emerging new equations then permit the *representation of antimatter beginning at the Newtonian level for the first time in an antiautomorphic way.* The extension of the isodual methods at the subsequent levels then permits the achievement of a *complete methodological equivalence in the treatment of matter and antimatter at all levels of study.*

2.2. Isotopies of Newtonian mechanics. Newton's equations have remained essentially unchanged since their inception in 1687 [24]. In this section we introduce, apparently for the first time, the *isotopies of Newton's equations*, or *isonewton's equations* for short, characterized by the isodifferential calculus of the preceding section for the purpose of broadening their original conception. The isotopies have been selected over a variety of other possibilities because of their axiom-preserving character as well as of the consequential broadening of classical and quantum mechanics outlined in subsequent sections.

The contemporary formulation of Newton's equations requires the tensorial space $S(t,x,v) = E(t,R_t)\times E(x,\delta,R)\times E(v,\delta,R)$ where $E(t,R_t)$ is the one-dimensional space representing time t, $E(x,\delta,R)$ is the conventional three-dimensional Euclidean space with local trajectories $x = x(t) = (x^k) = (x, y, z)$ and $E(v,\delta,R)$ is the *tangent space* TE (see Sect. 3.2) which, at this Newtonian level, can be considered as an independent space representing the contravariant velocities $v = (v^k) = dx^k/dt$. Newton's equations for a test body of mass m = const. ($\neq 0$) moving within a resistive medium (e.g., our atmosphere) can then be written

$$m\, dv_k / dt - F^{SA}{}_k(t, x, v) - F^{NSA}{}_k(t, x, v) = 0, \quad k = 1, 2, 3\ (= x, y, z), \tag{2.1}$$

where SA (NSA) stands for *variational selfadjointness (variational non-self-adjointness)*, i.e.. the verification (violation) of the necessary and sufficient conditions for the existence of a potential U(t, x, v) originally due to Helmholtz [13] (see monograph [29] for historical notes and systematic studies). It should be recalled that in Newtonian mechanics the potential U(y, x, v) must be linear in the velocities (to avoid a redefinition of the mass),

$$U(t, x, v) = U_k(t, x)\, v^k + U_o(t, x). \tag{2.2}$$

Eq.s (2.1) can then be written

$$\left\{ m\frac{dv_k}{dt} - \frac{d}{dt}\frac{\partial U(t,x,v)}{\partial v^k} + \frac{\partial U(t,x,v)}{\partial x^k} - F^{NSA}{}_k(t,x,v)\right\}^{NSA} =$$

$$= \left\{ m\frac{dv_k}{dt} - \frac{\partial U_k(t,x)}{\partial x^s}\frac{dv^s}{dt} + \frac{\partial U_o(t,x)}{\partial x^k} - F^{NSA}{}_k(t,x,v)\right\}^{NSA} = 0, \tag{2.3}$$

namely, they are not in general derivable from the conventional Lagrange's [17] or Hamilton's [12] equations in the local chart (t, x, v), those without external terms, as well known [29,30] (see later on for coordinate transforms). The extension to systems of n particles with masses m_k ($\neq 0$) is straighforward and will be ignored for brevity.

The representation space of the desired isotopic image of Newton's equations is given by the Kronecker product of isospaces $\hat S(\hat t,\hat x,\hat v) = \hat E(\hat t,\hat R_t)\times\hat E(\hat x,\hat\delta,\hat R)\times\hat E(\hat v,\hat\delta,\hat R)$ characterized by the one-dimensional *time isounit* $\hat 1_t = (\hat T_t)^{-1}$ and the three-dimensional *space isounit* $\hat 1 = (\hat 1^k{}_i) = (\hat T_k{}^i)^{-1}$. Since the velocities are independent from x, they carry in general a different isounit. Such further degree of freedom is however un-necessary and we shall assume the total isounit of isospace $\hat S(\hat t,\hat x,\hat v)$ be given by the several dimensional quantity $\hat 1_{tot} = \hat 1_t\times\hat 1\times\hat 1$.

The *isotime* $\hat t$, *isocoordinates* $\hat x^k(\hat t)$ and *isovelocities* $\hat v^k(\hat t)$ are related to the original respective quantities t, x^k and v^k, by the following relations

$$\hat t = t,\ \hat v^k \equiv v^k, \qquad \hat v_k = \hat\delta_{kj}\hat v^j = \hat T_k{}^i\delta_{ij}\hat v^j = \hat T_k{}^i v_i \neq v_k = \delta_{ki}x^i. \tag{2.4}$$

The desired isotopic lifting of Newton's equations (2.3) in isospace $\hat S(\hat t,\hat x,\hat v)$, here called *isonewton equations* and submitted here apparently for the first time, are given by

$$\hat\Gamma_k(\hat t,\hat x,\hat v) = \hat m\frac{\hat\partial\hat v_k}{\hat\partial\hat t} - \frac{\hat\partial}{\hat\partial\hat t}\frac{\hat\partial\hat U(\hat t,\hat x,\hat v)}{\hat\partial\hat v^k} + \frac{\hat\partial\hat U(\hat t,\hat x,\hat v)}{\hat\partial\hat x^k} =$$

$$\hat m\frac{\hat\partial\hat v_k}{\hat\partial\hat t} - \frac{\hat\partial\hat U_k(\hat t,\hat x)}{\hat\partial\hat x^i}\frac{\hat\partial\hat x^i}{\hat\partial\hat t} + \frac{\hat\partial\hat U_o(\hat t,\hat x)}{\hat\partial\hat x^k} = 0,$$

$$\hat U(\hat t,\hat x,\hat v) = \hat U_k(\hat t,\hat x)\,\hat v^k + \hat U_o(\hat t,\hat x), \tag{2.5}$$

where we have used properties (2.2) and $\hat m = m\times\hat 1_m$ ($\neq 0$) is the *isotopic mass*, that is, the image of the Newtonian mass in isospace with isounit $\hat 1_m$ which will be identified shortly.

Theorem 2.1: *All possible sufficiently smooth, regular and variationally nonself-adjoint Newton's equations* (2.3) *always admit in a neighborhood of a point* (t, x, v) *the representation in terms of the isotopic equations* (2.5)

$$\hat{m}\frac{\hat{\partial}\hat{v}_k}{\hat{\partial}t} - \frac{\hat{\partial}}{\hat{\partial}t}\frac{\partial\hat{O}(t,\hat{x},\hat{v})}{\partial\hat{v}^k} + \frac{\partial\hat{O}(t,\hat{x},\hat{v})}{\partial\hat{x}^k} =$$

$$= \hat{T}_k{}^i\left\{ m\frac{dv_i}{dt} - \frac{\partial U_i(t,x)}{\partial x^s}\frac{dx^s}{dt} + \frac{\partial U_o(t,x)}{\partial x^i} - F^{NSA}{}_i(t,x,v)\right\} = 0. \quad (2.6)$$

Proof. When projected in the original space $S(t, x, v)$, Eq.s (3.5) can be written

$$\hat{m}\times\hat{1}_t\frac{d(\hat{T}_k{}^i v_i)}{dt} - \hat{T}_t\frac{d}{dt}\hat{T}_k{}^i\frac{\partial\hat{O}(t,\hat{x},\hat{v})}{\partial\hat{v}^i} + \hat{T}_k{}^i\frac{\partial\hat{O}(t,\hat{x},\hat{v})}{\partial x^i} =$$

$$= \hat{m}\,\hat{T}_t\hat{T}_k{}^i\frac{dv_i}{dt} - \hat{T}_t\hat{T}_k{}^i\frac{\partial\hat{O}_i(t,x)}{\partial x^s}v^s + \hat{T}_k{}^i\frac{\partial\hat{O}_o(t,x)}{\partial x^i} + \hat{m}\hat{T}_t\frac{d\hat{T}_k{}^i}{dt}v_i = 0. \quad (2.7)$$

Sufficient conditions for identities (2.6) are then then given by

$$\hat{m}\,T_t\,dv_i/dt = m\,dv_i/dt, \quad (2.8a)$$

$$\hat{T}_t\frac{\partial\hat{O}_i(t,x)}{\partial x^s}v^s = \frac{\partial U_i(t,x)}{\partial x^s}v^s, \quad (2.8b)$$

$$\frac{\partial\hat{O}_o(t,x)}{\partial x^i} = \frac{\partial U_o(t,x)}{\partial x^i}, \quad (2.8c)$$

$$\hat{m}\,\hat{T}_t\frac{d\hat{T}_k{}^i(t,x,\ldots)}{dt}v_i = -\hat{T}_k{}^i F^{NSA}{}_i(t,x,v). \quad (2.8d)$$

which, under the assumed continuity and regularity conditions (see [29] for details) always admits a solution in the unknown quantities $\hat{m}$, $\hat{T}_t$, $\hat{T}_k{}^i$, $\hat{O}_k$ and $\hat{O}_o$ for given equations (2.3). In fact, system (2.8) is *overdetermined* and a solution exists for *diagonal* space isounit and *constant* time isounit,

$$\hat{1}^i_k = \delta^i_k\, e^{f_k(t,x,v)}, \qquad \hat{1}_t = \text{constant} > 0, \quad (2.9)$$

for which

$$\hat{m}\,T_t \equiv m, \quad \hat{m} = m\times\hat{1}_t, \quad \hat{O}_k(t,x) = \hat{1}^o{}_o U_k(t,x), \quad \hat{O}_o(t,x) = U_o(t,x),$$

$$f_k(t,x,v) = -m^{-1}\int_o^t dt\,F^{NSA}{}_k(t,x,v)/v_k, \quad (2.10)$$

where there are no repeated indices $\hat{m}$ is constant and the functions f_k are computed from Eq.s (2.10b). **q.e.d.**

The primary motivations for the submission of the isonewton's equations are expressed by the following properties with self-evident proofs which will be only illustrated.

Corollary 2.1.A: *The isonewton equations permit a representation of the actual, extended and nonspherical shape of the body considered and of its possible deformations via the generalized unit (or isotopic element) of the theory.*

Recall that Newton's equations can only approximate the body considered as a massive point, as well known since Newton's time [25]. The point-like representation of particles then persists under analytic representations via Hamilton's equations as well as under symplectic map to quantum mechanical formulations. A representation of the extended character of particles is reached in *second* quantization via the form factors. However, this representation is restricted to spherical shapes from the fundamental symmetry of quantum mechanics, the rotational symmetry. The latter symmetry is known to be a symmetry of *rigid bodies*. Form factors cannot therefore represent the *deformations* of particles under sufficiently intense external interactions which is studied via other rather complicated procedures.

A first motivation for the studies presented in this paper is to introduce a representation of actual *extended, nonspherical and deformable* shapes of particles at the primitive *Newtonian* level, which then persists under *classical* analytic representations and maps to *first* quantization. The isonewton equations do indeed achieve these objectives by setting the foundations for possible new advances in classical and quantum physics. The objective is achieved via the generalized unit of the theory which is evidently absent in the conventional Newtonian, classical and quantum formulations.

As a simple case, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes $n_1{}^2$, $n_2{}^2$, $n_3{}^2$ = constants. Such a shape is directly represented by the isotopic element of the theory in the simple diagonal form

$$\hat{T} = \mathrm{diag.}\,(n_1^{-2}, n_2^{-2}, n_3^{-2}), \quad n_k = \mathrm{const} > 0, \quad k = 1, 2, 3, \quad \hat{T}_t = 1. \qquad (2.11)$$

The representation of the shape in isospace $\hat{S}(\hat{t},\hat{x},\hat{v})$ is then embedded in the *isoderivatives* of the isotopic Newton equations and, when projected in the conventional space S(t, x, v) can be written

$$m\, \hat{T}_k^{\,i} \frac{d\, v_i}{dt} - \hat{T}_k^{\,i} \frac{\partial U_i(t, x)}{\partial x^s} v^s + \hat{T}_k^{\,i} \frac{\partial U_o(t, x)}{\partial x^i} = 0\,, \quad (2.12)$$

namely, the shape terms $\hat{T}_k^{\,i}$ are admitted as factors.

Note that *the representation of shape occurs only in isospace because, when projected in the conventional Euclidean space, the shape terms cancel out by recovering the conventional point-like character of Newton's equations.* This illustrates the effectiveness of the isotopy for the representation of shape. Moreover, the *nonspherical character of the shape emerges only in the projection in ordinary spaces, because all deformed spheres in ordinary spaces are mapped into the perfect sphere in isospace, the isosphere of Sect. I,*

$$\hat{x}^2 = (x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_3^{-2} x^3) \times \hat{1} \in \hat{R}(\hat{n},\hat{+},\hat{\times})\,. \qquad (2.13)$$

The representation of shapes more complex than the spheroidal ellipsoids is possible with non-diagonal isounits. The representation of the *deformations* of the original shape due to motion within resistive media or other effects, can be achieved via a suitable functional dependence of the $\hat{T}_k^{\,i}$ terms in velocities, pressure, etc. [37,38].

A simple application discussed in detail in [38] is given by a charged, spinning and spherical metallic shell which is subjected to a sufficiently intense external electromagnetic field represented by the known Lorentz force with potential $U = eA_k v^k + e\phi$, where e is the charge and (A_k, ϕ) are the familiar electromagnetic potentials. It is evident that the original spherical shape is deformed by the Lorentz force, with consequential alteration of its magnetic moment. Such a deformation is not representable by Newton's equations as well as by its analytic and quantum representations, but it is easily representable via our isotopic equations (2.5).

The operator image of this classical setting illustrates the relevance of the theory herein submitted. In first quantization, the constituents of a nuclear structure (protons and neutrons) are represented as point-like particles. As such, they maintain in the nuclear structure their intrinsic magnetic moments when in vacuum. However, this approach has not permitted an *exact* representation of the total magnetic moments of few-body nuclei (such as deuteron, tritium, etc.). The isotopic representation of protons and neutrons as they are in the physical reality (*extended and therefore elastic,* spinning, charge distributions) has instead permitted the achievement of an exact representation of said total magnetic moments because each particle experiences a (generally small) deformation of its shape when under the short-range strong forces of a nuclear structure, resulting in an alteration of the intrinsic magnetic moment in vacuum which is missing in conventional quantum treatments. In turn, such alteration permits the *exact* representation of the total magnetic moments of few-body nuclei as well as other intriguing implications and novel predictions [38].

Corollary 2.1.B: *The isonewton equations permit a novel representation of variationally nonselfadjoint forces via the isometric of the underlying geometry, according to the rules*

$$m\, dv_k / dt - F^{NSA}{}_k(t, x, v) = \hat{T}^i_k\, m\, d\, \hat{T}_i^{\,j} v_j / dt\,, \qquad (2.14)$$

while leaving unchanged the representation of conventional self-adjoint forces (except for the constant factor $\hat{T}_t$ of U_k).

In fact, the nonselfadjoint forces are embedded in the *covariant* coordinates in isospace $\hat{v}_i = T_i^{\,j} v_j$, where the v_j's are the covariant coordinates in conventional space. The novelty therefore lies on the fact that nonselfadjoint forces are represented by the isogeometry itself, thus providing another motivation for the isotopies.

The simplicity of representation (2.14) should be kept in mind and compared to the complexity of the conventional solution of the *inverse problem of Newtonian mechanics* [29], i.e., the representation of non-self-adjoint systems via a Lagrangian or a Hamiltonian. Moreover, under the assumed conditions, the latter exists in the fixed coordinates (t, x, v) of the observer only for a restricted class called *nonessentially nonselfadjoint* [loc. cit.], while isorepresentation (2.6) always exists in the given coordinates (t, x, v) under the same conditions.

When coordinate transformations are admitted, an *indirect analytic representation* (i.e., a representation in transformed coordinates (t′, x′, v′)) always exists for all local-differential, analytic and regular, nonselfadjoint Newtonian systems in a star-shaped region of the variables (this is the *Lie-Koening theorem* [30] as the analytic counterpart of the geometric *Darboux's theorem* of Sect. 3.2).

However, the latter representation has a number of *physical* drawbacks. First, the transformations $(t, x, v) \rightarrow (t', x', v')$ are nonlinear and, as such, the new coordinates are not realizable in laboratory. Also, their nonlinearity implies the loss of the original inertial character of the reference frame with consequential loss of conventional relativities (in fact, the Galilei and Einstein relativities are solely applicable to inertial systems as well known).

These are the reasons why, after completing the studies of ref.s [29,30], this author continued the search for a representation of nonselfadjoint systems which occurs in the given *inertial* reference frame of the observer, and it is universal, i.e., applicable to all systems occurring in the physical reality.

The following examples illustrate isorepresentation (2.6). The equation of the linearly damped particle in one dimension

$$m\, dv/dt + \gamma v = 0, \quad \gamma \in R(n,+,\times), \; \gamma > 0, \tag{2.15}$$

admits isorepresentation (3.6) with values

$$\hat{T} = \hat{S}_o e^{\gamma t/2m}, \quad \hat{T}_t = 1, \quad U_k = U_o = 0, \tag{2.16}$$

where $\hat{S}_o$ is a *shape factor*, e.g., of the spheroidal type (2.11) which is prolate in the direction of motion. In this way, the isotopic Newton equations represent: 1) the nonselfadjoint force $F^{NSA} = -\gamma v$ experienced by an object moving within a physical medium; 2) its extended character (which is necessary for the existence of the resistive force); and 3) the deformation of the original shape (in the case considered a perfect sphere) caused by the medium.

The equation for the linearly damped harmonic oscillator in one dimension

$$m\ddot{x} + \gamma \dot{x} + kx = 0, \quad k \in R(n,+,\times), \; k > 0, \tag{2.17}$$

admits isorepresentation (2.6) with the values

$$\hat{T} = \hat{S}_o e^{\gamma t/2m}, \quad U_o = -\tfrac{1}{2} k x^2, U_k = 0, \quad \hat{T}_t = 1, \tag{2.18}$$

where $\hat{S}_o$ represents the shape of the body oscillating within a resistive medium. The interested reader can construct a virtually endless variety of isorepresentations of non-self-adjoint forces. A systematic study will be conducted elsewhere.

Corollary 2.1.C: *The isonewton equations permit the representation of nonlocal-integral forces when completely embedded in the isounit of the theory.*

The above occurrence is permitted by the integro-differential topology of the Tsagas-Sourlas isomanifolds recalled in Sect. 1. Consider as an example the integro-differential equation

$$m\, dv/dt + \gamma v^2 \int_\sigma d\sigma\, \mathcal{F}(\sigma, ...) = 0, \quad \gamma > 0, \tag{2.19}$$

representing an extended object (such as a space-ship during re-entry in our atmosphere) with local-differential center-of-mass trajectory x(t) and corrective terms of integral type due to the shape (surface) σ of the body moving within a resistive medium, where $\mathcal{F}$ is a suitable kernel depending on σ as well as on other variables such as pressure, temperature, density, etc. The above equation admits isorepresentation (2.6) with the values

$$\hat{T} = \hat{S}_\sigma e^{\gamma m^{-1} x \int_\sigma d\sigma \mathcal{F}(\sigma, ...)}, \quad \hat{T}_t = 1, \quad U_k = U_o = 0, \tag{2.20}$$

where $\hat{S}_\sigma$ is the shape factor, which is admitted by the integro-differential topology of the isomanifold $\hat{M}(\hat{E})$ because all integral terms are embedded in the isounit. Similar isorepresentations can be easily constructed by the interested reader.

It should be recalled that the representation of nonlocal-integral terms is prohibited in Hamiltonian mechanics because the underlying geometry and topology are local-differential. In fact, the Lie-Koening Theorem requires a *local-differential approximation* of systems and it is inapplicable to integral systems of type (2.19).

In the author's opinion, the generalization of Newton's equations into a form admitting nonlocal-integral forces has the most important epistemological, mathematical and physical implications. Recall that contemporary mathematical and physical knowledge is generally restricted to point-like/local formulations. The isotopies therefore permit the study of more general nonlocal-integral systems beginning at the primitive Newtonian level. Mathematically, the representation of nonlocal-integral forces requires the study of new methods, such as new topologies, geometries and mechanics. Physically, the implications are equally important and they deal with the historical legacy due to Blockint'sev, Fermi, and others that the strong interactions have a nonlocal-integral component. In fact, all strongly interacting particles (hadrons) have a charge radius which is of the same order of the range of the strong interactions (about 10^{-13} cm). A necessary condition to activate the strong interactions is therefore that hadrons enter into mutual penetration of their charge distributions. But hadrons are some of the densest objects measured in laboratory until now. The historical legacy on the nonlocality of the strong interactions then follows (see [39] for details).

A quantitative treatment of the historical legacy of the nonlocality of strong interactions has been the primary motivation for this author to conduct his studies on the isotopies, with evident need to initiate the studies at the primitive Newtonian level, then passing to classical analytic representations and finally to operator treatment.

The isonewton equation on a curved space are submitted in Sect. 3.3. We introduce here, apparently for the first time, the *forward genonewton's equations*, i.e., the equations based on the genodifferential calculus of Sect. 1.5 defined on the *forward genospace* $\hat S^>(\hat t^>,\hat x^>,\hat v^>) = \hat E^>(\hat t^>,\hat R_t^>)\times\hat E^>(\hat x^>,\hat\delta^>,\hat R^>)\times\hat E^>(\hat v^>,\hat\delta^>,\hat R^>)$, which can be written

$$\hat m^> \frac{\hat\partial^> \hat v_k{}^>}{\hat\partial \hat t^>} - \frac{\hat\partial^>}{\hat\partial \hat t^>}\,\frac{\hat\partial^> \hat U(\hat t,\hat x,\hat v)}{\hat\partial\,\hat v^{k>}} + \frac{\hat\partial^> \hat U(\hat t,\hat x,\hat v)}{\hat\partial\,\hat x^{k>}} = 0, \qquad (2.21)$$

with corresponding *backward genonewton equations* here omitted for brevity. The preceding equations coincide with the isonewton's equations for motion in one dimension, but are particularly suited to *represent irreversible trajectories in more than one dimension.*

We also introduce the *forward hypernewton equations*, i.e., the equations based on the hyperdifferential calculus of Sect. 1.5 on the related hyperspace, which can be written

$$\{\hat m^>\}\frac{\{\hat\partial^>\}\{\hat v_k{}^>\}}{\{\hat\partial\hat t^>\}} - \frac{\{\hat\partial^>\}}{\{\hat\partial\hat t^>\}}\,\frac{\{\hat\partial^>\}\{\hat U(\hat t,\hat x,\hat v)^>\}}{\{\hat\partial\}\{\hat v^{k>}\}} + \frac{\{\hat\partial^>\}\{\hat U(\hat t,\hat x,\hat v)^>\}}{\{\hat\partial\}\{\hat x^{k>}\}} = 0, \qquad (2.22)$$

with corresponding *backward hypernewton equations*, where the brackets {...} represent a finite and ordered set. The latter equations are particularly suited to represent *a system of particles in irreversible conditions*, as one can verify with explicit examples.

The *isoduality of the ordinary Newton's equations* for potential systems, called *isodual Newton equations*, and introduced in this paper apparently for the first time, are defined on the *isodual isospace* $S^d(t^d,x^d,v^d) = E^d(t^d,R^d_t)\times E^d(x^d,\delta^d,R^d)\times E^d(v^d,\delta^d,R^d)$ with *isodual unit* $I^d_{tot} = I^d_t \times I^d \times I^d$, $I_t{}^d = -1$, $I^d = \mathrm{diag.}(-1,-1,-1)$, and can be written

$$m^d\,\frac{d^d v_k{}^d}{d^d t^d} - \frac{d^d}{d^d t^d}\,\frac{\partial^d U^d(t^d,x^d,v^d)}{\partial^d v^{kd}} + \frac{\partial^d U^d(t^d,x^d,v^d)}{\partial^d x^{kd}} = 0. \qquad (2.23)$$

It is an instructive exercise for the interested reader to prove that *the isodual Newton equations change sign under isoduality* (this requires the isoduality not only of all multiplications, but also of all quotients, see Sect. 1.3). However, such a *negative* value is referred to a *negative* unit, thus establishing their full equivalence to the *positive* value of the conventional equation referred to *positive* units. Note that under the above representation, antimatter possesses *negative masses*, and *moves backward in time.*

We can also introduce, apparently for the first time in this paper, the *isodual isonewton equations* which are defined don the *isodual isospace* $\hat S^d(\hat t^d,\hat x^d,\hat v^d) = \hat E^d(\hat t^d,\hat R^d_t)\times\hat E^d(\hat x^d,\hat\delta^d,\hat R^d)\times\hat E^d(\hat v^d,\hat\delta^d,\hat R^d)$ with *isodual isounit* $\hat 1^d_{tot} = \hat 1^d_t \times \hat 1^d \times \hat 1^d$, $\hat 1_t{}^d = -\hat 1_t$, $\hat 1^d = -\hat 1$, and can be written

$$\hat m^d\,\frac{\hat\partial^d \hat v_k{}^d}{\hat\partial^d\hat t^d} - \frac{\hat\partial^d}{\hat\partial^d\hat t^d}\,\frac{\hat\partial^d \hat U^d(\hat t^d,\hat x^d,\hat v^d)}{\hat\partial^d\hat v^{kd}} + \frac{\hat\partial^d \hat U^d(\hat t^d,\hat x^d,\hat v^d)}{\hat\partial^d\hat x^{kd}} = 0. \qquad (2.24)$$

with corresponding *isodual backward isonewton's equations.* Similarly, we have the *isodual forward and backward hypernewton's equations* which are omitted for brevity.

2.3. Variational isoselfadjointness. The fundamental methods of the Inverse Newtonian Problem are the conditions of variational self-adjointness in $E(t)\times E(x,\delta,R)\times E(v,\delta,R)$ [13,29]. In this section we shall identify, apparently for the first time, their image in isospace here called *conditions of variational isoself-adjointness.*

Theorem 2.2: *A necessary and sufficient condition for a system of ordinary second-order isodifferential equations in* $\hat E(\hat t)\times\hat E(\hat x,\hat\delta,\hat R)\times\hat E(\hat v,\hat\delta,\hat R)$

$$\hat\Gamma_k(\hat t,\hat x,\hat v,\hat a) = 0,\ k = 1,2,\ldots,n,\ \hat v = \hat d\hat x/\hat d\hat t,\ \hat a = \hat d\hat v/\hat d\hat t \qquad (2.25)$$

which are isodifferentiable at least up to the third order and regular in a region $\hat{\mathfrak R}$ *of points* $(\hat t,\hat x,\hat v,\hat a,\hat d\hat a/\hat d\hat t)$ *(i.e., det* $\hat\partial\hat\Gamma_i / \hat\partial\hat a^j(\hat{\mathfrak R}) \neq 0$*) to be variationally isoself-adjoint* (ISOSA) *in* $\hat{\mathfrak R}$ *is that all the following conditions*

$$\frac{\partial \hat{\Gamma}_i}{\partial \hat{a}^k} = \frac{\partial \hat{\Gamma}_k}{\partial \hat{a}^i},$$

$$\frac{\partial \hat{\Gamma}_i}{\partial \hat{v}^k} + \frac{\partial \hat{\Gamma}_k}{\partial \hat{v}^i} = 2\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\frac{\partial \hat{\Gamma}_i}{\partial \hat{a}^k} = \frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\partial \hat{\Gamma}_i}{\partial \hat{a}^k} + \frac{\partial \hat{\Gamma}_k}{\partial \hat{a}^i}\right)$$

$$\frac{\partial \hat{\Gamma}_i}{\partial \hat{x}^k} - \frac{\partial \hat{\Gamma}_k}{\partial \hat{x}^i} = \frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left[\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\partial \hat{\Gamma}_k}{\partial \hat{v}^i}\right) - \frac{\partial \hat{\Gamma}_k}{\partial \hat{v}^i}\right] =$$

$$= \frac{1}{2}\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\left(\frac{\partial \hat{\Gamma}_i}{\partial \hat{a}^k} - \frac{\partial \hat{\Gamma}_k}{\partial \hat{a}^i}\right), \qquad (2.26)$$

are identically verified in $\hat{\mathfrak{R}}$.

Proof. The proof is provided by an elementary isotopy of the conventional case, ref. [29], Theorem 2.1.2, p. 60, and consists in computing the isovariational forms of system (2.25), proving their uniqueness and showing that conditions (2.26) are necessary and sufficient for the isovariational forms to coincide with their adjoint. **q.e.d.**

The novelty of conditions (2.26) is illustrated by the following

Corollary 2.2.A: *Systems of ordinary isodifferential equations which are variationally isoselfadjoint in isospace are generally variational nonselfadjoint when projected in ordinary spaces.*

Proof. Conditions (2.26) imply no restriction on the isotopic terms $\hat{T}_k^{\ i}$ in isospace while the same terms are restricted by the ordinary conditions of self-adjointness in ordinary spaces **q.e.d.**

Theorem 2.3: *The isotopic Newton equations* (2.5) *are variationally iso-self-adjoint.*

Proof. The verification of the first set of conditions (2.25a) reads

$$\frac{\partial \hat{F}_i}{\partial \hat{a}^j} - \frac{\partial \hat{F}_j}{\partial \hat{a}^i} = \hat{T}_j^{\ m}\frac{\partial \hat{F}_i}{\partial \hat{a}^m} - \hat{T}_j^{\ m}\frac{\partial \hat{F}_j}{\partial \hat{a}^m} = \hat{T}_j^{\ m}\hat{T}_i^{\ m} - \hat{T}_i^{\ m}\hat{T}_j^{\ m} \equiv 0, \qquad (2.27)$$

and the same identities hold for all remaining conditions. **q.e.d.**

It is an instructive exercise for the interested reader to work out the isotopies of the remaining theorems for second-order ordinary differential equations (see [29], Sections 2.2 and 2.3).

We now introduce the conditions of variational isoselfadjointness for n-dimensional systems (2.25) in an equivalent 2n-dimensional first-order form. Let $T^*\hat{E}(\hat{x},\hat{\delta},\hat{R})$ be the isocotangent space (see Sect. 3.2 for a geometric treatment) which in this section can be characterized via the independent space $\hat{E}(\hat{p},\hat{\delta},\hat{R})$ with new, independent, covariant coordinates $\hat{p}_k$ and let the total representation space be $\hat{T}(t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$ with local chart $\hat{b} = (b^\mu) = (\hat{x}^k, \hat{p}_k)$, $\mu = 1, 2, ..., 2n$, $k = 1, 2, ..., n$. Assign sufficiently smooth and invertible prescriptions for the characterization of the new variables $\hat{p}_k$

$$\hat{p}_k = \hat{g}_k(\hat{t}, \hat{x}, \hat{v}), \qquad (2.28)$$

with unique system of implicit functions $v^k = \hat{f}^k(\hat{t}, \hat{x}, \hat{p})$ (see [29], Sect. 2.4, for the conventional case). By using the latter implicit functions, system (2.25) can be written in the equivalent 2n-dimensional form

$$\hat{\Gamma}_\mu(\hat{t}, \hat{b}, \hat{c}) = \hat{C}_{\mu\nu}(\hat{t}, \hat{b})\,\hat{c}^\nu + \hat{D}_\mu(\hat{t}, \hat{b}) = 0, \qquad \hat{c}^\nu = \hat{\partial}\hat{b}^\nu / \hat{\partial}\hat{t}. \quad (2.29)$$

Theorem 2.4: *A necessary and sufficient condition for system* (2.29) *which is at least twice isodifferentiable and regular* (det. $(\hat{C}_{\mu\nu})(\hat{\mathfrak{R}}) \neq 0$) *in a* (6n+1)*-dimensional region* $\hat{\mathfrak{R}}$ *of points* $(\hat{t}, \hat{b}, \hat{c}, \hat{\partial}\hat{c}/\hat{\partial}\hat{t})$ *to be isoselfadjoint in* $\hat{\mathfrak{R}}$ *is that all the following conditions*

$$\hat{C}_{\mu\nu} + \hat{C}_{\nu\mu} = 0,$$

$$\frac{\partial \hat{C}_{\mu\nu}}{\partial \hat{b}^\rho} + \frac{\partial \hat{C}_{\nu\rho}}{\partial \hat{b}^\mu} + \frac{\partial \hat{C}_{\rho\mu}}{\partial \hat{b}^\nu} = 0,$$

$$\frac{\partial \hat{D}_\mu}{\partial \hat{b}^\nu} + \frac{\partial \hat{D}_\nu}{\partial \hat{b}^\mu} = \frac{\partial \hat{C}_{\mu\nu}}{\partial \hat{t}}, \qquad (2.30)$$

are identically satisfying in $\hat{\mathfrak{R}}$.

Proof. The proof is also a simple isotopy of the proof of Theorem 2.7.2, p. 87, ref. [29]. Also, conditions (2.30) are uniquely derivable from conditions (2.26) when systems (2.25) are assumed to be 2n-dimensional and of first-order. **q.e.d.**

The following property is self-evident,

Corollary 2.4.A: *When systems (2.29) assume the "isocanonical form"*

$$\hat{\Gamma}_\mu(\hat{t}, \hat{b}, \hat{c}) = \omega_{\mu\nu}\, \hat{c}^\nu - \hat{\Xi}_\mu(\hat{t}, \hat{b}) = 0, \tag{2.31}$$

where $\omega_{\mu\nu}$ is the conventional canonical symplectic tensor

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{2.32}$$

the conditions of variational isoselfadjointness (2.30) reduce to

$$\frac{\partial \hat{\Xi}_\mu}{\partial \hat{b}^\nu} - \frac{\partial \hat{\Xi}_\nu}{\partial \hat{b}^\mu} = 0. \tag{2.33}$$

Note that a conventional canonical system which is selfadjoint is also isoselfadjoint. Additional and this illustrates the reason why a potential representation of a selfadjoint forces persists at the isotopic level. Additional properties of variational isoselfadjointness will be identified later on.

Let us recall the following meanings of the conditions of variational selfadjointness for 2n-dimensional systems of ordinary first-order differential equations

$$\Gamma_\mu(t, b, c) = C_{\mu\nu}(t, b)\; c^\nu + D_\mu(t, b) = 0, \quad b = (x^k, p_k), \; c^\nu = d\, b^\nu / dt, \tag{2.34}$$

on a conventional space (see [29,30] for detailed studies):

1) *Analytic meaning.* The conditions imply the direct derivability (i.e., derivability without change of local variables or use of integrating factors) of the equations from a first-order variational principle

$$\delta A = \delta \int_{t_1}^{t_2} dt\, [\, R_\mu(t, b)\, db^\mu - H(t, b)\,] = 0,$$

$$C_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu, \quad D_\mu = \partial_\mu H - \partial_t R, \quad \partial_\mu = \partial / \partial b^\mu, \partial_t = \partial / \partial t; \tag{2.35}$$

2) *Geometric meaning.* The two form

$$C = C_{\mu\nu} db^\mu \wedge db^\nu, \tag{2.36}$$

characterized by the covariant tensor $C_{\mu\nu}(b)$ is an exact symplectic form; and

3) *Algebraic meaning.* The brackets among two smooth functions A(b) and B(b)

$$[\, A, B\,] = (\, \partial_\mu A)\, C^{\mu\nu}(b)\, (\, \partial_\nu B\,), \tag{2.37}$$

characterized by the contravariant version of $C_{\mu\nu}$

$$C^{\mu\nu} = [\, (\, C_{\alpha\beta}\,)^{-1}\,]^{\mu\nu}, \tag{2.38}$$

are Lie.

In the next sections we show that the above properties persist in their entirety when formulated under isotopies in isospaces.

The above conditions of isoselfadjointness admit genotopic and hyperstructural. extensions as well as isodual images which are not studied here for brevity.

2.4. Isolagrangian and isohamiltonian mechanics. We now show the derivability of the isonewton equations from a first-order isovariational principle and then study the isotopies of Lagrange's [17] and Hamilton's [12] mechanics.

Proposition 2.1: *All Newtonian action functionals of second or higher order in Euclidean space $S(t,x,v) = E(t,R_t)\times E(x,\delta,R)\times E(v,\delta,R)$ whose integrand is sufficiently smooth and regular in a region $\Re$ of their variables can always be identically rewritten as first-order action isofunctionals in isospace $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t})\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta}\hat{R})$ which are bilinear in the velocities,*

$$\hat{A} = \int_{t_1}^{t_2} dt\, \mathcal{L}(t, x, v, a, ...) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{\partial}\hat{t}\, \hat{L}(\hat{t}, \hat{x}, \hat{v}),$$

$$\hat{L} = \tfrac{1}{2}\hat{m}\, \hat{v}^i \delta_{ij}\, \hat{v}^j - \hat{U}^i(\hat{t}, \hat{x})\, \delta_{ij}\, v^j - U_o(\hat{t}, \hat{x}) = \tfrac{1}{2}\hat{m}\, \hat{v}_k\, \hat{v}^k - \hat{U}_k(\hat{t}, \hat{x})\, v^k - \hat{U}_o(\hat{t}, \hat{x}), \tag{2.39}$$

In fact, identities (2.39a) are overdetermined because, for each given $\mathcal{L}$, there exist infinitely many choices of $\hat{m}$, $\hat{T}_t$, $\hat{T}^j_i$, $\hat{U}_k$ and $\hat{U}_o$. We shall assume that integral terms are admitted in the integrand provided that they are all embedded in the isometric.

The *isovariational calculus* is a simple extension of the isodifferential calculus. In fact, we can write the following isovariation along an admissible isodifferentiable path $\hat{P}$

$$\delta \hat{A}(P) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \left(\delta\hat{x}^k \frac{\hat{\partial}}{\hat{\partial}\hat{x}^k} + \delta\hat{v}^k \frac{\hat{\partial}}{\hat{\partial}\hat{v}^k} \right) \hat{L}(P) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, \left\{ \frac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{x}^k} - \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\hat{L}}{\hat{\partial}\hat{v}^k} \right\} (P) \tag{2.40}$$

where we have used isointegration by parts. The isotopy of the celebrated Euler [10] necessary condition can be formulated as follows.

Theorem 2.5 (Isoeuler Necessary Condition): *A necessary condition for an isodifferentiable path $\hat{P}_o$ in isospace $\hat{S}(\hat{t},\hat{x},\hat{v}) = \hat{E}(\hat{t},\hat{R}_t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta},\hat{R})$ to be an extremal of the action isofunctional $\hat{A}$ is that all the following isotopic equations*

$$\hat{L}_k(\hat{P}_o) = \left\{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{v}^k} - \frac{\hat{\partial}\hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{x}^k} \right\} (\hat{P}_o) = 0, \tag{2.41}$$

are identically verified along $\hat{P}_o$.

It is an instructive exercise for the interested reader to prove the following:

Corollary 2.5.A: *Isoequations* (2.41) *are variationally isoself-adjoint.*

The isotopies of the remaining aspects of the calculus of variations (see, e.g., Bliss [6]) with consequential isotopies of the optimal control theory are intriguing and significant, but they cannot be studied here for brevity. Eq.s (2.41), which are introduced in this paper apparently for the first time, are hereon called *isoeuler equations* when dealing with the calculus of isovariations, and *isolagrange equations* when dealing with analytic mechanics.

We shall say that the isonewton equations (2.5) admit a *direct isoanalytic representation*, when there exists one isolagrangian $\hat{L}(\hat{t},\hat{x},\hat{v})$ under which all the following identities occur

$$\left\{ \frac{\hat{d}}{\hat{d}\hat{t}} \frac{\hat{\partial}\hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{v}^k} - \frac{\hat{\partial}\hat{L}(\hat{t},\hat{x},\hat{v})}{\hat{\partial}\hat{x}^k} \right\}^{ISOSA} =$$

$$= \left\{ \hat{m} \frac{\hat{d}v_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\hat{U}_k(\hat{t},\hat{x})}{\hat{\partial}\hat{x}^i} \frac{\hat{d}\hat{x}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_o(\hat{t},\hat{x})}{\hat{\partial}\hat{x}^k} \right\}^{ISOA} =$$

$$= \hat{T}_k^{\,i} \left\{ m \frac{dv_i}{dt} - \frac{\partial U_i(t,x)}{\partial x^s} \frac{dx^s}{dt} + \frac{\partial U_o(t,x)}{\partial x^i} - F^{NSA}{}_i(t,x,v) \right\}^{NSA} = 0$$

$$\hat{L}(\hat{t},\hat{x},\hat{v}) = \tfrac{1}{2}\hat{m}\hat{v}^k\hat{v}_k - \hat{U}, \quad \hat{U}(\hat{t},\hat{x},\hat{v}) = \hat{U}_k(\hat{t},\hat{x})\hat{v}^k + \hat{U}_o(\hat{t},\hat{x}), \tag{2.42}$$

Theorem 2.6 (Universality of isolagrangian mechanics). *All possible sufficiently smooth and regular dynamical systems in a star-shaped neighborhood of a point of their variables always admit a direct isorepresentation via the isolagrange equations in isospace.*

Proof. The universality of the isorepresentation follows from the fact that conditions (2.8) always admit solution (2.10) in the unknown functions. **q.e.d.**

Note that Newtonian systems are usually referred to systems with local-differential forces depending ut most on velocities. Theorem 2.6 includes also non-Newtonian forces, e.g., when they are of integral type or acceleration-dependent. Discontinuous Newtonian forces, such as those of impulsive type, have been removed from the theorem because of lack of current knowledge on the topology of isospaces with discontinuous isounits (isospaces of Kadeisvili's Class V [15]), although such an extension is expected to exist, and its study is left to interested readers.

Note the simplicity of the construction of an isolagrangian representation as compared to the complexity of the construction of a conventional Lagrangian representation [29,30], when it exists.

We now introduce, apparently for the first time, the isotopies of the Legendre transform based on the isodifferential calculus (ref. [39] presents a different isotopies based on the isotopic degrees of freedom of the multiplication). For this purpose, we introduce the following isodifferentials in isospace $\hat{S}(\hat{t},\hat{x},\hat{p}) = \hat{E}(\hat{t})\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$

$$\hat{d}\hat{t} = \hat{1}_t\, dt, \quad \hat{d}\hat{x}^k = \hat{1}^k{}_i\, dx^i, \quad \hat{\partial}\hat{x}^i / \hat{\partial}\hat{x}^j = \hat{\delta}^i{}_j, \quad \text{etc.},$$

$$\hat{d}\hat{p}_k = \hat{T}_k{}^i\, d\hat{p}_i, \quad \hat{d}\hat{p}^k = \hat{1}^k{}_i\, d\hat{p}^i, \quad \hat{\partial}\hat{p}_i / \hat{\partial}\hat{p}_j = \hat{\delta}_i{}^j, \quad \text{etc.} \tag{2.43}$$

The total isounits and isotopic elements of the isospace $\hat{S}(\hat{t},\hat{x},\hat{p}) = \hat{E}(\hat{t},\hat{R}_t)\times\hat{E}(\hat{x},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$ are therefore given by

$$\hat{1}_2 = \hat{1}_t \times \hat{1} \times \hat{T}, \quad \hat{T}_2 = \hat{T}_t \times \hat{T} \times \hat{1} \tag{2.44}$$

It should be indicated that, in view of the independence of the variables $\hat{p}_k$ from $\hat{x}^k$, we can introduce a new isounit $\hat{W} = \hat{Z}^{-1}$ for the isospace $\hat{E}(\hat{p},\hat{\delta},\hat{R})$ which is different than the unit $\hat{1} = \hat{T}^{-1}$ of isospace $\hat{E}(\hat{x},\hat{\delta},\hat{R})$, in which case the total unit is $\hat{1}_2$

$= \hat{1}_t \times \hat{1} \times \hat{W}$. Selection (2.44) is based on the simplest possible case $\hat{W} \equiv \hat{1}$ which is recommendable from the geometric isotopies studied in the next section. Other alternatives belong the the problem of the degrees of freedom of the isotopic theories which is not studied in this paper for brevity.

We now introduce the *isocanonical momentum* via the following prescriptions

$$\hat{p}_k = \frac{\partial \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{v}^k} = \hat{m}\, \hat{v}_k - \hat{U}_k(\hat{t}, \hat{x}), \qquad (2.45)$$

under the condition of being regular in a (2n+1)–dimensional region $\hat{\mathfrak{R}}$ of points $(\hat{t}, \hat{x}, \hat{p})$

$$\text{Det.}\left(\frac{\partial^2 \hat{L}(\hat{t}, \hat{x}, \hat{v})}{\partial \hat{v}^i \partial \hat{v}^j} \right) (\hat{\mathfrak{R}}) \neq 0 . \qquad (2.46)$$

thus admitting a unique set of implicit isofunctions $\hat{v}^k = \hat{f}^k(\hat{t}, \hat{x}, \hat{p})$. The *isolegendre transform* can then be defined by

$$\hat{L}(\hat{t}, \hat{x}, \hat{v}(\hat{t}, \hat{x}, \hat{p})) = \hat{p}_k\, \hat{v}^k(\hat{t}, \hat{x}, \hat{p}) - \tfrac{1}{2}\hat{m}\, \hat{v}_i(\hat{t}, \hat{x}, \hat{p})\, \hat{v}^i(\hat{t}, \hat{x}, \hat{p}) +$$

$$+ \hat{U}_k(\hat{t}, \hat{x})\, \hat{v}^k(\hat{t}, \hat{x}, \hat{p}) + \hat{U}_o(\hat{t}, \hat{x}) = \hat{p}_k\, \hat{p}^k / 2\hat{m} + \hat{V}^k(\hat{t}, \hat{x})\, \hat{p}_k + \hat{V}^o(\hat{t}, \hat{x}) = \hat{H}(\hat{t}, \hat{x}, \hat{p}) . \qquad (2.47)$$

We are now equipped to study the isotopies of Hamilton's principle [12]. By using the unified variables $\hat{b} = \{ \hat{b}^\mu \} = \{ \hat{x}^k, \hat{p}_k \}$, $\hat{c}^\mu = \hat{\partial}\, \hat{b}^\mu / \hat{\partial} \hat{t}$, and by introducing the notation

$$\hat{R}^\circ = \{ \hat{R}^\circ_\mu \} = \{ \hat{p}_k, \hat{0} \}, \qquad \mu = 1, 2, ..., 2n, \qquad k = 1, 2, ..., n, \qquad (2.48)$$

the *isocanonical principle* assumes the form along an actual path $\hat{P}_o$

$$\delta \hat{A} = \delta \int_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\hat{p}_k\, \hat{d}\hat{x}^k / \hat{d}\hat{t} - \hat{H})\, (\hat{P}_o) = \delta \int_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\hat{R}^\circ_\mu\, \hat{c}^\mu - \hat{H})\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\delta\hat{p}_i \frac{\partial}{\partial \hat{p}_i} + \delta\hat{v}^i \frac{\partial}{\partial \hat{v}^i} + \delta\hat{x}^i \frac{\partial}{\partial \hat{x}^i})\, (p_k\, v^k - H)\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, [\, (\frac{\hat{d}\hat{x}^k}{\hat{d}\hat{t}} \frac{\partial \hat{p}_k}{\partial \hat{p}_i} - \frac{\partial \hat{H}}{\partial \hat{p}_i})\, \delta\hat{p}_i - [\, \frac{\hat{d}}{\hat{d}\hat{t}} (\hat{p}_k \frac{\partial \hat{v}^k}{\partial \hat{v}^i}) + \frac{\partial \hat{H}}{\partial \hat{x}^i})\, \delta\hat{x}^i\,]\, (\hat{P}_o) =$$

$$\hat{=} \int_{\hat{t}_1}^{\hat{t}_2} \hat{d}\hat{t}\, (\delta\hat{b}^\mu \frac{\partial}{\partial \hat{b}^\mu} + \delta\, \hat{c}^\mu \frac{\partial}{\partial \hat{c}^\mu})\, (\hat{R}^\circ_\mu\, \hat{d}\hat{b}^\mu - \hat{H}\, \hat{d}\hat{t})\, (\hat{P}_o) =$$

$$\hat{=} \int_{\hat{t}_1}^{\hat{t}_2} \{ (\frac{\partial \hat{R}^\circ_\nu}{\partial \hat{b}^\mu} - \frac{\partial \hat{R}^\circ_\mu}{\partial \hat{b}^\nu}) \frac{\hat{d}\hat{b}^\nu}{\hat{d}\hat{t}} - \frac{\partial \hat{H}}{\partial \hat{b}^\mu} \}\, (\hat{P}_o) * \delta\hat{b}^\mu = 0, \qquad (2.49)$$

Theorem 2.7 (Isohamilton Necessary Condition): *A necessary condition for an isofunctional in isocanonical form whose integrand is sufficiently smooth and regular in a region $\hat{\mathfrak{R}}$ of points $(\hat{t}, \hat{b}, \hat{c})$ to have an extremum along a path $\hat{P}_o$ is that all the following isoequations in disjoint notation*

$$\frac{\hat{d}\hat{x}^k}{\hat{d}\hat{t}} = \frac{\partial \hat{H}(\hat{t}, \hat{x}, \hat{p})}{\partial \hat{p}_k}, \qquad \frac{\hat{d}\hat{p}_k}{\hat{d}\hat{t}} = - \frac{\partial \hat{H}(\hat{t}, \hat{x}, \hat{p})}{\partial \hat{x}^k}, \qquad (2.50)$$

or in unified notation

$$\left(\frac{\partial \hat{R}^\circ_\nu}{\partial \hat{b}^\mu} - \frac{\partial \hat{R}^\circ_\mu}{\partial \hat{b}^\nu} \right) \frac{\hat{d}\hat{b}^\nu}{\hat{d}\hat{t}} - \frac{\partial \hat{H}(\hat{t}, \hat{b})}{\partial \hat{b}^\mu} = 0, \qquad (2.51)$$

hold along an actual path $\hat{P}_o$.

It is also instructive for the interested reader to prove the following:

Corollary 2.7.A: *Isotopic equations (2.51) are variationally isoself–adjoint.*

Eq.s (2.50) or (2.51), which are introduced in this paper apparently for the first time, are called *isohamilton equations.* They can be more simply written in the following respective covariant and contravariant forms

$$\omega_{\mu\nu}\,\frac{\hat{\partial}\,\hat{b}^{\nu}}{\hat{\partial}\hat{t}} = \frac{\partial\,\hat{H}(\hat{t},\hat{b})}{\partial\hat{b}^{\mu}},$$

$$\frac{\hat{\partial}\,\hat{b}^{\mu}}{\hat{\partial}\hat{t}} = \omega^{\mu\nu}\,\frac{\partial\,\hat{H}(\hat{t},\hat{b})}{\partial\hat{b}^{\nu}}, \tag{2.52}$$

where the quantities

$$(\omega_{\mu\nu}) = \left(\frac{\partial\hat{R}^{\circ}_{\nu}}{\partial\hat{b}^{\mu}} - \frac{\partial\hat{R}^{\circ}_{\mu}}{\partial\hat{b}^{\nu}}\right) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N}\end{pmatrix},$$

$$(\omega^{\alpha\beta}) = \left(\frac{\partial\hat{R}^{\circ}_{\nu}}{\partial\hat{b}^{\mu}} - \frac{\partial\hat{R}^{\circ}_{\mu}}{\partial\hat{b}^{\nu}}\right)^{-1} = \begin{pmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N}\end{pmatrix}, \tag{2.53}$$

are the *conventional* covariant and contravariant canonical tensors, respectively, which hold in view of the identities originating from properties (1.33) and values (2.48)

$$\partial\hat{R}^{\circ}_{\nu}\,/\,\partial\hat{b}^{\mu} \equiv \partial R^{\circ}_{\nu}\,/\,\partial b^{\mu}. \tag{2.54}$$

The equivalence of the isolagrangian and isohamiltonian equations under the assumed regularity and invertibility of the isolegendre transform can be proved as in the conventional case (see, e.g., [29], Sect. 3.8).

We now study the following additional property of isohamiltonian mechanics which is important for operator maps. The *isotopic Hamilton–Jacobi problem* (see, e.g., [29], p. 201 and ff. for the conventional case) is the identification of an isocanonical transform under which the Hamiltonian becomes null. The generating function of such a transform is the isocanonical action itself, resulting in the end-point contributions

$$\partial\hat{A} = \partial\int_{t_o}^{t}(\hat{p}_k\,\hat{\partial}\hat{x}^k - \hat{H}\,\hat{\partial}\hat{t}) = \left|\hat{p}_k\,d\hat{x}^k - \hat{H}\,\hat{\partial}\hat{t}\right|_{t_o}^{t}, \tag{2.55}$$

with *isotopic Hamilton–Jacobi equations*

$$\frac{\partial\hat{A}}{\partial\hat{t}} + \hat{H}(t,\hat{x},\hat{p}) = 0, \qquad \frac{\partial\hat{A}}{\partial\hat{x}^k} - \hat{p}_k = 0. \tag{2.56}$$

plus initial conditions $\partial\hat{A}/\partial\hat{x}^{\circ k} = \hat{p}_k^{\circ}$, where $\hat{x}^{\circ}$ and $\hat{p}^{\circ}$ are constants.

Note the *abstract identity between the conventional and isotopic mechanics.* Since the isounits are positive-definite, at the abstract level there is no distinction between dt and $\hat{\partial}\hat{t}$ or dx and $\hat{\partial}\hat{x}$, etc. The isolagrange and isohamilton equations therefore coincide at the abstract level with the conventional equations. This illustrates the axiom-preserving character of the isotopies.

Note the *direct universality* of the isohamiltonian mechanics in the fixed inertial frame of the observer should be compared with the corresponding *lack* of universality of the conventional Hamiltonian mechanics. A first direct universality was achieved by this author [30] via a step-by-step generalization of Hamiltonian mechanics called (for certain historical reasons) *Birkhoffian mechanics.* The latter mechanics is based on the most general possible *first-order* Pfaffian variational principle (4.11) in the unified variables $b = (b^{\mu}) = (x^k, p_k)$ in a conventional space S(t, x, p), i.e.,

$$\delta\int_{t_1}^{t_2}[R_{\mu}(b)\,db^{\mu} - H(t,b)\,dt\,] = 0, \tag{2.57}$$

yielding *Birkhoff's equations* [5,30] in covariant form

$$\left\{\Omega_{\mu\nu}(b)\frac{\partial\,b^{\nu}}{dt} - \frac{\partial\,H(t,b)}{\partial\,b^{\mu}}\right\}^{SA} = 0, \qquad \Omega_{\mu\nu}(b) = \frac{\partial R_{\nu}}{\partial b^{\mu}} - \frac{\partial R_{\mu}}{\partial b^{\nu}}. \tag{2.58}$$

with contravariant version

$$\frac{db^{\mu}}{dt} = \Omega^{\mu\nu}(b)\,\frac{\partial\,H(t,b)}{\partial b^{\nu}}, \qquad \Omega^{\mu\nu} = [\,(\Omega_{\alpha\beta})^{-1}\,]^{\mu\nu}. \tag{2.59}$$

The connection between the Birkhoffian and the isohamiltonian mechanics is intriguing. In fact, the Pfaffian action can always be identically rewritten as the isotopic action

$$\int_{t_1}^{t_2}[R_{\mu}(b)\,db^{\mu} - H(t,b)\,dt\,] \equiv \int_{\hat{t}_1}^{\hat{t}_2}[\hat{R}^{\circ}_{\mu}(\hat{b})\,\hat{\partial}\hat{b}^{\mu} - \hat{H}(\hat{t},\hat{b})\,dt\,],$$

$$\hat{b}^{\mu} \equiv b^{\mu}\quad \hat{H} \equiv H,\ \hat{\partial}\hat{t} = dt, \tag{2.60}$$

and the general, totally antisymmetric Lie tensor $\Omega^{\mu\nu}$ (see later on) always admits the factorization into the canonical Lie tensor $\omega^{\mu\nu}$ and a regular symmetric matrix $\hat{T}_{\mu}{}^{\nu}$

$$\Omega^{\mu\nu} \equiv \omega^{\alpha\beta} \hat{T}_{\beta}{}^{\nu}, \tag{2.61}$$

under which Birkhoff's equations (2.59) coincide with the isohamilton's equations (2.52) for $\hat{T}_t = I$.

Despite these similarities, it should be indicated that the isohamiltonian mechanics is considerably broader than the Birkhoffian mechanics. In fact, the former is based on an action of arbitrary order, while the latter necessarily requires a first-order action. Also, the former can represent integral forces, while the latter cannot (because the underlying geometry (the symplectic geometry in its most general possible exact realization) only admits local-differential systems. Finally, the former is based on a broader mathematics, the isodifferential calculus, while the latter is based on conventional mathematics.

An important application of the isohamiltonian mechanics is to provide a novel classical realization of the Lie-Santilli isotheory [3,14,16,19,43]. Recall that the conventional classical realization of the Lie product is given by the familiar Poisson brackets among two functions A(b) and B(b) in the cotangent bundle (phase space)

$$[A, B]_{Poisson} = \frac{\partial A}{\partial x^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial x^k}\frac{\partial A}{\partial p_k} = \frac{\partial A}{\partial b^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial b^\nu}. \tag{2.62}$$

From the selfadjointness of Birkhoff's equations [30] and the algebraic meaning of the conditions of selfadjointness recalled in Sect. 2.3, the most general possible (regular, unconstrained) brackets in cotangent bundle verifying the Lie algebra axioms are given by the *Birkhoffian brackets* (also called generalized Poisson brackets) [30]

$$[A, B]_{Birkhoff} = \frac{\partial A}{\partial b^\mu}\,\Omega^{\mu\nu}(b)\,\frac{\partial B}{\partial b^\nu}. \tag{2.63}$$

The novel brackets introduced in this paper are given by the following brackets among isofunctions $\hat{A}(\hat{b})$, $\hat{B}(\hat{b})$ on isocotangent bundle

$$[\hat{A}, \hat{B}]_{Isotopic} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{x}^k}\frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{x}^k}\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_k} =$$

$$= \frac{\partial A}{\partial x^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial x^k}\frac{\partial A}{\partial p_k}, \tag{2.64}$$

and they *formally coincide* with the conventional brackets (2.62). This illustrates Bruck's [7] statement to the effect that "the isotopies are so natural to creep in unnoticed".

However, one should remember that the underlying geometry is generalized. In fact, the isotopic brackets can be written

$$[A, B]_{Isotopic} = \frac{\partial A}{\partial \hat{x}_i}\hat{T}_i{}^k(t, r, p, \ldots)\,\delta_{kj}\,\frac{\partial B}{\partial \hat{p}_j} - \frac{\partial B}{\partial \hat{x}_i}\hat{T}_i{}^k(t, r, p, \ldots)\,\delta_{kj}\,\frac{\partial A}{\partial \hat{p}_j} \tag{2.65}$$

and, in the latter form, *they do not verify the Lie axioms*, thus showing their differences with the conventional brackets. This illustrates that *the isotopic theory of this paper verifies the Lie axioms only in isospace but not when projected in conventional spaces.* This occurrence should be compared to other realizations studied in ref.s [37,38] in which the Lie axioms are verified in isospace as well as in their projection in conventional spaces.

Moreover, one should keep in mind that we have selected the simplest possible isotopies for which the isounits of the independent variables $\hat{p}^k$ and x^k are the same. The use of different isounits for $\hat{p}^k$ and x^k evidently implies further differences between the isotopic and conventional brackets.

Note finally that the Lie-Santilli character of brackets (2.64) is assured by the iso-selfadjointness of the isohamilton equations.

Brackets (2.64) can be written in unified notation

$$[A, B]_{Isotopic} = \frac{\partial A}{\partial \hat{b}^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial \hat{b}^\nu} = \frac{\partial A}{\partial \hat{b}^\alpha}\,\hat{T}_\mu{}^\alpha\,\omega^{\mu\nu}\,\hat{T}_\nu{}^\beta\,\frac{\partial B}{\partial \hat{b}^\beta} = \frac{\partial A}{\partial \hat{b}^\alpha}\,\omega^{\alpha\beta}\frac{\partial B}{\partial \hat{b}^\beta} \tag{2.66}$$

where the last identity occurs in view of the properties for diagonal isounits

$$\hat{T}_\mu{}^\alpha\,\omega^{\mu\nu}\,\hat{T}_\nu{}^\beta = \omega^{\alpha\beta}. \tag{2.67}$$

It is also possible to show that isohamiltonian mechanics provides a classical realization of the Lie-Santilli isogroups [3,14,16,19,43], as shown by Kadeisvili in the adjoining paper [16].

We now introduce, apparently for the first time, the *isodual Hamilton equation*

$$\omega^d_{\mu\nu}\,\frac{d^d\, b^{\nu d}}{d^d t^d} = \frac{\partial^d\, H^d(t^d, b^d)}{\partial^d\, b^{\mu d}}, \tag{2.68}$$

which represent the *isodual Newton equations* for antiparticles with potential forces only, Eq.s (2.23).

It is an instructive exercise to prove that *the Hamiltonian changes sign under isoduality,* as necessary for compatibility, because all physical characteristics of antimatter in isodual representation must change sign, thus including the energy, $H^d = p^d \times^d p^d /^d 2^d + V^d(x^d) = - H$. It is also instructive to prove that *the isodual Hamilton equations change sign under isoduality,* as it is the case for Eq.s (2.23).

The *isodual isohamilton equations* can be constructed via a simple isoduality of Eq.s (3.51) here left to the interested reader for brevity. Note that the isohamilton equations have the same properties under isoduality of the conventional equations.

The *isodual Lagrange and isolagrange equations* can also be constructed via the same isoduality and their study is also omitted for brevity. In short, *all aspects of conventional and isotopic analytic mechanics admit a simple, yet significant antiautomorphic image for the characterization of antimatter which therefore acquires a full methodological equivalence with the treatment of matter.*

The isolagrange and isohamiltonian mechanics of this section admit genotopic and hyperstructural formulations, each of which with an antiautomorphic image via isoduality. These latter formulations then provide a direct isoanalytic representation of the corresponding iso-, geno- and hyper-newton equations of Sect. 2.2 and of their isoduals. The latter analytic representations can be constructed via the methods of this paper and their explicit forms is not presented for brevity (see [38] for detailed study of the genoanalytic equations).

2.5. Isotopies of quantum mechanics. A significance of isohamiltonian mechanics is that its map under the conventional (or symplectic) quantization *is not* quantum mechanics, but instead a broader theory originally submitted by the author under the name of *hadronic mechanics* because intended for the study of hadrons and their strong interactions, but also known as *isoquantum theory.* It is important for this paper to see that the isotopic operator theory preserves all the main features of the isotopic *Newton* equations, such as the representation of extended, nonspherical and deformable shapes, nonselfadjoint forces and nonlocal-integral/nonhamiltonian interactions. By comparison, conventional quantum mechanics can only represent in first quantization point-particles with local and potential interactions and can represent in second quantization via form factors only perfectly spherical and rigid shapes. The broadening of quantum mechanics

under isotopies is then evident. For a comprehensive presentation see monographs [37,38]. A summary of hadronic mechanics with application to superconductivity is available in paper [2].

The simplest possible map of Hamiltonian into quantum mechanics, called *naive quantization,* is characterized by the map of the canonical action functional $A \rightarrow - i\hbar \mathrm{Ln}\psi(t, x)$, which maps the conventional Hamilton-Jacobi equations into the Schrödinger equations.

But the isoaction of the isohamiltonian mechanics is of arbitrary order in conventional space and the preceding map is not therefore applicable. Animalu and Santilli (see [38] for details and references) have therefore introduced the map (called *naive isoquantization*)

$$\hat{A}(\hat{t}, \hat{x}) \quad \rightarrow \quad - i\, \hat{1}(\hat{t}, \hat{p})\, \mathrm{Ln}\, \hat{\psi}(\hat{t}, \hat{x}) , \tag{2.69}$$

which is essentially based on the lifting of the basic unit of quantum mechanics $\hbar = 1$ into the isounit $\hat{1}$ of the isonewtonian and isoanalytic mechanics, here assumed to be independence from $\hat{x}$ for simplicity (see ref. [38] for the general case).

The application of map (2.69) to iso-Hamilton/Jacobi equations (2.56) yields the *isotopic Schrödinger equations*

$$i\, \hat{\partial} \hat{\psi} / \hat{\partial} \hat{t} = \hat{H} \hat{T} \hat{\psi} = \hat{H} * \hat{\psi} , \quad \hat{p}_k \hat{T} \hat{\psi} = \hat{p}_k * \hat{\psi} = - i\, \hat{\partial} \hat{\psi} / \hat{\partial} \hat{x}^k . \tag{2.70}$$

which are defined on a *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $\hat{\psi}$, $\hat{\phi}$, ..., and *isoinner product* $< \hat{\psi} \,\hat{|}\, \hat{\phi} > = \hat{1} \int d\hat{x}^3 \hat{\psi}^\dagger \hat{T} \hat{\phi}$ over the isocomplex field $\hat{C}(\hat{c},+,\hat{\times})$ originally submitted by Myung and Santilli in 1982 [23] (see [38] for a detailed study and all related references).

The equivalent *isoheisenberg equation* for an observable $\hat{O}$ are given by

$$i\, \hat{\partial} \hat{O} / \hat{\partial} \hat{t} = [\hat{O} \hat{,} \hat{H}] = \hat{O} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{O} = \hat{O} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \hat{\times} \hat{O} , \tag{2.71}$$

and results to be defined on an enveloping algebra $\hat{\xi}$ of operators $\hat{A}$, $\hat{B}$, ..., and isounit $\hat{1} = \hat{T}^{-1}$ on $\hat{\mathcal{H}}$ equipped with the isoassociative product $\hat{A} \hat{\times} \hat{B} = \hat{A} \times \hat{T} \times \hat{B}$ over $\hat{C}(\hat{c},+,\hat{\times})$ originally submitted by Santilli in 1978 (see [38] for details and references). The operator image of the isobrackets (5.27) is therefore given by

$$[\hat{A} \hat{,} \hat{B}] = \hat{A} \times \hat{T} \times \hat{B} - \hat{B} \times \hat{T} \times \hat{A} , \tag{2.72}$$

which constitute the operator realization of the Lie-Santilli isoproduct.

The exponentiated form of Eq.s (2.72) yields the time evolution of isostates

$$\hat{\psi}' = \hat{U} \hat{\times} \hat{\psi} = (\hat{e}^{i \hat{H} t}) \hat{\times} \hat{\psi} = e^{i H T t}) \times \hat{\psi}, \qquad (2.73)$$

where $\hat{e}^{\alpha}$ is the *isoexponentiation* of an arbitrary (well behaved) quantity α, i.e., the exponentiation in $\hat{\xi}$ turned into a universal enveloping isoassociative algebra via the isotopic Poincaré–Birkhoff–Witt theorem first formulated in [28], and then studied in [30]

$$\hat{e}^{\alpha} = \hat{1} + \alpha / 1! + \alpha \hat{\times} \alpha / 2! + \ldots = (e^{\alpha T}) \hat{1}. \qquad (2.74)$$

The notion of unitarity is preserved under isotopies and expressed by the broader *isounitary transforms*, i.e., transforms verifying the rules on $\hat{\mathcal{H}}$

$$\hat{U} \hat{\times} \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\times} \hat{U} = \hat{1}. \qquad (2.75)$$

The important point is that unitarity and isounitarity coincide at the abstract level for Class I isotopies.

The isotopic group laws [28] can be written for isounitary transforms on an isoparameters $\hat{w} \in \hat{R}(\hat{n},+,\hat{\times})$

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) \hat{\times} \hat{U}(-\hat{w}) = \hat{U}(\hat{0}) = \hat{1}. \qquad (2.76)$$

and they also coincide with the conventional group laws at the abstract level.

Most importantly, *the condition of isohermiticity, i.e., hermiticity in isohilbert space, coincide with the conventional Hermiticity,* $H^{\hat{\dagger}} \equiv H$ (see [38] for detailed prof). This carries the importance consequence that *all observables of quantum mechanics, such as energy, linear momentum, angular momentum etc. remain observables under isotopies.*

Despite the above equivalences, quantum and hadronic mechanics are inequivalent because the latter can only be derived from the former via a *nonunitary transform.* In fact, the unit, associative product and Lie product are notoriously invariant under unitary transforms $U \times U^{\dagger} = U^{\dagger} \times U = I$. In order to reach their isotopic covering, one need the nonunitary transform

$$U \times U^{T} = \hat{1} = \hat{1}^{\dagger} \neq I, \quad \hat{T} = (U \times U^{\dagger})^{-1} = \hat{T}^{\dagger},$$

$$U \times I \times U^{\dagger} = \hat{1}, \qquad U \times A \times B \times U^{\dagger} = A' \times \hat{T} \times B$$

$$U \times (A \times B - B \times A) \times U^{\dagger} = A' \times \hat{T} \times B' - B' \times \hat{T} \times A'$$

$$A' = U \times A \times U^{\dagger}, \; B' = U \times B \times U^{\dagger} \qquad (2.77)$$

Once an isotopic structure is achieved via nonunitary transforms, *it does not* remain invariant under additional nonunitary transforms $W \times W^{\dagger} = K \neq I$, e.g., because the isounit is not invariant, $W \times \hat{1} \times W^{\dagger} = \hat{1}' \neq \hat{1}$. However, the latter transform can always be written in the isounitary form

$$W = \hat{W} \times \hat{T}^{1/2}, \qquad W \times W^{\dagger} = \hat{W} \hat{\times} \hat{W}^{\dagger} = W^{\dagger} \times W = \hat{W}^{\dagger} \hat{\times} \hat{W} = \hat{1}, \qquad (2.78)$$

which establishes the form-invariance of hadronic mechanics under its own transformation theory, that of isounitary transforms,

$$\hat{W} \times \hat{1} \hat{\times} \hat{W}^{\dagger} \equiv \hat{1}, \quad \hat{W} \hat{\times} \hat{A} \times \hat{B} \hat{\times} \hat{W}^{\dagger} = \hat{A}' \hat{\times} \hat{B}',$$

$$\hat{W} \hat{\times} (\hat{A} \hat{\times} \hat{B} - \hat{B} \hat{\times} \hat{A}) \hat{\times} \hat{W}^{\dagger} = \hat{A}' \hat{\times} \hat{B}' - \hat{B}' \hat{\times} \hat{A}',$$

$$\hat{A}' = \hat{W} \hat{\times} \hat{A} \hat{\times} \hat{W}^{\dagger}, \; \hat{B}' = \hat{W} \hat{\times} \hat{B} \hat{\times} \hat{W}^{\dagger}. \qquad (2.79)$$

Note that the isounit and isotopic element are left *numerically invariant,* and they preserve all properties of Planck's constant,

$$\hat{1}^{\hat{n}} = \hat{1} \hat{\times} \hat{1} \hat{\times} \ldots \hat{\times} \hat{1} \equiv \hat{1}, \quad \hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \quad \hat{1} \hat{/} \hat{1} \equiv \hat{1},$$

$$i \, \partial \hat{1} / \partial \hat{T} = \hat{1} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{1} = \hat{H} - \hat{H} \equiv 0. \qquad (2.80)$$

As one can see, the matrix $\hat{T}$ of the isotopic Newton equations is preserved in its entirety at the operator level, and this confirms the capability of the isotopic operator theory of representing nonspherical-deformable shapes, nonselfadjoint forces and nonlocal-integral interactions (see [38] for comprehensive studies).

The significance of the Lie-Santilli isotheory over the conventional formulation is illustrated by the appearance of the matrix $\hat{T}$ with arbitrary nonlinear-integral terms in the *exponent* of the isogroup, Eq. (2.73). This assures that the original linear, local and canonical theory is mapped under isotopies into nonlinear, nonlocal and noncanonical forms, as desired (see the following article by Kadeisvili [16] for details).

The abstract identity of quantum and hadronic mechanics should be kept in mind because it assures the *axiomatic consistence of hadronic mechanics.* Criticisms on hadronic mechanics may therefore result to be criticisms on the axiomatic structure of quantum mechanics itself. In fact, all properties of the latter are equally shared by the former, *both mechanics being different realizations of the same abstract axioms.*

By comparison, the reader should be aware of physical problematic aspects of other modifications of quantum mechanics, such as the so-called q-deformations, which do not possess an invariant unit (and, therefore, cannot be applied to actual experiments), Hermiticity is not generally preserved in time (thus preventing the existence of observables), the q-special functions are not invariant under the time evolution of the theory (thus preventing invariant data elaboration() and other problems (see [38] for details. All these problems are resolved by quantum mechanics, by therefore indicating that isotopies are preferable on methodological grounds over other generalizations.

Conventional quantum mechanics admits an antiautomorphic image under isoduality called *isodual quantum mechanics* [38], which is used for the representation of antiparticles and is characterized by: 1) isodual complex fields $C^d(c^d,+,\times^d)$, with isounit $\hbar^d = -\hbar = -1$, elements $c^d = -\bar{c}$. and product $\times^d = -\times$ (See Sect. 1.3 for more details); 2) Isodual enveloping operator algebra ξ^d with unit $\hbar^d = -\hbar = -1$ and isodual products $A^d \times^d B^d = -A\times B$, and 3) the *isodual Hilbert space* with isodual states $|\psi>^d = -<\psi|$ (or $\psi^d = -\psi^\dagger$) and *isodual inner product*

$$\mathcal{H}^d: \quad <\phi|\psi>^d = <\phi|\times(-\hbar)\times|\psi>\times(-\hbar)^{-1} \in C^d, \quad \hbar = 1. \qquad (2.81)$$

The dynamical equations are derived via *isodual naive quantization* which consists of the mapping of the isodual canonical action $A^d \to -i\hbar^d Ln^d \psi^d(t^d, x^d)$, where one should remember the isoselfduality of the imaginary unit ($i^d = -\bar{i} \equiv i$), under which the isodual Hamiltonian mechanics of Sect. 2.4 is mapped into the *isodual Schrödinger equation*

$$i\,\partial^d_t \psi^d = H^d \times^d \psi^d = E^d \times^d \psi^d, \qquad (2.82)$$

thus resulting in *negative eigenvalues* $E^d = -E$, as requested for consistency under duality as well as for compatibility with the classical counterpart (where the Hamiltonian is negative-definite).

The *isodual Heisenberg equation* is given by [38]

$$i\, d^d O^d / d^d t^t = O^d \times^d H^d - H^d \times^d O^d, \qquad (2.$$

The rest of the isodual mechanics can be derived accordingly (see [38] for details).

The mapping from quantum mechanics to its isodual has resulted to be equivalent to charge conjugation (see [38] for details). However, isoduality appears to be preferable over charge conjugation because it avoids the need for the "hole theory" and the uneasiness of its underlying assumptions. In short, antiparticles were discovered in the negative-energy solutions of Dirac's and other equations, but they behaved unphysically (when tacitly referred to our positive units of space and time), thus requesting the construction of the "hole theory" in second quantization. Santilli [38] has shown that *negative-energy solutions behave in a fully physical way when referred to negative units of space and time,* thus eliminating the need to conjecture the existence of many different infinite oceans of antiparticles, one per each antiparticle and each ocean inside all others, whose "holes" are the antiparticles we observe physically. As shown in this section, antiparticle acquire their identity under isoduality *beginning at the Newtonian level,* which identification then persists at all subsequent levels. Note that for charge conjugation antiparticles remain in our space-time, while under isoduality they exist in the *different* isodual space-time.

Note finally *the identity of the conventional and isodual inner products of the Hilbert space,* which may explain why isoduality has not been discovered until recently despite about one century of investigations. Note however that the identification of isoduality requires the prior knowledge of *new numbers,* those with negative units.

Isodual quantum mechanics admits simple isotopies into the *isodual hadronic mechanics* which is here omitted for brevity. Hadronic mechanics also admits *forward and backward genotopic formulations* [38] as well as *forward and backward hyperformulations* [40] each with its isodual, whose explicit form is not presented here for brevity.

Hadronic mechanics possesses nowadays applications and experimental verifications in nuclear physics, particle physics, astrophysics, superconductivity and other fields, which we cannot review here for brevity [37,38]. We merely mention that in all cases the isounit is assumed to be different than I only at very small distances (of the order of 1 fm) where nonlinear, nonlocal and noncanonical effects are expected to be significant. Therefore, at the scale, say, of the atomic structure, $\hat{1} \equiv I$ and hadronic mechanics recovers quantum mechanics identically.

3. Isotopic liftings of local-differential geometries

3.1. Introduction In Sect. 2 we have studied from an analytic profile the historical distinction between the *exterior problem*, consisting of point particles moving in the *homogeneous and isotropic vacuum*, and the *interior problem*, consisting of extended, nonspherical and deformable particles moving within *inhomogeneous and anisotropic physical media*.

On geometrical grounds, the above distinction was well known in the first part of this century. For instance, Schwarzschild wrote *two* papers, the celebrated paper [41] on the *exact* solution for the exterior problem, and the virtually unknown second paper [42] on an *approximate* solution for the interior problem. The same distinction was kept in the well written treatises in geometry of the first part of this century, but then it was progressively ignored up to the current virtual complete silence in the technical literature.

This is unfortunate for several reasons. To begin, *the exterior and interior problems are geometrically inequivalent*. For instance, it is known that the speed of light c_0 is constant in vacuum, while within physical media light has the locally varying speed $c = c_0/n$, where n is the familiar index of refraction. It is evident that geometries which have been built for the characterization of the *constant* speed of light c_0 are not effective for the characterization of locally *varying* speeds of light $c = c_0/n$, e.g., because of the need of different metrics.

The inequivalence of exterior and interior problems is confirmed by the fact that the equations of motion of the former are *variationally selfadjoint* [29], thus admitting a (first-order) Lagrangian or a Hamiltonian, while the equations of motion of the latter are *nonselfadjoint* [loc. cit.], thus being beyond the representational capabilities of a Lagrangian or a Hamiltonian. As an example, missiles in atmosphere have nowadays reached such speeds to experience drag forced proportional up to the *tenth power of the speed*. The expectation that such systems can be exactly represented via the Riemannian or other conventional geometries is outside the boundaries of science.

But the deepest inequivalence of the exterior and interior problems is of topological nature. In fact, the former admits an exact local-differential topology (such as the Zeeman topology for the Minkowski space), while the latter requires an integro-differential topology, as discussed in Sect.s 1 and 2. Thus, interior problems are outside any realistic capability of conventional geometries (for additional arguments on the inequivalence here considered see [38]).

Moreover, *the interior problem cannot be exactly reduced to a collection of elementary particles in exterior conditions in vacuum*, as generally assumed in the contemporary literature. Besides the lack at this writing of a consistent quantum gravity (which is evidently requested in the transition from the classical to the particle level), the expectation of recovering the exterior problem at the particle level has been disproved by the so-called *No-Reduction Theorems* [38], which essentially establish that, say, a space-ship during re-entry in atmosphere *with monotonically decreasing angular momentum*, simply cannot be reduced in a consistent way to a finite collection of point particles in vacuum, *each with conserved angular momentum* (for additional arguments on the irreducibility here considered, see [38]).

Even ignoring the preceding issues, *conventional local-differential geometries are usable in practical applications only for systems which are Hamiltonian in the frame of the observer* b = (x, p), which constitute a rather small class of physical systems. By ignoring nonlocal-integral terms and under suitable regularity and continuity conditions, *Darboux's theorem* [9] of the symplectic geometry does indeed permit the identification of new coordinates b'(b) = (x'(x, p), p'(x, p)) in which nonhamiltonian systems acquire a Hamiltonian form. However, as indicated in Sect. 2.4, the transformed coordinates are nonlinear functions of the original ones. As such, *Darboux's coordinates* b' = (x', p') *are not realizable in actual experiments*. Moreover, being nonlinear images of inertial frames, *Darboux's coordinates are highly noninertial*, thus implying the loss of contemporary relativities, such as Galilei's and the special relativities.

The above occurrences have created the problem of *constructing new geometries specifically built for the description of interior problems in the fixed inertial frame of the observer or, equivalently, for the geometrization of inhomogeneous and anisotropic physical media*. Moreover, to be effective in physical applications, the new geometries must satisfy certain physical requirements, such as admitting of the conventional geometries as particular cases, and permitting a clear separation between local-differential and nonlocal-integral effects (the latter being needed for experiments).

A number of new geometries resolving some of the preceding problems already exist in the literature. Without any claim of completeness, we here quote the *Finslerian geometry* [20] which is particular suited to represent anisotropic interior systems, such as a spinning body. Nevertheless, the geometries which are needed for interior problems must necessarily be both anisotropic and inhomogeneous, thus requiring a necessary broadening of the Finslerian geometry. At any rate, the latter is arbitrarily nonlinear only in the coordinates, thus being insufficient for interior problems.

We also mention the *higher-order Lagrangian formulation* by Miron and his collaborators (see, e.g., [22] and quoted references) which is arbitrarily nonlinear in the velocities and, thus, directly applicable for the representation of (local-differential) interior systems in the coordinates of the observer. However, higher order Lagrangian formulations do not have a conventional Hamiltonian counterpart which is needed for quantization. This illustrates the emphasis on the *first-order* character of the Lagrangian representation of Sect. 2

Similarly, we mention here the new *time-oriented Lorentzian geometry* by Papuc [25] which, as such, is also directly applicable for the geometrization of irreversible interior systems. The latter geometry is however arbitrarily nonlinear only in the coordinates besides being local-differential, thus creating the need for further generalizations.

A solution of the above problems was submitted by the author in memoirs [32,33] of 1988 (see monographs [37,38] for a recent presentation) via the *isogeometries* which were built via *the isotopic degrees of freedom of the product*. In this section we present, apparently for the first time, the isogeometries characterized by the *isodifferential calculus* of Sect. 1.5. The latter formulation results to be preferable over the former because it permit a much more transparent abstract unity of conventional and isotopic geometries which, in turn, permits a unified treatment of exterior and interior problems.

The *isoeuclidean and isominkowskian geometries* are studied in details in monograph [37]. Their reformulation in terms of the isodifferential calculus is elementary and therefore they will not be considered here for brevity.

In this section we shall first reformulate the *isosymplectic geometry* via the isodifferential calculus for the primary purpose of reaching an alternative to Darboux's theorem and, more generally, achieving the *direct universality* for interior systems, that is, the representation of all interior systems of the class admitted (universality), directly in the fixed inertial frame of the experimenter (direct universality). The reformulation also satisfies the above indicated conditions of admitting the conventional geometry as a particular case and possessing a clear separation between local-differential and nonlocal-integral terms. The isosymplectic geometry will emerge as the isogeometry underlying the *isohamiltonian mechanics* of Sect. 2 and of the *Lie-Santilli isotheory* of Sect. 1.9 and of the accompanying paper by Kadeisvili [16].

We shall then reformulate the *isoriemannian geometry* via the isodifferential calculus to achieve: a geometrization of locally varying speeds of light; a theory on the "origin" (rather than the "description") of the gravitational field; the "identification" (rather than the "unification") of the gravitational and electromagnetic field; the geodesic characterization of free motion within physical media (such as a leaf falling from the Pisa tower); and other objectives. The isominkowskian geometry is a particular case of the isoriemannian geometry at null curvature.

We shall then merely outline the isodual isogeometries for the antiautomorphic characterization of antimatter, for the purpose of confirming the expectation that *antimatter-antimatter systems* experience a gravitational *attraction* similar to that of *matter-matter systems*, although the theory predicts the *reversal of gravitational attraction (antigravity) for matter-antimatter systems.*

We shall finally point out the expectation that the isogeometries admit further broadening of genotopic and hyperstructural type, although without treatment at this time.

3.2. Isosymplectic geometry. The *isotopies of the symplectic geometry*, or *isosymplectic geometry* for short, were first identified by Santilli in memoir [32] of 1988 via the isotopic degrees of freedom of the product, they were then studied in various works, and presented in monograph [37]. In this section we shall present the isosymplectic geometry formulated via the isodifferential calculus of Sect. 1.5.

Unless otherwise stated, our formulation is local and in the fixed coordinates of the observer. All quantities are assumed to satisfy the needed continuity conditions, e.g., of being of class $\hat{C}^{\infty}$ and all neighborhoods of a point are assumed to be star-shaped or have an equivalent topology. For the conventional symplectic geometry one may inspect ref. [1] for the abstract treatment and ref. [20] for the local formulation. We shall first study the isosymplectic geometry of Class I representing matter and then study its antiautomorphic image under isoduality for the characterization of antimatter.

Let $\hat{M}(\hat{E}) = \hat{M}(\hat{E}(\hat{\delta},\hat{R})$ be an n-dimensional Tsagas-Sourlas isomanifold [44,45] on the isoeuclidean space $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isoreals $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ with n×n-dimensional isounit $\hat{1} = (\hat{1}^i{}_j)$, i, j = 1, 2, ..., n, of Kadeisvili Class I and local chart $\hat{x} = (\hat{x}^k)$. A *tangent isovector* $\hat{X}(\hat{m})$ at a point $\hat{m} \in \hat{M}(\hat{E})$ is an isofunction defined in the neighborhood $\hat{N}(\hat{m})$ of $\hat{m}$ with values in $\hat{R}$ satisfying the *isolinearity conditions*

$$\hat{X}_m(\hat{\alpha} \hat{\times} \hat{f} + \beta \hat{\times} g) = \hat{\alpha} \hat{\times} \hat{X}_{\hat{m}}(\hat{f}) + \beta \hat{\times} \hat{X}_{\hat{m}}(\hat{g}),$$

$$\hat{X}_{\hat{m}}(\hat{f} \hat{\times} \hat{g}) = \hat{f}(\hat{m}) \hat{\times} \hat{X}_{\hat{m}}(\hat{g}) + \hat{g}(\hat{m}) \hat{\times} \hat{X}_m(\hat{f}), \qquad (3.1)$$

for all $\hat{f}, \hat{g} \in \hat{M}(\hat{E})$ and $\hat{\alpha}, \hat{\beta} \in \hat{R}$, where $\hat{\times}$ is the isomultiplication in $\hat{R}$ and the use of the symbol ^ means that the quantities are defined on isospaces.

The collection of all tangent isovectors at $\hat{m}$ is called the *tangent isospace* and denoted $T\hat{M}(\hat{E})$. The *tangent isobundle* is the 2n-dimensional union of all possible tangent isospaces when equipped with an isotopic structure (see below).

The *cotangent isobundle* $T^*\hat{M}(\hat{E})$ is the 2n-dimensional dual of the tangent isobundle with local coordinates $\hat{b} = (\hat{b}^\mu) = (\hat{x}^k, \hat{p}_k)$, $\mu = 1, 2, ..., 2n$,. Since $\hat{p}$ is independent of $\hat{x}$, the isounits of the respective differentials are generally different, i.e., we can have $\hat{d}x = \hat{1}dx$ and $\hat{d}p = \hat{W}dp$, $\hat{1} \neq \hat{W}$, in which case the total isounit of $T^*\hat{M}(\hat{E})$ is the 2n-dimensional Cartesian product $\hat{1}_2 = \hat{1} \times \hat{W}$.

For reasons which will be clarified later on, in this note we assume the following particular form of the *isounit of the cotangent isobundle*

$$\hat{1}_2 = (\hat{1}_2{}^\mu{}_\nu) = \begin{pmatrix} \hat{1}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{T}_{n\times n} \end{pmatrix} = \hat{T}_2{}^{-1} = (\hat{T}_{2\mu}{}^\nu)^{-1} \qquad \hat{1} = \hat{T}^{-1}, \quad (3.2)$$

where $\hat{1}$ is the isounit of the coordinates $\hat{d}x = \hat{1}dx$, and $\hat{T}$ is the isounit of the momenta, $\hat{d}p = \hat{T}p = \hat{1}^{-1}dp$. In different terms, we select the particular case in which $\hat{W} = \hat{1}^{-1}$.

An *isobasis* of $T^*\hat{M}(\hat{E})$ is, up to equivalence, the (ordered) set of isoderivatives $\hat{\partial} = (\hat{\partial}/\hat{\partial}\hat{b}^\mu) = (\hat{T}_{2\mu}{}^\nu \partial/\partial b^\nu)$. A generic elements $\hat{X} \in T^*\hat{M}(\hat{E})$, called *vector isofield*, can then be written $\hat{X} = \hat{X}^\mu(\hat{m})\, \hat{\partial}/\hat{\partial}\hat{b}^\mu = \hat{X}^\mu\, \hat{T}_{2\mu}{}^\nu \partial/\partial b^\mu$.

The *fundamental one-isoform* on $T^*\hat{M}(\hat{E})$ is given in the local chart $\hat{b}$ by

$$\hat{\theta} = \hat{R}^\circ_\mu(\hat{b})\, \hat{d}\hat{b}^\mu = \hat{R}^\circ_\mu(b)\, \hat{1}_2{}^\mu{}_\nu\, d\hat{b}^\nu = \hat{p}_k\, \hat{d}\, \hat{x}^k = \hat{p}_k\, \hat{1}^k{}_i\, d\hat{x}^i, \quad \hat{R}^\circ = (\hat{p}, \hat{0}). \quad (3.3)$$

The above expression, which can be written $\hat{\theta} = p\hat{d}x = p_i \hat{1}^i{}_j dx^j$ to emphasize the differential origin of the isotopies, should be compared with the originally proposed one-isoform $\hat{\theta} = p\hat{\times}dx = p_k \hat{T}^k{}_i\, dx^i$ [13] obtained via the isotopic degrees of freedom of the product. The preference of the isodifferential calculus over the isomultiplication is then evident for a geometric unity of the conventional and isotopic formulations.

The space $T^*\hat{M}(\hat{E})$, when equipped with the above one-form, is an *isobundle* denoted $T_1{}^*\hat{M}(\hat{E})$. The *isoexact, nowhere degenerate, isocanonical isosymplectic two-isoform* is given by

$$\hat{\omega} = \hat{d}\,\hat{\theta} = \tfrac{1}{2}\, \hat{d}\, (\hat{R}^\circ_\mu\, \hat{d}\hat{b}^\mu) = \tfrac{1}{2}\, \omega_{\mu\nu}\, \hat{d}\hat{b}^\mu \wedge \hat{d}\hat{b}^\nu =$$

$$= \hat{d}\hat{x}^k \wedge \hat{d}\hat{p}_k = \hat{1}^k{}_i\, d\hat{x}^i \wedge \hat{T}_k{}^j\, d\hat{p}_j \equiv d\hat{x}^k \wedge d\hat{p}_k. \quad (3.4)$$

The isomanifold $T^*\hat{M}(\hat{E})$, when equipped with the above two-isoform, is called *isosymplectic isomanifold* in isocanonical realization and denoted $T_2{}^*\hat{M}(\hat{E})$. The *isosymplectic geometry* is the geometry of the isosymplectic isomanifolds.

The last identity in (3.4) show that *the two-isoform $\hat{\omega}$ formally coincides with the conventional symplectic canonical two-form* ω, and this illustrates the selection of isounit (3.2). The abstract identity of the symplectic and isosymplectic geometries is then evident. However, one should remember that: the underlying metric is isotopic; $\hat{p}_k = \hat{T}_k{}^i p_i$, where p_i is the variable of the conventional canonical realization of the symplectic geometry; and identity $\hat{\omega} \equiv \omega$ no longer holds for the more general isounits $\hat{1}_2 = \hat{1} \times \hat{W}$, $\hat{1} \neq \hat{W}^{-1}$.

Note that the *isosymplectic geometry has the Tsagas-Sourlas Integro-differential topology* and, as such, it can characterize interior systems when all nonlocal-integral terms are embedded in the isounit.

A *vector isofield* $\hat{X}(\hat{m})$ defined on the neighborhood $\hat{N}(\hat{m})$ of a point $\hat{m} \in T_2{}^*\hat{M}(\hat{E})$ with local coordinates $\hat{b}$ is called (locally) *isohamiltonian* when there exists an isofunction $\hat{H}$ on $\hat{N}(\hat{m})$ over $\hat{R}$ such that

$$\hat{X} \,\lrcorner\, \hat{\omega} = \hat{d}\,\hat{H}, \quad \text{i.e.,}$$

$$\omega_{\mu\nu}\, \hat{X}^\nu(\hat{m})\, \hat{d}\hat{b}^\mu = \hat{d}\hat{H}(\hat{m}) = (\hat{\partial}\hat{H} / \hat{\partial}\hat{b}^\mu)\, \hat{d}\hat{b}^\mu, \quad (3.5)$$

We are now equipped to present the main result of this paper, the isotopic alternative to Darboux's Theorem for the representation of nonlinear, nonlocal-integral and nonhamiltonian interior systems within the fixed coordinates of their experimental observation, which can be formulated as follows.

Theorem 3.1 (Direct Universality of the Isosymplectic Geometry for Interior Systems): *Under sufficient continuity and regularity conditions, all possible vector fields which are not (locally) Hamiltonian in the given coordinates are always isohamiltonian in the same coordinates, that is, there exists a neighborhood $N(\hat{m})$ of a point $\hat{m}$ of their variable $\hat{b} = (\hat{x}, \hat{p})$ under which Eq.s (3.5) hold.*

Proof. Let $\hat{X}^\mu(\hat{b})$ be a vector field which is nonhamiltonian in the chart $\hat{b}$, and consider the decomposition

$$\hat{X}(b) = \Gamma^\mu{}_\alpha(b)\, \hat{X}_o{}^\alpha(b), \quad (3.6)$$

where the 2n×2n matrix $(\hat{T}^{\mu}{}_{\alpha})$ is nowhere degenerate and $\hat{X}_o{}^{\alpha}$ is the maximal, local-differential and Hamiltonian sub-vector field, i.e., there exists a function H(b) and a neighborhood N(m) of a point m of b = (x, p) such that

$$\omega_{\alpha\beta}\, \hat{X}_o{}^{\beta}(m)\, db^{\alpha} = dH(m) = (\partial H / \partial b^{\alpha})\, \hat{d}\hat{b}^{\alpha}, \tag{3.7}$$

and all nonlocal-integral and nonhamiltonian terms are embedded in $\hat{T}$. Then, there always exists an isotopy such that

$$\omega_{\mu\nu}\, \hat{X}^{\nu}(\hat{m})\, \hat{d}\hat{b}^{\mu} = \omega_{\mu\alpha}\, \hat{T}^{\alpha}{}_{\beta}(\hat{m})\, \hat{X}_o{}^{\beta}(\hat{m})\, \hat{d}\hat{b}^{\mu} =$$

$$= \hat{d}\hat{H}(\hat{m}) = (\partial \hat{H} / \partial \hat{b}^{\mu})\, \hat{d}\hat{b}^{\mu} = \hat{T}_{\mu}{}^{\beta}\, \partial H / \partial b^{\beta})\, \hat{d}b^{\mu}. \tag{3.8}$$

In fact, the script $\hat{X}^{\mu}$ is only a unified formulation in 2n dimension of two separate terms each in n-dimension. Therefore, the quantity $\hat{T}$ has the structure

$$\hat{T} = \begin{pmatrix} \hat{A}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{B}_{n\times n} \end{pmatrix}. \tag{3.9}$$

The identification

$$\hat{1} = \begin{pmatrix} \hat{B}^{-1}{}_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \hat{A}^{-1}{}_{n\times n} \end{pmatrix}, \tag{3.10}$$

then implies

$$\hat{1}^{\mu}{}_{\alpha}\, \omega_{\mu\nu}\, \hat{T}^{\nu}{}_{\rho} \equiv \omega_{\alpha\rho}, \tag{3.11}$$

and identities (3.8) always exist. **q.e.d.**

Corollary 3.1.A: *For all Newtonian systems we have $\hat{A} = \hat{B}^{-1}$, i.e. , the 2n-dimensional isounit of the cotangent isobundle has structure* (3.2).

Proof. All Newtonian systems in the 2n-dimensional, first-order, vector field form can be written in disjoint n-component

$$\begin{pmatrix} dx/dt \\ dp/dt \end{pmatrix} = \begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \hat{X}(b) = (\hat{X}^{\mu}(b)) \tag{3.12}$$

where SA (NSA) stands for variational selfadjointness (nonselfadjointness), i.e., the integrability conditions for the existence (lack of existence) of a Hamiltonian. Thus $F^{SA} = -\partial H/\partial x$, with $H = p^2/2m + V(x)$, while there is no such Hamiltonian for F^{NSA}.

Then, isohamiltonian representation (3.8) explicitly reads

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p/m \\ F^{SA} + F^{NSA} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} p/m \\ F^{SA} \end{pmatrix} =$$

$$= \begin{pmatrix} -B\,F^{SA} \\ A\, p/m \end{pmatrix} = \begin{pmatrix} \partial H / \partial x \\ \partial H / \partial p \end{pmatrix} = \begin{pmatrix} B\, \partial H / \partial x \\ A\, \partial H / \partial p \end{pmatrix}. \tag{3.13}$$

From which we have the general solution

$$\hat{T} = B = 1 + F^{NSA}/F^{SA} = A^{-1} = \hat{1}^{-1}, \tag{3.14}$$

where the last identity follow from the fact that, since $\partial H/\partial p = p/m$, A remains arbitrary and can be therefore assumed to be $A = B^{-1}$. **q.e.d.**

The above results confirm, this time on independent geometric grounds, the corresponding results achieved in Sect. 2 on analytic grounds, thus confirming the overall unity of isotopic methods.

It is now important to verify that the above geometric isotopies do indeed preserve the remaining axiomatic properties of the symplectic geometry. For this it is sufficient to prove the preservation under isotopies of the Poincaré Lemma and of Darboux's Theorem [1,20,30].

To prove the preservation of the Poincaré Lemma one can easily construct isoforms $\hat{\Phi}_p$ of arbitrary order p. The proof of the following property is a simple isotopy of the conventional proof (see, e.g., [20]) via the use of the isodifferential calculus.

Lemma 3.1 (Isopoincaré Lemma): *Under the assumed smoothness and regularity conditions, isoexact p-isoforms are isoclosed, i.e.,*

$$\hat{d}\, \hat{\Phi}_p = \hat{d}\,(\hat{d}\hat{\phi}_{p-1}) \equiv 0. \tag{3.15}$$

The nontriviality of the above result is illustrated by the following

Corollary 3.1.A: *Isoexact p-isoform are not necessarily closed, i.e., their projection in the original tangent bundle does not necessarily verify the Poincaré Lemma.*

By comparison, we should mention that the original formulation of the isopoincaré lemma [32,37], that via the isotopic degrees of freedom of the product, did verify the Poincaré lemma in both the conventional and isotopic bundle.

To prove the preservation of the Darboux's Theorem [9], consider the *general one-isoform* in the local chart $\hat{b}$

$$\hat{\Theta}(\hat{b}) = \hat{R}_\mu(\hat{b})\,\hat{\partial}\hat{b}^\mu = \hat{R}_\mu(b)\,\hat{1}_2{}^\mu{}_\nu(t, b, db/dt, ...)\,db^\nu , \tag{3.16}$$

where

$$\hat{R} = \{ \hat{P}(\hat{x}, \hat{p}), \hat{Q}(\hat{x}, \hat{p}) \} . \tag{3.17}$$

The *general isosymplectic isoexact two-isoform* in the same chart is then given by

$$\hat{\Omega}(\hat{b}) = \tfrac{1}{2}\,\hat{\partial}\,(\hat{R}_\mu(\hat{b})\,\hat{\partial}\hat{b}^\nu) = \tfrac{1}{2}\,\hat{\Omega}_{\mu\nu}(\hat{t}, \hat{b}, \hat{\partial}\hat{b}/\hat{\partial}\hat{t}, ...)\,\hat{\partial}\hat{b}^\mu \wedge \hat{\partial}\hat{b}^\nu ,$$

$$\hat{\Omega}_{\mu\nu} = \frac{\hat{\partial}\hat{R}_\nu}{\hat{\partial}\hat{b}^\mu} - \frac{\hat{\partial}\hat{R}_\mu}{\hat{\partial}\hat{b}^\nu} = \hat{T}_{2\mu}{}^\alpha\,\frac{\partial\hat{R}_\nu}{\partial\hat{b}^\alpha} - \hat{T}_{2\nu}{}^\alpha\,\frac{\partial\hat{R}_\mu}{\partial\hat{b}^\alpha}. \tag{3.18}$$

One can see that, while at the canonical level the exact two-form ω and its isotopic extension $\hat{\omega}$ formally coincide, *this is no longer the case for exact, but arbitrary two forms* Ω and $\hat{\Omega}$ in the same local chart.

Note that the isoform $\hat{\Omega}$ is isoexact, $\hat{\Omega} = \hat{\partial}\hat{\Theta}$, and therefore isoclosed, $\hat{\partial}\hat{\Omega} \equiv 0$ (Lemma 3.1), in isospace over the isofield $\hat{R}$. However, if the same isoform $\hat{\Omega}$ is projected in ordinary space and called Ω, it is no longer necessarily exact, $\Omega \neq d\theta$ and, therefore, it is not generally closed, $d\Omega \neq 0$.

Recall that the Poincaré Lemma $d\Omega = d(d\Theta) = 0$ for the case of Birkhoffian two-form Ω (Sect. 2.4) provides the necessary and sufficient conditions for the tensor $\Omega^{\mu\nu} = [\,(\Omega_{\alpha\beta})^{-1}]^{\mu\nu}$ to be Lie [29]. It is easy to prove that this basic property persists under isotopy, although it characterizes the broader Lie-Santilli isotheory (Sect. 1.9). We therefore have the following

Lemma 3.2 (General Lie-Santilli Brackets): *Let* $\hat{\Omega}(\hat{b}) = \hat{\partial}\hat{\Theta} = \hat{\partial}(\hat{R}_\mu\hat{\partial}\hat{b}^\mu) = \hat{\Omega}_{\mu\nu}\hat{\partial}\hat{b}^\mu\wedge\hat{\partial}\hat{b}^\nu$ *be a general exact two-isoform. Then the brackets among sufficiently smooth and regular isofunctions* $\hat{A}(\hat{b})$ *and* $\hat{B}(\hat{b})$ *on* $T_2{}^*\hat{M}(\hat{E})$

$$[\,\hat{A}, \hat{B}\,]_{\text{Isot.}} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{b}^\mu}\,\hat{\Omega}^{\mu\nu}\,\frac{\hat{\partial}\hat{B}}{\hat{\partial}\hat{b}^\nu} ,$$

$$\hat{\Omega}^{\mu\nu} = \Big[\Big(\frac{\hat{\partial}\hat{R}_\alpha}{\hat{\partial}\hat{b}^\beta} - \frac{\hat{\partial}\hat{R}_\beta}{\hat{\partial}\hat{b}^\alpha}\Big)^{-1}\Big]^{\mu\nu} , \tag{3.19}$$

satisfy the Lie-Santilli axioms (Sect. 1.9) *in isospace (but not necessarily the same axioms when projected in ordinary spaces).*

The above theorem establishes that *the isosymplectic geometry is indeed the geometry underlying the Lie-Santilli isotheory,* as discussed in more details in the accompanying paper by Kadeisvili [16]. In particular, the isocanonical two-isoform characterizes the isocanonical realization of the Lie-Santilli brackets, Eq.s (2.64), while brackets (3.19) are the most general possible ones.

Even though we cannot use Darboux's theorem in practical applications for the reasons indicated in Sect.s 2.4 and 3.1, it is nevertheless important for completeness to prove that it admits a simple yet significant isotopies.

Theorem 3.2 (Isodarboux Theorem): *A 2n-dimensional cotangent isobundle* $T_2{}^*\hat{M}(\hat{E})$ *equipped with a nowhere degenerate, exact,* $\hat{C}^\infty$ *two-isoform* $\hat{\Omega}$ *in the local chart* $\hat{b}$ *is an isosymplectic manifold if and only if there exist coordinate transformations* $\hat{b} \to \hat{b}'(\hat{b})$ *under which* $\hat{\Omega}$ *reduces to the isocanonical two-isoform* $\hat{\omega}$, *i.e.,*

$$\frac{\hat{\partial}\hat{b}^\mu}{\hat{\partial}\hat{b}'^\alpha}\,\hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}'))\,\frac{\hat{\partial}\hat{b}^\nu}{\hat{\partial}\hat{b}'^\beta} = \omega_{\alpha\beta} . \tag{3.20}$$

Proof. Suppose that the transformation $\hat{b} \to \hat{b}'(b)$ occurs via the following intermediate transform $\hat{b} \to \hat{b}''(\hat{b}) \to \hat{b}'(b''(b))$. Then there always exists a transform $\hat{b} \to \hat{b}''$ such that

$$(\,\hat{\partial}\hat{b}^\rho / \hat{\partial}\hat{b}''^\sigma)\,(\hat{b}'') = \hat{1}^\rho{}_\sigma(\hat{b}(\hat{b}'')), \tag{3.21}$$

under which the general isosymplectic tensor $\hat{\Omega}_{\mu\nu}$ reduces to the Birkhoffian form when recompute in the $\hat{b}$ chart

$$\frac{\partial \hat{b}^{\mu}}{\partial \hat{b}''^{\alpha}} \hat{\Omega}_{\mu\nu}(\hat{b}(\hat{b}'')) \frac{\partial \hat{b}^{\nu}}{\partial \hat{b}''^{\beta}}_{|\hat{b}''} = \left(\frac{\partial \hat{R}_{\nu}}{\partial \hat{b}^{\alpha}} - \frac{\partial \hat{R}_{\mu}}{\partial \hat{b}^{\nu}} \right)_{|\hat{b}''} = \Omega_{\alpha\beta}{}_{|\hat{b}''} . \quad (3.22)$$

The existence of a second transform $\hat{b}'' \rightarrow \hat{b}'$ reducing $\Omega_{\alpha\beta}$ to $\omega_{\alpha\beta}$ is then known to exist (see, e.g., [30]). This proves the necessity of the isodarboux transform. The sufficiency is proved as in the conventional case [20]. **q.e.d.**

The nonlinear, nonlocal and noncanonical character of the isotopies is evident from the preceding analysis. It is important to point out that linearity is reconstructed in isospace and called *isolinearity*, as shown in Eq. (3.1). Locality is equally reconstructed in isospace, and called *isolocality*, because one– and two–isoforms are based on the local isodifferentials $\hat{\partial}\hat{x}$ and $\hat{\partial}\hat{p}$. Similarly, canonicity is reconstructed in isospace, and called *isocanonicity* as shown in Sect. 2.4 (see, e.g., isoaction principle of Sect. .2.4 which is precisely isocanonical).

The isotopies of the remaining aspects of the symplectic geometry (Lie derivative, global treatment, symplectic group, etc.) can be constructed along the preceding lines and are omitted for brevity. The isosymplectic geometry is also expected to admit a genotopic and hyperstructural extension, although they are not studied in this paper for brevity (for an initial formulation of the genosymplectic geometry, see [29], Ch. 7).

On closing we should mention that the preceding formulation of the isosymplectic geometry is solely restricted for the representation of *matter*. The characterization of antimatter is made via the antiautomorphic isodual map $\hat{1}_2 \rightarrow \hat{1}_2{}^d = -\hat{1}_2$. This results in the *isodual isosymplectic geometry* which is characterized by *isodual coordinates* $\hat{b}^d$, *isodual isodifferentials* $\hat{\partial}^d\hat{b}^d$, *isodual one–isoforms* $\hat{\theta}^d(\hat{b}^d)$, *isodual two–isoforms* $\hat{\omega}^d$, *isodual cotangent isobundle* $T^*\hat{M}^d(\hat{E}^d)$, and similar isodualities whose explicit construction is left to the interested reader for brevity.

In closing, we mention that the *isosymplectic quantization* were first studied by Lin [18] via the early formulation of the symplectic geometry. The same quantization via the isosymplectic geometry of this section is remarkable inasmuch as it leaves the formulation of the conventional symplectic quantization completely unchanged, and merely introduces broader *realizations* via the lifting of the unit $\hbar = 1$ into the isounit $\hat{1}$, of the differential dx into the isodifferentials $\hat{\partial}\hat{x}$, and of the canonical two–form ω into the isocanonical two–isoform $\hat{\omega}$, with the understanding that at the abstract level $\hbar \equiv \hat{1}$, $dx \equiv \hat{\partial}\hat{x}$, $\omega \equiv \hat{\omega}$ and all differences cease to exist. Yet, the isotopy is nontrivial because the the emerging new operator

theory, the hadronic mechanics of Sect. 2.5, is related to the conventional quantum mechanics via a *nonunitary* transform [38].

3.2. Isoriemannian geometry. The *isotopies of the Riemannian geometry*, or *isoriemannian geometry* for short, were submitted for the first time by Santilli in memoir [33] of 1988 via the isotopic degrees of freedom of the product. In this section we present, apparently for the first time, the isoriemannian geometry constructed via the isodifferential calculus of Sect. 1.5. As we shall see, the latter formulation is more conducive to a single, unified, abstract formulation of the conventional and isoriemannian geometries, thus permitting a unified treatment of the exterior and interior problems.

Our study is in local coordinates representing the fixed frame of the observer. All abstract, coordinate-free treatments are left to the interested mathematician. For the conventional geometry we assume all topological properties of Lovelock and Rund [20] of which we preserve the symbols for clarity in the comparison of the results. For the isotopic geometry we assume the topological properties by Tsagas and Sourlas [44,45] as outlined of Def. 1.6. Our presentation is made, specifically, for the (3+1)-dimensional space-time, with the understanding that the extension to arbitrary dimensions and signatures is elementary. For clarity we shall first study the isoriemannian geometry of Class I for the characterization of *matter* and then study its isodual image (Class II) for the characterization of *antimatter*.

Let $\Re = \Re(x,g,R)$ be a (3+1)-dimensional Riemannian space over the reals $R(n,+,\times)$ [20] with: local coordinates $x = (x^{\mu}) = (r, x^4)$, $x^4 = c_o t$, $\mu = 1, 2, 3, 4$, where c_o is the speed of light in vacuum; nowhere singular, symmetric and real-valued metric $g(x) = (g_{\mu\nu}) = g^t$; and tangent Minkowski space $M(x,\eta,R)$ with metric $\eta = \mathrm{diag.}\,(1, 1, 1, -1)$ over the reals R. Let the interval be written in the familiar expression $x^2 = x^{\mu}g_{\mu\nu}(x)x^{\nu} \in R$ with infinitesimal line element $ds^2 = dx^{\mu}g_{\mu\nu}(x)dx^{\nu}$ and related formalism (covariant derivative, Christoffel's symbols, etc. [20]).

Let $\hat{\Re} = \hat{\Re}(\hat{x},\hat{g},\hat{R})$ be the *isoriemannian space* of Definition 1.2 with local coordinates $\hat{x} = (\hat{x}^{\mu}) = (x^{\mu})$ and *isometric* $\hat{g}(x, v, a, \mu, \tau, ...) = \hat{T}(x, v, a, \mu, \tau, ...)g(x)$, where $\hat{T} = (\hat{T}_{\mu}{}^{\nu})$ is a nowhere singular, real valued and symmetric matrix of Class I with C^{∞} elements. The isospace $\hat{\Re}$ is defined over the isoreals $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ with *common* isounit $\hat{1} = (\hat{1}^{\mu}{}_{\nu}) = \hat{T}^{-1}$. We then have the *isoline element*

$$\hat{x}^2 = [\, \hat{x}^{\mu}\, \hat{g}_{\mu\nu}(\hat{x}, \hat{v}, \hat{a}, \mu, \tau, ...)\, \hat{x}^{\nu}\,] \times \hat{1} \in \hat{R} , \quad (3.23)$$

with infinitesimal version $\hat{\partial}\hat{s}^2 = (\, \hat{\partial}\hat{x}^{\alpha}\, \hat{g}_{\alpha\beta}\, \hat{\partial}\hat{x}^{\beta}\,) \times \hat{1} \in \hat{R}$.

The *isonormal coordinates* $\hat{y}$ occur when the isometric $\hat{g}$ is reduced, *not* to the Minkowski metric η, but rather to its isotopic image, i.e., $\hat{g} \to \hat{\eta} = \hat{T}\eta$ and, as such, they are the *conventional* normal coordinates (*principle of isoequivalence* [38]). In different terms, the correct tangent space is not the conventional space $M(x,\eta,R)$, but the *isominkowskian space* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$ characterized by the same isounit and isotopic element of $\hat{\mathfrak{R}}(\hat{x},\hat{g},\hat{R})$, that is, with the *isominkowskian metric* $\hat{\eta} = \hat{T}\eta$, where $\hat{T}$ is the same as in the isoriemannian metric $\hat{g} = \hat{T}g$. Under these conditions, the isonormal coordinates reduce the g-component in $\hat{g} = Tg$ to the η-component of $\hat{\eta} = \hat{T}\eta$ and this illustrates that isonormal coordinates coincide with the conventional normal coordinates.

To have an idea of the possible applications, we mention that the isounits of Class I can always be diagonalized, thus expressed in the form

$$\hat{T} = \text{diag.}\,(n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}),\ n_\mu > 0,\ m = 1, 2, 3, 4, \qquad (3.24)$$

As such, the above isounits permit the achievement of the central geometric objective of Sect. 3.1: *the geometrization of a locally varying speed of light* $c = c_o/n_4$ within physical media, which occurs via the fourth component of the isoline element

$$\hat{x}^4\,\hat{g}_{44}\,\hat{x}^4 = t\,c(\hat{x}, \mu, \tau, ...)\,g_{44}(x)\,t\,, \quad c = c_o\,/\,n_4(x, \mu, \tau, ...)\,, \qquad (3.25)$$

where g_{44} is the ordinary metric element and n_4 is the familiar index of refraction.

The general isotopic rule $\hat{g} = \hat{T}g$ then permits the lifting into interior conditions of any given exterior metric, such as Schwarzschild's metric [41]. Note that the latter metric can only represent the *constant* speed c_o and this illustrates the effectiveness of the isotopies for the geometric study of interior problems.

The representation of locally varying speeds of light technically occurs via the *isolight cone* [35,37] $\hat{d}\hat{s}^2 = \hat{d}\hat{x}^\mu \hat{g}_{\mu\nu} \hat{d}\hat{x}^\nu = 0$ which is the image in isospace of the *deformation* of the light cone in the tangent Minkowski space caused by variable speeds. It has been proved that *the isolight cone is a perfect cone in isospace* and, moreover, that *the characteristic angles of the conventional and isolight cones coincides* (see ref. [30], Ch. 8, for details). In other words, *the isotopies reconstruct the speed of light in vacuum as the maximal causal speed in isospaces over isofields, thus preserving all Einsteinian axioms in their entirety.*

The latter result is not a mere mathematical curiosity because it is important for specific applications, such as the correct calculations of gravitational horizons. In fact, the region outside these horizons is not empty, but filled up

instead by very large and dense chromospheres. It is well known that within these chromospheres the speed of light is locally varying with the density μ, temperature τ, etc., thus preventing the exact validity of the conventional light cone. The use of the isolight cone then permits more realistic studies of gravitational horizons because, while in isospace the maximal speed is c_o, the speed of light in its projection in the conventional space-time is the actual speed $c = c_o/n(x, \mu, \tau, ...)$.

Note that the conventional exterior geometrization in vacuum with constant speed of light c_o is a particular case of the isoriemannian geometry occurring for $\hat{1} = I$. This illustrates the *covering* character of the isoriemannian geometry over the conventional form.

In the first formulation of the isoriemannian geometry [33], differentials of contravariant isovector fields $\hat{X}^\beta$ on $\hat{\mathfrak{R}}$ where defined by $d\hat{X} = (\partial\hat{X})\hat{\times}d\hat{x} = (\partial_\mu\hat{X})\hat{T}_\nu{}^\mu d\hat{x}^\nu \neq dX = (\partial_\mu X)dx^\mu$, $\partial_\mu = \partial/\partial x^\mu$ The isodifferential calculus allows us to introduce the following alternative definition

$$\hat{d}\,\hat{X}^\beta = (\hat{\partial}_\mu \hat{X}^\beta)\,\hat{d}\hat{x}^\mu = \hat{T}_\mu{}^\rho\,(\partial_\rho \hat{X}^\beta)\,\hat{1}^\mu{}_\sigma\,d\hat{x}^\sigma \equiv (\partial_\mu \hat{X}^\beta)\,d\hat{x}^\mu = (\partial^\alpha \hat{X}^\beta)\,\hat{g}_{\alpha\beta}\,d\hat{x}^\beta, \qquad (3.26)$$

where the last expression is introduce to recall that the contractions are in isospace. The preceding expression then shows that *isodifferentials of isovector fields coincide at the abstract level with conventional differential of vector fields for all Class I isotopies.*

The *isocovariant differential* can be defined by

$$\hat{D}\,\hat{X}^\beta = \hat{d}\,\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\gamma\,\hat{X}^\alpha\,\hat{d}\hat{x}^\gamma, \qquad (3.27)$$

with corresponding *isocovariant derivative*

$$\hat{X}^\beta{}_{\hat{|}\mu} = \hat{\partial}_\mu\,\hat{X}^\beta + \hat{\Gamma}_\alpha{}^\beta{}_\mu\,\hat{X}^\alpha, \qquad (3.28)$$

where the *isochristoffel's symbols* are given by

$$\hat{\Gamma}_{\alpha\beta\gamma} = \tfrac{1}{2}(\partial_\alpha\,\hat{g}_{\beta\gamma} + \partial_\gamma\,\hat{g}_{\alpha\beta} - \partial_\beta\,\hat{g}_{\alpha\gamma}) = \hat{\Gamma}_{\gamma\beta\alpha},$$

$$\hat{\Gamma}_\alpha{}^\beta{}_\gamma = \hat{g}^{\beta\rho}\,\hat{\Gamma}_{\alpha\rho\gamma} = \hat{\Gamma}_\gamma{}^\beta{}_\alpha, \quad \hat{g}^{\beta\rho} = [(\hat{g}_{\mu\nu})^{-1}]^{\beta\rho}. \qquad (3.29)$$

One should note the abstract identity of the conventional and Class I isotopic connections. The extension to covariant isovector fields and covariant or contravariant isotensor fields is consequential (see [45]).

The isotopy of the proof of [20], pap. 80–81, yields to the following:

Lemma 3.3 (Isoricci Lemma): *Under the assumed conditions, the isocovariant derivatives of all isometrics on isoriemannian spaces are identically null,*

$$\hat{g}_{\alpha\beta\uparrow\gamma} \equiv 0, \qquad \alpha, \beta, \gamma = 1, 2, 3, 4. \tag{3.30}$$

The novelty of the isoriemannian geometry is then illustrated by the fact that *the Ricci property persists under an arbitrary dependence of the metric.*

Despite the similarities with the conventional case, the lack of equivalence of the Riemannian and isoriemannian geometries can be illustrated via the *isotorsion* [33]

$$\hat{\tau}_{\alpha}{}^{\beta}{}_{\gamma} = \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} - \hat{\Gamma}_{\gamma}{}^{\beta}{}_{\alpha}, \tag{3.31}$$

which is identically null for the isoriemannian geometry here considered, but its projection in the original space $\Re$ is not necessarily null. *Interior gravitational models treated with the isoriemannian geometry are therefore theories with null isotorsion but generally non-null torsion.* This property is needed for a realistic treatment of *interior* problems to avoid excessive approximation such as the *tacit* assumption of the existence of the "perpetual motion" within a physical medium which is implied by interior theories with null torsion.

The occurrence also illustrates the property, verified at subsequent levels later on, that departures from conventional geometric properties must be studied in the *projection* of isoriemannian spaces in the original spaces because, when treated in their respective spaces, the two geometries coincide. Stated in different terms, when using the conventional Riemannian geometry, exterior gravitation can only be studied in the spaces $\Re$. On the contrary, when using the isoriemannian geometry, interior gravitation can be studies in *two* different spaces, the isoriemannian spaces $\hat{\Re}$ and their projection into $\Re$.

Another way of identifying the differences between the Riemannian and isoriemannian geometries is by considering the following *isogeodesic equations*

$$\frac{\hat{D}\hat{x}_\beta}{\hat{D}\hat{s}} = \frac{\hat{\partial} v_\beta}{\hat{\partial}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma}(\hat{x}, \hat{v}, \hat{a}, ...)\frac{\hat{\partial}\hat{x}^\alpha}{\hat{\partial}\hat{s}}\frac{\hat{\partial}\hat{x}^\gamma}{\hat{\partial}\hat{s}} = 0, \tag{3.32}$$

where $\hat{v} = \hat{\partial}\hat{x}/\hat{\partial}\hat{s} = \hat{1}_s \times dx/ds$, $\hat{s}$ is the proper isotime and $\hat{1}_s$ the related (one–dimensional) isounit. The preceding equations must then be compared with the

conventional equations

$$\frac{Dx_\beta}{Ds} = \frac{dv_\beta}{ds} + \Gamma_{\alpha\beta\gamma}(x)\frac{dx^\alpha}{ds}\frac{dx^\gamma}{ds} = 0. \tag{3.33}$$

It is evident that the latter equations are at most quadratic in the velocities while the isotopic equations are arbitrarily nonlinear in the velocities, as it occurs already in a flat space (Sect. 2.5). Also, the latter equations are local–differential and variationally selfadjoint while the former admit nonlocal–integral terms and are variationally nonselfadjoint in $\Re$ although isoselfadjoint in $\hat{\Re}$.

We now introduce the *isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} = \partial_\delta \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} - \partial_\gamma \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\delta} + \hat{\Gamma}_{\rho}{}^{\beta}{}_{\delta}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\gamma} - \hat{\Gamma}_{\rho}{}^{\beta}{}_{\gamma}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\delta}; \tag{3.34}$$

the *isoricci tensor* $\hat{R}_{\mu\nu} = \hat{R}_{\mu}{}^{\beta}{}_{\nu\beta}$; the *isocurvature isoscalar* $\hat{R} = \hat{g}^{\alpha\beta}\hat{R}_{\alpha\beta}$; the *isoeinstein tensor* $\hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R}$; and the *isotopic isoscalar*

$$\hat{\Theta} = \hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}(\hat{\Gamma}_{\rho\alpha\delta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\beta} - \hat{\Gamma}_{\rho\alpha\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}) = \hat{\Gamma}_{\rho\alpha\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}(\hat{g}^{\alpha\delta}\hat{g}^{\gamma\beta} - \hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}). \tag{3.35}$$

the latter one being new for the isoriemannian geometry (see below).

Tedious but simple calculations then yield the following five basic identities of the isoriemannian geometry:

Identity 1: *Antisymmetry of the last two indices of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} = -\hat{R}_{\alpha}{}^{\beta}{}_{\delta\gamma}; \tag{3.36}$$

Identity 2: *Symmetry of the first two indices of the isocurvature tensor*

$$\hat{R}_{\alpha\beta\gamma\delta} \equiv \hat{R}_{\beta\alpha\gamma\delta}; \tag{3.37}$$

Identity 3: *Vanishing of the totally antisymmetric part of the isocurvature tensor*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta} + \hat{R}_{\gamma}{}^{\beta}{}_{\delta\alpha} + \hat{R}_{\delta}{}^{\beta}{}_{\alpha\gamma} \equiv 0; \tag{3.38}$$

Identity 4: *Isobianchi identity*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta\uparrow\rho} + \hat{R}_{\alpha}{}^{\beta}{}_{\rho\gamma\uparrow\delta} + \hat{R}_{\alpha}{}^{\beta}{}_{\delta\rho\uparrow\gamma} \equiv 0; \tag{3.39}$$

Identity 5: *Isofreud identity*

$$\hat{R}^{\alpha}{}_{\beta} - \tfrac{1}{2}\delta^{\alpha}{}_{\beta}\hat{R} - \tfrac{1}{2}\delta^{\alpha}{}_{\beta}\hat{\Theta} = \hat{U}^{\alpha}{}_{\beta} + \partial_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}, \tag{3.40}$$

where $\hat{\Theta}$ is the isotopic isoscalar and

$$\hat{U}^{\alpha}{}_{\beta} = -\tfrac{1}{2}\frac{\partial\hat{\Theta}}{\partial\hat{g}^{\alpha\beta}{}_{\uparrow\alpha}}\hat{g}^{\alpha\beta}{}_{\uparrow\beta},$$

$$\hat{V}^{\alpha\rho}{}_{\beta} = \tfrac{1}{2}[\hat{g}^{\gamma\delta}(\delta^{\alpha}{}_{\beta}\hat{\Gamma}_{\alpha}{}^{\rho}{}_{\delta} - \delta^{\rho}{}_{\beta}\hat{\Gamma}_{\gamma}{}^{\rho}{}_{\delta}) +$$

$$+(\delta^{\rho}{}_{\beta}\hat{g}^{\alpha\gamma} - \delta^{\alpha}{}_{\beta}\hat{g}^{\rho\gamma})\hat{\Gamma}_{\gamma}{}^{\delta}{}_{\delta} + \hat{g}^{\rho\gamma}\hat{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma} - \hat{g}^{\alpha\gamma}\hat{\Gamma}_{\beta}{}^{\rho}{}_{\gamma}], \tag{3.41}$$

A curiosity is that the conventional Riemannian geometry is generally thought to possess only *four* identities. In fact, the *fifth* identity, given in the above list by the Freud identity, is generally unknown in the contemporary technical literature in the field.

The latter identity was introduced by Freud [11] in 1939, treated in detail by Pauli [26] and then generally forgotten for a half a century, apparently, because of a conflict between the lack of source of Einstein's field equations in vacuum, $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R = 0$, and the evident need of a source in vacuum for the Freud identity, $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R - \frac{1}{2}\delta^{\alpha}{}_{\beta}\Theta = U^{\alpha}{}_{\beta} + \partial_{\rho}V^{\alpha\rho}{}_{\beta}$ (here written in a conventional space, see [39] for details).

Following a suggestion by the author, Rund [27] studied again the identity and proved that *the Freud identity is a bona fide identity for all Riemannian spaces irrespective of dimension and signature*, thus confirming the general need of a source also in vacuum (see below). In this paper we have presented the *isotopies* of the Freud identity, that is, its formulation in isoriemannian spaces, as characterized by the isodifferential calculus.

Note that *all conventional and isotopic identities coincide at the abstract level.* This confirms that the conventional and isotopic geometries (exterior and interior problems) can be treated at the abstract, realization-free level via one single set of axioms, as desired (Sect. 3.1), in which both the exterior and interior formulations are *realizations* of the same axioms.

The isotopy of the proof of the Theorem of [20], p. 321, leads to the following property first identified in 1988 [33] (see also [38]) and which is here recovered via the isodifferential calculus.

Theorem 3.3 (Fundamental Theorem for Interior Gravitation of Matter): *Under the assumed regularity and continuity conditions, the most general possible isolagrange equations $\hat{E}^{\alpha\beta} = 0$ of Class I along an actual path $\hat{P}_0$ on a (3+1)-dimensional isoriemannian space for the characterization of the interior gravitational problem of matter satisfying the properties: 1) Symmetry condition, $\hat{E}^{\alpha\beta} = \hat{E}^{\beta\alpha}$, 2) Contracted isobianchi identity, $\hat{E}^{\alpha\beta}{}_{\uparrow\beta} \equiv 0$;* and *3) The isofreud identity; are given by*

$$\hat{E}^{\alpha\beta} = \alpha\,\hat{g}^{\frac{1}{2}}(\hat{R}^{\alpha\beta} - \tfrac{1}{2}\hat{g}^{\alpha\beta}\hat{R} - \tfrac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta}) + \beta\,\hat{g}^{\alpha\beta} - \hat{g}^{\frac{1}{2}}\hat{D}^{\alpha\beta} = 0, \tag{3.42}$$

where: $\hat{g}^{\frac{1}{2}} = (\det\hat{g})^{1/2}$; α and β are constants; and $\hat{D}^{\alpha\beta}$ is a source tensor. For $\alpha = 1$ and $\beta = 0$ the interior isogravitation field equations can be written

$$\hat{R}^{\alpha\beta} - \tfrac{1}{2}\hat{g}^{\alpha\beta} - \tfrac{1}{2}\hat{g}^{\alpha\beta}\hat{\Theta} = \hat{t}^{\alpha\beta} - \hat{\tau}^{\alpha\beta} = \hat{U}^{\alpha}{}_{\beta} + \partial_{\rho}\hat{V}^{\alpha\rho}{}_{\beta}, \tag{3.43}$$

where $\hat{t}^{\alpha\beta}$ is a source tensor and $\hat{\tau}^{\alpha\beta}$ is a stress-energy tensor.

Note the appearance in Eq.s (3.43) of the isotopic isoscalar $\hat{\Theta}$ in the l.h.s and of source terms in the r.h.s., both originating from the isofreud identity. Additional studies not reported here for brevity (see [38], Ch. 9) have shown that the tensor $\hat{t}^{\alpha\beta}$ is nowhere null, of first order in magnitude and given by the electromagnetic tensor originating the mass of the elementary particles which constitute the body considered. Therefore, *the isotopies permit the "identification" of the gravitational and electromagnetic fields in the exterior problem* by eliminating the need of their "unification" (in the interior problem there are additional contributions from short range interactions) [38].

Also, *the isotopic formulation of the interior problem permits a theory on the "origin" of the gravitational field*, rather than its "description", in which all mass terms are replaced by the fields which originate the same at the particle level. This establishes the need of a source also for the field equations in vacuum, exactly as requested by the exterior Freud identity $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R - \frac{1}{2}\delta^{\alpha}{}_{\beta}\Theta = U^{\alpha}{}_{\beta} + \partial_{\rho}V^{\alpha\rho}{}_{\beta} = t^{\alpha\beta} - \tau^{\alpha\beta}$, which therefore emerges as the exterior limit of interior equations (3.43) [loc. cit.].

Equivalently, it has been proved that the equations $R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R = 0$ are incompatible with the primary, well established, electromagnetic origin of the mass of elementary particles, and are afflicted by other problematic aspects, such as: lack of weight in the *relativistic* description of particles in tangent Minkowski space; problematic aspects in the relativistic limit of conservation laws; lack of uniqueness in the PPN approximation with consequential serious ambiguities in the interpretation of experimental data; and others. These problematic aspects are apparently resolved by the above isotopic theory on the "origin" of the gravitational field based on the Freud identity and its exterior limit in vacuum [33].

We should finally note that the isotopic scalar is written in the l.h.s., rather than in the r.h.s. of the field equations because of geometric problems with Einstein's tensor $G_{\mu\nu} = R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R$ (lack of preservation of the Ricci Lemma under isotopies), which are resolved by the tensor $S_{\mu\nu} = R^{\alpha}{}_{\beta} - \frac{1}{2}\delta^{\alpha}{}_{\beta}R - \frac{1}{2}\delta^{\alpha}{}_{\beta}\Theta$ (see [38], Ch. 5 for details).

Theorem 3.3 applies, specifically and solely, for the characterization of *matter.* The corresponding representation of *antimatter* can be obtained via the antiautomorphic map called (isoduality) $\hat{1} > 0 \rightarrow \hat{1}^d = -\hat{1} < 0$ of the entire formalism, including the basic unit, numbers, spaces, etc. This results in the *isodual isoriemannian geometry,* that of Class II, which is characterized by *isodual isoreal isofields* $\hat{R}^d(\hat{n}^d,+,\hat{\times}^d)$, *isodual isoriemannian spaces* $\hat{\mathfrak{R}}^d(\hat{x}^d,\hat{g}^d,\hat{R}^d)$, *isodual curvature tensor* $\hat{R}^d{}_{\mu\nu}$, *isodual curvature scalar* $\hat{R}^d$, *isodual isotopic scaler* $\hat{\Theta}^d$, etc., whose study is omitted for brevity (see [16b] for all details).

The above isodual formulation essentially implies that *all conventionally positive quantities change sign under isoduality, including the energy-momentum tensor, curvature, etc., although these negative quantities are now referred to a negative unit.* In this way, the *gravitational* treatment of antimatter is brought in line with the *particle* treatment, that is, the characterization of antiparticles as historically discovered, that via *negative energy,* which now holds at all levels of study.

A first result is that *antimatter-antimatter have a gravitational "attraction" equivalent to that of matter-matter systems* because a *negative* curvature referred to a *negative* unit $\hat{1}^d$ is fully equivalent to a *positive* curvature referred to the *positive* unit $\hat{1}$.

However the isodual theory predicts that *matter-antimatter and antimatter-matter systems experience a gravitational "repulsion",* because in this case we have the projection of one system in the corresponding isodual space, resulting in *negative curvature referred to a positive unit, or viceversa, which represents repulsion* (see [38] for details and proposed experimental verifications).

An isovector field $\hat{X}^{\beta}$ on $\hat{\mathfrak{R}}$ is said to be transported by *isoparallel displacement* from a point $\hat{m}(\hat{x})$ on a curve $\hat{C}$ on $\hat{\mathfrak{R}}$ to a neighboring point $\hat{m}'(\hat{x} + \hat{\partial}\hat{x})$ on $\hat{C}$ if

$$\hat{D}\hat{X}^{\beta} = \hat{\partial}\hat{X}^{\beta} + \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma}\hat{X}^{\alpha}\hat{\partial}\hat{x}^{\gamma} \equiv 0\,. \qquad (3.44)$$

or in integrated form

$$\hat{X}^{\beta}(\hat{m}') - \hat{X}^{\beta}(\hat{m}) = \hat{\int}_{\hat{m}}^{\hat{m}'} \frac{\hat{\partial}\hat{X}^{\beta}}{\hat{\partial}\hat{x}^{\alpha}}\frac{\hat{\partial}\hat{x}^{\alpha}}{\hat{\partial}s}\hat{\partial}s\,. \qquad (3.45)$$

where one should note the isotopic character of the integration. The isotopy of the conventional case [20] then yield the following:

Lemma 3.4: *Necessary and sufficient conditions for the existence of an isoparallel transport along a curve $\hat{C}$ on a (3+1)-dimensional isoriemannian space are that all the following conditions are identically verified along $\hat{C}$*

$$\hat{R}_{\alpha}{}^{\beta}{}_{\gamma\delta}\,\hat{X}^{\alpha} = 0\,, \qquad \beta,\gamma,\delta = 1,2,3,4\,. \qquad (3.46)$$

Note, again, the abstract identity of the conventional and isotopic parallel transport. However, it is easy to see that the projection of the isoparallel transport in the conventional space $\mathfrak{R}$ is structurally different than the conventional parallel transport. In particular, if the latter is represented by an arrow, one would note a twisting action as occurring in the reality of motion within physical media, which is evidently absent in the exterior case.

Along similar lines, we say that a smooth path $\hat{x}_{\alpha}$ on $\hat{\mathfrak{R}}$ with isotangent $\hat{v}_{\alpha} = \hat{\partial}\hat{x}_{\alpha}/\hat{\partial}\hat{s}$ is an *isogeodesic* when it is solution of the isodifferential equations

$$\frac{\hat{D}\hat{x}_{\beta}}{\hat{D}\hat{s}} = \frac{\hat{\partial}\hat{v}_{\beta}}{\hat{\partial}\hat{s}} + \hat{\Gamma}_{\alpha\beta\gamma}\frac{\hat{\partial}\hat{x}^{\alpha}}{\hat{\partial}\hat{s}}\frac{\hat{\partial}\hat{x}^{\gamma}}{\hat{\partial}\hat{s}} = 0\,. \qquad (3.47)$$

It is easy to prove the following:

Lemma 3.5: *The isogeodesics of an isoriemannian space $\hat{\mathfrak{R}}$ are the curves verifying the isovariational principle*

$$\delta\hat{\int}[\,\hat{g}_{\alpha\beta}(\hat{x},\hat{v},\hat{a},\mu,\tau,\ldots)\,\hat{\partial}\hat{x}^{\alpha}\,\hat{\partial}\hat{x}^{\beta}\,]^{1/2} = 0. \qquad (3.48)$$

where again isointegration is understood.

Finally, we point out the property which is inherent in the notion of isotopies as realized in this paper according to which *geodesic trajectories in ordinary space coincide with the corresponding isogeodesic trajectories in* *isospace*. For instance, if a circle is originally a geodesic, its image under isotopy in isospace remains the perfect circle, the *isocircle* (Sect. 1.3), even though its projection in the original space is an ellipse. The same preservation in isospace occurs for all other curves.

The differences between a geodesic and an isogeodesic therefore emerge only when projecting the latter in the space of the former.

An empirical but conceptually effective rule is that *interior physical media "disappear" under their isoriemannian geometrization*, in the sense that actual trajectories under resistive forces due to physical media (which are not geodesics of a Riemannian space) are turned into isogeodesics in isospace with the shape of the geodesics in the absence of resistive forces.

It should be also noted that the isoriemannian geometry is a particular case of the broader *genoriemannian geometry* over genofields of Sect. 1.2 [37]. The latter is intriguing inasmuch as it establishes that the abstract axioms of the Riemannian geometry *do not* require the metric to be necessarily symmetric. In fact, the same axioms permit a *nonsymmetric metrics* $\hat{g}^{>} = \hat{T}^{>}g$, provided that the nonsymmetric component $\hat{T}^{>}$ is totally represented by the underlying isounit, $\hat{1}^{>} = (\hat{T}^{>})^{-1}$.

The genoriemannian geometry is particularly suited to represent *irreversibility*, in which case the ordering to the right represents motion forward in time and that to the left motion backward in time. The difference $\hat{1}^{>} \neq {}^{<}\hat{1}$ then assures an *axiomatic representation of irreversibility*, that is, a representation via the geometry itself which remains irreversible even for fully reversible Lagrangians (see [37,38] for details and applications).

Finally, we should mention that, in turn, the genoriemannian geometry is expected to be a particular case of the still broader *hyperiemannian geometry* on the hyperfields of Sect. 2.2, although no study has been done on the latter geometry at this writing.

In summary, a basic question raised in this section is: *why use in interior problems the Riemannian geometry with metric* $g(x)$ *when the same axioms permit metrics* $\hat{g}(\hat{x}, \hat{v}, \hat{a}, \ldots)$ *with an unrestricted functional dependence in the velocities and other variables* ? Equivalently, we can ask the question: *why use the simplest possible realization of the Riemannian axioms when a structurally more general realization exists for a more adequate representation of interior problems?* Still equivalently, *why use geometries implying the sole constancy of the speed of light when the same axioms in a more general realization geometrize locally varying speed of light as occurring in the physical reality ?*

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FOUNDATIONS OF THE LIE-SANTILLI ISOTHEORY

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Lie's theory in its current formulation is linear, local and canonical. As such, it is inapplicable to a growing number of nonlinear, nonlocal and noncanonical systems in various fields. In this paper we review and develop a generalization of Lie's theory proposed by the physicist R. M. Santilli in the late 1970's then at Harvard University and today called *Lie-Santilli isotopic theory* or *isotheory* for short. The latter theory is based on the so-called *isotopies* which are nonlinear, nonlocal and noncanonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie-Santilli isotheory is remarkable because it preserves the abstract axioms of Lie's theory while being applicable to nonlinear, nonlocal and noncanonical systems. We review the foundations of the Lie-Santilli isoenvelopes, isoalgebras, isogroups and isorepresentation theory; we introduce seemingly novel advances in their structure and interconnections; and we show that the Lie-Santilli isotheory provides the invariance of all infinitely possible, signature-preserving, nonlinear, nonlocal and noncanonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We finally indicate a number of applications and identify rather intriguing open mathematical problems.

1. Introduction

1.A. Limitations of Lie's theory. As it is well known, *Lie's theory* has permitted outstanding achievements in various disciplines. Nevertheless, in its traditional conception [30] and realization (see, e.g., [15]), Lie's theory is *linear, local-*

differential and canonical-Hamiltonian. As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and other founders of analytic dynamics between the *exterior dynamical problems* in vacuum and the *interior dynamical problems* within physical media. Exterior problems consist of particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the exact validity of conventional local-differential topologies (e.g., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the symplectic geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interior problems consist instead of extended, nonspherical and deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contact-resistive interactions (such as a space-ship during re-entry in Earth's atmosphere). In the latter case the forces are of local-differential type (e.g., potential forces acting on the center-of-mass) as well as of nonlocal-integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact-resistive interactions violate Helmholtz's conditions for the existence of a potential (the *conditions of variational selfadjointness* [109]), thus implying the noncanonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is *inapplicable* (and not "violated") for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1.B. The need for a suitable generalization of Lie's theory. Lie's theory is currently applied to nonlinear, nonlocal and noncanonical systems via their simplifications into more treatable forms, e.g., via the expansion of nonlocal-integral terms into power series in the velocities and then the transformation of the system into a coordinate frame in which it admits a Hamiltonian via the Lie-Koening or the Darboux Theorems [110].

However, however, nonlinear, nonlocal and nonhamiltonian systems cannot be consistently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwarzschild's *two* papers, the first celebrated paper [119] on the exterior problem and the second little known paper [120] on the interior problem). The same distinction was also kept in early well written treatises in the field (see, e.g., [4], [38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend is based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that *the exterior and interior problems are inequivalent, and the latter is not exactly reducible to the former.* The inequivalence is established by the fact that *the exterior problem is local-differential and variational selfadjoint* [109], while *the interior problem is nonlocal-integral and variationally nonselfadjoint* [loc. cit.]. This establishes the *inequivalence* on: topological grounds (because the conventional topologies are inapplicable to nonlocal conditions); analytic grounds (because of the lack of a first-order Lagrangian); geometric grounds (because of the inapplicability of conventional geometries to characterize, say, locally varying speeds of light); and other grounds (see monograph [116] for comprehensive studies).

The *irreducibility* of the interior to the exterior problem is established by the so-called *No-Reduction Theorems* [65] which prohibit the reduction of a macroscopic interior system (such as a satellite during re-entry) *with a monotonically decreasing angular momentum,* to a finite collection of elementary particles each one with *a conserved angular momentum* (see also [116] for comprehensive studies here omitted for brevity).

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily nonlinear (in coordinates and velocities), nonlocal-integral (in various quantities) and non-hamiltonian (variationally nonselfadjoint).

Additional insufficiencies of the current formulation of Lie's theory as well as of its underlying geometries and mechanics exist for the characterization of antimatter. In fact, we possess today effective methods for the characterization of antimatter only at the *operator* level via *charge conjugation*. These methods do not have a counterpart at the *classical* level because charge conjugation is *antiautomorphic* and no corresponding map exists in the classical realization of Lie's theory, as well as in its underlying carriers spaces, geometries and mechanics. There is therefore the need of achieving first a consistent antiautomorphic characterization of antimatter at the *classical-astrophysical* level, and then at the level of its elementary constituents.

Similar occurrences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a generalization of the conventional Lie theory which is directly applicable (i.e., applicable without approximation or transformations) to nonlinear, integro-differential and variationally nonselfadjoint equations for the characterization of matter, and then possesses a suitable antiautomorphic map for the effective characterization of antimatter.

1.C: Santilli's isotopies and isodualities of Lie's theory. In a seminal memoir [52] written in 1978 (see also memoir [53] and paper [54] written in the same year) when at Harvard University, the theoretical physicist Ruggero Maria Santilli proposed a step-by-step generalization of the conventional formulation of Lie's theory (that is, a generalization of envelopes, algebras, groups, representation theory, etc.) specifically conceived for nonlinear, integro-differential and noncanonical systems. The generalized theory was subsequently studied by Santilli in over one hundred papers, including studies on the structure of the theory and its applications in various fields (see representative papers [52–108]), and then additionally studied in ten monographs [109–118]. The new formulation of Lie's theory which has emerged from these studies is today called the *Lie-Santilli isotopic theory* or *isotheory* for short (see papers [1], [2], [8], [11], [12], [16]–[23], [25], [32], [33], [35]–[37], [40]–[43], [122]–[125], independent monographs [3], [24], [31], [121] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from other generalizations, is its *isotopic* nature intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies [52]–[54] are today referred to *maps of any given linear, local and canonical structure into its most general possible nonlinear, nonlocal and noncanonical forms which are capable of reconstructing linearity, locality and canonicity in certain generalized isospaces and isofields within the fixed inertial coordinates of the observer.*

These properties are remarkable, mathematically and physically, inasmuch

as they permit the preservation of the abstract Lie theory and the transition from exterior to interior problems via a more general *realization* of the same theory. We assume the reader is aware of the array of novel problems raised by the above definition of isotopies, such as the representation of *nonhamiltonian* vector fields in the coordinates of the observer *without* Darboux's transformations to an equivalent Hamiltonian form, because the latter, being nonlinear images of the coordinates of the observer, are not realizable in experiments as well as noninertial and, as such, are not usable in practical applications (see the preceding article [100] by Santilli for the solution of this and the other problems connected with the above definition).

It should be indicated that Santilli submitted his isotopic theory in memoir [52] *as a particular case* of a yet more general theory today called *Santilli's Lie-admissible theory* or *Lie-Santilli genotopic theory*, where the term *genotopies* was introduced (from its Greek meaning of "inducing configuration") to denote the characterization of covering Lie-admissible axioms.

In fact, Santilli initiated his research during his Ph. D. studies in theoretical physics at the University of Turin, Italy, by introducing in 1967 [47] a new notion of Lie-admissible algebra with its explicit realization. These early studies in Lie-admissibility were then continued in papers [49]–[53], [55]–[58], and numerous other, as well as in monographs [111], [112].

In essence the *first notion of Lie-admissibility* is due to the American mathematician A. A. Albert (see the historical notes of ref. [52]) and is referred to a nonassociative algebra U with elements a, b, ... and (abstract) product ab whose attached antisymmetric algebra U^-, which is the same vector space as U but equipped with the product $[a, b]_U = ab - ba$, is Lie. As such, the algebra U does not necessarily contain a Lie algebra in its classification, thus resulting to be inapplicable for the construction of mathematical and physical coverings of Lie's theory.

In fact, Albert was primarily concerned with the requirement that U should contain *Jordan algebras* as particular cases, and conducted his studies with the *quasiassociative algebra* with product $(a, b) = \lambda ab + (1-\lambda)ba$, where λ is a non-null scalar, which yield a commutative Jordan algebra for $\lambda = \frac{1}{2}$ and ab associative, but which does not admit a Lie algebra under a finite value of λ

The *second notion of Lie-admissibility* was introduced by Santilli in paper [47] as the preceding definition, plus the condition that the algebra U admits Lie algebras in their classification or, equivalently, that the product ab admits as a particular case the Lie product. This definition was presented via the realization of the *flexible Lie-admissible algebras* with product $(a, b) = \lambda ab - \mu ba$, where λ, μ and

$\lambda + \mu$ are non-null scalars, under the conditions that $[a, b]_U = (a, b) - (b, a) = (\lambda + \mu)(ab - ba)$ is Lie, plus the condition that the product (a, b) admits the Lie product as particular case. The latter conditions are easily met for $\lambda = \mu$ and ab associative.

To the author's best knowledge, paper [47] initiated in 1967 the studies in the so-called "q-deformations" subsequently conducted in the 1980's by a large number of authors with the simpler product $(a, b) = ab - qba$, $\lambda = 1$, $\mu = q$ (although papers in the latter field rarely quote [47]). Santilli also identified in paper [49] of 1969 the first Lie-admissible structure on record of classical dynamics for dissipative systems, thus illustrating the physical need of his "want of a Lie algebra content" [47].

Subsequently, in memoirs [52], [53] of 1978, Santilli introduced the realization of the *general Lie-admissible algebra* with the product $(a, b) = a{\times}R{\times}b - b{\times}S{\times}a$, where $a{\times}R$, $R{\times}b$, etc. are associative, and R, S, R+S are nonsingular but otherwise arbitrary *operators* with scalars values λ and μ as particular cases. He then discovered that the attached antisymmetric algebras *were not* conventionally Lie with the familiar commutator $a{\times}b - b{\times}a$, but were instead characterized by the product $[a, b]_U = (a, b) - (b, a) = a{\times}T{\times}b - b{\times}T{\times}a$, $T = R + S$, which he called *Lie-isotopic* [52], [53]. This resulted in the *third definition of Lie-admissibility*, today called *Albert-Santilli Lie-admissibility*, which refers to a *nonassociative algebra* U *which admit Lie-Santilli isoalgebras both in their attached antisymmetric form* U^- *as well as in their classification.*

Jointly, Santilli identified in the same memoirs a classical [52] and operator [53] realization of the general Lie-admissible algebras, thus establishing the foundations of a structural generalization of Lie-admissible type of analytic and quantum mechanics and of their interconnecting map, of which in this paper we shall merely study the isotopic particular case occurring for $R = S = T = T^\dagger \neq 0$.

Albert-Santilli notion of Lie-admissibility can be considered the birth of the Lie-Santilli isotheory, and can be found in Sect. 3 (particularly Sect. 3.7) of ref. [52] and in Sect. 4 (particularly Sect. 4.14) of ref. [53]. In fact, Santilli recognized that the antisymmetric brackets $(a, b)_U$ attached to the *nonassociative* algebra U with product $(a, b) = a{\times}R{\times}b - b{\times}S{\times}a$ can be *identically* rewritten as the antisymmetric brackets attached to an *associative* algebra $\hat{A}$ with product $a{\times}R{\times}b$, $[a, b]_U \equiv [a, b]_{\hat{A}}$.

The latter identity signaled the transition from studies within the context of *nonassociative algebras* (done by Santilli until 1978), to genuine studies for the generalization of *Lie's theory* (done from 1978 on), which are based on the lifting of the *associative* enveloping algebras, from the product $a{\times}b$, to the isotopic product $a\hat{\times}b = a{\times}T{\times}b$.

Santilli then discovered that the quantity $\hat{1} = T^{-1}$ is indeed the correct left and right unit of the isotopic envelope $\hat{A}$. *The Lie-Santilli isotheory can therefore be initially conceived as the image of the conventional theory under the lifting of the trivial unit* 1 *of conventional use to a well behaved but otherwise arbitrary unit* $\hat{1}$.

This conception permitted Santilli to identify all main lines of the isotheory already in the original proposal [52], which include: the isotopies of universal enveloping associative algebras (including the isotopies of the fundamental Poincaré-Birkhoff-Witt and Baker-Campbell-Hausdorff theorems); the isotopies of Lie algebras (including the isotopies of the celebrated Lie's first, second and third theorem); the isotopies of Lie transformations groups; and other isotopies.

The original proposal [52] also included the remarkable property of the Lie-Santilli isoalgebra of unifying compact and noncompact simple Lie algebras of the same dimension (see ref. [52], Definition 3.7.2 on the isotopic envelope characterizing *nonisomorphic* Lie algebras with the *same* basis and changing instead T, and the isotopic unification of O(2.1) and O(3) in p. 289). All subsequent developments, including this presentation, have essentially been refinements of these foundations introduced in the original proposal [52], [53].

By the early 1980's Santilli recognized that the available Lie, Lie-isotopic and Lie-admissible formulations could only be applied to matter and not to antimatter for the reasons indicated in Sect. 1.B. He then reinspected his isotopies and in papers [62], [63] (written in 1983 but published in 1985 because of quite unreasonable editorial obstructions by various physics journals reviewed in p. 26 of [62]) he discovered that, once the elementary unit +1 is abandoned in favor of an arbitrary quantity $\hat{1}$, the latter unit admits in a natural way *negative values*. He also discovered that the map $\hat{1} > 0 \rightarrow \hat{1}^d = -\hat{1} < 0$ is *antiautomorphic* precisely as the charge conjugation, and called it *isoduality* in the sense of being a form of duality which necessarily requires the isotopic generalization of the unit.

In the same papers [62], [63] he reformulated the Lie-isotopic theory for negative units $\hat{1}^d$ which is today called *isodual Lie-Santilli isotheory*, and introduced a number of novel notions, such as *isorotational symmetry* $\hat{O}(3)$ and its *isodual* $\hat{O}^d(3)$ which leave invariant the conventional *ellipsoids* with positive semiaxes, and the new *isodual ellipsoids with negative semiaxes*, respectively. He then proved the isomorphism $\hat{O}(3) \approx O(3)$ (and the anti-isomorphism between $\hat{O}^d(3)$ and O(3)), thus disproving the rather popular belief that the rotational symmetry is broken for the ellipsoidical deformations of the sphere (which is correct only under the assumption of realizing Lie's theory in its simplest conceivable form, but incorrect otherwise, as illustrated in Sect. 3.E).

Despite these advances and as admitted in private communications, Santilli abstained from indicating in papers [62], [63] the applicability of the isodual theory for the characterization of antimatter because of its rather deep implications such as a causal motion backward in time, the prediction of antigravity for antiparticles in the field of matter, and others.

After due studies, the above reservation were resolved, and Santilli first applied his isodual theory for the characterization of antimatter in monographs [113], [114] of 1991. The equivalence between isoduality and charge conjugation was first proved in paper [84] of 1994. Some of the far reaching implications of isoduality were studied in papers [86], [87] of the same year. The first comprehensive treatment of isoduality appeared in the 1994 edition of monograph [116]. The mathematical and physical studies based on isoduality are now rapidly expanding.

The culmination of Santilli's isotopies and isodualities can be seen in the emergence of new notions of space-time and internal symmetries for matter, and their isodual for antimatter which, in turn, culminate in the isotopies and isodualities of the Poincaré symmetry, first proposed by Santilli in paper [59] of 1983 (see paper [79] of 1993 for the latest comprehensive study including its isospinorial covering). The isotopies of the SU(3) symmetry were first studied in paper [34] of 1984 and those of the quark theory in paper [90] of 1995.

The new space-time isosymmetries imply corresponding new classical and quantum mechanics and have far reaching implications, such as: the first exact-numerical representation of the magnetic moment of the deuteron [85] (which has escaped quantum mechanics for three quarters of a century despite all possible relativistic and tensorial corrections); the first exact-numerical representation of the synthesis of the neutron inside new stars from protons and electrons only [95] (which cannot be treated quantitatively by quantum mechanics and quark theories); the consequential prediction of a new source of clean, *subnuclear* energy called "hadronic energy" [88] (all predictive capacities for new energies based on the conventional Poincaré symmetry were exhausted during the first half of this century); and other novel applications, verifications and predictions [116], [118].

In view of the above advances, Santilli received various honors, including the Nomination in 1989 by the Estonia Academy of Sciences among the most illustrious applied mathematicians of all times, jointly with Gauss, Hamilton, Cayley, Lie, Frobenius, Poincaré, Cartan, Riemann, and others, the only member of Italian origin to enter in the list (see the charts of pages 6-7 of ref. [31]). Quite appropriately, the Nomination lists Santilli's first paper [47] on Lie-admissibility written at the University of Turin, Italy, from which everything else follows.

This paper is written by a theoretical physicist for mathematicians and it is solely devoted to the Lie-Santilli isotheory with a few indication of its isodual. A study of the broader Lie-Santilli genotheory is contemplated as a future work. To avoid un-necessary repetition, all background notions on isotopies are referred to the preceding article by Santilli [100] in this Journal. Therefore, in Sect. 2 we shall only indicate certain rudimentary notions. The isotopies and isodualities of Lie's theory are presented in Sect. 3 jointly with new developments, such as a study of the transition from isogroups to the corresponding isoalgebras which are permitted by the recent advanced of paper [100].

As an illustration of the capabilities of the Lie-Santilli isotheory, we review in Sect.s 3.D-3.F the "direct universality" of the Poincaré-Santilli isosymmetry, that is, the achievement of the symmetries of all infinitely possible, well behaved, nonlinear, nonlocal and noncanonical generalization of the minkowskian line element (universality), directly in the coordinates of the observer (direct universality). This universality includes as particular case the symmetry of all possible gravitational models in (3+1)-dimension with consequential unification of the special and general relativities and emergence of a novel quantization of gravity via the *unit* of relativistic quantum mechanics without any need of a Hamiltonian [79], [98]. A number of intriguing open mathematical problems will be identified during the course of our analysis and in the final section.

A comprehensive mathematical presentation of the Lie-Santilli isotheory up to 1992 is available the monograph by Sourlas and Tsagas [121]. A historical perspective is available in the monograph by Lõhmus, Paal and Sorgsepp [31]. The study of continuity properties under isotopies was initiated by Kadeisvili [22]. The first identification of isomanifolds (today called *Tsagas-Sourlas isomanifolds*) was done in ref. [122] which is a topological complement of these algebraic studies.

In this paper we can only quote contributions on the generalization of Lie's theory *based on the broadening of the unit* and we regret our inability at this time to quote the rather numerous contributions on *different* generalization based on the *conventional* unit. The author would be grateful to any colleague who cares to bring to his attention additional relevant literature for quotation in future works.

2. Elements of Isotopies and Isodualities

2.A: Statement of the problem. Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing: the theory of numbers; differential and exterior calculus; vector and metric spaces; geometry, algebra and topology; functional analysis; and others. Santilli's isotopies of Lie's theory require simple, yet unique and significant isotopic liftings of *all* these mathematical methods, without any exception known to this author.

The most recent presentation on the isotopies of contemporary mathematical methods is presented in the preceding article by Santilli [100] in this Journal. To avoid un-necessary repetitions, we shall herein assume the entirety of the content of paper [100] and refer to it as I, Sect. I.3 or Eq. (I.3.33). Additional studies via a different type of isotopies are available in monographs [115], [116] with numerous applications.

In this section we shall mainly recall the fundamental notions, and refer to paper I for all details.

2.B. Isotopies and isodualities of the unit and of related mathematical structures. The fundamental isotopies from which all others can be uniquely derived are given by the liftings of the n-dimensional unit $I = \mathrm{diag.}(1, 1, ..., 1)$ of the current formulation of Lie's theory into a matrix $\hat{1}$ of the same dimension of I, but with unrestricted functional dependence of its elements in the local coordinates x, their derivatives with respect to an independent variable of arbitrary order, $\dot{x}$, $\ddot{x}$, ... as well as any needed additional quantity [52], [53],

$$I \rightarrow \hat{1} = \hat{1}(x, \dot{x}, \ddot{x}, ...) . \qquad (2.1)$$

The *isotopies* occur when $\hat{1}$ preserves all the topological characteristics of I, such as nowhere-degeneracy, real-valuedness and positive-definiteness.

Once the unit is generalized, there is the natural emergence of the map [62], [63]

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1} , \qquad (2.2)$$

called by Santilli *isoduality* which provides an antiautomorphic image of all formulations based on $\hat{1}$.

The above liftings were classified by Kadeisvili [22] into:

Class I (generalized units that are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite, characterizing the *isotopies* properly speaking);

Class II (the same as Class I although $\hat{1}$ is negative-definite, characterizing *isodualities*);

Class III (the union of Class I and II);

Class IV (Class III plus singular isounits); and

Class V (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures studied in this paper also admit the same classification which will be omitted for brevity. Hereon we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Classes IV and V have not been studied until now, to our best knowledge.

Lie's theory is constructed over ordinary fields $F(a,+,\times)$ hereon assumed to be of characteristic zero (the fields of real $\Re$, complex C and quaternionic numbers Q) with generic elements a, addition $a_1 + a_2$, multiplication $a_1 a_2 := a_1 \times a_2$, additive unit 0, $a + 0 = 0 + a \equiv a$, and multiplicative unit I, $a \times I = I \times a \equiv a$, $\forall\, a, a_1, a_2 \in F$.

The Lie-Santilli isotheory is based on a generalization of the very notion of numbers and, consequently of fields (see the review in Sect. I.2.2 of [100], ref. [73] mathematical studies and monograph [115] for comprehensive treatment).

Consider a Class I lifting of the unit I of F, $I \rightarrow \hat{1}$ with $\hat{1}$ being *outside* the original set, $\hat{1} \notin F$. In order for $\hat{1}$ to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication ab into the so-called *isomultiplication* [52]

$$a\, b = a \times b \;\Rightarrow\; a * b := a \times T \times a = a\, T\, b , \qquad (2.3)$$

where the quantity T is fixed and called the *isotopic element*. Whenever $\hat{1} = T^{-1}$, $\hat{1}$ is the correct left and right unit of the theory, $\hat{1} * a = a * \hat{1} = a$, $\forall\, a \in F$, in which case (only) $\hat{1}$ is called the *isounit*. In turn, the liftings $I \rightarrow \hat{1}$ and $\times \rightarrow *$, imply the generalization of fields into the Class I structure

$$\hat{F}_I = \{ (\hat{a},+,*) \,|\, \hat{a} = a\hat{1};\; * = \times T \times;\, \hat{1} = T^{-1} \}, \qquad (2.4)$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [73] (in paper [100] the isoproduct is denoted with the new symbol $\hat{\times}$, while in this paper we have preserved the symbol $*$ used in the preceding literature in the field for easiness in the comparison of the results).

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have: $a + b \rightarrow \hat{a} + \hat{b} = (a + b)\hat{1}$; $a_1 \times a_2 \rightarrow \hat{a}_1 * \hat{a}_2 = \hat{a}_1 T \hat{a}_2 = (a_1 a_2)\hat{1}$; $a^{-1} \rightarrow \hat{a}^{\hat{1}} = a^{-1}\hat{1}$; $a / b = c \rightarrow \hat{a} \hat{/} \hat{b} = \hat{c}$, $a^{½} \rightarrow \hat{a}^{\hat{½}} = a^{½}\hat{1}^{½}$; etc. Thus, conventional squares $a^2 = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^{\hat{2}} = \hat{a} * \hat{a}$. The *isonorm* is

$$\hat{|}\, \hat{a}\, \hat{|} = (\bar{a}\, a)^{1/2}\, \hat{1} = |a|\, \hat{1} \in \hat{F}, \qquad (2.5)$$

where $\bar{a}$ denote the conventional conjugation in F and $|a|$ the conventional norm. Note that the *isonorm is positive-definite* for isofields of Class I, and *negative-definite* for isofields of Class II.

The isotopic character of the lifting $I \rightarrow \hat{1}$ is confirmed by the fact that the isounit $\hat{1}$ verifies all axioms of I, $\hat{1} * \hat{1} * \ldots * \hat{1} = \hat{1}$, $\hat{1} / \hat{1} = \hat{1}$, $\hat{1}^{\hat{1}} = \hat{1}$, etc.

The *isodual isofields* are the antihomomorphic image of $\hat{F}(\hat{a},+,*)$ induced by the map $\hat{1} \rightarrow \hat{1}^d = -\hat{1}$ and are given by the Class II structures

$$\hat{F}^d_{II} = \{ (\hat{a}^d,+,*^d) \mid \hat{a}^d = \bar{a}\hat{1}^d,\ *^d = \times T^d \times,\ T^d = -T,\ \hat{1}^d = -\hat{1} \}, \qquad (2.6)$$

in which the elements $\hat{a}^d = \bar{a}\hat{1}^d$ are called *isodual isonumbers*. For real numbers we have $n^d = -n$, for complex numbers we have $c^d = -\bar{c}$, where $\bar{c}$ is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^d = -q^\dagger$, where $\dagger$ is the Hermitean conjugate.

It is to be observed that the imaginary number i is *isoselfdual*, i.e., invariant under isoduality, $i^d = -\bar{i} = i$, and the conjugation of a complex number is given by $(n + i \times m)^d = n^d + i^d \times^d m^d = -n + (-\bar{i})(-\times)(-m) = -n + im$. The *isodual isosum* is given by $\hat{a}^d + \hat{b}^d = (\bar{a} + \bar{b})\hat{1}^d$, while for the *isodual isomultiplication* we have $\hat{a}^d *^d \hat{b}^d = \hat{a}^d T^d \hat{b}^d = -\hat{a}^d T \hat{b}^d = (\bar{a}\bar{b})\hat{1}^d$.

An important property is that the norm of isodual isofields is negative-definite,

$$\uparrow \hat{a}^d \uparrow^d = |\bar{a}| \hat{1}^d = -\uparrow \hat{a} \uparrow. \qquad (2.7)$$

The latter property has nontrivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, angular momentum, etc., are negative-definite.* For these reasons, isodual theories provide a novel and intriguing characterization of antimatter [61].

Note also that, *as a necessary condition for isotopies (isodualities) all isofields* $\hat{F}_I(\hat{a},+,*)$ *(isodual isofields* $\hat{F}_{II}{}^d(\hat{a}^d,+,*^d)$*) are isomorphic (antiisomorphic) to the original field* $F(a,+,*)$. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing to the freedom in the generalized unit [26].

As an illustrative example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{1} = I\, e^{t N \int d^3x\, \psi^\dagger_\uparrow(r)\, \phi_\downarrow(r)}, \qquad (2.8)$$

where t represents time, N is a positive real constant, and $\psi_\uparrow$ and $\bar{\phi}_\downarrow$ are the wavefunctions of the two electrons of the Cooper pair with related orientation of their spin. *Animalu's isounit* (2.8) therefore represents the *nonlocal-integral* contributions due to the wave overlapping of the two electrons in the Cooper pairs.

We also recall the still more general *genofields* [73] at the foundation of Albert-Santilli Lie-admissibility, characterized first by an isotopy of conventional fields, and then by the differentiation of the isomultiplications to the right $\hat{a} > \hat{b} = \hat{A} \times R \times \hat{b}$ from that to the left $\hat{a} < \hat{b} = \hat{a} \times S \times \hat{b}$, $\hat{a} > \hat{b} \neq \hat{a} < \hat{b}$, $R \neq S$. The important property is that all abstract axioms of a field are verified per *each* ordered isomultiplication thus yielding one *genofield* $\hat{F}^>(\hat{a},+,>)$ for the multiplication to the right and a different one $^<\hat{F}(\hat{a},+,<)$ for the multiplication to the left.

A still more general formulation is currently under study via the *hyperstructures* [105], [106]. In essence, the genotopic elements R and S are irreducible and fixed in the genotopic products $\hat{a} > \hat{b}$ and $\hat{a} < \hat{b}$. In the transition to the hyperstructure, the genotopic element R and S assume finite or infinite and ordered or non-ordered sets of values.

We finally recall that *all liftings of the sum are prohibited in this study*, e.g., because they would imply the divergence of exponentiation and all quantities defined via an infinite series [73].

The isotopies and isodualities of fields outlined above admit corresponding lifting of all conventional mathematical quantities defined on them, such as vector and metric spaces, functional analysis, differential calculus, etc. as presented in [100], whose knowledge is hereon tacitly assumed.

2.C. Isolinearity, isolocality and isocanonicity. In Sect. 1 we pointed out that the primary limitations of Lie's theory are that the theory is linear, local and canonical. The primary objectives of the Lie-Santilli isotheory are the achievement a covering theory which is structurally nonlinear, nonlocal and noncanonical, yet capable of reconstructing linearity, locality and canonicity on isospaces over isofields called *isolinearity, isolocality and isocanonicity.*

In turn, the preservation at the abstract level of the original linearity, locality and canonicity will prove to be crucial for the achievement of mathematical and physical consistency under nonlinear, nonlocal and nonhamiltonian interactions.

Let $S(x,R)$ be a conventional, real vector space with local coordinates x over the reals $R(n,+,\times)$, and let $x' = A(w)\, x$, $w \in F$, $x'^t = x' A^t(w)$ be a conventional *right, and left, linear, local and canonical transformation* on $S(x,R)$, where t denotes anspose.

The isotopic lifting $S(x,R) \rightarrow \hat{S}(\hat{x},\hat{R})$ [100] requires a corresponding necessary otopy of the transformation theory. In fact, it is instructive for the interested ader to verify that the application of conventional transformations to the ospace $\hat{S}(\hat{x},\hat{R})$ implies the *loss* of linearity, transitivity and other basic properties.

For these and other reasons, Santilli submitted in the original proposals [52] ne isotopy of the transformation theory, called *isotransformation theory*, which characterized by *isotransforms*

$$\hat{x}' = \hat{A}(\hat{w}) * \hat{x} = \hat{A}(\hat{w})\, T\, \hat{x}, \quad \hat{x}'^{t} = x^{t} * \hat{A}^{\dagger}(\hat{w}) = \hat{x}^{t}\, T\, \hat{A}^{\dagger}(\hat{w}), \tag{2.9a}$$

$$T = T^{\dagger} = \text{fixed}, \quad \hat{x} \in \hat{S}(\hat{x},\hat{R}), \quad \hat{w} \in \hat{R}(\hat{n},+,\hat{\times}), \quad \hat{1} = T^{-1}. \tag{2.9b}$$

vhere the isotopic element T is assumed to be of Class I.

The most dominant aspect in the transition from the conventional to the sotopic transforms is that, while the former are linear, local and canonical, the atter are *nonlinear* in the coordinates as well as other quantities and their derivatives of arbitrary order, *nonlocal-integral* in all needed quantities, and *noncanonical when projected in the original spaces* $S(x,F)$. In fact, from the unrestricted nature of the isotopic element T, the projection of isotransform (2.9) in $S(x, R)$ reads (for $\hat{x} = (\hat{x}^k) = (x^k)$)

$$x' = \hat{A}(\hat{w})\, T(t, x, \dot{x}, \ddot{x}, \ldots)\, x . \tag{2.10}$$

But the conventional and isotopic transforms coincide at the abstract level where we have no distinction between the modular action "Ax" and its isotopic form "A*x". We therefore have the following

Proposition 2.1 [115]: *Isotransforms (2.9) of Class I are "isolinear" when formulated on isospaces $\hat{S}(\hat{x},\hat{F})$ because they verify the conditions of linearity in isospaces,*

$$\hat{A} * (\hat{a} * \hat{x} + \hat{b} * \hat{y}) = \hat{a} * (\hat{A} * \hat{x}) + \hat{b} * (\hat{A} * \hat{y}), \tag{4.2.4a}$$

$$\forall\, \hat{x}, \hat{y} \in \hat{S}(\hat{x},\hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}(\hat{a},+,\hat{\times}), \tag{4.2.4b}$$

and coincide with linear transforms at the abstract level.

More directly, we can say that a Lie algebra is linear because it can be

interpreted as a linear vector space over a conventional field. By the same token, we can that that a *Lie-Santilli isoalgebra is isolinear because it can be interpreted as an isolinear vector space over an isofield.*

Note that conventional transforms are characterized by the *right modular associative action* Ax of A on $x \in S(x,R)$. Isotransforms (2.9) are then characterized by the *right isomodular associative action* action $\hat{A} * \hat{x}$ of $\hat{A}$ on $\hat{x} \in \hat{S}(\hat{x},\hat{R})$.

The situation for locality and canonicity follows the same lines [100]. It is known that Lie's theory is local because it possesses a local-differential topology. By the same token, we can say that *the Lie-isotopic theory is isolocal because it possesses the Tsagas-Sourlas isolocal topology* ([100], Sect. 1.6).

Similarly, we can say that the Lie-Santilli isotheory is isocanonical because it is derivable from a first-order action which is canonical in isospace over isofields ([100], Sect. 2).

The following property is important for the understanding of isotopic theories:

Proposition 2.2 [115]: *All possible nonlinear, nonlocal and noncanonical transforms on a vector space $S(x,R)$*

$$x' = B(w, x, \ldots)\, x , \quad x \in S(x,R), \quad w \in R(n,+,\times), \tag{2.11}$$

can always be rewritten in an identical isolinear, isolocal and isocanonical form, that is, there always exists at least one isotopy of the base field, $R(n,+,\times) \rightarrow \hat{R}(\hat{n},+,*)$, and a corresponding isotopy of the space $S(x,R) \rightarrow \hat{S}(\hat{x},\hat{R})$ under which

$$x' = B(w, x, \ldots)\, x \equiv \hat{A}(\hat{w}) * x , \quad \hat{T} = \hat{A}^{-1} B . \tag{2.12}$$

The above property is at the foundation of the "direct universality" of the Lie-Santilli isotheories, that is, its applicability to all possible nonlinear, nonlocal and noncanonical systems (universality) in the frame of the experimenter (direct universality). In order to apply a Lie-Santilli isosymmetry to a nonlinear, nonlocal and noncanonical system, one has merely to identity one of its possible isolinear, isolocal and isocanonical *identical reformulation in the same system of (contravariant) local coordinates*. The applicability of the methods studied in Sect. 3 then follows.

the *"isodual isotransforms"* of Class II are given by the image of isotransforms (2.9) under isoduality, and, as such, are defined on the isodual isospace $\hat{S}^{d}(\hat{x}^{d}, \hat{R}^{d})$ [100]

$$\hat{x}^{d} = \hat{A}^{d}(\hat{w}^{d}) *^{d} \hat{x}^{d} = -\hat{A}^{d}(\hat{w}^{d}) * \hat{x}^{d}, \quad x \in \hat{S}^{d}(x, \hat{R}^{d}), \quad \hat{w}^{d} \in \hat{R}^{d}(\hat{n}, +, *^{d}), \tag{2.13a}$$

$$\hat{x}^{d\,t\,\prime} = \hat{x}^{d} *^{d} \hat{A}^{d\,t}(\hat{w}^{d}) = -\hat{x}^{d} * \hat{A}^{d\,\dagger}(\hat{w}^{d}) \tag{2.13b}$$

where $\hat{A}^{t}$ and $\hat{A}^{d\dagger}$ will be identified later on in Sect. 3.

Isodual isotransforms characterize the isodual Lie isotopic theory which, in turn, characterizes the isodual symmetries for the treatment of antiparticles.

3. Isotopies and Isodualities of Lie's Theory

As recalled in Sect. 1, Lie's theory (see [13], [15]) is centrally dependent on the basic n-dimensional unit $I = \text{diag.}(1, 1, ..., 1)$ in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie-Santilli isotheory [52], [53], [110] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{1}(x, \dot{x}, \ddot{x}, ...)$.

One can therefore see from the very outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields, isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

3.A. Isotopies and isodualities of universal enveloping associative algebras. Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements A, B, C,... , trivial associative product AB and unit I. Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *isoassociative envelopes*. They coincide with ξ as vector spaces but are equipped with the isoproduct so as to admit $\hat{1}$ as the correct (right and left) unit

$$\hat{\xi}: \quad A * B = ATB, \quad T \text{ fixed}, \quad I * A = A * I \equiv A \quad \forall A\ \hat{\xi}, \quad \hat{1} = T^{-1}. \tag{3.1}$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N-dimensional Lie algebra L with ordered basis (X_k), $k = 1, 2, ..., N$, $[\xi(L)]^{-} \sim L$ over F, and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincaré-Birkhoff-Witt theorem [15]. A fundamental result due to Santilli ([52], see also [110], p. 154-163) is the following:

Theorem 3.1. *The cosets of $\hat{1}$ and the standard, isotopically mapped monomials*

$$\hat{1}, \quad X_k \qquad X_i * X_j \quad (i \leq j), \qquad X_i * X_j * X_k \quad (i \leq j \leq k), \tag{3.2}$$

form a basis of the universal enveloping isoassociative algebra $\hat{\xi}(L)$ of a Lie algebra L.

A first important consequence is that the isotopies of conventional exponentiation are given by the expression, called isoexponentiation, for $\hat{w} \in \hat{F}$,

$$e_{\hat{\xi}}^{i\hat{w}*X} = \hat{1} + (i\hat{w}*X)/1! + (i\hat{w}*X)*(i\hat{w}*X)/2! + ... = \hat{1}\,(e^{iwTX}) = (e^{iXTw})\,\hat{1}. \tag{3.3}$$

The implications of Theorem 3.1 also emerge at the level of functional analysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structurally dependent on the conventional exponentiation. As a result, they must be lifted under isotopies into the expressions [23]

$$f(x) = (1/2\pi)\int_{-\infty}^{+\infty} g(k) * e_{\hat{\xi}}^{ikx}\, dk, \quad g(k) = (1/2\pi)\int_{-\infty}^{+\infty} f(x) * e_{\hat{\xi}}^{-ikx}\, dx, \tag{3.4}$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a gaussian in isofunctional analysis is given by [23]

$$f(x) = N * e_{\hat{\xi}}^{-x/2a^2} = N e^{-x^2 T/2a^2} \rightarrow g(k) = N * e_{\hat{\xi}}^{-k^2 a^2/2} = N' e^{-k^2 T a^2/2}. \tag{3.5}$$

As a result, the widths are of the type $\Delta x \sim aT^{-\frac{1}{2}}$, $\Delta k \sim a^{-1}T^{-\frac{1}{2}}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x\, \Delta k \sim 1$ in favor of the local *isouncertainties* [61b]

$$\Delta x\, \Delta k \sim \hat{1}, \tag{3.6}$$

which illustrate the nontriviality of the the isotopy.

The *isodual isoenvelopes* $\hat{\xi}^{d}$ are characterized by the isodual basis $X_k{}^{d} = -X_k$ defined with respect to the isodual isounits $\hat{1}^{d} = -\hat{1}$ and isodual isotopic element

$T^d = -T$ over the isodual isofields $\hat{F}^d$. The *isodual isoex-ponentiation* is then given by

$$e_{\hat{\xi}^d}^{\,i^d w^d \times^d X^d} = \hat{1}^d\,(e^{iwTX}) = -\,e_{\hat{\xi}}^{\,iwX}, \qquad (3.7)$$

and plays an important role for the characterization of antiparticles as possessing negative-definite energy and moving backward in time (as necessary when using isodual isofields).

It is easy to see that Theorem 3.1 holds, as originally formulated [52], for envelopes now called of Class III, thus unifying isoenvelopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, the theorem was conceived to unify with one single Lie algebra basis X_k nonisomorphic compact and noncompact algebras of the same dimension N (see the example of Section 3.E).

The isotopy $\xi \rightarrow \hat{\xi}$ is not a conventional map because the local coordinates x, the infinitesimal generators X_k and the parameters w_k are not changed by assumption, while the underlying unit and related associative product are changed. Also, in the operator realization the Lie and Lie-Santilli isotheory can be linked by nonunitary transformations $UU^\dagger = \hat{1} \neq I$ [116], for which

$$I \rightarrow \hat{1} = U\,I\,U^\dagger,\; A\,B \rightarrow U\,A\,B\,U^\dagger = A' * B' = A'\,T\,B',\; T = (U\,U^\dagger)^{-1}. \quad (3.8)$$

where $A' = UAU^\dagger$, $B' = UBU^\dagger$. The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations,

$$H\,|\,b> = E\,|\,b>, \quad H = H^\dagger, \quad E \in \Re(n,+,\times),$$

and their isotopic form in the same Hamiltonian [116]

$$H * |\,\hat{b}> = H\,T\,|\,\hat{b}> = \hat{E} * |\,\hat{b}> \equiv E'\,|\,\hat{b}>, \quad H = H^\dagger,$$

where $E' \in \Re(n,+,\times)$, with consequential *different eigenvalues for the same operator* H, $E' \neq E$ (see Section 3.E for an example). We therefore expect that *the weights of the Lie and Lie-Santilli theories are different.*

3.B. Isotopies and isodualities of Lie algebras. A (finite-dimensional) isospace $\hat{L}$ over the isofield $\hat{F}$ of isoreal $\hat{\Re}(\hat{n},+,*)$ or isocomplex numbers with isotopic element T and isounit $\hat{1} = T^{-1}$ is called a *Lie-Santilli algebra* over $\hat{F}$ (see the original contributions [52], [53], [110], [115], [116] independent studies [3], [24], [31], [121] and references quoted therein), sometimes called *isoalgebra* (when no confusion with the isotopies of non-Lie algebras arises), when there is a composition $[A\hat{,}B]$ in $\hat{L}$, called *isocommutator*, which is isolinear (i.e., satisfies condition (2.40)) and such that for all A, B, C $\in \hat{L}$

$$[A\hat{,}B] = -\,[B\hat{,}A], \qquad (3.9a)$$

$$[A\hat{,}[B\hat{,}C]] + [B\hat{,}[C\hat{,}A]] + [C\hat{,}[A\hat{,}B]] = 0, \qquad (3.9b)$$

$$[A * B\hat{,}C] = A * [B\hat{,}C] + [A\hat{,}C] * B. \qquad (3.9c)$$

The isoalgebras are said to be: *isoreal (isocomplex)* when $\hat{F} = \hat{\Re}$ ($\hat{F} = \hat{C}$), and *isoabelian* when $[A\hat{,}B] \equiv 0\ \forall$ A, B $\in \hat{L}$. A subset $\hat{L}_0$ of $\hat{L}$ is said to be an *isosubalgebra* of $\hat{L}$ when $[\hat{L}_0\hat{,}\hat{L}_0] \subset \hat{L}_0$ and an *isoideal* when $[\hat{L}\hat{,}\hat{L}_0] \subset \hat{L}_0$. A maximal isoideal which verifies the property $[\hat{L}\hat{,}\hat{L}_0] = 0$ is called the *isocenter* of $\hat{L}$. For the isotopies of conventional notions, theorems and properties of Lie algebras see [74].

We recall the *isotopic generalizations of the celebrated Lie's First, Second and Third Theorems* introduced in ref. [47], but which we do not review here for brevity (see [52]). For instance, the isotopic second theorem reads

$$[X_i\hat{,}X_j] = X_i * X_j - X_j * X_i = X_i\,T(x,\ldots)\,X_j - X_j\,T(x,\ldots)\,X_i = \hat{C}_{ij}{}^k(x,\dot{x},\ddot{x},\ldots) * X_k, \qquad (3.10)$$

where the $\hat{C}$'s are called the *structure functions*, generally have an explicit dependence on the underlying isospace (see the example of Section 3.E), and verify certain restrictions from the Isotopic Third Theorem.

Let L be an N-dimensional Lie algebra with conventional commutation rules and structure constants $C_{ij}{}^k$ on a space S(x,F) with local coordinates x over a field F, and let L be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then $\hat{L}$ can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\hat{\xi}(L)]^-$ attached to the isoassociative envelope $\hat{\xi}(L)$ ([52]). In this way, an infinite number of isoalgebras $\hat{L}$, depending on all possible isounits $\hat{1}$, can be constructed via the isotopies of one single Lie algebra L. It is easy to prove the following result:

Theorem 3.2. *The isotopies* L $\rightarrow \hat{L}$ *of an N-dimensional Lie algebra* L *preserve the original dimensionality.*

In fact, the basis e_k , $k = 1, 2, ..., N$ of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted $\hat{e}_k$. Let the commutation rules of L be given by $[e_i , e_j] = C_{ij}{}^k e_k$.

The isocommutation rules of the isotopes $\hat{L}$ are

$$[\hat{e}_i \hat{,} \hat{e}_j] = \hat{e}_i T \hat{e}_j - \hat{e}_j T \hat{e}_i = \hat{C}_{ij}{}^k(x, \dot{x}, \ddot{x}, ...) \hat{e}_k , \qquad (3.11)$$

where $\hat{C} = CT$. One can then see in this way the necessity of lifting the structure "constants" into structure "functions", as correctly predicted by the Isotopic Second Theorem.

The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. To begin, we here introduce the *general isolinear and isocomplex Lie–Santilli algebras* denoted $G\hat{L}(n,\hat{C})$ as the vector isospaces of all n×n complex matrices over $\hat{C}$. It is easy to see that they are closed under isocommutators as in the conventional case. The *isocenter* of $G\hat{L}(n,\hat{C})$ is then given by $\hat{a} * \hat{1}$, $\forall\ \hat{a} \in \hat{\mathfrak{R}}$. The subset of all complex n×n matrices with null trace is also closed under isocommutators. We shall call it the *special, complex, isolinear isoalgebra* and denote it with $S\hat{L}(n,\hat{C})$. The subset of all antisymmetric n×n real matrices X, $X^t = - X$, is also closed under isocommutators, it is called the *isoorthogonal algebra*, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non–exceptional, Lie–Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ each one admitting realizations of Classes I, II, III, IV and V (of which only Classes I, II and III are studied herein). In fact, $\hat{A}_{n-1} = S\hat{L}(n,\hat{C})$; $\hat{B}_n = \hat{O}(2n+1, \hat{C})$; $\hat{C}_n = S\hat{P}(n,\hat{C})$; and $\hat{D}_n = \hat{O}(2n,\hat{C})$. One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of *homomorphism, automorphism and isomorphism* of two isoalgebras $\hat{L}$ and $\hat{L}'$, as well as of *simplicity* and *semisimplicity* are the conventional ones. Similarly, all properties of Lie algebras based on the addition, such as the *direct and semidirect sums*, carry over to the isotopic context unchanged (because of the preservation of the conventional additive unit 0).

An *isoderivation* $\hat{D}$ of an isoalgebra $\hat{L}$ is an isolinear mapping of $\hat{L}$ into itself satisfying the property

$$\hat{D}([A , B]) = [\hat{D}(A) \hat{,} B] + [A \hat{,} \hat{D}(B)] \qquad \forall\ A, B \in \hat{L} . \qquad (3.12)$$

If two maps $\hat{D}_1$ and $\hat{D}_2$ are isoderivations, then $\hat{a}*\hat{D}_1 + \hat{b}*\hat{D}_2$ is also an isoderivation, and the isocommutators of $\hat{D}_1$ and $\hat{D}_2$ is also an isoderivation. Thus, the set of all isoderivations forms a Lie–Santilli algebra as in the conventional case.

The isolinear map $a\hat{d}(\hat{L})$ of $\hat{L}$ into itself defined by

$$a\hat{d}\ A(B) = [A \hat{,} B] , \qquad \forall\ A, B \in \hat{L} , \qquad (3.13)$$

is called the *isoadjoint map*. It is an isoderivation, as one can prove via the iso–Jacobi identity. The set of all $a\hat{d}(A)$ is therefore an isolinear isoalgebra, called *isoadjoint algebra* and denoted $\hat{L}_a$. It also results to be an isoideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(o)} = \hat{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(o)} \hat{,} \hat{L}^{(o)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isoideals of $\hat{L}$. $\hat{L}$ is then called *isosolvable* if, for some positive integer n, $\hat{L}^{(n)} \equiv 0$. Consider also the sequence $L_{(o)} = L$, $\hat{L}_{(1)} = [\hat{L}_{(o)} \hat{,} \hat{L}]$, $\hat{L}_{(2)} = [\hat{L}_{(1)} \hat{,} \hat{L}]$, etc. Then $\hat{L}$ is said to be *isonilpotent* if, for some positive integer n, $\hat{L}_{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosolvable, but the converse is not necessarily true.

Let the *isotrace* of a matrix be given by the element of the isofield [61]

$$T\hat{r}\ A = (\mathrm{Tr}\ A) \hat{1} \in \hat{F} , \qquad (3.14)$$

where Tr A is the conventional trace. Then $T\hat{r}\,(A * B) = (T\hat{r}\ A) * (T\hat{r}\ B)$ and $T\hat{r}\,(B\ A\ B^{-1}) = T\hat{r}\ A$. Thus, the $T\hat{r}\ A$ preserves the axioms of Tr A, by therefore being a correct isotopy. Then the isoscalar product

$$(A \hat{,} B) = T\hat{r}\,[(A\hat{d}X) * (A\hat{d}\ B)] , \qquad (3.15)$$

is here called the *isokilling form*. It is easy to see that $(A \hat{,} B)$ is symmetric, bilinear, and verifies the property $(A\hat{d}X(Y) \hat{,} Z) + (Y \hat{,} Ad\ X(Z)) = 0$, thus being a correct, axiom–preserving isotopy of the conventional Killing form.

Let e_k , $k = 1, 2, ..., N$, be the basis of L with one–to–one invertible map $e_k \rightarrow \hat{e}_k$ to the basis of $\hat{L}$. Generic elements in $\hat{L}$ can then be written in terms of local coordinates x, y, z, $A = x^i \hat{e}_i$ and $B = y^j \hat{e}_j$, and

$$C = z^k \hat{e}_k = [A \hat{,} B] = x^i\ y^j\ [\hat{e}_i \hat{,} \hat{e}_j] = x^i\ x^j\ \hat{C}_{ij}{}^k\ \hat{e}_k . \qquad (3.16)$$

Thus,

$$[A\hat{d}\ A(B)]^k \doteq [A \hat{,} B]^k = \hat{C}_{ij}{}^k x^i x^j . \qquad (3.17)$$

We now introduce the *isocartan tensor* $\tilde{g}_{ij}$ of an isoalgebra $\hat{L}$ via the definition $(A \hat{,} B) = \tilde{g}_{ij} x^i y^j$ yielding

$$\tilde{g}_{ij}(x, \dot{x}, \ddot{x}, ...) = \tilde{C}_{ip}{}^k \tilde{C}_{jk}{}^p . \qquad (3.18)$$

Note that the isocartan tensor has the general dependence of the isometric tensor of Section 2.C, thus confirming the inner consistency among the various branches of the isotopic theory. In particular, the isocartan tensor is generally *nonlinear, nonlocal and noncanonical* in all variables $x, \dot{x}, \ddot{x}, ...$. This clarifies that isotopic generalization of the Riemannian spaces studied in ref. [60] $R(x,g,\mathfrak{R}) \Rightarrow \hat{R}(x,\hat{g},\hat{\mathfrak{R}})$, $\hat{g} = \hat{g}(s, x, \dot{x}, \ddot{x}, ...)$, has its origin in the very structure of the Lie-isotopic theory.

The isocartan tensor also clarifies another fundamental point of Section 1, that the isotopies naturally lead to an arbitrary dependence in the velocities and accelerations, exactly as needed for realistic treatment of the problems identified in Section 1, and that their restriction to the nonlinear dependence on the coordinates x only, as generally needed for the exterior (e.g., gravitational) problem, would be manifestly un-necessary.

The isotopies of the remaining aspects of the structure theory of Lie algebras can be completed by the interested reader. Here we limit ourselves to recall that when the isocartan form is positive- (or negative-) definite, $\hat{L}$ is compact, otherwise it is noncompact. Then it is easy to prove the following

Theorem 3.3. *The Class III liftings $\hat{L}$ of a compact (noncompact) Lie algebra L are not necessarily compact (noncompact).*

The identification of the remaining properties which are not preserved under liftings of Class III is an instructive task for the interested reader. For instance, if the original structure is irreducible, its isotopic image is not necessarily so even for Class I, trivially, because the isounit itself can be reducible, thus yielding a reducible isotopic structure.

Let $\hat{L}$ be an isoalgebra with generators X_k and isounit $\hat{1} = T^{-1} > 0$. From Equations (3.7) we then see that the *isodual Lie-Santilli algebras* $\hat{L}^d$ of $\hat{L}$ is characterized by the isocommutators

$$[X_i \hat{,} X_j]^d = -[X_i \hat{,} X_j] = \tilde{C}_{ij}{}^{k\,(d)} X^d{}_k , \quad \tilde{C}_{ij}{}^{k(d)} = -\tilde{C}_{ij}{}^k . \qquad (3.19)$$

$\hat{L}$ and $\hat{L}^d$ are then (anti) isomorphic. Note that the isoalgebras of Class III contain all Class I isoalgebras $\hat{L}$ and all their isoduals $\hat{L}^d$. The above remarks therefore show that the Lie-Santilli theory can be naturally formulated for Class III, as implicitly done in the original proposal [47]. The formulation of the same theory for Class IV or V is however considerably involved on technical grounds thus requiring specific studies.

The notion of isoduality applies also to conventional Lie algebras L, by permitting the identification of the *isodual Lie algebras* L^d via the rule [62]

$$[X_i , X_j]^d = X^d{}_i I^d X^d{}_j - X^d{}_j I^d X^d{}_i = -[X_i , X_j] =$$

$$= C_{ij}{}^{k\,(d)} X^d{}_k , \quad C_{ij}{}^{k(d)} = -C_{ij}{}^k . \qquad (3.20)$$

Note the necessity of the isotopies for the very construction of the isodual of conventional Lie algebras. In fact, they require the nontrivial lift of the unit $I \Rightarrow I^d = (-I)$, with consequential necessary generalization of the Lie product $AB - BA$ into the isotopic form $ATB - BTA$.

For realizations of the Lie-Santilli isoalgebras in classical and operator mechanics, we refer the reader for brevity to ref.s [115], [116].

3.C. Isotopies and isodualities of Lie groups. A *right Lie-Santilli isogroup* $\hat{G}$ (see the original contribution [52], independent monographs [3], [24], [31], [121] and papers quoted therein) on an isospace $\hat{S}(x,\hat{F})$ over an isofield $\hat{F}$, $\hat{1} = T^{-1}$ (of isoreal $\hat{\mathfrak{R}}$ or isocomplex numbers $\hat{C}$), also called *isotransformation group* or *isogroup*, is a group which maps each element $x \in \hat{S}(x,\hat{F})$ into a new element $x' \in \hat{S}(x,\hat{F})$ via the isotransformations $x' = \hat{U} * x = \hat{U}Tx$, T fixed, such that: (1) The map $(U, x) \rightarrow \hat{U} * x$ of $\hat{G} \times \hat{S}(x,\hat{F})$ onto $\hat{S}(x,\hat{F})$ is isodifferentiable; (2) $\hat{1} * \hat{U} = \hat{U} * \hat{1} = \hat{U}\ \forall\ \hat{U} \in \hat{G}$; and (3) $\hat{U}_1 * (\hat{U}_2 * x) = (\hat{U}_1 * \hat{U}_2) * x$, $\forall\ x \in \hat{S}(x,\hat{F})$ and $\hat{U}_1 , \hat{U}_2 \in \hat{G}$. A *left isotransformation group* is defined accordingly.

The notions of *connected or simply connected transformation groups* carry over to the isogroups in their entirety. We consider hereon the connected isotransformation groups. Right or left isogroups are characterized by the following laws [47]

$$\hat{U}(0) = \hat{1}, \quad \hat{U}(\hat{w}) * \hat{U}(\hat{w}') = \hat{U}(\hat{w}') * \hat{U}(\hat{w}) = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) * \hat{U}(-\hat{w}) = \hat{1}, \ \hat{w} \in \hat{F}. \qquad (3.21)$$

Their most direct realization of the isotransformation groups is that via isoexponentiation (3.3),

$$\hat{U}(\hat{w}) = \prod_k e_\xi^{\,i\hat{w}_k * X_k} = \prod_k e_\xi^{\,i X_k * \hat{w}_k} =$$
$$= \hat{1}\{\prod_k e^{iw_k T X_k}\} = \{\prod_k e^{iX_k T w_k}\}\hat{1}, \qquad (3.22)$$

where the X's and w's are the infinitesimal generator and parameters, respectively, of the original algebra L. Equations (3.22) hold for some open neighborhood of N of the isoorigin of $\hat{L}$ and, in this way, characterize some open neighborhood of the isounit of $\hat{G}$. Then the isotransformations can be reduced to an ordinary transform for computational convenience,

$$x' = \hat{U} * x = \{\prod_k e_\xi^{\,i X_k * w_k}\} * x = \{\prod_k e^{iX_k T w_k}\} x, \qquad (3.23)$$

with the understanding that, on rigorous mathematical grounds, only the isotransform is correct.

Still another important result obtained in [52] is the proof that conventional group composition laws admit a consistent isotopic lifting, resulting in the following *isotopy of the Baker–Campbell–Hausdorff Theorem*

$$\{e_\xi^{X}\} * \{e_\xi^{X}\} = e_\xi^{X_3}, X_3 = X_1 + X_2 + [X_1 \hat{,} X_2]/2 + [(X_1 - X_2)\hat{,}[X_1 \hat{,} X_2]]/12 + \ldots \qquad (3.24)$$

Note the crucial appearance of the isotopic element $T(x, \dot{x}, \ddot{x}, \ldots)$ in the *exponent* of the isogroup. This ensures a structural generalization of Lie's theory of the desired nonlinear, nonlocal and noncanonical form. For details see [49] and [74].

The structure theory of isogroups is also mostly unexplored at this writing. In the following we shall point out that the conventional structure theory of Lie groups does indeed admit a consistent isotopic lifting. The isotopies of the notions of weak and strong continuity of [22] are a necessary pre-requisite. Let $\hat{L}$ be a (finite-dimensional) Lie-Santilli algebra with (ordered) basis $\{X_k\}$, k = 1, 2, ..., N. For a sufficiently small neighborhood N of the isoorigin of $\hat{L}$, a generic element of $\hat{G}$ can be written

$$\hat{U}(w) = \prod{}^*_{k=1,2,\ldots,N} e_\xi^{\,iX_k w_k}, \qquad (3.25)$$

which characterizes some open neighborhood M of the isounit $\hat{1}$ of $\hat{G}$. The map

$$\Phi_{\hat{U}_1}(\hat{U}_2) = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{1}}, \qquad (3.26)$$

for a fixed $\hat{U}_1 \in \hat{G}$, characterizes an *inner isoautomorphism* of $\hat{G}$ onto $\hat{G}$. The corresponding isoautomorphism of the algebra $\hat{L}$ can be readily computed by considering the above expression in the neighborhood of the isounit $\hat{1}$. In fact, we have

$$\hat{U}'_2 = \hat{U}_1 * \hat{U}_2 * \hat{U}_1^{-\hat{1}} \cong \hat{U}_2 + w_1 w_2 [X_2 \hat{,} X_1] + o^{(2)}. \qquad (3.27)$$

The reduction of the isogroups to isoalgebras requires the knowledge of isodifferentials $\hat{\partial} w = T dw$ and isoderivatives $\hat{\partial}/\hat{\partial} w = \hat{1} dw$ [100] under which we have the following expression in one dimension:

$$i^{-1} \frac{\hat{\partial}}{\hat{\partial} w} \hat{U}\Big|_{w=0} = X * e_\xi^{\,iwX}\Big|_{w=0} = X. \qquad (3.28)$$

Thus, to every inner isoautomorphism of $\hat{G}$, there corresponds an inner isoautomorphism of $\hat{L}$ which can be expressed in the form:

$$(\hat{L})_i^{\,j} = \hat{C}_{ki}^{\,j} w^k. \qquad (3.29)$$

The isogroup $\hat{G}_a$ of all inner isoautomorphisms of $\hat{G}$ is called the *isoadjoint group*. It is possible to prove that the Lie–Santilli algebra of $\hat{G}_a$ is the isoadjoint algebra $\hat{L}_a$ of $\hat{L}$. This establishes that the connections between algebras and groups carry over in their entirety under isotopies.

We mentioned before that the direct sum of isoalgebras is the conventional operation because the addition is not lifted under isotopies (otherwise there will be the loss of distributivity, see [73]). The corresponding operation for groups is the semidirect product which, as such, demands care in its formulation.

Let $\hat{G}$ be an isogroup and $\hat{G}_a$ the group of all its inner isoautomorphisms. Let $\hat{G}^o{}_a$ be a subgroup of $\hat{G}_a$, and let $\hat{\Lambda}(\hat{g})$ be the image of $\hat{g} \in \hat{G}$ under $\hat{G}^o{}_a$. The *semidirect isoproduct* $\hat{G} \hat{\times} \hat{G}^o{}_a$ of $\hat{G}$ and $\hat{G}^o{}_a$ is the isogroup of all ordered pairs

$$(\hat{g}, \hat{\Lambda}) * (\hat{g}', \hat{\Lambda}) = (\hat{g} * \hat{\Lambda}(\hat{g}'), \hat{\Lambda} * \hat{\Lambda}'), \qquad (3.30)$$

with total isounit given by $(\hat{1}, \hat{1}_{\hat{\Lambda}})$ and inverse $\hat{g}, \hat{\Lambda} s)^{-\hat{1}} = (\hat{\Lambda}^{-\hat{1}}(\hat{g}^{-\hat{1}}), \hat{\Lambda}^{-\hat{1}})$. The above notion plays an important role in the isotopies of the inhomogeneous space–time symmetries outlined later on.

Let $\hat{G}_1$ and $\hat{G}_2$ be two isogroups with respective isounits $\hat{1}_1$ and $\hat{1}_2$. The *direct isoproduct* $\hat{G}_1 \hat{\otimes} \hat{G}_2$ of $\hat{G}_1$ and $\hat{G}_2$ is the isogroup of all ordered pairs $(\hat{g}_1, \hat{g}_2)$, $\hat{g}_1 \in \hat{G}_1, \hat{g}_2 \in \hat{G}_2$, with isomultiplication

$$(\hat{g}_1, \hat{g}_2) * (\hat{g}'_1, \hat{g}'_2) = (\hat{g}_1 * \hat{g}'_1, \hat{g}_2 * \hat{g}'_2), \tag{3.31}$$

total isounit $(\hat{1}_1, \hat{1}_2)$ and inverse $(\hat{g}_1^{-\hat{1}}, \hat{g}_2^{-\hat{1}})$. The isotopies of the remaining aspects of the structure theory of Lie groups can then be investigated by the interested reader.

Let $\hat{G}$ be an N-dimensional isotransformation group of Class I with infinitesimal generators X_k, $k = 1, 2, ..., N$. The *isodual Lie-Santilli group* $\hat{G}^d$ of $\hat{G}$ ([52], [53]) is the N-dimensional isogroup with generators $X_k{}^d = -X_k$ constructed with respect to the isodual isounit $\hat{1}^d = -\hat{1}$ over the isodual isofield $\hat{F}^d$. By recalling that $w \in F \Rightarrow w^d \in F^d$, $w^d = -w$, a generic element of $\hat{G}^d$ in a suitable neighborhood of $\hat{1}^d$ is therefore given by

$$\hat{U}^d(\hat{w}^d) = e_{\xi^d}{}^{i^d \hat{w}^d * X^d} = -e_{\xi}{}^{i\hat{w} * X} = -\hat{U}(\hat{w}). \tag{3.32}$$

The above antiautomorphic conjugation can also be defined for conventional Lie group, yielding the *isodual Lie group* G^d of G with generic elements $U^d(w^d) = e_{\xi^d}{}^{iw^d X} = -e_{\xi}{}^{iwX}$.

The symmetries significant for this paper are: the conventional form G, its isodual G^d, the isotopic form $\hat{G}$ and the isodual isotopic form $\hat{G}^d$. These different forms are useful for the respective characterization of particles and antiparticles in vacuum (exterior problem) or within physical media (interior problem) [116].

It is hoped that the reader can see from the above elements that the entire conventional Lie's theory does indeed admit a consistent and nontrivial lifting into the covering Lie-Santilli formulation. Particularly important are the isotopies of the conventional representation theory, known as the *isorepresentation theory* [116], which naturally yields the most general known, nonlinear, nonlocal and noncanonical representations of Lie groups. Studies along these latter lines were initiated by Santilli with the isorepresentations of $\hat{SO}(2)$ and of $\hat{SO}(3)$ [76], by Klimyk and Santilli Klimyk [27], and others.

A classical realization of the Lie-Santilli isogroups can be formulated on the isotangent bundle $T*\hat{E}(r,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, with local chart $a = (r^k, p_k)$, $\mu = 1, 2, 3, 4, 5, 6$, $k = 1, 2, 3$, and isounit [11-71]

$$\hat{1}_2 = \text{diag.} (\hat{1}, \hat{T}) \tag{3.33}$$

the Hamilton-Santilli equations

$$\hat{\partial} a^\mu / \hat{\partial} t = \hat{\Xi}^\mu = \omega^{\mu\alpha} T_{2\alpha}{}^\nu \frac{\partial H}{\partial a^\nu}, \tag{3.34}$$

where $\omega^{\mu\alpha}$ is the familiar canonical Lie tensor. Eq.s (3.34) can be isoexponentiated and, after factorization of the isounit, can be written

$$a(t) = (\hat{e}^{t\hat{\Xi}\partial/\partial a^\mu}) * a(0) = (e^{t\omega^{\mu\alpha} T_{2\alpha}{}^\nu (\partial H/\partial a^\mu)\, \partial/\partial a^\alpha})\, a(0), \tag{3.35}$$

where we have ignored the factorization of the isounit in the isoexponent for simplicity. The computation of the Lie-santilli isoalgebra is consequential and coincides at the abstract level with the conventional formulation in terms of vector fields.

An operator realization of the Lie-Santilli isogroups is given by *isounitary transformations* $x' = \hat{U} * x$ on an isohilbert space $\hat{\mathcal{H}}$ [100] with

$$\hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{1}, \tag{3.36}$$

with realization via an *isohermitean operator* H

$$\hat{U} = \hat{e}^{iHt} = (e^{iHTt})\hat{1}, \tag{3.37}$$

The use of the bimodular isotransforms and the techniques studied in this section, then characterize the corresponding Lie-Santilli isoalgebra, thus confirming the interconnection and mutual compatibility between isoalgebras and isogroups in exactly the same manner as that for the conventional theory.

The above classical and operator realizations are also interconnected in a unique and unambiguous way by the isoquantization [100].

3.D. Santilli's fundamental theorem on isosymmetries. We are now equipped to review without proof the following important result [62]:

Theorem 3.5. *Let* G *be an N-dimensional Lie symmetry group of an m-dimensional metric or pseudo-metric space* $S(x,g,F)$ *over a field* F

$$G: \; x' = A(w)\, x, \quad (x'-y')^\dagger A^\dagger g A (x-y) \equiv (x-y)^\dagger g (x-y),$$

$$A^\dagger g A = A g A^\dagger = g . \qquad (3.38)$$

Then the infinitely possible isotopes $\hat{G}$ *of* G *of Class* III *characterized by the same generators and parameters of* G *and new isounits* $\hat{1}$ (*isotopic elements* T), *automatically leave invariant the iso-composition on the isospaces* $\hat{S}(x,\hat{g},\hat{F})$, $\hat{g} = Tg$, $\hat{1} = T^{-1}$,

$$\hat{G}: x' = \hat{A}(w) * x, \; (x'-y')\dagger * \hat{A}\dagger \,\hat{g}\, \hat{A} * (x-y) =$$

$$= (x-y)\dagger \,\hat{g}\, (x-y), \;\; \hat{A}\dagger \hat{g} \hat{A} = \hat{A} \hat{g} \hat{A}\dagger = \hat{1}\hat{g}\hat{1}, \qquad (3.39)$$

The "direct universal" of the resulting isosymmetries for all infinitely possible isotopies $g \to \hat{g} = T(t, x, \dot{x}, \ddot{x}, ...)g$ is then evident owing to the completely unrestricted functional dependence of the isotopic element T. One should also note the *insufficiency* of the so-called *trivial isotopy*

$$X_k \;\to\; X'_k = X_k \hat{1}, \qquad (3.40)$$

for the achievement of the desired form-invariance. In fact, under the above mapping the isoexponentiation becomes

$$e_{\xi}^{\,i X'_k * w_k} = (e^{\,iX'_k T w_k})\hat{1} = (e^{\,iX_k w_k})\hat{1}, \qquad (3.41)$$

namely, we have the disappearance precisely of the isotopic element T in the exponent which provides the invariance of the isoseparation.

3.E. Isotopies and isodualities of the rotational symmetry. We now illustrate the Lie-Santilli isotheory with the first mathematically and physically significant case, the *isotopies of the rotational symmetry,* also called *isorotational symmetry.* They were first achieved by Santilli in paper [62], [63] and then studied in details in [monographs [115], [116], including the isotopies of SU(2), their isorepresentations, the iso-Clebsh-Gordon coefficients, etc.

Consider the lifting of the perfect sphere in Euclidean space $E(r,\delta,\mathfrak{R})$ with local coordinates $r = (x, y, z)$, and metric $\delta = \text{diag.}(1, 1, 1)$ over the reals $\mathfrak{R}$,

$$r^2 = r^t \delta r = xx + yy + zz, \qquad (3.42)$$

into the most general possible ellipsoid of Class III on isospace $\hat{E}^{III}(r,\hat{\delta},\hat{\mathfrak{R}})$, $\hat{\delta} = T\delta$, $T = \text{diag.}(g_{11}, g_{22}, g_{33})$, $\hat{1} = T^{-1}$.

$$r^{\hat{2}} = r^t \hat{\delta} r = x g_{11} y + y g_{22} y + z g_{33} z,$$

$$\hat{\delta}^\dagger = \hat{\delta}, \; g_{kk} = g_{kk}(t, r, \dot{r}, \ddot{r}, ...) \neq 0, \qquad (3.43)$$

The invariance of the original separation r^2 is the conventional rotational symmetry O(3). The isotopic techniques permit the construction, in the needed explicit and finite form, of the isosymmetries $\hat{O}(3)$ of all infinitely possible generalized invariants $r^{\hat{2}}$ via the following steps: (1) Identification of the basic isotopic element T in the lifting $\delta \to \hat{\delta} = T\delta$ which, in this particular case, is given by the new metric $\hat{\delta}$ itself, $T \equiv \hat{\delta}$, and identification of the fundamental unit of the theory, $\hat{1} = T^{-1}$; (2) Consequential lifting of the basic field $\mathfrak{R}(n,+,\times) \Rightarrow \hat{\mathfrak{R}}(\hat{n},+,*)$; (3) Identification of the isospace in which the generalized metric $\hat{\delta}$ is defined, which is given by the three-dimensional isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, $\hat{\delta} = T\delta$, $\hat{1} = T^{-1}$; (4) Construction of the $\hat{O}(3)$ symmetry via the use of the original parameters of O(3) (the Euler's angles θ_k, k = 1, 2, 3), the original generators (the angular momentum components $M_k = \epsilon_{kij} r^i p_j$) in their fundamental (adjoint) representation, and the new metric $\hat{\delta}$; and (5) Classification, interpretation and application of the results.

The explicit construction of $\hat{O}(3)$ is straightforward. According to the Lie-Santilli theory, the connected component $\hat{SO}(3)$ of $\hat{O}(3)$ is given by [63]

$$\hat{SO}(3): \;\; r' = \hat{R}(\theta) * r, \; \hat{R}(\theta) = \textstyle\prod^{*}_{k=1,2,3} e_{\xi}^{\,iM_k\theta_k} =$$

$$= (\textstyle\prod_{k=1,2,3} e^{\,iM_k T\theta_k})\hat{1}, \qquad (3.44)$$

while the discrete component is given by the *isoinversions* [loc. cit.] $r' = \hat{\pi} * r = \pi r = -r$, where π is the conventional inversion.

Under the assumed conditions on the isotopic element T, the convergence of isoexponentiations is ensured by the original convergence, thus permitting the explicit construction of the isorotations, with example around the third axis [53]

$$x' = x\cos[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] + y g_{22}(g_{11}g_{22})^{-\frac{1}{2}} \sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}],$$

$$y' = -x g_{11}(g_{11}g_{22})^{-\frac{1}{2}} \sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] + y\cos[\theta_3[g_{11}g_{22})^{\frac{1}{2}}],$$

$$z' = z. \qquad (3.45)$$

(see [116] for general isorotations). One should note that the argument of the trigonometric functions as derived via the above isoexponentiation coincides with the isoangle of the isotrigonometry in $\hat{E}(r,\hat{\delta},\hat{\Re})$ (see paper [60]) thus confirming the remarkable compatibility and interconnections of the various branches of the the isotopic theory.

The computation of the isoalgebras $\hat{o}(3)$ of $\hat{O}(3)$ is then straighforward. When the M_k's are assumed to be in their regular representation we have [63]

$$\hat{o}(3): [M_i \hat{,} M_j] = M_i\, T\, M_j - M_j\, T\, M_i = \hat{C}_{ij}{}^k * M_k, \qquad (3.46)$$

where $\hat{C}_{ij}{}^k = \epsilon_{ijk}\, g_{kk}{}^{-1}\hat{1}$. The above isoalgebra illustrates the explicit dependence of the structure *functions.* The proof of the isomorphism $\hat{o}(3) \approx o(3)$ was done [loc. cit.] via a suitable reformulation of the basis under which the structure functions recover the value $\hat{\epsilon}_{ijk} = \epsilon_{ijk}\,\hat{1}$ .The isocenter of $s\hat{o}(3)$ is characterized by the *isocasimir invariants*

$$C^{(0)} = \hat{1}, \quad C^{(2)} = M^{\hat{2}} = M * M = \textstyle\sum_{k=1,2,3} M_k T\, M_k . \qquad (3.47)$$

In hadronic mechanics [116] one of the possible realizations is the following. The linear momentum operator has the isotopic form

$$p_k * |\hat{\psi}\rangle = -i\, \hat{\nabla}_k |\hat{\psi}\rangle = -i\, \hat{1}_k{}^i\, \nabla_i |\hat{\psi}\rangle .$$

(see [II-71] for a different realization). The fundamental isocommutation rules are then given by

$$[r^i \hat{,} p_j] = i\,\hat{\delta}^i{}_j = i\,\hat{1}\,\delta^i{}_j, \quad [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0 .$$

However, in their contravariant form the coordinates are given by $r_k = \hat{\delta}_{kl}\, r^l$. As a result $\hat{\nabla}_i\, r_j = \delta_{ij}$ (where the delta is the *conventional* Kronecker delta). In this case the fundamental isocommutation rules are given by

$$[r_i \hat{,} p_j] = i\,\hat{\delta}^i{}_j = i\,\hat{1}\,\delta^i{}_j, \quad [r_i \hat{,} r_j] = [p_i \hat{,} p_j] = 0,$$

namely, their eigenvalues *coincide* with the quantum ones. The operator isoalgebra $\hat{o}(3)$ with generators $M_k = \epsilon_{kij}\, r_i\, p_j$ is then given by

$$\hat{o}(3): \quad [M_i \hat{,} M_j] = M_i\, T\, M_j - M_j\, T\, M_i = i\,\hat{\epsilon}_{ij}{}^k * M_k ,$$

where $\hat{\epsilon}_{ij}{}^k = \epsilon_{ijk}\hat{1}$, namely *the product of the algebra is generalized, but the structure constants are the conventional ones* (see [61] for details). The above results illustrates again the abstract identity of quantum and hadronic mechanics.

Note the nonlinear–nonlocal–noncanonical character of isotransformations (3.45) owing to the unrestricted functional dependence of the diagonal elements g_{kk}. Note also the extreme simplicity of the final results. In fact, the explicit symmetry transformations of separation (3.43) are provided by just plotting the given g_{kk} values into transformations (3.45) without any need of any additional computation. Note finally that the above invariance includes as particular case the general isosymmetry $\hat{O}(3)$ of (the space–component of) gravitation which, since it is locally Euclidean, remains isomorphic to O(3).

As an example, the symmetry of the space–component of the Schwarzschild line element is given by plotting the following values

$$g_{11} = (1 - M/r)^{-1}, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta, \qquad (3.48)$$

(see next section for the full (3+1)–dimensional case).

Despite this simplicity, the implications of the above results are nontrivial. On physical grounds, the isounit $\hat{1} > 0$ permits a direct representation of the nonspherical shapes, as well as all their infinitely possible deformations. By recalling that O(3) is a *theory of rigid bodies,* $\hat{O}(3)$ results to be a *theory of deformable bodies* [63] with fundamentally novel physical applications in the theory of elasticity, nuclear physics, particle physics, crystallography, and other fields [115], [116].

On mathematical grounds, we have equally intriguing novel insights. To see them, one must first understand the background isogeometry $\hat{E}^{III}(r,\hat{\delta},\hat{\Re})$ which unifies all possible conics in $E(r,\delta,\Re)$ [115], as mentioned earlier. To be explicit in this important point, the geometric differences between (oblate or prolate) ellipsoids and (elliptic or hyperbolic) paraboloids have mathematical sense when projected in our Euclidean space $E(r,\delta,\Re)$. However all these surfaces are geometrically unified with the perfect isosphere in $\hat{E}(r,\hat{\delta},\hat{\Re})$.

These geometric occurrences permits the unification of O(3) and O(2.1), as well as of all their infinitely possible isotopes,as formulated in the original proposal [52]. In fact, the classification of all possible isosymmetries $\hat{O}(3)$, achieved in the original derivation [53], includes:

(1) The compact O(3) symmetry evidently for $\hat{\delta} = \delta = \mathrm{diag.}\,(1, 1, 1)$;
(2) The noncompact O(2.1) symmetry evidently for $\hat{\delta} = \mathrm{diag.}\,(1, 1, -1)$;

(3) The isodual $O^d(3)$ of $O(3)$ holding for $\hat\delta = \text{diag.}(-1, -1, -1)$;
(4) The isodual $O^d(2.1)$ of $O(2.1)$ holding for $\hat\delta = \text{diag.}(-1, -1, 1)$;
(5) The infinite family of compact isotopes $\hat O(3) \sim O(3)$ with $\hat I > 0$ for $\hat\delta = \text{diag.}(b_1^2, b_2^2, b_3^2)$, $b_k > 0$;
(6) The infinite family of noncompact isotopes $\hat O(2.1) \sim O(2.1)$ for $\hat\delta = \text{diag.}(b_1^2, b_2^2, -b_3^2)$;
(7) The infinite family of compact isodual isotopes $\hat O^d(3) \sim O^d(3)$ for $\hat\delta = \text{diag.}(-b_1^2, -b_2^2, -b_3^2)$;
(8) The infinite family of isodual isotopes $\hat O^d(2.1) \sim O^d(2.1)$ for $\hat\delta = \text{diag.}(-b_1^2, -b_2^2, b_3^2)$.

Even greater differentiations between the Lie and Lie–Santilli theories occur in their representations because of the change in the eigenvalue equations due to the nonunitarity of the map indicated in Sect. 3A, from the familiar form $H\psi = E°\psi$, to the isotopic form $H*\hat\psi = \hat E*\hat\psi \equiv E\hat\psi$, $E° \neq E$), thus implying generalized weights, Cartan tensors and other structures studied earlier.

The first differences emerge in the spectrum of eigenvalues of $\hat o(2)$ and $o(2)$. In fact, the $o(2)$ algebra on a conventional Hilbert space *solely* admits the spectrum $M = 0, 1, 2, 3$ (as a necessary condition of unitarity). For the covering $\hat o(2)$ isoalgebra on an isohilbert space with isotopic element $T = \text{Diag.}(g_{11}, g_{22})$, the spectrum is instead given by

$$\hat M = g_{11}^{-1/2} g_{22}^{-1/2} M$$

and, as such, it can acquire *continuous* values in a way fully consistent with the condition, this time, of *isounitarity*. For the general $\hat O(3)$ case see also the detailed studied of ref.s [116].

Similarly, the unitary irreducible representations of su(2) are characterize the familiar eigenvalues

$$J_3 \hat\psi = M \psi, \quad J^2 \psi = J(J+1)\psi, \quad M = J, J-1, \ldots, -J, \quad J = 0, \tfrac{1}{2}, 1, \ldots \quad (3.49)$$

Three classes of irreducible isorepresentation of $\hat{su}(2)$ were identified in [76] which, for the adjoint case, are given by the following generalizations of Pauli's matrices: (1) *Regular isopauli matrices*

$$\hat\sigma_1 = \Delta^{-1/2}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix} \quad \hat\sigma_2 = \Delta^{-1/2}\begin{pmatrix} 0 & -i g_{11} \\ +i g_{22} & 0 \end{pmatrix},$$

$$\hat\sigma_3 = \Delta^{-1/2}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}, \quad (3.50a)$$

$$T = \text{diag.}(g_{11}, g_{22}), \quad \Delta = \det T = g_{11} g_{22} > 0,$$

$$[\hat\sigma_i, \hat\sigma_j]_\xi = i\, 2\Delta^{1/2} \epsilon_{ijk} \hat\sigma_k ; \quad (3.50b)$$

$$\hat\sigma_3 * |\hat b> = \pm \Delta^{1/2} |\hat b>, \quad \hat\sigma^2 * |\hat b> = 3\Delta |\hat b>. \quad (3.50c)$$

(2) *Irregular isopauli matrices*

$$\hat\sigma_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat\sigma_2' = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$

$$\hat\sigma_3' = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat 1 \sigma_3, \quad (3.51a)$$

$$[\hat\sigma_1' \hat{,} \hat\sigma_2']_\xi = 2 i \hat\sigma_3',$$

$$[\hat\sigma_2', \hat\sigma_3']_\xi = 2 i \Delta \hat\sigma_1', \qquad [\hat\sigma_3', \hat\sigma_1']_\xi = 2 i \Delta \hat\sigma_2', \quad (3.51b)$$

$$\hat\sigma_3' * |\hat b> = \pm \Delta |\hat b>,$$

$$\hat\sigma^2 * |\hat b> = \Delta(\Delta + 2) |\hat b>. \quad (3.51c)$$

(3) *Standard isopauli matrices*

$$\hat\sigma_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat\sigma_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix},$$

$$\hat\sigma_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \quad (3.52a)$$

$$T = \text{diag.}(\lambda, \lambda^{-1}), \quad \lambda \neq 0, \qquad \Delta = \det T = 1,$$

$$[\hat\sigma''_i, \hat\sigma''_j]_\xi = i \epsilon_{ijk} \hat\sigma''_k, \quad (3.52b)$$

$$\hat\sigma''_3 * |\hat b> = \pm |\hat b>, \qquad \hat\sigma''^2 * |\hat b> = 3 |\hat b>. \quad (3.52c)$$

The primary differences in the above isorepresentations are the following. For the case of the regular isorepresentations, the isotopic contributions can be factorized with respect to the conventional Lie spectrum. For the irregular case this is no longer possible. Finally, for the standard case we have conventional spectra of eigenvalues under a generalized structure of the matrix representations, as indicated by the appearance of a completely unrestricted, integro-differential function λ.

The regular and irregular representations of $\hat{o}(3)$ and $s\hat{u}(2)$ are applied to the angular momentum and spin of particles under extreme physical conditions, such as an electron in the core of a collapsing star. The standard isorepresentations are applied to conventional particles evidently because of the preservation of conventional quantum numbers [116]. The appearance of the isotopic degrees of freedom then permit novel physical applications, that is, applications beyond the capacity of Lie's theory even for the simpler case of preservation of conventional spectra (see Section 3.G).

The spectrum-preserving map from the conventional representations J_g of a Lie-algebra L with metric tensor g to the covering isorepresentations $\mathfrak{J}_{\hat{g}}$ of the Lie-Santilli algebra $\hat{L}$ with isometric $\hat{g} = Tg$ and isounit $\hat{1} = T^{-1}$ is important for physical application. It is called the *Klimyk rule* [27] and it given by

$$\mathfrak{J}_{\hat{g}} = J_g P, \qquad P = k\hat{1}, \quad k \in \hat{F}, \tag{3.53}$$

under which Lie algebras are turned into Lie-Santilli isoalgebras

$$J_i J_j - J_j J_i = C_{ij}{}^k J_k \equiv (\mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i) k^{-1} T = C_{ij}{}^k k^{-1} T \mathfrak{J}_k,$$

that is,

$$\mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i = C_{ij}{}^k \mathfrak{J}_k,$$

thus showing the preservation of the original structure constants.

However, by no means, the Klimyk rule can produce *all* Lie-Santilli isoalgebras, because the latter are generally characterized by *nonunitary* transforms of conventional algebras, with a general variation of the structure constants.

Nevertheless, the Klimyk rule is sufficient for a number of physical applications where the preservation of conventional quantum numbers is important, because it permits the identification of one specific and explicit form of

standard isorepresentations with "hidden" degrees of freedom represented by the isotopic element T available for specific uses. For instance, the standard isopauli matrices permit the reconstruction of the exact isospin symmetry in nuclear physics under electromagnetic and weak interactions [76], or the construction of the *isoquark theory* [100] with all conventional quantum numbers, yet an *exact confinement* (with an identically null probability of tunnel effects for free quarks because of the incoherence between the interior and exterior Hilbert spaces), and other novel applications.

3.F. Isotopies and isodualities of the Lorentz and Poincare´ symmetries. One of the most fundamental results achieved by santilli as a culmination of all his efforts [47]-[118] is a structural generalization of the current formulation of the Poincaré symmetry with far reaching mathematical and physical implications which we can only indicate in this section.

The generalized symmetry has been entirely and solely studied by Santilli up to this writing, and it is called the *Poincaré-Santilli isosymmetry*, with no additional studies by other scholars known to this author on the study of the isosymmetry itself (thus, excluding studies on its *applications* which are numerous). In fact, Santilli proposed: the classical isotopies of the Lorentz symmetry in Ref. [59] of 1983; their operator image in paper [60] of the same year; their rotational component in papers [62], [63] of 1985 reviewed in the preceding section; a comprehensive classical study in memoir [67]; a comprehensive operator counterpart in memoir [72] of 1992 (with the first experimental verification to the Bose-Einstein correlation); the comprehensive classical and operator study in paper [79] of 1993; specific studied on the spinorial case were conducted in paper [95] with additional experimental verifications; a detailed classical treatment in monograph [114] and the operator treatment in monograph [116].

Consider the line element in Minkowski space $x^2 = x^\mu \eta_{\mu\nu} x^\nu$, $\mu, \nu = 1, 2, 3, 4$, with local coordinates $x = \{x^1, x^2, x^3, x^4\}$, $x^4 = c_0 t$, and metric $\eta = \text{diag.}(1, 1, 1, -1)$. Its simple invariance group, the six-dimensional Lorentz group L(3.1), is characterized by the (ordered sets of) parameters given by the Euler's angles and speed parameter, $w = \{w_k\} = \{\theta, v\}$, $k = 1, 2, \ldots, 6$, and generators $X = \{X_k\} = \{M_{\mu\nu}\}$, in their known fundamental representation (see, e.g., [31], [32]).

Suppose now that the Minkowskian line element is lifted into the most general possible nonlinear-integral form verifying the conditions of Class III

$$x^2 = x^\mu \hat{g}_{\mu\nu}(x, \dot{x}, \ddot{x}, \ldots) x^\nu, \qquad \det \hat{g} \neq 0, \qquad \hat{g} = \hat{g}^\dagger, \tag{3.54}$$

which represent: all modifications of the Minkowski metric as encountered, e.g., in particle physics; conventional exterior gravitational line elements with $\hat{g} = \hat{g}(x)$, such as the full Schwarzschild line element; all its possible generalizations for the interior problem; etc.

The explicit form of the simple, six-dimensional invariance of generalized line element x^2 was first constructed by Santilli [59] by following the space-time version of Steps 1 to 5 of the preceding section. Step 1 is the identification of the fundamental isotopic element T via the factorization of the Minkowski metric, $\hat{g} = T\eta$ which, under the assumed conditions, can always be diagonalized into the form

$$T = \text{diag.}(g_{11}, g_{22}, g_{33}, g_{44}), \quad T = T^{\dagger}, \quad \det T \neq 0. \tag{3.55}$$

The fundamental isounit of the theory is then given by $\hat{1} = T^{-1}$.

Step 2 is the lifting of the conventional numbers into the isonumbers via the isofields $\hat{\mathfrak{R}}(\hat{n},+,*)$, $\hat{n} = n\,\hat{1}$ (which are different than those of $\hat{O}(3)$ because of the different dimension of the isounit).

Step 3 is the construction of the isospaces in which the isometric $\hat{g}$ is properly defined, which are given by the isominkowski spaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$. The reader should keep in mind that, when $\hat{g}$ is a conventional Riemannian metric, isospaces $\hat{M}(x,\hat{g},\hat{\mathfrak{R}})$ are not the Riemannian spaces $R(x,\hat{g},\hat{\mathfrak{R}})$ because the basic units of the two spaces are different.

Step 4 is also straightforward. The *Lorentz-Santilli isosymmetry* $\hat{L}(3.1)$ is characterized by the isotransformations

$$\hat{O}(3.1): \qquad x' = \hat{\Lambda}(\hat{w}) * x = \tilde{\Lambda}(w)\, x, \tag{3.56}$$

verifying the basic properties

$$\hat{\Lambda}^{\dagger}\hat{g}\hat{\Lambda} = \hat{\Lambda}\hat{g}\hat{\Lambda}^{\dagger} = \hat{1}\,\hat{g}\,\hat{1}, \quad \text{or}$$

$$\tilde{\Lambda}^{\dagger}\hat{g}\tilde{\Lambda} = \tilde{\Lambda}\hat{g}\tilde{\Lambda}^{\dagger} = \hat{g}, \tag{3.57a}$$

$$\hat{\text{Det}}\,\hat{\Lambda} = [\,\text{Det}(\hat{\Lambda}\,T)\,] = \pm\hat{1}. \tag{3.57b}$$

It is easy to see that $\hat{L}(3.1)$ preserves the original connectivity properties of $L(3.1)$ (see [61] for a detailed study). The connected component $S\hat{O}(3.1)$ of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\,\hat{\Lambda} = +\hat{1}$ and has the structure [loc. cit.]

$$\hat{\Lambda}(w) = \textstyle\prod^{*}_{k=1,2,\ldots,6} e_{\xi}^{\,i X_k * \hat{w}_k} =$$

$$= \{\textstyle\prod_{k=1,2,\ldots,6} e^{\,i X_k T w_k}\}\hat{1}, \tag{3.58}$$

where the parameters are the conventional ones, the generators X_k are also the conventional ones in their fundamental representation and the isotopic element T is given by Equations (3.23). The discrete part of $\hat{L}(3.1)$ is characterized by $\hat{\text{Det}}\,\hat{\Lambda} = -\hat{1}$, and it is given by the *space-time isoinversions* [loc. cit.]

$$\hat{\pi} * x = \pi x = (-r, x^4), \quad \hat{\tau} * x = \tau x = (r, -x^4). \tag{3.59}$$

Again, under the assumed conditions for T, the convergence of infinite series (3.58) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry transformations in the needed explicit, finite form. Their space components have been given in the preceding Section 3.E. The additional *Lorentz-Santilli isoboosts* were explicitly computed for the first time in [59], yielding the expression for all possible isometrics $\hat{g}$

$$x'^1 = x^1, \quad x'^2 = x^2, \tag{3.60a}$$

$$x'^3 = x^3\cosh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,] - x^4 g_{44}(g_{33}g_{44})^{-\frac{1}{2}}\sinh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,] =$$

$$= \hat{\gamma}(x^3 - g_{33}^{-1/2} g_{44}^{1/2}\beta x^4), \tag{3.60b}$$

$$x'^4 = -x^3 g_{33}(g_{33}g_{44})^{-\frac{1}{2}}\sinh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,] + x^4\cosh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,]$$

$$= \hat{\gamma}(x^4 - g_{33}^{1/2} g_{44}^{-1/2}\beta x^3), \tag{3.60c}$$

where $x^4 = c_o t$, $\beta = v/c_o$,

$$\beta = v^k g_{kk} v^k / c_o g_{44} c_o, \tag{3.61a}$$

$$\cosh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,] = \hat{\gamma} = (1 - \beta^2)^{-\frac{1}{2}},$$

$$\sinh[\,v(g_{33}g_{44})^{\frac{1}{2}}\,] = \beta\hat{\gamma}. \tag{3.61b}$$

Again, one should note: (A) the unrestricted character of the functional dependence of the isometric $\hat{g}$; (B) the remarkable simplicity of the final results whereby the explicit symmetry transformations are merely given by plotting the values $\hat{g}_{\mu\mu}$ in Equations (3.60); and (C) the generally nonlinear–nonlocal–noncanonical character of the isosymmetry.

The isocommutation rules when the generators $M_{\mu\nu}$ are in their regular representation can also be readily computed and are given by [loc. cit.]

$$\hat{o}(3.1): [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = \hat{g}_{\nu\alpha} M_{\beta\mu} - \hat{g}_{\mu\alpha} M_{\beta\nu} - \hat{g}_{\nu\beta} M_{\alpha\mu} + \hat{g}_{\mu\beta} M_{\alpha\nu}, \quad (3.62)$$

with isocasimirs

$$C^{(0)} = \hat{1}, \quad C^{(1)} = \tfrac{1}{2} M_{\mu\nu} T M^{\mu\nu} = M*M - N*N, \quad (3.63a)$$

$$C^{(3)} = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} M_{\mu\nu} T M_{\rho\sigma} = -M*N, \; M =$$

$$= (M_{12}, M_{23}, M_{31}), \quad N = (M_{01}, M_{02}, M_{03}) \quad (3.63c)$$

The classification of all possible isotopes $\hat{SO}(3.1)$ was also done in the original construction [59] via the realizations of the isotopic element

$$T = \text{diag.}(\pm b_1^2, \pm b_2^2, \pm b_3^2, \pm b_4^2), \quad b_\mu > 0, \quad (3.64)$$

where the b's are the characteristic functions of the interior medium, resulting in:

(1) The conventional orthogonal symmetry SO(4) for T = diag. (1, 1, 1, −1);
(2) The conventional Lorentz symmetry SO(3.1) for T = diag. (1, 1, 1, 1);
(3) The conventional de Sitter symmetry SO(2.2) for T = diag. (1, 1, −1, 1);
(4) The isodual $SO^d(4)$ for T = diag. (−1, −1, −1, 1);
(5) The isodual $O^d(3.1)$ for T = − diag. (1, 1, 1, 1);
(6) The isodual $SO^d(2.2)$ for T = diag. (−1, −1, 1, −1);
(7) The infinite family of isotopes $\hat{SO}(4) \approx SO(4)$ for $T = \text{diag.}(b_1^2, b_2^2, b_3^2, -b_4^2)$;
(8) The infinite family of isotopes $\hat{SO}(3.1) \approx SO(3.1)$ for $T = \text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$;
(9) The infinite family of isotopes $\hat{SO}(2.2) \approx SO(2.2)$ for $T = \text{diag.}(-b_1^2, b_2^2, b_3^2, b_4^2)$;
(10) The infinite family of isoduals $\hat{SO}^d(4) \approx SO^d(4)$ for $T = \text{diag.}(-b_1^2, -b_2^2, -b_3^2, b_4^2)$;
(11) The infinite family of isoduals $\hat{SO}^d(3.1) \approx SO(3.1)$ for $T = -\text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$;
(12) The infinite family of isoduals $\hat{SO}^d(2.2) \approx SO^d(2.2)$ for $T = \text{diag.}(b_1^2, -b_2^2, -b_3^2, -b_4^2)$.

On the basis of the above results, Santilli submitted the *conjecture* that *all simple Lie algebra of the same dimension over a field of characteristic zero in Cartan classification can be unified into one single abstract isotopic algebra of the same dimension.*

The above conjecture was proved by Santilli for the cases n = 3 and 6. A theorem unifying all possible fields into the isoreals was proved by Kadeisvili et al [26] in the expectation of such general unification. The conjecture has been recently studied by Tsagas [124] for the non-exceptional case.

In the above presentation we have shown that the lifting of the Lorentz symmetry can be naturally formulated for Class III. Nevertheless, whenever dealing with physical applications, the isotopic element is restricted to have the positive- or negative-definite structure $T = \pm\text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2)$, thus restricting the isotopies to $\hat{SO}(3.1) \approx SO(3.1)$ and $\hat{SO}^d(3.1) \approx SO^d(3.1)$.

The operator realization of the latter Lorentz-Santilli isoalgebra is the following. The linear four-momentum admits the isotopic realization [116]

$$p_\mu * |\hat{\psi}> = -i \hat{\partial}_\mu |\hat{\psi}> = -i T_\mu{}^\nu \partial_\nu |\hat{\psi}>.$$

Also, for $x_\mu = \eta_{\mu\nu} x^\nu$ (where η is the conventional Minkowski metric), one can show that $\hat{\partial}_\mu x_\nu = \hat{\eta}_{\mu\nu}$. The fundamental relativistic isocommutation rules are then given by [loc. cit.]

$$[x_\mu \hat{,} p_\nu] = i\,\hat{\eta}_{\mu\nu}, \quad [x_\mu \hat{,} x_\nu] = [p_\mu \hat{,} p_\nu] = 0,$$

The isocommutation rules are then given by

$$\hat{o}(3.1): [M_{\mu\nu} \hat{,} M_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}), \quad (3.62)$$

thus confirming the isomorphism $\hat{SO}(3.1) \approx SO(3.1)$ for all positive-definite T.

The *Poincaré-Santilli isosymmetry* [79]

$$\hat{P}(3.1) = \hat{L}(3.1) \times \hat{T}(3.1), \quad (3.65)$$

and its isodual $\hat{P}^d(3.1)$ have been constructed in their classical [114] and operator [116] forms as well as in their isospinorial form $\hat{\mathcal{P}}(3.1) = \hat{SL}(2.\hat{C}) \times \hat{T}(3.1)$ [95]. We here limit ourselves to a brief outline of the nonspinorial case mainly to illustrate the advances in the structure of isoalgebras and isogroups studied in this paper.

A generic element of $\hat{P}(3.1)$ can be written $\hat{A} = (\hat{\Lambda}, \hat{a})$, $\hat{\Lambda} \in \hat{O}(3.1)$, $\hat{a} \in \hat{T}(3.1)$ with isocomposition

$$\hat{A}' * \hat{A} = (\hat{\Lambda}', \hat{a}') * (\hat{\Lambda}, \hat{a}) = (\hat{\Lambda} * \hat{\Lambda}', \hat{a} + \hat{\Lambda}' * a'), \tag{3.66}$$

The realization important for physical applications is that via conventional generators in their adjoint representation for a system of n particles of non-null mass m_a

$$X = (X_k) = (M_{\mu\nu} = \sum_a (x_{a\mu}\, p_{a\nu} - x_{a\nu}\, p_{a\mu}),$$
$$P = \sum_a p_a), \; k = 1, 2, .10, \tag{3.67}$$

and conventional parameters $w = (w_k) = (v, \theta, a)$, where v represents the Lorentz parameters, θ represents the Euler's angles, and a characterizes conventional space-time translations.

The connected component of the isopoincaré group is given by

$$\hat{P}(3.1): \quad x' = \hat{A} * x, \qquad \hat{A} = \prod{}^*_k e_\xi^{\,iX_k w_k} = (\prod_k e^{\,iX_k T w_k})\hat{1}, \tag{3.68}$$

where the isotopic element T and the Lorentz generators $M_{\mu\nu}$ have the same realization as for $\hat{O}(3.1)$. The primary different with isosymmetries $\hat{O}((3.1)$ is the appearance of the isotranslations

$$\hat{T}(3.1) * x = (e_\xi^{\,iP\eta a}) * x = e_\xi^{\,iP\hat{g}a}) * x = x + \hat{a}, \; \hat{T}(3.1) * p \equiv 0. \tag{3.69}$$

Theorem 3.6 (General Poincare´– Santilli Isogroup [95]: *The "general Poincaré–Santilli isogroup' of Class III as characterized by the isotheory is given by the twelve-dimensional isotransforms*

$$x' = \hat{A} * x \qquad \text{Lorentz–Santilli isotransf.}, \tag{3.70a}$$

$$x' = x + a_0 B(s, x, \dot{x}, \ddot{x}, ...), \; \text{isotransl.}, \tag{3.70b}$$

$$x' = \hat{\pi}_r * x = (-r, x^4), \; \text{space isoinv.}, \tag{3.70c}$$

$$x' = \hat{\pi}_t * x = (r, -x^4), \qquad \text{time isoinv.}, \tag{3.70d}$$

$$\hat{1} \to \hat{1}^d = -\hat{1}, \;\; \hat{\eta} \to \hat{\eta}^d = -\eta, \quad \text{isoselfduality} \tag{3.70e}$$

$$\hat{1} \to \hat{1}' = n^2 \hat{1}, \; \hat{\eta} \to \hat{\eta}' = \hat{\eta} / n^2, \; \text{isoselfdilation.} \tag{3.70e}$$

where the B-functions are given by the expansions

$$B_\mu = b_\mu + a^\alpha [b_\mu \hat{,} P_\alpha] / 1! + a^\alpha a^\beta [[b_\mu \hat{,} P_\alpha] \hat{,} P_\beta] / 2! + ... \tag{3.71}$$

Isotransforms (3.70a)–(3.70d) are a direct consequence of the preceding analysis. The last two isotransforms (3.70e) and (3.70f) originate from the *isoscalar character* of the line element, that is, its structure $(x - y)^2$ = number×isounit $\in \hat{R}(\hat{n},+,*)$. In fact, the same isotransforms cannot be defined for the conventional line element $(x - y)^2$ = number $\in R(n,+,\times)$. Note that the isoduality and isodilation of the unit *do not exist* for the conventional transform, and this explains the reason of the transition from a ten- to a twelve-dimensional structure.

The operator realization of the Poincaré–Santilli isosymmetry is characterized by the isocommutation rules

$$[M_{\mu\nu} \hat{,} M_{\alpha\beta}] = i(\hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu}), \tag{3.72a}$$
$$[M_{\mu\nu} \hat{,} P_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu),$$
$$[P_\mu \hat{,} P_\nu] = 0, \; \mu, \nu, \alpha, \beta = 1, 2, 3, 4, \tag{3.72b}$$

and the isocenter is characterized by the isocasimirs

$$C^{(0)} = \hat{1}, \quad C^{(1)} = P^2 = P\, T\, P = P_\mu \hat{g}^{\mu\nu} P_\nu, \tag{3.73a}$$

$$C^{(2)} = \hat{W}^2 = \hat{W}_\mu \hat{g}^{\mu\nu} \hat{W}_\nu, \tag{3.73b}$$

$$\hat{W}_\mu = \epsilon_{\mu\alpha\beta\rho} J^{\alpha\beta} * P^\rho. \tag{3.73c}$$

The *restricted isotransformations* occur when the isotopic element T is constant.

An important application of the isotranslation is the characterization of the so-called *isoplane-waves* on $\hat{M}(x,\hat{\eta},\hat{\Re})$

$$\hat{\psi}(x) = e_\xi^{\,ipx} = \hat{1}\, e^{\,ipTx} = \hat{1}\, e^{\,ip_\mu \hat{g}^{\mu\nu} x_\nu} = \hat{1}\, e^{\,i(p_k b_k^2 x_k - p_4 b_4^2 x_4)} \tag{3.74}$$

which are solutions of the isotopic field equations, represents electromagnetic waves propagating within inhomogeneous and anisotropic media such as out atmosphere and offer quite intriguing predictions for experimentally verifiable <novel> effects, that is, effects beyond the predictive or descriptive capacities of the Poincaré symmetry.

As one can see, the verification of total conservation laws (for a system assumed as isolated from the rest of the universe), is intrinsic in the very structure of the isosymmetry. In fact, the generators are the conventional ones and, since they are invariant under the action of the group they generate, they characterize conventional total conservation laws. The simplicity of reading off the total conservation laws from the generators of the isosymmetry should be compared with the rather complex proof in conventional gravitational theories.

The *isodual Poincaré-Santilli isosymmetry* $\hat{P}^d(3.1)$ is characterized by the isodual generators $X_k{}^d = - X_k$, the isodual parameters $w_k{}^d = - w_k$, and the isodual isotopic element $T^d = -T$, resulting in the change of sign of isotransforms. This implies a novel *law of universal invariant under isoduàlity* which essentially state that any system which is invariant under a given symmetry is automatically invariant under its isodual. In turn, this law apparently permits novel advances in the study of antiparticles [116].

The significance of the Lie-Santilli isotheory for gravitation is illustrated by the following important property of the isosymmetry $\hat{P}(3.1)$ which evidently follows from of Theorem 3.5:

Theorem 3.7 (Universal Poincare´– Santilli Isosymmetry) [95]: *The general Poincaré-Santilli isotransforms of Theorem 3.6 with Class III isounits* $\hat{1}(x, \dot{x}, \ddot{x}, ...) = T^{-1} > 0$ *leave invariant all infinitely possible* (3+1)*-dimensional intervals with isometrics* $\hat{\eta}(x, \dot{x}, \ddot{x}, ...) = T(x, \dot{x}, \ddot{x}, ...)\eta$, *where* η *is the Minkowski metric,*

$$(x - y)^{\hat{2}} = [(x - y)^{\mu}\, \hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, ...)\, (x - y)^{\nu}]\, \hat{1}, \qquad (3.75)$$

It is an instructive exercise for the interested reader to verify that the isodistance (3.75) is indeed left invariant by all isotransforms (3.70).

As it is well known, each invariance of a space-time separation has profound physical implications. In fact, Theorem 3.7 essentially characterizes a covering of the special relativity for interior conditions worked out by Santilli at the classical [114] and operator [116] level, and known as *Santilli's isospecial relativity.*

The latter is a *covering* of the conventional relativity because: 1) it applies for much broader systems (nonlocal-integral and variationally nonselfadjoint systems); 2) it is constructed via structurally more general methods (the isotopic ones), and 3) it contains the conventional relativity as a particular case for $\hat{1} = I$.

Yet, the two relativities coincide at the abstract level by conception and construction for Class I isotopies [114], [116]. This ultimate identity of the special and isospecial relativities assures the axiomatic consistency of the new relativity because criticisms on the latter ultimately result to be criticisms on Einstein's theories.

To outline some of the main result and implications of the isospecial relativity, the Lorentz-Santilli isosymmetry has numerous applications for *interior conditions*, such as [116]: direct representation of locally varying speeds of light

$$c = c_o\, g_{44}{}^{1/2} = c_o / n,$$

where n is the ordinary index of refraction; exact-numerical representation of the difference in cosmological redshift between quasars and their associated galaxies, which is reduced to the *decrease of the speed of light* within the quasar's huge chromospheres; and several others in various fields.

The isoinversions permit the regaining of exact discrete symmetries when conventionally broken, such as the regaining of the exact space-parity under *weak* interactions by embedding the symmetry breaking terms in the isounit.

The invariance under isoduality (isoselfduality) assures the consistency of the isodual representation of antimatter, evidently because the same invariant holds identically for both matter and antimatter.

Moreover, the invariance under isotopic dilation (isoselfdilation) confirms the direct representation of the locally varying character of the speed of light. For instance, light propagating within homogeneous and isotropic water is represented by the isotopic element T with elements $g_{\mu\mu} = n^2$. The isoinvariant then reduces identically to the conventional invariant

$$(x - y)^{\hat{2}} = [(x - y)^{\mu}(\eta_{\mu\mu} / n^2)(x - y)^{\nu}](n^2 I) \equiv$$

$$\equiv (x - y)^{\mu}\, \eta_{\mu\mu}\, (x - y)^{\mu}\, I.$$

This permits the resolution of the problematic aspects of the special relativity in water, such as the apparent violation of the principle of causality, or the violation of the relativistic sum of speeds (because the sum of two light speeds in water does not yield the speed of light in water for the conventional Lorentz symmetry, but the sum is correct for the Lorentz–Santilli isotransforms).

Rather intriguingly, the quantity n in isoinvariance (3.70f) is non–null but otherwise arbitrary. Santilli's isospecial relativity therefore predicts in a natural way that the speed of light is a *local* quantity which arbitrarily smaller or bigger than the speed of light in vacuum. In fact, $c = c_0/n$ is smaller than c_0 in ordinary media such as water, but it is predicted to be bigger than c_0 in other conditions, such as in the hyperdense media inside hadrons or inside stars. For all these aspects and related references, see [116].

The above results are only the beginning of the implications of Theorem 3.7. In fact, the implications of Theorem 3.7 for gravitation alone are far reaching, and we can only indicate them here without treatment.

To begin, the theorem includes as particular cases the conventional Riemannian metric $\hat{\eta}(x, \dot{x}, \ddot{x}, ...) = g(x)$,. *The Poincaré–Santilli isosymmetry therefore provides the universal invariance of all infinitely possible exterior gravitation in vacuum.* More generally, Theorem 3.7 includes all infinitely possible signature–preserving modifications of the Minkowski and Riemannian metrics for *interior* problems. The simplicity of this universal invariance should also be kept in mind and compared with the known complexity of other approaches to nonlinear symmetries. In fact, one merely *plots* the $g_{\mu\mu}$ elements in isotransforms (3.45), (3.60), (3.70) without any need to compute anything, because the invariance of general separation (3.75) is ensured by the theorem. For numerous examples, see [95], [116].

Moreover, Theorem 3.7 implies *the unification of the special and general relativities.* [116]. After all, the unification is a necessary prerequisite for the very achievement of the universal symmetry of gravitation. Santilli achieves the unification by factorizing the Minkowski metric in any given exterior Riemannian metric g(x),

$$g(x) = T_{gr}(x)\,\eta\,, \tag{3.77}$$

and then by embedding the *gravitational isotopic element* $T_{gr}(x)$ in the *gravitational isounit*

$$\hat{1}_{gr}(x) = [\,T_{gr}(x)\,]^{-1}\,. \tag{3.78}$$

The Poincaré–santilli isosymmetry with the above isounit then evidently unifies the general and special relativity.

Note *the necessity of the representation of gravitation in the isominkowskian space* $\hat{M}(\hat{x},\hat{\eta},\hat{R})$, $\hat{\eta}(x) = T_{gr}(x)\eta$, $\hat{1}_{gr} = [T_{gr}(x)]^{-1}$, *for the achievement of such a unification.* In fact, no isosymmetry can be formulated in Riemannian spaces, as clear from the review of this section. This implies the formulation of gravity in an *isoflat* space. In fact, the space $\hat{M}(\hat{x},\hat{\eta},\hat{R})$, being an isotopy of the Minkowski space, preserves the geometric properties of the latter, including flatness, yet possesses the same metric as the Riemannian space, thus permitting a novel characterization of gravity.

Another implication of Theorem 3.6 is *a novel quantization of gravity* [116] which is based on *the embedding of gravitation in the unit of relativistic quantum mechanics without any need of a Hamiltonian.* In fact, the quantization is achieved via the lifting of the four–dimensional Minkowskian unit I = diag. (1, 1, 1, 1) of relativistic quantum mechanics into $\hat{1}_{gr}(x)$. As the reader can verify, the operator treatment of the Poincaré–santilli isosymmetry reviewed above *is* a quantum version of gravity for $\hat{1}(x, \dot{x}, \ddot{x}, ...) = \hat{1}_{gr}(x)$. The commutativity of the linear momentum, Eq.s (3.72b) confirms the novel achievement of a *flat* representation of gravity in terms of the *Riemannian* metric which emerges as the *structure functions* $g(x) \equiv \hat{\eta}$ of Eq.s (3.72).

The *isotopic* quantization gravity, called by Santilli *quantum–iso–gravity,* has itself rather deep implications. Recall that quantum gravity is afflicted by serious problems of consistency, such as the lack of invariance of the unit with consequential inapplicability to actual measures, the general lack of preservation of Hermiticity in time with consequential lack of observables, etc. [116]. Quantum–iso–gravity avoids all these problems *ab initio.* In fact, the isotopies assure that quantum–iso–gravity is as axiomatically consistent as relativistic quantum mechanics. After all, the two theories coincide at the abstract level because, from the local Minkowskian character of gravity, $T_{gr}(x)$ is necessarily positive–definite.

Moreover, *Theorem 3.7 predicts antigravity for elementary antiparticles in the field of matter* [86], [116]. In essence, calculations show that the gravitational force for *antimatter–antimatter systems in vacuum* characterized by $P^d(3.1)$ is *attractive* in the same way as for *matter–matter systems in vacuum* characterized by P(3.1). However, *antimatter–matter systems in vacuum experience a gravitational repulsion,* because they are characterized by the *projection* of $P^d(3.1)$ in the space of P(3.1) (see [116] for details. Note that these results are derived by the simplest possible case of Theorem 3.7, that in vacuum for $\hat{1} = I$ and $\hat{1}^d = -I$.

Theorem 3.7 has even greater implications in cosmology, because it implies a new conception of the universe called *Santilli's isocosmology* [116], which is based on the universal isosymmetry $U = \hat{P}_I(3.1)\times\hat{P}^d{}_{II}(3.1)$ and implies that, at the limit of equal amounts of matter and antimatter, all total physical characteristics of the universe are identically null, including null total energy, null total mass, null total time, etc.

This renders the act of creation of the universe more mathematically treatable then the "big bang" and other models, because the total characteristics remain null before and after the creation in Santilli's isocosmology, while for the "big bang" and other models we have the creating of the immensity of the universe literally from "nothing" with evident large discontinuity at creation.

Recall that the Poincaré symmetry provides the invariance only of relativistic classical and quantum systems. The significance of the Poincaré–Santilli isosymmetry is then illustrated by the fact that it provides the invariance of all (well behaved) infinitely possible, linear or nonlinear, local or nonlocal, Hamiltonian or nonhamiltonian, relativistic or gravitational, exterior or interior, classical or operator, and local or cosmological systems.

3.G: Mathematical and physical applications. Lie's theory is known to be at the foundation of virtually all branches of mathematics. The existence of intriguing and novel applications in mathematics originating from the Lie–Santilli theory is then beyond scientific doubts.

With the understanding that mathematical studies are at their first infancy, the isotopies have already identified new branches of mathematics besides isoalgebras, isogroups and isorepresentations. We here mention: the new branch of number theory dealing with isonumbers; the new branch of functional isoanalysis dealing with isospecial isofunctions, isotransforms and isodistribution; the new branch of topology dealing with the integro-differential topology; the new branch of the theory of manifold dealing with isomanifolds and their intriguing properties; and so on. It is hoped that interested mathematicians will contributed to these novel mathematical advances which have been identified and developed until now mainly by physicists, except a few exceptions.

Lie's theory in its traditional linear-local-canonical formulation is also known to be at the foundation of all branches of contemporary physics. Profound physical implications due to the covering, nonlinear-nonlocal-noncanonical Lie-Santilli theory cannot therefore be dismissed in a credible way.

With the understanding that these latter applications too are at the beginning and so much remains to be done, we have recalled after Theorem 3.7 some of the implications of the isotheory. We refer the interested reader to monographs [116], [118] for several additional applications and experimental verifications in nuclear physics, particle physics, astrophysics, cosmology,

superconductivity, theoretical biology and other fields.

The indication of existing contributions directly or indirectly related to the topic of this paper would be appreciated.

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