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Number 1

LIE ALGEBRAS AND FLEXIBLE LIE-ADMISSIBLE ALGEBRAS

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ABOUT THE TOPIC

Lie's theory, with its diversification into algebras, groups and geometries, constitutes one of the most fundamental branches of contemporary mathematics.

In applied mathematics, Lie's theory is equally fundamental. For instance, contemporary physical theories (such as classical mechanics, statistical mechanics and quantum mechanics) constitute different realizations of Lie's theory beginning from their most fundamental dynamical part, the time evolution.

This book deals with a generalization of Lie algebras (beyond grading-supersymmetric extensions) which was proposed by A. A. Albert in 1948 under the name of Lie—admissible algebras, and subsequently developed by a number of mathematicians and theoreticians.

The contents of this book is therefore of fundamental relevance for pure as well as applied mathematics. On the former grounds, the Lie—admissible algebras permit the broadening of the mathematical structure of all branches of mathematics dealing with Lie algebras. On the latter grounds, the Lie—admissible algebras permit the generalization of physical theories for a deeper understanding of nature.

The book is authored by one of the foremost mathematical leaders in the field.

The book is indispensable for all mathematicians interested in fundamental advances, as well as for all theoreticians interested in the broadening of the structure of contemporary physical theories.

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TO

Karen, Peggy, Jane, and Michael

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PREFACE

Following the long history of Lie algebras in physics, and since the introduction of Jordan algebras by physicist P. Jordan, there have been considerable efforts to generalize the formalism of quantum mechanics by means of other nonassociative algebras, such as Jordan, noncommutative Jordan, and octonion algebras.

Since A. A. Albert introduced Lie–admissible algebras in 1948, very little has been known about the structure of these algebras until the recent emergence of Lie–admissible algebras as a fundamental methodological tool in theoretical physics and mechanics. As far as I am aware, R. M. Santilli is the first physicist who became interested in Lie-admissible algebras. In an article written in 1967, he states: "As is known, there have been attempts to introduce new algebraic structures in physics other than Lie algebras (L.A.). One of the most interesting attempts is the Jordan investigation on the r–number algebras today called (commutative) Jordan algebras (C.J.A.), which however have not been successful in their physical applications. We personally think that a possible reason for this disappointment in elementary particle physics may be the want of L.A. content in the C.J.A. In other words, L.A. should not be abandoned, but might be expanded. For instance, the validity of L.A. for free particles is well known. It may be interesting to investigate the possible validity of new algebraic structures of an interacting or decaying region but only in such a way that the standard procedures corresponding to the free states remain unchanged, that is, preserving in any case a well–defined L.A. content."

Both nonflexible and flexible Lie–admissible algebras arise in classical and quantum mechanics as a generalization of conventional mechanics. However, the general Lie–admissible algebras are too broad and diversified to provide at this moment a fruitful structure theory. On the other hand, the structure of certain classes of flexible Lie–admissible algebras is closely related to the theory of Lie algebras of characteristic zero. Especially, the structure and representation of semisimple Lie algebras of characteristic zero serve as a reasonable model in the classification of certain classes of simple flexible Lie–admissible algebras.

This monograph is based on lectures delivered for a first year graduate course at Seoul National University while I was visiting under the project of the SNU–U.S.AID Graduate Program for Basic Sciences in 1979–1980. My intention was to provide elements of Lie algebras and flexible Lie-admissible algebras. The subject matter in Lie algebras was designed to set up the groundwork for the structure of certain classes of flexible Lie–admissible algebras rather than to provide a comprehensive account of the general theory of Lie algebras. Realizing that the majority of the audience had different motivations and backgrounds, and had no experience in Lie algebras, I tried to convey the material in a self–contained manner, without unnecessary difficulty. Accordingly, the amount of material in Lie algebras grew up to a moderate book length.

The treatment of Lie algebras followed the three well–known books: N. Jacobson, Lie algebras (Interscience, New York, 1962); D. J. Winter, Abstract Lie algebras (MIT Press, 1972); J. E. Humphreys, Introduction to Lie algebras and representation theory (Springer–Verlag, New York, 1972). The basic classical theorems, including the theorems of Weyl and Levi, are drawn from Winter's book which provides quick access to the classification of split semisimple Lie algebras of characteristic zero. The classification and construction of simple Lie algebras were treated as in Jacobson's book. The discussion for root systems followed the classical approach rather than the abstract one. The exposition of representation theory was based on Humphreys' book.

I had to omit many standard topics in Lie algebras, such as cohomology, theorems of Ado and Iwasawa, classification over non–algebraically closed fields, character formulas, and multiplicity formulas, which the interested reader can pursue in the books of Jacobson and Humphreys, cited above. For the discussion of certain flexible Lie–admissible algebras of prime characteristic in Chapter 7, I stated some elementary results on classical Lie algebras without proof which can be found in G. B. Seligman's book, Modular Lie algebras (Springer–Verlag, New York, 1967).

Chapters 6 and 7 are devoted to the structure of some special classes of flexible Lie–admissible algebras. The main objective in Chapter 6 is to classify finite–dimensional flexible Lie–admissible algebras over an algebraically closed field of characteristic zero whose associated Lie algebras are reductive. The central idea for this classification is the notion of adjoint operators which was first introduced by E. P. Wigner in 1941, for the Lie algebra of the SU(2) group. The general case of adjoint operators was later studied in particle physics, notably by Okubo. The main result is that simple Lie algebras of type A_n $(n \geq 2)$ alone allow a non–zero symmetric adjoint operator and otherwise, all adjoint operators are multiples of the adjoint mapping. Thus, only simple Lie algebras of type A_n $(n \geq 2)$ result in new simple flexible Lie–admissible algebras.

Chapter 7 is concerned with the structure of flexible Lie–admissible algebras of arbitrary characteristic. It is shown that the algebras turn out to be Lie algebras when their associated Lie algebras are classical in the sense of Seligman or generalized Witt algebras. In the final section, we investigate some basic structure of the mutation of an associative algebra which originated from Santilli's generalization of classical and quantum mechanics. When the mutation parameters are invertible, flexibility in a mutation of an associative algebra is virtually equivalent to all other nonassociative identities.

The material presented here is far from complete. I have had to omit many recent advances made for the structure of flexible Lie–admissible algebras which is at present quite diversified in nature. It is hoped that a collective work on Lie–admissible algebras will be published in a readable form in the near future.

In writing this monograph, I am indebted to a number of people. I should like to express my gratitude to R. M. Santilli who first brought to my attention the relevance of Lie–admissible algebras to physics in 1977. Since that time, his continual encouragement has been most influential in pursuing the study of Lie–admissible algebras. It is his suggestion to publish this monograph. I also owe a great debt to S. Okubo for making available the current developments in physics relating to Lie–admissible algebras and for many invaluable communications. The majority of material in Chapters 6 and 7 is drawn from joint works with Okubo. I wish to thank my friend and teacher M. L. Tomber who first aroused my interest in Lie–admissible algebras. I thank G. M. Benkart and J. M. Osborn for numerous conversations which have been immensely helpful in writing the last two chapters.

To the SNU–U.S.AID Graduate Program for Basic Sciences and the Department of Mathematics of Seoul National University who arranged my visit during which the majority of this monograph was written, I express my sincere thanks for financial support and generous hospitality. I would like to acknowledge also occasional support from DOE contract DE–AC02–80ER10651, and extensions A001, and A002. Special thanks are also due to J. S. Cross for a careful reading of the manuscript. It is a pleasure to acknowledge the great help provided by the editorial staff of the Hadronic Press.

I am, of course, solely responsible for the errors or shortcomings that remain.

November 1, 1982
Cedar Falls

Hyo C. Myung

1. NONASSOCIATIVE ALGEBRAS

1.1. Basic definitions

An (nonassociative) <u>algebra</u> A over a field F is a vector space over F with a multiplication $A \times A \to A$, denoted by xy, such that

$$(\alpha x + \beta y)z = \alpha(xz) + \beta(yz),$$

$$z(\alpha x + \beta y) = \alpha(zx) + \beta(zy).$$

$\alpha, \beta \in F$, $x, y, z \in A$.

Denote the associator (x,y,z) and commutator $[x,y]$ in A by

$$(x,y,z) = (xy)z - x(yz),$$

$$[x,y] = xy - yx.$$

If A is finite-dimensional over F, let $u_1,\ldots,u_n$ be a basis for A. Let

$$u_i u_j = \sum_{k=1}^{n} \gamma_{ij}^{k} u_k . \qquad (1.1)$$

The n^3 constants $\gamma_{ij}^{k} \in F$ are called the <u>structure constants</u> for A, which determine a unique element in the space $F^{n^3} = F \times\ldots\times F$ (n^3 times). Conversely any element $(\gamma_{ij}^{k}) \in F^{n^3}$ determines a unique nonassociative algebraic structure on the underlying vector space A via (1.1). Thus the set of algebras with underlying vector space A over F is identified with F^{n^3}.

<u>Problem 1.1.1.</u> Determine all algebras of dimension 1 or 2.

The definitions of the terms, such as left or right ideal (two-sided)ideal, homomorphism, kernel, quotient algebra, isomorphism theorems, and direct sum in an algebra can be stated exactly the same as in an associative algebra. An element $1 \in A$ is called a <u>unit element</u> for A if $1x = x1 = x$ for $x \in A$. For an algebra A over F, let $A_1 = F \oplus A$ be the vector space direct sum of F and A. Define a multiplication in A_1 by

$$(\alpha + a)(\beta + b) = \alpha\beta + (\beta a + \alpha b + ab), \qquad (1.2)$$

$\alpha,\beta \in F$, $a,b \in A$. Then A_1 becomes an algebra over F with unit element $1 \in F$.

Let A, B be algebras over F and let $B \otimes_F A = B \otimes A$ be the tensor product of A and B. Defining a multiplication in $B \otimes A$ by

$$(y_1 \otimes x_1)(y_2 \otimes x_2) = (y_1 y_2) \otimes (x_1 x_2),\ x_i \in A,\ y_i \in B$$

makes $B \otimes A$ into an algebra over F. If B has 1, $1 \otimes A$ is a subalgebra of $B \otimes A$ which is isomorphic to A and is identified with A. If A and B are finite-dimensional over F,

$$\dim_F B \otimes A = (\dim B)(\dim A).$$

In particular, if $B = K$ is an extension field of F, then in $A_K \equiv K \otimes A$ we identify x with $1 \otimes x$, $x \in A$, and A_K becomes an algebra over K via

$$\alpha(\Sigma\, \beta_i x_i) = \Sigma\, \alpha\beta_i x_i\ ,$$

$\alpha, \beta_i \in K$, $x_i \in A$. If A is finite-dimensional over F and $u_1, \ldots, u_n$ is a basis for A over F, it is readily seen that it is also a basis for A_K over K and so $\dim_K A_K = n$. A_K is called the <u>scalar extension</u> of A to K. While the scalar extension A_K is often useful, it should be stressed that some algebraic properties in A may be collapsed in A_K. An algebra A is called <u>simple</u> if $AA \neq 0$ and A has no proper iedals.

<u>Exercise 1.1.1.</u> Give an example of an algebra A such that A is simple over F but A_K is not over K for some extension K of F.

1.2. Modules

Definition 1.2.1. Let S be a set. An S-module over F is a vector space V over F together with a mapping $V \times S \to V$, denoted by $(x,s) \to xs$, such that $(\alpha x + \beta y)s = \alpha(xs) + \beta(ys)$, $\alpha, \beta \in F$, $x,y \in V$, $s \in S$. //

For an S-module V over F and a subset $W \subset V$ we let WS be the subspace of V spanned by xs, $x \in W$, $s \in S$. An S-submodule of V is a subspace W of V such that $WS \subset W$. If W is an S-submodule of V, the vector space V/W becomes an S-module over F via $(x + W)s = xs + W$, $x \in V$, $s \in S$. An S-homomorphism from an S-module V into an S-module V' over F is a linear transformation $f: V \to V'$ over F such that $f(xs) = f(x)s$, $x \in v$, $s \in S$. The isomorphism theorems for S-modules are straightforward generalizations of the usual ones.

Definition 1.2.2. (1) An S-complement of an S-submodule W of V is an S-submodule W' of V such that $V = W \oplus W'$.

(2) An S-module V is S-completely reducible if $VS = V$ and every S-submodule of V has an S-complement.

(3) V is S-irredcible if $VS = V$ and V has no proper S-submodules. //

Lemma 1.2.1. Let V be S-completely reducible and let W be an S-submodule of V. Then W and V/W are S-completely reducible.

Proof. (1) Let W_0 be an S-submodule of W and let W_0' be an S-complement of W_0 in V, so that $V = W_0 \oplus W_0'$. One sees that $W = W \cap (W_0 \oplus W_0')$ $= W_0 \oplus (W \cap W_0')$. Since $VS = WS \oplus W'S = V = W \oplus W'$, $WS = W$. Thus W is S-completely reducible.

(2) Let $\overline{V} = V/W$ and let $\overline{P}$ be an S-submodule of $\overline{V}$. If P denotes the inverse image of $\overline{P}$ under the natural S-homomorphism $: V \to \overline{V}$, $V = P \oplus P'$ for an S-submodule P' of V. Thus $\overline{V} = \overline{P} \oplus \overline{P'}$ since $P \supset W$. Clearly $VS = V$ implies $\overline{V}S = \overline{V}$ and so $\overline{V}$ is S-completely reducible. //

Lemma 1.2.2. Every nonzero S-completely reducible module V has a nonzero S-irreducible S-submodule.

Proof. Pick an $x \neq 0$ in V. If every nonzero S-submodule of V contains x, the proof is done. If there is an S-submodule not containing x, by Zorn's lemma one chooses a maximal S-submodule W of V such that $x \notin W$. Thus $V = W \oplus W'$, W' an S-complement, and $W' \neq 0$. Then W' is S-irreducible, since if not, $W' = W_1 \oplus W_2$ for some proper S-submodules W_1, W_2 (Lemma 1.2.1); but then $x \in (W + W_1) \cap (W + W_2)$ by the maximality of W, so that $x \in W$, a contradiction. //

Theorem 1.2.3. The following are equivalent.

1) V is S-completely reducible ;

2) $V = \Sigma V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V ;

3) $V = \Sigma \oplus V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V.

Proof. 1) $\Rightarrow$ 2): Assume V is S-completely reducible and let $W = \Sigma V_i$ where $\{V_i\}$ is the collection of all S-irreducible S-submodules of V.
Let $V = W \oplus W'$, W' an S-complement of W. Since W' has no nonzero S-irreducible S-submodules, by Lemmas 1.2.1 and 1.2.2, $W' = \{0\}$. Thus $V = \Sigma V_i$.

2) $\Rightarrow$ 3) : Let $\{W_i\}$ be a family of S-submodules. Call $\{W_i\}$ direct if ΣW_i is direct. Now, assume $V = \Sigma V_i$ for some family $\{V_i\}$ of S-irreducible S-submodules of V. By Zorn's lemma, one can choose a maximal direct family $\{V_k\}$ of S-submodules from the family $\{V_i\}$. Thus $V = \Sigma \oplus V_k$, for otherwise $V_i \not\subset \Sigma \oplus V_k$ for some i . But then $V_i \cap \Sigma \oplus V_k \neq V_i$ and so $V_i \cap \Sigma \oplus V_k = \{0\}$ since V_i is S-irreducible. Thus $V_i \oplus \Sigma \oplus V_k$ is direct and this contradicts the maximality of $\{V_k\}$.

3) $\Rightarrow$ 1) : Let $V = \Sigma \oplus V_i$ for some family of S-irreducible S-submodules of V. Let W be an S-submodule of V. By Zorn's lemma, one can pick a maximal S-submodule W' with $W \cap W' = 0$. For each i, $(W \oplus W') \cap V_i = 0$ or V_i. In the first case, $W \oplus W' \oplus V_i$ is direct and so $W \cap (W' \oplus V_i) \neq 0$, so $V_i = \{0\}$ by the maximality of W'. In the second case, $V_i \subset W \oplus W'$ and so $V = W \oplus W'$. //

1.3. Jordan and Fitting decompositions

If V is an S-module over F, then for $T \in S$, T_V denotes the linear transformation of V defined by $xT_V = xT$, $x \in V$. We say that $T \in S$ (or T_V) is split over F if the eigenvalues of T_V are in F or equivalently the minimum polynomial of T_V is factored over F into linear factors. Also, S is split over F if every element in S is split over F.

Definition 1.3.1. For $T \in S$ and $\alpha \in F$, let $V_\alpha(T) = \{x \in V \mid x(T_V - \alpha)^n = 0 \text{ for some } n > 0\}$. //

<u>Theorem 1.3.1</u>. Let V be a finite-dimensional S-module over F. If $T \varepsilon S$ is split over F then

$$V = \Sigma_{\alpha \varepsilon F} \oplus V_\alpha(T) .$$

<u>Proof</u>. Let $f(X)$ be the minimum polynomial of T_V and let $f(X) = \Pi(X - \alpha_i)^{e_i}$ with $\alpha_i \varepsilon F$ distinct. Setting $h_i(X) = f(X)/(X - \alpha_i)^{e_i}$, since the $h_i(X)$ are relatively prime, one finds polynomials $g_i(X)$ such that $\Sigma g_i(X)h_i(X) = 1$. Let

$$V_i = Vh_i(T_V) .$$

Since $x = \Sigma (xg_i(T_V))\, h_i(T_V)$ for $x \varepsilon V$, we have $V = \Sigma V_i$. Also, $V_i \subset V_{\alpha_i}(T)$ since $V_i(T_V - \alpha_i)^{e_i}$ $= Vh_i(T_V)(T_V - \alpha_i)^{e_i} = Vf(T_V) = \{0\}$.
Thus $V = \Sigma V_{\alpha_i}(T)$ and it remains to show that the sum is direct. Let $x \varepsilon V_{\alpha_i}(T) \cap \Sigma_{j \neq i} V_{\alpha_j}(T)$. Then the ideal $J = \{h(X) \varepsilon F[X] \mid xh(T_V) = 0\}$ contains $(X - \alpha_i)^{k_i}$ and $\Pi_{j \neq i}(X - \alpha_j)^{k_j}$ for some k_i, k_j.
Since these polynomials are relatively prime, $J = F[X]$ and $1 \varepsilon J$, so $x = x1 = 0$. //

<u>Remark 1.3.1</u>. In fact we have shown that $Vh_i(T_V) = V_{\alpha_i}(T)$, $i = 1,2,\ldots$. //

Definition 1.3.2. Let $\dim V < \infty$ and let T be an element in $\operatorname{Hom} V$ which is split over F. Then the semisimple part of T is the element $T_s \in \operatorname{Hom} V$ such that $T_s \mid_{V_\alpha(T)} = \alpha I_{V_\alpha(T)}$ for $\alpha \in F$. The nilpotent part of T is $T_n = T - T_s$. The decomposition $T = T_s + T_n$ is called the Jordan decomposition of T. If $T = T_s$, T is called semisimple. If $T = T_n$, T is called nilpotent. //

Let W be a T-stable subspace of V. It is evident that $W = \Sigma W_\alpha(T)$ where $W_\alpha(T) = V_\alpha(T) \cap W$. Hence if T is semisimple on V then so is $T|_W$ on W.

Since $xT_n = x(T - \alpha)$, $x \in V_\alpha(T)$, T_n and T_s stablize each $V_\alpha(T)$ and so T_n and T_s commute. Note that T is nilpotent if and only if $T^n = 0$ for some $n > 0$.

Lemma 1.3.2. Let $T \in \operatorname{Hom} V$ be split over F. Let $T = S + N$, where S is semisimple, N is nilpotent, and $SN = NS$. Then $S = T_s$ and $N = T_n$.

Proof. Since $SN = NS$, N and S stablize each $V_\alpha(T)$. If $xS = \beta x$ for $x \in V_\alpha(T)$, $xN = x(T - \beta)$ and so S has only one eigenvalue α on $V_\alpha(T)$. Since S is semisimple, this implies $S \mid_{V_\alpha(T)} = \alpha I_{V_\alpha(T)}$, $\alpha \in F$ and $S = T_s$. Thus $N = T - S = T - T_s = T_n$. //

Theorem 1.3.3. Let $T \in \operatorname{Hom} V$ be split over F. Define $\operatorname{ad} T \in \operatorname{Hom}(\operatorname{Hom} V)$ by

$$S \operatorname{ad} T = [S,T] = ST - TS, \; S \in \operatorname{Hom} V .$$

Then $\operatorname{ad} T$ is split over F, and $(\operatorname{ad} T)_s = \operatorname{ad} T_s$ and $(\operatorname{ad} T)_n = \operatorname{ad} T_n$.

Proof. First, note that $\operatorname{ad}[S,T] = [\operatorname{ad} S, \operatorname{ad} T]$. Thus $\operatorname{ad} T_s \operatorname{ad} T_n = \operatorname{ad} T_n \operatorname{ad} T_s$ since $T_sT_n = T_nT_s$. In view of Lemma 1.3.2, it suffices to show that $\operatorname{ad} T_s$ is split over F and semisimple, and $\operatorname{ad} T_n$ is nilpotent. One can choose a basis $e_1,\ldots,e_n$ for V consisting of eigenvectors for T_s, so that $e_iT_s = \alpha_ie_i$ for $1 \leqq i \leqq n$. Let $\{E_{ij} \mid 1 \leqq i, j \leqq n\}$ be the basis for $\operatorname{Hom}_F V$ such that $e_k E_{ij} = \delta_{ik}e_j$ where δ_{ik} is the Kronecker delta. Since $T_s = \Sigma\, \alpha_iE_{ii}$, we have $E_{ij} \operatorname{ad} T_s = (\alpha_j - \alpha_i)E_{ij}$. Thus $\operatorname{ad} T_s$ is split over F and semisimple. Let $T_n{}^q = 0$. Since $Y(\operatorname{ad} T_n)^m = \Sigma\, T_n^rYT_n^{m-r}$, $(\operatorname{ad} T_n)^{2q} = 0$ and $\operatorname{ad} T_n$ is nilpotent. //

Definition 1.3.3. Let $T \in \operatorname{Hom}_F V$, $\dim V < \infty$. Let $V_0 = V_0(T) = \bigcup_{i=0}^{\infty} \ker T^i$ and $V_* = V_*(T) = \bigcap_{i=1}^{\infty} VT^i$. Then V_0 and V_* are called the Fitting 0-component and 1-component of V with respect to T. //

<u>Fitting Lemma 1.3.4</u>. For any $T \varepsilon \mathrm{Hom}_F V$, $V = V_0 \oplus V_*$, V_0 and V_* are T-stable, and T is bijective on V_*.

<u>Proof</u>. Note that $V \supset VT \supset VT^2 \supset \ldots$ and $\ker T \subset \ker T^2 \subset \ldots$ Since $\dim V < \infty$, there exist p,q such that $VT^p = VT^{p+1} = \ldots$ and $\ker T^q = \ker T^{q+1} = \ldots$ If $t = \max(p,q)$ then $V_0 = \ker T^t$ and $V_* = VT^t$. For $x \varepsilon V$, one gets $sT^t = yT^{2t}$ for some $y \varepsilon V$ and $x = (x - yT^t) + yT^t$ with $x - yT^t \varepsilon V_0$ and $yT^t \varepsilon V_*$, so $V = V_0 + V_*$. If $z \varepsilon V_0 \cap V_*$, $z = uT^t$, $u \varepsilon V$, and $0 = zT^t = uT^{2t}$. Since $\ker T^t = \ker T^{2t}$, $uT^t = z = 0$ and so $V = V_0 \oplus V_*$. Since $V_*T = VT^{t+1} = VT^t = V_*$, T is surjective on V_* and so bijective on V_*. //

Let $f(X)$ be the characteristic polynomial of T on V and let $f(X) = \Pi f_i(X)^{e_i}$ be the prime factorization of $f(X)$ with $f_1(X) = x^{e_1}$. Put

$$V_i = \{x \varepsilon V \mid xf_i(T)^k = 0 \text{ for some } k > 0\} .$$

Then, as in the proof of Theorem 1.3.1, we see that

$$V = \Sigma \oplus V_i . \qquad (1.3)$$

The decomposition (1.3) is called the <u>primary decomposition</u> of V relative to T. It is routine to check that T is bijective on each V_i for $i \geqq 2$, $V_1 = V_0(T)$ and

$\Sigma_{i \geq 2} V_i = V_*(T)$. Since $V = V_0 \oplus V_*$ is direct, $f(X)$ is the product of the characteristic polynomials of T on V_0 and V_* . Thus $\dim V_0 = e_1$, the multiplicity of the eigenvalue 0 in $f(X)$. //

1.4. Derivations

Let A be an algebra over F . For $a \in A$, let L_a and R_a be the elements in $\mathrm{Hom}_F A$ defined by $xL_a = ax$, $xR_a = xa$ for $x \in A$. We call L_a and R_a the left and right multiplications in A by a . The adjoint mapping ad a by a is the element in Hom A defined by $x \text{ ad } a = [x,a] = xa - ax$, $x \in A$. If S is a subset of A , denote $L_S = \{L_a \mid a \in S\}$.

For an algebra A over F , denote by A^- the algebra with multiplication $[x,y] = xy - yx$ defined on the same vector space as A . If char $F \neq 2$, define A^+ as the algebra over F with multiplication $x \cdot y = \frac{1}{2}(xy + yx)$, called the Jordan product, with the same underlying space as A .

Definition 1.4.1. An algebra A over a field F with multiplication $[xy]$ is called a Lie algebra if A satisfies

1) $[xx] = 0, x \in A$,
2) the Jacobi identity $[[xy] z] + [[yz] x] + [[zx] y] = 0$ $x,y,z \in A$. //

Note that if char $F \neq 2$, $[xx] = 0$ is equivalent to the anticommutative law $[xy] = - [yx]$. Various types of Lie algebras will be discussed later.

Definition 1.4.2. An algebra A is said to be Lie-admissible if the associated algebra A^- is a Lie algebra, that is, A^- satisfies the Jacobi identity

$$[[x,y],z] + [[y,z],x] + [[z,x],y] = 0 . \quad //$$

It is readily seen that any Lie and associative algebras are Lie-admissible. Various classes of (nonassociative) Lie-admissible algebras will be explored later. To express the Lie-admissible condition into a more convenient form, we introduce the notation

$$S(x,y,z) \equiv (x,y,z) + (y,z,x) + (z,x,y)$$

where (x,y,z) is the associator of x,y,z in A . Then, by a direct computation, one checks that the identity

$$[xy,z] + [yz,x] + [zx,y] = S(x,y,z) \qquad (1.4)$$

holds in any algebra A . Thus A satisfies the identity

$$S(x,y,z) - S(x,z,y) = [[x,y],z] + [[y,z],x] + [[z,x],y] .$$

Therefore, we have

Lemma 1.4.1. An algebra A over F is Lie-admissible if and only if A satisfies $S(x,y,z) = S(x,z,y), x,y,z \in A$. //

While the algebraic origin of Lie-admissible algebras arises as a byproduct of the study of nonassociative algebras defined by identities, the analytic orgin of Lie-admissibility stems from a nonassociative product formed in terms of partial differential equations in a space of differentiable functions which are defined on a C^{∞}-manifold. Lie-admissible algebras have been utilized to construct a nonassociative quantization of forces or couplings not derivable from a potential. Thus Lie-admissible algebras have direct physical relevances in both classical and quantum mechanics (see R.M. Santilli, "Lie-admissible approach to the hadronic structure," Vol. 1,2, Hadronic Press, Nonantum, Mass. 1979).

It has been observed that the general Lie-admissible algebras are too broad to obtain a fruitful structure theory (see H.C. Myung, "On nonflexible Lie-admissible algebras", Hadronic J. 1,(1978), 1021-1143).

<u>Definition 1.4.3</u>. A Lie-admissible algebra A is called flexible Lie-admissible if A is a flexible algebra, that is, A satisfies the flexible law

$$(xy)x = x(yx) \quad . \qquad // \tag{1.6}$$

Note that the associative and Lie algebras are flexible Lie-admissible. The flexible law is also written as $(x,y,x) = 0$ which is equivalent to

$$(x,y,z) = -(z,y,x) \tag{1.7}$$

if char $F \neq 2$. The study of flexible Lie-admissible algebras was first initiated by A.A.Albert ("Power-associative rings" Trans. Amer. Math. Soc. 64(1948), 552-597), and the structure of these algebras have been investigated by Laufer and Tomber, Myung, and Okubo (for a review, see H.C.Myung, "Lie-admissible algebras", Hadronic J. 1(1978), 169-193). Applications of flexible Lie-admissible algebras to physics have been recently pointed out in particular reference to a generalization of the Heisenberg equation by a number of authors (see Santilli's monographs cited above; C.N. Ktorides, H.C. Myung and R.M. Santilli, "Elaboration of the recently proposed test of Pauli's principle under strong interactions", Phys. Rev. D22 (1980), 892-907; H.C. Myung and R.M. Santilli, "Further studies on the recently proposed experimental test of Pauli's exclusion principle for the strong interactions", Proc. of the 2nd Workshop on Lie-admissible Formulations, held at Harvard University, August 1979, Hadronic J. 3(1979), 194-255, S. Okubo, "Non-associative quantum mechanics via flexible Lie-admissible algebras, Proc. of the 3rd Workshop on Current Problems in High Energy Particle Theory, held at Florence, Italy, 1979, Edit. R. Casabuni, G. Domokos and S.K. Domokos, John Hopkins Univ. Press, (1979), 103-120, and "A generalization of Hurwitz theorem and flexible Lie-admissible algebras", Proc. of the 2nd Workshop on Lie-admissible Formulations, Hadronic J. 3(1979), 1-52).

Definition 1.4.3. Let A be an algebra over a field F. Then an element $D \in \mathrm{Hom}_F A$ is called a derivation of A if

$$(xy)D = (xD)y + x(yD) \ , \quad x,y \in A \ . \tag{1.8}$$

Denote by $\mathrm{Der}\, A$ the set of derivations of A. //

In terms of the left and right multiplications L_x and R_x, (1.8) is expressed as

$$L_x D = L_{xD} + DL_x \quad \text{or} \quad L_{xD} = [L_x, D] , \tag{1.9}$$

$$R_y D = DR_y + R_{yD} \quad \text{or} \quad R_{yD} = [R_y, D] \tag{1.10}$$

for $x,y \in A$. Since the Jacobi identity in A^- is equivalent to the fact that all adjoint mappings ad x are derivations of A^-, in view of (1.9) or (1.10) A is Lie-admissible if and only if

$$\text{ad}[x,y] = [\text{ad } x, \text{ad } y] \tag{1.11}$$

for all $x,y \in A$. In this case, notice that ad x is not necessarily a derivation of A (why?).

Suppose that A is an algebra over a field F of char $\neq 2$. For $x \in A$, define $T_x \in \text{Hom } A$ by $T_x = \frac{1}{2}(R_x + L_x)$. We contend that the following identities are equivalent.

$$(xy)x = x(yx) \text{ ; the flexible law ,} \tag{1.12}$$

$$(x,y,z) + (z,y,x) = 0 , \tag{1.13}$$

$$(x,y,z) + (z,y,x) + (x,z,y) + (y,z,x) = (y,x,z) + (z,x,y) , \tag{1.14}$$

$$[x,y.z] = [x,y].z + y.[x,z] . \tag{1.15}$$

As noted earlier, (1.12)$\Leftrightarrow$(1.13) . Clearly, (1.13)$\Leftrightarrow$(1.14), while (1.14)$\Leftrightarrow$(1.12) with $x = z$. By a direct expansion, one sees (1.14)$\Leftrightarrow$(1.15) . Since (1.15) is to say that each $\operatorname{ad} x$ is a derivation of A^+ , we can state

Lemma 1.4.2. Let $\operatorname{char} F \neq 2$. Then A is flexible if and only if $\operatorname{ad} A \subseteq \operatorname{Der} A^+$ if and only if $T_{[x,y]} = [T_x, \operatorname{ad} y]$ $x,y \in A$. //

Theorem 1.4.3. Let A be an algebra over a field F of $\operatorname{char} \neq 2$. Then A is flexible Lie-admissible if and only if $\operatorname{ad} A \subseteq \operatorname{Der} A$.

Proof. If A is flexible and Lie-admissible, then, in view of (1.11) and Lemma 1.4.2, we have

$$\operatorname{ad}[x,y] + 2T_{[x,y]} = [\operatorname{ad} x, \operatorname{ad} y] + [2T_x, \operatorname{ad} y]$$

which implies

$$R_{[x,y]} = [R_x, \operatorname{ad} y] \tag{1.16}$$

for $x,y \in A$, so, by (1.10), $\operatorname{ad} A \subseteq \operatorname{Der} A$. If $\operatorname{ad} A \subseteq \operatorname{Der} A$, $\operatorname{ad} A \subseteq \operatorname{Der} A^-$ and A is Lie-admissible. Also, (1.16) implies the flexible law with $x = y$. //

Theorem 1.4.4. For any algebra A over F, Der A is a subalgebra of $(\mathrm{Hom}_F\ A)^-$. That is, Der A is closed under the Lie product $[D,E] = DE - ED$.

Proof. By the Jacobi identity and (1.9),

$$
\begin{aligned}
[L_x,[D,E]] &= [[L_x,D],E] + [D,[L_x,E]] \\
&= [L_{xD},E] + [D,L_{xE}] \\
&= L_{xDE} - L_{xED} \\
&= L_{x[D,E]} . \quad //
\end{aligned}
$$

Lemma 1.4.5. For $D \in$ Der A and $\alpha,\beta \in F$,

$$(xy)(D - \alpha - \beta)^n = \sum_{m=0}^{n} \binom{n}{m} x(D-\alpha)^m\ y(D-\beta)^{n-m}$$

for all positive integers n and $x,y \in A$.

Proof. By induction, if $n = 1$,

$$
\begin{aligned}
(xy)(D - \alpha - \beta) &= (xD)y - \alpha(xy) + x(yD) - \beta(xy) \\
&= x[y(D - \beta)] + [x(D - \alpha)]y .
\end{aligned}
$$

Assume $(xy)(D - \alpha - \beta)^{n-1} = \sum \binom{n-1}{m} x(D - \alpha)^m\ y(D - \beta)^{n-m-1}$.

Then $(xy)(D - \alpha - \beta)^n$

$$= \sum \binom{n-1}{m} x(D - \alpha)^{m+1}\ y(D - \beta)^{n-(m+1)}$$

$$+ \sum \binom{n-1}{m} x(D - \alpha)^m\ y(D - \beta)^{n-m}$$

$$= \Sigma \binom{n}{m} x(D - \alpha)^m y(D - \beta)^{n-m}$$

since $\binom{n-1}{m-1} + \binom{n-1}{m} = \binom{n}{m}$ for $n > m$. //

Corollary 1.4.6. (Leibniz's rule). For $D \in \text{Der } A$,

$$(xy)D^n = \Sigma \binom{n}{m} xD^m yD^{n-m} . \quad //$$

Let $p > 0$ be a prime. Since $p \mid \binom{p}{i}$ for $1 \leq i < p$, we have from Corollary 1.4.6

Corollary 1.4.7. If $\text{char } F = p > 0$ then $D^p \in \text{Der } A$ for $D \in \text{Der } A$. //

Corollary 1.4.8. Let A be a finite-dimensional algebra over F. Let $D \in \text{Der } A$. Then $A_\alpha(D)\, A_\beta(D) \subseteq A_{\alpha+\beta}(D)$ for $\alpha, \beta \in F$ and $A_0(D)$ is a subalgebra of A. If D is split over F, then $A_0(D)A_*(D) \subseteq A_*(D)$ and $A_*(D)A_0(D) \subseteq A_*(D)$.

Proof. Let $x \in A_\alpha(D)$, $y \in A_\beta(D)$ and pick an r such that $x(D - \alpha)^r = y(D - \beta)^r = 0$. Letting $n = 2r$ in Lemma 1.4.5, $(xy)(D - \alpha - \beta)^n = 0$ and $A_\alpha(D)A_\beta(D) \subseteq A_{\alpha+\beta}(D)$. So, $A_0(D)$ is a subalgebra of A. If D is split over F, by the remark following the primary decomposition we have $A_*(D) = \Sigma_{\alpha \neq 0} A_\alpha(D)$ and the second assertion follows from this. //

Lemma 1.4.9. Let A be finite-dimensional over F and let $D \in \mathrm{Hom}_F A$ be semisimple and split over F. Then $D \in \mathrm{Der}\, A$ if and only if $A_\alpha(D)A_\beta(D) \subseteq A_{\alpha+\beta}(D)$ for all $\alpha, \beta \in F$.

Proof. Let $x \in A_\alpha(D)$, $y \in A_\beta(D)$. If $xy \in A_{\alpha+\beta}(D)$ then since D is semisimple, $(xy)D = (\alpha + \beta)xy$ $= (\alpha x)y + x(\beta y) = (xD)y + x(yD)$. Since $A = \Sigma_\alpha A_\alpha(D)$, $D \in \mathrm{Der}\, A$. The converse follows from Corollary 1.4.8. //

Corollary 1.4.10. Let $\dim A < \infty$ and let $D \in \mathrm{Der}\, A$ be split over F. Then D_s and D_n are in $\mathrm{Der}\, A$.

Proof. If D is split over F, so is D_s. Since $A = \Sigma_\alpha \oplus A_\alpha(D) = \Sigma_\alpha \oplus A_\alpha(D_s)$ and $A_\alpha(D) \subset A_\alpha(D_s)$, $A_\alpha(D) = A_\alpha(D_s)$. Thus $D_s \in \mathrm{Der}\, A$ by Lemma 1.4.9, and $D_n = D - D_s \in \mathrm{Der}\, A$. //

Lemma 1.4.11. Let $D \in \mathrm{Der}\, A$ be nilpotent and let $\mathrm{char}\, F = 0$. Then $\exp D = e^D = \Sigma_{m=0}^{\infty} D^m/m!$ is an automorphism of A.

Proof. Let n be such that $D^{n+1} = 0$. Then

$$xe^D ye^D = \sum_{m=0}^{2n} \sum_{i=0}^{m} \frac{xD^i}{i!} \frac{yD^{m-i}}{(m-i)!} .$$

By Leibniz's rule

$$(xy)e^D = \sum_{m=0}^{n} (xy)\frac{D^m}{m!} = \sum_{m=0}^{n}\sum_{i=0}^{m} \frac{1}{i!}\binom{m}{i} xD^i yD^{m-i}$$

$$= \sum_{m=0}^{2n}\sum_{i=0}^{m} \frac{xD^i}{i!}\frac{yD^{m-i}}{(m-i)!}$$

since $xD^i = yD^i = 0$ for $i > n$. //

1.5. Solvability and radical

Let A be an algebra over F. For subspaces B, C of A, denote by BC the subspace of A spanned by bc, $b \in B$, $c \in C$.

Definition 1.5.1. The subalgebras $A^{(i)}$, $i \geq 0$, are recursively defined by $A^{(0)} = A$, $A^{(i)} = A^{(i-1)}A^{(i-1)}$ for $i > 1$. For a subalgebra B of A, $B^{(i)}$ is similarily defined. Then B is called solvable if $B^{(i)} = 0$ for some $i > 0$. //

Note that if A is Lie or associative, $A^{(i)}$ is an ideal of A by the Jacobi identity.

Lemma 1.5.1. Let B be an ideal of A. Then A is solvable if and only if B and A/B are solvable.

Proof. Clearly $(A/B)^{(i)} = (A^{(i)} + B)/B$ and $B^{(i)} \subseteq A^{(i)}$. Thus if A is solvable then so are B and A/B. If B and A/B are solvable, choose i, j such that $(A/B)^{(i)} = 0$ and $B^{(j)} = 0$. Thus $A^{(i)} \subseteq B$ and $A^{(i+j)} = (A^{(i)})^{(j)} = 0$. //

As an immediate consequence of Lemma 1.5.1, we have

Corollary 1.5.2. If B is a solvable ideal of A and C is a solvable subalgebra then $B + C$ is solvable. //

If B and C are maximal solvable ideals of A, then by Corollary 1.5.2, $B + C$ is solvable, so $B = B + C = C$. Therefore, if A is finite-dimensional, A contains a unique maximal solvable ideal which is called the (solvable) radical of A and denoted by Rad A.

Definition 1.5.2. An algebra A over F is called semisimple if A has no nonzero solvable ideal. //

If A is finite-dimensional, then $A/\text{Rad } A$ is semisimple and A is semisimple if and only if $\text{Rad } A = 0$. Denote by $M(A)$ the (associative) subalgebra of $\text{Hom}_F A$ generated by L_x, R_x, $x \in A$.

Lemma 1.5.3. A is simple if and only if A is $M(A)$-irreducible. A is the direct sum of simple ideals of A if and only if A is $M(A)$-completely reducible.

Proof. The first assertion as well as one direction of the second is obvious. Let A be $M(A)$-completely reducible. Then $A = \Sigma \oplus A_i$ where the A_i are $M(A)$-irreducible $M(A)$-submodules of A. So, A_i is an ideal of A. Note that $A_iA_j = 0$ for $i \neq j$ since $A_iA_j \subset A_i \cap A_j$. Thus an ideal of A_i is an ideal of A and $A_i^2 = A_iA \neq 0$ since $A_iM(A) = A_i$. Hence each A_i has no proper ideal and $A_i^2 = A_i$, so A_i is simple. //

Corollary 1.5.4. Let $\dim A < \infty$ and let $A = \Sigma \oplus A_i$ with A_i simple ideals of A. Then every ideal B of A is a sum of some A_i's. In particular, the A_i are the only simple ideals of A.

Proof. By Lemma 1.5.3, A is $M(A)$-completely reducible. Thus B has an $M(A)$-complement C, so that C is an ideal of A and $A = B \oplus C$. Since $A = A^2$, $A = BA \oplus CA$ and $B = BA = \Sigma BA_i$. Since $B \cap A_i$ is an ideal of A_i, $B \cap A_i = 0$ or A_i. Thus B is the sum of those A_i's with $B \cap A_i = A_i$ ($B \cap A_i = A_i$ implies $BA_i = A_i$). //

If A is a direct sum of simple ideals, A is semisimple by Corollary 1.5.4. The converse is not true; however, we will see in the next section that the converse holds when A has a suitable nondegenerate bilinear form.

1.6. Algebras with invariant forms

Let $(,)$ be symmetric bilinear form on a vector space V over F. For $S \subseteq V$, define $S^{\perp} = \{x \in V \mid (x,y) = 0 \text{ for all } y \in S\}$. Call $V^{\perp}$ the radical of V. If $V^{\perp} = 0$, $(,)$ is nondegenerate. If we define $(,)'$ on $V/V^{\perp}$ by $(x + V^{\perp}, y + V^{\perp})' = (x,y)$, $x,y \in V$, $(,)'$ is nondegenerate on $V/V^{\perp}$.

Definition 1.6.1. Let A be an algebra over F. An invariant form $(,)$ on A is a symmetric bilinear form on A satisfying the associative law

$$(xy,z) = (x,yz),\ x,y \in A. \qquad (1.17)$$

If $(,)$ is nondegenerate, call A a symmetric algebra.

Note that if $(,)$ is an invariant form on A, then $A/A^{\perp}$ is a symmetric algebra. Observe that if B is an ideal of A, so is $B^{\perp}$ by (1.17).

Theorem 1.6.1 (Dieudonné). Let A be a finite-dimensional symmetric algebra over a field F such that there is no nonzero ideal B with $B^2 = 0$. Then $A = \Sigma \oplus A_i$ where the A_i are simple ideals of A and symmetric with $(A_i, A_j) = 0$ for $i \neq j$.

Proof. Let $(\, , \,)$ be a nondegenerate invariant form on A. Let B be any ideal of A. First we show that $C = B \cap B^{\perp} = 0$. Since $C \subset B$ and $CA \subset B^{\perp}$, by (1.17) $(xy,z) = (x,yz) = 0$ for $x,y \in C$ and $z \in A$. Thus $C^2 \subset A^{\perp} = 0$ and $C^2 = 0$, so $C = 0$. Since $\dim A = \dim B + \dim B^{\perp}$, this proves $A = B \oplus B^{\perp}$. Hence A is $M(A)$-completely reducible and by Lemma 1.5.3 $A = A_1 \oplus \ldots \oplus A_n$ where the A_i are simple ideals of A.

Since $(A_i \, , A_j) = (A_i^2 \, , A_j) = (A_i \, , A_iA_j) = 0$ for $i \neq j$, each A_i is symmetric too. //

If $A = \Sigma \oplus A_i$ where the A_i are simple symmetric ideals of A then A is semisimple by Corollary 1.5.4 and is symmetric.

Exercise 1.6.1. Let A be a finite-dimensional symmetric algebra with $(\, , \,)$ over a field F of char $\neq 2$. Prove :

(1) If A satisfies third-power-associativity $xx^2 = x^2x$ for all $x \in A$ then A is flexible.

(2) If char $F \neq 2,3,5$ and A satisfies $xx^2 = x^2x$ and $x^2x^2 = xx^3$ for all $x \in A$ then A satisfies the Jordan identity

$$(x^2y)x = x^2(yx) . \qquad (1.18)$$

A flexible algebra satisfying (1.18) is called a noncommutative Jordan algebra

Exercise 1.6.2. Let char $F \neq 2$. An algebra A over F is called Jordan-admissible if the algebra A^+ is a Jordan algebra; that is, A^+ satisfies $x^2 \cdot (y \cdot x) = (x^2 \cdot y) \cdot x$. Prove that A is a noncomutative Jordan algebra if and only if A is flexible Jordan-admissible.

Definition 1.6.2. Let L be a finite-dimensional Lie algebra over F and V be a finite-dimensional vector space over F. Then a representation f of L acting on V is a homomorphism of L into $(\mathrm{Hom}_F V)^-$. That is, $f([xy]) = [f(x), f(y)]$, $x, y \in L$. Also, define $(x,y) = (x,y)_f = \mathrm{Tr} f(x) f(y)$ and call (x,y) the trace form of L with respect to f. The mapping $\mathrm{ad} : L \to (\mathrm{Hom}\, L)^-$ defined by $x \to \mathrm{ad}\, x$ becomes a representation acting on L, called the adjoint representation of L. The trace form $K(,)$ of L with respect to ad is called the Killing form of L. //

Lemma 1.6.2. The trace form $(,) = (,)_f$ is an invariant form on L.

Proof. $([x,y],z) = \mathrm{Tr}\, f([xy])\, f(z)$

$$= \mathrm{Tr}[f(x), f(y)]\, f(z)$$

$= \mathrm{Tr}\ f(x)\ f(y)\ f(z) - \mathrm{Tr}\ f(y)\ f(x)\ f(z)$

$= \mathrm{Tr}\ f(x)\ f(y)\ f(z) - \mathrm{Tr}\ f(x)\ f(z)\ f(y)$

$= \mathrm{Tr}\ f(x)\ [f(y), f(z)]$

$= (x, [yz])$. //

<u>Lemma 1.6.3</u>. Let the Killing form $K(,)$ be nondegenerate on L . Then $L = \Sigma \oplus L_i$ where the L_i are simple ideals of L and $K(,)$ is nondegenerate on each L_i . In particular, L is semisimple.

<u>Proof</u>. Let B be an ideal of A with $B^2 = 0$. Then B ad B ad $L = 0$ and L ad B ad $L \subset B$. Thus $K(b,x) = \mathrm{Tr}(\mathrm{ad}\ b\ \mathrm{ad}\ x) = 0$ for all $b \in B$, $x \in L$ and so $B \subset L^{\perp} = 0$. The result then follows from Theorem 1.6.1. //

2. POLYNOMIAL MAPPINGS

2.1. The Zariski topology

Let V and V' be vector spaces over a field F and let $\dim V = m$, $\dim V' = n$. Let $\{e_i\}$ and $\{e'_j\}$ be basis for V and V', respectively. A mapping $f : V \to V'$ is called a polynomial mapping with respect to $\{e_i\}$, $\{e'_j\}$ if

$$f(\Sigma\, \alpha_i e_i) = \Sigma\, f_j(\alpha_1, \ldots, \alpha_m) e'_j ,$$

where $f_j \in F[X_1, \ldots, X_m]$, $j = 1, 2, \ldots, n$. A polynomial mapping $f : V \to F$ is called a polynomial function on V. Note that $\mathrm{Hom}_F(V, V')$ consists of polynomial mappings. Denote by $F[V]$ the set of polynomial functions on V. Then $F[V]$ becomes a commutative associative algebra over F with multiplication defined by $(fg)(x) = f(x)g(x)$, $x \in V$.

Henceforth we assume that F is infinite. Thus $F[V]$ is an F-algebra which is an integral domain.

If $f \in F[V]$, define $V_f = \{x \in V \mid f(x) \neq 0\}$.

Then $V_1 = V$, $V_0 = \phi$, and $V_{f_1} \cap \ldots \cap V_{f_n} = V_{f_1 f_2 \ldots f_n}$.

The Zariski topology in V is the topology for V having the V_f, $f \in F[V]$, as a basis of open sets.
Note that this topology is not Hausdorff. Let P be a subset of $F[X_1,\ldots,X_m]$ and identify V with F^m.
Let

$$\mathcal{Z}(P) = \{x = (x_1,\ldots,x_m) \in V \mid f(x) = 0 \text{ for all } f \in P\}.$$

Then the Zariski closed sets in V are the sets $\mathcal{Z}(P)$. Let I be the ideal of $F[X_1,\ldots,X_m]$ generated by P. Clearly $\mathcal{Z}(P) = \mathcal{Z}(I) = \mathcal{Z}(\{f_1,\ldots,f_r\})$ by the Hilbert Basis Theorem, where $f_1,\ldots,f_r \in I$. Thus the closed sets are the affine varieties in V over F.

Lemma 2.1.1. Every nonempty open set in V is dense in V.

Proof. Let U_1 and U_2 be nonempty open sets in V. It is enough to show $U_1 \cap U_2 \neq \phi$. Assume $U_i \neq V$, $i = 1,2$, and let f_1, f_2 be nonconstant polynomials such that $U_i \supset V_{f_i}$.

Then $V_{f_1 f_2} = V_{f_1} \cap V_{f_2} \neq 0$ since $f_1 f_2 \neq 0$ and F is infinite. Thus $U_1 \cap U_2 \neq \phi$. //

Lemma 2.1.2. Any polynomial mapping of V into V' is continuous with respect to the Zariski topology.

Proof. Let $\dim V = m$ and $\dim V' = n$, and let $f : V \to V'$ be a polynomial mapping. Let S be a closed subset of V', so $S = \mathfrak{Z}(\{g_1,\ldots,g_r\})$. Let $f(x) = \Sigma_j f_j(\alpha_1,\ldots,\alpha_m)\, e'_j$, where $x = \Sigma\alpha_i e_i \in V$.
Put

$$h_i(X_1,\ldots,X_m) = g_i(f_1(X_1,\ldots,X_m),\ldots,f_n(X_1,\ldots,X_m)),$$
$$i = 1,2,\ldots,r.$$

It is routine to check that $f^{-1}(S) = \mathfrak{Z}(\{h_1,\ldots,h_r\})$ and hence $f^{-1}(S)$ is closed. //

2.2. Differentials

Let $\{e_1,\ldots e_m\}$ and $\{e'_1,\ldots,e'_n\}$ be basis for V and V', respectively.

Definition 2.2.2. Let $p : V \to V'$ be a polynomial mapping. Define the mapping $\sigma_p : F[V'] \to F[V]$ by $(\sigma_p(f))(x) = f(p(x))$, $x \in V$, $f \in F[V']$. //

Clearly, σ_p is an F-algebra homomorphism. Let π_i be the element in $F[V]$ given by $\pi_i(\Sigma\alpha_j e_j) = \alpha_i$. Call π_i the i-th proiection. Then $F[V]$ is generated by $\pi_1,\ldots,\pi_m$, i.e.,

$$F[V] = F[\pi_1,\ldots,\pi_m] \; . \tag{1.21}$$

For each $y \in V$, define the mapping $\tau_y : F[V] \to F$ by $\tau_y(f) = f(y)$, $f \in F[V]$. Then τ_y is an F-algebra homomorphism. Any F-algebra homomorphism $\lambda: F[V] \to F$ is described in this way. Indeed, let $\alpha_i = \lambda(\pi_i)$, $i = 1,\ldots,m$, and let $y = \Sigma\alpha_i e_i$. By (1.21) any $f \in F[V]$ is of the form $f = f(\pi_1,\ldots,\pi_m)$ and hence $\lambda(f) = f(\alpha_1,\ldots,\alpha_m) = \tau_y(f)$. Also, $\tau_x = \tau_y$ if and only if $x = y$. Thus we have

Lemma 2.2.1. For any F-algebra homomorphism $\lambda : F[V] \to F$, there is a unique $y \in V$ such that $\tau_y = \lambda$. //

Lemma 2.2.2. Let F be a perfect field. Let $p : V \to V'$ be a polynomial mapping. If $d_a p$ is surjective for some $a \in V$ then $\sigma_p : F[V'] \to F[V]$ is an isomorphism.

Proof. Let $\dim V = m$ and $\dim V' = n$. Write $\beta = (\beta_1,\ldots,\beta_m)$ and $X = (X_1,\ldots,X_m)$, so that $g(\beta) = g(\beta_1,\ldots,\beta_m)$, $g \in F[X] = F[X_1,\ldots,X_m]$. Suppose that σ_p is not injective. Then, there is an $f = f(Y_1,\ldots,Y_n) \neq 0$ such that $\sigma_p(f) = 0$, so that

$0 = f(p(y)) = f(p_1(\beta),\ldots,p_n(\beta))$ for all $y = \Sigma\, \beta_i e_i \in V$ where $p(y) = \Sigma\, p_i(\beta)e_i'$. Since F is infinite, this implies $f(p_1(X),\ldots,p_n(X)) = 0$ and so $p_1(X),\ldots,p_n(X)$ are algebraically dependent. Let f be of minimal degree giving the algebraic dependence. By the chain rule we have

$$0 = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial Y_j}\right)(p_1(X),\ldots,p_n(X))\frac{\partial p_j}{\partial X_i}\,, \quad i = 1,\ldots,m. \tag{1.22}$$

Since $d_a p$ is surjective, $n \leq m$ and the matrix $(\partial p_j/\partial X_i)$ has rank n. Thus the system (1.22) has the only trivial solution,

$$\frac{\partial f}{\partial Y_j}(p_1(X),\ldots,p_n(X)) = 0\,, \quad j = 1,2,\ldots,n\,.$$

Since $\deg\,(\partial f/\partial Y_j) < \deg f$, it follows that $\partial f/\partial Y_j = 0$ for $j = 1,\ldots,n$. Thus, if $\operatorname{char} F = 0$, f is a nonzero constant, which is absurd. If $\operatorname{char} F = p > 0$, f is a polynomial in $Y_1^p,\ldots,Y_n^p$ and so $f = g^p$ since F is perfect. This contradicts the minimality of f. //

2.3. Extension of homomorphisms

Theorem 2.3.1. Let A' be an F-algebra which is an integral domain, K be an algebraically closed extension field of F and R be an F-subalgebra of A' containing 1. Let $S = R[z_1,\ldots,z_n]$, $z_i \in A'$. Let f be any nonzero element in S. Then there exists a nonzero element $g \in R$ such that if $\delta : R \to K$ is any F-algebra homomorphism with $g\delta \neq 0$, then δ can be extended to an F-algebra homomorphism $\tau : S \to K$ such that $f\tau \neq 0$.

Proof.(Chevalley). To proceed by induction, let $S = R[z]$.

Case 1 : z is transcendental over R. Since $f \in R[z]$, $f = a_0 + \ldots + a_k z^k$ with $a_k \neq 0$. Put $g = a_k$ and let $\delta : R \to K$ be an F-algebra homomorphism with $g\delta \neq 0$. Let $\bar{f}(X) = a_0\delta + \ldots + (a_k\delta) X^k \in K[X]$. Then $\bar{f}(X) \neq 0$ and there is an $\alpha \in K$ such that $\bar{f}(\alpha) \neq 0$. Since z is transcendental over R, the mapping $\tau : R[z] \to K$ defined by $(\Sigma\, b_i z^i)\tau = \Sigma\, (b_i\delta)\alpha^i$ gives an F-algebra homomorphism such that $\tau|_R = \delta$ and $f\tau = \bar{f}(\alpha) \neq 0$

<u>Case 2</u> : z is algebraic over R . Let $p(x)$ be a polynomial in $R[X]$ of minimal degree with $p(z) = 0$ and let $p(X) = a_0 + \dots + a_nX^n$ with $a_n \neq 0$. Then $p(X) = a_nh(X)$ where $h(X)$ is monic in $Q[X]$, Q the quotient field of R . Thus $h(X)$ is the minimum polynomial of z over Q . Let $k(X) = b_0 + \dots + b_mX^m$ be a minimum polynomial of f over R .
Then $b_0 \neq 0$ and set $g = ab_0 \neq 0$. We show that g is a desired element. Let $\delta : R \to K$ be an F-algebra homomorphism with $g\delta \neq 0$. Let $\alpha \in K$ be a root of $p^\delta(X) = a_0\delta + \dots + (a_n\delta)X^n \neq 0$. For each $q(z) = \Sigma\, q_iz^i \in R[z]$, define $\tau : R[z] \to K$ by

$q(z)\tau = q^\delta(\alpha) = \Sigma\,(q_i\delta)\alpha^i$. Then $\tau\,|_R = \delta$. We show that τ is well-defined. Suppose that $q(z) = 0$ for some $q(X) \in R[X]$. Then $q(x) = c(X)h(X)$, $c(X) \in Q[X]$ and so

$$a_nq(X) = c(X)a_nh(X) = c(X)p(X) \ . \tag{1.23}$$

Letting $c(X) = c_0 + c_1X + \dots + c_tX^t$ and $q(X) = q_0 + q_1X + \dots + q_sX^s$, one computes from (1.23)

$$a_nq_s = a_nc_t \Rightarrow q_s = c_t \in R \ ,$$

$$a_nq_{s-1} = c_{t-1}a_n + c_ta_{n-1} \Rightarrow c_{t-1}a_n \in R \ ,$$

$$a_nq_{s-2} = c_{t-2}a_n + c_ta_{n-2} + c_{t-1}a_{n-1} \ .$$

Multiplying the last equation by a_n, $c_{t-2}a_n^2 \varepsilon R$ and continuing this we have that $c_{t-i}a_n^i \varepsilon R$, $i = 1,2,\ldots,t$.

Since $a_n \varepsilon R$, this gives $c_j a_n^{t+1} \varepsilon R$ for all $j = 0,1,\ldots,t$. Thus

$$a_n^{t+2}q(X) = a_n^{t+1}c(X)p(X) = \overline{c}(X)p(X) \varepsilon R[X]$$

and $[a_n^{t+2}q(z)]\tau = (a_n\delta)^{t+2}(q^\delta(\alpha)) = \overline{c}^\delta(\alpha)p^\delta(\alpha) = 0$.

Since $a_n\delta \neq 0$, $q^\delta(\alpha) = q(z)\tau = 0$, so τ is well-defined. Evidently, $f\tau \neq 0$ since $b_0 \neq 0$.

Suppose that the result holds for $n - 1$ and let $B = R[z_n]$. Then $S = B[z_1,\ldots,z_{n-1}]$. Let $f \neq 0$ be in S. Then there is a $v \neq 0$ in B such that if $\rho : B \to K$ is any F-algebra homomorphism with $v\rho \neq 0$, ρ can be extended to an F-algebra homomorphism $\tau : S \to K$ with $f\tau \neq 0$. The case $n = 1$ applied to B with $v \neq 0$ gives the desired homomorphism τ. //

<u>Theorem 2.3.2.</u> Let F be algebraically closed and perfect. Let $p : V \to V'$ be a polynomial mapping such that $d_a p$ is surjective for some $a \varepsilon V$. For any nonzero $f \varepsilon F[V]$, there exists a nonzero element $g \varepsilon F[V']$ such that $p(V_f) \supseteq V'_g$, or equivalently, there exists $0 \neq g \varepsilon F[V']$ such that if $g(y) \neq 0$ then $p(x) = y$ for some $x \varepsilon V$ with $f(x) \neq 0$.

<u>Proof</u>. Let $R = F[V']\sigma_p$ where $\sigma_p : F[V'] \to F[V]$ is an F-algebra isomorphism by Lemma 2.2.2. Then $S \equiv F[V] = F[\pi_1,\ldots,\pi_m] = R[\pi_1,\ldots,\pi_m]$. By Theorem 2.3.1, there exists a nonzero $g' \in R$ such that any homomorphism $\delta : R \to F$ with $g'\delta \neq 0$ is extended to an F-algebra homomorphism $\tau : S \to F$ such that $f\tau \neq 0$.
Choose a $g \neq 0$ in $F[V']$ with $g\sigma_p = g'$. Let $y \in V'$ be any element such that $g(y) \neq 0$, i.e., $y \in V'_g$.
Define $\delta : R \to F$ by $(h\sigma_p)\delta = h(y)$, $h \in F[V']$.
Then δ is an F-algebra homomorphism with $g'\delta \neq 0$.
Let τ be an extension of σ to an F-algebra homomorphism : $S \to F$ such that $f\tau \neq 0$. By Lemma 2.2.1 $\tau = \tau_x$ for some $x \in V$, so that $f\tau = \tau_x(f) = f(x) \neq 0$.
For any $h \in F[V']$,

$$(h\sigma_p)\tau = (h\sigma_p)\delta = h(y)$$

$$= (h\sigma_p)\tau_x = h\sigma_p(x) = h(p(x)) .$$

Thus $p(x) = y$ and $y \in p(V_f)$. //

<u>Exercise 2.3.1</u>. Let K be an algebraically closed field containing F and let $F[X] = F[X_1,\ldots,X_n]$ be the polynomial ring in $X_1,\ldots,X_n$ over F. Let S be a subset of $F[X]$. The set $\mathfrak{Z}_K(S) = \{v \in K^n | f(v) = 0$ for all $f \in S\}$ is called an (Zariski) F-closed set in K^n

If I is an ideal of $F[X]$, the radical $\sqrt{I}$ of I is defined by $\sqrt{I} = \{f \in F[X] \mid f^m \in I \text{ for } m > 0\}$.
Clearly, $\sqrt{I}$ is an ideal of $F[X]$ containing I. Using Theorem 2.3.1 prove the Hilbert Nullstellensatz : For any ideal I of $F[X]$, $\sqrt{I} = \mathcal{J}(\mathcal{Z}_K(I))$ where $\mathcal{J}(E) = \{f \in F[X] \mid f(v) = 0 \text{ for all } v \in E\}$ for a subset E of K^n.

3. LIE ALGEBRAS OF CHARACTERISTIC 0

3.1. Introduction

The theory of Lie algebras of characteristic 0 is an outgrowth of the Lie theory of continuous groups in which local problems concerning Lie groups are reduced to corresponding problems on Lie algebras. During the development of the structure of Lie algebras for many years, Lie algebras brought applications to many branches of mathematics, such as group theory, differential geometry, differential equations, topology, and physics. Besides being useful in many parts of mathematics, the theory of Lie algebras is the most widely and successfully studied area of nonassociative algebras, mainly because of the elegance and completeness of the structure and representation theories for semisimple Lie algebras of characteristic 0 .

In this chapter, we briefly discuss the classical theorems on Lie algebras which are essential for the general structure, the classification and representations of semisimple Lie algebras of characteristic 0 . The development of the material is designed to set up the groundwork for the structure of flexible Lie-admissible algebras rather than to provide the general theory of Lie algebras for a comprehensive account.

Definition 3.1.1. Let B, S be subsets of a Lie algebra L over a field F . The centralizer of S in B is the set $C_B(S) = \{x \in B \mid [xS] = 0\}$. The centralizer of S is $C_L(S)$ and the center of L is $C(L) = C_L(L)$. For a subspace B of L , the normalizer of B (in L) is the set $N_L(B) = N(B) = \{x \in L \mid [xB] \subseteq B\}$. //

Definition 3.1.2. A Lie module V for L is an L-module over F such that $v[xy] = (vx)y - (vy)x$, $v \in V$, $x,y \in L$. If $f : L \to (\mathrm{Hom}_F V)^-$ is a representation of L , then V together with the module operation $vx = vf(x)$ for $v \in V$, $x \in L$ is a Lie module for L . The Lie module obtained in this way is called the Lie module afforded by f . By an irreducible representation f of L , we mean that the L-module V afforded by f is L-irreducible.

Conversely, if V is a Lie module for L over F, then the mapping $f : L \to (\mathrm{Hom}_F V)^-$ defined by $vf(x) = vx$ for $x \in L$ becomes a representation of L acting on V, which is called the representation afforded by V. //

Henceforth, for a Lie algebra L, all L-modules are referred to as Lie modules for L. Note that the kernel of the ad representation of L is the center $C(L)$. Also, a subalgebra B of L is an ideal of $N_L(B)$, and if B and C are ideals of L then so is $[BC] = [CB]$. These are consequences of the Jacobi identity.

Let V be an L-module and suppose that V is also a Lie algebra such that $[vv']x = [(vx)v'] + [v(v'x)]$. $v, v' \in V$, $x \in L$. Define a product in $L \oplus V$ by

$$[x + v, x' + v'] = [xx'] + (vx' - v'x + [vv'])$$

for $x, x' \in L$ and $v, v' \in V$. Then $L \oplus V$ becomes a Lie algebra. Indeed, clearly $L \oplus V$ is anticommutative. If $x \in L$, $R_x|_L$ and $R_x|_V$ are derivations. Thus, to see $R_x \in \mathrm{Der}\,(L \oplus V)$, one checks

$$[[v,y],x] = (vy)x = (vx)y + v[yx]$$

$$= [[v,x],y] + [v,[y,x]]$$

for $v \in V$, $x,y \in L$. For $v \in V$, $R_v|_V$ is clearly a derivation and it remains to check that

$$[[x,y],v] = -v[xy] = -(vx)y + (vy)x = [[x,v],y] + [x,[y,v]]$$

and likewise R_v acts on $v'y$ as a derivation. Thus $R_{L\oplus V} \subseteq \mathrm{Der}\,(L \oplus V)$ and $L \oplus V$ is a Lie algebra which is called the <u>split extension</u> of L by V. In particular, the split extension of Der L by L is called the <u>holomorph</u> of L. The Lie algebra $L \oplus V$ with $[VV] = 0$ is the split <u>null</u> extension of L by V.

3.2. <u>Nilpotent Lie algebras</u>

The structure of Lie algebras can be described in terms of certain nilpotent subalgebras (Cartan subalgebra). In this section we develop some fundamental properties of nilpotent Lie algebras. Here we assume that the characteristic of F is arbitrary.

<u>Definition 3.2.1</u>. For a Lie algebra L, the descending central series $L \supseteq L^2 \supset \ldots \supset L^n \supset \ldots$ is recursively defined by $L^1 = L$, $L^{i+1} = [L^i L]$, $i = 0,1,2,\ldots$

If $L^i = 0$ for some i, L is called nilpotent. //

The derived series $L^{(i)}$ and the descending central series L^i are analogous to those of a group. If B is an ideal of L, then $B^{(i)}$ and B^i are ideals of L. Note also that any homomorphic images of nilpotent Lie algebras are nilpotent. By induction, one easily sees that $[L^i\ L^j] \subset L^{i+j}$ for integers $i, j \geq 0$. This implies that the products of n elements of L in any association are contained in L^n. In particular, $L^{(n)} \subset L^{2^n}$ and hence any nilpotent Lie algebra is solvable ; however the convers is not true (why ?). If C is the center of L and L/C is nilpotent then L is nilpotent since $L^i \subset C$ implies $L^{i+1} = 0$. Since C is the kernel of ad, we have

<u>Lemma 3.2.1</u>. L is nilpotent if and only if ad L is nilpotent. //

A stronger version of Lemma 3.2.1 is the celebrated theorem of Engel that L is nilpotent if and only if ad L consists of nilpotent linear transformations, which we show below.

<u>Definition 3.2.2</u>. The ascending central series of L is the sequence $C^i(L)$, $i \geq 0$, defined recursively by

$$C^0(L) = 0,\ C^i(L) = \{x \in L \mid [xL] \subset C^{i-1}(L)\} \quad i \geq 1 .$$

Obviously, $C^1(L) = C(L)$ and $C^i(L)/C^{i-1}(L)$

$$= C(L/C^{i-1}(L)) \ . \qquad //$$

Lemma 3.2.2. $L^{i+1} = 0$ if and only if $C^i(L) = L$ for $i \geqq 0$.

Proof. If $L^{i+1} = 0$, $L^i \subset C^1(L)$ and so $L^{i-1} \subset C^2(L)$, $L \subset C^i(L)$. If $C^i(L) = L$, $L^2 \subset C^{i-1}(L)$ and continuing this gives $L^{i+1} \subset C^0(L) = 0$. //

Definition 3.2.3. Let N be a subalgebra of L . Then $C_L^i(N)$ is the series of subspaces of L defined inductively by

$$C_L^0(N) = 0 \ , \ C_L^i(N) = \{x \in L \mid [xN] \subset C_N^{i-1}(N)\} \ .$$

Clearly, $C^i(L) = C_L^i(L)$. //

Lemma 3.2.3. Let T be a nilpotent linear transformation on V . If $W \subset U$ are T-stable subspaces of V then $W \subset U$ can be refined as $W = U_0 \subset U_1 \subset \ldots \subset U_{r-1} \subset U_r = U$ with $U_i T \subset U_{i-1}$, $i = 1,\ldots,r$.

Proof. If we let $\overline{U} = U/W$, T induces a nilpotent transformation $\overline{T}$ on $\overline{U}$ by $(a + W)\overline{T} = aT + W$, $a \in U$. Thus $\overline{U}\,\overline{T}^r = \overline{0}$ for some $r > 0$ and, letting $\overline{U}_i = \overline{U}\,\overline{T}^{r-i}$, $i = 0,1,\ldots,r$, we have $\overline{0} \subset \overline{U}_1 \subset \ldots \subset \overline{U}_r$

$= \bar{U}$ with $\bar{U}_i\bar{T} = \bar{U}_{i-1}$. Let U_i be the inverse image of $\bar{U}_i$ by the natural homomorphism. Then $W \subset U_1 \subset \ldots \subset U_r =$ with $U_i T \subset U_{i-1}$, as desired. //

Theorem 3.2.4. Let N be a subalgebra of L such that $ad_L N$ consists of nilpotent linear transformations. Then $C_L^i(N) = L$ for some i .

Proof. If $\dim N = 1$, it is trivial. Assume that the result holds for such subalgebras of dimension < dim N . Let H be a maximal proper subalgebra of N . Then there exists an $m > 0$ such that $C_L^m(H) = L$, so $C_N^m(H) = N$ since $C_L^m(H) \cap N = C_N^m(H)$. Thus one can choose a j such that $C_N^j(H) \subset H$ but $C_N^{j+1}(H) \not\subset H$, and let $x \in C_N^{j+1}(H) - H$. Then $Fx \oplus H$ is a subalgebra of N since $[xH] \subset H$, so $N = Fx + H$ and H is an ideal of N . We show by induction on i that $C_L^i(H)$ is ad x-stable Thus, if $C_L^{i-1}(H)$ is ad x-stable, $y \in C_L^i(H)$ and $h \in H$, then

$$[[yx]h] = [[yh]x] + [y[xh]]$$

$$\in [C_L^{i-1}(H)x] + [C_L^i(H)[xh]] \subset C_L^{i-1}(H)$$

since $[xH] \in H$, so $y \text{ ad } x \in C_L^i(H)$. Finally, since ad : is nilpotent and stablizes the spaces in the chain

$0 = C_L^0(H) \subset \ldots \subset C_L^m(H) = L$, by Lemma 3.2.3 this chain has a refinement $0 = B^0 \subset B^1 \subset \ldots \subset B^n = L$ such that $[B^i x] \subset B^{i-1}$. Moreover, $[B^i H] \subset B^{i-1}$ since $[C_L^i(H)H] \subset C_L^{i-1}(H)$. If follows that $[B^i N] \subset B^{i-1}$ for $1 \leqq i \leqq n$. Thus $C_L^n(N) = L$. //

If L is nilpotent, clearly $\operatorname{ad} x$ is nilpotent for all $x \in L$. If $\operatorname{ad} L$ consists of nilpotent transformations then by Theorem 3.2.4 $L = C_L^i(L) = C^i(L)$ for some i and by Lemma 3.2.2 L is nilpotent. Therefore we have the following theorem of Engel.

Theorem 3.2.5 (Engel). L is nilpotent if and only if $\operatorname{ad} L$ consists of nilpotent transformations. //

Remark. Let A be an algebra over F. For a subalgebra B of A and a positive integer n, define B^n as the linear span of all product of n elements in B in all possible associations. Then B is called nilpotent if $B^n = 0$ for some $n > 0$. We have observed that a nilpotent Lie algebra is also nilpotent in this sense. An algebra A is called power-associative if the subalgebra generated by $x \in A$ is associative for every $x \in A$, or equivalently, $x^m x^n = x^{m+n}$ for all integers $m,n \geqq 0$, $x \in A$. Then, an element $x \in A$ is said to be nilpotent if $x^m = 0$ for some $m > 0$, and

A is called <u>nil</u> if every element is nilpotent. An anticommutative algebra A is trivially nil with $x^2 = 0$, $x \in A$. An algebra A is called <u>alternative</u> if it satisfies the alternative laws $x^2 y = x(xy)$ and $yx^2 = (yx)x$ for all $x,y \in A$. Unlike anticommutative algebras (so Lie algebras), for other nonassociative algebras such as Jordan or alternative algebras A, the nilpotence of x, L_x and R_x are equivalent. Moreover, if A is finite-dimensional (char $F \neq 2$), solvability, nilpotence and nility of A are equivalent (R.D.Schafer, "An introduction to nonassociative algebras", Academic Press, N.Y., 1966). Therefore, for finite-dimensional alternative and Jordan algebras, Engel's Theorem holds. A.A.Albert conjectured in 1948 that the solvability and nilpotence are equivalent in a commutative power-associative algebra. However, D.Suttles disproved this conjecture in 1972 by constructing a 5-dimensional commutative power-associative algebra A which is solvable but not nilpotent. The algebra A has a basis $\{a,b,c,d,e\}$ with multiplication given by $ab = c$, $ac = d$, $ae = -c$, $bc = e$, $bd = c$ and all other products are 0. It is easily seen that $A^2A^2 = 0$ but A is not nilpotent. Subsequently, M. Gerstenhaber and H. C. Myung showed that dimesion 5 is least possible by demonstrating that any commutative power-associative nil algebra of dimension $\leqq 4$ is nilpotent ("On commutative power-associative nilalgebras of low dimension", Proc. Amer. Math. Soc. 48(1975), 29-32). In particular, Suttles' example shows that Engel's Theorem does not hold for commutative algebras. //

If one takes $N = L$ in Theorem 3.2.4 then the proof establishes the following result.

Corollary 3.2.6. A maximal proper subalgebra of a nilpotent Lie algebra L is an ideal of L of codimension 1. //

Definition 3.2.4. Let N be a set and let B be an N-module. Then $B_0^i(N)$ is defined recursively by

$$B_0^0(N) = 0 \ , \quad B_0^i(N) = \{x \in B \mid xN \subset B_0^{i-1}(N)\} \ . \qquad //$$

If N is a subalgebra of L and $B = L$ then $B_0^i(\mathrm{ad}\ N) = C_L^i(N)$. In the following we prove two variants of Engel's Theorem.

Theorem 3.2.7. Let V be a finite-dimensional vector space over F. Let N be a subalgebra of $(\mathrm{Hom}_F\ V)^-$ consisting of nilpotent linear transformations. Then $V_0^i(N) = V$ for some i. Thus, relative to a suitable basis for V, the subalgebra N is represented by nil triangular matrices.

Proof. Since $y(\mathrm{ad}\ x)^n$ is a linear combination of terms $x^m y x^{n-m}$ for $x,y \in \mathrm{Hom}_F V$, $x^{2n} = 0$ implies $(\mathrm{ad}\ x)^n = 0$. Thus, $\mathrm{ad}_N N$ consists of nilpotent linear transformations. Let $L = N \oplus V$ be the split null extension of N by V. Then $\mathrm{ad}_L N$ also consists of nilpotent linear transformations. Thus by Theorem 3.2.4 $L = C_L^i(N) = L_0^i(N)$ for some i, where $L_0^i(N) = L_0^i(\mathrm{ad}\ N)$. Since $V \subset L$, it follows by induction that $V \cap L_0^i(N) = V_0^i(N)$, $i = 0,1,2,\ldots$. Therefore, $V = V_0^i(N)$, and this gives a chain $V_0^1(N) \subset \ldots \subset V_0^{i-1}(N) \subset V_0^i(N) = V$ such that $V_0^k(N)N \subset V_0^{k-1}(N)$, $k = 1,2,\ldots,i$. Hence one can choose a basis for V relative to which N is represented by nil triangular matrices. //

Theorem 3.2.8. (Engel). Let $f : L \to (\mathrm{Hom}_F V)^-$ be a representation where V is finite-dimensional. Suppose that $f(x)$ is nilpotent for $x \in L$. Then there exists a nonzero vector $v \in V$ such that $vf(x) = 0$ for all $x \in L$.

Proof. In Theorem 3.2.6, take $f(L) = N$. Since $V = V_0^i(N)$ for some i, $V_0^1(N) \neq 0$ and any nonzero vector in $V_0^1(N)$ gives the desired condition. //

Definition 3.2.5. Let B be a finite-dimensional L-module over F. For a function $a : L \to F$, define

$$B_a(L) = \{x \in B \mid x(T - a(T))^{n(T)} = 0 \text{ for } T \in L\}.$$

If L is a nilpotent Lie algebra and $B_a(L) \neq 0$, then $B_a(L)$ is called the weight space for L in B with respect to a and a is called a weight of L in B. Note that $B_a(L) = \bigcap_{T\epsilon L} B_{a(T)}(T)$ for any function $a : L \to F$. Let $B_*(L) = \Sigma_{T\epsilon L} B_*(T)$. Then $B_0(L)$ and $B_*(L)$ are called the Fitting components of B with respect to L. //

Theorem 3.2.9. Let B be a finite-dimensional L-module over F. Suppose that L is nilpotent and that each $x \epsilon L$ is split over F. Then B is a direct sum $B = \Sigma_a \oplus B_a(L)$ of weight spaces for L in B and each $B_a(L)$ is an L-submodule of B.

Proof. The proof is by induction on $\dim B$. If $\dim B = 1$, there is nothing to prove. One may take L to be a subalgebra of $(\mathrm{Hom}_F B)^-$. Then $\mathrm{ad}_L x$ is nilpotent for $x \epsilon L$. Thus $0 = (\mathrm{ad}_L x)_s = \mathrm{ad}_L x_s$ by Theorem 1.3.3 and $[y,x_s] = 0$ for $x,y \epsilon L$. If x_s for each $x \epsilon L$ acts on B as a single scalar $a(x)$, then $B = B_a(L)$ since $x - a(x) = x_n$. Thus we may assume that there is an x_s which is not a scalar on B.

Hence $B = \Sigma \oplus B_{\alpha_i}(x_s)$ and $B_{\alpha_i}(x_s) \equiv B_i \neq 0$, $1 \leq i \leq m$ with $m > 1$, so $\alpha_1,\ldots,\alpha_m$ are distinct.

Since $[L, x_s] = 0$, each B_i is L-stable and by induction applied to B_i, we have $B_i = \Sigma_a \oplus B_{ia}(L)$ where $B_{ia}(L)$ is L-stable. Since $\Sigma_i B_{ia}(L) \subset B_a(L)$, $B = \Sigma_a B_a(L)$. Note that $B_i = B_{\alpha_i}(x_s) = B_{\alpha_i}(x)$. Since the α_i are distinct, it follows from this that if $B_a(L) \neq 0$, $a(x) = \alpha_i$ for some i since $a(x)$ is an eigenvalue of x. Thus $B_a(L) \subset B_i$ for some i and moreover it is easily seen that $B_a(L) = B_{ia}(L)$. //

In view of Theorem 3.2.9, if B is finite-dimensional, there are only finitely many weights for L in B.

<u>Theorem 3.2.10</u>. Let N be a nilpotent subalgebra of L and let B be a finite-dimensional L-module. Regard L as an N-module via ad. Then $B_a(N)L_b(N) \subset B_{a+b}(N)$ for all functions $a,b : N \to F$.

<u>Proof</u>. Let $\overline{B} = L \oplus B$ be the split null extension of L by B. Regard $\overline{B}$ as an N-module via ad. Since L and B are N-submodules of $\overline{B}$, $\overline{B}_a(N) = L_a(N) \oplus B_a(N)$. Noting that $[\overline{B}_a(N), \overline{B}_b(N)] \subset \overline{B}_{a+b}(N)$ by Corollary 1.4.8, we have

$$[\overline{B}_a(N), \overline{B}_b(N)] = [L_a(N)L_b(N)] + B_a(N)L_b(N) + B_b(N)L_a(N)$$

$$\subset L_{a+b}(N) \oplus B_{a+b}(N) .$$

Since B is an ideal of $\overline{B}$, this in particular implies $B_a(N)L_b(N) \subset B_{a+b}(N)$. //

Exercise 3.2.1. Let N be a nilpotent ideal in L and let B be a finite-dimensional L-module. Prove that $B_a(N)$ is an L-submodule for every $a : N \to F$.

3.3. Cartan subalgebras

The Cartan subalgebras of a Lie algebra L are certain nilpotent subalgebras that are central objects for the structure of L . We show here that they exist if F is infinite and that the decomposition of L into weight spaces for a split Cartan subalgebra provides a rough description of a multiplication table for L .

Definition 3.3.1. A subalgebra H of L is a Cartan subalgebra (CSA) of L if H is nilpotent and $H = L_0(\text{ad } H)$, the Fitting 0-component of L relative to ad H . //

The following characterization of a CSA is often convenient.

Theorem 3.3.1. A subalgebra H of L is a CSA of L if and only if H is nilpotent and $H = N_L(H)$.

Proof. Let $N = N_L(H)$. Suppose that H is a CSA of L, so $H = L_0(\text{ad } H)$. It follows that $N(\text{ad } x)^n = 0$ for $x \in H$ and $n > 0$, since $N \text{ ad } x \subset H$ for $x \in H$. Thus $N \subset L_0(\text{ad } H) = H$ and $N = H$. For the converse, show that if $H \subsetneq L_0(\text{ad } H)$, $H \subsetneq N$. Let $L_0 = L_0(\text{ad } H)$. Since ad H stablizes L_0 and H, and ad $x (x \in H)$ is nilpotent on L_0 and H, ad H gives rise to a Lie algebra of nilpotent linear transformations on $L_0/H \neq 0$. Thus by Engel's Theorem 3.2.8 there exists $\overline{0} \neq x + H \in L_0/H$ such that $(x + H) \text{ ad } H = 0$ and hence $x \in N$ but $x \notin H$, i.e., $H \subsetneq N$. //

Exercise 3.3.1. Show that a CSA H of L is a maximal nilpotent subalgebra of L.

Definition 3.3.2. An element $h \in L$ such that $\dim L_0(\text{ad } h)$ is minimal is called a regular element. Denote by L_{reg} the set of regular elements in L. //

Note that $\dim L_0(\text{ad } h) \geq 1$ since $[hh] = 0$ and that $L_{reg} \neq \phi$. Henceforth we assume that F is infinite.

Theorem 3.3.2. If $x \in L_{reg}$ then $H = L_0(\text{ad } x)$ is a CSA of L .

Proof. Let $L = H \oplus L_*(\text{ad } x)$ be the Fitting decomposition of L relative to ad x . Let $L_* = L_*(\text{ad } x)$. By Corollary 1.4.8 $L_* \text{ ad } H \subset L_*$. Thus if we let $f(y) = \det(\text{ad } y|_{L_*})$ for $y \in H$, then f is a polynomial function on H , and $H_f = \{y \in H | f(y) \neq 0\}$ is a nonempty Zariski open set in H since ad x is nonsingular on L_* by Lemma 1.3.4 and so $f(x) \neq 0$. Hence H_f is dense in H . Since $L = L_* \oplus H$ and ad y stablizes H and L_* for $y \in H$, it follows that $L_0(\text{ad } y) \subseteq L_0(\text{ad } x) = H$ for $y \in H_f$. (Note that ad y is nonsingular for $y \in H_f$) . Since dim H is minimal, $L_0(\text{ad } y) = H$ for all $y \in H_f$. Thus if dim $H = n$, $(\text{ad } y|_H)^n = 0$ for $y \in H_f$. If we let $g(y) = (\text{ad } y|_H)^n$ for $y \in H$, g is a polynomial mapping of H into $\text{Hom}_F H$ and so is continuous by Lemma 2.1.2. Since each point is closed in Hom H , $\{y \in H | g(y) = 0\}$ is a closed subset of H containing H_f and coincides with H since H_f is dense in H . Thus $(\text{ad } y|_H)^n = 0$ for all $y \in H$ and by Engel's Theorem H is nilpotent. Since $x \in H$, this implies that $H = H_0(\text{ad } H) \subset L_0(\text{ad } H) \subset L_0(\text{ad } x) = H$. Thus $L_0(\text{ad } H) = H$ and H is a CSA of L . //

Corollary 3.3.3. Let H be a CSA of L . If $a \in H \cap L_{reg}$ then $H = L_0(\text{ad } a)$.

Proof. Since H is nilpotent, $H \subset L_0(\text{ad } a)$. But by Theorem 3.3.2 $L_0(\text{ad } a)$ is a CSA of L and $H = L_0(\text{ad } a)$ since a CSA of L is a maximal nilpotent subalgebra of L (Exercise 3.3.1) . //

In view of Corollary 3.3.3, if two CSA of L have a regular element in common, they coincide.

Corollary 3.3.4. If F is infinite, L has a CSA.

Proof. Since $L_{reg} \neq \phi$, the result follows from Theorem 3.3.2. //

Remark. It can be shown that any solvable Lie algebra over an arbitrary field F has a CSA (D.J. Winter, "Abstract Lie algebras", MIT Press, Cambridge, MA, 1972). Also, C.W.Barnes has shown that if F has at least $\dim L$ elements, L has a CSA ("On Cartan subalgebras of Lie algebras", Math. Z., 101(1967), 350-355). //

Theorem 3.3.5. If F is infinite, L_{reg} is a Zariski open set in L .

Proof. Note that $L_{reg} \neq \phi$ and $x \in L_{reg}$ if and onl if $\dim L_*(\text{ad } x)$ is maximal since $L = L_0(\text{ad } x) \oplus L_*(\text{ad } x)$

Let $s = \max\{\dim L_*(\text{ad } x) | x \in L\}$ and let $n = \dim L$. Then $L_*(\text{ad } x) = L(\text{ad } x)^n$ (see the proof of Fitting Lemma 1.3.4). Thus $x \in L_{reg}$ if and only if $s = \text{rank}(\text{ad } x)^n$ if and only if $\det_B M(x) \neq 0$ for some $s \times s$ minor $M(x)$ of $(\text{ad } x)^n$ where B is a fixed basis for L. Since the mapping $x \to \det_B M(x)$ is a polynomial function on L, it is continuous and $\{x \in L | \det_B M(x) \neq 0\}$ is open in L. Let $M_1(x), \ldots, M_r(x)$ be all the $s \times s$ minors of $(\text{ad } x)^n$ and let $U_i = \{x \in L | \det_B M_i(x) \neq 0\}$, $i = 1, 2, \ldots, r$. Then $L_{reg} = U_1 \cup \ldots \cup U_r$ and L_{reg} is open. //

Since L_{reg} is dense in L, almost all elements in L are regular. If H is a CSA of L, we always regard L as an H-module via the ad representation.

<u>Definition 3.3.3</u>. We say that a CSA H of L is split over F if $\text{ad}_L x$ is split over F for $x \in H$. Also, L is split over F if L has a split CSA over F. If H is a CSA of L, a root of H in L is a function $a : H \to F$ such that $L_a(H) (= L_a(\text{ad } H)) \neq 0$. //

Thus, the roots of H in L are the weights of H in L where L is regarded as an H-module via ad. We will show later that the roots of a split CSA are linear functions if the characteristic is 0.

<u>Theorem 3.3.6</u>. Let H be a split CSA of L. Then

1) $[L_a(H)\ L_b(H)] \subset L_{a+b}(H)$ for all $a,b : H \to F$;

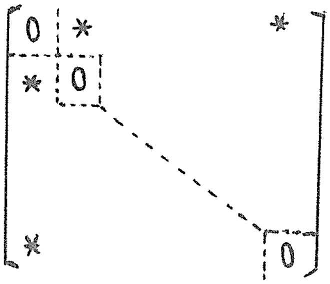
2) $[H\ L_a(H)] \subset L_a(H)$ for all $a : H \to F$ and if $a \neq 0$ then $[H\ L_a(H)] = [L_a(H)\ H] = L_a(H)$;

3) $K(L_a(H), L_b(H)) = 0$ for $a,b : H \to F$ with $a + b \not\equiv 0$;

4) $L = \Sigma_a \oplus L_a(H)$ where a ranges over the roots of H in L ;

5) $K(H, L_a(H)) = 0$ for all $a : H \to F$ with $a \neq 0$.

<u>Proof</u>. 1) and the first part of 2) are consequences of Theorem 3.2.10 while 4) follows from Theorem 3.2.9. For the second part of 2), let $h \in H$ with $a(h) \neq 0$. If $x \text{ ad } h = 0$ for some $x \in L_a(H)$, then $x(\text{ad } h - a(h))^n = 0$ implies $a(h)^n x = 0$ or $x = 0$, so ad h is injective. For 3), let $a + b \not\equiv 0$ and let $L = \Sigma_c \oplus L_c(H)$. Let $B = \bigcup_c B_c$ be a basis for L such that B_c is a basis of $L_c(H)$. For $x \in L_a(H)$ and $y \in L_b(H)$, $L_c(H) \text{ ad } x \text{ ad } y \subset L_{a+b+c}(H)$ where $L_{a+b+c}(H) = 0$ or $c \neq a + b + c$. Thus the matrix of ad x ad y relative to B is



Hence $\mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y) = K(x,y) = 0$. 5) is a special case of 3). //

3.4. Solvable Lie algebras

In this section we assume that B is a finite-dimensional L-module over a field F of characteristic 0 .

Theorem 3.4.1. Let $h \in L$ be such that $h \in (L_0(\mathrm{ad}\ h))^{(1)}$. Then $B = B_0(h)$.

Proof. We may assume that F is algebraically closed. Indeed, for the scalar extension B_K of B to the algebraic closure K of F we have $B \cap (B_K)_0(h) = B_0(h)$. Let $h = \Sigma [x_i y_i]$ where $x_i, y_i \in L_0(\mathrm{ad}\ h)$. By Theorem 3.2.10 $B_\alpha(h) L_0(\mathrm{ad}\ h) \subset B_\alpha(h)$ where $B = \Sigma_\alpha \oplus B_\alpha(h)$. Hence each $B_\alpha(h)$ is stable under x_i, y_i .

Let $f : L \to (\mathrm{Hom}_F\ B)^-$ be the representation of L afforded by B. Then

$$f(h)|_{B_\alpha(h)} = \Sigma f([x_i y_i])|_{B_\alpha(h)} = \Sigma [f(x_i), f(y_i)]|_{B_\alpha(h)}$$

and hence $\mathrm{Tr}\ f(h)|_{B_\alpha(h)} = \alpha \cdot \dim B_\alpha(h) = 0$. Since char $F = 0$, this implies $\alpha = 0$ unless $B_\alpha(h) = 0$, so $B = B_0(h)$. //

<u>Corollary 3.4.2</u>. If ad h is nilpotent and $h \in L^{(1)}$ then $B = B_0(h)$. //

<u>Theorem 3.4.3</u>. Let L be solvable. Then $B = B_0(L^{(1)})$.

<u>Proof</u>. Suppose not and take a counterexample with $\dim L + \dim B$ minimal. Choose n maximal such that $L^{(n)} \neq 0$, and let $A = L^{(n)}$. Then A is an abelian ideal of L. If $n = 0$, $L^{(1)} = 0$ and so $B_0(L^{(1)}) = B$, contrary to the supposition. Thus $n \geq 1$. Also $L\ \mathrm{ad}[xy] \subset L^{(n)}$ for $x,y \in L^{(n-1)}$ and hence $L(\mathrm{ad}[xy])^2 = 0$ since A is abelian. Therefore $L = L_0(\mathrm{ad}[xy])$ and by Corollary 3.4.2 $B = B_0([xy])$ for $x,y \in L^{(n-1)}$. Since A is abelian, this implies that $B = B_0(A)$. Let $W = \{v \in B | vA = 0\}$. Since $B = B_0(A) \neq 0$, $W \neq 0$ by Engel's Theorem 3.2.8.

Suppose $W = B$. Then $BA = 0$ and B is regarded as an L/A-module with $v(x + A) = vx$, $x \in L$ and $v \in B$. By the minimality of $\dim B + \dim L$, we must have $B = B_0((L/A)^{(1)})$ and this implies $B = B_0(L^{(1)})$, a contradiction. If $W \neq B$ then since $(WL)A \subseteq (WA)L + W[LA] = 0$, W is L-stable and B/W becomes an L-module with $(v + W)x = vx + W$, $v \in B$, $x \in L$. Thus $W = W_0(L^{(1)})$ and $B/W = (B/W)_0(L^{(1)})$ by the minimality of $\dim B + \dim L$. For $v \in B$, $(v + W)x^m = 0$ for some $m > 0$ and $vx^m \in W$, so $vx^{m+n} = 0$ for $x \in L^{(1)}$. Hence $B = B_0(L^{(1)})$, a contradiction. //

<u>Corollary 3.4.4</u>. Let L be solvable. Then the set $N = \{x \in L \mid B = B_0(x)\}$ is an ideal of L which contains $L^{(1)}$.

<u>Proof</u>. By Theorem 3.4.3, $L^{(1)} \subset N$. Let H be a maximal subalgebra of L such that $L^{(1)} \subset H$ and $B = B_0(H)$. Thus, in view of Theorem 3.2.7, $B_0^m(H) = B$ for some $m > 0$. We show that $N = H$, whence N is an ideal of L since $[NL] \subseteq L^{(1)} \subset N$. So, let $x \in N$ be any element and consider the series $0 = B_0^0(H) \subset \ldots \subset B_0^m(H) = B$. For $v \in B_0^i(H)$,

$(vx)H \subset (vH)x + v[Hx] \subset B_0^{i-1}(H)x + vH$ since $[Hx] \subset L^{(1)} \subset H$.

Thus by induction we see that each $B_0^i(H)$ is x-stable. By Lemma 3.2.3 this series has a refinement $0 = B^0 \subset B^1 \subset \ldots \subset B^n = B$ such that $B^i x \subset B^{i-1}$ and $B^i H \subset B^{i-1}$ (since $B_0^i(H)H \subset B_0^{i-1}(H)$) . Hence $B^i(Fx + H) \subset B^{i-1}$ for all i and $B(Fx + H)^n = 0$, so $B = B_0(Fx + H)$. By the maximality of H we have $H = Fx + H$ and $H = N$. //

Theorem 3.4.5.(Lie). Let L be a solvable Lie algebra over F and let $B \neq 0$ be a finite-dimensional irreducible L-module over F . Suppose that L has a split CSA H such that every $x \in H$ is split on B over F . Then $\dim B = 1$.

Proof. In view of Theorem 3.4.3, $B = B_0(L^{(1)})$ and so by Theorem 3.2.7 $B_0^1(L^{(1)}) \equiv W \neq 0$. Since $L^{(1)}$ is an ideal of L , W is an L-submodule and hence $W = B$ by the L-irreducibility of B . Thus $BL^{(1)} = 0$ and B is regarded as an A-module with $v(x + L^{(1)}) = vx$, $v \in B$ and $x \in L$, where $A = L/L^{(1)}$. Put $L_* = \Sigma_{a \neq 0} L_a(\text{ad } H)$. Then by Theorem 3.3.6 $L = H \oplus L_*$ and $[HL_*] = L_* = [L_*H]$. Thus $L = H + L^{(1)}$ and $A = (H + L^{(1)})/L^{(1)}$, so every element in A is split on B over F . By Theorem 3.2.9 $B = \Sigma \oplus B_a(A)$ where each $B_a(A)$ is an A-submodule. But then since

$BL^{(1)} = 0$, every A-submodule of B is an L-submodule of B , so $B = B_a(A)$ for $B_a(A) \neq 0$. Now, let $x \in A$ be any element. Then there is a $v \neq 0$ in B such that $vx = a(x)v$ and so $B_x \equiv \{v \in B | vx = a(x)v\} \neq 0$. It follows that B_x is an A-submodule of B , since A is abelian. Thus $B_x = B = B_a(A)$ for $x \in A$ and this happens only if $\dim B = 1$, since B is A-irreducible. //

Definition 3.4.1. Let V be an S-module where S is a set. An S-composition series of V is a sequence of S-submodules of V

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

such that each V_i/V_{i-1} is S-irreducible. //

If V is a finite-dimensional S-module, V has an S-composition series. Indeed, let V_m be a proper S-submodule of V of maximal dimension. Then $0 \subset V_m \subset V$ and continuing this with V_m , one arrives at an S-composition series.

Let L be a solvable subalgebra of $(\mathrm{Hom}_F B)$ such that each $x \in L$ is split on B over F where B is a finite-dimensional vector space over F . Note that each $\mathrm{ad}_L x$ is split over F by Theorem 1.7.7

Then B has an L-composition series

$$0 = B_0 \subsetneq B_1 \subsetneq \cdots \neq B_m = B \ .$$

Since each $B_i/B_{i-1} \neq 0$ is L-irreducible, by Theorem 3.4.5 $\dim B_i/B_{i-1} = 1$, $i = 1,2,\ldots,m$. Thus $\dim B_i = i$, $i = 1,2,\ldots,m$. Therefore, one can choose a basis for B relative to which the matrix of each $x \in L$ is in an upper triangular form :

$$\begin{bmatrix} \alpha_1 & & & * \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_m \end{bmatrix} . \qquad (1.24)$$

We refer to this situation as the simultaneous triangulability of L. We state this as

Theorem 3.4.6. (Lie). Let B be a finite-dimensional vector space over a field F of characteristic 0. Suppose that L is a "split" solvable subalgebra of $(\mathrm{Hom}_F B)^-$. Then L is simultaneously triangulable. //

Corollary 3.4.7. Let L be a "split" nilpotent Lie algebra and let B be a finite-dimensional L-module. Then each weight of L in B is linear and vanishes on $L^{(1)}$. In particular, each root of a split CSA of a Lie algebra is linear. (char $F = 0$).

Proof. By Theorem 3.2.9 $B = \Sigma_a \oplus B_a(L)$ where each $B_a(L)$ is an L-submodule of B and the only eigenvalue of $x|_{B_a(L)}$ is $a(x)$ for $x \in L$. Thus by Theorem 3.4.6 the matrix (1.24) of $x|_{B_a(L)}$ has the form

$$\begin{bmatrix} a(x) & & * \\ & \ddots & \\ 0 & & a(x) \end{bmatrix}$$

for $x \in L$. Thus there is an element $v \neq 0$ in $B_a(L)$ such that $vx = a(x)v$ for all $x \in L$. Hence

$v(x + y) = a(x + y)v = vx + vy = [a(x) + a(y)]\, v$,

$v(\alpha x) = a(\alpha x)v = \alpha(a(x))v$ and $v[xy] = (vx)y - (vy)x$

$= [a(x)a(y) - a(y)a(x)]\, v = a([xy])v$, so $a([xy]) = 0$,

$\alpha \in F$, $x,y \in L$. //

Corollary 3.4.8. Let L be a split solvable subalgebra of $(\mathrm{Hom}_F\, B)^-$. Then $L^{(1)}$ is nilpotent.

Proof. By Theorem 3.4.6 $L^{(1)}$ is simultaneously nil triangulable. //

Corollary 3.4.9. If L is solvable then $L^{(1)}$ is nilpotent.

Proof. We may assume that F is algebraically closed, since solvability is preserved under scalar extension. Since $\mathrm{ad}\, L$ is a solvable subalgebra of $(\mathrm{Hom}\, L)^{-}$, $[\mathrm{ad}\, L, \mathrm{ad}\, L] = \mathrm{ad}\, L^{(1)}$ is nilpotent and so is $L^{(1)}$ by Lemma 3.2.1. //

Corollary 3.4.10. Let L be the same as in Corollary 3.4.8. Then L has a common eigenvector $v \in B$, that is, $vx = \lambda(x)v$, $x \in L$, $\lambda(x) \in F$.

Proof. This is clear from Theorem 3.4.6. //

3.5. Conjugacy of Cartan subalgebras

In this section we assume that F is of characteristic 0. If H is a split CSA of L then by Theorem 3.3.6 L is expressed as a direct sum of root spaces for H in L;

$$L = H \oplus L_\alpha \oplus L_\beta \oplus \cdots \oplus L_\rho \qquad (1.25)$$

where $L_\alpha = L_\alpha(H)$ and $\alpha, \beta, \ldots, \rho$ are the nonzero roots of H in L.

We refer to (1.25) as the Cartan decomposition of L relative to H. Note also that the roots of H in L are linear (Corollary 3.4.7) and so polynomial functions on H. Let

$$H^0 = \{h \in H \mid \alpha(h)\beta(h) \ldots \rho(h) \neq 0\} .$$

Then H^0 is a nonempty Zariski open set in H. Letting $L_* = L_\alpha + \ldots + L_\rho$, L_* is ad h-stable for $h \in H$ and if $h \in H^0$ then all the eigenvalues of $\mathrm{ad}\, h|_{L_*}$ are nonzero and so $\mathrm{ad}\, h|_{L_*}$ is nonsingular. Therefore we have $H = L_0(\mathrm{ad}\, h)$ for $h \in H^0$. Conversely, if $H = L_0(\mathrm{ad}\, h)$ for $h \in H$ then $\mathrm{ad}\, h|_{L_*}$ is nonsingular since $L = H \oplus L_*$, and hence $h \in H^0$. Thus we have

Lemma 3.5.1. $H^0 = \{h \in H \mid \mathrm{ad}\, h|_{L_*}$ is nonsingular$\}$

$= \{h \in H \mid H = L_0(\mathrm{ad}\, h)\}$. //

If $\mathrm{ad}\, x$ is nilpotent for $x \in L$ then $\exp \mathrm{ad}\, x$ is an automorphism of L. Denote by $\mathrm{Aut}_e(L)$ the group of automorphisms of L generated by $\exp \mathrm{ad}\, x$ for $x \in L$ where $\mathrm{ad}\, x$ is nilpotent. Each element in $\mathrm{Aut}_e(L)$ is called an invariant automorphism of L. For a nonzero root α of H, let $x \in L_\alpha$. Since $L_\beta(\mathrm{ad}\, x)^n \in L_{\beta+n\alpha}$ and there are only finitely many roots of H, $L_\beta(\mathrm{ad}\, x)^n = 0$ for some $n > 0$ and any root β of H.

Thus $\exp \text{ad } L_\alpha \subset \text{Aut}_e(L)$ for nonzero roots α of H.

Theorem 3.5.2. Let L be a finite-dimensional Lie algebra over an algebraically closed field F of characteristic 0. Let H and H_1 be any CSA of L. Then there exists an invariant automorphism η of L such that $H = H_1\eta$.

Proof. Let $L = H + L_\alpha + L_\beta + \ldots + L_\rho$ be the Cartan decomposition of L relative to H and let $L_* = L_\alpha + L_\beta + \ldots + L_\rho$. Let $\{h_1, \ldots, h_\ell, e_{\ell+1}, \ldots, e_n\}$ be a basis for L where $\{h_1, \ldots, h_\ell\}$ is a basis of H and the e_j consist of basis of $L_\alpha, \ldots, L_\rho$. For indeterminates $X_1, X_2, \ldots, X_n$, let

$$(\Sigma_i X_i h_i)\exp(\text{ad } X_{\ell+1}e_{\ell+1}) \ldots \exp(\text{ad } X_n e_n)$$

$$= \Sigma_i p_i(X_1, \ldots, X_n)h_i + \Sigma_j p_j(X_i, \ldots, X_n)e_j .$$

Then the mapping $p : L \to L$ defined by

$$p(\Sigma \alpha_i h_i + \Sigma \alpha_j e_j) = \Sigma p_i(\alpha_1, \ldots, \alpha_n)h_i + \Sigma p_j(\alpha_1, \ldots, \alpha_n)e_j$$

is a polynomial mapping. For any elements $h_0 \in H^0 = \{x \in H \mid \alpha(x) \ldots \rho(x) \neq 0\}$ and $e \in L_*$ with $e = \Sigma \alpha_j e_j$ we have

$$p(h_0 + t(h + e)) = (h_0 + th)\exp(\text{ad } t\alpha_{\ell+1}e_{\ell+1}) \ldots \exp(\text{ad } t\alpha_n e_n)$$

$$\equiv h_0 + th + t[h_0 e] \qquad (\text{mod } t^2)$$

where $h \in H$ and t is an indeterminate. But then the Taylor formula applied to $p(h_0 + t(h + e))$ implies that

$(d_{h_0} p)(h + e) = h + [h_0 e]$. Since $\operatorname{ad} h_0|_{L_*}$ is

nonsingular by Lemma 3.5.1, this shows that $d_{h_0} p$ is

nonsingular on L and so is surjective. Since $H^0 + L_*$ is a nonempty open set in L, by Theorem 2.3.2 $p(H^0 + L_*) \supset U$ for some open set $U \neq \phi$ in L, so $U \subset H^0 \mathrm{Aut}_e(L)$. The same argument applied to H_1 assures that $U_1 \subset H_1^0 \mathrm{Aut}_e(L)$ for some nonempty open set U_1. Since $U \cap U_1 \neq \phi$, we see that $h = k\eta$ for some $h \in H^0$, $k \in H_1^0$ and $\eta \in \mathrm{Aut}_e(L)$. But by Lemma 3.5.1, $H = L_0(\operatorname{ad} h)$ $= L_0(\operatorname{ad} (k\eta)) = L_0(\operatorname{ad} k)\eta = H_1\eta$. //

Theorem 3.5.2 in particular implies that all CSA of L have the same dimension. Thus by Corollary 3.3.3 and Lemma 3.5.1 we have

<u>Corollary 3.5.3</u>. $H \cap L_{reg} = H^0$. //

3.6. Cartan's criteria

We discuss important criteria for solvability and semisimplicity of L in terms of a trace form. These criteria give the basis for the structure of semisimple Lie algebras. We assume that all L-modules are finite-dimensional over a field F of characteristic 0 and f denotes the representation of L afforded by B.

Theorem 3.6.1. Let x,y,h be elements in L such that $[xy] = h$, $x \in L_{-\alpha}(\mathrm{ad}\ h)$ and $y \in L_{\alpha}(\mathrm{ad}\ h)$ where $\alpha \in F$. Suppose that $B \neq B_0(h)$. Then $\mathrm{Tr}\, f(h)^2 \neq 0$ and $\mathrm{Tr}\, f(h)^2 = r\,\alpha^2$ for a positive rational number r. In particular, $\alpha \neq 0$.

Proof. As in Theorem 3.4.1. we assume that F is algebraically closed. Let $B = \Sigma \oplus B_\beta(h)$. Then β is the only eigenvalue of $f(h)|_{B_\beta(h)}$ if $B_\beta(h) \neq 0$.

Letting $W_\beta = \sum_{i \in Z} \oplus B_{\beta+i\alpha}(h)$, by Theorem 3.2.10 W_β is stable under x and y. Thus

$$0 = \mathrm{Tr}[f(x), f(y)]\big|_{W_\beta} = \mathrm{Tr}\, f(h)|_{W_\beta} = \Sigma d_i(\beta + i\alpha)$$

where $d_i = \dim B_{\beta+i\alpha}$. Hence $\beta = -\dfrac{\Sigma\, i d_i}{\Sigma\, d_i}\alpha$

and $\beta = r_\beta \alpha$ for some rational r_β. Since $B \neq B_0(h)$, $\dim B_\beta(h) \neq 0$ for some $\beta \neq 0$ and, for such β, $r_\beta \neq 0$ and so $\alpha \neq 0$. Since β^2 is the only eigenvalue of $f(h)^2|_{B_\beta(h)}$, we now have $\mathrm{Tr}\, f(h)^2 = \Sigma_\beta\, d_\beta \beta^2 = \Sigma d_\beta (r_\beta \alpha)^2$ where $d_\beta = \dim B_\beta(h)$. //

Let K be an extension field of F and L_K be the scalar extension of L to K. Then we note that $L_K^{(i)} = (L_K)^{(i)}$ and so L is solvable if and only if L_K is. Note also that f has a unique extension f_K to L_K, which is afforded by B_K, and that $\ker f_K = (\ker f)_K$.

Theorem 3.6.2. (Cartan's criterion for solvability). Let $(x,y) = \mathrm{Tr}\, f(x)f(y)$. Then L is solvable if and only if $\ker f$ is solvable and $(x,y) = 0$ for all $x,y \in L^{(1)}$.

Proof. By the foregoing remark, we can assume that F is algebraically closed. If L is solvable, $\ker f$ and $f(L)$ are solvable. Since $f(L)$ is a solvable subalgebra of $(\mathrm{Hom}_F B)^-$, by Lie's Theorem 3.4.6 $f(L)$ is simultaneously triangulable. Thus, relative to some basis for B, the elements of $f(L)^{(1)} = f(L^{(1)})$ are upper triangular nilpotent matrices, so $(x,y) = \mathrm{Tr}\, f(x)f(y) = 0$ for $x,y \in L^{(1)}$.

Conversely, suppose that $\ker f$ is solvable and $(x,y) = 0$ for $x,y \in L^{(1)}$. We proceed by induction on $\dim L$. If $\dim L = 1$, the assertion is trivial. It suffices to show that $f(L)$ is solvable since $f(L) \cong L/_{\ker f}$ and $\ker f$ is solvable (Lemma 1.5.1). Thus, we assume that L is a subalgebra of $(\mathrm{Hom}_F\ B)^-$ and $f(x) = x$ for $x \in L$. Let H be a CSA of L and let $L = H + \Sigma_{\alpha \neq 0} L_\alpha$ be the Cartan decomposition of L relative to H. Let $x \in L_{-\alpha}$, $y \in L_\alpha$, $h = [xy]$. Then $[xy] = h \in H$ and since $h \in L^{(1)}$, $(h,h) = \mathrm{Tr}\ h^2 = 0$. Thus by Theorem 3.6.1 $B = B_0(h)$ and h is nilpotent. So, $[L_{-\alpha}L_\alpha] \subset N \equiv \{x \in H | x \text{ is nilpotent}\}$ for all roots α. Now, by Corollary 3.4.4, N is an ideal of H. Letting $J = N + \Sigma_{\alpha \neq 0} L_\alpha$, J is an ideal of L since $[HJ] \subset J$ and $[L_{-\alpha}L_\alpha] \subset N$. If $J \neq L$, by induction J is solvable and $L/J = (H + J)/J \cong H/_{H \cap J} = H/_N$ is solvable, so is L. If $J = L$ then $H = N$ and x is nilpotent for $x \in H$, so $\mathrm{ad}\ x$ is nilpotent by Theorem 1.3.3. Thus $L = L_0(\mathrm{ad}\ H) = H$ since H is a CSA. //

Notice that, in Theorem 3.6.2, $(x,y) = 0$ for $x,y \in L^{(1)}$ if and only if $(x,x) = 0$ for $x \in L^{(1)}$.

Theorem 3.6.3 (Cartan's criterion for semisimplicity). If L is semisimple then the trace form of any $1-1$ representation of L is nondegenerate.

L is semisimple if and only if the Killing form is nondegenerate.

Proof. If f is a $1-1$ representation of L, let $(x,y) = \mathrm{Tr}\, f(x)f(y)$. Thus $\ker f = 0$, and $L^{\perp} = \{x \in L \mid (x,y) = 0 \text{ for } y \in L\}$ is an ideal of L. Since $(x,y) = 0$ for $x,y \in (L^{\perp})^{(1)}$, by Theorem 3.6.2. $L^{\perp}$ is solvable. Hence if L is semisimple, $(,)$ is nondegenerate. Note that if L is semisimple, ad representation of L is $1-1$. The converse follows from Lemma 1.6.3. //

Corollary 3.6.4. Let J be an ideal of L. Then L/J is semisimple if and only if $\mathrm{Rad}\, L \subset J$.

Proof. If L/J is semisimple, then $(J + \mathrm{Rad}\, L)/J$ is a solvable ideal of L/J and so $\mathrm{Rad}\, L \subset J$. Suppose that $\mathrm{Rad}\, L \subset J$. Let $\bar{L} = L/\mathrm{Rad}\, L$ and $\bar{J} = J/\mathrm{Rad}\, L$. Then $L/J \cong \bar{L}/\bar{J}$ and since $\bar{L}$ is semisimple by Lemma 1.5.1, the Killing form on $\bar{L}$ is nondegenerate by Theorem 3.6.3. Lemma 1.6.3 then assures that $\bar{L}$ is a direct sum of simple ideals $\bar{L}_i$ of $\bar{L}$. On the other hand, by Corollary 1.5.4 $\bar{J}$ is a sum of some $\bar{L}_j$'s and hence $\bar{L}/\bar{J}$ is a direct sum of simple ideals. In fact, $\bar{L}/\bar{J}$ is a direct sum of simple ideals $(\bar{L}_i + \bar{J})/\bar{J}$ where $\bar{L}_i \not\subset \bar{J}$. Thus L/J is semisimple. //

Corollary 3.6.5. Any homomorphic image of a semisimple Lie algebra is semisimple.

Proof. Let $f : L \to f(L)$ be a homomorphism of a semisimple Lie algebra. Then $f(L) \cong L/_{\ker f}$ is semisimple by Corollary 3.6.4. //

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of L. Then we note that the Killing form $K(\ ,\ )$ of L is nondegenerate if and only if the matrix $(K(x_i, x_j))$ is nonsingular. For an extension field K of F, since $K(\ ,\ )$ is regarded as the Killing form on L_K and $\{x_1, \ldots, x_n\}$ is a basis for L_K, by Theorem 3.6.3 we have

Corollary 3.6.6. If L is a semisimple Lie algebra over F (of char 0), then any scalar extension L_K of L is semisimple. //

3.7. The theorems of Weyl and Levi

In this section we prove two important theorems on the existence of complements. One is the Weyl's theorem that any finite-dimensional L-module for a semisimple Lie algebra L is L-completely reducible. Another one is the Levi's theorem that if L is a finite-dimensional Lie algebra then L contains a semisimple subalgebra S such that $L = S \oplus \text{Rad } L$. Here we assume that the characteristic is 0. As before, B denotes a finite-dimensional L-module and f is the representation afforded by B.

Suppose also that L has a nondegenerate invariant form $(\, , \,)$, and let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be dual bases for L, that is, $(e_i, f_j) = \delta_{ij}$.

<u>Definition 3.7.1</u>. The Casimir operator for L with respect to $(\, , \,)$, $\{e_i\}$ and $\{f_j\}$ is the element $T = \Sigma f(e_i)f(f_i)$ in $\text{Hom}_F B$. //

<u>Lemma 3.7.1</u>. The Casimir operator T commutes with every element in $f(L)$.

<u>Proof</u>. For $x \in L$, let $[xe_i] = \Sigma_j \alpha_{ij} e_j$ and $[xf_i] = \Sigma_j \beta_{ij} f_j$. Then $\alpha_{ik} = \Sigma_j \alpha_{ij}(e_j, f_k) = ([xe_i], f_k)$ $= -([e_i x], f_k) = -(e_i, [xf_k]) = -\Sigma_j \beta_{kj}(e_i, f_j)$ $= -\beta_{ki}$. Using this, we compute

$$[f(x), T] = \Sigma_i [f(x), f(e_i)f(f_i)]$$

$$= \Sigma_i f(e_i)[f(x),f(f_i)] + \Sigma_i [f(x),f(e_i)] f(f_i)$$

$$= \Sigma_{i,j}\alpha_{ij} f(e_j)f(f_i) + \Sigma_{i,j}\beta_{ij} f(e_i)f(f_j)$$

$$= 0 . \quad //$$

Assume now that L is a nonzero semisimple Lie algebra and the representation f is faithful, that is, $\ker f = 0$. Then the trace form $(x,y) = \mathrm{Tr}\, f(x)f(y)$ is nondegenerate by Theorem 3.6.3. Let $\{e_i\}$ and $\{f_j\}$ be dual basis for L relative to $(\, , \,)$ and let T be the corresponding Casimir operator $T = \Sigma f(e_i)f(f_i)$. Then $\mathrm{Tr}\, T = \Sigma\, \mathrm{Tr}\, f(e_i)f(f_i) = \Sigma(e_i,f_i) = \dim L = n \neq 0$. We use this to prove

<u>Fitting's Lemma 3.7.2. (for semisimple Lie algebras)</u>. Let L be semisimple and let B be a finite-dimensional L-module. Then B is a unique direct sum $B = B_0 \oplus B_*$ where B_0 and B_* are L-submodules of B such that $B_0L = 0$ and $B_*L = B_*$.

<u>Proof</u>. We prove by induction on $\dim B$. If $B = 0$, it is trivial, while if $L = 0$, we set $B_0 = B$ and $B_* = 0$. Assume that $B \neq 0$ and $L \neq 0$. We may further assume that the representation f afforded by B is faithful. Indeed, letting $J = \ker f$, we make B an L/J-module via $v(x + J) = vx,\ v \in B,\ x \in L$.

The decomposition of B for L/J gives the desired conditions for L. Note also that L/J is semisimple by Corollary 3.6.5. For brevity, we set $f(x) = x$, $x \varepsilon L$. Let T be the Casimir operator for L as above. Let $B = B^0 \oplus B^*$ be the Fitting decomposition of B relative to T. Thus $B^0 = B_0(T)$ and $B^* = B_*(T)$. Since $\mathrm{Tr}\, T \neq 0$, T is not nilpotent on B and hence $\dim B_0 < \dim B$. Since $[T,L] = 0$ by Lemma 3.7.1, B^0 and B^* are L-submodules, so by induction we have $B^0 = B_0 \oplus W$ with $B_0 L = 0$ and $WL = W$. If we let $B_* = W + B^*$, then $B = B_0 \oplus B_*$ and $B_* L = B_*$ since $B^*T = B^*$ and $B^*T \subset (B^*L)L \subset B^*L$.

The uniqueness of B_* follows from $BL = B_*$ while that of B_0 amounts to $\{v \varepsilon B_* | vL = 0\} = 0$. Noting $\dim W < \dim B$, by induction we have $\{u \varepsilon W | uL = 0\} = 0$ since $WL = W$, so $W_0 = 0$. Since $B_* = W \oplus B^*$ and T is bijective on B^*, it follows that $\{v \varepsilon B_* | vL = 0\} = 0$. //

<u>Theorem 3.7.3.</u> Let B be an L-module and let $\mathrm{Rad}\, L$ be abelian. Suppose that U is a proper L-submodule of B such that $U\, \mathrm{Rad}\, L = 0$ and $B\, \mathrm{Rad}\, L \subset U$. Then there exists a projection $\pi : B \to U$ such that $L = C_\pi + \mathrm{Rad}\, L$ and $C_\pi \cap \mathrm{Rad}\, L = (\ker f) \cap \mathrm{Rad}\, L$, where $C_\pi = \{x \varepsilon L | [\pi, f(x)] = 0\}$ and f is the representation afforded by B.

Proof. Let $A = \text{Rad } L$ and $N = \{g \in \text{Hom}_F (B,U) | g|_U = k.\text{id}_U\}$. We make N an L-module with $gx = gf(x) - f(x)g = [g,f(x)]$, $g \in N$, $x \in L$. It suffices to find a $\pi \in N$ such that $\pi|_U = \text{id}_U$ and $\pi L = \pi A$. Indeed, let $C_\pi = \{x \in L | \pi x = [\pi, f(x)] = 0\}$. If $x \in L$, $\pi x = \pi y$ for some $y \in A$, so $x - y \in C_\pi$ and $L = C_\pi + A$. To see $C_\pi \cap A = (\ker f) \cap A$, cleary $(\ker f) \cap A \subseteq C_\pi \cap A$ and if $x \in C_\pi \cap A$ then, for $v \in B$, we have $vx = (v\pi)x + vx - (v\pi)x = (v\pi)x + vx - (v\pi)f(x) = (v\pi)x + vx - (vx)\pi = (v\pi)x + vx(1 - \pi) \in Ux + U(1 - \pi) = 0$. To find such a π, let $M = f(A)$. Then M is an L-submodule of N with $MA = 0$ $(=[M,f(A)])$. Make $\overline{N} \equiv N/M$ an $\overline{L}$-module via $(g + M)(x + A) = gx + M$, $g \in N$, $x \in L$, where $\overline{L} = L/A$. By Fitting's Lemma 3.7.2, $\overline{N} = \overline{N}_0 \oplus \overline{N}_*$ where $\overline{N}_0\overline{N} = 0$ and $\overline{N}_*\overline{L} = \overline{N}_*$. Letting $N_0 = \{g_0 \in N | g_0 L \subseteq M\}$, it follows that $N = N_0 + NL + M$. Since $NL = [N, f(L)]$ and each element in N is a scalar on U, M and NL are contained in $M' = \{g \in N \mid g|_U = 0\}$. Thus $N = N_0 + M'$ and since $N \neq M'$, $N_0 \neq 0$. One chooses a $\pi \neq 0$ in N_0. Then $\pi|_U = c\ \text{id}_U$ and we may assume $c = 1$, so π is a projection from B onto U. By choice, $\pi L \subset M$. For $x \in A$, $\pi x = \pi f(x) - f(x)\pi = -f(x)$ since $B f(x) \subset U$ and $\pi|_U = \text{id}_U$. Hence $M \subset \pi A$ and $M = \pi A$ since $\pi A \subset \pi L \subset M$. Thus $\pi A = \pi L$, as desired. //

Corollary 3.7.4. Let L be semisimple and let B be a finite-dimensional L-module. Then, for any proper L-submodule U of B, there is a projection π from B onto U such that $[\pi, f(L)] = 0$.

Proof. If Rad $L = 0$ then $C_\pi = L$ in Theorem 3.7.3. //

Theorem 3.7.5. (Weyl). Let L be a finite-dimensional semisimple Lie algebra over a field of characteristic 0. Then any finite-dimensional L-module B is L-completely reducible.

Proof. Let U be any proper L-submodule. By Corollary 3.7.4 there exists a projection π from B onto U such that $L = C_\pi$ or $[\pi, f(L)] = 0$. Letting $U' = B(1 - \pi)$, U' is an L-complement of U in B. //

Theorem 3.7.6. (Levi). Let L be a finite-dimensional Lie algebra over a field of charactersitic 0. Then there exists a semisimple subalgebra S of L such that $L = S \oplus \text{Rad } L$.

Proof. Case 1 : Rad L is abelian. Let $A = \text{Rad } L$ and let C be the center of L. By Theorem 3.7.3 there exists a projection π from L onto A such that $L = C_\pi + A$ and $C_\pi \cap A = A \cap \ker \text{ad} = C$. Note that C_π/C is semisimple, since $L/A = (C_\pi + A)/A \cong C_\pi/C_\pi \cap A$.

Regard C_π as a C_π/C-module via $x(y + C) = [xy]$, $x \in C_\pi$, $y + C \in C_\pi/C$. In view of Theorem 3.7.5, C_π has a C_π/C-complement S of C. It follows that S is a semisimple subalgebra of L, since $C_\pi = S \oplus C$ and $C_\pi/C \cong S$. Also, $L = C_\pi + A = S \oplus C + A = S \oplus A$, since $A \cap S$ is a solvable ideal of S.

Case 2 : $A = \operatorname{Rad} L$. Show by induction on $\dim L$. If $L = 0$ or $A = 0$, there is nothing to prove. Thus, we may assume that $A^{(1)} \neq 0$. Noting that $\operatorname{Rad}(L/A^{(1)}) = A/A^{(1)}$ is abelian, by Case 1, we have $L/A^{(1)} = B/A^{(1)} \oplus A/A^{(1)}$ where B is a subalgebra containing $A^{(1)}$ and $B/A^{(1)}$ is semisimple. By Corollary 3.6.4 $\operatorname{Rad} B = A^{(1)}$ and by induction $B = S \oplus A^{(1)}$ for some semisimple subalgebra S. But then $L = B + A + A^{(1)} = B + A = S \oplus A$. //

The decomposition $L = S \oplus \operatorname{Rad} L$ in Theorem 3.7.6 is called a Levi decomposition for L and the semisimple subalgebra S is called a Levi factor of L. It is known that all Levi factors of L are conjugate ; that is, if S_1, S_2 are Levi factors, there exists an invariant automorphism η of L such that $S_1 = S_2\eta$ (Theorem of Mal̦cev-Harish-Chandra).

Corollary 3.7.7. Let $\operatorname{Rad} L$ be the center Z of L. Let B be an L-module where each element in Z is split on B over F and is semisimple. Then B is L-completely reducible.

Proof. By Theorem 4.2.9, decompose $B = \Sigma_a \oplus B_a(Z)$ as the direct sum of weight spaces relative to Z. Since each element in Z is semisimple, each $x \in Z$ is the scalar $a(x)$ on B_a. Let $L = S \oplus Z$ be a Levi decomposition of L. Since $[LZ] = 0$, each $B_a(Z)$ is L-stable. Thus, by Weyl's Theorem $B_a(Z)$ is S-completely reducible. Since each element in Z is a scalar on $B_a(Z)$, every S-irreducible submodule of $B_a(Z)$ is L-stable and so is an L-irreducible submodule of $B_a(Z)$. The result then follows from Theorem 1.2.3. //

Schur's Lemma 3.7.8. Let V be a finite-dimensional vector space over an algebraically closed field F and let $f : L \to (\mathrm{Hom}_F V)^-$ be an irreducible representation of L. If P is an element in $\mathrm{Hom}_F V$ such that $[P, f(L)] = 0$ then P is a scalar on V.

Proof. Let $v \neq 0$ be an eigenvector of P with eigenvalue $\lambda \in F$. Then $(vx)P = (vP)x = \lambda(vx)$, $x \in L$. Let V_0 be the subspace of V spanned by v and the elements obtained by repeated application of the elements in L to v. Then V_0 is L-stable and $V_0 = V$ since V is L-irreducible. Since $[P, f(L)] = 0$, P acts on V_0 as the scalar λ. //

Therefore, if f is an irreducible representation of a semisimple Lie algebra L then, by Lemma 3.7.1, the Casimir operator of L is a scalar.

<u>Theorem 3.7.9. (Zassenhaus)</u>. If the Killing form of L is nondegenerate (in arbitrary characteristic), then $\operatorname{Der} L = \operatorname{ad} L$.

<u>Proof</u>. Let $D \in \operatorname{Der} L$. Then $x \to \operatorname{Tr}(\operatorname{ad} x)D$ is a linear mapping of L into F . Since $K(,)$ is nondegenerate, there exists an element $d \in L$ such that $K(d,x) = \operatorname{Tr}(\operatorname{ad} x)D$ for all $x \in L$. Let $E = D - \operatorname{ad} d$. Then $\operatorname{Tr}(\operatorname{ad} x)E = \operatorname{Tr}(\operatorname{ad} x)D - K(d,x) = 0$ and so $\operatorname{Tr}(\operatorname{ad} x)E = \operatorname{Tr} E(\operatorname{ad} x) = 0$ for $x \in L$. We now compute $K(xE,y) = \operatorname{Tr}(\operatorname{ad} xE)\operatorname{ad} y = \operatorname{Tr}[\operatorname{ad} x,E]\operatorname{ad} y$ (since $E \in \operatorname{Der} L) = \operatorname{Tr}\{(\operatorname{ad} x)E \operatorname{ad} y - E \operatorname{ad} x \operatorname{ad} y\}$ $= \operatorname{Tr}(E \operatorname{ad} y \operatorname{ad} x - E \operatorname{ad} x \operatorname{ad} y) = -\operatorname{Tr} E \operatorname{ad}[xy] = 0$ for $x,y \in L$. Thus $xE = 0$ for $x \in L$ and $E = 0$ or $D = \operatorname{ad} d \in \operatorname{ad} L$. //

A derivation of L in $\operatorname{ad} L$ is called <u>inner</u>. In view of Theorem 3.6.3 we have

<u>Corollary 3.7.10</u>. If L is semisimple and of characteristic 0 then all derivations of L are inner. //

Let V be a finite-dimensional vector space over an algebraically closed field F of characteristic 0. To ascertain the semisimplicity of L, there is often a much simpler method than Theorem 3.6.3. A Lie algebra $L \neq 0$ is called <u>reductive</u> if $\text{Rad } L = Z(L)$. Abelian and semisimple Lie algebras are obvious examples for this. Also, $(\text{Hom } V)^-$ is reductive.

<u>Theorem 3.7.11</u>. (1) Let L be reductive. Then $L = [LL] \oplus Z(L)$ and $[LL]$ is semisimple or 0.

(2) Let $L \subset (\text{Hom } V)^-$ be a Lie algebra acting irreducibly on V. Then L is reductive with $\dim Z(L) \leqq 1$. If, in addition, $\text{Tr } x = 0$ for $x \in L$, then L is semisimple.

<u>Proof</u>. (1) Suppose that L is reductive but not abelian, so $L' = L/Z(L)$ is semisimple. Since $\text{ad } L \cong L'$, by Weyl's Theorem $\text{ad } L$ acts completely reducibly on L and hence $L = M \oplus Z(L)$, M an ideal of L. In particular, $[LL] = [MM] \subset M$. But since $L' = [L'L']$, $[LL]\pi = L' \cong M$ where π is the natural map : $L \to L'$. Hence $M = [LL]$ and $L = [LL] \oplus Z(L)$.

(2) Let $A = \text{Rad } L$. By Corollary 3.4.10 there is a vector $v \neq 0$ in V such that $vz = \lambda(z)v$, $z \in A$, $\lambda(z) \in F$. If $x \in L$ then $[xz] \in A$, $z \in A$

and so (*) $(vx)z = v[xz] + (vz)x = \lambda(z)vx + \lambda([xz])v$. Since L acts irreducibly, V is spanned by v and the elements obtained by repeated application of the elements of L to v . It therefore follows from (*) that the matrices of all $z \in A$ relative to a suitable basis of V are triangular with $\lambda(z)$ the only diagonal entry. Since $[AL] \subset A$ have trace 0 , this implies that $\lambda([AL]) = 0$. Thus by (*) we conclude that $z \in A$ acts diagonally on V as the scalar $\lambda(z)$. In particular, $A = Z(L)$, L is reductive and $\dim A \leq 1$. If $\mathrm{Tr}\, x = 0$ for $x \in L$ then L contains no nonzero scalars and hence $A = 0$, so L is semisimple. //

We remark that Theorem 3.7.11 (1) is also a consequence of the Levi's Theorem (why?).

For example, let $s\ell(V) = \{x \in \mathrm{Hom}\, V \mid \mathrm{Tr}\, x = 0\}$. Then $\mathrm{Hom}\, V = s\ell(V) \oplus F1$. Since $\mathrm{Hom}\, V$ acts irreducibly on V (why?) , so does $s\ell(V)$ on V . Thus by Theorem 3.7.11 (2) $s\ell(V)$ is semisimple. In fact, it can be seen that $s\ell(V)$ is simple.

4. SPLIT SEMISIMPLE LIE ALGEBRAS OF CHARACTERISTIC 0

4.1. Introduction

In this chapter we discuss the classification of simple Lie algebras over an algebraically closed field F of characteristic 0. This classification was first given by Killing and E.Cartan, and its simplification was later made by Weyl, Coxeter, and Dynkin. The restriction on the base field F can be relaxed by assuming that a semisimple Lie algebra L has a split CSA H. This is due to the fact that the algebraically closedness in classical treatments is used only to ensure the existence of the Cartan decomposition $L = H \oplus L_\alpha \oplus \ldots \oplus L_\delta$ relative to H. The classification is carried out by obtaining more stringent information on the Cartan decomposition and then by associating a simple system of roots with each split semisimple Lie algebra. The classification is then reduced to that of simple systems of roots. We finally outline the existence of a simple Lie algebra corresponding to each type of simple system of roots.

Throughout this chapter we assume that L denotes a split semisimple Lie algebra over a field F of characteristic 0 and all L-modules and representations of L are finite-dimensional.

4.2. Maximal tori

We first describe the CSA of L as maximal tori in L. Let x be an element in L such that $\operatorname{ad} x$ is split over F. Since the Jordan components $(\operatorname{ad} x)_s$ and $(\operatorname{ad} x)_n$ are derivations of L by Corollary 1.4.10, in view of Corollary 3.7.10, there exist elements $x_s, x_n \in L$ such that $(\operatorname{ad} x)_s = \operatorname{ad} x_s$ and $(\operatorname{ad} x)_n = \operatorname{ad} x_n$. Since L has center 0, we have

$$x = x_s + x_n \tag{1.26}$$

and clearly $[x_s x_n] = 0$. The decomposition (1.26) is unique subject to the fact that $\operatorname{ad} x_s$ is semisimple, $\operatorname{ad} x_n$ is nilpotent and $[x_s x_n] = 0$, or equivalently, $\operatorname{ad} x_s = (\operatorname{ad} x)_s$ and $\operatorname{ad} x_n = (\operatorname{ad} x)_n$ in view of Lemma 1.3.2. The decomposition (1.26) is referred to as the <u>abstract Jordan decomposition</u> of x.

If $x = x_s$, x is called <u>semisimple</u> while x is <u>nilpotent</u> if $x = x_n$.

<u>Lemma 4.2.0</u>. Let V be a finite-dimensional vector space over F (of arbitrary char.). Let $x \in \text{Hom } V$ split over F . Then there exist polynomials $p(T), q(T)$ without constant terms such that $x_s = p(x)$ and $x_n = q(x)$ where $x = x_s + x_n$ is the (usual) Jordan decomposition of x . Moreover, if $U \subset W \subset V$ are subspaces of V and $Wx \subset U$ then x_s and x_n map W into U also.

<u>Proof</u>. Let $\Pi(T - a_i)^{m_i}$ be the characteristic polynomial of x , with the a_i distinct. Let $V = \Sigma \oplus V_i$ be the primary decomposition of V relative to x , so that $V_i = \ker(x - a_i)^{m_i}$ (see Section 1.3), being x-stable. By the Chinese Remainder Theorem, we find a polynomial $p(T)$ satisfying the congruences : $p(T) \equiv a_i \pmod{(T - a_i)^{m_i}}$, $p(T) \equiv 0 \pmod{T}$ (if 0 is an eigenvalue of x, the last congruence is superfluous). Set $q(T) = T - p(T)$ and $x_s = p(x)$, $x_n = q(x)$, so x, x_s, x_n all commute with each other. Hence the V_i are x_s- , x_n-stable. From $p(T) \equiv a_i \pmod{(T - a_i)^{m_i}}$, it follows that $V_i(x_s - a_i) = 0$, so $x_s|_{V_i} = a_i\text{id}$ and x_s is semisimple. Hence $x = x_s + x_n$ is the Jordan decomposition of x (Lemma 1.3.2).

By definition, $x_n = x - x_s$ is nilpotent. The last part is now obvious. //

Theorem 4.2.0. Let $L \subset (\mathrm{Hom}_F V)^-$ be a semisimple Lie algebra where V is finite-dimensional over F (of char 0). Let $x \in L$ be split over F. Then L contains the semisimple part x_s and the nilpotent part x_n of x.

Proof. Let $E = \mathrm{Hom}_F V$. Note that $\mathrm{ad}\, x = \mathrm{ad}\, x_s + \mathrm{ad}\, x_n$ is the (usual) Jordan decomposition for $\mathrm{ad}\, x$ (Theorem 1.3.3). Set $N = N_E(L)$, the normalizer of L in E. Since $L\, \mathrm{ad}\, x \subset L$, by Lemma 4.2.0 $L\, \mathrm{ad}\, x_s \subset L$ and $L\, \mathrm{ad}\, x_n \subset L$, so $x_s, x_n \in N$. Let $\mathfrak{W}$ be the family of L-submodules of V. If $W \in \mathfrak{W}$, define $L_W = \{y \in E \mid Wy \subset W \text{ and } \mathrm{Tr}\, y|_W = 0\}$. Since $L = [LL]$, clearly $L \subset L_W$ for all $W \in \mathfrak{W}$. Set $L_0 = N \cap (\bigcap_{W \in \mathfrak{W}} L_W)$. Then L_0 is a subalgebra of N and contains L as an ideal (since L is an ideal of N). Note that $x_s, x_n \in L_0$ by Lemma 4.2.0 (since $Wx \subset W$ for all $W \in \mathfrak{W}$ and $\mathrm{Tr}\, x_s|_W = \mathrm{Tr}\, (x - x_n)|_W = 0$).

We contend that $L = L_0$. Regard L_0 as an L-module via ad, so by Weyl's Theorem we have $L_0 = L \oplus M$ with $[L,M] \subset M$. But since $[L,L_0] \subset L$ (since $L_0 \subset N$), $[L,M] =$ Let W be any irreducible L-submodule of V. If $y \in M$, then $[L,y] = 0$, so Schur's Lemma implies that $y|_W$ is a scalar.

But $\mathrm{Tr}\ y|_W = 0$ as $y \in L_W$, so $y|_W = 0$. Since V is a direct sum of irreducible L-submodules of V (Weyl's Theorem), $y = 0$ for $y \in M$, so $L = L_0$ and $x_s, x_n \in L$. //

Corollary 4.2.0. Let L be the same as in Theorem 4.2.0 and let $x \in L$ be split over F. Then the usual and abstract Jordan decompositions of x coincide.

Proof. Let $x = x_s' + x_n' = x_s + x_n$ be respectively the abstract and usual Jordan decompositions of x. Then $\mathrm{ad}\ x_s' = \mathrm{ad}\ x_s$ and $\mathrm{ad}\ x_n' = \mathrm{ad}\ x_n$ (Lemma 1.3.2; uniqueness, and Theorem 1.3.3). Since $x_s, x_n \in L$ by Theorem 4.2.0, $x_s' = x_s$ and $x_n' = x_n$. //

Theorem 4.2.1. Let f be a representation of L. Let x be an element in L such that $\mathrm{ad}\ x$ is split over F. Then $f(x_s) = f(x)_s$ and $f(x_n) = f(x)_n$, that is, $f(x) = f(x_s) + f(x_n)$ is the usual Jordan decomposition of $f(x)$. Thus the usual and abstract Jordan decompositions of $f(x)$ coincide.

Proof. Since $\mathrm{ad}\ x_s = (\mathrm{ad}\ x)_s$ is semisimple on L, L is spanned by eigenvectors of $\mathrm{ad}\ x_s$. Thus $f(L)$ is spanned by eigenvectors of $\mathrm{ad}_{f(L)} f(x_s)$. Since $\mathrm{ad}\ x_n$ is nilpotent on L, so is $\mathrm{ad}_{f(L)} f(x_n)$ on $f(L)$.

From $[x_s x_n] = 0$, we have $[ad_{f(L)} f(x_s), ad_{f(L)} f(x)_n] = 0$ and thus by Lemma 1.3.2,

$$ad_{f(L)} f(x_s) = (ad_{f(L)} f(x))_s \ , \ ad_{f(L)} f(x_n) = (ad_{f(L)} f(x))_n \ .$$

Since $f(L)$ is semisimple by Corollary 3.6.5, we have that $f(x_s) = f(x)_s$ and $f(x_n) = f(x)_n$, where $f(x)_s$ and $f(x)_n$ are the abstract Jordan components of $f(x)$. It follows from Corollary 4.2.0 that these are in fact the usual Jordan components. //

Definition 4.2.1. Let T be a subalgebra of L such that $ad_L x$ is split over F for $x \varepsilon T$. Then T is called a torus of L if T is abelian and consists of semisimple elements. //

Theorem 4.2.2. A subalgebra T of L is a maximal torus of L if and only if T is a split CSA of L.

Proof. It suffices to show that any maximal torus T of L is contained in a CSA of L and a split CSA of L is a torus of L.

Suppose that T is a maximal torus of L. Let $C = C_L(T) = \{x \varepsilon L \mid [xT] = 0\}$, the centralizer of T in L. Let H be a CSA of C. By Theorem 3.3.1 $T \subset N_C(H) = H$ and so $L_0(ad\ H) \subset L_0(ad\ T)$. Noting that $C_L(T) \subset L_0(ad\ T)$ and each element in $ad\ T$ is semisimple, we see that $L_0(\text{ad } T) = C_L(T)$. Thus $L_0(\text{ad } H) \subset C_0(\text{ad } H) = H$ since H is a CSA of C . Hence $L_0(\text{ad } H) = H$ and H is a CSA of L .

Suppose that H is a split CSA of L and T is the set of semisimple elements in H . We show that T is a torus and $H = T$. Let $x \varepsilon H$. Since $\text{ad } x|_H$ is nilpotent, so is $\text{ad } x_s|_H = (\text{ad } x)_s|_H$ on H . Thus $\text{ad } x_s|_H = 0$ and $x_s \varepsilon C_L(H) \subset N_L(H) = H$. In particular,

$$T \subset C_L(H) \quad \text{or} \quad [TH] = 0 . \tag{1.27}$$

Hence $x_s \varepsilon T$ for all $x \varepsilon H$. Let $x,y \varepsilon T$. Since $[xy] = 0$ by (1.27), x and y span an abelian subalgebra A . By Theorem 3.2.9, L is decomposed into $L = \Sigma \oplus L_a(\text{ad } A)$ as an A-module via ad . Since ad x and ad y are semisimple, they are scalars on $L_{a(x)}(\text{ad } x)$ and $L_{a(y)}(\text{ad } y)$. Noting that $L_a(\text{ad } A) \subset L_{a(x)}(\text{ad } x)$, we find that the matrices of ad x and ad y are diagonal. Thus any linear combination of x and y is semisimple and T is an abelian subalgebra. Let $N = \{x \varepsilon H \mid \text{ad}_L x \text{ is nilpotent}\} = \{x \varepsilon H \mid x \text{ is nilpotent}\}$. By Corollary 3.4.4 N is an ideal of H . If $x \varepsilon H$, $x_n = x - x_s \varepsilon H$ and so $H = T \oplus N$. It remains to show $N = 0$. Since $\text{ad}_L N$ is simultaneously nil triangulable by Engel's Theorem, $K(N,N) = 0$. Noting that $[\text{ad } x , \text{ad } y] = 0$ for $x \varepsilon N$ and $y \varepsilon T$, we see that

$K(N,T) = 0$ since ad x is nilpotent. But since $K(\, , \,)$ is nondegenerate on H by Theorem 3.3.6, $N = 0$ and $H = T$. //

Corollary 4.2.3. Let H be a split CSA of L and $L = H + \Sigma_{\alpha \neq 0} L_\alpha$ be the Cartan decomposition. Then

(1) $L_\alpha = \{x \in L \mid [xh] = \alpha(h)x \, , \, h \in H\}$ for each root α of H.

(2) $H = C_L(H)$.

Proof. (1) By Theorem 4.2.2 ad h $(h \in H)$ is semisimple, so $(\text{ad } h)|_{L_\alpha}$ is the scalar $\alpha(h)$.

(2) Clearly $H \subset C_L(H)$ since H is abelian. Let $x \in C_L(H)$ and let $x = \Sigma_\alpha x_\alpha$, $x_\alpha \in L_\alpha$. If $\alpha \neq 0$, choose an $h \in H$ with $\alpha(h) \neq 0$. Thus $[xh] = 0$ implies $x_\alpha = 0$ since the sum is direct. //

4.3. Representations of $s\ell(2)$

Let S be the 3-dimensional Lie algebra with basis $\{x,y,h\}$ such that

$$[xh] = 2x, \ [yh] = -2y, \ [xy] = h \ . \qquad (1.28)$$

Assume that F is a field of characteristic 0. Then S is simple and is isomorphic to the Lie algebra $s\ell(2,F)$ of 2×2 trace 0 matrices over F. In fact, S is a simple Lie algebra of least dimension (why ?). Clearly, $H = Fh$ is a CSA of S. The 3-dimensional Lie algebra given by (1.28) is called the <u>split 3-dimensional simple Lie algebra</u>.

Let V be an S-module with the afforded representation $f \neq 0$. We first suppose that F is algebraically closed. Henceforth, we denote by $\lambda, \mu, \ldots$ weights of a CSA H in V. For brevity, we identify $vt = vf(t)$, $v \in V$, $t \in S$. Let

$$V = \Sigma_\lambda \oplus V_\lambda(h) \qquad (1.29)$$

be the weight space decomposition of V relative to $H = Fh$. In view of Theorem 4.2.1 $h(= f(h))$ is semisimple, so $V = V_\lambda(h) = \{v \in V \mid vh = \lambda v\}$.

<u>Lemma 4.3.1</u>. If $v \in V_\lambda$ then $vx \in V_{\lambda+2}$ and $vy \in V_{\lambda-2}$.

<u>Proof</u>. $(vx)h = v[xh] + (vh)x = (2 + \lambda)vx$ and likewise $(vy)h = (\lambda - 2)vy$. //

Since V is finite-dimensional, in view of (1.29) there exists $V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$. For such λ, any nonzero element in V_λ is called a <u>maximal vector</u> of the weight λ . Note that $vx = 0$ for all $v \in V_\lambda$ by Lemma 4.3.1.

<u>Lemma 4.3.2</u>. Let $v_0 \in V_\lambda$ be a maximal vector, $v_{-1} = 0$ and let $v_i = (1/i!)v_0 y^i$ $(i \geqq 0)$. Then

(1) $v_i h = (\lambda - 2i)v_i$,

(2) $v_i y = (i + 1)v_{i+1}$,

(3) $v_i x = (i - \lambda - 1)v_{i-1}$ $(i \geqq 0)$.

<u>Proof</u>. (1) follows from Lemma 4.3.1 while (2) is just the definition of v_i . For (3), use induction on i . If $i = 0$ then it is clear since $v_{-1} = 0$.

$$\begin{aligned} iv_i x &= (v_{i-1}y)x \\ &= v_{i-1}[yx] + (v_{i-1}x)y \\ &= -v_{i-1}h + (i - 2 - \lambda)v_{i-2}y \quad \text{(induction and (2))} \\ &= -(\lambda - 2(i - 1))v_{i-1} + (i - 1)(i - 2 - \lambda)v_{i-1} \\ &\qquad \text{((1) and (2))} \\ &= i(i - \lambda - 1)v_{i-1} . \quad // \end{aligned}$$

In view of Lemma 4.3.2 (1), the nonzero v_i are linearly independent. Since $\dim V < \infty$, one can choose the least integer m such that $v_m \neq 0$ but $v_{m+1} = 0$;

evidently $v_{m+i} = 0$ for all $i > 0$. Lemma 4.3.2 (1) - (3) show that $v_0, v_1, \ldots, v_m$ span an S-submodule of V. Suppose that V is S-irreducible. Then $v_0, v_1, \ldots, v_m$ form a basis for V. Since $v_{m+1} = 0$ and $v_m \neq 0$, it follows from (3) that $\lambda = m$. Thus the weight m of a maximal vector is a nonnegative integer, called the <u>highest weight</u> of h in V. Each weight space V_μ is one-dimensional and so $\lambda = m = \dim V - 1$ is uniquely determined by V. Since $\dim V_\lambda = 1$, v_0 is unique up to scalar multiple. It follows from (1) that the weights of h form an arithmetic progression with difference 2; that is, the weights are $m, m-2, \ldots -(m-2), -m$ with the lowest weight $-m$. Therefore, if m is even, the weight 0 occurs once and only once while if m is odd, 1 occurs as a weight once and only once. It is clear that both 0 and 1 cannot occur as weights.

Since any irreducible S-module of dimension $m + 1$ is described by Lemma 4.3.2 (1) - (3), irreducible S-modules are isomorphic if and only if they have the same dimension. For any nonnegative integer m, let V be a vector space over F with a basis $v_0, v_1, \ldots, v_m$. Define the actions of S on V by the relations (1) - (3) in Lemma 4.3.2. It is easy to check that V becomes an irreducible S-module (Exercise 4.3.1). We summarize these in

Theorem 4.3.3. Let V be a finite-dimensional irreducible S-module. Then

(1) V is the direct sum of weight spaces V_μ of h, $\mu = m, m-2, \ldots, -(m-2), -m$, where $m + 1 = \dim V$, $\dim V_\mu = 1$ for each μ and m is the highest weight of h.

(2) V has, up to nonzero scalar multiple, a unique maximal vector whose weight is m and highest.

(3) Exactly one of 0 and 1 occurs as a weight once and only once.

(4) For any nonnegative integer m, there exists a unique irreducible S-module of dimension $m + 1$. This module is described by (1) - (3) in Lemma 4.3.2. //

Note that any nonzero weight vector v of h with $vx = 0$ is a maximal vector of V.

Corollary 4.3.4. Let V be a finite-dimensional S-module. Then the eigenvalues of h on V are all integers and if k is an eigenvalue of h then the arithmetic progression $k, (k-2), \ldots, -(k-2), -k$ are all eigenvalues of h. Moreover, the number of irreducible S-submodules in V is precisely $\dim V_0(h) + \dim V_1(h)$.

<u>Proof</u>. By Weyl's Theorem 3.7.5, V is decomposed as a direct sum of irreducible S-modules. Thus by Theorem 4.3.3 all eigenvalues of h are integers. If k is an eigenvalue of h on V, k occurs as an eigenvalue of h on an irreducible summand of V and hence $k, k-2, \ldots, -(k-2), -k$ are all eigenvalues of h. Since each weight space in an irreducible S-module is one-dimensional, by Theorem 4.3.3 (3), the number of irreducible S-submodules is $\leqq \dim V_0(h) + \dim V_1(h)$. If $v \in V_0(h)$ then $vh = 0$ and, by Lemma 4.3.2 (1), v is a linear combination of eigenvectors of h having 0 as eigenvalues. Thus $\dim V_0(h)$ equals the number of irreducible S-submodules of V in which 0 occurs as weight. Likewise, $\dim V_1(h)$ is the number of irreducible S-submodule of V in which 1 occurs as weight. //

<u>Remark</u>. If F is not algebraically closed, let K be the algebraic closure of F. Regard V_K as an S_K-module. Then the above discussion shows that the eigenvalues of h in V_K are all integers ; but these are also the eigenvalues of h in V. //

<u>Exercise 4.3.1</u>. Let $V(m)$ be the irreducible S-module with basis $v_0, v_1, \ldots, v_m$ given by Lemma 4.3.2 (1) - (3) . Find the matrices of $x(= f(x))$, y and h relative to this basis. Utilize these matrices to verify

that the relations (1) -(3) of Lemma 4.3.2 define an $(m+1)$-dimensional irreducible S-module.

Notice that the matrix of $f(h)$ is diagonal, while those of $f(x)$ and $f(y)$ are respectively upper and lower nil triangular matrices, so nilpotent.

Exercise 4.3.2. Let U and W be L-modules where L is any Lie algebra. Prove that the tensor product $U \otimes W$ becomes an L-module via

$$(u \otimes w)x = (ux) \otimes w + u \otimes (wx),\ u \in U,\ w \in W,\ x \in L .$$

Let $V(m)$ be the irreducible S-module as in Exercise 4.3.1 Prove that the S-module $V(3) \otimes V(7)$ is decomposed into the direct sum of irreducible S-modules : $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. The general case of this is the Clebsch-Gordan formula : If $n \leq m$ then $V(m) \otimes V(n) \cong V(m+n) \oplus V(m+n-2) \oplus \ldots \oplus V(m-n)$, $n+1$ summands in all. This formula plays a role in elementary particle physics, angular momentum, spin, etc.

4.4. Properties of roots and root spaces

Let H be a split CSA of L and let

$$L = H + \Sigma_{\alpha \neq 0} L_\alpha \tag{1.30}$$

be the Cartan decomposition of L relative to H. Henceforth, we denote the Killing form $K(\ ,\ )$ by $(\ ,\ )$. Let H^* be the dual space of H. Since $(\ ,\ )$ is nondegenerate on H, the mapping $h \to (h,\ )$ is an isomorphism of H onto H^*. Thus, for each $\phi \in H^*$, there exists a unique $t_\phi \in H$ such that $\phi = (t_\phi,\ )$. Also, the mapping $\phi \to t_\phi$ is an isomorphism of H^* onto H, and

$$\psi(t_\phi) = (t_\psi,\ t_\phi), \quad , \psi \in H^* . \tag{1.31}$$

Denote by Φ the set of all nonzero roots of H in L. We define a bilinear form $(\ ,\ )$ on H^* by $(\phi, \psi) = (t_\phi, t_\psi)$. Then $(\ ,\ )$ is nondegenerate on H^* since $(\ ,\ )$ is nondegenerate on H.

<u>Theorem 4.4.1</u>. (1) Φ spans H^*.

(2) If $\alpha \in \Phi$ then $-\alpha \in \Phi$.

(3) For $\alpha \in \Phi$, let $x \in L_\alpha$ and $y \in L_{-\alpha}$. Then $xy = -(x,y)t_\alpha$.

(4) If $\alpha \in \Phi$ then $[L_\alpha L_{-\alpha}]$ is one-dimensional with basis t_α .

(5) $\alpha(t_\alpha) = (\alpha,\alpha) = (t_\alpha,t_\alpha) \neq 0$ for $\alpha \in \Phi$.

(6) For $\alpha \in \Phi$ and each nonzero $x_\alpha \in L_\alpha$, there exists an element $y_\alpha \in L_{-\alpha}$ such that x_α, y_α , $h_\alpha = [x_\alpha y_\alpha]$ span the split 3-dimensional simple Lie algebra where $h_\alpha = 2t_\alpha/(\alpha,\alpha)$ with $h_\alpha = -h_{-\alpha}$. In particular, h_α is uniquely determined by α .

Proof. (1) Let $\bar{H}$ be the subspace of H^* spanned by Φ . Let $\{\alpha_1,\ldots,\alpha_r\} \subset \Phi$ be a basis for $\bar{H}$. Then the mapping $h \to \phi(h) = \Sigma_{i=1}^{r}\alpha_i(h)\alpha_i$ gives a linear mapping of H into $\bar{H}$. Thus, if $\dim \bar{H} < \dim H$, there exists a nonzero $h \in H$ such that $\phi(h) = 0$, so $\alpha_1(h) = \ldots = \alpha_r(h) = 0$ and $\alpha(h) = 0$ for $\alpha \in \Phi$. This, by Corollary 4.2.3 (1), implies that h is in the center of L . This is absurd since L is semisimple.

(2) Suppose that $-\alpha \notin \Phi$. Then $\alpha + \beta \neq 0$ for all $\beta \in \Phi$ and hence $(L_\alpha,L_\beta) = 0$ for $\beta \in \Phi$ by Theorem 3.3.6(3). Therefore, $(L_\alpha,L) = 0$ and this is impossible since (,) is nondegenerate.

(3) Let $x \in L_\alpha$ and $y \in L_{-\alpha}$. For $h \in H$, we have $(h,[xy]) = (-[xh],y) = -\alpha(h)(x,y) = -(t_\alpha,h)(x,y) = (-(x,y)t_\alpha,h)$. The nondegeneracy of (,) on H then implies $[xy] = -(x,y)t_\alpha$.

(4) In view of (3), it suffices to show $[L_\alpha, L_{-\alpha}] \neq 0$. Let $x \neq 0$ be in L. Suppose that $(x, L_{-\alpha}) = 0$. Then by Theorem 3.3.6 (3), $(x, L_\beta) = 0$ for $\beta \in \Phi$ and so $(x, L) = 0$, contrary to the nondegeneracy of $(\ ,\ )$. Thus there exists a $y \neq 0$ in $L_{-\alpha}$ with $(x,y) \neq 0$ and $[xy] \neq 0$ by (3).

(5) Suppose that $\alpha(t_\alpha) = (\alpha,\alpha) = 0$. Then $[xt_\alpha] = [yt_\alpha] = 0$ for all $x \in L_\alpha$ and $y \in L_{-\alpha}$. As in (3), find $x \in L_\alpha$ and $y \in L_{-\alpha}$ with $(x,y) \neq 0$. We may assume that $(x,y) = -1$. Then, by (3), $[xy] = t_\alpha$ and x, y, t_α span a solvable Lie algebra N. But then $\mathrm{ad}_L\, z$ is nilpotent for $z \in [NN]$ by Corollary 3.4.4. Since $[NN] = Ft_\alpha \subset H$, $\mathrm{ad}_L\, t_\alpha$ is both semisimple and nilpotent, so $\mathrm{ad}_L\, t_\alpha = 0$ and $t_\alpha \in$ center of L, contrary to the semisimplicity of L.

(6) Given $x_\alpha \neq 0$ in L_α. As in (4), choose a $y \neq 0$ in $L_{-\alpha}$ such that $(x_\alpha, y) \neq 0$. Since $(\alpha,\alpha) \neq 0$, we set

$$y_\alpha = \frac{-2y}{(x_\alpha, y)(\alpha,\alpha)}, \quad h_\alpha = \frac{2t_\alpha}{(\alpha,\alpha)}. \tag{1.32}$$

Then

$$(x_\alpha, y_\alpha) = \frac{-2}{(\alpha,\alpha)} \tag{1.33}$$

and hence $[x_\alpha y_\alpha] = -(x_\alpha, y_\alpha) t_\alpha = 2t_\alpha/(\alpha,\alpha) = h_\alpha$ by (3), (1.32) and (1.33). Also, $[x_\alpha h_\alpha] = (2/(\alpha,\alpha))[x_\alpha t_\alpha] = 2x_\alpha$ and likewise $[y_\alpha h_\alpha] = -2y_\alpha$. Since $\alpha(h) = (t_\alpha, h)$ for $h \varepsilon H$ and $(\ ,\ )$ is nondegenerate on H, we have $t_{-\alpha} = -t_\alpha$ and so $h_{-\alpha} = -h_\alpha$. Since t_α is uniquely determined by α, so is h_α, by (1.32). //

<u>Corollary 4.4.1</u>. Let H be a split CSA of L and V be an L-module. Then V is a split H-module and $V_\lambda(H) = \{v \varepsilon V \mid vh = \lambda(h)v,\ h \varepsilon H\}$. Thus $f(h)$, $h \varepsilon H$, is split over F and semisimple where f is the representation afforded by V.

<u>Proof</u>. By Theorem 4.4.1(6), there is a basis $h_{\alpha_1}, \ldots, h_{\alpha_\ell}$ of H such that x_{α_i}, y_{α_i}, h_{α_i} span a split 3-dimensional simple Lie algebra S_i. Regard V as an S_i-module. Then the eigenvalues of $f(h_{\alpha_i})$ are integers, so $f(h_{\alpha_i})$ is split over F and acts diagonally on V (Section 4.3, Remark). Since H is abelian, by a standard argument in linear algebra there is a basis $\{v_1, v_2, \ldots, v_n\}$ of V relative to which every $f(h_{\alpha_i})$ has a diagonal matrix. Since the h_{α_i} form a basis of H, it follows that every $f(h)(h \varepsilon H)$ has a diagonal matrix relative to $\{v_j\}$. Thus $v_j h = \lambda_j(h) v_j$, $h \varepsilon H$, $j = 1,2,\ldots,n$. For each weight λ of H, it is easily seen that $V_\lambda(H)$ is spanned by the v_i such that $\lambda = \lambda_i$. //

Theorem 4.4.2. (1) $\alpha \in \Phi$ implies that $\dim L_\alpha = 1$. In particular, $S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ is the split 3-dimensional simple Lie algebra and, for any nonzero $x_\alpha \in L_\alpha$, there is a unique $y_\alpha \in L_{-\alpha}$ such that $[x_\alpha y_\alpha] = h_\alpha$, where $H_\alpha = [L_\alpha L_{-\alpha}]$.

(2) If $\alpha \in \Phi$ then the only scalar multiples of α which are roots are α and $-\alpha$.

Proof. For a fixed $\alpha \in \Phi$, let $M = H + \Sigma_{c \in F^*} L_{c\alpha}$ where $F^* = F - \{0\}$. Let S_α be the split 3-dimensional simple Lie algebra spanned by $x_\alpha, y_\alpha, h_\alpha$ given in Theorem 4.4.1 (6). Regard M as an S_α-module via ad. Then by Corollary 4.3.4 the weights of h_α in M are integers which must be 0 and $c\alpha(h_\alpha)$ for $L_{c\alpha} \neq 0$, $c \in F^*$. But then by (1.32) $\alpha(h_\alpha) = 2$ and so

$$0 \neq c\alpha(h_\alpha) = 2c = \text{an integer} \tag{1.34}$$

for $c \in F^*$ with $L_{c\alpha} \neq 0$. If we let $H' = \ker \alpha$ then $H = H' \oplus Fh_\alpha$. since $\alpha(h_\alpha) = 2$, and H' is an S_α-submodule of M in which all S_α-irreducible summands are one-dimensional with the only weight 0. Also, S_α itself is an irreducible S_α-submodule having the weights 0, ± 2 of h_α. By (1.34), $H' \oplus S_\alpha$ exhausts the occurance of weight 0. Thus, in view of Theorem 4.3.3, the even weights of h_α in M must occur in $H' \oplus S_\alpha$.

Thus 0 , ± 2 are the only even weights of h_α in M . Therefore, 2α is not a root ; for, if it is a root then $2\alpha(h_\alpha) = 2 \cdot 2 = 4$ would be an even weight of h_α . It follows from this that $\frac{1}{2}\alpha$ is not a root. This in turn implies that 1 can not be a weight of h_α . Indeed, if it is a weight then $L_{c\alpha} \neq 0$ for some $c \in F^*$ and by (1.34) $2c = 1$, so $\frac{1}{2}\alpha$ is a root, a contradiction. Therefore, $M_1(\text{ad } h_\alpha) = 0$ and by Corollary 4.3.4 $M = H + S_\alpha$, so $L_{c\alpha} = 0$ unless $c = \pm 1$. Thus the only roots of the form $c\alpha$ are $\pm\alpha$. Since $L_\alpha, L_{-\alpha} \subset M = H + S_\alpha$, $L_\alpha = Fx_\alpha$ and $L_{-\alpha} = Fy_\alpha$; $\dim L_\alpha = 1$. By Theorem 4.4.1 (4) $[L_\alpha L_{-\alpha}] = H_\alpha = Fh_\alpha$ and $[xy] \neq 0$ for $x \neq 0$ in L_α and $y \neq 0$ in $L_{-\alpha}$. This proves (1) and (2) . //

<u>Theorem 4.4.3</u>. Let V be an L-module and let λ be any weight of H in V . For $\alpha \in \Phi$, let r and q be the largest integers for which $\lambda - r\alpha$ and $\lambda + q\alpha$ are weights. The set of all weights of the form $\lambda + i\alpha$ forms an arithmetic progression with first term $\lambda - r\alpha$, difference α , and last term $\lambda + q\alpha$. Also, $r,q \geqq 0$ and

$$\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = r - q . \tag{1.35}$$

If we let $\lambda' = \lambda - [2(\lambda,\alpha)/(\alpha,\alpha)]\alpha$ then λ' is also a weight and $\dim V_\lambda(H) = \dim V_{\lambda'}(H)$.

<u>Proof</u>. Let S_α be the split 3-dimensional simple Lie algebra spanned by the canonical basis $x_\alpha, y_\alpha, h_\alpha$ as in Theorem 4.4.1 (6). Set $L^{(\alpha)} = H + S_\alpha$ and let $H_0 = \ker \alpha$. Then $L^{(\alpha)} = H_0 \oplus S_\alpha$ and $Z(L^{(\alpha)})$ = center of $L^{(\alpha)}$ is H_0 . Regard V as an $L^{(\alpha)}$-module. Then by Corollary 4.4.1 each $h \in H$ is split over F and semisimple. Thus by Corollary 3.7.7 V is the direct sum of irreducible $L^{(\alpha)}$-submodules and, furthermore, each $h \in H_0$ is a scalar on every irreducible summand W of V (see the proof of Corollary 3.7.7). Thus W is S_α-irreducible also. Let v_0 be a maximal vector of W , so that $v_0 x_\alpha = 0$ and $v_0 h_\alpha = m v_0$ where $m = \dim W - 1$ (see Theorem 4.3.3). Since $h \in H_0$ is a scalar on W , we can let $v_0 h = \mu(h) v_0$ for $h \in H$ where μ is a weight of H . Thus

$$v_0 h_\alpha = \mu(h_\alpha) v_0 = \frac{2(\mu,\alpha)}{(\alpha,\alpha)} v_0$$

by (1.31) and (1.32), so that $2(\mu,\alpha)/(\alpha,\alpha) = m$. Then, as in Lemma 4.3.2, $v_0 y_\alpha^i$, $i = 0,1,\ldots,m$, form a basis for W . An easy induction on i shows

$$(v_0 y_\alpha^i)h = (\mu - i\alpha)(h) v_0 y_\alpha^i , \quad h \in H . \qquad (1.36)$$

Hence $\mu - i\alpha$, $i = 0,1,\ldots,m$, are distinct weights of H in W and since $\dim W = m + 1$, these are all the weights in W . Therefore, any weight λ of H in W is of the form $\mu - i\alpha$ and

$$\dim W_\lambda(H) = 1 \ . \tag{1.37}$$

Let λ be any weight in V and let $V = \Sigma \oplus V_i$ be the direct sum of irreducible $L^{(\alpha)}$-submodules V_i of V. Then λ is a weight in one of the V_i. Indeed, if $vh = \lambda(h)v$, $h \in H$, $v \neq 0$ then $v = \Sigma v_i$ and $vh = \Sigma v_i h = \lambda(h)v = \Sigma\lambda(h)v_i$, so that $v_i h = \lambda(h)v_i$, $h \in H$. Thus λ is a weight in V_i where $V_\lambda(H) \cap V_i \neq 0$ and hence by (1.37) $\dim V_\lambda(H)$ is the number of V_i's in which λ is a weight. Let W be an irreducible $L^{(\alpha)}$-submodule of V in which λ is a weight. As before, $v_0 h = \mu(h)v_0$ where v_0 is a maximal vector of W and $2(\mu,\alpha)/(\alpha,\alpha) = m = \dim W - 1$, so $\lambda = \mu - k\alpha$ with $0 \leqq k \leqq m$ by (1.36). Let q be the largest integer such that $\lambda + q\alpha$ is a weight, so $q \geqq 0$ since λ is a weight. Let W' be an irreducible $L^{(\alpha)}$-submodule of V in which $\lambda + q\alpha$ is a weight. Choose a weight vector $v_0' \neq 0$ in W' such that $v_0'h = (\lambda + q\alpha)(h)v_0'$. Since $\lambda + (q+1)\alpha$ is not a weight $v_0'x = 0$ and so v_0' is a maximal vector of W' with the highest weight $(\lambda + q\alpha)(h_\alpha) = 2(\lambda + q\alpha, \alpha)/(\alpha,\alpha) = 2(\lambda,\alpha)/(\alpha,\alpha) + 2q \equiv s = \dim W' - 1$ of h_α. Since $\mu = \lambda + k\alpha$ is a weight, $0 \leqq k \leqq q$ and $m = 2(\mu,\alpha)/(\alpha,\alpha) = 2(\lambda + k\alpha,\alpha)/(\alpha,\alpha) = 2(\lambda,\alpha)/(\alpha,\alpha) + 2k$. Thus

$$k - m = -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} - k \ ,$$

$$q - s = -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} - q \ ,$$

so $k - m \geqq q - s$. In view of (1.36), the weights of H in W and W' are respectively

$$\mu = \lambda + k\alpha, \lambda + (k - 1)\alpha, \ldots, \lambda + (k - m)\alpha \ , \qquad (1.38)$$

$$\lambda + q\alpha, \lambda + (q - 1)\alpha, \ldots, \lambda + (q - s)\alpha \ . \qquad (1.39)$$

Since $k \leqq q$ and $q - s \leqq k - m$, (1.38) is contained in (1.39). Thus all weights of the form $\lambda + i\alpha$ are in (1.39). Indeed, if $\lambda + i\alpha$ is a weight, then let j be the largest integer such that $\lambda + i\alpha + j\alpha$ is a weight ; but then $i + j = q$. Thus $\lambda + i\alpha$ is in (1.39). Letting $r = -(q - s)$, $r \geqq 0$ since $q - s \leqq k - m \leqq 0$ and the last term in (1.39) is $\lambda - r\alpha$, so that r is the largest integer such that $\lambda - r\alpha$ is a weight.

Let $\lambda' = \lambda - [2(\lambda,\alpha)/(\alpha,\alpha)]\alpha = \lambda + (2q - s)\alpha$. Since $-r = q - s \leqq 2q - s = q + (q - s) \leqq q(q - s \leqq 0)$, λ' occurs in (1.39). The argument just used shows that λ is a weight in an irreducible $L^{(\alpha)}$-submodule W if and only if λ' is a weight in W . Then, by (1.37) and the foregoing remark, $\dim V_\lambda(H) = \dim V_{\lambda'}(H)$. //

Notice that $\lambda - i\alpha$ is a weight for any integer i between 0 and $r - q$ where r , q are as in (1.35).

<u>Corollary 4.4.4</u>. (1) Let $\alpha,\beta \in \Phi$ with $\beta \neq \pm\alpha$. Let r and q be respectively the largest integers for which $\beta - r\alpha$ and $\beta + q\alpha$ are roots. Then all $\beta + i\alpha \in \Phi$

for $-r \leq i \leq q$ with $r,q \geq 0$ and

$$\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = r - q \; . \qquad (1.40)$$

(2) If $\alpha \in \Phi$ then $\beta - \beta(h_\alpha)\alpha \in \Phi$.

(3) If $\alpha,\beta \in \Phi$ and $\alpha + \beta \in \Phi$ then $[L_\alpha L_\beta] = L_{\alpha+\beta}$.

(4) L is generated by the root spaces L_α , $\alpha \in \Phi$.

Proof. Since $\beta \neq \pm\alpha$, $\beta + i\alpha$ can not be zero. Thus (1) and (2) are consequences of Theorem 4.4.3 when one notes that if $\beta = \pm\alpha$, $\beta - \beta(h_\alpha)\alpha = \pm\alpha$.

For (3), let $K = \Sigma_{i=-r}^{q} L_{\beta+i\alpha}$ and regard K as an S_α-module. Then the weights of h_α are integers which must be $\beta(h_\alpha) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$. Since these are all distinct, not both 0 and 1 can occur as a weight of h_α . Therefore, by Corollary 4.3.4, K is S_α-irreducible with highest weight $\beta(h_\alpha) + 2q$. Let $0 \neq x \in L_{\beta+q\alpha}$ be a maximal vector in K . By Lemma 4.3.2, $\frac{1}{i!}x(\text{ad } y_\alpha)^i$, $i = 0,1,\ldots,\beta(h_\alpha) + 2q = r + q$, form a basis for K and these elements respectively span the weight spaces $L_{\beta+q\alpha},\ldots,L_\beta,\ldots,L_{\beta-r\alpha}$. In particular, $(1/q!)x(\text{ad } y_\alpha)^q$ spans L_β and we have

$$\frac{1}{q!}x(\text{ad } y_\alpha)^q \text{ ad } x_\alpha = -(r+1)\frac{1}{(q-1)!}x(\text{ad } y_\alpha)^{q-1}$$

by Lemma 4.3.2 (3). Since $\alpha + \beta \in \Phi$, $q \geq 1$ and the right side is nonzero. Thus $[L_\alpha L_\beta] \neq 0$ and $[L_\alpha L_\beta] = L_{\alpha+\beta}$ since $\dim L_{\alpha+\beta} = 1$.

(4) Since $\alpha \to t_\alpha$ is an isomorphism of H^* onto H , by (1.32) and Theorem 4.4.1 (1) the h_α for $\alpha \in \Phi$ span H . //

Difinition 4.4.1. For roots $\alpha, \beta \in \Phi$, the integers $\beta(h_\alpha)$ given by (1.40) are called the Cartan integers. The arithmetic progression of roots

$$\beta - r\alpha\ , \beta - (r-1)\alpha, \ldots, \beta, \beta + \alpha, \ldots, \beta + q\alpha \qquad (1.41)$$

in Corollary 4.4.1 (1) is called the α-string through β. //

Theorem 4.4.5. For $\alpha, \beta \in \Phi$, the α-string through β contains at most four roots. Hence the Cartan integer $2(\beta,\alpha)/(\alpha,\alpha) = 0$, ± 1, ± 2, ± 3 .

Proof. If $\beta = \pm\alpha$, The result is trivial. Suppose that $\beta \neq \pm\alpha$ and that the α-string through β contains at least five roots. By relabelling these, we may assume that $\beta - 2\alpha$, $\beta - \alpha, \beta, \beta + \alpha, \beta + 2\alpha$ are roots. Then $2\alpha = (\beta + 2\alpha) - \beta$ and $2(\beta + \alpha) = (\beta + 2\alpha) + \beta$ are not roots. Hence the β-string through $\beta + 2\alpha$ consists of the one term $\beta + 2\alpha$. Thus the Cartan integer $2(\beta + 2\alpha, \beta)/(\beta,\beta) = 0$ since $r = q = 0$. Similarly $\beta - 2\alpha - \beta$ and $\beta - 2\alpha + \beta$ are not roots, so that

$2(\beta - 2\alpha,\beta)/(\beta,\beta) = 0$. Adding these gives a contradiction, since $(\beta,\beta) \neq 0$ by Theorem 4.4.1 (5). Since the α-string through β, $\beta - r\alpha,\ldots,\beta,\ldots,\beta + q\alpha$, contains exactly $r + q + 1$ roots, $r + q + 1 \leqq 4$ and so $0 \leqq r$, $q \leqq 3$. This implies that $2(\alpha,\beta)/(\alpha,\alpha) = r - q = 0,\pm 1,\pm 2,\pm 3$. //

Since the characteristic of F is zero, we may identify the prime field of F with the rational number field Q .

Definition 4.4.2. Denote by H_0^* the Q-subspace of H^* spanned by Φ . Thus H_0^* is a vector space over Q spanned by Φ . For convenience, denote $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ for $\alpha,\beta \in H^*$ when $(\alpha,\alpha) \neq 0$. //

Notice that $\langle\beta,\alpha\rangle$ is linear only in the first variable.

Theorem 4.4.6. $\dim H_0^* = \dim H^*$.

Proof. Let $\dim H^* = \ell$. Since Φ spans H^* , Φ contains a basis $\alpha_1,\ldots,\alpha_\ell$ of H^* . For $\beta \in \Phi$, let $\beta = \Sigma_{i=1}^{\ell} c_i\alpha_i$, $c_i \in F$. It suffices to show that all c_i are in Q . We have

$$(\beta,\alpha_j) = \Sigma_i\, c_i(\alpha_i, \alpha_j) ,$$

$$\langle\beta,\alpha_j\rangle = \Sigma_i \langle\alpha_i, \alpha_j\rangle c_i , \quad j = 1,2,\ldots,\ell . \qquad (1.42)$$

Since all $<\beta,\alpha_j>$ and $<\alpha_i,\alpha_j>$ are integers, it remains to show that the system (1.42) has a solution. But then, since $(\ ,\ )$ is nondegenerate, $\det(<\alpha_i,\alpha_j>)$ $= [2^\ell/\Pi_j(\alpha_j,\alpha_j)]\cdot\det((\alpha_i,\alpha_j)) \neq 0$. Thus (1.42) has a solution . //

<u>Theorem 4.4.7</u>. (α,β) is rational for all $\alpha,\beta \in H_0^*$ and $(\ ,\ )$ is positive definite on H_0^* ; that is, $(\beta,\beta) > 0$ for all $\beta \neq 0$ in H_0^* .

<u>Proof</u>. Let $h, k \in H$. Since H is abelian and ad h, ad k are semisimple, $(h,k) = \text{Tr ad } h \text{ ad } k$ $= \Sigma_{\alpha\varepsilon\Phi}\ \alpha(h)\alpha(k)$. Thus we have for $\lambda,\mu \in H^*$

$$(\lambda,\mu) = (t_\lambda,t_\mu) = \Sigma_{\alpha\varepsilon\Phi}\ \alpha(t_\lambda)\alpha(t_\mu)$$

$$= \Sigma_{\alpha\varepsilon\Phi}\ (t_\alpha,t_\lambda)(t_\alpha,t_\mu) \qquad \text{(by(1.31))}$$

$$= \Sigma_{\alpha\varepsilon\Phi}\ (\alpha,\lambda)(\alpha,\mu)\ . \qquad (1.43)$$

Thus, if $\beta \in \Phi$ then

$$(\beta,\beta) = \Sigma_{\alpha\varepsilon\Phi}\ (\alpha,\beta)^2\ . \qquad (1.44)$$

Let the β-string through α be $\alpha - r_{\alpha\beta}\beta,\ldots,\alpha,\ldots,\alpha + q_{\alpha\beta}\ \beta$. Then $p_{\alpha\beta} \equiv 2(\alpha,\beta)/(\beta,\beta) \in Z$ and $(\alpha,\beta) = \frac{1}{2}P_{\alpha\beta}(\beta,\beta)$. It follows from this and (1.44) that $(\beta,\beta) = \frac{1}{4}\Sigma_{\alpha\varepsilon\Phi}\ P^2_{\alpha\beta}\ (\beta,\beta)^2$.

Since $(\beta,\beta) \neq 0$, $\Sigma_\alpha\ p^2_{\alpha\beta} \neq 0$ and so $(\beta,\beta) = 4/\Sigma_\alpha\ p^2_{\alpha\beta}$ is rational. Thus $(\alpha,\beta) = \frac{1}{2}p_{\alpha\beta}\ (\beta,\beta)$ is rational for $\alpha,\beta \in \Phi$. Since Φ spans H^*_0 , it follows from (1.43) that $(\lambda,\mu) \in Q$ for $\lambda,\mu \in H^*_0$. By (1.43), $(\lambda,\lambda) = \Sigma_{\alpha\in\Phi}\ (\alpha,\lambda)^2 \geqq 0$ for $\lambda \in H^*_0$. Thus if $(\lambda,\lambda) = 0$, $(\alpha,\lambda) = 0$ for all $\alpha \in \Phi$ and so $\lambda = 0$. Hence $(\lambda,\lambda) > 0$ for $0 \neq \lambda \in H^*_0$. //

Corollary 4.4.8. Let $\alpha,\beta \in \Phi$ and let $\beta \neq \pm\alpha$. If $(\alpha,\beta) < 0$ then $\alpha + \beta \in \Phi$. If $(\alpha,\beta) > 0$ then $\alpha - \beta \in \Phi$

Proof. This is immediate from Corollary 4.4.4 (1).

4.5. Simple systems of roots

Let $\alpha_1,\alpha_2,\ldots,\alpha_\ell$ be a Q-basis of roots of H^*_0 . A vector $\rho = \Sigma^\ell_{i=1}\ c_i\alpha_i \in H^*_0$ is called positive if the first nonzero λ_i is positive. If $\sigma,\rho \in H^*_0$, we write $\sigma > \rho$ if $\sigma - \rho > 0$. Then H^*_0 is totally ordered in this way and we refer to this ordering in H^*_0 as the lexicographic ordering determined by the ordered set $(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ of roots.

Lemma 4.5.1. Let $\rho_1, \rho_2, \ldots, \rho_k \in H_0^*$ and suppose that the $\rho_i > 0$ and $(\rho_i, \rho_j) \leqq 0$ for $i \neq j$. Then $\rho_1, \ldots, \rho_k$ are linearly independent over Q.

Proof. Suppose $\rho_k = \Sigma_{i=1}^{k-1} c_i \rho_i = \Sigma\, c'_q \rho_q + \Sigma\, c''_s \rho_s$

where $1 \leqq q, s \leqq k - 1$, $c'_q > 0$, $c''_s \leqq 0$. Set $\sigma = \Sigma\, c'_q \rho_q$

and $\tau = \Sigma\, c''_s \rho_s$. Since $\rho_k > 0$, $\sigma \neq 0$ and we have

$(\sigma, \tau) = \Sigma\, c'_q c''_s (\rho_q, \rho_s) \geqq 0$. Thus $(\rho_k, \sigma) = (\sigma, \sigma) + (\sigma, \tau) > 0$;

but $(\rho_k, \sigma) = \Sigma\, c'_q (\rho_k, \rho_q) \leqq 0$, a contradiction. Hence the ρ_i are linearly independent. //

Definition 4.5.1. A root $\alpha \in \Phi$ is called simple (relative to the given ordering of H_0^*) if $\alpha > 0$ and α can not be expressed in the form $\beta + \gamma$ where β, γ are positive roots. Denote by Π and Φ^+ the sets of simple roots and of positive roots, respectively. //

Thus the set Φ^- of negative roots is equal to $-\Phi^+ = \{-\alpha \mid \alpha \in \Phi^+\}$ and $\Phi = \Phi^+ \bigcup \Phi^-$.

Theorem 4.5.2. (1) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$ then $\alpha - \beta$ is not a root.

(2) If $\alpha, \beta \in \Pi$ and $\alpha \neq \beta$ then $(\alpha, \beta) \leqq 0$.

(3) Π is a basis for H_0^* over Q. If $\beta \in \Phi^+$ then $\beta = \Sigma_{\alpha \in \Pi} k_\alpha \alpha$ where the k_α are nonnegative integers.

(4) If $\beta \in \Phi^+$ and $\beta \notin \Pi$ then there exists an $\alpha \in \Pi$ such that $\beta - \alpha \in \Phi^+$.

Proof. (1) Let $\alpha, \beta \in \Pi$. If $\alpha - \beta \in \Phi^+$, $\alpha = \beta + (\alpha - \beta)$, contrary to the simplicity of α. If $\alpha - \beta \in \Phi^-$ then we write $\beta = (\beta - \alpha) + \alpha$ to obtain a contradiction.

(2) Let $\alpha - r\alpha, \ldots, \beta, \ldots, \beta + q\alpha$ be the α-string through β. Then $<\beta, \alpha> = r - q$. Thus $(\alpha, \beta) \leq 0$, since $r = 0$ by (1) and $(\alpha, \alpha) > 0$.

(3) It follows from Lemma 4.5.1 and (2) that Π is linearly independent. Since Φ^+ is a finite set, by induction we may assume that every $\gamma \in \Phi^+$ with $\beta > \gamma > 0$ is of the desired form and that $\beta \notin \Pi$. Thus $\beta = \beta_1 + \beta_2$ $\beta_i \in \Phi^+$ and since $\beta > \beta_i > 0$, $\beta = \Sigma_{\alpha \in \Pi} k_\alpha \alpha$, k_α nonnegative integers. It follows from this that every $\beta \in \Phi^-$ is of the form $\Sigma_\alpha k_\alpha \alpha$, k_α nonpositive integers. Since Φ spans H_0^*, this implies that Π is a basis for H_0^*.

(4) Suppose that $\beta \in \Phi^+$ and $\beta \notin \Pi$. In view of Lemma 4.5.1, (2) and (3), there must exist an $\alpha \in \Pi$ such that $(\beta, \alpha) > 0$. Thus $<\beta, \alpha> = r - q > 0$ and $r > 0$, so $\beta - \alpha \in \Phi$. If $\beta - \alpha < 0$ then $\alpha - \beta > 0$ and $\alpha = \beta + (\alpha - \beta)$. Thus $\beta - \alpha \in \Phi^+$ and $\beta = (\beta - \alpha) + \alpha$. //

Definition 4.5.2. If Π contains ℓ elements then we denote $\Pi = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ and call this the simple system of roots for L relative to H and the given ordering in H_0^*. //

The classification of simple Lie algebras reduces to that of simple systems of roots. Theorem 4.5.2(3) in fact characterizes a simple system of roots.

<u>Corollary 4.5.3</u>. Let $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ be an ordered set of roots where $\ell = \dim H$. Then Π is a simple system of roots relative to a certain ordering in H_0^* if and only if every $\beta \in \Phi$ has the form $\Sigma\, k_i \alpha_i$ where the k_i are integers and either all $k_i \geqq 0$ or all $k_i \leqq 0$.

<u>Proof</u>. One direction is simply Theorem 4.5.2(3). For the other direction, we first note that Π is a basis for H_0^*, since Φ spans H_0^*. We introduce an ordering in H_0^* by : $\Sigma\, c_i \alpha_i > 0$ if $c_1 = \ldots = c_k = 0$, $c_{k+1} > 0$, $k < \ell$. Then Φ^+ consists of those roots $\beta = \Sigma\, k_i \alpha_i$ with the $k_i \geqq 0$ and some $k_i > 0$. Since Π is a basis for H_0^*, the α_i are simple. Since any simple system consists of ℓ roots, Π is the simple system of roots defined by the ordering. //

<u>Definition 4.5.3</u>. Let $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ be a simple system of roots and let $A_{ji} = \langle \alpha_i, \alpha_j \rangle$. Then the matrix (A_{ji}) is called the Cartan matrix of Φ (or Π). If $\beta = \Sigma\, k_i \alpha_i \in \Phi$, we define the level of β as $|\beta| = |\Sigma k_i|$. //

If Φ has more than one root length $(||\alpha|| = \sqrt{(\alpha,\alpha)}$ = length of α), we speak of longer and shorter roots. For $\alpha,\beta \in \Phi$ the angle θ between α and β is given by $||\alpha||\ ||\beta|| \cos\theta = (\alpha,\beta)$. Hence $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$ $= 2(||\beta||/||\alpha||)\cos\theta$ and $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta$. This shows that $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ have the same sign, and if $\alpha \neq \pm\beta$ then $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle < 4$ since α,β are linearly independent and so $0 \leq \cos^2\theta < 1$. Since $\langle\alpha,\beta\rangle$, $\langle\beta,\alpha\rangle$ are integers, they must be $0, \pm1, \pm2, \pm3$. Thus, assuming $||\beta|| \geq ||\alpha||$, we have the following possibilities

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	θ	$\|\|\beta\|\|^2/\|\|\alpha\|\|^2$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

In particular, if α_i,α_j are in $\Pi (i \neq j)$ then $\langle\alpha_i,\alpha_j\rangle = A_{ji} \leq 0$ (Theorem 4.5.2(2)) and both A_{ij} and A_{ji} are 0, or one is -1 while the other is -1, -2, or -3. Hence we have

<u>Lemma 4.5.4</u>. Any Cartan matrix (A_{ji}) has all diagonal entries $A_{ii} = 2$ and if $i \neq j$ then either

$A_{ij}A_{ji} = 0$ or one of A_{ij}, A_{ji} is -1 while the other is -1, -2, or -3. If $i \neq j$ then the angle θ_{ij} between α_i, α_j is 90^0, 120^0, 135^0, or 150^0. //

For each $\alpha_i \in \Pi$, choose the canonical basis x_{α_i}, y_{α_i}, h_{α_i} as in Theorem 4.4.1(6) and denote

$$x_i = x_{\alpha_i}, \quad y_i = y_{\alpha_i}, \quad h_i = h_{\alpha_i}.$$

Then $[x_i h_i] = 2x_i$, $[y_i h_i] = -2y_i$ and $[x_i y_i] = h_i$. Since $\alpha_i - \alpha_j$ is not a root for $i \neq j$, $[x_i y_j] = 0$ and $[x_i h_j] = \alpha_i(h_j)x_i = A_{ji}x_i$ and $[y_i h_j] = -A_{ji}y_i$ by (1.32). Therefore, we have

$$\begin{aligned} [h_i h_j] &= 0, \\ [x_i y_j] &= \delta_{ij} h_i, \\ [x_i h_j] &= A_{ji} x_i, \\ [y_i h_j] &= -A_{ji} y_i, \quad i,j = 1,2,\ldots,\ell. \end{aligned} \tag{1.45}$$

Theorem 4.5.5. Φ is determined by the simple system Π and the Cartan matrix (A_{ji}). That is, the sequences $(k_1,\ldots,k_\ell)$ such that $\Sigma\, k_i\alpha_i \in \Phi$ are determined by (A_{ji}).

Proof. It suffices to determine Φ^+. The proof proceeds by induction on level. The roots in Φ^+ of level one are the $\alpha_i \in \Pi$. Suppose that the roots in Φ^+ of level $\leqq n$ are determined by (A_{ij}). By Theorem 4.5.2(4) any root $\beta \in \Phi^+$ of level $n + 1$ is of the form $\alpha = \alpha + \alpha_j$, $\alpha \in \Phi^+$ of level n, $\alpha_j \in \Pi$. Since if $\alpha = \alpha_j$ then β is not a root, we may assume that $\alpha = \Sigma\, k_i\alpha_i$ with some $k_i > 0$ for $i \neq j$ and the k_i are determined by A_{ij} . Let $\alpha - r\alpha_j, \ldots, \alpha, \alpha + \alpha_j, \ldots, \alpha + q\alpha_j$ be the α_j-string through α . Since β is in this string, it suffices to show that r and q are determined by A_{ij} Since $\alpha - \alpha_j, \ldots, \alpha - r\alpha_j$ are positive roots of level $< n$, these are determined by A_{ij} and so is r by A_{ij} . Thus $q = r - \langle\alpha,\alpha_j\rangle = r - \Sigma\, k_i A_{ji}$ is determined by A_{ij} . //

Since $\alpha + \alpha_j$ is a root if and only if $q > 0$, the last step in the proof gives a method of ascertaining whether or not $\alpha + \alpha_j$ is a root.

<u>Theorem 4.5.6</u>. Let $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ be a simple system of roots for L relative to H and let x_i, y_i, h_i be as in (1.45). Then the 2ℓ elements x_i, y_i generate L . For each $\beta \in \Phi^+$ we can choose an expression of $\beta = \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_k}$, so that $\alpha_{i_1} + \cdots + \alpha_{i_m} \in \Phi^+$ for every $m \leqq k$. Then the elements

$$h_i,\ [x_{i_1}x_{i_2} \cdots x_{i_k}]\ ,\ [y_{i_1}y_{i_2} \cdots y_{i_k}] \qquad (1.46)$$

determined by the $\beta \in \Phi^+$ form a basis for L , where

$$[\cdots [x_1x_2] \cdots x_r] = [x_1x_2 \cdots x_r] .$$

Proof. By induction on level, the second conclusion follows from Theorem 4.5.2(4). Let β be any positive root. In view of this and Corollary 4.4.4(3), the expression for β implies that $[x_{i_1} \dots x_{i_k}] \neq 0$ and $[y_{i_1} \dots y_{i_k}] \neq 0$; hence these elements respectively span L_β and $L_{-\beta}$. Thus the elements in (1.46) form a basis of L since the h_i form a basis of H . //

Notice that if there is no positive root of level n then Φ^+ contains no roots of level $n + i$ for $i \geq 0$. This is due to the fact that if $\beta \in \Phi^+$ is of level $n + 1$, by Theorem 4.5.2(4) $\alpha = \beta - \alpha_j \in \Phi^+$ for some $\alpha_j \in \Pi$ and $|\alpha| = n$.

Example. Let L be a semisimple Lie algebra with the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} \langle\alpha_1,\alpha_1\rangle & \langle\alpha_1,\alpha_2\rangle \\ \langle\alpha_2,\alpha_1\rangle & \langle\alpha_2,\alpha_2\rangle \end{pmatrix} . \qquad (1.47)$$

We determine Φ from this matrix. By (1.47) $2(\alpha_2,\alpha_1)/(\alpha_1,\alpha_1) = -3$ and $2(\alpha_1,\alpha_2)/(\alpha_2,\alpha_2) = -1$. Since $\alpha_1 - \alpha_2$ is not a root, these imply that the α_1-string through α_2 and the α_2-string through α_1 are respectively

α_2 , $\alpha_2 + \alpha_1$, $\alpha_2 + 2\alpha_1$, $\alpha_2 + 3\alpha_1$,

α_1 , $\alpha_1 + \alpha_2$

by (1.40). The only root in Φ^+ of level 2 is $\alpha_1 + \alpha_2$. Since $\alpha_1 + 2\alpha_2 \notin \Phi$, $\alpha_2 + 2\alpha_1$ is the only positive root of level 3. Similarly, $\alpha_2 + 3\alpha_1$ is the only one in Φ^+ of level 4. Since $2(\alpha_2 + 3\alpha_1, \alpha_2)/(\alpha_2, \alpha_2) = 2 - 3 = -1$ $(\alpha_2 + 3\alpha_1) + \alpha_2 = 3\alpha_1 + 2\alpha_2$ is a root. Noting that $\alpha_1 + 4\alpha_2$ is not a root, $3\alpha_1 + 2\alpha_2$ is the only root in Φ^+ of level 5. In view of Theorem 4.5.2(4), there are no roots of level $\geqq 6$. Hence Φ consists of

$$\begin{aligned} &\pm\alpha_1, \ \pm\alpha_2, \ \pm(\alpha_1 + \alpha_2), \ \pm(\alpha_2 + 2\alpha_1) \\ &\pm(\alpha_2 + 3\alpha_1), \ \pm(3\alpha_1 + 2\alpha_2) \end{aligned} \tag{1.48}$$

and thus $\dim L = 14$. //

In general, this procedure applies to construct Φ from any given Cartan matrix.

<u>Exercise 4.5.1</u>. Let $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ be a simple system of roots. Let x_i, y_i, h_i be the generators for L as in (1.45) and let L have the basis as in (1.46). Then the multiplication table for this basis has rational coefficients which are determined by the Cartan matrix (A_{ij}).

<u>Exercise 4.5.2</u>. (The isomorphism theorem). Let L, L' be semisimple Lie algebras with split CSA H and H' of the same dimension ℓ. Let $(\alpha_1, \ldots, \alpha_\ell)$ and $(\alpha_1', \ldots, \alpha_\ell')$ be simple systems of roots for L and L' respectively. Suppose the Cartan matrices $(\langle\alpha_i, \alpha_j\rangle)$ and $(\langle\alpha_i', \alpha_j'\rangle)$

are identical. Let x_i, y_i, h_i , x_i', y_i', h_i' be the generators for L and L' as in (1.45). Then there exists a unique isomorphism of L onto L' mapping x_i on x_i', y_i on y_i', h_i on h_i' .

Exercise 4.5.3. Let F be algebraically closed. Then L has a basis h_i, x_α, $\alpha \in \Phi$, $i = 1,2,\ldots,\ell = \dim H$ such that if $\beta \neq \pm\alpha$ and $\alpha,\beta \in \Phi$ then

$$[x_\alpha x_\beta] = N_{\alpha\beta} x_{\alpha+\beta}$$

where $x_\alpha \in L_\alpha$ and $N_{\alpha\beta} \in F$ satisfies the condition $N_{\alpha\beta} = N_{-\alpha,-\beta}$.

Definition 4.5.4. A simple system of roots $\Pi = (\alpha_1,\ldots,\alpha_\ell)$ is called irreducible if it is impossible to partition Π into the disjoint union of proper subsets Π' , Π'' such that $A_{ij} = 0$ for every $\alpha_i \in \Pi'$, $\alpha_j \in \Pi''$. //

Theorem 4.5.7. L is simple if and only if the associated simple system Π of roots is irreducible.

Proof. Suppose that $\Pi = (\alpha_1,\ldots,\alpha_k) \bigcup (\alpha_{k+1},\ldots,\alpha_\ell)$.

$1 \leq k < \ell$, so that $A_{ij} = 0$, $i \leq k$, $j > k$. Let L_1 be the subalgebra generated by the x_j, y_j, h_j, $j \leq k$, as in (1.45). One easily checks that $L_1 = H_i + \Sigma L_\gamma$ where H_1 is the subspace spanned by $h_1,\ldots,h_k$ and the summation runs over the roots γ which are linearly dependent on the α_j. Thus $0 \neq L_1 \neq L$. If $r > k$ and $j \leq k$ then $A_{jr} = 0$ and since $\alpha_j - \alpha_r$ is not a root, this implies that $\alpha_j + \alpha_r$ is not a root. Hence $[x_j x_r] = [y_j x_r] = 0$ as well as $[h_j x_r] = 0$ since $(\alpha_r, \alpha_j) = 0$. Therefore, x_r is in $N(L_1)$, the normalizer of L_1 and likewise $y_r \in N(L_1)$. Since the x_i, y_i generate L, it follows that $L \subset N(L_1)$, so L_1 is an ideal of L and L is not simple. Conversely, suppose that L is not simple and $L = L_1 \oplus L_2$ where the L_i are proper ideals of L. Let $\alpha \in \Phi$ and $x_\alpha \in L_\alpha$. Then $x_\alpha = x_\alpha^{(i)} + x_\alpha^{(2)}$, $x_\alpha^{(i)} \in L_i$ and since $[x_\alpha h] = \alpha(h)x_\alpha$, we have $[x_\alpha^{(i)} h] = \alpha(h)x_\alpha^{(i)}$. Since $\dim L_\alpha = 1$, this implies that $L_\alpha \subset L_1$ or $L_\alpha \subset L_2$ Noting that $[L_1 L_2] = 0$ and $[L_\alpha L_{-\alpha}] \neq 0$, we have either $L_\alpha + L_{-\alpha} \subset L_1$ or $L_\alpha + L_{-\alpha} \subset L_2$. Thus we may order the generators x_i, y_i, so that $x_1, y_1, \ldots, x_k, y_k \in$ $x_{k+1}, y_{k+1}, \ldots, x_\ell, y_\ell \in L_2$. Since the L_i are proper, $1 \leq k < \ell$ and $0 = [x_j [x_r y_r]] = [x_j h_r] = A_{rj} x_j$ for $j \leq k$, $r > k$. Hence $A_{jr} = A_{rj} = 0$ and Π is not irreducible. //

4.6. Classification

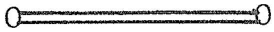
In view of the results in the previous section, determination of the simple Lie algebras reduces to that of the Cartan matrices associated with irreducible simple systems of roots. Instead, we classify all connected Dynkin diagrams corresponding to irreducible simple systems of roots. It is readily seen that a Dynkin diagram essentially determines the associated Cartan matrices. Throughout we fix a lexicographic ordering in H_0^* determined by $\Pi = (\alpha_1, \ldots, \alpha_\ell)$.

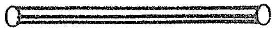
We call $||\alpha||^2 = (\alpha, \alpha) (\alpha \in \Phi)$ the <u>weight</u> of α . If we let a shortest root α have weight 1 then the weight of any root β with $(\alpha, \beta) \neq 0$ is 1,2, or 3 (see the remark following Definition 4.5.3).

<u>Definition 4.6.1</u>. The Coxeter graph of Φ (or Π) is a graph having ℓ vertices, the i th joined (connected) to the j th ($i \neq j$) by $A_{ij}A_{ji}$ lines. The Dynkin diagram of Φ (or L) is the Coxeter graph having vertices $\alpha_1, \ldots, \alpha_\ell$, together with weight attached to each vertex. //

For example,

A_2 :  ,

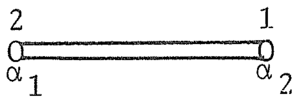
B_2 :  ,

G_2 :  ,

are Coxeter graphs having two vertices while their associated Dynkin diagrams are

A_2 : 

,

B_2 : 

,

G_2 : 

.

By virtue of Lemma 4.5.4, it is possible to determine the Cartan matrix (A_{ji}) from the Dynkin diagram. Indeed, if α_i and α_j are not joined then $A_{ij} = A_{ji} = 0$. If α_i and α_j are connected then $A_{ij} \neq 0$, $A_{ji} \neq 0$ and hence $A_{ji}/A_{ij} = (\alpha_i, \alpha_i)/(\alpha_j, \alpha_j)$ and $A_{ij}A_{ji}$ can be determined from the diagram. For example, the Dynkin diagram of G_2 gives $A_{12}/A_{21} = (\alpha_2, \alpha_2)/(\alpha_1, \alpha_1) = 3$ and $A_{12}A_{21} = 3$, which imply $A_{21} = -1$, $A_{12} = -3$ by Lemma 4.5.4. Hence the Cartan matrix is the matrix in (1.47). Likewise, the Cartan matrix for A_2 is

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} .$$

<u>Definition 4.6.2</u>. Let E be a Euclidian space over the reals. A set $\mathfrak{A}$ of ℓ linearly independent unit vectors $u_1, u_2, \ldots, u_\ell$ in E is called an admissible set (a.s.) if it satisfies

$$(u_i, u_j) \leqq 0 , \quad 4(u_i, u_j)^2 = 0, 1, 2, \text{ or } 3,$$

$$i \neq j, \quad i,j = 1,\ldots,\ell . \quad // \qquad (1.49)$$

If $\mathfrak{A}$ is an a.s. then we can attach a graph Γ to $\mathfrak{A}$ which has ℓ vertices $u_1,\ldots,u_\ell$, the i th joined to the j th by $4(u_i, u_j)^2$ lines $(i \neq j)$. A simple system of roots $\Pi = (\alpha_1,\ldots,\alpha_\ell)$ gives rise to an a.s. when we set $u_i = \alpha_i/||\alpha_i||$, since $A_{ij}A_{ji} = 4(u_i, u_j)$. Thus the Coxeter graph of Π is regarded as a graph of an a.s. An a.s. is said to be <u>irreducible</u> if it satisfies the conditions in Definition 4.5.4. The graph of an a.s. $\mathfrak{A}$ is called <u>connected</u> if, for any $u,v \in \mathfrak{A}$, there exists a sequence $u = u_{i_1}, u_{i_2}, \ldots, u_{i_k} = v$ in $\mathfrak{A}$ such that u_{i_j} and $u_{i_{j+1}}$ are joined in the graph. This definition of connectedness applies to Coxeter graphs and Dynkin diagrams. It is easily seen that the graph of an a.s. $\mathfrak{A}$ is connected if and only if $\mathfrak{A}$ is irreducible. Recall that the classification of simple Lie algebras is equivalent to that of connected Dynkim diagrams. Since the weights can be re-introduced, it suffices to classify the graphs of irreducible a.s. This classification proceeds in several steps. Let $\mathfrak{A}$ be an a.s. with graph Γ.

<u>D 1</u>. If some of the u_i in $\mathfrak{A}$ are discarded then the remaining ones still form an admissible set whose graph is obtained from Γ by omitting the corresponding vertices and all incident lines. //

<u>D 2</u>. The number of pairs of vertices in Γ connected by at least one line is less than ℓ.

<u>Proof</u>. Let $u = \Sigma_1^{\ell} u_i$. Then $0 < (u,u) = \ell + 2 \sum_{i<j} (u_i, u_j)$ by (1.49). If $(u_i, u_j) \neq 0$, $2(u_i, u_j) \leq -1$. Hence the number of pairs u_i, u_j with $(u_i, u_j) \neq 0$ is less than ℓ. //

<u>D 3</u>. Γ contains no cycles (a cycle is a sequence of vertices $u_1, \ldots, u_k$ such that u_i is joined to u_{i+1}, $1 \leq i \leq k-1$, and u_1 is joined to u_k.)

Proof. A cycle is a graph of an a.s. by D 1, which violates D 2 . //

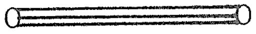
D 4. The number of lines issuing from a vertex in Γ does not exceed three.

Proof. Let u be a vertex and let $v_1,\ldots,v_k$ be the vertices joined to u . No two v_i are connected since there are no cycles. Hence $(v_i, v_j) = 0$, $i \neq j$. Set $v_0' = u - \Sigma_1^k (u, v_i)v_i$ and $v_0 = v_0'/||v_0'||$. Then $(v_0, v_0) = 1$ and $v_0,v_1,\ldots,v_k$ are mutually orthogonal. Thus $u = \Sigma_0^k (u, v_i)v_i$ where $(u, v_0) \neq 0$, since u and the v_i are linearly independent. It follows from this that

$$(u,u) = (u,v_0)^2 + (u,v_1)^2 + \ldots + (u,v_k)^2 = 1 .$$

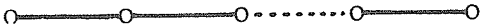
Hence $\Sigma_1^k (u,v_i)^2 < 1$ and $\Sigma_1^k 4(u,v_i)^2 < 4$. Since $4(u,v_i)^2$ is the number of lines joining u and v_i , we have the desired result. //

D 5. The only connected graph Γ of an a.s. which contains a triple line is the Coxeter graph

G_2 :  .

Proof. This follows from D 4 . //

D 6. Let $\{u_1,\ldots,u_k\} \subset \mathfrak{A}$ have subgraph

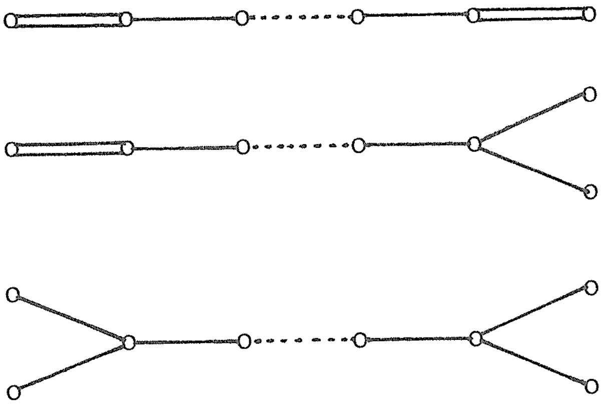


(a simple chain in Γ) . If $\mathfrak{A}' = (\mathfrak{A} - \{u_1,\ldots,u_k\}) \cup \{u\}$, $u = \Sigma_1^k u_i$, then $\mathfrak{A}'$ is an a.s.

Proof. We have $2(u_i, u_{i+1}) = -1$, $i = 1,\ldots,k-1$. Thus $(u,u) = k + 2\Sigma_{i<j} (u_i, u_j)$. Since there are no cycles, $(u_i, u_j) = 0$ if $i < j$ unless $j = i + 1$. Hence $(u, u) = k - (k - 1) = 1$ and u is a unit vector. Let $v \in \mathfrak{A}$, $v \neq u_i$. Since there are no cycles, v is joined to at most one of the u_i , say u_j . Then

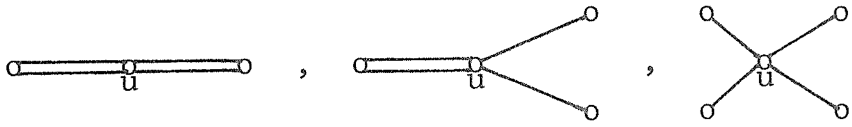
$(v,u) = (v, \Sigma_1^k u_i) = (v, u_j)$ and $4(v,u)^2 = 4(v, u_j)^2 = 0,1,2,$ or 3 . Hence $\mathfrak{A}'$ is an a.s. //

D 7. Γ contains no subgraph of the form



Proof. These graphs contain simple chains whose vertices are denoted by $u_1,\ldots,u_k$ as in D 6. Also, let

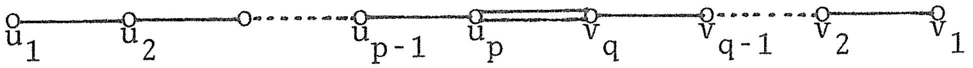
$u = \Sigma_1^k u_i$. As in the proof of D 6, if $v \in \mathfrak{A}$, $v \neq u_i$ and v is joined to u_j then $4(v,u)^2 = 4(v,u_j)^2$. Hence the graph of $\mathfrak{A}'$ in D 6 contains subgraphs of the form

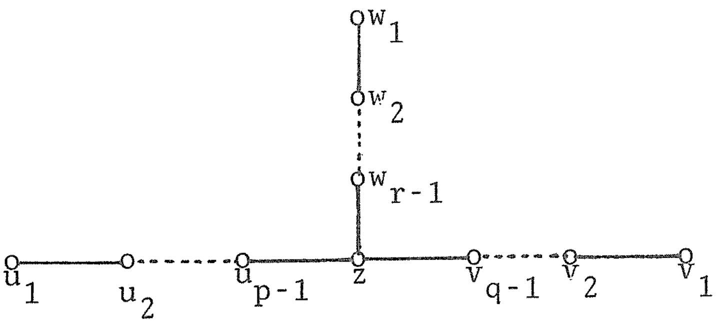


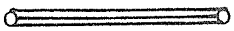
Since $\mathfrak{A}'$ is an a.s., this violates D 4. //

D 8. Any conncted graph Γ of an a.s. has one of the following forms

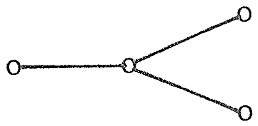
 ; A_ℓ





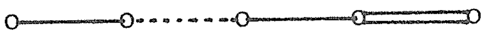
 ; G_2

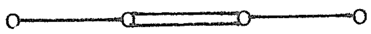
<u>Proof</u>. If Γ contains a triple line, it must be G_2 by D 5. If Γ contains a double line, it contains only one such line by D 7. If Γ has a double line, it can not have a node (branch point)



(again by D 7). Let Γ have only single line ; if Γ has no node, it must be a simple chain since there are no cycles. Also, by D 7 Γ cannot contain more than one node. Thus the four graphs above are the only possiblities

<u>D 9</u>. The only connected Γ of the second type in D 8 are the Coxeter graphs $B_\ell (= C_\ell)$ and F_4 :

 ; $B_\ell = C_\ell$

 ; F_4

<u>Proof</u>. Set $u = \Sigma_1^p iu_i$ and $v = \Sigma_1^q jv_j$. Since $2(u_i,u_{i+1}) = 2(v_j,v_{j+1}) = -1$, $(u,u) = \Sigma_1^p i^2 - \Sigma_1^{p-1} i(i+1)$ $= p^2 - p(p-1)/2 = p(p+1)/2$. Likewise $(v,v) = q(q + 1$ Also, $(u,v) = pq(u_p,v_q) = (pq/2)2(u_p,v_q)$. Since

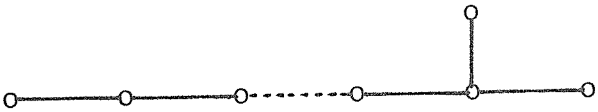
$4(u_p, v_q)^2 = 2$, this gives $(u,v)^2 = p^2q^2/2$. Thus by Schwarz's inequality, $p^2q^2/2 < [p(p+1)/2][q(q+1)/2]$. Since $pq > 0$, this gives $(p+1)(q+1) > 2pq$, which is equivalent to $(p-1)(q-1) < 2$. Hence the only possibilities for $p, q > 0$ are

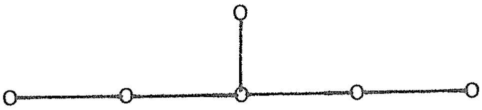
$p = 1$, q arbitrary ; $q = 1$, p arbitrary,

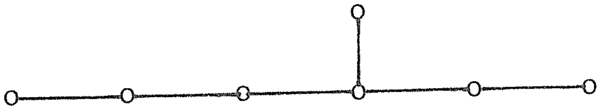
$p = q = 2$.

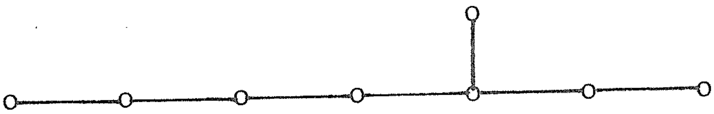
The first two cases give the Coxeter graphs $B_\ell = C_\ell$ while the second gives F_4 . //

<u>D 10</u>. The only connected Γ of the third type in D 8 are Coxeter graphs D_ℓ , E_6 , E_7 and E_8 :

 ; D_ℓ

 ; E_6

 ; E_7

 ; E_8

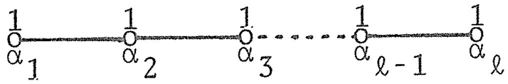
Proof. Set $u = \Sigma_1^{p-1} iu_i$, $v = \Sigma_1^{q-1} jv_j$ and $w = \Sigma_1^{r-1} kw_k$. Then u, v, w are mutually orthogonal and z, u, v, w are linearly independent. As in the proof of D 4, find a unit vector u_0 orthogonal to u, v, w . Then $z = (z,u_0)u_0 + [(z,u)/(u,u)]u + [(z,v)/(v,v)]v + [(z,w)/(w,w)]w$ where $(z,u_0) \neq 0$. Since $(z,z) = 1$, this implies $1 = (z,u_0)^2 + \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3$ or $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 < 1$ where $\theta_1, \theta_2, \theta_3$ are respectively angles between z and u, v and w . Now $\cos^2\theta_1 = (u,z)^2/(u,u) = \frac{1}{4}(p-1)^2/(p(p-1)/2)$ (in the proof of D 9, replace p by $p-1$) $= (p-1)/2p = \frac{1}{2}(1 - \frac{1}{p})$. Similarly, $\cos^2\theta_2 = \frac{1}{2}(1 - 1/q)$ and $\cos^2\theta_3 = \frac{1}{2}(1 - 1/r)$, so that we have

$$\tfrac{1}{2}(1 - 1/q + 1 - 1/q + 1 - 1/r) < 1 \quad \text{or} \quad p^{-1} + q^{-1} + r^{-1} > 1$$

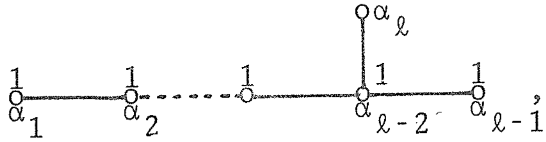
We may assume $p \geqq q \geqq r$ ($\geqq 2$) . Then $p^{-1} \leqq q^{-1} \leqq r^{-1}$ and so $3r^{-1} > 1$. Since $r \geqq 2$, this gives $r = 2$. Thus $p^{-1} + q^{-1} > \frac{1}{2}$, so $2q^{-1} > \frac{1}{2}$ and $q < 4$. Hence $2 \leqq q < 4$. If $q = 2$ then $p^{-1} > 0$ which holds for all p . If $q = 3$ then $p^{-1} + q^{-1} > \frac{1}{2}$ gives $p^{-1} > 1/6$ and $p < 6$. Hence, in this case, $p = 3,4,5$. Thus the possible triples (p,q,r) are : $(p,2,2)$ for D_ℓ ; $(3,3,2)$ for E_6 ; $(4,3,2)$ for E_7 : $(5,3,2)$ for E_8 . //

D 1 - D 10 show that the connected graphs of a.s. of vectors in E are among the Coxeter graphs of types A - G . In particular, the Coxeter graph of an irreducible simple system of roots $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ must be one of these types. Therefore, for the classification of the connected Dynkin diagrams, it remains to introduce the weights.

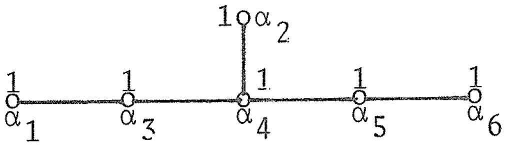
As before, let a shortest root in Π have weight 1 . If the Coxeter graph has only single lines then $A_{ij}A_{ji} = (\alpha_i, \alpha_i)/(\alpha_j, \alpha_j) = 1$ if α_i is joined to α_j . Thus all $(\alpha_i, \alpha_i) = 1$ by the connectedness of the graph. Hence the Dynkin diagrams for A_ℓ, D_ℓ, E_6, E_7, E_8 are

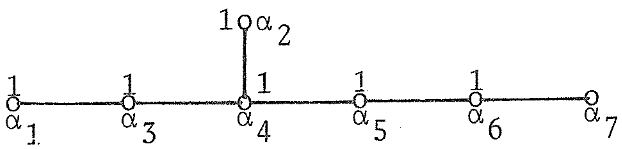
A_ℓ : 

, $\ell \geq 1$

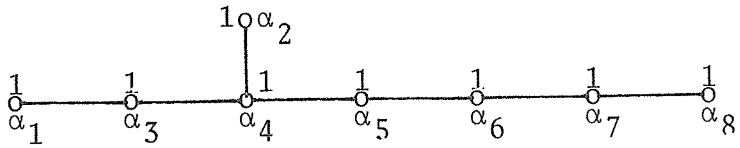
D_ℓ : 

, $\ell \geq 4$

E_6 : 

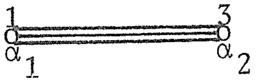
E_7 : 

E_8 :



For G_2, if we let $(\alpha_2, \alpha_2) = 3$, $(\alpha_1, \alpha_1) = 1$. Thus

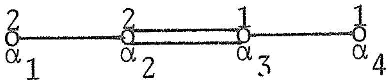
G_2 :



.

For F_4, if we let $(\alpha_3, \alpha_3) = 1$ then $A_{23}A_{32} = 2$ implies $(\alpha_2, \alpha_2)/(\alpha_3, \alpha_3) = 2$ or $(\alpha_2, \alpha_2) = 2$. Hence

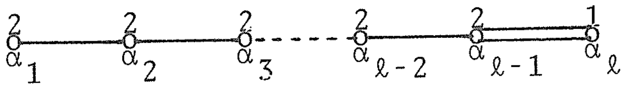
F_4 :



.

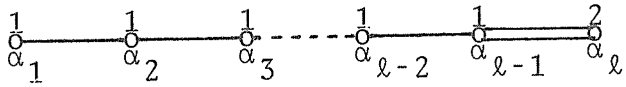
For B_ℓ and C_ℓ, we have

B_ℓ :



, $\ell \geqq 2$

C_ℓ :



, $\ell \geqq 3$

We summarize these in

<u>Theorem 4.6.1</u>. The only connected Dynkin diagrams are $A_\ell (\ell \geqq 1)$, $B_\ell (\ell \geqq 2)$, $C_\ell (\ell \geqq 3)$, $D_\ell (\ell \geqq 4)$ and the five "exceptional" ones, G_2, F_4, E_6, E_7, E_8. //

Exercise 4.6.1. Verify the following table of the Cartan matrices and their determinants. (Note that the Cartan matrix depends on orderings of Π.)

Cartan matrix	determinant
A_ℓ : $\begin{pmatrix} 2 & -1 & 0 & & & \cdot & \cdot & \cdot & & 0 \\ -1 & 2 & -1 & 0 & & \cdot & \cdot & \cdot & & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdot & \cdot & \cdot & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & -1 & 2 \end{pmatrix}$	$\ell + 1$
B_ℓ : $\begin{pmatrix} 2 & -1 & 0 & \cdot & \cdot & \cdot & & & & 0 \\ -1 & 2 & -1 & 0 & \cdot & \cdot & & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -2 \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & 0 & -1 & 2 \end{pmatrix}$	2
C_ℓ : $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ 0 & -1 & 2 & -1 & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & \cdot & \cdot & -1 & 2 & -1 \\ 0 & 0 & & & \cdot & \cdot & \cdot & 0 & -2 & 2 \end{pmatrix}$	2
D_ℓ : $\begin{pmatrix} 2 & -1 & 0 & & \cdot & \cdot & \cdot & & & 0 \\ -1 & 2 & -1 & & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdot & \cdot & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & & \cdot & \cdot & & -1 & 2 & -1 & -1 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 2 & 0 \\ 0 & 0 & & \cdot & \cdot & & 0 & -1 & 0 & 2 \end{pmatrix}$	4
E_6 : $\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	3

	Cartan matrix	determinant
E_7 :	$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$	2
E_8 :	$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$	1
F_4 :	$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$	1
G_2 :	$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$	1

4.7. Construction of the algebras

Having classified all connected Dynkin diagrams, we here construct simple Lie algebras whose Dynkin diagrams are those types described in Section 4.6. We first construct the simple algebras of types $A_\ell (\ell \geq 1)$, $B_\ell (\ell \geq 2)$, $C_\ell (\ell \geq 3)$ and $D_\ell (\ell \geq 4)$ which are called <u>classical</u>, because they correspond to the linear Lie groups that Weyl has called the classical groups. These algebras are the only infinite classes of simple Lie algebras. The construction of the five exceptional simple Lie algebras of types G_2, F_4, E_6, E_7, E_8 is more sophisticated and related to simple alternative and Jordan algebras. The treatment here follows Jacobson (Lie algebra, Interscience, New York, 1962).

In each of the constructions, we first show that the algebra is semisimple. The simplicity then follows from irreducibility of simple system of roots, or equivalently, from the connectedness of the corresponding Dynkin diagram (Theorem 4.5.7). For this, Theorem 3.7.11 and the following are fundamental. Let L be semisimple and let L contain an abelian subalgebra H such that $L = H \oplus \Sigma_\alpha \oplus Fx_\alpha$ where α represents a nonzero mapping of H into F such that $[x_\alpha h] = \alpha(h)x_\alpha$, $h \in H$. Then H is a split CSA of L. This follows from the fact that the centralizer of H in L is H and hence H is a maximal torus of L (see Theorem 4.2.2).

<u>$A_\ell (\ell \geq 1)$</u>. Let V be an $(\ell + 1)$-dimensional vector space over F. We identify $\mathrm{Hom}_F V = F_{\ell+1}$, the algebra

of $(\ell + 1) \times (\ell + 1)$ matrices over F. Let $L = s\ell(\ell + 1, F) = \{x \in F_{\ell+1} \mid \mathrm{Tr}\, x = 0\}$. By comparing dimension, one sees that $\mathrm{Hom}\, V = s\ell(\ell + 1, F) \oplus F1$. We have observed in Section 3.7 that $s\ell(\ell + 1, F) = L$ is semisimple. Choose a basis for L as

$$h_k = e_{kk} - e_{\ell+1,\ell+1}, \quad k \leq \ell \; ; \; e_{ij}, \quad i \neq j = 1, \ldots, \ell+1 .$$

Set $h = \Sigma_1^\ell \omega_k h_k$. Then the set of the h's is an abelian subalgebra H of dimension ℓ and we have

$$[e_{rs}, h] = (\omega_s - \omega_r) e_{rs},$$

$$[e_{\ell+1,r}, h] = (\gamma + \omega_r) e_{\ell+1,r}, \quad \gamma = \Sigma_1^\ell \omega_i,$$

$$[e_{r,\ell+1}, h] = -(\gamma + \omega_r) e_{r,\ell+1}, \quad r \neq s = 1, \ldots, \ell,$$

where the e_{ij} are the usual matrix units. The $\ell^2 + \ell$ linear functions $h \to \omega_s - \omega_r$, $h \to \gamma + \omega_r$, $h \to -(\gamma + \omega_r$ are distinct and are nonzero weights of $\mathrm{ad}_L H$. This gives $L = H + \Sigma_\alpha L_\alpha$, α running over these weights. It then follows that H is a split CSA and the α are now the nonzero roots. Set

$$\alpha_1 = \omega_1 - \omega_2, \quad \alpha_2 = \omega_2 - \omega_3, \ldots, \alpha_{\ell-1} = \omega_{\ell-1} - \omega_\ell,$$

$$\alpha_\ell = \gamma + \omega_\ell .$$

Then $\alpha_k + \alpha_{k+1} + \ldots + \alpha_\ell = \gamma + \omega_k$, $k = 1,\ldots,\ell-1$ and $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j = \omega_i - \omega_{j+1}$, $1 \leqq i \leqq j \leqq \ell - 1$. This shows that every root has the form $\Sigma k_i\alpha_i$ where the k_i are integers and $k_i \geqq 0$ for all i or $k_i \leqq 0$ for all i. Thus the α_i form a simple system of roots by Corollary 4.5.3. The same relation also shows that $\alpha_i + \alpha_{i+1}$ is a root, $1 \leqq i \leqq \ell - 1$, while $\alpha_i + 2\alpha_{i+1}$ is not a root and $\alpha_i + \alpha_j$ is not a root for $j > i + 1$. Hence $A_{i+1,i} = A_{i,i+1} = -1$, $1 \leqq i < \ell$ and $A_{ij} = 0$ for $j > i + 1$ or $j < i - 1$. Thus the Dynkin diagram is the connected simple chain of type A_ℓ and L is simple by Theorem 4.5.7.

<u>$B_\ell (\ell \geqq 1)$.</u> Let $\dim V = 2\ell + 1$ and let (x,y) be a symmetric nondegenerate bilinear form on V whose matrix is

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_\ell \\ 0 & 1_\ell & 0 \end{pmatrix} .$$

The orthogonal algebra $\mathfrak{O}(V)$ or $\mathfrak{O}(2\ell + 1, F)$ consists of all elements a in $\mathrm{Hom}\, V$ such that $(va, w) = -(v, wa)$ for $v,w \in V$. Denote $L = \mathfrak{O}(V)$ and partition any element $a \in L$ in the same way as s:

$$a = \begin{pmatrix} \alpha_{11} & v_1 & v_2 \\ u_1 & a_{11} & a_{12} \\ u_2 & a_{21} & a_{22} \end{pmatrix} .$$

The orthogonal condition is equivalent to the relation $as = -sa'$ which in turn translates into the following set of conditions, where a' denotes the transpose of a.

$$u_1 = -v_2' , \quad u_2 = -v_1' , \quad a_{22} = -a_{11}' ,$$
$$a_{12}' = -a_{12} , \quad a_{21}' = -a_{21} , \quad \alpha_{11} = 0 . \tag{1.50}$$

In particular, (1.50) shows that $\mathrm{Tr}\, a = 0$ for $a \in L$ where we have identified $\mathrm{Hom}\, V = F_{2\ell+1}$. The relations (1.50) imply that L has a basis consisting of the elements

$$h_i = e_{i+1,i+1} - e_{i+\ell+1,i+\ell+1} , \tag{1.51a}$$

$$e_{\omega_i-\omega_j} = e_{j+1,i+1} - e_{i+\ell+1,j+\ell+1} , \quad i \neq j \tag{1.51b}$$

$$e_{\omega_i+\omega_j} = e_{i+\ell+1,j+1} - e_{j+\ell+1,i+1} , \quad i < j \tag{1.51c}$$

$$e_{-\omega_i-\omega_j} = e_{j+1,i+\ell+1} - e_{i+1,j+\ell+1} , \quad i < j \tag{1.51d}$$

$$e_{\omega_i} = e_{1,i+1} - e_{i+\ell+1,1} , \tag{1.51e}$$

$$e_{-\omega_i} = e_{i+1,1} - e_{1,i+\ell+1} \tag{1.51f}$$

where $i,j = 1,\ldots,\ell$ and we have set $h = \Sigma_1^\ell \omega_k h_k$.

In fact (1.51a) comes from the relation $a_{22} = -a_{11}'$. (1.51e) and (1.51f) correspond to $u_2 = -v_1'$ and $u_1 = -v_2'$. (1.51b) is from $a_{22} = -a_{11}'$ while (1.51c) follows from $a'_{21} = -a_{21}$ Finally, (1.51d) is obtained by $a'_{12} = -a_{12}$. The total number of basis elements is $2\ell^2 + \ell$.

We show that V is L-irreducible. Note that a subspace W of V is L-stable if and only if W is stable under the (associative) subalgebra L^* of Hom V generated by 1 and L. Since Hom V acts irreducibly on V, it suffices to show that $L^* =$ Hom V. From the unit matrix 1 we get all scalars. From the matrices in (1.51a) we can then generate all diagonal matrices. Then multiplying various matrices in (1.51b-f) by suitable e_{ii} gives all the off-diagonal matrix units e_{ij}. Hence $L^* =$ Hom V and by Theorem 3.7.11(2) L is semisimple since $L \subset s\ell(V)$.

The linear forms which are subscripts in (1.51) can be identified with the linear mappings $h \to \alpha(h)$, where $h = \Sigma\omega_i h_i$ and we get $[e_\alpha h] = \alpha(h)e_\alpha$ where $\alpha = \omega_i - \omega_j$, $\omega_i + \omega_j$, etc. Thus H is a split CSA and the α are the nonzero roots. We now assume $\ell \geqq 2$ and we set

$$\alpha_1 = \omega_1 - \omega_2 ,$$

$$\alpha_2 = \omega_2 - \omega_3 ,$$

$$\cdots\cdots\cdots$$

$$\alpha_{\ell-1} = \omega_{\ell-1} - \omega_\ell ,$$

$$\alpha_\ell = \omega_\ell .$$

One checks that this is a simple system of roots with

Dynkin diagram of type B_ℓ . Hence L is simple of type $B_\ell (\ell \geqq 2)$.

<u>$C_\ell (\ell \geqq 1)$.</u> Let $\dim V = 2\ell$, with basis $(v_1,\ldots,v_{2\ell})$ Define a nondegenerate skew-symmetric form (x,y) on V by the matrix

$$q = \begin{pmatrix} 0 & 1_\ell \\ -1_\ell & 0 \end{pmatrix} .$$

(It can be shown that even dimensionality is a necessary condition for existence of such a form). Denote by $L \equiv sp(V)$ or $sp(2\ell,F)$, called the <u>symplectic algebra</u>, the set of elements $a \in \operatorname{Hom} V$ such that $(va,w) = -(v,wa)$ $v,w \in V$. If we identify the elements $a \in L$ with the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \quad a_{ij} \in F_\ell ,$$

then the symplectic condition is equivalent to the condition $aq = -qa'$, which implies

$$a_{22} = -a_{11}' , \quad a_{12}' = a_{12} , \quad a_{21}' = a_{21} .$$

Hence $\operatorname{Tr} a = 0$ for $a \in L$ and L has the basis

$$h_i = e_{ii} - e_{\ell+i,\ell+i} ,$$

$$e_{\omega_j-\omega_i} = e_{ij} - e_{j+\ell,i+\ell}\ , \quad i \neq j$$

$$e_{-\omega_i-\omega_j} = e_{i,j+\ell} + e_{j,i+\ell}\ , \quad i < j$$

$$e_{\omega_i+\omega_j} = e_{i+\ell,j} + e_{j+\ell,i}\ ,$$

$$e_{-2\omega_i} = e_{i,i+\ell}\ ,$$

$$e_{2\omega_i} = e_{i+\ell,i}\ ,$$

where $i,j = 1,2,\ldots,\ell$. As before, we can show that $H = \{\Sigma\omega_i h_i\}$ is a split CSA of L and $\alpha = \omega_j - \omega_i$, $-\omega_j - \omega_i$, etc. are the nonzero roots. As for B_ℓ , V is L-irreducible and L is semisimple. If $\ell \geqq 3$ then the roots

$$\alpha_1 = \omega_1 - \omega_2, \ldots, \alpha_{\ell-1} = \omega_{\ell-1} - \omega_\ell\ ,$$

$$\alpha_\ell = 2\omega_\ell$$

form a simple system of roots whose Dynkin diagram is connected and of type C_ℓ . Hence L is simple of type C_ℓ.

<u>$D_\ell (\ell \geqq 2)$</u>. Let $\dim V = 2\ell$, with basis $(v_1, v_2, \ldots, v_{2\ell})$. The construction is identical with that for B_ℓ , except that $\dim V = 2\ell$ is even. Thus the orthogonal algebra $L \equiv \mathcal{O}(2\ell, F)$ consists of matrices $a \in F_{2\ell}$ such that $at = -ta'$ where

$$t = \begin{pmatrix} 0 & 1_\ell \\ 1_\ell & 0 \end{pmatrix} .$$

Hence L is the set of matrices

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} , \quad a_{ij} \in F_\ell$$

such that $a_{22} = -a_{11}'$, $a_{12}' = -a_{12}$ and $a_{21}' = -a_{21}$.

Thus L has the basis

$$h_i = e_{ii} - e_{\ell+i,\ell+i} ,$$

$$e_{\omega_i-\omega_j} = e_{ji} - e_{i+\ell,j+\ell} , \quad i \neq j$$

$$e_{\omega_i+\omega_j} = e_{i+\ell,j} - e_{j+\ell,i} , \quad i < j$$

$$e_{-\omega_i-\omega_j} = e_{j,i+\ell} - e_{i,j+\ell} , \quad i < j$$

where $i,j = 1,\ldots,\ell$. $H = \{\Sigma\omega_i h_i\}$ is a split CSA and the subscripts are the nonzero roots. As before, we can show that V is L-irreducible and so L is semisimple since $\mathrm{Tr}\ a = 0$ for $a \in L$. If $\ell \geqq 4$ then

$$\alpha_1 = \omega_1 - \omega_2,\ldots,\alpha_{\ell-1} = \omega_{\ell-1} - \omega_\ell , \quad \alpha_\ell = \omega_{\ell-1} + \omega_\ell$$

form a simple system of roots which has the Dynkin diagram D_ℓ . Hence L is simple of type D_ℓ .

Each algebra of types $A_\ell(\ell \geqq 1)$, $B_\ell(\ell \geqq 2)$, $C_\ell(\ell \geqq 3)$ and $D_\ell(\ell \geqq 4)$ is unique up to isomorphism and no two of these algebras are isomorphic.

This follows from Exercise 4.5.2 and the classification of connected Dynkin diagrams. When the restriction on ℓ is relaxed, we have the following isomorphism relations for the low dimensional orthogonal and symplectic algebras.

Exercise 4.7.1. Prove that $A_1 \simeq B_1 \simeq C_1$, $B_2 \simeq C_2$, $A_3 \simeq D_3$ and $D_2 \simeq A_1 \oplus A_1$.

The construction of the five exceptional simple algebras is related to simple alternative and Jordan algebras. Thus a complete discussion will take us too far afield. Here we outline the construction for G_2, F_4, and E_6. The situation for E_7 and E_8 is more complicated ; however, we intend to state the results of a "remarkable" uniform method by Tits for the construction of the exceptional algebras.

To proceed, we state some definitions in order. Recall that an algebra A is called alternative if it satisfies the alternative law $x^2y = x(xy)$ and $yx^2 = (yx)x$. An algebra A over F with unit element 1 is called a composition algebra if it has a nondegenerate quadratic form $n(x)$ on A satisfying the composition law $n(xy) = n(x)n(y)$, $x \in A$.

Exercise 4.7.2. Prove that any composition algebra A(of char $\neq 2$) is a quadratic algebra with involution

$x \to \bar{x}$ in the sense that every element $x \in A$ satisfies the quadratic equation

$$x^2 - t(x)x + n(x)1 = 0 ,$$

$x \to \bar{x}$ linear, $\overline{xy} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$, where $t(\alpha 1) = 2\alpha$ and $n(\alpha 1) = \alpha^2, \alpha \in F$. In this case, $t : A \to F$ is linear and is called the <u>trace form</u> of A.

<u>Exercise 4.7.3</u>. Prove that any simple alternative quadratic algebra (char $\neq 2$) of finite dimension is necessarily a composition algebra.

We construct a well-known simple alternative algebra. Let V be a 3-dimensional vector space over F with basis i, j, k. Let $(\, , \,)$ be the inner product on V and define an anticommutative multiplication $a \times b$ on V by

$$i \times i = j \times j = k \times k = 0 ,$$

$$i \times j = k , \quad j \times k = i , \quad k \times i = j .$$

In this way, V becomes a simple Lie algebra. Now, let C be the set of 2×2 matrices

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} , \quad \alpha, \beta \in F , \quad a, b \in V .$$

Define a multiplication in C as

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix}\begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma - (a,d) & \alpha c + \delta a + b \times d \\ \gamma b + \beta d + a \times c & \beta\delta - (b,c) \end{pmatrix} .$$

The remaining operations are defined in the obvious manner. The 8-dimensional algebra C so defined is called the <u>split Cayley algebra</u> or <u>vector matrix algebra</u>.

<u>Exercise 4.7.4</u>. (1) Prove that C is a simple alternative quadratic algebra with trace $t(x) = \alpha + \beta$ where α, β are the diagonal entries of x .

(2) Show that $C_0 = \{x \in C \mid t(x) = 0\}$ is equal to $[C,C] = \{ [x,y] = xy - yx \mid x,y \in C\}$.

Due to the following famous theorem of Hurwitz, C is the only nonassociative composition algebra.

<u>Exercise 4.7.5</u>. Any composition algebra A over F (of char $\neq 2$) is one of the following : $F1$; $F \oplus F$; a separable quadratic field K over F ; a quaternion algebra Q over F ; a Cayley algebra C over F . Hence the possible dimensions for A are 1, 2, 4, 8 .

<u>Exercise 4.7.6</u>. Show that, in any alternative algebra A , the mappings

$$D_{a,b} = [L_a, L_b] + [L_a, R_b] + [R_a, R_b] , \quad a,b \in A$$

are derivations on A and that $D_{a,b} = -D_{b,a}$.

Exercise 4.7.7. Let V be the 3-dimensional space defined above. Let $T \in \mathrm{Hom}\, V$ be of trace 0 and let T^* be the adjoint of T relative to $(,)$. Prove that the mapping

$$\begin{bmatrix} \alpha & a \\ b & \beta \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & aT \\ -bT^* & 0 \end{bmatrix}$$

is a derivation on C and the set L_0 of these derivations is isomorphic to $A_2 = s\ell(3,F)$.

G_2. We retain the previous notations. Let $L = \mathrm{Der}\, C$, the Lie algebra of derivations on C . Since $C_0 = [C,C]$ (Exercise 4.7.4), $C = C_0 \oplus F1$ and $C_0 L \subset C_0$. Also, $CL \subset C_0$ since $1L = 0$. Thus L acts faithfully on the 7-dimensional space C_0 . Moreover, it is shown that L acts irreducibly on C_0 and that any derivation in L has the form $D_{e_1,a_{12}} + D_{e_2,b_{21}} + D_0$ where

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} , \qquad e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$a_{12} = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} , \qquad b_{21} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$

and $D_{a,b}$ and L_0 are the same as in Exercises 4.7.6 and 4.7.7, respectively. Then $\operatorname{Tr} D = 0$ for $D \in L$ and hence L is semisimple by Theorem 3.7.11. We identify L_0 with $A_2 = s\ell(3,F)$. If H is a CSA of L_0, H is also a CSA of L. Therefore, we can take H to be the set of matrices of the form $\omega_1 h_1 + \omega_2 h_2$, $h_1 = e_{11} - e_{33}$, $h_2 = e_{22} - e_{33}$. Then H is a CSA of L and the nonzero roots are : $\pm\omega_1$, $\pm\omega_2$, $\pm(\omega_1 - \omega_2)$, $\pm(\omega_1 + \omega_2)$, $\pm(2\omega_1 + \omega_2)$, $\pm(\omega_1 + 2\omega_2)$. The roots $\alpha_1 = \omega_1 - \omega_2$, $\alpha_2 = \omega_2$ form a simple system of roots with Dynkin diagram G_2. Hence L is simple of type G_2.

Recall that an algebra J over F is called <u>Jordan</u> if it is commutative, i.e., $x \cdot y = y \cdot x$, and satisfies the Jordan identity $[(x \cdot x) \cdot y] \cdot x = (x \cdot x) \cdot (y \cdot x)$. Note that the Cayley algebra C has an involution $x \to \bar{x}$ such that $\bar{1} = 1$ and $\bar{x} = -x$ for $x \in C_0$. Let M_3^8 be the set of 3×3 matrices of the form

$$x = \begin{bmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{bmatrix}, \quad \xi_i \in F, \quad a,b,c \in C$$

which are hermitian relative to the involution. If $x,y \in M_3^8$, then $x \cdot y \equiv \frac{1}{2}(xy + yx) \in M_3^8$ where xy is the usual matrix product. It is shown that M_3^8 with

$x \cdot y$ becomes a central simple Jordan algebra. An algebra A over F is called <u>central simple</u> if A_K is simple for any scalar extension K of F. M_3^8 is the only "exceptional" central simple Jordan algebra (of char $\neq 2$), because all other central simple Jordan algebras are isomorphic to subalgebras of A^+ for some associative algebra A but M_3^8 is not.

<u>Exercise 4.7.8</u>. In any Jordan algebra J, the mapping $D_{x,y} = [R_x, R_y]$ for $x,y \in J$ is a derivation on J.

<u>Exercise 4.7.9</u>. Let L_0 be the set of derivations on M_3^8 which kill $e_1 = \mathrm{diag}(1,0,0)$, $e_2 = \mathrm{diag}(0,1,0)$, $e_3 = \mathrm{diag}(0,0,1)$. Show that the Lie algebra L_0 is isomorphic to D_4, the orthogonal algebra $\mathcal{O}(8,F)$.

<u>Exercise 4.7.10</u>. Prove that $\mathrm{Tr}\, R_x = 9\mathrm{Tr}\, x$ for $x \in M_3^8$ where $\mathrm{Tr}\, x = \xi_1 + \xi_2 + \xi_3$. Use this to show that $M_3^8 D \subset J_0$ for all $D \in \mathrm{Der}\, M_3^8$, where

$J_0 = \{x \in M_3^8 \mid \mathrm{Tr}\, x = 0\}$.

<u>Exercise 4.7.11</u>. Prove that every derivation D on M_3^8 is inner (char $\neq 2, 3$):

$$D = \Sigma [R_{x_i}, R_{y_i}], \quad x_i, y_i \in M_3^8 .$$

<u>F_4</u>. Let $L = \text{Der } M_3^8$ and let J_0 be the same as in Exercise 4.7.10. Then it is shown that L acts faithfully and irreducibly on J_0. In view of Exercise 4.7.11, $\text{Tr } x = 0$ for $x \in L$ and by Theorem 3.7.11 L is semisimple. When L_0 is identified with D_4 by Exercise 4.7.9, then the CSA H of L_0 corresponding to the CSA of D_4 is a split CSA for L. Then the nonzero roots of H in L are $\pm\omega_i$, $\pm\omega_i\pm\omega_j$, $\pm\Lambda_i$, $\pm M_i$, where

$$\Lambda_i = \tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4) - \omega_i ,$$

$$M_1 = \tfrac{1}{2}(\omega_1 + \omega_2 + \omega_3 + \omega_4) ,$$

$$M_2 = \tfrac{1}{2}(\omega_1 + \omega_2 - \omega_3 - \omega_4) ,$$

$$M_3 = \tfrac{1}{2}(\omega_1 - \omega_2 + \omega_3 - \omega_4) ,$$

$$M_4 = \tfrac{1}{2}(\omega_1 - \omega_2 - \omega_3 + \omega_4) .$$

The roots $\alpha_1 = \omega_4$, $\alpha_2 = \omega_3 - \omega_4$, $\alpha_3 = \omega_4$, $\alpha_4 = \tfrac{1}{2}(\omega_1 - \omega_2 - \omega_3 - \omega_4)$ form a simple system of roots with Dynkin diagram of type F_4. Thus L is simple of type F_4. Note that there are exactly 48 roots and $\dim L = 52$.

<u>Exercise 4.7.12</u>. Prove that $\text{Der } M_3^8 + R_{J_0}$ acts irreducibly on M_3^8 (char $\neq 2,3$) where J_0 is as in Exercise 4.7.10 and $R_{J_0} = \{R_x \mid x \in J_0\}$.

<u>E_6</u>. Let $L = \text{Der}\, M_3^8 + R_{J_0}$. By Exercise 4.7.8, $[R_x, R_y] \in \text{Der}\, M_3^8$ for $x, y \in M_3^8$ and $[R_x\ D] = R_{xD}$ for $D \in M_3^8$. Thus $[L, L] \subset L$ and L is a Lie algebra. Since $\text{Tr}\ z = 0$ for $z \in L$, by Exercise 4.7.12 L is semisimple. Let H be the CSA of $\text{Der}\, M_3^8$ (of F_4). Then $H' = FR_{e_1 - e_3} + FR_{e_2 - e_3} + H$ is a split CSA of L where e_1, e_2, e_3 are the same as in Exercise 4.7.9. The nonzero roots of H' in L are $\pm\omega_i \pm\omega_j$, $i < j = 1, 2, 3, 4$, $\pm(\omega_i \pm \frac{1}{2}(\omega_6 - \omega_5))$, $\pm(\Lambda_i \pm \frac{1}{2}(\omega_7 - \omega_5))$, $\pm(M_i \pm \frac{1}{2}(\omega_7 - \omega_6))$, where M_i and Λ_i are as in type F_4. The roots $\alpha_1 = -\omega_1 + \frac{1}{2}(\omega_6 - \omega_5)$, $\alpha_2 = \omega_3 - \omega_4$, $\alpha_3 = \omega_1 - \omega_2$, $\alpha_4 = \omega_2 - \omega_3$, $\alpha_5 = \omega_3 + \omega_4$, $\alpha_6 = -M_1 + \frac{1}{2}(\omega_7 - \omega_6)$ form a simple system of roots with Dynkin diagram of type E_6. Hence L is simple of type E_6. Notice that there are 72 roots and $\dim L = 78$.

Let B be a finite-dimensional algebra over F with a nonzero linear form t and let B have a unit element 1. Assume $t(1) \neq 0$. Thus if we let $B_0 = \{x \in B \mid t(x) = 0\}$, then we have the vector space direct sum $B = B_0 + F1$. Hence, for $x, y \in B$, xy is uniquely expressed as

$$xy = \frac{1}{t(1)}\, t(xy)1 + x{*}y \qquad (1.52)$$

where $x*y \in B_0$, since $t(x*y) = 0$.

<u>The Tits' construction</u>. We outline a partial results of the Tits' construction. Let A be a composition algebra over F of char $\neq 2,3$, so that A is a quadratic algebra with trace form t. Let A have a unit element u. Let $J = M_3^8$ with unit element e. Set

$$A_0 = \{a \in A \mid t(a) = 0\}, \quad J_0 = \{x \in J \mid \mathrm{Tr}\, x = 0\}.$$

Note that $t(u) = 2$ and $\mathrm{Tr}\, e = 3$, and the bilinear forms $\langle x,y\rangle = \mathrm{Tr}\, x\cdot y$ and $(a,b) = t(ab)$ are respectively nondegenerate on J and A. The relation (1.52) turns out as

$$ab = \tfrac{1}{2}\, t(ab)u + a*b, \quad a,b \in A,$$

$$x\cdot y = \frac{1}{3}\,(\mathrm{Tr}\, x\cdot y)e + x*y, \quad x,y \in J.$$

It is easy to check that $a*b$ defines an anticommutative product in A_0 while $x*y$ defines a commutative one in J_0. Let

$$L \equiv \mathrm{Der}\, A + A_0 \otimes_F J_0 + \mathrm{Der}\, J \tag{1.53}$$

be the vector space direct sum of $\mathrm{Der}\, A$, $\mathrm{Der}\, J$ and the tensor product $A_0 \otimes J_0$ over F. We define a multiplication $[\ ,\ ]$ in L which is bilinear and anticommutative, which agrees with the ordinary Lie product in $\mathrm{Der}\, A$ and $\mathrm{Der}\, J$, and which satisfies

$$[\mathrm{Der}\ A\ ,\ \mathrm{Der}\ J] = 0\ ,$$

$$[a \otimes x,\ D] = aD \otimes x,\ D \in \mathrm{Der}\ A,\ a \in A_0,\ x \in J_0,$$

$$[a \otimes x,\ D] = a \otimes xD,\ D \in \mathrm{Der}\ J,\ a \in A_0,\ x \in J_0,$$

$$[a \otimes x,\ b \otimes y] = \frac{1}{12}\,\mathrm{Tr}(xy)D_{a,b} + (a*b) \otimes (x*y)$$

$$+ \tfrac{1}{2}\, t(ab)\ [R_x,\ R_y]\ ,$$

$$a,b \in A_0\ ;\ x,y \in J_0\ ,$$

where $D_{a,b}$ is as in Exercise 4.7.6. Utilizing the fact that $D_{a,b} = -D_{b,a}$ and $a*b = -b*a$ for $a,b \in A_0$, it can be shown that L is a Lie algebra under the product just defined.

Let K be the algebraic closure of F. By Exercise 4.7.5 the composition algebra $\bar{A} \equiv A_K$ is one of the following : Ku, $Ku \oplus Kv_1$ with $v_1^2 = u$, a split quaternion algebra Q, a split Cayley algebra C. Then the corresponding Lie algebra L_K formed by (1.53) is listed as :

	Ku	$Ku + Kv_1$	Q	C
L_K	F_4	E_6	E_7	E_8

Hence L is central simple. Here we note that

$\mathrm{Der}(Ke) = \mathrm{Der}(Ku + Kv_1) = 0$ while $\mathrm{Der}(Q) \cong s\ell(2)$ and $\mathrm{Der}\, C \cong G_2$. Thus when $A = Ku + Kv_1$ (so $A_0 = Kv_1$), L in (1.53) corresponds to E_6. In fact, $L = Kv_1 \otimes J_0 + \mathrm{Der}\, J$ is isomorphic to $E_6 = R_{J_0} + \mathrm{Der}\, J$ via $v_1 \otimes x + D \to R_x + D$.

In (1.53), J can be replaced by other central simple Jordan algebras. For example, if $J = Ke$ and $A = C$ then L in (1.53) corresponds to G_2. The relation (1.53) enables us to enumerate $\dim L$ via

$$\dim L = \dim \mathrm{Der}\, A + (-1 + \dim A)(-1 + \dim J) + \dim \mathrm{Der}\, J .$$

In particular,

$$\dim E_6 = 0 + 1 \cdot 26 + 52 = 78 ,$$

$$\dim E_7 = 3 + 3 \cdot 26 + 52 = 133 , \quad \text{and}$$

$$\dim E_8 = 14 + 7 \cdot 26 + 52 = 248 .$$

The dimensions of irreducible representations we presented here for $A_\ell - D_\ell$, G_2, F_4 and E_6 are minimal. In fact, E. Cartan has computed the minimal dimension of irreducible representations of the exceptional algebras.

L	dim L	minimal dim. of irreducible representation
A_ℓ	$\ell(\ell+2)$	$\ell + 1$
B_ℓ	$\ell(2\ell+1)$	$2\ell + 1$
C_ℓ	$\ell(2\ell+1)$	2ℓ
D_ℓ	$\ell(2\ell-1)$	2ℓ
G_2	14	7
F_4	52	26
E_6	78	27
E_7	133	56
E_8	248	248

Thus the "simplest" matrix realization of E_8 will involve 248×248 matrices.

Jacobson first identified G_2 with Der C , C a split Cayley algebra (Cayley numbers and normal simple Lie algebras of type G , Duke Math. J. 5(1939), 775-783), while the description of F_4 and E_6 as Der M_3^8 and R_{J_0} + Der M_3^8 is due to Chevalley and Schafer (The exceptional simple Lie algebras F_4 and E_6 , Proc. Nat. Acad. Sci. U.S.A. 36(1950), 137-141). The Tits' construction has been reinvestigated many times ;

1. J. Tits, "Algèbres alternatives, algèbres Jordan et algèbres de Lie exceptionelles, I.", Nederl. Akad.

Wetensch, Proc. Ser. A 69 = Indag. Math. 28(1966), 223-237.

2. J. R. Faulkner and J. C. Ferra, "Exceptional Lie algebras and related algebraic and geometric structures", Bull. London Math. Soc., 9(1977), 1-35.

3. N. Jacobson, "Exceptional Lie algebras", Lecture Notes in Pure and Appl. Math. I, Marcel Dekker, Inc. N.Y., 1971.

4. R. D. Schafer, "An introduction to nonassociative algebras", Academic Press, New York, 1966.

5. R. D. Schafer, "On the simplicity of the Lie algebras, E_7 and E_8", Nederl. Akad. Wetensch. Proc. Ser. A 69 = Indag. Math. 28(1966), 64-69.

The Tits' construction allowed Schafer (Paper 5. cited above) to show that there are indeed simple Lie algebras of types E_7 and E_8 .

5. REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS

5.1. Universal enveloping algebras

We discuss an associative algebra defined by a universal property. This algebra has a unit element and is generated by a Lie algebra L. The purpose is to prove the Poincaré-Birkhoff-Witt Theorem (PBW Theorem) that this algebra has a basis consisting of 1 and the "standard" monomials in an ordered basis of L. Universal enveloping algebras along with PBW Theorem are basic tools for representation theory of Lie algebras. In this section, L denotes a Lie algebra (not necessarily finite-dimensiona over an arbitrary field F.

Definition 5.1.1. An (nonassociative) algebra A is said to be graded if $A = \Sigma_{i=0}^{\infty} \oplus A^i$, where the A^i

are subspaces of A and $A^iA^j \subseteq A^{i+j}$. The elements of A^i are called homogeneous of degree i . Every element $a \in A$ is uniquely expressed as $a = \Sigma a_i$, $a_i \in A^i$, $a_i = 0$ for all but a finite number of i . Then a_i is called the homogeneous part of a of degree i . A left (right) ideal B of A is called homogeneous if $B = \Sigma_i (B \cap A^i)$. An algebra A is said to be filtered if for each $i \in Z^+$ = the set of nonnegative integers, there is defined a subspace A_i such that $A_i \subset A_j$ if $i \leqq j$, $A = \bigcup_i A_i$ and $A_iA_j \subset A_{i+j}$. //

If A is graded as $A = \Sigma_i \oplus A^i$ and we set $A_i = \Sigma_{j \leqq i} A^j$ then this defines a filteration in A . Let A now be an algebra with unit element 1 and let A be generated by a subspace V . Denote by V^n the subspace spanned by all the products of any n elements in V in all possible associations. Thus $A = F1 + V + V^2 + \ldots + V^n + \ldots$. If we set $A_i = F1 + \ldots + V^i$ with $A_0 = F1$ then these A_i define a filteration in A .

Suppose that A is filtered by subspaces A_i . We set $G^m \equiv G^m(A) = A_i/A_{i-1}$ and take the vector space direct sum $G \equiv G(A) = \Sigma_{i=1}^{\infty} \oplus G^i$ where we set $A_{-1} = 0$. A multiplication in G is defined by

$$(x_i + A_{i-1})(x_j + A_{j-1}) = x_ix_j + A_{i+j-1} \qquad (1.54)$$

for $x_i \in A_i$, $x_j \in A_j$. If $x_i \equiv y_i \pmod{A_{i-1}}$ and $x_j \equiv y_j \pmod{A_{j-1}}$ then clearly $x_i x_j \equiv y_i y_j \pmod{A_{i+j-1}}$ and hence (1.54) is well-defined. This makes G a graded algebra, called the graded algebra associated with A .

We turn to special cases of graded algebras. Let V be a vector space over F . Let $T^0 V = F$, $T^1 V = V$, $T^m V = V \otimes \ldots \otimes V$ (m times). Set $T \equiv T(V) = \Sigma_{i=0}^{\infty} \oplus T^i V$ and introduce an associative product in T by the obvious manner, $(v_1 \otimes \ldots \otimes v_k)(w_1 \otimes \ldots \otimes w_m) = v_1 \otimes \ldots \otimes v_k \otimes w_1 \otimes \ldots \otimes w_m \in T^{k+m} V$. This makes T(V) an associative graded algebra with 1 which is generated by 1 and any basis of V . Call T(V) the tensor algebra on V . T(V) is the universal associative algebra on a basis of V in the following sense ; given an associative algebra A with 1 and an F-linear map $\phi : V \to A$, there exists a unique homomorphism of F-algebras $\psi : T(V) \to A$ such that $\psi(1) = 1$ and $\psi i = \phi$ where i is the inclusion map $V \to T(V)$.

Let I be the ideal in T(V) generated by $x \otimes y - y \otimes x$, $x, y \in V$. Call the algebra $S(V) \equiv T(V)/I$ the symmetric algebra on V and denote by $\sigma : T(V) \to S(V)$ the natural map. We note that the generators of I are in $T^2 V$ and hence $I = (I \cap T^2 V) \oplus (I \cap T^3 V) \oplus \ldots$. Thus σ is injective on $T^0 V = F$, $T^1 V = V$. Since I is homogeneous, setting $S^m V = \sigma(T^m)$ makes S(V) a commutative associative graded algebra $S(V) = \Sigma_{i=0}^{\infty} \oplus S^m V$. Hence $S(V)$ is the universal commutative associative algebra on a basis of V in the above sense. Let $\{x_i \mid i \in P\}$ be a fixed basis of V and let $\{z_i \mid i \in P\}$ be a set of indeterminates which is bijective to $\{x_i\}$. It is readily seen that $S(V)$ is canonically isomorphic to the polynomial algebra over F in $\{z_i\}$.

Definition 5.1.2. Let L be a Lie algebra over F . A universal enveloping algebra of L is a pair (U,i) of an associative algebra U with 1 and a homomorphism $i : L \to U^-$ satisfying the following property : If A is any associative algebra with 1 and $j : L \to A^-$ is any homomorphism then there exists a unique homomorphism of algebras $\phi : U \to A$ such that $\phi(1) = 1$ and $\phi i = j$. //

Theorem 5.1.1. 1) Let (U,i) and (V,j) be universal enveloping algebras of L . Then there exists a unique isomorphism ϕ of U onto V such that $\phi i = j$.

2) U is generated by 1 and $i(L)$

3) Let L_1, L_2 be Lie algebras with universal enveloping algebras (U_1,i_1), (U_2,i_2) respectively and let τ be a homomorphism of L_1 into L_2 . Then there is a unique homomorphism τ' of U_1 into U_2 such that $i_2\tau = \tau' i_1$.

Proof. Proofs are straightforward and standard. //

We show that there always exists a universal enveloping algebra (U,i) for L. Let $T(L)$ be the tensor algebra on L and let J be the ideal in $T(L)$ generated by all

$$(*) \qquad x \otimes y - y \otimes x - [xy], \quad x,y \in L .$$

Set $U \equiv U(L) = T(L)/J$ and let $\pi : T(L) \to U(L)$ be the natural homomorphism. Since $J \subset \Sigma_{i>0} T^i L$, π is injective on $T^0 L = F$. We will show later that π is injective on L too. Let i be the restriction of π to L. We claim that $(U(L), i)$ is a universal enveloping algebra of L. Indeed, let $j : L \to A^-$ be as in Definition 5.1.2. By the universal property of $T(L)$ we have an algebra homomorphism $\phi' : T(L) \to A$ which extends j and sends 1 to 1. Since ϕ' vanishes on the elements in $(*)$, $J \subset \ker \phi'$ and so ϕ' induces an algebra homomorphism ϕ of $U(L)$ into A such that $\phi(a + J) = \phi'(a)$, $a \in U(L)$, so $\phi i = j$ and ϕ is unique since 1 and $i(L)$ generate $U(L)$.

Note that if L is abelian then $I = J$ and $U(L)$ is just the symmetric algebra $S(L)$ on L.

To prove the PBW Theorem we proceed in steps. Let $\{x_i \mid i \in p\}$ be a fixed ordered basis of L and let

$\{z_i \mid i \in P\}$ be a set of indeterminates. Thus $S(L)$ is identified with the polynomial algebra over F in $\{z_i\}$. For brevity, write $T = T(L)$, $S = S(L)$, $U = U(L)$, $T^m = T^mL$ and $S^m = S^mL$. For each sequence $M = (i_1,\ldots,i_m)$ of indices, let $z_M = z_{i_1} \ldots z_{i_m} \in S^m$ and let $x_M = x_{i_1} \otimes \ldots \otimes x_{i_m} \in T^m$ where m is called the <u>length</u> of M. Say that M is increasing if $i_1 \leqq i_2 \leqq \cdots \leqq i_m$ in the given order of P. View ϕ as increasing and $z_\phi = 1$. Thus $\{z_M \mid M \text{ increasing}\}$ is a basis for S. Define filterations on S and T by $S_m = S^0 \oplus S^1 \oplus \ldots \oplus S^m$ and $T_m = T^0 \oplus T^1 \oplus \ldots \oplus T^m$, respectively. In the following, write $i \leqq M$ if $i \leqq j$ for all $j \in M$.

<u>Lemma 5.1.2</u>. For each $m \in Z^+$, there exists a unique linear map $f_m : L \otimes S_m \to S$ satisfying ;

(A_m). $f_m(x_i \otimes z_M) = z_i \cdot z_M$ for $i \leqq M$, $z_M \in S_m$.

(B_m). $f_m(x_i \otimes z_M) \equiv z_i \cdot z_M \pmod{S_k}$ for $k \leqq m$, $z_M \in S_k$.

(C_m). $f_m(x_i \otimes f_m(x_j \otimes z_N)) = f_m(x_j \otimes f_m(x_i \otimes z_N))$

$+ f_m([x_i x_j] \otimes z_N)$ for all $z_N \in S_{m-1}$.

Moreover, the restriction of f_m to $L \otimes S_{m-1}$ is f_{m-1}.

Proof. To verify existence and uniqueness of f_m, we proceed by induction on m. If $m = 0$ then only $z_M = 1$ occurs, so we let $f_0(x_i \otimes 1) = z_i$. Then f_0 is extended to a linear map : $L \otimes S_0 \to S$ and (A_0), (B_0), (C_0) are clearly satisfied. The uniqueness of f_0 follows from (A_0).

Assume the existence of a unique f_{m-1} satisfying (A_{m-1})-(C_{m-1}). We show how to extend f_{m-1} to f_m. For this it suffices to define $f_m(x_i \otimes z_M)$ when M is an increasing sequence of length m.

Case 1 : $i \leq M$. To meet (A_m), we must define $f_m(x_i \otimes z_M) = z_i \cdot z_M$. Then (B_m) is automatically satisfied.

Case 2 : $i \leq M$ fails. In this case, the first index j in M is less than i. Thus if we let $M = (j, N)$ then $j < i$ and $j \leq N$ where N has length $m - 1$. By (A_{m-1}), $f_{m-1}(x_j \otimes z_N) = z_j z_N = z_M$, and since $j \leq N$, by Case 1 $f_m(x_j \otimes z_i z_N) = z_j z_i z_N$ is already defined. Thus the left side of $(C_m) = f_m(x_i \otimes f_m(x_j \otimes z_N)) = f_m(x_i \otimes z_M)$. On the other hand, (B_{m-1}) implies that $f_m(x_i \otimes z_N)$ $= f_{m-1}(x_i \otimes z_N) \equiv z_i z_N \pmod{S_{m-1}}$. Thus the right side of (C_m) is already defined as

$$u \equiv z_j z_i z_N + f_{m-1}(x_j \otimes y) + f_{m-1}([x_i x_j] \otimes z_N) , \quad y \in S_{m-1} .$$

If we define $f_m(x_i \otimes z_M) = u$ then $f_m(x_i \otimes z_M) \equiv z_i z_M \pmod{S_m}$. Hence by induction (B_m) is satisfied.

Therefore, by Cases 1 and 2, f_m can be defined in a unique way, so as to satisfy (A_m) and (B_m) as well as to extend f_{m-1}. It remains to verify (C_m) for arbitrary i, j. If $j < i$ and $j \leq N$ then this is the situation treated in Case 2, so that $f_m(x_i \otimes z_M) = f_m(x_i \otimes f_m(x_j \otimes z_N))$ has been defined to satisfy (C_m). If $i < j$ and $i \leq N$, then since $[x_i x_j] = -[x_j x_i]$, we can reverse the above case. When $i = j$, (C_m) is trivial. It remains to consider the case where neither $i \leq N$ nor $j \leq N$ is true. Write $N = (k,R)$ where $k \leq R$, $k < i$, $k < j$. For convenience, abbreviate $f_m(x \otimes z)$ by xz whenever $x \in L$, $z \in S_m$. By induction hypothesis we have that $x_j z_N = x_j(x_k z_R)$ $= x_k(x_j z_R) + [x_j x_k]\, z_R$ (by (C_{m-1})) and $x_j z_R = z_j z_R + w$,

$w \in S_{m-2}$, by (B_{m-1}). Since $k \leq (j,R)$ and $k < i$, by the case treated above (C_m) applies to $x_i(x_k(z_j z_R))$. By induction, (C_m) also applies to $x_i(x_k w)$. Therefore, (C_m) applies to $x_i(x_k(x_j z_R)) = x_i(x_k(z_j z_R + w))$

$= x_i(x_k(z_j z_R)) + x_i(x_K w)$. Consequently,

$$(**) \quad x_i(x_j z_N) = x_i(x_k(x_j z_R)) + x_i([x_j x_k] z_R)$$

$$= x_k(x_i(x_j z_R)) + [x_i x_k]\ (x_j z_R)$$

$$+ [x_j x_k]\ (x_i z_R) + [x_i\ [x_j x_k]]\, z_R\ , \quad \text{by } (C_{m-1}) .$$

Note that i, j are interchangeable throughout this argument. Thus if we interchange i, j in $(**)$ and subtract the two resulting relations, we have

$$x_i(x_j z_N) - x_j(x_i z_N)$$

$$= x_k [x_i(x_j z_R) - x_j(x_i z_R)] + [x_i[x_j x_k]]z_R - [x_j[x_i x_k]]$$

$$= x_k([x_i x_j] z_R) + [x_i[x_j x_k]] z_R + [x_j[x_k x_i]] z_R \quad \text{(by } (C_{m-}$$

$$= [x_i x_j](x_k z_R) + \{[x_k[x_i x_j]] + [x_i[x_j x_k]] + [x_j[x_k x_i]]$$

$$= [x_i x_j] z_N$$

by (C_{m-1}), (A_{m-1}) and the Jacobi identity.

This proves (C_m). //

Lemma 5.1.3. There exists a representation $g : L \to (\mathrm{Hom}_F\, S)^-$ satisfying :

(1) $z_M g(x_i) = -z_i z_M$ for $i \leq M$.

(2) $z_M g(x_i) = -z_i z_M \pmod{S_m}$ if M has length m.

Proof. Note that $S = \bigcup_{m=0}^{\infty} S_m$. By Lemma 5.1.2. one can define a linear map $f : L \otimes S \to S$ by setting $f(x \otimes z_M) = f_m(x \otimes z_M)$, if M has length m. Since $f_m|_{L \otimes S_{m-1}} = f_{m-1}$, f is well-defined and satisfies (A_m)-(C_m) for all m. If we define $z_M g(x_i) = -f(x_i \otimes z_M)$

then it follows from (C_m) that $g : L \to (\mathrm{Hom}\ S)^-$ is a representation. Also, (1) and (2) are consequences of (A_m) and (B_m). //

Lemma 5.1.4. Let $t \in T_m \cap J$. Then the homogeneous part t_m of t of degree m lies in I, where $J = \ker \pi$, $\pi : T \to U$ natural map and $I = \ker \sigma$, $\sigma : T \to S$ natural map.

Proof. Write $t_m = \Sigma_{i=1}^{r} \alpha_i x_{M(i)}$, where each $M(i)$ is of length m. Let $g : L \to (\mathrm{Hom}\ S)^-$ be the representation as in Lemma 5.1.3. By the universal property of T, g is extended to an algebra homomorphism : $T \to \mathrm{Hom}\ S$, denoted by the same g, with $J \subset \ker g$. Thus $g(t) = 0$. In view of Lemma 5.1.3, the term of highest degree of $1g(t) = 0$ is precisely $\pm\Sigma_{i=1}^{r} \alpha_i z_{M(i)} = 0$. Hence $t_m \in I$. //

If we set $U_m = \pi(T_m)$ with $U_{-1} = 0$ then U is filtered by the U_m since $U_m U_n \subset U_{m+n}$ and $U_{m-1} \subset U_m$. Let $G = \Sigma_{i=0}^{\infty} \oplus G^i$ be the graded algebra associated with this filteration where $G^m = U_m/U_{m-1}$ (see (1.54)). Let $\phi_m : T_m \xrightarrow{\pi} U_m \xrightarrow{nat.} G^m$ be the composite. Since $\pi(T_m - T_{m-1}) = \pi(T^m) = U_m - U_{m-1}$, ϕ_m is surjective.

Thus the maps ϕ_m combine to yield a linear map $\phi : T \to G$ which is surjective and sends 1 to 1 .

Lemma 5.1.5. $\phi : T \to G$ is an algebra homomorphism. Moreover, $\phi(I) = 0$, so ϕ induces a homomorphism ω of $S = T/I$ onto G .

Proof. Let $x \in T^m$, $y \in T^n$. Then $\phi(x \otimes y)$ $= \phi_{m+n}(x \otimes y) = \pi(x \otimes y) + U_{m+n-1} = (\pi(x) + U_{m-1})(\pi(y) + U_{n-1})$ $= \phi(x)\phi(y)$ by (1.54). If $x,y \in L$ then $\pi(x \otimes y - y \otimes x) \in U_2$ but $\pi(x \otimes y - y \otimes x) = \pi([xy]) \in U_1$. Hence $\phi(x \otimes y - y \otimes x) \in U_1/U$ $= 0$ and $I \subset \ker \phi$. //

Theorem 5.1.6. The homomorphism $\omega : S \to G$ is an isomorphism which sends S^m isomorphically onto G^m .

Proof. By Lemma 5.1.5, it suffices to show that ω is injective. Let $\omega(t+I) = \overline{0}$ for $t \in T^m$. Then $\omega(t+I) = \phi(t) = \phi_m(t) = \pi(t) + U_{m-1} = U_{m-1}$ and hence $\pi(t) \in U_{m-1}$, so $\pi(t) = \pi(t')$ for some $t' \in T_{m-1}$. Thus $t-t' \in J$ and $t-t' \in T_m \cap J$. By Lemma 5.1.4, the homogeneous part, being t , of $t-t'$ of degree m , must lie in I . Hence ω is injective. //

Corollary 5.1.7. Let W be a subspace of T^m . If the natural map $\sigma : T^m \to S^m$ maps W isomorphically onto S^m , then π is injective on W and $U_m = \pi(W) \oplus U_{m-1}$.

Proof. Consider the diagram

$$\begin{array}{ccc} T^m & \xrightarrow{\ \pi\ } & U_m \\ \sigma \downarrow & & \downarrow \text{nat.} \\ S^m & \xrightarrow[\ \omega\ ]{} & G^m \end{array}$$

Clearly, $\omega(t+I) = \phi(t) = \pi(t) + U_{m-1}$, $t \in T^m$, and so the diagram is commutative. Thus, by Theorem 5.1.6, $\omega\sigma$ isomorphically sends W onto G^m. If $\pi(t) + U_{m-1} = U_{m-1}$ for $t \in W$ then $\omega\sigma(t) = \overline{0}$ in G^m and $t = 0$. This in particular implies that π is injective on W and $\pi(W) + U_{m-1}$ is direct. Since $\omega\sigma$ is surjective on W, so is $t\ (\in W) \to \pi(t) + U_{m-1}$, and $U_m = \pi(W) \oplus U_{m-1}$. //

Corollary 5.1.8. The natural map $i : L \to U(L)$ is injective (so L may be identified with $i(L)$). Thus any Lie algebra has a faithful representation.

Proof. We have observed that σ isomorphically maps $W \equiv T'(=L)$ onto S'. Hence the result follows from Corollary 5.1.7. //

Theorem 5.1.9. (PBW Theorem). Let $\{x_j \mid j \in P\}$ be any given ordered basis for L. Then the elements

$x_{j_1} x_{j_2} \ldots x_{j_m} = \pi(x_{j_1} \otimes x_{j_2} \otimes \ldots \otimes x_{j_m})$, $m \in Z^+$,

$j_1 \leqq j_2 \leqq \ldots \leqq j_m$, form a basis for $U(L)$.

Proof. Let W be the subspace of T^m spanned by all $x_{j_1} \otimes \ldots \otimes x_{j_m}$, $m \in Z^+$, $j_1 \leq \cdots \leq j_m$. Then σ maps isomorphically W onto S^m . Hence by Corollary 5.1.' π is injective on W and $U_m = \pi(W) \oplus U_{m-1}$ for all $m \in Z^+$. //

Definition 5.1.3. For $j_1 \leq \cdots \leq j_m$, $m \in Z^+$, the basis element $x_{j_1} \ldots x_{j_m}$ is called a standard monomial of degree m . The standard monomial of degree 0 is 1 . A basis of $U(L)$ consisting of all standard monomials in $\{x_j \mid j \in P\}$ is referred to as a PBW basis of $U(L)$. //

Because of the identification $i(x) = x$, $x \in L$, the PBW Theorem can be restated as : For any ordered basis $\{x_j \mid j \in P\}$ of L , the standard monomials in this basis form a basis of $U(L)$. Also $U(L)$ is characterized as : If ϕ is any homomorphism of L into A^- as in Definition 5.1.2 then ϕ can be extended to a unique homomorphism (denoted by ϕ) of $U(L)$ into A , sending 1 to 1 . In particular, if f is a representation of L into $(\mathrm{Hom}\ V)^-$, f is extended to a unique representation f of $U(L)$ into $\mathrm{Hom}\ V$ and hence f affords a unique $U(L)$-module structure on V . Conversely if V is a $U(L)$-module then the restriction of the module action to L defines an L-module structure on V .

<u>Theorem 5.1.10</u>. Let H be a subalgebra of L and extend an ordered basis $\{h_i\}$ of H to an ordered basis $\{h_i, x_j\}$ of L .

(1) U(H) is the subalgebra of U(L) generated by 1 and H . Hence U(L) is a free U(H)-module with basis consisting of standard monomials $x_{j_1} x_{j_2} \dots x_{j_m}$, $j_1 \leqq \dots \leqq j_m$, $m \in Z^+$.

(2) Let H be an ideal of L . Then L/H can be identified with (L+R)/R where R is the ideal of U(L) generated by H , and U(L)/R is the universal enveloping algebra U(L/H) of L/H . Moreover, the standard monomials

$$(***) \quad h_{i_1} \dots h_{i_k} \; x_{j_1} \dots x_{j_m} , \quad i_1 \leqq \dots \leqq i_k ,$$
$$j_1 \leqq \dots \leqq j_m , \quad k > 0 , \quad m \in Z^+ ,$$

form a basis for R .

<u>Proof</u>. (1) Let B be the subalgebra of U(L) generated by H . Then the standard monomials $h_{i_1} h_{i_2} \dots h_{i_k}$, $i_1 \leqq \dots \leqq i_k$, $k \in Z^+$, form a basis for B . Thus by PBW Theorem B = U(H) .

(2) We first show that $(U(L)/R, i)$ is a universal enveloping algebra of L/H where $i : a + H \to a + R,\ a \in L$. Let $j : L/H \to A^-$ be as in Definition 5.1.2. Define $j' : L \to A^-$ by $j'(a) = j(a + H)$. Then j' is extended to a unique homomorphism $j' : U(L) \to A$. Since $H \subset \ker j'$, $R \subset \ker j'$. Thus j' induces a homomorphism $\phi : U(L)/R \to A$ such that $\phi(a + R) = j'(a)$. Now $\phi i(a + H) = \phi(a + R) = j'(a) = j(a + R)$, so $\phi i = j$. The uniqueness follows from the fact that $i(L/H)$ generates $U(L)/R$. Thus, by Corollary 5.1.8, we identify L/H with the subalgebra $(L+R)/R$ of $(U(L)/R)^-$. Hence $(L+R)/R$ has a basis $\{x_j + R\}$. By the PBW Theorem $x_{j_1} \dots x_{j_m} + R,\ j_1 \leqq \dots \leqq j_m,\ m \in Z^+$, form a basis of $U(L)/R$. Therefore, if we let D be the subspace of $U(L)$ spanned by the standard monomials $x_{j_1} \dots x_{j_m},\ m \in Z^+$ then $R \cap D = 0$. Noting that the elements in (***) lie in R and these together with $x_{j_1} \dots x_{j_m},\ j_1 \leqq \dots \leqq j_m$ form a PBW basis for $U(L)$, it follows that the elements in (***) form a basis of R. //

Let A be an associative algebra with 1. Then an associative algebra B is called a <u>left quotient division ring</u> for A if (1) B is a division ring, (2) A is a subring of B and (3) every element of B has the form $a^{-1}b$, $a,b \in A$.

In a ring theoretical aspect, a universal enveloping algebra has the following interesting properties.

Exercise 5.1.1. Prove :

(1) The universal enveloping algebra U of any Lie algebra L has no nonzero zero divisor.

(2) If L is finite-dimensional then U has the a.c.c. on left or right ideals and U has a left or right quotient division ring. //

5.2. Free Lie algebras

A free Lie algebra L over F on a set X is defined by the universal property :

Definition 5.2.1. Let L be a Lie algebra over F generated by X . Then L is said to be free on X if, given any mapping ϕ of X into a Lie algebra K , ϕ can be extended to a unique homomorphism $\psi : L \to K$. //

It is easy to verify that such a Lie algebra L is unique up to isomorphism. For the existence of L, we let V be a vector space over F with X as basis and form the tensor algebra $T(V)$ on V. Let L be the Lie subalgebra of $T(V)^-$ generated by X. We claim that L is free on X. Thus, let ϕ be any mapping of X into a Lie algebra K. Extend ϕ to a linear map $\phi : V \to K \subset U(K)$. Then ϕ is extended to an algebra homomorphism $\phi' : T(V) \to U(K)$. The restriction ψ of ϕ' to L gives the desired $\psi : L \to K$ since ψ maps X into K and L is generated by X. Observe that if L is free on X then any vector space V can be made into an L-module by assigning to each $x \in X$ an element of $(\mathrm{Hom}\ V)^-$ and extending canonically.

Theorem 5.2.1. (Witt). Let X be any set and V be a vector space over F with X as basis. Then the Lie subalgebra L of $T(V)^-$ generated by X is the free Lie algebra on X and the tensor algebra $T(V)$ is the universal enveloping algebra of L.

Proof. The first part has been proved. Let $\phi : L \to A^-$ be as in Definition 5.1.2. Since $V \subset L$, $\phi|_V$ is extended to a unique algebra homomorphism $\psi : T(V) \to A$, sending 1 to 1. Noting that $\psi|_X = \phi|_X$ and X generates L, $\psi|_L = \phi$ since $\psi : T(V)^- \to A^-$

is a homomorphism. Thus ψ is an extension of ϕ and is unique, since L also generates $T(V)$. Hence $T(V) = U(L)$. //

A free Lie algebra along with universal enveloping algebras is a basic tool for the construction of semisimple Lie algebras by means of the Cartan matrix, the canonical generators x_i, y_i, h_i and the relations (1.45). For detail, consult Jacobson's or Humphreys' book.

5.3. The Weyl group and roots

We return to a split semisimple Lie algebra L with split CSA H . We retain the notations in Chapter 4. Thus Φ stands for the set of roots $\neq 0$, H_0^* for the Q-subspace of H^* spanned by Φ , Π for a simple system of roots relative to a given ordering in H_0^* , Φ^+ (more precisely Φ_Π^+) for the set of positive roots relative to Π , and $(\, , \,)$ for the positive definite form on H_0^* . Let E be the Euclidean space of dimension

$\ell = \dim H^*$ that is the scalar extension of H_0^* to R, the reals. Thus $(\ ,\ )$ is canonically extended to a positive definite form on E. Note that Π is a basis of E. Also, Π defines a <u>partial</u> order in E which is compatible with the order $\alpha > 0$, $\alpha \in \Phi^+$; namely, define $\alpha \leqq \beta$ in E if $\beta - \alpha$ is a sum of roots in Π or $\alpha = \beta$.

<u>Definition 5.3.1</u>. A hyperplane in E is a subspace of E of codimension 1. A reflection in E is an element in $\mathrm{Hom}\, E$ which leaves pointwise fixed some hyperplane P in E and sends any vector α orthogonal to P to $-\alpha$. //

Clearly, a reflection σ is nonsingular with its own inverse and ± 1 are the only eigenvalues of σ. Also, σ is orthogonal, that is, $(\sigma\alpha, \sigma\beta) = (\alpha, \beta)$ for $\alpha, \beta \in E$ (note $E = R\alpha \oplus P$ where $\alpha \perp P$ and P is a hyperplane fixed pointwise by σ). Any nonzero vector α determines a reflection σ_α, with <u>reflecting hyperplane</u> $P_\alpha = \{\beta \in E \mid (\beta, \alpha) = 0\}$ and $\sigma_\alpha(\alpha) = -\alpha$. This σ_α is explicitly given as

$$\sigma_\alpha(\beta) = \beta - [2(\beta,\alpha)/(\alpha,\alpha)]\,\alpha = \beta - \langle\beta,\alpha\rangle\,\alpha \qquad (1.55)$$

where $\langle\beta,\alpha\rangle = 2(\beta,\alpha)/(\alpha,\alpha)$, $\alpha \neq 0$. When $\alpha \in \Phi$, the reflection σ_α is particularly important, since $\sigma_\alpha(\Phi) = \Phi$ by Corollary 4.4.4(2).

Definition 5.3.2. If $\alpha \in \Phi$, σ_α is called the Weyl reflection by α. The subgroup WG of the general linear group GL(E) generated by all Weyl reflections σ_α, $\alpha \in \Phi$, is called the Weyl group of Φ. //

Since WG permutes Φ and Φ is finite, we can identify WG with a subgroup of the symmetric group on Φ. The Weyl group plays a crucial role in the sequel.

Lemma 5.3.1. (1) If $\sigma \in GL(E)$ leaves Φ invariant, fixes pointwise a hyperplane P in E, and sends some nonzero $\alpha \in \Phi$ to $-\alpha$ then $\sigma = \sigma_\alpha$ and $P = P_\alpha$.

(2) If $\sigma \in GL(E)$ leaves Φ invariant then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ for all $\alpha \in \Phi$ and $\langle\beta,\alpha\rangle = \langle\sigma(\beta),\sigma(\alpha)\rangle$ for all $\alpha, \beta \in \Phi$.

(3) Let E' be a subspace of E. If a reflection σ_α leaves E' invariant then $\alpha \in E'$ or $E' \subset P_\alpha$.

Proof. (1) Let $\tau = \sigma_\alpha\sigma$. Then $\tau(\Phi) = \Phi$ and $\tau(\alpha) = \alpha$. Since σ fixes P pointwise and $E = P \oplus R\alpha$, for a suitable basis of E the matrix of τ is upper triangular having all diagonal entries 1. Thus the minimal polynomial of τ is of the form $(T - 1)^m$. Since Φ is finite, for each $\beta \in \Phi$, not all

$\beta, \tau(\beta), \tau^2(\beta), \ldots$ are distinct, so $\tau^k(\beta) = \beta$ for some $k > 0$. Choose a sufficiently large k so that τ^k fixes all $\beta \in \Phi$. Since Φ spans E, $\tau^k = 1$, so the minimal polynomial of τ divides $T^k - 1$: This forces the minimum polynomial of τ to be $T - 1$, so $\tau = 1$, and $\sigma = \sigma_\alpha$ since $\sigma_\alpha^{-1} = \sigma_\alpha$. //

(2) Since $\sigma_\alpha(\beta) \in \Phi$, $\sigma\sigma_\alpha\sigma^{-1}(\sigma(\beta)) = \sigma\sigma_\alpha(\beta) \in \Phi$. But $\sigma\sigma_\alpha(\beta) = \sigma(\beta - \langle\beta,\alpha\rangle\, \alpha) = \sigma(\beta) - \langle\beta,\alpha\rangle\, \sigma(\alpha)$. Since $\sigma(\Phi) = \Phi$, this shows that $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant, while it fixes the hyperplane $\sigma(P_\alpha)$ pointwise and sends $\sigma(\alpha)$ to $-\sigma(\alpha)$. By (1), $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$, but then $\sigma_{\sigma(\alpha)}(\sigma(\beta)) = \sigma(\beta) - \langle\sigma(\beta),\sigma(\alpha)\rangle\, \sigma(\alpha)$. Comparing this with the above equation gives the desired relation.

(3) Suppose that $\alpha \notin E'$. For $\beta \in E'$, $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\, \alpha$. If $(\alpha,\beta) \neq 0$, one would have $\alpha \in E'$. Thus $E' \subset P_\alpha$. //

<u>Exercise 5.3.1</u>. Prove that the union of finitely many hyperplanes P_α, $\alpha \in \Phi$, can not exhaust E.

<u>Exercise 5.3.2</u>. Let $\gamma_1, \ldots, \gamma_\ell$ be a basis of E and let $P^+_{\gamma_i} = \{\beta \in E \mid (\beta,\gamma_i) > 0\}$ be the positive open half-space, $i = 1,2,\ldots,\ell$. Prove that $\bigcap_{i=1}^{\ell} P_i \neq \phi$.

To obtain more stringent information about the effect of the Weyl group to the root system, we put several terminologies in order. For each vector $\gamma \varepsilon E$, set $\Phi^+(\gamma) = \{\alpha \varepsilon \Phi \mid (\gamma,\alpha) > 0\}$. Thus $\Phi^+(\gamma) = \Phi \cap P^+_\gamma$ (Exercise 5.3.2).

Definition 5.3.3. Let $K = E - \bigcup_{\alpha\varepsilon\Phi} P_\alpha$ ($\neq \phi$; Exercise 5.3.2). Any vector $\gamma \varepsilon K$ is called regular. Call $\alpha \varepsilon \Phi^+(\gamma)$ decomposable if $\alpha = \beta_1 + \beta_2$ for some $\beta_i \varepsilon \Phi^+(\gamma)$, indecomposable otherwise. If γ is regular then $\Pi(\gamma)$ denotes the set of all indecomposable roots in $\Phi^+(\gamma)$. Each connected component of K is called a (open) Weyl chamber of E. Therefore each regular $\gamma \varepsilon E$ belongs to precisely one Weyl chamber, denoted by $C(\gamma)$. //

Exercise 5.3.3. (1) Prove that there are only finitely many Weyl chambers of E.

(2) Let $\Pi = (\alpha_1,\ldots,\alpha_\ell)$ be a simple system of roots. Prove that $\bigcap_{i=1}^{\ell} P^+_{\alpha_i}$ is a Weyl chamber (compare with Exercise 5.3.2).

Theorem 5.3.2. Let $\gamma \varepsilon E$ be regular. Then $\Pi(\gamma)$ is a simple system of roots relative to some ordering in H_0^* (or E) and $\Phi^+(\gamma)$ is the set of positive roots (relative to $\Pi(\gamma)$). Moreover any simple system of roots Π has the form $\Pi(\gamma)$ for some regular $\gamma \varepsilon E$.

Proof. (1) Show that each $\beta \in \Phi^+(\gamma)$ is of the form $\Sigma_{\alpha\in\Pi(\gamma)} k_\alpha \alpha$, where all k_α are nonnegative integers. Suppose not and let $\alpha \in \Phi^+(\gamma)$ be a counterexample with (γ,α) minimal. Thus $\alpha \notin \Pi(\gamma)$ and $\alpha = \beta_1 + \beta_2$, $\beta_i \in \Phi^+(\gamma)$. Hence $(\gamma,\alpha) = (\gamma,\beta_1) + (\gamma,\beta_2)$ Since $(\gamma,\beta_i) \geqq 0$, $i = 1,2$, by the minimality of (γ,α), β_1 and β_2 are nonnegative Z-linear combinations of $\Pi(\gamma)$ and so is α, a contradiction.

(2) Show that $\Pi(\gamma)$ is linearly independent. First, let $\alpha, \beta \in \Pi(\gamma)$ and let $\alpha \neq \beta$. By Corollary 4.4.8 $(\alpha,\beta) \leqq 0$. Suppose $\Sigma\, r_\alpha \alpha = 0$, $\alpha \in \Pi(\gamma)$, $r_\alpha \in R$. Partitioning the indices α as the disjoint union of α's and β's where $r_\alpha > 0$ and $r_\beta < 0$, we rewrite the sum as $\Sigma\, r_\alpha \alpha = \Sigma\, t_\beta \beta$, $r_\alpha > 0$, $t_\beta = -r_\beta > 0$, Let $\mu = \Sigma\, r_\alpha \alpha$. Then $(\mu,\mu) = \Sigma_{\alpha,\beta} r_\alpha t_\beta (\alpha,\beta) \leqq 0$ and since $(\mu,\mu) > 0$, this is absurd unless all r_α and t_β are 0.

Since $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$, in view of Corollary 4.5. (1) and (2) assure that $\Pi(\gamma)$ is a simple system of roots relative to some ordering in H_0^* such that $\Phi^+(\gamma)$ is the set of positive roots.

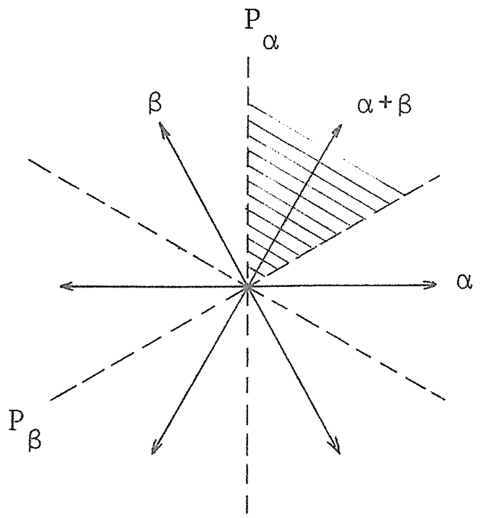
(3) Claim that any simple system of roots Π has the form $\Pi(\gamma)$ for some regular $\gamma \in E$. By Exercise 5.3.

choose a $\gamma \in \bigcap_{\alpha \in \Pi} P_\alpha^+$, so γ is regular. Clearly, $\Phi^+ \subset \Phi^+(\gamma)$ and $\Phi^- \subset -\Phi^+(\gamma)$, and hence $\Phi^+ = \Phi^+(\gamma)$. Since each root in Π is indecomposable, $\Pi \subset \Pi(\gamma)$. But both Π and $\Pi(\gamma)$ are bases of E, $\Pi = \Pi(\gamma)$. //

Notice that the map $\gamma \to (\gamma,\alpha)$ is continuous for each α and P_α^+ or P_α^- is an open convex set in E. Thus if $C(\gamma) = C(\gamma')$ for regular γ, γ' then (γ,α) and (γ',α) have the same sign for each $\alpha \in \Phi$. Hence $\Phi^+(\gamma) = \Phi^+(\gamma')$ or $\Pi(\gamma) = \Pi(\gamma')$. Conversely, if $\Pi(\gamma) = \Pi(\gamma')$ then by Theorem 5.3.2 $\Phi^+(\gamma) = \Phi^+(\gamma')$ and so γ, γ' lie in the same component, or $C(\gamma) = C(\gamma')$. This, in view of Theorem 5.3.2, shows that $C(\gamma) \to \Pi(\gamma)$ gives a 1-1 correspondence between the set of Weyl chambers and the set of simple systems of roots. If $\Pi = \Pi(\gamma)$, write $C(\gamma) = C(\Pi)$ and call this the <u>fundamental Weyl chamber relative to</u> Π. Thus if $\gamma' \in C(\Pi) = C(\gamma)$ then $(\gamma',\alpha) > 0$ for all $\alpha \in \Pi$ by the above remark. Thus $\gamma' \in \bigcap_{\alpha \in \Pi} P_\alpha^+$ and $C(\Pi) = \bigcap_{\alpha \in \Pi} P_\alpha^+$ (this in particular proves Exercise 5.3.3).

<u>Example</u>. Let L be of type A_2, so $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$ where α, β are simple. Note that Φ has only one root length and hence the angle between α and β is 120^0 (see the table following Definition 4.5.3). Also, $C(\Pi) = \{\gamma \in K \mid$ the angle between γ and α, β is $< 90^0\}$. The Weyl chambers

with the shaded one being fundamental relative to $\{\alpha,\beta\}$ can be dipicted as follows.



Since each $\sigma \in WG$ is a homeomorphism on E, $\sigma(C(\gamma)) = C(\alpha(\gamma))$ and σ sends a Weyl chamber onto another. Also, we have $\sigma(\Pi(\gamma)) = \Pi(\sigma(\gamma))$ since $(\sigma(\gamma), \sigma(\alpha)) = (\gamma,\alpha)$. Thus these actions of WG are compatible with the above correspondence between Weyl chambers and simple systems of roots.

<u>Lemma 5.3.3</u>. Let $\alpha \in \Pi$. Then σ_α permutes the positive roots other than α., i.e., $\sigma(\Phi^+ - \{\alpha\}) = \Phi^+ - \{\alpha\}$.

<u>Proof</u>. Let $\beta \in \Phi^+ - \{\alpha\}$ and $\beta = \Sigma_{\gamma\in\Pi} k_\gamma \gamma (k_\gamma \in Z^+)$

Hence $k_\gamma \neq 0$ for some $\gamma \neq \alpha$ and so $\sigma_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\, \alpha$ has term γ with the cofficient $k_\gamma > 0$. This forces $\sigma_\alpha(\beta)$ to be positive and moreover, $\sigma_\alpha(\beta) \neq \alpha$ since $\sigma_\alpha(-\alpha) = \alpha$. //

Corollary 5.3.4. Let $\delta = \frac{1}{2}\Sigma_{\beta>0}\, \beta$. Then $\sigma_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Pi$. //

Lemma 5.3.4. Let $\alpha_1, \alpha_2, \ldots, \alpha_t \in \Pi$ (not necessarily distinct), and set $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \cdots \sigma_{t-1}(\sigma_t) < 0$ then for some index $1 \leqq s < t$, $\sigma_1 \cdots \sigma_t = \sigma_1 \cdots \sigma_{s-1}\sigma_{s+1} \cdots \sigma_{t-1}$.

Proof. Write $\beta_i = \sigma_{i+1} \cdots \sigma_{t-1}(\alpha_t)$, $0 \leqq i \leqq t-2$, $\beta_{t-1} = \alpha_t$. Since $\beta_0 < 0$ and $\beta_{t-1} > 0$, we find a smallest index s for which $\beta_s > 0$. Then $\sigma_s(\beta_s) = \beta_{s-1} < 0$ and by Lemma 5.3.3 we must have $\alpha_s = \beta_s$. Set $\sigma = \sigma_{s+1} \cdots \sigma_{t-1}$. Then $\sigma(\alpha_t) = \beta_s = \alpha_s$ and the relation $\sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$ for $\sigma \in WG$ gives $\sigma_s = (\sigma_{s+1} \cdots \sigma_{t-1})\sigma_t(\sigma_{t-1} \cdots \sigma_{s+1})$ (Lemma 5.3.1(2)), which yields the lemma. //

If $\sigma = \sigma_1 \cdots \sigma_t$ in Lemma 5.3.4 is a shortest expression, then the Lemma forces $\sigma_1 \cdots \sigma_{t-1}(\alpha_t)) > 0$ and this gives $\sigma(\alpha_t) < 0$ since $\alpha_t = -\sigma_t(\alpha_t)$. Thus we have

Corollary 5.3.5. If $\sigma = \sigma_1 \cdots \sigma_t$ is an expression for $\sigma \in WG$ in terms of reflections by simple roots with t minimal then $\sigma(\alpha_t) < 0$. //

Theorem 5.3.6. Let Π be a simple system of roots.

(1) Let $\gamma \in E$ be any regular vector. Then there exists $\sigma \in WG$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Pi$, i.e., $\sigma(C(\gamma)) = C(\Pi)$. Thus WG acts transitively on Weyl chambers.

(2) If Π' is another simple system of roots then $\sigma(\Pi') = \Pi$ for some $\sigma \in WG$ (WG acts transitively on simple systems of roots).

(3) If α is any root, $\sigma(\alpha) \in \Pi$ for some $\sigma \in WG$

(4) WG is generated by all σ_α, $\alpha \in \Pi$.

(5) If $\sigma(\Pi) = \Pi$ for some $\sigma \in WG$ then $\sigma = 1$ (WG acts simply transitively on simple systems of roots).

Proof. Let WG' be the subgroup of WG generated by all σ_α, $\alpha \in \Pi$. We prove (1) - (3) for WG' and then show that WG = WG'.

(1) Let $\delta = \frac{1}{2}\Sigma_{\alpha>0}\, \alpha$ and choose $\sigma \in WG'$ for which $(\sigma(\gamma), \delta)$ is maximal. Let $\alpha \in \Pi$ be any simple

root. By the choice of σ, $(\sigma(\gamma),\delta) \geqq (\sigma_\alpha\sigma(\gamma),\delta)$ $= (\sigma(\gamma),\sigma_\alpha(\delta)) = (\sigma(\gamma),\delta-\alpha) = (\sigma(\gamma),\delta) - (\sigma(\gamma),\alpha)$ (Corollary 5.3.4) and so $(\sigma(\gamma),\alpha) \geqq 0$. But since γ is regular, we can not have $(\sigma(\gamma),\alpha) = 0 = (\gamma,\sigma^{-1}(\alpha))$ and so $(\sigma(\gamma),\alpha) > 0$, $\alpha \in \Pi$. This means that $\sigma(\gamma) \in C(\Pi)$ or $\sigma(C(\gamma)) = C(\sigma(\gamma)) = C(\Pi)$, as desired.

(2) Let $\Pi' = \Pi(\gamma)$ for some regular γ (Theorem 5.4.2). By (1), there exists $\sigma \in WG'$ such that $C(\sigma(\gamma)) = C(\Pi)$. Hence $\Pi(\sigma(\gamma)) = \sigma(\Pi(\gamma)) = \sigma(\Pi') = \Pi$.

(3) In view of (2), it suffices to show that each root belongs to at least one simple system of roots. If $\beta \neq \pm\alpha$ and $\beta \in \Phi$ then since α, β are linearly independent, $P_\alpha \neq P_\beta$ for all $\beta \neq \pm\alpha$. Indeed, if $P_\alpha = P_\beta$, write $E = P_\alpha \oplus R\beta = P_\beta \oplus R\alpha$ and this implies that α, β are linearly dependent. Therefore, $P_\alpha \neq \bigcup_{\beta\neq\pm\alpha} P_\beta$, so we can choose a $\gamma \in P_\alpha$ but $\gamma \notin P_\beta$ for all $\beta \neq \pm\alpha$. Since $\gamma \to (\gamma,\beta)$ is continuous from E into R, in a sufficiently small neighborhood of γ we choose a $\gamma' \in E$ such $(\gamma',\alpha) = \varepsilon > 0$ while $|(\gamma',\beta)| > \varepsilon$ for all $\beta \neq \pm\alpha$. Then γ' is regular and $\alpha \in \Pi(\gamma')$; if $\alpha = \beta_1 + \beta_2$ for $\beta_i \in \Phi^+(\gamma')$ then $(\gamma',\alpha) = (\gamma',\beta_1) + (\gamma',\beta_2) > 2\varepsilon$, a contradiction.

(4) For $WG' = WG$, it is enough to show that

each $\sigma_\alpha \in WG'$ for $\alpha \in \Phi$. By (3), find $\sigma \in WG'$ such that $\beta = \sigma(\alpha) \in \Pi$. Then $\sigma_\beta = \sigma_{\sigma(\alpha)} = \sigma\sigma_\alpha\sigma^{-1}$ (Lemma 5.3.: so $\sigma_\alpha = \sigma^{-1}\sigma_\beta\sigma \in WG'$.

(5) Let $\sigma(\Pi) = \Pi$ and $\sigma \neq 1$. By (4), let $\sigma = \sigma_1 \dots \sigma_t$ be a minimal expression by simple reflections. Then $t \geq 1$ and by Corollary 5.3.5 $\sigma(\sigma_t) < 0$, $\alpha_t \in \Pi$, a contradiction. //

Definition 5.3.4. If $\sigma \in WG$ is expressed as $\sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ with $\alpha_i \in \Pi$ and t minimal, we call the expression reduced and write $\ell(\sigma) = t$, called the length of σ relative to Π . View $\ell(1) = 0$. Denote by $n(\sigma)$ the number of $\sigma \in \Phi^+$ such that $\sigma(\alpha) < 0$. /

Lemma 5.3.7. $\ell(\sigma) = n(\sigma)$ for all $\sigma \in WG$.

Proof. If $\ell(\sigma) = 0$ then $\sigma = 1$ and $n(\sigma) = 0$. To proceed by induction on $\ell(\sigma)$, assume that the equality holds for all $\tau \in WG$ with $\ell(\tau) < \ell(\sigma)$. Express σ in a reduced form as $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ and let $\alpha = \alpha_t$. By Corollary 5.3.5 $\sigma(\alpha) < 0$. Since σ_α permutes the positive roots in $\Phi^+ - \{\alpha\}$ by Lemma 5.3.3, we have $n(\sigma\sigma_\alpha) = n(\sigma) - 1$. But clearly $\ell(\sigma\sigma_\alpha) = \ell(\sigma) - 1 < \ell(\sigma)$, so by induction $\ell(\sigma\sigma_\alpha) = n(\sigma\sigma_\alpha)$ and $\ell(\sigma) = n(\sigma)$. //

<u>Theorem 5.3.8</u>. Let $\overline{C(\Pi)}$ be the closure of the fundamental Weyl chamber relative to $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$.

(1) Let $\gamma \in \overline{C(\Pi)}$ and let $\sigma \in WG$ be any element. Then $\gamma - \sigma(\gamma) = \Sigma\, c_i \alpha_i$ where the c_i are nonnegative real numbers and the α_i are those simple roots occuring in a reduced form of σ .

(2) If $\gamma \in C(\Pi)$ and $\sigma(\gamma) = \gamma$ then $\sigma = 1$.

(3) If $\gamma \in \overline{C(\Pi)}$ and $\sigma(\gamma) = \gamma$ then $\sigma = 1$ or $\gamma \in \overline{C(\Pi)} - C(\Pi)$.

(4) $\sigma(C(\Pi)) \cap C(\Pi) = \phi$ for all $\sigma \neq 1$ in WG .

(5) Let $\gamma \in \overline{C(\Pi)}$ be such that $<\sigma(\gamma), \alpha> \in Z$ for all $\alpha \in \Pi$. Then $\sigma(\gamma) \leqq \gamma$ for all $\sigma \in WG$ (relative to the partial order in E). The equality holds only when $\sigma = 1$ or $\gamma \in \overline{C(\Pi)} - C(\Pi)$.

(6) Every vector $\gamma \in E$ is WG-conjugate to one and only one vector in $\overline{C(\Pi)}$.

<u>Proof</u>. Let σ be any element in WG and assume $\sigma \neq 1$. Write $\sigma = \sigma_{i(1)} \cdots \sigma_{i(t)}$ in a reduced form where $\sigma_i = \sigma_{\alpha_i}$, $\alpha_i \in \Pi$. Note that if $\gamma \in \overline{C(\Pi)}$, $(\gamma, \alpha) \geqq 0$ for all $\alpha \in \Phi^+$.

(1) Note that each partial product $\sigma_{i(t)}\cdots\sigma_{i(s)}$ $(1 \leqq s \leqq t)$ in σ^{-1} is also reduced. Thus, by Corollary 5 $\sigma_{i(t)}\cdots\sigma_{i(s+1)}\alpha_{i(s)} = -\sigma_{i(t)}\cdots\sigma_{i(s+1)}\sigma_{i(s)}\alpha_{i(s)} > 0$ and hence $(\sigma_{i(s+1)}\cdots\sigma_{i(t)}\gamma,\alpha_{i(s)}) = (\gamma,\sigma_{i(t)}\cdots\sigma_{i(s+1)}\alpha_{i(s)})$ is nonnegative real for any $1 \leqq s \leqq t$, since $\gamma \in \overline{C(\Pi)}$. Let $c_{i(s)} = \langle\sigma_{i(s+1)}\cdots\sigma_{i(t)}\gamma,\alpha_{i(s)}\rangle$. Then we have $\sigma(\gamma) = \sigma_{i(2)}\cdots\sigma_{i(t)}\gamma - \langle\sigma_{i(2)}\cdots\sigma_{i(t)}\gamma,\alpha_{i(1)}\rangle\alpha_{i(1)}$ and continuing this we get $\sigma(\gamma) = \gamma - \sum_s c_{i(s)}\alpha_{i(s)}$.

(2) If $\sigma(\gamma) = \gamma$ then $c_{i(t)} = \langle\gamma,\alpha_{i(t)}\rangle = 0$ in (1), so $(\gamma,\alpha_{i(t)}) = 0$ and $\gamma \notin C(\Pi)$ unless $\sigma = 1$. This also proves (3).

(4) Suppose $\sigma(C(\Pi)) \cap C(\Pi) \neq \phi$ and let γ be a vector in this intersection. Thus $\gamma = \alpha(\gamma')$ for some $\gamma' \in C(\Pi)$. But then $C(\Pi) = C(\gamma) = \sigma(C(\gamma'))$ and this corresponds to $\Pi = \Pi(\gamma) = \sigma(\Pi(\gamma')) = \sigma(\Pi)$. By Theorem 5.3.6(5) this forces $\sigma = 1$.

(5) Under the assumption, we have all $c_{i(s)} \in Z^+$ in (1) and hence $\sigma(\gamma) \leqq \gamma$ for all $\sigma \in WG$. The second part follows from (3).

(6) Let $\gamma \in E$. Then $\gamma \in \overline{C(\gamma_0)}$ for some regular γ_0. By Theorem 5.3.6(1) there exists a $\sigma \in WG$ such that $\sigma C(\gamma_0) = C(\Pi)$ and so $\sigma\overline{C(\gamma_0)} = \overline{C(\Pi)}$. Hence γ is

WG-conjugate to a vector in $\overline{C(\Pi)}$. For the uniqueness, it suffices to show that $\sigma(\gamma) = \gamma'$ for $\gamma, \gamma' \in \overline{C(\Pi)}$, $\sigma \in WG$ only when $\gamma = \gamma'$. Since Π is a basis of E , this follows from (1). //

We call $\overline{C(\Pi)}$ the <u>fundamental</u> <u>domain</u> for the action of WG (in the sense of Lemma 3.5.8(6)). Recall that the irreducibility of Π is equivalent to the simplicity of L (Theorem 4.5.7). We show that this is also equivalent to the irreducibility of Φ in the sense of Definition 4.5.4.

<u>Lemma 5.3.9</u>. Π is irreducible if and only if Φ is irreducible.

<u>Proof</u>. Let Π be irreducible and let $\Phi = \Phi_1 \cup \Phi_2$ (disjoint) with $(\Phi_1, \Phi_2) = 0$. Then $\Pi = (\Pi \cap \Phi_1) \cup (\Pi \cap \Phi_2)$ and since Π is irreducible, $\Pi \subset \Phi_1$ or $\Pi \subset \Phi_2$; but $\Pi \subset \Phi_1$ implies $(E, \Phi_2) = 0$ since Π spans E . Hence $\Phi_2 = \phi$. Conversely, suppose that Φ is irreducible and let $\Pi = \Pi_1 \cup \Pi_2$ with $(\Pi_1, \Pi_2) = 0$. Since each root is WG-conjugate to a simple root, we can write $\Phi = \Phi_1 \cup \Phi_2$ where Φ_i is the set of roots having a conjugate in Π_i . Noting that $(\alpha, \beta) = 0$ implies $\sigma_\alpha\sigma_\beta = \sigma_\beta\sigma_\alpha$ and that WG is generated by the σ_α, $\alpha \in \Pi$, we can write each $\sigma \in WG$ as $\sigma_1\sigma_2$ where σ_i is a product of reflections

by roots in Π_i. Since σ_2 fixes each root in Π_1, $\sigma(\alpha)$ is an integral linear combination of roots in Π_1 for each $\alpha \in \Pi_1$. This shows that $\Phi_1 \subset E_1$, the subspace spanned by Π_1, and likewise $\Phi_2 \subset E_2$. Thus we have $(\Phi_1, \Phi_2) = 0$ and this forces $\Phi_1 = \phi$ or $\Phi_2 = \phi$, so $\Pi_1 = \phi$ or $\Pi_2 = \phi$. //

Lemma 5.3.10. Let Φ be irreducible. Regard E as a WG-module. Then E is WG-irreducible and in particular, the WG-orbit of a root α spans E (i.e., $WG(\alpha) = E$).

Proof. Let E' be a nonzero WG-stable subspace of E and let $E'' = E'^{\perp}$ be the radical of E' in E under $(\ ,\ )$. Since $(\ ,\ )$ is nondegenerate, $E = E' \oplus E''$ and E'' is clearly WG-stable. Let $\alpha \in \Phi$. By Lemma 5.3. either $\alpha \in E'$ or $E' \subseteq P_\alpha$, since $\sigma_\alpha(E') = E'$. Thus $\alpha \notin E'$ implies $\alpha \in E''$, so each root lies in E' or E''. Since Φ is irreducible and spans E, this implies that $E' = E$. For the second part, note that $WG(\alpha) \neq 0$ and $WG(\alpha)$ is WG-stable. //

Lemma 5.3.11. Let Φ be irreducible. Then at most two root lengths occur in Φ, and all roots of the same length are WG-conjugate.

Proof. Let α, β be any roots. Since the inner product is invariant under WG and $WG(\alpha) = E$ (Lemma 5.3.10), we may assume $(\alpha, \beta) \neq 0$. Then the possible ratios $||\alpha||^2/||\beta||^2$ are 1, 2, 3, ½, 1/3 (the table following Definition 4.5.3). Suppose that there are three root lengths and let α, β, γ be roots with $||\alpha||^2 < ||\beta||^2 < ||\gamma||^2$. Then $||\beta||^2/||\alpha||^2 = 2, 3$ and $||\gamma||^2/||\alpha||^2 = 2, 3$, and these give $||\gamma||^2/||\beta||^2 = 3/2$, a contradiction. Let α, β have the same length. As in the above, we may assume that $(\alpha,\beta) \neq 0$ and $\alpha \neq \beta$ (if $\alpha = \beta$, we are done). The same table shows $<\alpha,\beta> = <\beta,\alpha> = \pm 1$. Replacing β by $-\beta = \sigma_\beta(\beta)$, if necessary, we may assume $<\alpha,\beta> = 1$. Hence $(\sigma_\alpha\sigma_\beta\sigma_\alpha)(\beta) = \sigma_\alpha\sigma_\beta(\beta-\alpha) = \sigma_\alpha(-\beta-\alpha+\beta) = \alpha$. //

In view of Theorem 5.3.6(3), the first part of Lemma 5.3.11 is also a consequence of the classification of Dynkin diagrams in Section 4.6. If L is a simple Lie algebra of types A_ℓ, D_ℓ, E_6, E_7, E_8 then Φ has only one root length and for the remaining types Φ has two root lengths. If Φ has two root lengths then we can now speak of long and short roots.

Theorem 5.3.12. Let Φ be irreducible.

(1) Let β' be a positive root such that $(\beta',\alpha) \geqq 0$ for all $\alpha \in \Pi$. Then $\beta' = \Sigma k_\alpha \alpha$ $(\alpha \in \Pi)$ with $k_\alpha > 0$ for all $\alpha \in \Pi$.

(2) Relative to the partial ordering $<$ in E, there is a unique maximal root β and β satisfies $(\beta,\alpha) \geqq 0$ for all $\alpha \in \Pi$. Furthermore, β is the unique positive root of maximal level.

(3) If β' is any positive root with $(\beta',\alpha) \geqq 0$ for all $\alpha \in \Pi$ then $\beta = \beta'$ or $\beta - \beta' \in \Phi^+$.

(4) β is long (if Φ has one root length, regard all roots long).

Proof. (1) Let $\beta' = \Sigma k_\alpha \alpha$ $(\alpha \in \Pi)$ and set $\Pi_1 = \{\alpha \in \Pi \mid k_\alpha > 0\}$ and $\Pi_2 = \{\alpha \in \Pi \mid k_\alpha = 0\}$. Suppose $\Pi_2 \neq \phi$. Since Π is irreducible (Lemma 5.3.9) and $(\alpha,\alpha') \leqq 0$ for all $\alpha \neq \alpha'$ in Π (Theorem 4.5.2), we have $(\alpha,\alpha') < 0$ for some $\alpha \in \Pi_2$ and $\alpha' \in \Pi_1$. But this leads to $(\alpha,\beta') < 0$, a contradiction.

(2) Let β be maximal in the ordering. Clearly $\beta > 0$. If $(\alpha,\beta) < 0$ for some $\alpha \in \Pi$, $\alpha + \beta$ would be a root (Corollary 4.4.8) and this contradicts the maximality of β. Thus $(\alpha,\beta) \geqq 0$ for all $\alpha \in \Pi$ and $(\alpha,\beta) > 0$ for some $\alpha \in \Pi$ since Π spans E. Let β_0

be another maximal root, so $(\alpha', \beta_0) \geqq 0$ for all $\alpha' \in \Pi$ and by (1) $\beta_0 = \Sigma k_\alpha \alpha$, $k_\alpha > 0$ for all $\alpha \in \Pi$. When $(\alpha, \beta) > 0$ for some $\alpha \in \Pi$, we conclued that $(\beta_0, \beta) > 0$. It follows from Corollary 4.4.8 that $\beta - \beta_0 \in \Phi$ or $\beta = \beta_0$. If $\beta - \beta_0 \in \Phi$, either $\beta < \beta_0$ or $\beta_0 < \beta$, a contradiction. Hence β is unique. Let β_1 be a positive root with maximal level. Then β_1 is evidently maximal in the ordering (any root β_2 with $\beta_1 < \beta_2$ has level $> |\beta_1|$). Thus $\beta = \beta_1$.

(3) As in the last part of the proof of (2), we have $(\beta, \beta') > 0$ and $\beta = \beta'$ or $\beta - \beta' \in \Phi^+$.

(4) Let α be any root. In view of Theorem 5.3.8(6) we may assume that $\alpha \in \overline{C(\Pi)}$, so that $(\alpha, \alpha') \geqq 0$ for all $\alpha' \in \Pi$. By (3) $\beta - \alpha \geqq 0$ and so $(\gamma, \beta - \alpha) \geqq 0$ for any $\gamma \in \overline{C(\Pi)}$. By (1) $(\beta, \beta - \alpha) \geqq 0$, so $(\beta, \beta) \geqq (\alpha, \beta)$ while $(\alpha, \beta - \alpha) \geqq 0$, showing $(\alpha, \beta) \geqq (\alpha, \alpha)$. Thus $(\beta, \beta) \geqq (\alpha, \alpha)$. //

<u>Exercise 5.3.4</u>. Verify that the simple Lie algebras have the following maximal roots :

A_ℓ : $\alpha_1 + \alpha_2 + \dots + \alpha_\ell$,

B_ℓ : $\alpha_1 + 2\alpha_2 + \dots + 2\alpha_\ell$,

C_ℓ : $2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{\ell-1} + \alpha_\ell$,

D_ℓ : $\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{\ell-2} + \alpha_{\ell-1} + \alpha_\ell$,

E_6 : $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$,

E_7 : $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$,

E_8 : $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$,

F_4 : $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$,

G_2 : $3\alpha_1 + 2\alpha_2$.

Exercise 5.3.5. Depict the root systems Φ for $A_1 \oplus A_1$, B_2 and G_2 in the plane. Prove that the Weyl groups of $A_1 \oplus A_1$, A_2 , B_2 and G_2 are respectively dihedral groups of order 4, 6, 8, 12 .

The results discussed in this sections all hold for an "abstract" root system Φ . In other words, the only conditions to be imposed on Φ are :

(R1) Φ is finite, spans E and does not contain 0

(R2) If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3) $\sigma_\alpha(\Phi) \subset \Phi$ for all reflections σ_α , $\alpha \in \Phi$.

(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in Z$.

For details, consult Humphreys' book.

5.4. Integral functions

Let H be a split CSA of a split semisimple Lie algebra L and $\Pi = (\alpha_1, \ldots, \alpha_\ell)$ be a simple system of roots relative to some ordering in H_0^*. Denote by $h_1, h_2, \ldots, h_\ell$ be the basis of H as in (1.45), corresponding to Π (Section 4.5).

Definition 5.4.1. An element $\lambda \in H^*$ is called integral if $\langle\lambda, \alpha\rangle \in Z$ for all $\alpha \in \Phi$. Denote by Λ the set of all integral functions in H^*. Define $\lambda \in \Lambda$ to be dominant if $\langle\lambda, \alpha_i\rangle = \lambda(h_i) \in Z^+$ for all $\alpha_i \in \Pi$ and to be strongly dominant if all $\langle\lambda, \alpha_i\rangle$ are positive. Denote by Λ^+ the set of dominant integral functions in H^*. //

One can imitate the proof of Theorem 4.4.6 to ensure that every element in Λ is a rational linear combination of roots in Π and hence $\Lambda \subset H_0^*$. Thus Λ is canonically embedded in E with the inner product $(\, , \,)$. An element $\lambda \in \Lambda$ is often called a weight in E in the abstract sense (see Humphreys' book). In view of Theorem 4.4.3, any weight of H in a finite-dimensional

L-module is integral. It will be shown later that any $\lambda \in \Lambda^+$ is a weight of H in some finite-dimensional irreducible L-module.

Since Π is a basis of E, the vectors $2\alpha_i/(\alpha_i,\alpha_i)$, $\alpha_i \in \Pi$, form a basis of E. Let $\Omega = \{\lambda_1,\lambda_2,\ldots,\lambda_\ell\}$ be the dual basis of this basis relative to the inner product, so that $2(\lambda_i,\alpha_j)/(\alpha_j,\alpha_j) = \delta_{ij}$, $i,j = 1,2,\ldots,\ell$

<u>Definition 5.4.2</u>. Call Ω the fundamental system of dominant weights (FSDW) relative to Π.

<u>Lemma 5.4.1</u>. The Weyl group WG leaves Λ invariant and Λ is a free Z-module with basis Ω. Moreover, $\lambda \in \Lambda^+$ if and only if $\lambda = \Sigma\, m_i\lambda_i$ with all $m_i \in Z^+$.

<u>Proof</u>. Let $\sigma \in$ WG and $\lambda \in \Lambda$. Then $<\sigma(\lambda),\sigma(\alpha)> = <\lambda,\alpha> \in Z$ for all $\alpha \in \Phi$ since WG preserves the inner product. Noting $\sigma(\Phi) = \Phi$, we have $\sigma(\lambda) \in \Lambda$. Let $\lambda \in \Lambda$ and let $m_i = <\lambda,\alpha_i>$, $\alpha_i \in \Pi$. Then $0 = <\lambda - \Sigma\, m_i\lambda_i,\alpha>$ for each $\alpha \in \Pi$ and $(\lambda - \Sigma\, m_i\lambda_i,\alpha) = 0$. Since Π spans E, we get $\lambda = \Sigma\, m_i\lambda_i$, $m_i \in Z$. Evidently, $\lambda \in \Lambda^+$ if and only if all $m_i = <\lambda,\alpha_i> \in Z^+$.

Note that $\Phi \subset \Lambda$. Let Λ_r be the Z-span of Π in E. Clearly, Λ_r is the Z-span of Φ also and is a free Z-module with basis Π which is a (additive) subgroup of Λ. Call the quotient group Λ/Λ_r the <u>fundamental group</u> of Φ.

Suppose that L is simple. Then the order of Λ/Λ_r can be directly computed from the Cartan matrix of L relative to Π. In fact, let $\alpha_i = \Sigma\, m_{ij}\lambda_j$. Then $\langle\alpha_i,\alpha_k\rangle = \Sigma_j\, m_{ij}\langle\lambda_j,\alpha_k\rangle = m_{ik}$, so (m_{ij}) is the Cartan matrix of L. Thus each λ_j is expressed as $\lambda_j = \frac{1}{d}\Sigma_i\, k_{ji}\alpha_i$ where $\frac{1}{d}(k_{ji})$ is the inverse matrix of $(\langle\alpha_i,\alpha_j\rangle)$, d is the determinant of the Cartan matrix and $k_{ji} \in Z$. As shown in Exercise 4.6.1, d is a positive integer and hence is the order of Λ/Λ_r. We state this as

<u>Lemma 5.4.2</u>. If L is simple then the fundamental group Λ/Λ_r of Φ has order : $\ell + 1$ for A_ℓ ; 2 for B_ℓ, C_ℓ, E_7 ; 4 for D_ℓ ; 3 for E_6 ; 1 for E_8, F_4, G_2. //

The explicit expressions of the λ_i in terms of simple roots are useful information for the structure of certain classes of simple flexible Lie-admissible algebras, which we discuss later. For examples, the algebra G_2 has the Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ with inverse $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$, so $\lambda_1 = 2\alpha_1 + \alpha_2$ and $\lambda_2 = 3\alpha_1 + 2\alpha_2$. The

Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ of A_2 has the inverse $1/3\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, so $\lambda_1 = \frac{1}{3}(2\alpha_1 + \alpha_2)$ and $\lambda_2 = \frac{1}{3}(\alpha_1 + 2\alpha_2)$ Through somewhat laborious computations, one can furnish the following table where $\Sigma\, c_i\alpha_i$ is abbreviated $(c_i, c_2, \ldots, c_\ell)$.

<u>Exercise 5.4.1</u>. Verify the following table.

$$A_\ell \;:\; \lambda_i = \frac{1}{\ell+1}\Big[(\ell-i+1)\alpha_1 + 2(\ell-i+1)\alpha_2 + \ldots + (i-1)(\ell-i+1)\alpha_{i-1} + i(\ell-i+1)\alpha_i + i(\ell-1)\alpha_{i+1} + \ldots + i\alpha_\ell\Big]$$

$$B_\ell \;:\; \lambda_i = \alpha_1 + 2\alpha_2 + \ldots + (i-1)\alpha_{i-1} + i(\alpha_i + \alpha_{i+1} + \ldots + \alpha_\ell) \quad (i < \ell)$$

$$\lambda_\ell = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + \ell\alpha_\ell)$$

$$C_\ell \;:\; \lambda_i = \alpha_1 + 2\alpha_2 + \ldots + (i-1)\alpha_{i-1} + i(\alpha_i + \ldots + \alpha_{\ell-1} + \tfrac{1}{2}\alpha_\ell + \tfrac{1}{2}i(\alpha_{\ell-1} + \alpha_\ell) \quad (i < \ell - 1)$$

$$\lambda_{\ell-1} = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + (\ell-2)\alpha_{\ell-2} + \tfrac{1}{2}\ell\alpha_{\ell-1} + \tfrac{1}{2}(\ell-2)\alpha_\ell$$

$$\lambda_\ell = \tfrac{1}{2}(\alpha_1 + 2\alpha_2 + \ldots + (\ell-2)\alpha_{\ell-2} + \tfrac{1}{2}(\ell-2)\alpha_{\ell-1} + \tfrac{1}{2}\ell\alpha_\ell)$$

E_6 : $\lambda_1 = \frac{1}{3}(4, 3, 5, 6, 4, 2)$

$\lambda_2 = (1, 2, 2, 3, 2, 1)$

$\lambda_3 = \frac{1}{3}(5, 6, 10, 12, 8, 4)$

$\lambda_4 = (2, 3, 4, 6, 4, 2)$

$\lambda_5 = \frac{1}{3}(4, 6, 8, 12, 10, 5)$

$\lambda_6 = \frac{1}{3}(2, 3, 4, 6, 5, 4)$

E_7 : $\lambda_1 = (2, 2, 3, 4, 3, 2, 1)$

$\lambda_2 = \frac{1}{2}(4, 7, 8, 12, 9, 6, 3)$

$\lambda_3 = (3, 4, 6, 8, 6, 4, 2)$

$\lambda_4 = (4, 6, 8, 12, 9, 6, 3)$

$\lambda_5 = \frac{1}{2}(6, 9, 12, 18, 15, 10, 5)$

$\lambda_6 = (2, 3, 4, 6, 5, 4, 2)$

$\lambda_7 = \frac{1}{2}(2, 3, 4, 6, 5, 4, 3)$

E_8 : $\lambda_1 = (4, 5, 7, 10, 8, 6, 4, 2)$

$\lambda_2 = (5, 8, 10, 15, 12, 9, 6, 3)$

$\lambda_3 = (7, 10, 14, 20, 16, 12, 8, 4)$

$\lambda_4 = (10, 15, 20, 30, 24, 18, 12, 6)$

$\lambda_5 = (8, 12, 16, 24, 20, 15, 10, 5)$

$\lambda_6 = (6, 9, 12, 18, 15, 12, 8, 4)$

$\lambda_7 = (4, 6, 8, 12, 10, 8, 6, 3)$

$\lambda_8 = (2, 3, 4, 6, 5, 4, 3, 2)$

F_4 : $\lambda_1 = (2, 3, 4, 2)$

$\lambda_2 = (3, 6, 8, 4)$

$\lambda_3 = (2, 4, 6, 3)$

$\lambda_4 = (1, 2, 3, 2)$

G_2 : $\lambda_1 = (2, 1)$

$\lambda_2 = (3, 2)$

Let L be simple and regard L as an L-module via ad . Then the weights of H are the roots of H . Let λ_0 be the unique maximal root of H relative to the partial ordering < in E . Comparing the list in Exercise 5.3.4 with the table above, one finds the following interesting result.

<u>Theorem 5.4.3</u>. Let L be a simple Lie algebra with split CSA H and let λ_0 be the maximal root of H relative to the partial ordering < in E . Then

$\lambda_0 = \lambda_1 + \lambda_\ell$ for $A_\ell (\ell \geqq 1)$,

$\lambda_0 = \lambda_1$ for E_7, F_4 ,

$\lambda_0 = \lambda_2$ for $B_\ell (\ell \geqq 3)$, $D_\ell (\ell \geqq 4)$, E_6, G_2 ,

$\lambda_0 = 2\lambda_1$ for $C_\ell (\ell \geqq 2)$,

$\lambda_0 = \lambda_8$ for E_8 . //

In Theorem 5.4.3 we adopted the ordering of Π as in Theorem 4.6.1.

<u>Exercise 5.4.2</u>. Let $\gamma_1,\ldots,\gamma_\ell$ be an "obtuse" basis of E (i.e., all $(\gamma_i, \gamma_j) \leqq 0$ for $i \neq j$). Prove that the dual basis $\gamma_1^*,\ldots,\gamma_\ell^*$ is "acute" (i.e., $(\gamma_i^*, \gamma_j^*) \geqq 0$ for $i \neq j$).

<u>Lemma 5.4.4</u>. Each $\lambda \in \Lambda$ is WG-conjugate to one and only one dominant integral function. If λ is dominant then $\sigma(\lambda) \leqq \lambda$ for all $\sigma \in WG$, and if λ is strongly dominant then $\sigma(\lambda) = \lambda$ only when $\sigma = 1$.

<u>Proof</u>. The first part follows from Theorem 5.3.8(6), since $WG(\Lambda) \subset \Lambda$. Recall that Theorem 5.3.8(1), so (5), requires the only conditions that $(\lambda,\alpha) \geqq 0$ for all $\alpha \in \Pi$ and $WG(\Lambda) \subset \Lambda$. This proves the second part while the last part is immediate from Theorem 5.3.8(2). //

<u>Lemma 5.4.5</u>. Let $\lambda \in \Lambda^+$. The number of dominant $\mu \in \Lambda^+$ such that $\mu \leqq \lambda$ is finite.

<u>Proof</u>. Recall that each $\lambda_i \in \Omega$, FSDW, is a rational linear combination of the $\alpha_i \in \Pi$. It is easy to see that these rational coefficients are all nonnegative. In fact, let $\lambda_i = \Sigma\, r_{ij}\alpha_j$. Then $(\lambda_i, \lambda_k) = \Sigma\, r_{ij}(\alpha_j, \lambda_k) = r_{ik}(\alpha_k, \alpha_k)$ and $r_{ik} \geqq 0$

by Exercise 5.4.2. Thus, by Lemma 5.4.1, $\lambda = \Sigma r_i\alpha_i$ and $\mu = \Sigma s_i\alpha_i$, $r_i, s_i \in Q^+$. If $\mu \leqq \lambda$ then all $r_i - s_i \in Z^+$. This allows only finitely many possibilities for the s_i (why ?). //

5.5. Weights and standard cyclic modules

Let L, H, Π and F be the same as in Section 5.4. If V is a finite-dimensional L-module, recall that, in view of Corollary 4.2.3(1), H acts diagonally on $V = \Sigma_{\lambda \varepsilon H^*} V_\lambda(H)$ where $V_\lambda = V_\lambda(H)$ $= \{v \varepsilon V \mid vh = \lambda(h)v$ for all $h \varepsilon H\}$, and that $\lambda \varepsilon H^*$ is called a weight of H in V if $V_\lambda \neq 0$. We extend the notion of weight to an arbitrary L-module.

Definition 5.5.1. Let V be an L-module of arbitrary dimension and let $V_\lambda = \{v \varepsilon V \mid vh = \lambda(h)v$ for all $h \varepsilon H\}$, $\lambda \varepsilon H^*$. If $V_\lambda \neq 0$, we call V_λ a weight space and call λ a weight of H in V. //

Unlike the finite-dimensional case (Theorem 3.2.9), if $\dim L = \infty$ then there is no guarantee that V is the sum of its weight spaces. However, the sum V' of all weight spaces V_λ is direct : The argument is the same as in showing that eigenvectors of distinct eigenvalues for a single linear transformation are linearly independent (check!). If V is an arbitrary L-module then various situations can happen, as shown by

<u>Exercise 5.5.1</u>. (1) If V is an irreducible L-module having at least one (nonzero) weight space, prove that V is the direct sum of its weight spaces.

(2) Let V be an irreducible L-module. Then V has a weight space if and only if $vU(H)$ is finite-dimensional for all $v \in V$, or if and only if vA is finite-dimensional for all $v \in V$ where A is the subalgebra with 1 generated by an arbitrary $h \in H$ in $U(H)$, the universal enveloping algebra of H.

(3) Let $L = s\ell(2,F)$ with the canonical basis x, y, h as in (1.28). Show that $1 - x$ is not invertible in $U(L)$, so $1 - x$ lies in a maximal right ideal I of $U(L)$. Let $V = U(L)/I$, so V is an irreducible L-module. Prove that $1 + I, h + I, h^2 + I, \ldots$ are all linearly independent in V (so $\dim V = \infty$),

using the relations

$$(x-1)^r h^s \equiv \begin{cases} 0 \pmod{I}, & r > s \\ (-2)^r \, r! \cdot 1 \pmod{I}, & r = s . \end{cases}$$

Show that V has no weight spaces. [See F.W.Lemire, Existence of weight space decompositions for irreducible representations of simple Lie algebras, Canad. Math. Bull. 14 (1971), 113-115.]

Due to Weyl's Theorem on complete reducibility (Theorem 3.7.5), the determination of finite-dimensional L-modules reduces to that of finite-dimensional irreducible L-modules. The objective in this section is to prove that there exists a one-one correspondence between the set Λ^+ of dominant integral functions in H^* and the isomorphism classes of finite-dimensional irreducible L-modules. The correspondence is given by assigning to each $\lambda \in \Lambda^+$ a cyclic L-module generated by a single element which has λ as the highest weight. For this, we proceed in steps. A universal enveloping algebra and a maximal solvable subalgebra of L play main roles.

<u>Definition 5.5.2</u>. A Borel subalgebra of any Lie algebra L is a maximal solvable subalgebra of L. //

While we do not intend to elaborate on Borel subalgebras, we are interested in a special type of Borel subalgebra (for detail, e.g., the conjugacy of Borel subalgebras under an invariant automorphism, see Humphreys' book, p. 83).

Let Π be a simple system of roots and let $L = H + \Sigma_{\alpha \neq 0} L_\alpha$ be the Cartan decomposition of L.

<u>Lemma 5.5.1</u>. Set $B(\Pi) = H + \Sigma_{\alpha > 0} L_\alpha$. Then $B(\Pi)$ is a Borel subalgebra of L.

<u>Proof</u>. Let $N = [B(\Pi)B(\Pi)]$. Then, since L is semisimple, $N = \Sigma_{\alpha > 0} L_\alpha$. We contend that N is nilpotent. Note that $[NN] = \Sigma_{\alpha+\beta>0} L_{\alpha+\beta}$ and the level of a positive root $\alpha + \beta$, $\alpha, \beta \in \Phi^+$, increases by at least one. Since Φ^+ is finite, continuing this brings the descending central series of N to zero. This in particular implies that $B(\Pi)$ is solvable. Let J be a subalgebra of L properly containing $B(\Pi)$. Then the weight space decomposition of J relative to ad H must involve a weight γ (in fact, a root) with $\gamma \notin \Phi^+$, so γ is a negative root. But this forces K to contain the 3-dimensional split simple Lie algebra $S_{-\gamma}$ as in Theorem 4.4.2(1), so K can not be solvable. //

Call $B(\Pi)$ a <u>standard Borel subalgebra</u> relative to H. Note that $B(\Pi)$ depends on the choice of Π.

<u>Lemma 5.5.2</u>. Let V be an arbitrary L-module. Then

(1) $V_\lambda L_\alpha \subset V_{\lambda+\alpha}$, $\lambda \varepsilon H^*$, $\alpha \varepsilon \Phi$.

(2) The sum $V' = \Sigma_{\lambda \varepsilon H^*} V_\lambda$ is direct and V' is an L-submodule of V.

(3) If $\dim V < \infty$ then $V = V'$.

<u>Proof</u>. If $x \varepsilon L_\alpha$, $v \varepsilon V_\lambda$ and $h \varepsilon H$, then $(vx)h = v[xh] + (vh)x = (\alpha(h) + \lambda(h))vx$, showing $V_\lambda L_\alpha \subset V_{\lambda+\alpha}$. The first part of (2) is just the foregoing remark while the second part follows from (1). (3) is clear from Corollary 4.2.3(1). //

Recall that if V is an L-module then the L-module action on V is extended to define a unique unital right $U(L)$-module V where $U(L)$ is the universal enveloping algebra of L.

<u>Definition 5.5.2</u>. Let V be an arbitrary L-module. A maximal vector of weight λ in V is defined to be a nonzero vector $v^+ \varepsilon V_\lambda$ such that $v^+ L_\alpha = 0$ for all $\alpha \varepsilon \Phi^+$ (or just $\alpha \varepsilon \Pi$; Theorem 4.5.6). If $V = v^+ U(L)$

for a maximal vector of weight λ then V is called standard cyclic of weight λ and λ is called the highest weight of V. //

Notice that the notion of maximal vector depends on the choice of Π and the existence of such vectors is not guaranteed in case $\dim V = \infty$. However, we have

Lemma 5.5.3. Let V be a finite-dimensional L-module. Then a nonzero $v^+ \in V$ is a maximal vector of weight λ if and only if v^+ is a common eigenvector of $B(\Pi)$. Hence V has a maximal vector, and if, in addition, V is irreducible then V is standard cyclic of weight λ. If L is simple and regarded as an L-module via ad, then $0 \neq v^+ \in L$ is a maximal vector of weight β if and only if β is the unique maximal root and $v^+ \in L_\beta$.

Proof. Any maximal vector of weight λ is evidently a common eigenvector for $B(\Pi)$. Let $v^+ \neq 0$ be a common eigenvector for $B(\Pi)$. Then $v^+h = \lambda(h)v^+$ for $h \in H$, so λ is a weight. Also, $v^+L_\alpha \subset V_\lambda$, $\alpha > 0$ but by Lemma 5.5.2(1) $v^+L_\alpha \subset V_{\lambda+\alpha}$; this is absurd unless $v^+L_\alpha = 0$ for $\alpha > 0$. Since $B(\Pi)$ is solvable, by Corollary 3.4.10 $B(\Pi)$ has a common eigenvector in V. If V is irreducible then $v^+U(L)$, being a nonzero L-submodule, must be V. The last part is clear (Theorem 5.3.12). //

Therefore, the study of standard cyclic L-modules embraces that of finite-dimensional irreducible L-modules. Recall that the partial ordering $\lambda > \mu$ in E if and only if $\lambda - \mu$ is a sum of positive roots. This ordering is also defined in H* . We show below that the highest weight λ associated with a maximal vector is highest in the sense that $\mu \leqq \lambda$ for all weights μ in V . Recall also that for a fixed nonzero $x_\alpha \in L_\alpha$ $(\alpha > 0)$ there is a unique $y_\alpha \in L_{-\alpha}$ for which $[x_\alpha y_\alpha] = h_\alpha$ (Theorem 4.4.2(1)) where $h_\alpha \in H$ is as in (1.32). The following describes the structure of a standard cyclic module.

<u>Theorem 5.5.4</u>. Let V be a standard cyclic L-module with maximal vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \beta_2, \ldots, \beta_m\}$. Then,

(1) V is spanned by the vectors $v^+ y_{\beta_1}^{i_1} \ldots y_{\beta_m}^{i_m}$ $(i_j \in Z^+)$. In particular, V is the direct sum of its weight spaces (compare with Lemma 5.5.2(3)).

(2) The weights of V are of the form $\mu = \lambda - \Sigma_{i=1}^{\ell} k_i\alpha_i$ $(k_i \in Z^+)$. Thus all weights μ satisfy $\mu \leqq \lambda$.

(3) For each $\mu \in H^*$, $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$.

(4) V is an indecomposable L-module.

(5) If W is an L-submodule of V then $W = \sum_{\mu} \oplus W \cap V_{\mu}$ is the weight space decomposition for W. Hence V contains a unique maximal (proper) L-submodule.

(6) Every nonzero homomorphic image of V is also standard cyclic of highest weight λ.

<u>Proof</u>. (1) and (2). Let $L = B + N_-$ where $N_- = \Sigma_{\alpha<0} L_\alpha$ and $B = B(\Pi)$. Note that N_- is a subalgebra of L. It follows from Theorem 5.1.10(1) that $V = v^+U(L) = v^+U(B)U(N_-) = Fv^+U(N_-)$, since v^+ is a common eigenvector for B. By the PBW theorem $U(N_-)$ has a basis consisting of the standard monomials $y_{\beta_2}^{i_1} \ldots y_{\beta_m}^{i_m}$, so the vectors

$$(*) \qquad v^+ y_{\beta_1}^{i_1} y_{\beta_2}^{i_2} \ldots y_{\beta_m}^{i_m}, \quad i_j \in Z^+$$

form a basis for V. Thus V is the direct sum of weight spaces V_μ and since each vector in $(*)$ has weight $\lambda - \Sigma_j i_j\beta_j$ by Lemma 5.5.2(1), rewriting each β_j as a sum of simple roots, we have (2).

(3) If $\mu = \lambda - \Sigma_i k_i\alpha_i$ is a weight then $\dim V_\mu$ is the number of vectors $(*)$ for which $\Sigma_j i_j\beta_j = \Sigma k_i\alpha_i$ $(k_i \in Z^+)$. This number is finite when $\Sigma k_i\alpha_i$ is fixed.

The only vector in $(*)$ having weight $\mu = \lambda$ is v^+, so $\dim V_\lambda = 1$.

(4) Suppose that $V = V_1 \oplus V_2$, V_i L-submodules of V. Let $v^+ = v_1^+ + v_2^+$, $v_i^+ \in V_i$. Then $v^+x = v_1^+x + v_2^+x$ is a scalar multiple of v^+ for all $x \in B$ (Lemma 5.5.3). If $v_i^+ \neq 0$, this forces v_i^+ to be a maximal vector of weight λ. Since $\dim V_\lambda = 1$ by (3), this in turn implies that v_i^+ is a scalar multiple of v^+. Since $V = V_1 + V_2$ is direct, we have $v_1^+ = 0$ or $v_2^+ = 0$, so $v^+ \in V_2$ or $v^+ \in V_1$; that is, $V = V_2$ or $V = V_1$ since v^+ generates V.

(5) Clearly $\Sigma_\mu \oplus W \cap V_\mu \subset W$. Let $v \in W$. Then $v \in W \cap (V_{\mu_1} + \ldots + V_{\mu_n}) \equiv W'$ and regard W' as an H-module. It is clear that any weight of H in W' is a weight in V and hence is among $\mu_1, \ldots, \mu_n$. This implies that if $W' = \Sigma \oplus W'_\mu$ is the weight space decomposition (this decomposition exists since W' is finite-dimensional (Theorem 3.2.9)) then $W'_\mu = W \cap V_\mu$ and $W' = \sum_{i=1}^{n} \oplus W \cap V_{\mu_i}$. Thus $W \subset \Sigma_\mu \oplus W \cap V_\mu$. If W is proper then $W \cap V_\lambda = 0$ since if $W \cap V_\lambda \neq 0$, $V_\lambda \subset W$ (by part (3)) and $W = V$ as v^+ generates V. Therefore $W \subset \Sigma_{\mu \neq \lambda} V_\mu$ and the sum of all proper L-submodules of V is still proper.

(6) Let $f : V \to V' = f(V)$ be a nontrivial L-module homomorphism. Then $V' = f(v^+)U(L)$ and $f(v^+) \neq 0$ since $V' \neq 0$, and clearly $f(v^+)$ is a maximal vector of weight λ. //

<u>Corollary 5.5.5</u>. Let L be simple and let β be the maximal root relative to Π. Then $\alpha \leqq \beta$ for all $\alpha \in \Phi$.

<u>Proof</u>. Note that L is standard cyclic of highest weight β (Lemma 5.5.3) and the weights of H are just the roots in Φ. The result follows from Theorem 5.5.4(2). //

<u>Corollary 5.5.6</u>. Let V be as in Theorem 5.5.4. Suppose, in addition, that V is L-irreducible. Then v^+ is the unique maximal vector in V up to nonzero scalar multiples.

<u>Proof</u>. Let w^+ be another maximal vector in V. Then $w^+U(L) = V$ since V is L-irreducible. If w^+ has weight λ', Theorem 5.5.4 applied to v^+, w^+ implies that $\lambda' \leqq \lambda$ and $\lambda \leqq \lambda'$, so $\lambda = \lambda'$. But then by Theorem 5.5.4(3) w^+ is a scalar multiple of v^+. //

5.6. Finite-dimensional modules

Since finite-dimensional irreducible L-modules are closely related to standard cyclic L-modules, it is natural to consider the following basic questions as to :

(a) Existence of irreducible standard cyclic modules.

(b) Uniqueness of such modules.

(c) Which of such modules are finite-dimensional.

(d) Enumeration of $\dim V_\mu$ if μ is a weight.

In this section we treat Questions (a), (b), (c) and we can obtain the results for these along the expected line. Due to the uniqueness (b), we may denote by $V(\lambda)$ the irreducible standard cyclic module of highest weight λ. A rigorous treatment of the Question (d) involves a somewhat laborious procedure which we omit here. $\dim V_\mu$ is known as the <u>multiplicity</u> of μ in $V(\lambda)$. There are two well-known formulas for this, Freudenthal's formula and Kostant's formula. Related to this is also Weyl's formula which enables us to enumerate $\dim V(\lambda)$. Finally, consider the L-module $V(\lambda') \otimes V(\lambda'')$ where λ', λ'' are dominant integral. In view of Weyl's Theorem 3.7.5, $V(\lambda') \otimes V(\lambda'')$ is the direct sum $\Sigma_\lambda \oplus V(\lambda)$

of irreducible standard cyclic L-modules $V(\lambda)$. Steinberg's formula is designed to enumerate the number of times $V(\lambda)$, $\lambda \in \Lambda^+$, occurs in $V(\lambda') \otimes V(\lambda'')$. For the development of these formulas in detail, the reader may consult Jacobson's book or Humphreys' book.

We first consider the uniqueness question.

Theorem 5.6.1. Let V and W be standard cyclic L-modules of highest weight λ . If V and W are irreducible then V and W are isomorphic as L-modules.

Proof. Consider the L-module $U = V \oplus W$ and let v^+, w^+ be respective maximal vectors of weight λ in V , W . Clearly, $u^+ = (v^+,w^+)$ is a maximal vector of weight λ in U . Let X be the L-submodule of U generated by u^+ , so X is standard cyclic of weight λ . Let $p : X \to V$, $p' : X \to W$ be the maps induced by the projections of U onto V and W . Clearly, p and p' are L-module homomorphisms, and since $p(u^+) = v^+$ and $p'(u^+) = w^+$, p, p' are surjective. Thus, it suffices to verify that p, p' are injective, so that X is isomorphic to both V and W as L-modules. Note that $\ker p' = V \cap X$ where $(v,0)$ $(v \in V)$ is identified with v . Since $\ker p'$ is an L-submodule of V and V is irreducible, $V \cap X = 0$ or $V \cap X = V$. If $V \cap X = V$ then $(v^+,0) \in X$ is a maximal vector of weight λ in X

(since it is killed by $B(\Pi)$), so $(v^+,0)$ is a scalar multiple of (v^+,w^+) (Theorem 5.5.4(3)). This is impossible, so p' (and likewise p) is injective. //

Since an irreducible standard cyclic L-module of highest weight λ is uniquely determined by the weight λ (up to isomorphism), we denote this module (if exists) by $V(\lambda)$. As to the existence question, the central idea stems from the fact that when any standard cyclic L-module is viewed as a $B(\Pi)$-module, it contains a one-dimensional $B(\Pi)$-submodule spanned by a maximal vector. Hence we begin with a one-dimensional space in the following existence theorem.

Theorem 5.6.2. For any $\lambda \in H^*$, there exists an irreducible standard cyclic L-module $V(\lambda)$ of weight λ

Proof. Let $D_\lambda = Fv^+$ be a one-dimensional vector space with basis v^+ and let $B = B(\Pi)$. Make D_λ a B-module by the rule $v^+(h + \Sigma_{\alpha>0}\, x_\alpha) = v^+h = \lambda(h)v^+$, $h \in H$, $x_\alpha \in L_\alpha$; it is easy to check that this is a B-module action on D_λ. This also makes D_λ a right U(B)-module. Regarding U(L) as a left U(B)-module, we form the tensor product $Z(\lambda) = D_\lambda \otimes_{U(B)} U(L)$. At this moment, $Z(\lambda)$ is simply an abelian group ; however, we make $Z(\lambda)$ a right U(L)-module via the natural right action in $U(L)$: $(v^+ \otimes t)t' = v^+ \otimes tt'$, $t,t' \in U(L)$.

We contend that $Z(\lambda)$ is standard cyclic of weight λ. Clearly, $v^+ \otimes 1$ generates $Z(\lambda)$ and since $U(L)$ is a free $U(B)$-module (Theorem 5.1.10(1)), $v^+ \otimes 1 \neq 0$. Therefore $v^+ \otimes 1$ is a maximal vector of weight λ. In view of Theorem 5.5.4(5), $Z(\lambda)$ contains a maximal (proper) L-submodule $Y(\lambda)$ and $V(\lambda) = Z(\lambda)/Y(\lambda)$ is irreducible standard cyclic of weight λ (Theorem 5.5.4(6)). //

<u>Remark</u>. Let $Z(\lambda) = D_\lambda \otimes_{U(B)} U(L)$ be the same as above.

(1) By PBW Theorem, $U(L) \cong U(B) \otimes_F U(N_-)$, where $N_- = \Sigma_{\alpha<0} L_\alpha$. Then $Z(\lambda)$ is viewed as a right $U(N_-)$-module, so that $Z(\lambda) \cong F \otimes U(N_-)$ (as right $U(N_-)$-modules) via $kv^+ \otimes (b \otimes t) \to k1 \otimes bt$, $k \in F$, $b \in U(B)$, $t \in U(N_-)$ (again by PBW Theorem).

(2) An alternative construction of $Z(\lambda)$ is as follows. Let $I(\lambda)$ be the right ideal in $U(L)$ generated by all L_α $(\alpha > 0)$ and all $h_\alpha - \lambda(h_\alpha)1$ $(\alpha \in \Phi)$. Since $v^+h = \lambda(h)v^+$ and $v^+L_\alpha = 0$, $\alpha > 0$, $v^+I(\lambda) = 0$. Define a mapping $g : U(L)/I(\lambda) \to Z(\lambda)$ by $g(t + I(\lambda)) = v^+ \otimes t$, $t \in U(L)$. If $t \in I(\lambda)$ then $v^+ \otimes t = v^+ \otimes (\Sigma b_i t_i) = \Sigma v^+ b_i \otimes t_i = 0$ where $b_i \in U(B) \cap I(\lambda)$ (note $U(L) = U(B)U(N_-)$). Thus g is well-defined, and so a $U(L)$-module homomorphism. Suppose that $v^+ \otimes t = 0$

for $t \in U(L)$. Write $t = \Sigma b_i t_i$ where the t_i are $U(B)$-basis elements of $U(L)$ so $b_i \in U(B)$ (PBW Theorem). Then $0 = v^+ \otimes t = \Sigma v^+ b_i \otimes t_i$, so $v^+ b_i \otimes t_i = 0$ for all i since $U(L)$ is $U(B)$-free. This implies that the $b_i \in U(B) \cap I(\lambda)$, so $t \in I(\lambda)$. Thus g is injective and an isomorphism. //

Finally, we attack a necessary and sufficient condition for an irreducible standard cyclic L-module $V(\lambda)$ to be finite-dimensional. Corresponding to each $\alpha_i \in \Pi$, let S_i denote the 3-dimensional simple Lie algebra spanned by the canonical basis x_i, y_i, h_i . As before, let Λ and Λ^+ be respectively the integral and dominant integral functions in H^* . It is easy to see that a necessary condition for $V(\lambda)$ to be finite-dimensional is that $\lambda \in \Lambda^+$:

Theorem 5.6.3. If V is a finite-dimensional irreducible L-module of highest weight λ then λ is dominant integral.

Proof. For each $i = 1, 2, \ldots, \ell$, regard V as an S_i-module. Let v^+ be a maximal vector for L in V of weight λ . Since V is a finite-dimensional S_i-module and $\lambda(h_i)$ is a weight of h_i , v^+ is a maximal vector of weight $\lambda(h_i)$ in an irreducible

S_i-submodule V' of V. Thus $\lambda(h_i)$ is a nonnegative integer (see Section 4.3). //

Indeed, that $\lambda \in \Lambda^+$ is a sufficient condition also. The proof of this is carried out in steps. Let A be an associative algebra and let $x, a \in A$. One can easily prove by induction the formula

$$(*) \qquad a^k x = xa^k - \binom{k}{1}x'a^{k-1} + \binom{k}{2}x''a^{k-2} - \dots \pm x^{(k)}$$

where $x^{(0)} = x$, $x' = [x, a]$ and $x^{(i+1)} = (x^{(i)})'$.

Lemma 5.6.4. Let $U(L)$ be the universal enveloping algebra of L. For $k \geqq 0$, $1 \leqq i$, $j \leqq \ell$, the following identities hold in $U(L)$.

(1) $[y_i^{k+1}, x_j] = 0$, $i \neq j$;

(2) $[y_i^{k+1}, x_i] = (k + 1)(k1 - h_i)y_i^k$.

Proof. (1) $[y_i, x_j] = 0$ if $i \neq j$, since $\alpha_j - \alpha_i$ is not a root (Theorem 4.5.2(1)).

(2) If $k = 0$, it is trivial. Assume that $[y_i^k, x_i] = k[(k - 1)1 - h_i]y_i^{k-1}$. First, note that $h_i' = [h_i, y_i] = 2y_i$, $h_i'' = [[h_i, y_i], y_i] = 0$, $h_i^{(t)} = 0$ for $t \geqq 2$ in $(*)$. Using this, we compute

$$[y_i^{k+1}, x_i] = y_i^k[y_i, x_i] + [y_i^k, x_i]y_i$$

$$= -y_i^k h_i + k[(k-1)1 - h_i]y_i^k \quad \text{(by inductio}$$

$$= 2ky_i^k - h_i y_i^k + k(k-1)y_i^k - kh_i y_i^k \quad \text{(by (*}$$

$$= (k+1)(k1 - h_i)y_i^k . \qquad //$$

Lemma 5.6.5. Let $V = V(\lambda)$ be an irreducible standard cyclic L-module of weight λ where λ is dominant integral.

(1) If v^+ is a maximal vector or weight λ then, for each $1 \leqq i \leqq \ell$, $v^+ y_i^{m_i+1} = 0$ where $m_i = \lambda(h_i)$.

(2) For each $1 \leqq i \leqq \ell$, V is the sum of finite-dimensional S_i-submodules.

(3) For any $v \in V$ and each $1 \leqq i \leqq \ell$, there exist positive integers r , s such that $vx_i^r = vy_i^s = 0$.

Proof. (1) Set $w = v^+ y_i^{m_i+1}$. The m_i are nonnegative integers by the assumption. If $i \neq j$, $wx_j = 0$ by Lemma 5.6.4(1). Using Lemma 5.6.4(2), we have

$$v^+ y_i^{m_i+1} x_i = v^+[y_i^{m_i+1}, x_i] + v^+ x_i y_i^{m_i+1}$$

$$= (m_i + 1)(m_i v^+ - m_i v^+)y_i^{m_i+1}$$

$$= 0 ,$$

since $v^+x_i = 0$ and $v^+h_i = \lambda(h_i)v^+ = m_iv^+$. Thus $wx_i = 0$ for $i = 1,\ldots,\ell$. If $w \neq 0$, w would be a maximal vector in V . But, by Lemma 5.5.2(1), w has weight $\lambda - (m_i + 1)\alpha_i \neq \lambda$, which is contrary to Corollary 5.5.6. Hence $w = 0$.

(2) We first show that V contains a nonzero finite-dimensional S_i-submodule for each i . Let W be the subspace spanned by $v^+, v^+y_i,\ldots,v^+y_i^{m_i}$. By (1), W is y_i-stable. Since these vectors are weight vectors in V , W is h_i-stable. These in turn imply that, in view of Lemma 5.6.4(2), W is also x_i-stable, so $W \neq 0$ is a finite-dimensional S_i-submodule. Now, let V' be the sum of all finite-dimensional S_j-submodules. Then $V' \neq 0$. Let W be any finite-dimensional S_i-submodule and W' be the span of the subspaces WL_α , $\alpha \in \Phi$, and Wh_j , $j = 1,\ldots,\ell$. Then W' is finite-dimensional, since Φ is finite. One can directly check that W' is S_i-stable, so $W' \subset V'$. This implies that V' is L-stable, so $V = V'$ since V is irreducible.

(3) Let $v \in V$ be any vector. By (2), for each $1 \leqq i \leqq \ell$, $v \in W$ for some finite-dimensional S_i-submodule W . Then, by the discussion in Section 4.3, x_i and y_i act on W as nilpotent linear transformations. //

Lemma 5.6.6. Let V and λ be the same as in Lemma 5.6.5. Then the set $\Delta(V)$ of weights in V is permuted by the Weyl group WG, i.e., $WG(\Delta(V)) \subset \Delta(V)$.

Proof. Let σ_i be the simple reflection by $\alpha_i \in \Pi$. Since the σ_i generate WG (Theorem 5.3.6(4)), it suffices to show that $\sigma_i(\Delta(V)) \subset \Delta(V)$, $i = 1,\ldots,\ell$. Let μ be any weight in V. By Lemma 5.6.5(2), μ is a weight of h_i in a finite-dimensional S_i-submodule. Hence $\mu(h_i)$ is an integer.

Let $v \neq 0$ be a weight vector of μ in V, so that $vh = \mu(h)v$, $h \in H$. Suppose that $\mu(h_i) \geqq 0$. By Lemma 5.6.5(3), choose q such that $w = vx_i^q \neq 0$, $vx_i^{q+1} = 0$ and m such that $wy_i^m \neq 0$, $wy_i^{m+1} = 0$. Since $wx_i = 0$, as in the proof of Lemma 5.6.5(2) the subspace W spanned by $w, wy_i,\ldots,wy_i^m$ is S_i-stable, so W is an irreducible S_i-submodule of highest weight m, i.e., $wh_i = mw$ (see Section 4.3). Since $v \in V_\mu$, by Lemma 5.5.2(1) we have

$$wh = (vx_i^q)h = (\mu + q\alpha_i)(h)\, w,$$

$$(**) \quad (wy_i^k)h = (\mu + q\alpha_i - k\alpha_i)(h)wy_i^k,$$

$h \in H$. Hence $(\mu + q\alpha_i)(h_i) = \mu(h_i) + 2q = m$ and it follows from $(**)$ that the sequence

$$\mu + q\alpha_i,\ \mu + (q-1)\alpha_i,\ldots,\mu + (q-m)\alpha_i$$

consists of weights of H in V. Since $\mu(h_i) = m - 2q \geqq 0$ and $q \geqq 0$, $q - m \leqq 2q - m \leqq q$, so $\sigma_i(\mu) = \mu - \langle\mu, \alpha_i\rangle\alpha_i = \mu - \mu(h_i)\alpha_i = \mu + (2q - m)\alpha_i$ occurs in the sequence. Thus $\sigma_i(\mu) \in \Delta(V)$. If $\mu(h_i) \leqq 0$, we set $x_i' = y_i$, $y_i' = x_i$, $h_i' = -h_i$. Then x_i', y_i', h_i' form a canonical basis for S_i. The same argument can be applied to x_i', y_i', h_i' to show that $\sigma_i(\mu) \in \Delta(V)$. //

We are now ready to verify the sufficient condition.

<u>Theorem 5.6.7</u>. If $\lambda \in H^*$ is dominant integral then the irreducible standard cyclic L-module $V = V(\lambda)$ is finite-dimensional.

<u>Proof</u>. Since each weight space V_μ is finite-dimensional and $V = \Sigma_\mu \oplus V_\mu$ (Theorem 5.5.4), we only need to verify that $\Delta(V)$ is finite. Let $\Lambda_0^+ = \{\mu_0 \in \Lambda^+ \mid \mu_0 \leqq \lambda\}$. Then, by Lemma 5.4.5, Λ_0^+ is finite. Note that $\Delta(V) \subset \Lambda$ (cf. the proof of Lemma 5.6.6). Since WG is finite, the set $WG(\Lambda_0^+)$ of WG-conjugates of Λ_0^+ is finite. By Lemma 5.4.4, each $\mu \in \Delta(V)$ is WG-conjugate to an element $\mu_0 \in \Lambda^+$, i.e., $\mu = \sigma(\mu_0)$, $\sigma \in WG$. It then follows from Lemma 5.6.6 that $\mu_0 \in \Delta(V)$, so $\mu_0 \leqq \lambda$, since λ is highest. Therefore, $\mu \in WG(\Lambda_0^+)$ and $\Delta(V) \subset WG(\Lambda_0^+)$; hence $\Delta(V)$ is finite. //

Corollary 5.6.7. The map $\lambda \to V(\lambda)$ induces a one-one correspondence between Λ^+ and the isomorphism classes of finite-dimensional irreducible L-modules.

Proof. In view of Theorems 5.6.2 and 5.6.7, the map $\lambda \to V(\lambda)$ is well-defined. By Theorem 5.6.3, the map is surjective (cf. Lemma 5.5.3). Finally, it follows from Corollary 5.5.6 that the map is injective. //

6. FLEXIBLE LIE–ADMISSIBLE ALGEBRAS WITH A^- SEMISIMPLE

6.1. Adjoint operators

In this chapter we classify finite-dimensional flexible Lie-admissible algebras A over an algebraically closed field F of characteristic 0 when A^- is semisimple. The central idea for this classification is the notion of adjoint operators which was first introduced by Wigner (E. P. Wigner, "On representations of certain finite groups", Amer. J. Math. 63(1941), 57-63) for the Lie algebra of the $SU(2)$ group. The general case of adjoint operators has been studied in particle physics, recently by Okubo (S. Okubo, "Gauge groups without triangular anomaly", Phys. Rev. D16(1977), 3528-3534 and "Casimir invariants and vector operators in simple and classical Lie algebras", J. Math. Phys. 18(1977), 2382-2394).

Definition 6.1.1. Let L be an arbitrary Lie algebra over F (of any char.). Let $f : L \to (\mathrm{Hom}_F V)^-$ be a representation; V not necessarily finite-dimensional. A linear mapping $\delta : L \to \mathrm{Hom}\, V$ is called an adjoint operator (ad-operator ; L-invariant) of L in f (or in V) if δ satisfies

$$\delta[xy] = [\delta(x), f(y)] \tag{2.1}$$

for all $x,y \in L$. The set $V_f(L,V)$ of all ad-operators of L in f forms a subspace of $\mathrm{Hom}_F (L, \mathrm{Hom}_F V)$. If $\dim V = d < \infty$, $V_f(L,V)$ is often denoted by $V_f(L,d)$. If $f = \mathrm{ad}$ then we write $V_0(L)$ for $V_f(L,L)$. //

Note that $f \in V_f(L,V)$, so if $f \neq 0$, $\dim V_f(L,V) \geqq 1$. If L is finite-dimensional, we let $\{x_1, x_2, \ldots, x_N\}$ be a basis of L . Let L have the multiplication table

$$[x_i, x_j] = \sum_{k=1}^{N} b_{ij}^k \, x_k \tag{2.2}$$

where the $b_{ij}^k \in F$ are the structure constants of L . For $\delta \in V_f(L,V)$, let $T_i = \delta(x_i)$, $i = 1,2,\ldots,N$. Then the ordered N-tuple $(T_i) = (T_1,\ldots,T_N)$ satisfies the relations

$$[T_i, f(x_j)] = \sum_k b_{ij}^k \, T_k \tag{2.3}$$

Conversely, let $(T_i) = (T_1,\ldots,T_n) \in (\mathrm{Hom}\ V)^N$, the N-fold Cartesian product space. If (T_i) satisfies (2.3) then the unique linear mapping $\delta : L \to \mathrm{Hom}\ V$ defined by $\delta(x_i) = T_i$ $(i = 1,2,\ldots,N)$ is, by (2.2) and (2.3), an ad-operator of L in f. Thus the mapping $\delta \to (\delta(x_i))$ is an isomorphism of $V_f(L,V)$ to the vector space (the subspace of $(\mathrm{Hom}\ V)^N$) consisting of $(T_1,\ldots,T_N)$ satisfying (2.3).

When L is simple and $f = \mathrm{ad}$, we can explicitly enumurate $\dim V_0(L)$. This virtually leads to the classification of flexible Lie-admissible algebras A with A^- semisimple. Accordingly, $\delta \in V_f(L,V)$ is often identified with $(\delta(x_1),\ldots,\delta(x_N)) \in (\mathrm{Hom}\ V)^N$. In case $f = \mathrm{ad}$, we have

Definition 6.1.2. Let L be an arbitrary Lie algebra. An ad-operator $0 \neq \delta \in V_0(L)$ is said to be symmetric if δ satisfies

$$y\delta(x) = x\delta(y), \quad x,y \in L,$$

while δ is called skew-symmetric if it obeys

$$y\delta(x) = -x\delta(y), \quad x,y \in L. \quad //$$

Note that any Lie algebra L has a skew-symmetric ad-operator, since $\mathrm{ad} \in V_0(L)$ is skew-symmetric.

We will see later that not every Lie algebra possesses a symmetric ad-operator.

Lemma 6.1.1. Let $L = s\ell(\ell+1,F)$ with $\ell \geqq 2$ be the special linear algebra over F of char 0 . Then the linear mapping $\theta : L \to \mathrm{Hom}_F L$ defined by

$$y\theta(x) = xy + yx - \frac{2}{\ell+1}(\mathrm{Tr}\ xy)I \ , \quad x, y \in L$$

is a symmetric ad-operator of L , where xy is the usual matrix product and I is the $(\ell + 1)\times(\ell + 1)$ identity matrix.

Proof. It is clear that θ is well-defined and is nonzero. In fact, if $\theta = 0$ then $xy + yx = (2/(\ell+1))(\mathrm{Tr}$ for all $x, y \in L$. Since $\ell \geqq 2$, we can choose matrix units e_{ij}, e_{jk} with $i < j < k$. But then $e_{ij}e_{jk} = e_{ik}$ while $e_{jk}e_{ij} = 0$; this forces $e_{ik} \in FI$, a contradicti By direct computation one sees that $z\theta[x,y] = z[\theta(x),\mathrm{ad}\ y$ for all $x, y, z \in L$, so θ is a symmetric ad-operator of L . //

In contrast, note that if $\ell = 1$ then $xy + yx = (\mathrm{Tr}\ xy)I$ for all $x, y \in s\ell(2,F)$. Specifically, it will be shown that the simple Lie algebra of type $A_\ell(\ell \geqq$ alone allows a symmetric ad-operator.

Let A be an algebra over F of char $\neq 2$.

Denote by R_x and L_x the right and left multiplications in A by x. Set

$$\text{ad } x = R_x - L_x \quad \text{and} \quad \tau(x) = R_x + L_x . \qquad (2.4)$$

We have observed that A is Lie-admissible if and only if $\text{ad}[x, y] = [\text{ad } x, \text{ad } y]$, $x, y \in A$ ((1.11), Section 1.4), and that A is flexible if and only if $\tau[x, y] = [\tau(x), \text{ad } y]$ for $x, y \in A$ (Lemma 1.4.2). Therefore we have

Theorem 6.1.2. Let A be an algebra over F of char $\neq 2$. A necessary and sufficient condition that A is flexible Lie-admissible is that ad is a skew-symmetric ad-operator in $V_0(A^-)$ and τ defined by (2.4) is a symmetric ad-operator in $V_0(A^-)$. //

6.2. Highest adjoint weights

We utilize representation theory of semisimple Lie algebras to obtain more information on $V_f(L,V)$ which is useful to enumurate $\dim V_f(L,V)$. Let L be a split semisimple Lie algebra over F of char 0 and H

be a split CSA of L . Denote by V a finite-dimensional irreducible L-module with afforded representation $f \neq 0$. Henceforth, for brevity, we write $f(x) = x$, $x \in L$.

Let $\Pi = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a simple system of roots and let $h_i = h_{\alpha_i}$, $i = 1, \ldots, \ell$. Then the h_i form a basis of H . In view of Theorem 4.4.1, for each $\alpha \in \Phi$ we can choose basis elements $x_\alpha \in L_\alpha$, $x_{-\alpha} \in L_{-\alpha}$ such that $[x_\alpha x_{-\alpha}] = t_\alpha$, where t_α is uniquely determined by α . For $\Psi \in H^*$, we can write $t_\Psi = \Sigma_{i=1}^{\ell} \Psi^i h_i$ where the $\Psi^i \in F$ are uniquely determined by Ψ . Therefore, by (1.31) we have

$$(\Psi, \phi) = (t_\Psi, t_\phi) = \Sigma_{i=1}^{\ell} \Psi^i \phi(h_i) \tag{2.5}$$

for Ψ, $\phi \in H^*$. The $x_\alpha (\alpha \in \Phi)$, $h_1, h_2, \ldots, h_\ell$ form a basis of L and have the multiplication table

$$\begin{aligned}
&[h_i h_j] = 0 , \quad i,j = 1,2,\ldots,\ell , \\
&[x_\alpha h_i] = \alpha(h_i) x_\alpha , \\
&[x_\alpha x_\beta] = N_{\alpha,\beta}\, x_{\alpha+\beta} , \quad \alpha + \beta \neq 0 , \\
&[x_\alpha x_{-\alpha}] = \Sigma_{i=1}^{\ell} \alpha^i h_i
\end{aligned} \tag{2.6}$$

where α, $\beta \in \Phi$ and $N_{\alpha,\beta} \in F$, $N_{\alpha,\beta} = 0$ if $\alpha + \beta \notin \Phi$
If F is algebraically closed then even more can be done
$N_{\alpha,\beta}$ can be so chosen that $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ (Exercise 4.5.3

Denote this basis by a short hand notation $(t_p) = (h_i, x_\alpha, x_{-\alpha})$ in a fixed order with α representing all positive roots. Accordingly, we denote by $(T_p) = (H_i, E_\alpha, E_{-\alpha})$ an ad-operator of L in V. Thus, by (2.3), H_i $(i = 1,\ldots,\ell)$, $E_\alpha, E_{-\alpha} (\alpha \in \Phi^+)$ are elements in $\mathrm{Hom}_F V$ satisfying the commutation relations

$$\begin{aligned}
&[H_i, h_j] = 0 , \\
&[E_\alpha, h_i] = [x_\alpha, H_i] = \alpha(h_i)E_\alpha , \\
&[E_{-\alpha}, h_i] = [x_{-\alpha}, H_i] = -\alpha(h_i)E_{-\alpha} , \qquad\qquad (2.7)\\
&[E_\alpha, x_\beta] = N_{\alpha,\beta}\, E_{\alpha+\beta} , \quad \alpha + \beta \neq 0 , \\
&[E_\alpha, x_{-\alpha}] = [x_\alpha, E_{-\alpha}] = \Sigma_{i=1}^{\ell}\, \alpha^i H_i .
\end{aligned}$$

Note that if $\delta \in V_f(L,V)$, we can identify δ with $(H_i, E_\alpha, E_{-\alpha})$ via $\delta(h_i) = H_i$, $\delta(x_\alpha) = E_\alpha$, $\alpha \in \Phi$.

Recall that V is a standard cyclic L-module $V(\lambda)$ of maxmimal vector v^+ with highest weight λ (Lemma 5.5.3). Let $(H_i, E_\alpha, E_{-\alpha}) \in V_f(L,V)$. Since $[H_i, h] = 0$ for $h \in H$,

$$(v^+H_i)h = (v^+h)H_i = \lambda(h)v^+H_i , \quad h \in H .$$

Hence v^+H_i has weight λ, and since $\dim V_\lambda = 1$ (Theorem 5.5.4(3)), we have

$$v^+H_i = \gamma_i v^+ , \quad i = 1,2,\ldots,\ell \qquad\qquad (2.8)$$

where $\gamma_i \in F$ is uniquely determined by H_i . This leads to

Definition 6.2.1. For an ad-operator $(T_p) = (H_i, E_\alpha, E_{-\alpha})$ in $V_f(L,V)$, the highest ad-weight γ of (T_p) for H in V is defined as the element $\gamma \in H^*$ such that $\gamma(h_i) = \gamma_i$ $(i = 1,2,\ldots,\ell)$ with $\gamma_i \in F$ in (2.8). Call $(H_i) = (H_1,\ldots,H_\ell)$ the Cartan part of (T_p) relative to Π . //

If $\delta \in V_f(L,V)$ then the Cartan part of δ is $(\delta(h_i))$.

Lemma 6.2.1. Let $(H_i, E_\alpha, E_{-\alpha}) \in V_f(L,V)$. Then, for any $\alpha \in \Phi^+$, $v^+E_\alpha = 0$ and $v^+E_{-\alpha} \in V_{\lambda-\alpha}$.

Proof. In view of (2.7), $(v^+E_\alpha)h = v^+[E_\alpha,h] + (v^+h)E_\alpha = (\lambda + \alpha)(h)v^+E_\alpha$. Thus if $v^+E_\alpha \neq 0$ then $\lambda + \alpha$ would be a weight and this is absurd since λ is highest. Similarly, one gets $(v^+E_{-\alpha})h = (\lambda - \alpha)(h)v^+E_{-\alpha}$ so $v^+E_{-\alpha} \in V_{\lambda-\alpha}$. //

Lemma 6.2.2. Let γ be the highest ad-weight of $(H_i, E_\alpha, E_{-\alpha}) \in V_f(L,V)$. If $(\alpha,\lambda) = 0$ for $\alpha \in \Phi^+$ then $(\alpha,\gamma) = 0$.

Proof. Let r, q be the least nonnegative integers such that $\lambda - (r + 1)\alpha$ and $\lambda + (q + 1)\alpha$ are not weight

Since λ is highest and $\alpha > 0$, by Theorem 4.4.3 $q = 0$. Therefore, if $(\lambda,\alpha) = 0$ then $<\lambda,\alpha> = r - q = r$ gives $r = 0$, so that $\lambda - \alpha$ is not a weight (Theorem 4.4.3). Hence, by Lemma 6.2.1, $v^+E_{-\alpha} = 0$ and by (2.5) and (2.7) we have

$$\begin{aligned} 0 = (v^+E_{-\alpha})x &= v^+([E_{-\alpha},x_\alpha] + x_\alpha E_{-\alpha}) \\ &= -v^+(\Sigma_i\ \alpha^i H_i) \\ &= -(\Sigma_i\ \alpha^i\gamma_i)v^+ \qquad (\text{by}(2.8)) \\ &= -(\Sigma\ \alpha^i\gamma(h_i))v^+ \\ &= -(\alpha,\gamma)v^+ \qquad (\text{by}(2.5)). \qquad // \end{aligned}$$

<u>Lemma 6.2.3</u>. Let $(T_p) \in V_f(L,V)$. If $v^+T_p = 0$ for all p then $(T_p) = 0$.

<u>Proof</u>. Let $U(L)$ be the universal enveloping algebra of L. Then $V = v^+U(L)$. Let $(t_p) = (h_i, x_\alpha, x_{-\alpha})$ be the basis of L. Since $T_p t_q - t_q T_p = \sum_k b^k_{pq}\ T_k$,

$v^+T_p = 0$ for all p implies that $(v^+L)T_p = 0$ for all p. Noting $[xy,T_p] = x[y,T_p] + [x,T_p]y$ for $x,y \in L$, so $(v^+xy)T_p = 0$, it follows by induction that $VT_p = v^+U(L)T_p = 0$. //

<u>Theorem 6.2.4</u>. Let γ be the highest ad-weight of $(T_p) \in V_f(L,V)$. If $\gamma = 0$ then $(T_p) = 0$.

Proof. Let $(T_p) = (H_i, E_\alpha, E_{-\alpha})$ and let $N = \sum_{\alpha<0} L_\alpha$. We first contend that

$$V_0 \equiv \sum_{\alpha>0} v^+ E_{-\alpha} U(N)$$

is an L-submodule of V. Let $P = \sum_{\alpha>0} L_\alpha$. Then by PBW Theorem $U(L) = U(N)U(H)U(P) = U(H)U(P)U(N)$, so $V_0 U(L) = \sum_{\alpha>0} v^+ E_{-\alpha} U(H)U(P)U(N)$. Observing that $E_{-\alpha} h = [E_{-\alpha}, h] + hE_{-\alpha} = -\alpha(h)E_{-\alpha} + hE_{-\alpha}$, by induction we have that $v^+ E_{-\alpha} U(H) \subset Fv^+ E_{-\alpha}$, since v^+ is a weight vector of H. Hence $V_0 U(L) = \sum_{\alpha>0} v^+ E_{-\alpha} U(P)U(N)$ and it remains to show that $v^+ E_{-\alpha} x_\beta \in V_0$ for all $\alpha, \beta \in \Phi^+$. Since $v^+ E_{-\alpha} x_\beta = v^+ [E_{-\alpha}, x_\beta] + v^+ x_\beta E_{-\alpha} = v^+ [E_{-\alpha}, x_\beta]$, it suffices to verify

$$v^+ [E_{-\alpha}, x_\beta] \in V_0 \quad \text{for } \alpha, \beta \in \Phi^+ .$$

If $\beta - \alpha < 0$ then $v^+ [E_{-\alpha}, x_\beta] = N_{\beta,-\alpha} v^+ E_{-(\alpha-\beta)} \in V_0$.

If $\beta - \alpha > 0$ then $v^+ [E_{-\alpha}, x_\beta] = N_{\beta,-\alpha} v^+ E_{\beta-\alpha} = 0$ by Lemma 6.2.1. Finally, if $\alpha = \beta$ then

$$\begin{aligned} v^+ [E_{-\alpha}, x_\alpha] &= -v^+ \sum_i \alpha^i H_i \\ &= -\sum_i \alpha^i v^+ H_i \\ &= -(\alpha, \gamma) v^+ = 0 \end{aligned}$$

since $\gamma = 0$ by the assumption. This proves that V_0 is an L-submodule.

Since V_0 is spanned by weight vectors $v^+E_{-\alpha}$, $v^+E_{-\alpha}x_{-\beta},\ldots$ with weights $\lambda - \alpha$, $\lambda - \alpha - \beta,\ldots$ (Lemmas 6.2.1 and 5.5.2) which are lower than λ, V_0 can not contain v^+. This implies $V_0 = 0$, since V is irreducible. In particular, $v^+E_{-\alpha} = 0$ for $\alpha > 0$ while $v^+E_{\alpha} = 0$ by Lemma 6.2.1. Hence $v^+T_p = 0$ for all p since $v^+H_i = \gamma_i v^+ = 0$, $i = 1,2,\ldots,\ell$. This, in view of Lemma 6.2.3, proves $(T_p) = 0$. //

Therefore, an ad-operator of L in V is uniquely determined by its highest ad-weight and so by its Cartan part. Denote by H^*_Γ the subspace of H^* spanned by all highest ad-weights of H in V.

<u>Corollary 6.2.5</u>. $\dim V_f(L, V) = \dim H^*_\Gamma$.

<u>Proof</u>. Let $\gamma^{(j)} \in H^*_\Gamma$ $(j = 1,2,\ldots,m)$ and let δ_j be the ad-operator in $V_f(L, V)$ with highest ad-weight $\gamma^{(j)}$. Suppose that $\Sigma_{j=1}^m c_j\gamma^{(i)} = 0$. Then the Cartan part of $\delta = \Sigma_{j=1}^m c_j\delta_j$ is given by

$$\delta(h_k) = \Sigma_{j=1}^m c_j\delta_j(h_k), \quad k = 1,\ldots,\ell .$$

As these act on v^+, we have $v^+\delta(h_k) = \Sigma_{j=1}^m c_j v^+\delta_j(h_k)$ $= \sum_j c_j\gamma^{(j)}(h_k) = 0$. Hence the highest ad-weight γ of δ is zero, so by Theorem 6.2.4 $\delta = 0$.

This proves $\dim V_f(L,V) \leqq \dim H_\Gamma^*$. Conversely, let $\Sigma c_j \delta_j = 0$ for $\delta_j \in V_f(L,V)$. Let $\gamma^{(j)}$ be the highest ad-weight of δ_j . Then $\Sigma c_j \delta_j(h_k) = 0$, $k = 1,\ldots,\ell$, and $\Sigma c_j v^+ \delta_j(h_k) = \Sigma c_j \gamma^{(j)}(h_k) = 0$, so $\Sigma c_j \gamma^{(j)} = 0$. This proves $\dim V_f(L,V) \geqq \dim H_\Gamma^*$. //

6.3. The adjoint dimension

We utilize the results in Section 6.3 to enumurate $\dim V_f(L,V)$ in terms of the highest weight of H in V .

Definition 6.3.1. $\dim V_f(L,V)$ is called the adjoint dimension of L in V (or in f) . The adjoint dimension of L in V is customarily denoted by $n_A(f)$ in physics literature. //

It is clear from Corollary 6.2.5 that $n_A(f) \leqq \ell$, $\ell = \dim H^*$. Since V is irreducible, it is standard cyclic of highest weight λ and λ is dominant integral (Theorem 5.6.3). Let $\Omega = (\lambda_1,\ldots,\lambda_\ell)$ be the fundamental system of dominant weights (FSDW) relative to Π . Thus $\langle\lambda_i,\alpha_j\rangle = \delta_{ij}$ $(i,j = 1,\ldots,\ell)$ and

$$\lambda = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_\ell\lambda_\ell \qquad (2.9)$$

where

$$m_j = \langle\lambda, \alpha_j\rangle \qquad (2.10)$$

$(j = 1,\ldots,\ell)$ are nonnegative integers (Lemma 5.4.1).

Let $V_i (i = 1,\ldots,m)$ be L-modules with afforded representations f_i. Then we can make the tensor product $V' = V_1 \otimes\ldots\otimes V_m$ an L-module via

$$(v_1 \otimes\ldots\otimes v_m)x = \sum_{k=1}^{m} v_1 \otimes\ldots\otimes v_k x \otimes\ldots\otimes v_m ,$$

$x \in L$ (cf. Exercise 4.3.2). The representation afforded by the L-module V' is customarily denoted by $f' = f_1 \otimes\ldots\otimes f_m$.

<u>Lemma 6.3.1</u>. Let V be a finite-dimensional irreducible L-module of highest weight λ and let $\lambda_1,\ldots,\lambda_n$ be dominant integral functions on H such that $\lambda = \lambda_1 + \ldots + \lambda_n$. Then $\lambda_1,\ldots,\lambda_n$ are highest ad-weights of some ad-operators of L in V.

<u>Proof</u>. Since the λ_i are dominant integral, by Theorem 5.6.7 there exist finite-dimensional irreducible L-modules V_i of highest weights λ_i. Let v_i^+ be a maximal vector in V_i of weights λ_i. Then $v_1^+ \otimes\ldots\otimes v_n^+$ is clearly a maximal vector in the L-module $V' = V_1 \otimes\ldots\otimes V_n$

of weight λ. By Weyl's Theorem, write $V' = \Sigma \oplus W_j$ as a direct sum of irreducible L-submodules W_j. Let $v_1^+ \otimes \ldots \otimes v_n^+ = \Sigma w_j^+$, $w_j^+ \in W_j$. Then any $w_j^+ \neq 0$ is also maximal vector in W_j of weight λ. Since W_j is irreducible, if $w_j^+ \neq 0$ then W_j is a standard cyclic L-module of weight λ generated by w_j^+. Thus W_j is isomorphic to V as L-module (Theorem 5.6.1). Hence V can be identified with one of these W_j. Let f be the representation afforded by V.

For each $j = 1,2,\ldots,n$, define a linear mapping $g_j : L \to \text{Hom } V'$ by

$$(v_1 \otimes \ldots \otimes v_n) g_j(x) = v_1 \otimes \ldots \otimes v_j x \otimes \ldots \otimes v_n ,$$

$x \in L$, $v_j \in V_j$. Let $i : V \to V'$ be the injection and let $p : V' \to V$ be the projection. Then i and p are L-module homomorphisms. Finally, define a linear mapping $\delta_j : L \to \text{Hom } V$ by $\delta_j(x) = i \circ g_j(x) \circ p$, $x \in L$. Then

$$(v_1 \otimes \ldots \otimes v_n)(\delta_j(x) f(y) - f(y)\delta_j(x))$$

$$= (v_1 \otimes \ldots \otimes v_j x \otimes \ldots \otimes v_n) p f(y)$$

$$- (\sum_k v_1 \otimes \ldots \otimes v_k y \otimes \ldots \otimes v_n) \delta_j(x)$$

$$= (v_1 \otimes \ldots \otimes v_j x \otimes \ldots \otimes v_n) f(y) p$$

$$- \Big[(v_1 \otimes \ldots \otimes v_j x \otimes \ldots \otimes v_n) f(y)$$

$$- v_1 \otimes \ldots \otimes v_j [xy] \otimes \ldots \otimes v_n\Big] p$$

$$= (v_1 \otimes \ldots \otimes v_n)\delta_j[xy]$$

for $v_1 \otimes \ldots \otimes v_n \in V$. Hence δ_j is an ad-operator of L in V , $j = 1,2,\ldots,n$. It follows that each λ_j is the highest ad-weight of δ_j . //

Theorem 6.3.2. (Adjoint Dimension Theorem). The adjoint dimension $n_A(f)$ of L in V is equal to the number $m^+(f)$ of nonzero m_j's in (2.9).

Proof. Denote $q = n^+(f)$. Reordering the roots in Π , if necessary, we may assume that $m_j = 0$ in (2.9) for $q + 1 \leqq j \leqq \ell$. Thus λ is expressed as $\lambda = \Sigma_{j=1}^{q} m_j\lambda_j$. Since $\langle\lambda_i,\alpha_j\rangle = \delta_{ij}$, we have $(\alpha_j,\lambda) = 0$ for $q + 1 \leqq j \leqq \ell$. Let γ be any highest ad-weight. Since Ω is a basis of H^* , one can write $\gamma = \Sigma_{j=1}^{\ell} c_j\lambda_j$, $c_j \in F$. Then by Lemma 6.2.2 $(\alpha_j,\gamma) = 0$ for $q + 1 \leqq j \leqq \ell$ and this implies $c_j = 0$, $q + 1 \leqq j \leqq \ell$. It follows from Corollary 6.2.5 that $n_A(f) \leqq q = n^+(f)$.

Since the $m_j\lambda_j$ are dominant integral and $m_1\lambda_1,\ldots,m_q\lambda_q$ are linearly independent, it follows from Lemma 6.3.1 that $n_A(f) \geqq q$. //

Theorem 6.3.2 has been proved over the complex number field by Okubo, employing an analytic method. The present algebraic proof is due to Okubo and Myung ("Adjoint operators in Lie algebras and the classification of simple flexible Lie-admissible algebras", Trans. Amer. Math. Soc., 264(1981), 459-472).

Let L be a Lie algebra over arbitrary F and let $U(L)$ be the universal enveloping algebra of L. A linear mapping $\delta : L \to U(L)$ is called an ad-operator of L (in $U(L)$) if δ satisfies

$$\delta[xy] = [\delta(x), y]$$

for all $x, y \in L$. Denote by $V(L)$ the space of all ad-operators of L. Let K be the quotient field of the center C of $U(L)$ (C is an integral domain ; cf. Exercise 5.1.1). Set $V(L)_K = K \otimes_C V(L)$. When L is a simple Lie algebra over the complex number field, Okubo has shown that $\dim V(L)_K \geqq \dim H$ for the algebra L of type A_n, B_n, C_n, D_n, or G_2 ("Casimir invariants and vector operators in simple and classical Lie algebras", J. Math. Phys. 18(1977), 2382-2394). Okubo and Myung have conjectured that $\dim V(L)_K = \dim H$ over a field of characteristic 0 ("On the classification of simple flexib Lie-admissible algebras", Hadronic J. 1(1978), 504-567).

If $f = \mathrm{ad}$ and L is simple then Theorem 6.3.2 is strengthened to

Theorem 6.3.3. Let L be a simple Lie algebra. Then the adjoint dimension $n_A(\mathrm{ad})$ of L in ad is 1 or 2.

Proof. Since $ad \in V_0(L)$ and $ad \neq 0$, $\dim V_0(L) = n_A(ad) \geqq 1$. The highest weight λ_0 in ad is the unique maximal root (so positive) of H (Theorem 5.3.12(2) and Corollary 5.5.5). Write

$$\lambda_0 = m_1\lambda_1 + m_2\lambda_2 + \ldots + m_\ell\lambda_\ell \qquad (2.11)$$

as in (2.9) where $m_j = \langle\lambda_0, \alpha_j\rangle \in Z^+$. Since λ_0 is a positive root, $m_j = 0, 1, 2,$ or 3 (Theorem 4.4.5). Let

$$\lambda_0 = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_\ell\alpha_\ell \, . \qquad (2.12)$$

Then all $c_i \in Z^+$. Since $\langle\lambda_i, \alpha_j\rangle = \delta_{ij}$, it follows from (2.12) that $c_i = 2(\lambda_0, \lambda_i)/(\alpha_i, \alpha_i)$, so (2.12) becomes

$$\lambda_0 = \sum_i \frac{2(\lambda_i, \lambda_0)}{(\alpha_i, \alpha_i)} \alpha_i \, ,$$

which leads to

$$(\lambda_0, \lambda_0) = \sum_i (\lambda_i, \lambda_0)\langle\lambda_0, \alpha_i\rangle = \sum_i m_i(\lambda_i, \lambda_0)$$

since $m_i = \langle\lambda_0, \alpha_i\rangle$. We now rewrite this as

$$2 = \sum_i m_i\langle\lambda_i, \lambda_0\rangle \, . \qquad (2.13)$$

Since each $\lambda_i \in \Lambda^+$, λ_i is the highest weight in some irreducible L-module (Theorem 5.6.7). Since $\lambda_0 \in \Phi^+$, this implies that $\langle\lambda_i, \lambda_0\rangle \in Z^+$, $i = 1, 2, \ldots, \ell$. We further claim that all $\langle\lambda_i, \lambda_0\rangle > 0$. Since λ_0 is the highest weight in ad, each root in Π (being a weight) is expressed as $\lambda_0 - \sum_i k_i\alpha_i$ with the $k_i \in Z^+$ (Theorem 5.5.4(2)). Thus each c_i in (2.12) is positive, so all $\langle\lambda_i,\lambda_0\rangle > 0$ since $c_i = 2(\lambda_0,\lambda_i)/(\alpha_i,\alpha_i)$. Hence, by (2.13) we have $\sum_{j=1}^{\ell} m_j \leqq 2$ and this with (2.11) allows the only possibilities

$$\lambda_0 = \lambda_i , \quad \lambda_0 = 2\lambda_j , \quad \text{or} \quad \lambda_0 = \lambda_p + \lambda_q . \qquad (2.14)$$

Thus $n^+(\text{ad}) = 1$ or 2 and by Theorem 6.3.2 $n_A(\text{ad}) = 1$ or 2. //

In view of (2.14), one notes that the result in Theorem 5.4.3 is no accident and determines the relations more explicitly. Hence, by virtue of the Adjoint Dimension Theorem and Theorem 5.4.3, we can state

<u>Theorem 6.3.4</u>. Let L be a split simple Lie algebra over a field F of char 0. Then the adjoint dimension of L in ad is 2 for L of type $A_\ell (\ell \geqq 2)$ and is 1 for all other types. //

When L is of type $A_\ell(\ell \geqq 2)$, we have observed that $V_0(L)$ has two linearly independent operators ad and θ defined as in Lemma 6.1.1. Thus ad and θ form a basis of $V_0(L)$. Moreover, since θ is symmetric, we have

Corollary 6.3.5. If L is of type $A_\ell (\ell \geq 2)$ then any symmetric ad-operator of L in ad is a scalar multiple of θ, constructed in Lemma 6.1.1. Also, any skew-symmetric element in $V_0(L)$ is a scalar multiple of ad. //

Remark. The result of Corollary 6.3.5 agrees with those by Djoković and by Krämer who computed directly the multiplicity of ad in the tensor product ad⊗ad by means of Steinberg's formula or its variant. The relations in Theorem 5.4.3 is noted and utilized by Kac for the classification of simple Lie superalgebras. Also, Okubo has obtained Theorem 5.4.3 by invoking (2.14) and the table in Exercise 5.4.1 with slightly different ordering of Π. References to these are cited below.

1. D. Ž. Djoković, "Classification of some 2-graded Lie algebras", J. Pure and Appl. Algebra, 1(1976), 210-230.

2. M. Krämer, "Eine Klassification Bestimmter untergruppen Kompakter Zusammen-Hängender Lie gruppen", Comm. in Algebra, 3(8)(1975), 691-737.

3. V. G. Kac, "Lie superalgebra", Advances in Math. 26(1977), 8-96.

4. S. Okubo, "Gauge groups without triangular anomaly", Phys. Rev. D16(1977), 3528-3534.

6.4. The classification

We show that the classification of flexible Lie-admissible algebras A over F of char $\neq 2$ is equivalent to finding all symmetric elements in $V_0(A^-)$. The classification under consideration is then an easy consequence of Lemma 6.1.1 and Corollary 6.3.5.

Theorem 6.4.1. Let L be an arbitrary Lie algebra over F of char $\neq 2$ and let θ be any symmetric element in $V_0(L)$. Denote by $L(\theta)$ the algebra defined on the vector space L but with multiplication given by

$$xy = \tfrac{1}{2}(x\theta(y) + [xy]) .$$

Then $L(\theta)$ is flexible Lie-admissible such that $L(\theta)^-$ is isomorphic to L. Moreover, $\theta(x) = \tau(x)$ in $L(\theta)$ where τ is defined by (2.4).

Proof. Let L_x and R_x be the left and right multiplications by x in $L(\theta)$. Then $\mathrm{ad}_{L(\theta)}\, x = R_x - L_x$ and clearly $\mathrm{ad}_{L(\theta)} = \mathrm{ad}_L$, since θ is symmetric. Hence $L(\theta)$ is Lie-admissible such that $L(\theta)^- = L$ and $\theta = \tau$ in $L(\theta)$. Noting that

$$L_xR_x = \tfrac{1}{4}(\theta(x) - \mathrm{ad}\; x)(\theta(x) + \mathrm{ad}\; x) ,$$

$$R_x L_x = \tfrac{1}{4}(\theta(x) + \text{ad } x)(\theta(x) - \text{ad } x) ,$$

we have $L_x R_x - R_x L_x = \tfrac{1}{2}[\theta(x), \text{ ad } x] = \tfrac{1}{2}\theta[x, x] = 0$ since θ is an ad-operator of L in ad . Thus $L(\theta)$ is flexible. //

In view of this result, the classification of flexible Lie-admissible algebras A reduces to the determination of all symmetric elements in $V_0(A^-)$. However, at present, $V_0(A^-)$ is determined only when A^- is reductive with a simple Levi factor. We first classify A when A^- is simple.

<u>Theorem 6.4.2</u>. Let A be a finite-dimensional flexible Lie-admissible algebra such that A^- is a split simple Lie algebra over F of char 0 . Then either A is itself a Lie algebra isomorphic to A^- or A^- is simple Lie algebra of type A_ℓ $(\ell \geqq 2)$. In the latter case, A is either a Lie algebra or isomorphic to an algebra with multiplication given by

$$x*y = \mu xy + (1 - \mu)yx - \frac{1}{\ell+1} \text{Tr}(xy)I \qquad (2.15)$$

which is defined on the space of $(\ell + 1)\times(\ell + 1)$ trace 0 matrices over F , where xy is the matrix product, $\mu \neq \tfrac{1}{2}$ is a fixed scalar in F and I is the identity matrix.

Proof. If A^- is not of type $A_\ell (\ell \geqq 2)$, then $\dim V_0(A^-) = 1$. Since $0 \neq \text{ad} \in V_0(A^-)$, the symmetric ad-operator $\tau(x) = R_x + L_x$ in $V_0(A^-)$ must be zero for all $x \in A$. Thus A is a Lie algebra. If A^- is of type $A_\ell (\ell \geqq 2)$ then we set $A^- = s\ell(\ell+1,F)$. By Theorem 6.1.2, τ defined by $\tau(x) = R_x + L_x$ is a symmetric element in $V_0(A^-)$. It follows from Corollary 6.3.5 that $\tau = c\theta$ for some $c \in F$, where θ is given by

$$y\theta(x) = xy + yx - \frac{2}{\ell+1}(\text{Tr } xy)I ,$$

$x,y \in s\ell(\ell+1,F)$ and xy is the matrix product. Thus, by Theorem 6.4.1, A is isomorphic to an algebra defined on $A^- = s\ell(\ell+1,F)$ but with multiplication given by

$$\begin{aligned} x*y &= \tfrac{1}{2}x(c\theta(y) + \text{ad } y) \\ &= \tfrac{1}{2}\{(c+1)xy + (c-1)yx - \frac{2c}{\ell+1}(\text{Tr } xy)I\} . \end{aligned}$$

If $c = 0$ then A is isomorphic to $A^- = s\ell(\ell+1,F)$. If $c \neq 0$ then, dividing both sides by c and setting $\mu = \tfrac{1}{2}(1 + \frac{1}{c}) \neq \tfrac{1}{2}$, we obtain the algebra described by (2.15). //

When A^- is semisimple, it can be shown that A is a direct sum of ideals A_i such that A_i^- is simple. More generally, we can prove

Theorem 6.4.3. Let A be a flexible algebra over F of char $\neq 2$ (not necessarily finite-dimensional) such that A^- is a direct sum of simple Lie algebras A_i^-. Then each A_i is an ideal of A, so that A is the direct sum of simple flexible Lie-admissible algebras A_i.

Proof. We first show that each A_k is a subalgebra of A. Let $B_k^- = \Sigma_{i \neq k} A_i^-$. Then $A^- = A_k^- \oplus B_k^-$ and $A_k \subset C_{A^-}(B_k^-)$, the centralizer of B_k^- in A^-, since $[A_k^-, B_k^-] = 0$. Noting that each ad x is a derivation of A, the centralizer of any subset of A in A^- is a subalgebra of A. Thus it suffices to verify $A_k = C_{A^-}(B_k^-) \equiv C$. Hence, let $x \in C$ and let $x = a + b$, $a \in A_k$, $b \in B_k$. Then, for every $b' \in B_k$, $0 = [x, b'] = [a, b'] + [b', b] = [b', b]$, so $b \in C$. Suppose $b \neq 0$. Then, write $b = \Sigma_{i \neq k} b_i$ with $b_j \neq 0$ for some $j \neq k$. It follows from $[b, B_k^-] = 0$ that $[b_j, A_j^-] = 0$, since $[A_i^-, A_j^-] = 0$ for $i \neq j$. This shows that the center of A_j^- is not 0, and this is absurd since A_j^- is simple. Therefore $x = a$ is in A_k, showing that $A_k = C$.

To show that each A_k is an ideal of A, observe first that $[B_k^-, B_k^-] = B_k^-$ since $[A_i^-, A_i^-] = A_i^-$ for all i. Let $a \in A_k$ and $b \in B_k$, and write $b = \Sigma_i [x_i, y_i]$, $x_i, y_i \in B_k$. Since ad z is a derivation of A, we have

$$ab = \sum_i a[x_i, y_i] = \sum_i [ax_i, y_i] - \sum_i [a, y_i] x_i = \sum_i [ax_i, y_i] \in B_k$$

since B_k^- is an ideal of A^-. Using $[A_k^-, A_k^-] = A_k^-$, we can similarily show that $ab \in A_k$. Hence $ab = 0$ and A_k is an ideal of A. //

If A^- is a finite-dimensional semisimple Lie algebra of char 0 then A^- is a direct sum of simple ideals (Lemma 1.6.3 and Theorem 3.6.3). Therefore, in view of Theorems 6.4.2 and 6.4.3, we have

Corollary 6.4.4. Let A be a finite-dimensional flexible Lie-admissible algebra over a field F of char 0 such that A^- is a split semisimple Lie algebra. Then A is the direct sum of simple Lie algebras and simple flexible algebras defined by (2.15). //

Corollary 6.4.4 completes the classification of flexibl Lie-admissible algebras in the characteristic zero case whicl was proposed by Albert in 1948.

Theorem 6.4.3 first proved by Weiner when A is power-associative, and later by Laufer and Tomber when F is algebraically closed of char 0 and A is finite-dimensional over F. The present general form of Theorem 6.4.3 is due to Myung.

The simple algebras described by (2.15) are neither power-associative nor have unit elements. As an application of Theorem 6.4.2 we can easily prove the following result of Laufer and Tomber (the reference cited below).

Corollary 6.4.5. Let A be a finite-dimensional power-associative flexible Lie-admissible algebra over a field F of char 0 such that A^- is semisimple. Then A is a Lie algebra isomorphic to A^-.

Proof. We may assume that F is algebraically closed since any scalar extension of A^- is semisimple also (Corollary 3.6.6). By Corollary 6.4.4 it suffices to show that the algebra defined by (2.15) can not be power-associative. Thus, let $A^- = s\ell(\ell + 1, F)$ with $\ell \geq 2$ and let A have the multiplication given by (2.15). If A is power-associative, it is easily checked that $(x*x)*(x*x) = [(x*x)*x]*x$ implies

$$(\mathrm{Tr}\ x^2)x^2 - (\mathrm{Tr}\ x^3)x - \frac{1}{\ell+1}(\mathrm{Tr}\ x^2)^2 I = 0$$

for all $x \in A$. This is absurd since, for example, the diagonal matrix $x = \mathrm{diag}\{1,2,-3,0,\ldots,0\}$ does not satisfy this identity. Thus A can not be power-associative. //

1. A. A. Albert, "Power-associative rings", Trans. Amer. Math. Soc. 64(1948), 552-597.

2. P. J. Laufer and M. L. Tomber, "Some Lie admissible algebras", Canad. J. Math. 14(1962), 287-292.

3. H. C. Myung, "Lie-admissible algebras", Hadronic J. 1(1978), 169-193.

4. L. M. Weiner, "Lie admissible algebras", Univ. Nac. Tucumān Rev. Ser. A., 11(1957), 10-24.

6.5. The reductive case

Recall that a Lie algebra L of characteristic 0 is called reductive if $Z(L) = \text{Rad } L$ and that $L = S \oplus Z$, where Z is the center of L and S is a semisimple subalgebra (a Levi-factor) of L (Theorem 3.7.11). Since $S = [LL]$, S is the unique Levi factor of L. The classification in Section 6.4 can be applied to determine all flexible Lie-admissible algebras A such that A^- is reductive and the Levi factor of A^- is simple.

Lemma 6.5.1. Let L be a finite-dimensional simple Lie algebra over an algebraically closed field F of char 0. Then any invariant bilinear form $(\, , \,)$ on L (not necessarily symmetric or nondegenerate) is a scalar multiple of the Killing form; that is,

$$(x, y) = k \text{ Tr ad } x \text{ ad } y , \quad x, y \in L , \quad k \in F .$$

Proof. Recall the invariant condition of $(\, , \,)$:

$$([xy], z) = (x, [yz]) \, , \quad x, y, z \in L \, .$$

Let $x \in L$ be any element. Since the Killing form is nondegenerate, there exists an $x' \in L$ such that $(x, y) = \operatorname{Tr} \operatorname{ad} x' \operatorname{ad} y$ for all $y \in L$. Then the mapping $x \to x'$ defines a linear mapping $T : L \to L$, so that

$$(x, y) = \operatorname{Tr} \operatorname{ad} xT \operatorname{ad} y \, , \quad x, y \in L \, .$$

Then we have

$$\begin{aligned} (x, [yz]) &= \operatorname{Tr} \operatorname{ad} xT \operatorname{ad} [yz] \\ &= \operatorname{Tr}(\operatorname{ad} xT \operatorname{ad} y \operatorname{ad} z - \operatorname{ad} xT \operatorname{ad} z \operatorname{ad} y) \\ &= \operatorname{Tr} \operatorname{ad} [xT, y] \operatorname{ad} z \\ &= ([xy], z) \\ &= \operatorname{Tr} \operatorname{ad} [xy]T \operatorname{ad} z \, . \end{aligned}$$

Thus the nondegeneracy of the Killing form implies

$$[xy]T = [xT, y] \, , \quad x, y \in L$$

and this is equivalent to $[T, \operatorname{ad} y] = 0$ for all $y \in L$. Hence by Schur's Lemma 3.7.8 we have $T = k$, a scalar. //

Let L be a reductive Lie algebra with a simple

Levi factor S and center Z. Henceforth we denote the elements of S by $x,y,z,\ldots$ while those of Z are labelled by $a,b,c,\ldots$. Then the symmetric elements in $V_0(L)$ are determined as follows.

Theorem 6.5.2. Let L be a finite-dimensional reductive Lie algebra over an algebraically closed field F of char 0 such that the Levi factor S is simple. Then any symmetric element θ in $V_0(L)$ satisfies the relations

$$x\theta(y) = x\theta_S(y) + [\mathrm{Tr}\ \mathrm{ad}\ x\ \mathrm{ad}\ y]c \tag{2.16a}$$

$$x\theta(a) = a\theta(x) = \eta(a)x , \tag{2.16b}$$

$$a\theta(b) = b\theta(a) = \theta_Z(a)b \in Z , \tag{2.16c}$$

where θ_S and θ_Z are respectively symmetric elements in $V_0(S)$ and $V_0(Z)$, $x, y \in S$, $a, b \in Z$, c is fixed in Z, and η is a linear form on Z. Conversely, given any symmetric elements $\theta_S \in V_0(S)$ and $\theta_Z \in V_0(Z)$, c fixed in Z and linear form η on Z, the linear mapping $\theta : L \to \mathrm{Hom}_F L$ satisfying (2.16a)-(2.16c) is a symmetric element in $V_0(L)$.

Proof. Since $L = S \oplus Z$, $\mathrm{Hom}_F L = \mathrm{Hom}_F S \oplus \mathrm{Hom}_F (S, Z) \oplus \mathrm{Hom}_F (Z, S) \oplus \mathrm{Hom}_F Z$. Thus it is convenien to express a linear mapping $\theta : L \to \mathrm{Hom}\ L$ by the matrix notation

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}, \quad \theta(x) = \begin{pmatrix} \theta_{11}(x) & \theta_{12}(x) \\ \theta_{21}(x) & \theta_{22}(x) \end{pmatrix} \qquad (2.17)$$

for $x \in L$ where the θ_{ij} are the component linear mappings of θ :

$$\theta_{11} : L \to \text{Hom } S , \quad \theta_{12} : L \to \text{Hom } (S,Z) ,$$

$$\theta_{21} : L \to \text{Hom } (Z,S) , \quad \theta_{22} : L \to \text{Hom } Z .$$

In particular,

$$\text{ad } x = \begin{pmatrix} \text{ad}_S\, x & 0 \\ 0 & 0 \end{pmatrix}, \quad x \in S \qquad (2.18)$$

while $\text{ad } Z = 0$ since Z is the center of L .

Suppose now that θ is a symmetric element in $V_0(L)$. Equating $0 = \theta[ax] = [\theta(a), \text{ad } x]$ with (2.17), (2.18), we have

$$(\text{ad}_S\, x)\theta_{11}(a) = \theta_{11}(a)\text{ad}_S\, x ,$$

$$(\text{ad}_S\, x)\theta_{12}(a) = 0 ,$$

$$\theta_{21}(a)\text{ad}_S\, x = 0 .$$

Since ad_S is an irreducible representation of L , by Schur's Lemma the first relation gives

$$\theta_{11}(a) = \eta(a)I_S$$

where I_S is the identity mapping on S and η is a linear form on Z. It follows from the second relation that $S\theta_{12}(a) = 0$ (since $[SS] = S$), so $\theta_{12}(a) = 0$. Similarily, $\theta_{21}(a) = 0$. Thus

$$\theta(a) = \begin{pmatrix} \eta(a) I_S & 0 \\ 0 & \theta_{22}(a) \end{pmatrix}, \quad a \in Z \qquad (2.19)$$

while $\theta[ab] = [\theta(a), \text{ad } b] = 0$ is an identity. By (2.17), the symmetric condition $a\,\theta(x) = x\,\theta(a)$ leads to $a\,\theta_{21}(x) = x\,\theta_{11}(a)$ and $a\,\theta_{22}(x) = x\,\theta_{12}(a)$, which reduce to

$$\theta_{22}(x) = 0, \quad a\,\theta_{21}(x) = \eta(a)x \qquad (2.20)$$

by (2.19). Similarily, equating $\theta[xy] = [\theta(x), \text{ad } y]$ gives the relations

$$\theta_{11}[xy] = [\theta_{11}(x), \text{ad}_S\, y], \qquad (2.21a)$$

$$\theta_{12}[xy] = -(\text{ad}_S\, y)\theta_{12}(x), \qquad (2.21b)$$

$$\theta_{21}[xy] = \theta_{21}(x)\, \text{ad}_S\, y. \qquad (2.21c)$$

It follows from $x\theta(y) = y\theta(x)$ that

$$x\theta_{11}(y) = y\theta_{11}(x), \qquad (2.21d)$$

$$x\theta_{12}(y) = y\theta_{12}(x). \qquad (2.21e)$$

If we put $\theta_S = \theta_{11}|_S$ then (2.21a) and (2.21d) show that θ_S is a symmetric element in $V_0(S)$. Set $\theta_Z = \theta_{22}|_Z$. Then $a\theta(b) = b\theta(a)$ implies that θ_Z is a symmetric element in $V_0(Z)$. We also have from (2.21b)

$$z\theta_{12}[xy] = -[zy]\theta_{12}(x) . \qquad (2.22)$$

Note that Z always has a symmetric nondegenerate bilinear form (,) . Since $\theta_{12} : L \to \mathrm{Hom}\,(S, Z)$ is linear, one can define a bilinear form $(\ ,\ )_a$ on S by

$$(x, y)_a = (a, y\theta_{12}(x)) \qquad (2.23)$$

for each $a \in Z$. Then by (2.22) we have

$$\begin{aligned} ([xy], z)_a &= (a, z\theta_{12}[xy]) \\ &= -(a, [zy]\theta_{12}(x)) \\ &= -(x, [zy])_a = (x, [yz])_a , \end{aligned}$$

so (,) is invariant. Thus, by Lemma 6.5.1,

$$(x, y)_a = \alpha(a)\ \mathrm{Tr\ ad}\ x\ \mathrm{ad}\ y , \qquad (2.24)$$

$\alpha(a) \in F$. From (2.23) and (2.24) it is easily checked that α is a linear form on Z . Since (,) is nondegenerate on Z , we have $\alpha(a) = (a, c)$ for some $c \in Z$. Again, by the nondegeneracy of (,) , this together with (2.23) and (2.24) implies

$$y\theta_{12}(x) = (\mathrm{Tr}\ \mathrm{ad}\ x\ \mathrm{ad}\ y)c\ .$$

Since $y\theta(x) = y\theta_{11}(x) + y\theta_{12}(x) = y\theta_S(x) + y\theta_{12}(x)$, this gives (2.16a). We have $a\theta(x) = a\theta_{21}(x) + a\theta_{22}(x) = a\theta_{21}(x) = \eta(a)x$ by (2.20). On the other hand, $x\theta(a) = x\theta_{11}(a) + x\theta_{12}(a) = x\theta_{11}(a) = \eta(a)x$ by (2.19). This gives (2.16b). Finally, $b\theta(a) = a\theta(b) = b\theta_{21}(a) + b\theta_{22}(a) = b\theta_{22}(a) = b\theta_Z(a)$ by (2.19) and this is (2.16c).

Conversely, let $\theta : L \to \mathrm{Hom}\ L$ be a linear mapping satisfying (2.16a)-(2.16c). Then since $\theta(a)$ is a scalar on S, $[\theta(a), \mathrm{ad}\ y] = 0$, so $\theta[x + a, y + b] = \theta[xy] = \theta_S[xy] = [\theta_S(x), \mathrm{ad}\ y] = [\theta(x) + \theta(a), \mathrm{ad}\ (y + b)] = [\theta(x + a), \mathrm{ad}\ (y + b)]$. Thus $\theta \in V_0(L)$ and evidently θ is symmetric. //

<u>Theorem 6.5.3</u>. Let A be a finite-dimensional flexible Lie-admissible algebra over an algebraically closed field F of char 0 such that $A^- = S \oplus Z$ is reductive with S simple. If S is of type A_ℓ $(\ell \geqq 2)$ then A is isomorphic to the algebra defined on $s\ell(\ell + 1, F) \oplus Z$ with multiplication given by

$$x*y = \mu xy + (1 - \mu)yx - \frac{1}{\ell+1}(\mathrm{Tr}\ xy)I + (\mathrm{Tr}\ \mathrm{ad}\ x\ \mathrm{ad}\ y)c, \quad (2.25a$$

$$x*a = a*x = \eta(a)x\ , \quad (2.25b$$

$$a*b = b*a \in Z\ . \quad (2.25c$$

If S is not of type $A_\ell (\ell \geqq 2)$ then A is isomorphic to an algebra with multiplication given by

$$x*y = [xy] + (\mathrm{Tr}\ \mathrm{ad}\ x\ \mathrm{ad}\ y)c \qquad (2.25d)$$

and the remaining multiplication is the sames as in (2.25b) and (2.25c). Here $[xy]$ denotes the multiplication in S while xy indicates the matrix product. The notations are the same as in Theorem 6.5.2.

Proof. In view of Theorems 6.1.2 and 6.4.1, A is isomorphic to an algebra defined on the space A^- but with multiplication given by $x*y = x\theta(y) + [x, y]$ where θ is a symmetric element in $V_0(A^-)$. But then θ is determined by (2.16a)-(2.16c). Thus if S is of type $A_\ell (\ell \geqq 2)$ then θ_S is determined by Lemma 6.1.1 and (2.25a) follows from (2.16a) and Theorem 6.4.2. Also, (2.25b) and (2.25c) are consequences of (2.16b) and (2.16c). If S is not of type $A_\ell (\ell \geqq 2)$ then $\theta_S = 0$ (Theorem 6.3.4). This gives (2.25d). //

Notice that the quaternion algebra and $\mathrm{Hom}_F V$ are included in the classification of Theorem 6.5.3. It is easy to determine all flexible Lie-admissible algebras A of dimension $\leqq 4$ such that A^- is non-solvable.

Corollary 6.5.4. Let A be a flexible Lie-admissible algebra over an algebraically closed field F of char 0 such that $\dim A \leqq 4$ and A^- is non-solvable. Then

(1) A is a Lie algebra isomorphic to $s\ell(2, F)$.

(2) A is a 4-dimensional algebra with basis x, h, y whose multiplication table is given by

	x	h	y	a
x	0	$2x$	$h-\gamma a$	αx
h	$-2x$	$2\gamma a$	$2y$	αh
y	$-h-\gamma a$	$-2y$	0	αy
a	αx	αh	αy	βa

where α, β, γ are scalars in F.

<u>Proof</u>. It is easy to see that if $\dim A \leqq 2$ then A^- is solvable. Thus $\dim A = 3$ or 4. If $\dim A = 3$ then it is routine to check that A^- is the split 3-dimensional simple Lie algebra, so A is a Lie algebra isomorphic to $s\ell(2, F)$ (Theorem 6.4.2). If $\dim A = 4$, then by a dimension argument we see that $A^- = S \oplus Fa$ is a Levi decomposition for A^- where S is isomorphic to $s\ell(2, F)$ Let x, h, y be the canonical basis of S, so that

$$[xh] = 2x , \quad [yh] = -2y , \quad [xy] = h .$$

Then A^- is reductive (why ?). Also, ad x, ad h, ad y have the matrices (relative to x, h, y) :

$$\mathrm{ad}\ x = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathrm{ad}\ h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\mathrm{ad}\ y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus the matrix of the Killing form relative to x, h, y is

$$\begin{pmatrix} 0 & 0 & -4 \\ 0 & 8 & 0 \\ -4 & 0 & 0 \end{pmatrix}.$$

The multiplication table given in (2) then follows from (2.25b)-(2.25d), since $\dim Z = 1$ and Z is a subalgebra of A. //

The classification in Corollary 6.5.4 has been obtained by Myung under the assumption that A is a nilalbebra ("A class of almost commutative nilalgebras", Canad. J. Math. 26(1974), 1192-1198). The table above also includes the pseudo-quaternion algebra P_4 introduced by Okubo ("Pseudo-quaternion and pseudo-octonion algebras", Hadronic J. 1(1978), 1250-1278). For this, let S be a 3-dimensional simple Lie algebra over the field F (as in Corollary 6.5.4) with basis x_1, x_2, x_3 such that

$$[x_1x_2] = x_3 , \quad [x_2x_3] = x_1 , \quad [x_3x_1] = x_2 ,$$

or

$$[x_ix_j] = \sum_{k=1}^{3} \varepsilon_{ijk}\, x_k , \quad i, j, k = 1, 2, 3$$

where ε_{ijk} is the totally antisymmetric Levi-Civita symbol with $\varepsilon_{123} = 1$. Note that S is isomorphic to the split 3-dimensional simple Lie algebra (in fact, this is the unique Lie algebra L of dimension 3 such that $[LL] = L$) Thus any non-solvable Lie algebra of dimension 4 is of the form $S \oplus Fe$. Let A be a flexible Lie-admissible algebra such that $A^- \simeq S \oplus Fe$. Then A is determined by (2.25b)-(2.25d). Compute the matrices relative to x_1, x_2, x_3 as

$$\text{ad } x_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{ad } x_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\text{ad } x_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that the Killing form of S has the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Thus if we set $\alpha = -\beta = 1$ and $c = -\frac{1}{2}e$ in the table, we have a 4-dimensional flexible Lie-admissible algebra with multiplication given by

$$
\begin{aligned}
x_i * x_j &= \sum_{k=1}^{3} \varepsilon_{ijk}\, x_k + \delta_{ij} e , \\
x_j * e &= e * x_j = x_j , \quad e * e = -e , \quad j = 1, 2, 3 .
\end{aligned}
\tag{2.26}
$$

This algebra is the pseudo-quaternion algebra P_4 .

Remark. (1) Benkart and Osborn ("Flexible Lie-admissible algebras", J. Algebra, 71(1981), 11-31.) have determined all flexible Lie-admissible algebras A of char 0 such that Rad A^- is a direct summand of A^- . Thus the case A^- is reductive is included in their classification.

(2) In spite of these results, the classification of the general simple flexible Lie-admissible algebras appears to be very difficult, if not impossible. To support this view, Okubo has recently provided a method to construct a (infinite) class of simple flexible Lie-admissible algebras from a given simple algebra ("A generalization of Hurwitz theorem and flexible Lie-admissible algebras", Hadronic J. 3(1979), 1-52). These algebras are related to quasi-classical Lie algebras. An arbitrary Lie algebra L is called quasi-classical if it has a symmetric nondegenerate bilinear form $(\ ,\ )$ satisfying the invariant condition $([xy], z) = (x, [yz])$, $x, y, z \in L$. It is easy to see that any

reductive Lie algebra is quasi-classical. However, the converse is not true ; in fact, there is a solvable Lie algebra which is quasi-classical.

(3) The material in this chapter is drawn from Okubo and Myung ("Adjoint operators in Lie algebras and the classification of simple flexible Lie-admissible algebras", Trans. Amer. Math. Soc., 264(1981), 459-472.).

6.6. An extension of the PBW Theorem

We digress to discuss an extension of the PBW Theorem to a flexible Lie-admissible algebra. Specifically, given a Lie algebra L, we construct a flexible Lie-admissible algebra generated by L which satisfies a "universal-like" property.

Let A be any algebra over a field F of char $\neq 2$ and let $\lambda \neq \mu$ be fixed scalars in F. Denote by $A(\lambda,\mu)$ the algebra defined on the vector space A but with multiplication given by

$$x \circ y = \lambda xy + \mu yx ,$$

$x, y \in A$, where xy indicates the product in A . Call $A(\lambda,\mu)$ the (λ,μ)-mutation of A . Let $(x,y,z)^{o}$ and $[x,y]^{o}$ be the associator and commutator in $A(\lambda,\mu)$. Thus,

$$(x,y,z)^{\hat{o}} = (xoy)oz - xo(yoz) ,$$

$$[x,y]^{o} = xoy - yox .$$

Since $[x,y]^{o} = (\lambda - \mu)[x,y]$, A is Lie-admissible if and only if $A(\lambda,\mu)$ is .

Henceforth, assume that A is an associative algebra over F . It is easily checked that $(x,y,z)^{o}$ is written as

$$(x,y,z)^{o} = \lambda\mu[[z,x],y] = \frac{\lambda\mu}{(\lambda-\mu)^2}[[z,x]^{o},y]^{o} , \qquad (2.27)$$

$x, y, z \in A$. Setting $x = z$ in (2.27) implies that $A(\lambda,\mu)$ is flexible. If we put $x^2 = z$, (2.27) gives the Jordan identity $(x,y,xox) = 0$, so $A(\lambda,\mu)$ is flexible Jordan-admissible (noncommutative Jordan).

Remark. The (λ,μ)-mutation of an associative algebra is nothing but a quasi-associative algebra introduced by Albert (Ref. 1 cited in Section 6.4). Denote the $(\lambda,1-\lambda)$-mutation of A by $A(\lambda)$. A nonassociative algebra A over F is called quasi-associative if there exist an extension field K of F

and an associative algebra B over K such that $A_K = B(\lambda$ for some $\lambda \in K$. Note that if $\lambda = 0$, 1 then A_K is associative, so is A. If $\lambda = \frac{1}{2}$ then A_K is Jordan, so is A. If $\lambda \neq -\mu$, the product $x \circ y$ in $A(\lambda,\mu)$ is given by

$$x \circ y = (\lambda + \mu)\left[\frac{\lambda}{\lambda+\mu} xy + (1 - \frac{\lambda}{\lambda+\mu})yx\right].$$

Since an algebra with product xy is isomorphic to an algebra with product αxy ($\alpha \neq 0$ fixed in F) via $x \to \frac{1}{\alpha} x$, we have that if A is associative then $A(\lambda,\mu)$ $(\lambda \neq -\mu)$ is quasi-associative. Indeed, the identity (2.27) characterizes quasi-associativity. It is shown that an algebra A over F, which is neither associative nor Jordan, is quasi-associative if and only if A satisfies the identity $(x,y,z) = \alpha[[z,x],y]$ for some $\alpha \neq \frac{1}{2}$ in F (Y. Ko and H. C. Myung, "On Lie-admissible algebras associated with invariant bilinear forms", Bull. Korean Math. Soc., 16(1980), 77-84.). //

Let L be any Lie algebra over F and let λ, μ be fixed scalars in F with $\lambda \neq \mu$. Let $T \equiv T(L)$ $= F1 \oplus L \oplus L \otimes L \oplus \ldots$ be the tensor algebra on L. Take the (λ,μ)-mutation $T(\lambda,\mu)$ of T. Denote by $T_{\lambda,\mu}$ the subalgebra of $T(\lambda,\mu)$ generated by 1 and L. Let R be the ideal of $T_{\lambda,\mu}$ generated by the elements

$$[xy] - [x,y]^{o} = [xy] - (\lambda-\mu)(x\otimes y - y\otimes x) ,$$

$x, y \in L$, where $[xy]$ is the product in L and $[x,y]^o = xoy - yox$. We form the quotient algebra

$$U_{\lambda,\mu} \equiv U(L)_{\lambda,\mu} = T_{\lambda,\mu}/R .$$

Then $U_{\lambda,\mu}$ is flexible Lie-admissible and satisfies (2.27). Let $\phi : T_{\lambda,\mu} \to U_{\lambda,\mu}$ be the natural homomorphism. Clearly, ϕ maps F isomorphically into $U_{\lambda,\mu}$ and hence $U_{\lambda,\mu}$ contains the scalars. In fact, we show that ϕ is injective on L also.

Theorem 6.6.1. Let L be a Lie algebra over F and B be an associative algebra with unit element over F. If f is a homomorphism of L into B^- then there exists a unique homomorphism g of $U_{\lambda,\mu}$ into $B(\lambda,\mu)$ such that $g\phi = \frac{1}{\lambda-\mu} f$.

Proof. Since the tensor algebra $T(L)$ is a universal associative algebra generated by 1 and L, the linear mapping $\frac{1}{\lambda-\mu} f$ is extended to a unique homomorphism g' of $T(L)$ into B. Since g' is a homomorphism of $T(L)(\lambda,\mu)$ into $B(\lambda,\mu)$, g' induces a homomorphism of $T_{\lambda,\mu}$ into $B(\lambda,\mu)$. Then we have

$$g'([xy] - (\lambda-\mu)(x\otimes y - y\otimes x))$$

$$= \frac{1}{\lambda-\mu} f([xy]) - (\lambda-\mu)g'(x\otimes y - y\otimes x)$$

$$= \frac{1}{\lambda-\mu}[f(x),f(y)] - (\lambda-\mu)(g'(x)g'(y) - g'(y)g'(x))$$

$$= \frac{1}{\lambda-\mu}[f(x),f(y)] - \frac{1}{\lambda-\mu}[f(x),f(y)] = 0 .$$

Hence $R \subset \ker g'$. Therefore, there is a unique homomorphism $g : U_{\lambda,\mu} = T_{\lambda,\mu}/R \to B(\lambda,\mu)$ such that $g\phi = g'$. As this is restricted to L, we have $g\phi = \frac{1}{\lambda-\mu} f$. Such g is unique since 1 and $\phi(L)$ generate $U_{\lambda,\mu}$. //

Corollary 6.6.2. The natural map $\phi : T_{\lambda,\mu} \to U_{\lambda,\mu}$ is injective on L.

Proof. Take B in Theorem 6.6.1 to be the universal enveloping algebra $U(L)$ of L. By Corollary 5.1.8, the natural map $i : L \to U(L)$ is injective. Thus if we set $f = i$, it follows from Theorem 6.6.1 that ϕ is injective on L. //

Due to Theorem 6.6.1, $U_{\lambda,\mu}$ is called a universal enveloping (λ,μ)-mutation algebra (UE(λ,μ)-MA) of L. In view of Corollary 6.6.2, we identify $a = a + R$, $a \in L$ so $U_{\lambda,\mu}$ is generated by 1 and L. Theorem 6.6.1 now reads as : If f is a representation of L into an associative algebra B with unit element then the representation $\frac{1}{\lambda-\mu} f$ of L into the flexible Lie-admissible algebra $B(\lambda,\mu)$ is extended to a unique homomorphism of $U_{\lambda,\mu}$ into $B(\lambda,\mu)$, which we denote by $\frac{1}{\lambda-\mu} f$.

Let $\{x_i \mid i \in I\}$ be an ordered basis of L. Then each element in $U_{\lambda,\mu}$ is a linear combination of nonassociative monomials in x_i, $i \in I$,

$$1 \;, \quad x_{i_1} o x_{i_2} o \ldots o x_{i_r} \;, \quad i_k \in I$$

in some association of the product. Thus the unique homomorphism $\frac{1}{\lambda-\mu} f$ of $U_{\lambda,\mu}$ into $B(\lambda,\mu)$ is given by

$$\frac{1}{\lambda-\mu} f(x_{i_1} o x_{i_2} o \ldots o x_{i_r}) = \frac{1}{(\lambda-\mu)^r} f(x_{i_1}) o f(x_{i_2}) o \ldots o f(x_{i_r})$$

where both sides have the same type of association. In particular, if $B = U(L)$, the universal enveloping algebra of L then there is a unique homomorphism $g : U_{\lambda,\mu} \to U(L)(\lambda,\mu)$ such that

$$g(x_{i_1} o x_{i_2} o \ldots o x_{i_r}) = \frac{1}{(\lambda-\mu)^r} x_{i_1} o x_{i_2} o \ldots o x_{i_r} \;,$$

where the right side is a product in $U(L)(\lambda,\mu)$. It is not known whether g is injective.

Let B be an associative algebra with 1. If $b, a_1, a_2, \ldots, a_k \in B$ then, using the fact that each ad x is a derivation of B, one easily checks

$$[b,a_1a_2\ldots a_k] = \sum_{i=0}^{k-1} a_1\ldots a_i[b,a_{i+1}]a_{i+2}\ldots a_k \tag{2.28}$$

where $a_1a_2\ldots a_i = 1$ if $i = 0$. Call an element of the form $[[a,b],c]$ a double commutator in a, b, c. Thus we can state

Lemma 6.6.3. Let $b,c,a_1,a_2,\ldots,a_k$ be elements in B. Then $[[b,a_1a_2\ldots a_k],c]$ is a sum of products each of which contains exactly two commutators or one double commutator in $b,c,a_1,\ldots,a_k$. //

Let $\{x_i \mid i \in I\}$ be an ordered basis of L, L a Lie algebra. A monomial $x_{i_1}ox_{i_2}o\ldots ox_{i_r}$ in $T(L)_{\lambda,\mu}$ (or in $U_{\lambda,\mu}$) in some association is called standard if $i_1 \leqq i_2 \leqq \ldots \leqq i_r$ and call r the degree of the monomial. Since the product is not associative, an ordered subset $\{x_{i_1},\ldots,x_{i_r}\}$ with $i_1 \leqq \ldots \leqq i_r$ gives rise to different standard monomials in $U_{\lambda,\mu}$. For examples,

$$((x_{i_1}ox_{i_2})ox_{i_3})ox_{i_4}\ , \quad (x_{i_1}ox_{i_2})o(x_{i_3}ox_{i_4})\ ,$$

$$(x_{i_1}o(x_{i_2}ox_{i_3}))ox_{i_4}\ , \quad \text{etc.}$$

To obtain an analogue of the PBW Theorem for $U_{\lambda,\mu}$, we first need to develop machinery to interchange two basis elements in a monomial. Let u, v be monomials in

$T_{\lambda,\mu}$ and let x, y be basis elements of L. Then

$$(x\circ u)\circ y = (u\circ x)\circ y + [x,u]^{\circ}\circ y = (u\circ x)\circ y + (\lambda-\mu)[x,u]\circ y ,$$

and by (2.28) the last term is a linear combination of monomials (in $T(L)$) of degree $\leqq 1 + \deg(x)$ modulo J where J is the ideal of $T(L)$ generated by the elements $[xy] - (x\otimes y - y\otimes x)$, $x,y \in L$. A similar observation can be made for a product of the form $(u\circ x)\circ(v\circ y)$. Thus it suffices to consider the following two types of products in $T_{\lambda,\mu}$.

$$(u\circ x)\circ y , \quad (u\circ x)\circ(y\circ v)$$

where u and v are monomials in $T_{\lambda,\mu}$ of degree m and n, and x, y are basis elements. We make repeated use of (2.27) and the fact that $\operatorname{ad} a$ is a derivation of $T_{\lambda,\mu}$ and $T(L)$. First we compute

$$\begin{aligned}(u\circ x)\circ y &= u\circ(x\circ y) + \lambda\mu[[y,u],x] \\ &= u\circ(y\circ x) + u\circ[x,y]^{\circ} + \lambda\mu[[y,u],x] \\ &= (u\circ y)\circ x + u\circ[x,y]^{\circ} - \lambda\mu[[x,u],y] \\ &\quad + \lambda\mu[[y,u],x] .\end{aligned}$$

In the last expression, the second term can be written as a sum of standard $T(L)$-monomials modulo J of degree $\leqq m + 1$

and by Lemma 6.6.3 the last two terms are linear combination of standard $T(L)$-monomials modulo J of degree $\leq m$.

Similarly, we compute

$$(uox)o(voy) = ((uox)oy)ov - \lambda\mu[[v,uox],y]$$

$$= ((uox)oy)ov - \lambda\mu[uo[v,x] + [v,u]ox,y]$$

$$= ((uox)oy)ov - \lambda\mu uo[[v,x],y] - \lambda\mu[u,y]o[v$$

$$- \lambda\mu[[v,u],y]ox - \lambda\mu[v,u]o[x,y] .$$

Now, x and y in $((uox)oy)ov$ can be interchanged by the previous calculation. As before, the remaining terms are linear combinations of standard $T(L)$-monomials modulo J of degree $\leq m + n + 1$. Note that $(uox)o(yov)$ is a $T_{\lambda,\mu}$-monomial of degree $m + n + 2$.

Therefore, we have proved the following generalization of the PBW Theorem

Theorem 6.6.4. Every element in a universal enveloping (λ,μ)-mutation algebra $U_{\lambda,\mu}$ of L is a linear combination of R cosets of 1 and standard $T_{\lambda,\mu}$-monomials, and J cosets of 1 and standard $T(L)$-monomials. //

Remark. The standard monomial expression in Theorem 6.6.4 is quite crude. However, when

$L = su(2)$, $U(su(2))(\lambda,\mu)$ has physical applications and has been utilized to explain the non-conservative nature of angular momentum under strong interactions (C. N. Ktorides, H. C. Myung and R. M. Santilli, "Elaboration of the recently proposed test of Pauli's principle under strong interactions", Phys. Rev. D22(1980), 892-907.).

7. LIE-ADMISSIBLE ALGEBRAS OF ARBITRARY CHARACTERISTIC

7.1. Classical Lie algebras

The aim is to classify certain classes of flexible Lie-admissible algebras A over F of char $\neq 2, 3$, according to the structure of the Lie algebras A^- which closely resembles that of semisimple Lie algebras of char 0

Unlike the characteristic zero case, the classificatic of simple Lie algebras of prime char is a long standing open problem. However, there is a class of Lie algebras over F of char $\neq 2, 3$, called classical Lie algebras, which satisfy properties similar to those of semisimple Lie algebras of char 0 .

Definition 7.1.1. Let L be a finite-dimensional Lie algebra over a field F of char $\neq 2, 3$. Then L

is called classical if :

(1) the center of L is 0 ;

(2) $[LL] = L$;

(3) L has an abelian CSA H (called a classical CSA), relative to which :

(a) L has the Cartan decomposition $L = H + \Sigma_{\alpha \neq 0} L_\alpha$ such that $[xh] = \alpha(h)x$, $x \in L_\alpha$, $h \in H$;

(b) if $\alpha \neq 0$ is a root, $\dim [L_\alpha L_{-\alpha}] = 1$;

(c) if α, β are roots and $\beta \neq 0$ then not all $\alpha + k\beta$ are roots. //

Clearly, split semisimple Lie algebras of char 0 are classical in this sense. In fact, it can be shown that if F is algebraically closed and L has nondegenerate Killing form then L is classical relative to any CSA (see G. B. Seligman, "Modular Lie algebras", Springer-Verlag, New York, 1967 ; hereafter this is referred to as [S]). Note that the results in Chapter 1 and in Sections 3.1-3.3 are independent of characteristic. Thus by Lemma 1.6.3 each of the following propositions implies the next :

(a) L has nondegenerate Killing form.

(b) L is a direct sum of simple Lie algebras.

(c) L is semisimple.

In the characteristic zero case, these are all equivalent (Theorem 3.6.3) while all reverse implications fail in prime characteristic (see [S]). However, if L is classica then (b) and (c) are equivalent ([S] , p. 37).

We classify flexible Lie-admissible algebras A when A^- is classical. In fact, we can do so under slightly weaker conditions. We list, without proofs, some known results about classical Lie algebras. The reader may find proofs in Seligman's book.

Theorem 7.1.1. Let L be a classical Lie algebra with classical CSA H . Let Φ be the set of nonzero roots of H in L . Then

(1) if $\alpha \in \Phi$ then $\dim L_\alpha = 1$,

(2) Only $0, \pm\alpha$ are roots among the integral multiples of α ,

(3) if $\alpha, \beta \in \Phi$ and $\alpha + \beta \in \Phi$ then $[L_\alpha L_\beta] = L_{\alpha+\beta}$. //

If α, β are roots with $\beta \in \Phi$ then we let r, q be the least nonnegative integers such that $\alpha - (r + 1)\beta$ and $\alpha + (q + 1)\beta$ are not roots. Then the integer $r - q$ is called the <u>Cartan integer</u> of the ordered pair of roots α, β and is denoted by $A_{\alpha,\beta}$.

<u>Theorem 7.1.2</u>. Let $A_{\alpha,\beta}$ be the Cartan integer of α, β $(\beta \neq 0)$. Then

(1) $-3 \leqq A_{\alpha,\beta} \leqq 3$,

(2) $A_{\alpha,\beta} \leqq -2$ implies that $\alpha - \beta$ is not a root,

(3) $A_{\alpha,-\beta} = -A_{\alpha,\beta} = A_{-\alpha,\beta}$,

(4) $A_{0,\beta} = 0$,

(5) $A_{\alpha,\beta} < 0$ implies that $\alpha + \beta$ is a root,

(6) if $\alpha,\beta \in \Phi$ then $A_{\alpha,\beta} = 0$ implies $A_{\beta,\alpha} = 0$. //

<u>Definition 7.1.2</u>. Let L be a classical Lie algebra with classical CSA H . Then a set $\Pi = (\alpha_1,\alpha_2,\ldots,\alpha_m)$ of roots is called a fundamental system of roots relative to H if :

(1) $\alpha_i - \alpha_j$ is not a root for $i \neq j$,

(2) if $\alpha \in \Phi$ then one of the following holds :

(a) $\alpha = 0$;

(b) $\alpha = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_k}$, $\alpha_{i_j} \in \Pi$;

and $\alpha_{i_1} + \ldots + \alpha_{i_q}$ is a root for all $1 \leq q \leq k$.

(3) every diagonal minor of the matrix (A_{ij}) is positive where $A_{ij} = A_{\alpha_i, \alpha_j}$.

We say that two roots $\alpha, \beta \in \Pi$ lie in the same component if there are roots $\mu_1, \ldots, \mu_k \in \Pi$ such that $\alpha = \mu_1$, $\beta = \mu_k$ and $A_{\mu_i, \mu_{i+1}} \neq 0$, $1 \leq i \leq k-1$. This relation is an equivalence relation on Π and the equivalence classes are called the <u>components</u> of Π . If Π consists of a single component the Π is said to be <u>connected</u>. //

<u>Theorem 7.1.3</u>. For each $\alpha \in \Phi$, there exist nonzero elements $x_\alpha \in L_\alpha$, $y_\alpha \in L_{-\alpha}$ and $h_\alpha \in [L_\alpha L_{-\alpha}]$ such that $[x_\alpha y_\alpha] = h_\alpha$, $[x_\alpha h_\alpha] = x_\alpha$ and $[y_\alpha h_\alpha] = -y_\alpha$. If $\Pi = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a fundamental system of roots then for any root α, α or $-\alpha$ is a sum of roots in Π , so L has a basis consisting of elements of the form

$$h_i \ , \quad [\ldots[x_{i_1} x_{i_2}] \ldots x_{i_k}] \ , \quad [\ldots[y_{i_1} y_{i_2}] \ldots y_{i_k}] \ ,$$

$$i = 1, 2, \ldots, m \ , \quad \{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, m\}$$

where $h_i = h_{\alpha_i}$ and $x_{i_j} = x_{\alpha_{i_j}}$. //

<u>Theorem 7.1.4</u>. Let L be a classical Lie algebra. Then L is a direct sum of simple classical Lie algebras. L is simple if and only if Π is connected. //

7.2. Flexible Lie-admissible algebras with A^- classical

Let A be a flexible algebra over F of char $\neq 2$, and denote by A^+ the algebra with multiplication $x \cdot y = \frac{1}{2}(xy + yx)$ defined on the vector space A. Then the right multiplication $T(x)$ in A^+ by x is expressed as $T(x) = \frac{1}{2}(R(x) + L(x))$ where $R(x)$ and $L(x)$ are the right and left multiplications in A by x. Recall that A is flexible if and only if all $\mathrm{ad}\, x$ are derivations of A^+ (Lemma 1.4.2) ; that is, $[x, y \cdot z] = y \cdot [x,z] + [x,y] \cdot z$ for all $x,y,z \in A$. This is also expressed as

$$\mathrm{ad}\, y \cdot z = (\mathrm{ad}\, y)T(z) + (\mathrm{ad}\, z)T(y) , \qquad (2.29)$$

$x,y,z \in A$.

Let $h \in A$ be a power-associative element. Then we get from (2.29)

$$\mathrm{ad}\, h^2 = 2(\mathrm{ad}\, h)T(h) = 2T(h)\mathrm{ad}\, h \qquad (2.30)$$

and $\mathrm{ad}\, h^3 = (\mathrm{ad}\, h)T(h^2) + (\mathrm{ad}\, h^2)T(h)$, which imply

$$\mathrm{ad}\, h^3 = (\mathrm{ad}\, h)(2T(h)^2 + T(h^2)) . \qquad (2.31)$$

From (2.29) and (2.30) we obtain

$$\text{ad } h^4 = 2(\text{ad } h^2)T(h^2) = 4(\text{ad } h)T(h)T(h^2)$$

$$= 4T(h^2)T(h)\text{ad } h \qquad (2.32)$$

$$= (\text{ad } h)T(h^3) + (\text{ad } h^3)T(h) ,$$

$$\text{ad } h^5 = (\text{ad } h^2)T(h^3) + (\text{ad } h^3)T(h^2) . \qquad (2.33)$$

Lemma 7.2.1. Let A be a flexible algebra over F of char $\neq 2$ and let $h \in A$ be a power-associative element. Then

(1) If $x(\in A)$ is a common eigenvector of ad h and ad h^2 then $[x,h^3] = [x,h^4] = 0$ imply $[x,h^2] = 0$.

(2) If x is a common eigenvector of ad h, ad h^2 $R(h)$, and $R(h^2)$ then $[x,h^4] = [x,h^5] = 0$ imply $[x,h^3] = 0$.

Proof. (1) Let $x \text{ ad } h = \lambda x$ and $x \text{ ad } h^2 = \mu x$, $\lambda,\mu \in F$. If $\lambda = 0$ then it follows from (2.30) that $\mu = 0$, or $[x,h^2] = 0$. Suppose $\lambda \neq 0$. Then by (2.31) we get $xT(h^2) = -2xT(h)^2$ since $[x,h^3] = 0$. Hence by (2.32), $0 = x \text{ ad } h^4 = 4xT(h^2)T(h) \text{ ad } h = -8xT(h)^3 \text{ ad } h$ $= -8\lambda xT(h)^3$, since $T(h) \text{ ad } h = (\text{ad } h)T(h)$ (the flexible law is equivalent to the identity $R(x)L(x) = L(x)R(x)$). Thus $xT(h)^3 = 0$ but by (2.30) $xT(h) = \mu(2\lambda)^{-1}x$, so $\mu = 0$; $[x,h^2] = 0$.

(2) Let $x \text{ ad } h = \lambda x$, $x \text{ ad } h^2 = \mu x$, $xR(h) = \nu x$ and $xR(h^2) = \omega x$, $\lambda, \mu, \nu, \omega \in F$. If $\lambda = 0$ then we can use (2.31) to conclude $[x,h^3] = 0$. Now suppose $\lambda \neq 0$. If $\mu = 0$ then by (2.30) $xT(h) = 0$, so by (2.31) we get $x \text{ ad } h^3 = x(\text{ad } h)T(h^2) = \lambda\omega x$, since $[x,h^2] = 0$; $xR(h^2) = xL(h^2)$. Hence, by (2.33), this implies that $0 = x(\text{ad } h^3)T(h^2) = \lambda\omega xT(h^2) = \lambda\omega^2 x$ (recall $\mu = 0$). Thus $\omega = 0$ and $[x,h^3] = 0$. Finally, assume $\mu \neq 0$. By (2.32),

$$0 = x \text{ ad } h^4 = 2x(\text{ad } h^2)T(h^2) = 2\mu xT(h^2) ,$$

so $xT(h^2) = 0$. Since $\lambda \neq 0$, this gives $x(\text{ad } h)T(h^2) = 0$. Therefore, by (2.31), we have

$$x \text{ ad } h^3 = 2x(\text{ad } h)T(h)^2 = 2\lambda xT(h)^2 = 2\lambda\bar{\nu}^2 x$$

since $xT(h) = \tfrac{1}{2}hx + \tfrac{1}{2}xh = \tfrac{1}{2}(\nu - \lambda)x + \tfrac{1}{2}\nu x = (\nu - \tfrac{1}{2}\lambda)x$, where $\bar{\nu} = \nu - \tfrac{1}{2}\lambda$. Since $xT(h^2) = 0$, this and (2.33) imply

$$\begin{aligned} 0 = x \text{ ad } h^5 &= x(\text{ad } h^2)T(h^3) + x(\text{ad } h^3)T(h^2) \\ &= \mu xT(h^3) + 2\lambda\bar{\nu}^2 xT(h^2) \\ &= \mu xT(h^3) , \end{aligned}$$

so $xT(h^3) = 0$ since $\mu \neq 0$. Therefore, it follows from (2.32) that

$$0 = x \text{ ad } h^4 = x(\text{ad } h)T(h^3) + x(\text{ad } h^3)T(h)$$

$$= \lambda xT(h^3) + 2\lambda\bar{\nu}^2 xT(h)$$

$$= 2\lambda\bar{\nu}^3 x$$

since $xT(h) = \bar{\nu}x$. Thus $\bar{\nu} = 0$ and $x \text{ ad } h^3 = 0$ since we have computed $x \text{ ad } h^3 = 2\lambda\bar{\nu}^2 x$. //

In the remainder of this section, we assume that A is a finite-dimensional flexible Lie-admissible algebra over F . Let H be a CSA of A^- . Then $H = A_0^-(\text{ad } H)$ and since $\text{ad } h$ $(h \in H)$ is a derivation of A , we have

Lemma 7.2.2. Any CSA of A^- is a subalgebra of A .

If H is a split CSA of A^- then let

$$A^- = H + \sum_{\alpha \neq 0} A_\alpha$$

be the Cartan decomposition of A^- relative to H . Then

$$[A_\alpha, A_\beta] \subseteq A_{\alpha+\beta} \ .$$

Since $\text{ad } H \subset \text{Der } A$ (Theorem 1.4.3), in view of Corollary 1.4.8 $A_{\alpha(h)}(\text{ad } h)A_{\beta(h)}(\text{ad } h) \subset A_{(\alpha+\beta)(h)}(\text{ad } h)$.

Noting that $A_\alpha = \bigcap_{h \in H} A_{\alpha(h)}(\text{ad } h)$, we have

Lemma 7.2.3. Let A_α be the root space of A^-

relative to a split CSA H of A^-, corresponding to a root α. Then

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta}, \quad \alpha, \beta, \text{ roots.} \quad //$$

<u>Lemma 7.2.4</u>. Let H be a split CSA of A^- such that $h^2 = hh = 0$ for all $h \in H$. If $\alpha \neq 0$ is a root of H such that $\operatorname{ad} h$ is a scalar on A_α for all $h \in H$ then $A_\alpha T(H) = 0$.

<u>Proof</u>. Let $h \in H$. Since $\alpha(h)$ is the only eigenvalue of $\operatorname{ad} h$ on A_α, $\operatorname{ad} h = \alpha(h)$ on A_α. By (2.30) $(\operatorname{ad} h)T(h) = 0$ for all $h \in H$. Hence if $\alpha(h) \neq 0$, then $0 = A_\alpha(\operatorname{ad} h)T(h) = \alpha(h)A_\alpha T(h)$ gives $A_\alpha T(h) = 0$. Assume $\alpha(h) = 0$. Since $\alpha \neq 0$, choose an $h' \in H$ with $\alpha(h') \neq 0$. We linearize $(\operatorname{ad} h)T(h) = 0$ to get $(\operatorname{ad} h)T(h') + (\operatorname{ad} h')T(h) = 0$, so $0 = A_\alpha(\operatorname{ad} h)T(h') + A_\alpha(\operatorname{ad} h')T(h) = A_\alpha(\operatorname{ad} h')T(h) = \alpha(h')A_\alpha T(h)$. Thus $A_\alpha T(h) = 0$. //

Recall that a power-associative element $x \in A$ is called nilpotent if $x^m = 0$ for some $m > 0$. If $\dim A = n$, then $x, x^2, \ldots$ span a nilpotent commutative associative algebra B of dimension $\leq n$. Thus the right multiplication $R(x)$ in B by x is nilpotent, so $R(x)^n = 0$ and this gives $x^{n+1} = 0$.

We prove the following structure theorem on flexible Lie-admissible algebras.

Theorem 7.2.5. Let A be a finite-dimensional flexible Lie-admissible algebra over a field F of char $\neq 2$. Suppose that A^- has a split abelian CSA H satisfying :

(1) H is nil in A ; that is, every $h \in H$ is nilpotent in A ;

(2) $\dim A_\alpha = 1$ for each root $\alpha \neq 0$;

(3) the center A^- is zero.

Then A is a Lie algebra isomorphic to A^- .

Proof. We first show that H is a nil subalgebra of A such that $h^2 = 0$ for all $h \in H$. By the foregoin remark there is a positive integer t such that $h^t = 0$ for all $t \in H$ (for example, $t = 1 + \dim H$).

Suppose $t \geq 3$ and let n be the least integer such that $3n \geq t$. Let h be any element in H . Set $g = h^n$. Then $g^3 = 0$. Since $\dim A_\alpha = 1$ for $\alpha \neq 0$, each element of A_α is a common eigenvector of ad g and ad g^2 . Since H is abelian and $g^3 = 0$, by Lemma 7.2.1(1) we have that g^2 is in the center of A^- , so $g^2 = 0$ or $h^{2n} = 0$. If $2n > 4$, let m be the least integer with $3m \geqq 2n$. Then $n > m$, since if $m \geqq n$ then $3m \geqq 3n > 2n$ and $n = m$, so $3(n - 1) = 2n + (n - 3) \geqq 2n$ ($2n > 4$ gives $n \geqq 3$), a contradiction. Then the argument just used implies $h^{2m} = 0$. Hence, by repeated applications of this, we have either $h^4 = 0$ or $h^2 = 0$. Since $\dim A_\alpha = 1$ for $\alpha \neq 0$, by Lemma 7.2.3 every element of A_α is an eigenvector of $R(h')$ for all $h' \in H$. Therefore, Lemma 7.2.1(2) implies $h^3 = 0$, since A^- has center 0. Thus by the above $h^2 = 0$. Hence $0 = (h_1 + h_2)^2 = h_1^2 + 2h_1h_2 + h_2^2 = 2h_1h_2$ for all $h_1, h_2 \in H$, since H is abelian. That is, $HH = 0$. If $A_\alpha = Fx$ for $\alpha \neq 0$ and $[x,h] = \alpha(h)x$, then by Lemma 7.2.4 we have

$$xh = -hx = \tfrac{1}{2}\alpha(h)x, \quad h \in H. \tag{2.34}$$

Let α, β be any nonzero roots of H. If $\alpha + \beta$ is not a root then by Lemma 7.2.3

$$A_\alpha A_\beta = A_\beta A_\alpha = 0. \tag{2.35}$$

Suppose $\alpha + \beta$ is a root. If $\alpha + \beta = 0$, choose an $h \in H$ with $\alpha = \alpha(h) \neq 0$ and let $xh = \sigma x$ and $A_\beta = Fy$, so $hx = (\sigma - \alpha)x$. Since xy and yx are in H, from the flexible law $(h,x,y) + (y,x,h) = 0$ (see (1.7)) or $(hx)y - h(xy) + (yx)h - y(xh) = 0$ we have $(\sigma - \alpha)xy - \sigma yx = 0$ or $xy = \sigma\alpha^{-1}[x,y]$. If $\alpha + \beta$ is a nonzero root then by Lemma 7.2.3 $A_\alpha A_\beta \subset A_{\alpha+\beta} = Fz$. Therefore, for any roots α and $\beta \neq 0$, we can let

$$xy - yx = \lambda z \ , \quad xy = \mu z \ , \quad yx = (\mu - \lambda) z \qquad (2.36)$$

for $x \in A_\alpha$, $\lambda, \mu \in F$ and some $z \neq 0$ in $A_{\alpha+\beta}$. Choose an $h \in H$ with $\beta(h) \neq 0$ and let $\alpha = \alpha(h)$ and $\beta = \beta(h)$. By Lemma 7.2.4 we get

$$xh = -hx = \tfrac{1}{2}\alpha x \ , \quad zh = -hz = \tfrac{1}{2}(\alpha + \beta) z \qquad (2.37)$$

for $x \in A_\alpha$ and some $z \neq 0$ in $A_{\alpha+\beta}$. We utilize the flexible law $(hx)y - h(xy) + (yx)h - y(xh) = 0$ together with (2.36) and (2.37) to obtain

$$\tfrac{1}{2}[-\alpha\mu + (\alpha+\beta)\mu + (\mu-\lambda)(\alpha+\beta) - \alpha(\mu-\lambda)] z = 0 \ .$$

Since $z \neq 0$, this gives $\beta(2\mu - \lambda) = 0$, so $\lambda = 2\mu$. It therefore follows from (2.34) - (2.37) that $xy = -yx = \tfrac{1}{2}[x,y]$ for all $x, y \in A$. Thus A is a Lie algebra isomorphic to A^- . //

If A^- is classical then A satisfies the condition of Theorem 7.2.5(Theorem 7.1.1). Thus we have

Corollary 7.2.6. Suppose that A^- is a classical Lie algebra having a classical CSA which is nil in A . Then A is a Lie algebra isomorphic to A^- . //

Let A be a power-associative algebra over F. An ideal I of A is said to be <u>nil</u> if every element of I is nilpotent. If I and J are nil ideals of A then so is $I + J$. In fact, let $a \in I$, $b \in J$ with $a^m = 0$. Then $(a + b)^m = a^m + c = c \in J$, so $(a + b)^{mn} = c^n = 0$ for some $n > 0$. Hence $I + J$ is nil and so A contains a unique maximal nil ideal $N(A)$, called the <u>nil radical</u> of A. It is shown that if A is a finite-dimensional flexible strictly power-associative algebra over F of char $\neq 2,3$ and $N(A) = 0$ then A has a unit element 1 (R. H. Oehmke, "On flexible algebras", Ann. of Math. (2) 68(1958), 221-230). Thus, if, in addition, A^- is a simple Lie algebra then A must be a nilalgebra. Suppose that A^- is a semisimple Lie algebra over a field F of char 0. Then A^- is a direct sum of simple Lie algebras, so by Theorem 6.4.3 A is a direct sum of simple flexible Lie-admissible algebras A_i with A_i^- simple. Therefore, A is a nilalgebra and so Corollary 6.4.5 is also a consequence of Theorem 7.2.5.

<u>Remark</u>. A necessary and sufficient condition that A be a nilalgebra has been given in terms of a CSA of A^-. It is shown that a finite-dimensional flexible Lie-admissible power-associative algebra over F of char 0 is a nilalgebra if and only if A^- contains a CSA H which

has a basis consisting of nilpotent elements (H. C. Myung, "Flexible Lie-admissible algebras with nil-basis", Hadronic J. 2(1979), 360-369). //

In Theorem 7.2.5, the conditions that H is abelian, $\dim A_\alpha = 1$ for $\alpha \neq 0$, and center $A^- = 0$ are not strong enough to imply that H is nil in A. Let A be a 3-dimensional algebra over F with multiplication given by

$$xh = x \ , \quad yh = \tfrac{1}{2}(\alpha+1)y \ , \quad hy = \tfrac{1}{2}(1-\alpha)y \ , \quad h^2 = h$$

and all other products are 0, where $\alpha \neq 0,1$ in F. It is easy to see that A is flexible Lie-admissible and A^- is given by

$$[x,y] = 0 \ , \quad [x,h] = x \ , \quad [y,h] = \alpha y \ ,$$

so that A^- is a solvable Lie algebra. Note that $A^- = Fh + Fx + Fy$ is the Cartan decomposition of A^- relative to CSA $H = Fh$, and $A_1 = Fx$, $A_\alpha = Fy$ for roots 1 and α. The center of A^- is 0 but H is not nil in A. This example also shows that the Lie algebra A^- in Theorem 7.2.5 need not be semisimple.

<u>Theorem 7.2.7</u>. Let A be a 4-dimensional flexible algebra over an algebraically closed field F of char 0 such that A^- is a non-solvable Lie algebra with a CSA H

which is nil in A. Then A is either a Lie algebra or a nil algebra of nil-index 3 with basis x, y, h, a whose multiplication table is given by

$$xy = h + \tfrac{1}{2}a \ , \quad yx = -h + \tfrac{1}{2}a \ , \quad xh = -hx = 2x \ ,$$

$$yh = -hy = -2y \ , \quad h^2 = -a$$

and all other products are 0.

<u>Proof</u>. The general case of A has been classified in Corollary 6.5.4. Then $H = Fh + Fa$ is a CSA of A^- which must be nil in A by the assumption. Thus $b^3 = 0$ for all $b \in H$ since $\dim H = 2$, and $a^2 = \beta a$ implies $\beta = 0$, so $a^2 = 0$. Using this and $h^3 = 0$, we have $0 = (h + a)^3 = h^2 a + (ha)a + (ha)h = \alpha^2 h + 2\alpha\gamma a$, so $\alpha = 0$. If $\gamma = 0$ then A is a Lie algebra. If $\gamma \neq 0$ then we replace a by $-\frac{1}{2\gamma}a$ to obtain the algebra given in the table. It is easy to check $u^3 = 0$ for all $u \in A$. //

The algebra A in Theorem 7.2.7 shows that the condition of center $A^- = 0$ is essential in Theorem 7.2.5.

7.3. Embedding of Lie algebras

We discuss a necessary and sufficient condition that a classical Lie algebra is embedded into a flexible Lie-admissible algebra. This condition is given in terms of a Cartan subalgebra of its associated Lie algebra.

Lemma 7.3.1. Any flexible Lie-admissible algebra A of char $\neq 2$ satisfies the identity

$$[x,y]y = \tfrac{1}{2}([x,y^2] + [[x,y],y]) .$$

Proof. Recall $\operatorname{ad} x$ is a derivation of A for all $x \in A$. The flexible law $(x,y,y) + (y,y,x) = 0$ implies $y(yx) = (xy)y - [x,y^2]$. Thus we have

$$\begin{aligned}[][[x,y],y] &= (xy)y - (yx)y - y(xy) + y(yx) \\ &= (xy)y - 2y(xy) + y(yx) \\ &= 2(xy)y - [x,y^2] - 2y(xy) \\ &= 2[xy,y] - [x,y^2] \\ &= 2[x,y]y - [x,y^2] ,\end{aligned}$$

so $[x,y]y = \tfrac{1}{2}([x,y^2] + [[x,y],y])$. //

Lemma 7.3.2. Let L be a Lie algebra over an arbitrary field F. Let S be a finite-dimensional subalgebra of L and H be a CSA of S. Then, for an ideal S of L, $[SR] = 0$ if and only if $[HR] = 0$.

Proof. We may assume that F is algebraically closed ; if not, one takes the scalar extension of S to the algebraic closure of F. Let $S = H + \sum_{\alpha \neq 0} S_\alpha$ be the Cartan decomposition of S relative to H. Then, for each $h \in H$ with $\alpha(h) \neq 0$ $(\alpha \neq 0)$, $\operatorname{ad} h : S_\alpha \to S_\alpha$ is surjective (see the proof of Theorem 3.3.6(2)), so $[S_\alpha h] = S_\alpha$. If $[HR] = 0$ and $\alpha \neq 0$ then, by the Jacobi identity,

$$[S_\alpha R] = [[S_\alpha h]R] \subset [[Rh]S_\alpha] + [[S_\alpha R]h] = 0$$

since R is an ideal of L. Thus $[SR] = 0$. //

Theorem 7.3.3. Let A be a flexible Lie-admissible algebra over a field F of char $\neq 2$. Let S be a finite-dimensional subalgebra of the Lie algebra A^- and H be a CSA of S. Then S is a subalgebra of A if and only if $HH \subset S$.

Proof. As before, we may assume that F is algebraically closed. Let $S = H + \Sigma_{\alpha \neq 0} S_\alpha$ be the Cartan decomposition of S relative to H. Suppose that $HH \subset S$.

We first show that $S_\alpha H \subset S$ for $\alpha \neq 0$. Thus, choose an $h \in H$ with $\alpha(h) \neq 0$. Then $[S_\alpha, h] = S_\alpha$ for $\alpha \neq 0$ since $\operatorname{ad} h : S_\alpha \to S_\alpha$ is surjective. For any element $x \in S_\alpha$, we have by Lemma 7.3.1

$$[x,h]h = \tfrac{1}{2}([x,h^2] + [[x,h],h]) \in [S,HH] + [[S,H],H] \subset S$$

Since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha h \subset S$. Now, let k be any element in H. Then one gets

$$\begin{aligned}[x,h]k &= [x,hk] - h[x,k] \\ &= [x,hk] - [h,[x,k]] - [x,k]h \\ &\in [S,HH] + [H,[S,H]] + [S_\alpha,H]h \subset S + S_\alpha h \subset S .\end{aligned}$$

Again, since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha H \subset S$ for $\alpha \neq 0$. Since $HH \subset S$, we have $SH \subset S$.

For any $\alpha \neq 0$, let h be an element of H such that $\alpha(h) \neq 0$. Let $x \in S_\alpha$, $y \in S$. Then

$$y[x,h] = [x,yh] - [x,y]h \in [S,SH] + SH \subset S .$$

Since $[S_\alpha, h] = S_\alpha$, this shows that $SS_\alpha \subset S$ for $\alpha \neq 0$. It follows from $SH \subset S$ that $SS \subset S$ and S is a subalgebra of A. //

In view of the known structure of A when A^- is classical, Theorem 7.3.3 enables us to give a condition for A that a classical Lie algebra is embedded into A as a subalgebra.

<u>Corollary 7.3.4</u>. Let A be the same as in Theorem 7.3.3. Let S be a finite-dimensional subalgebra of A^- which is classical and H be a classical CSA of S. Then S is a Lie algebra under the multiplication in A if and only if $HH \subset S$ and H is nil in A (char $F \neq 2,3$).

<u>Proof</u>. One direction is trivial. If $HH \subset S$ and H is nil in A then by Theorem 7.3.3 S is a subalgebra of A, so the result follows from Corollary 7.2.6. //

In the case of characteristic 0, we have the following stronger result.

<u>Corollary 7.3.5</u>. Let F be of characteristic 0 and let S be a finite-dimensional semisimple subalgebra of A^-. Suppose that S contains a split CSA H such that $HH \subset S$. Then S is a subalgebra of A, and is isomorphic to a direct sum of simple Lie algebras and simple algebras given by (2.15) defined on $s\ell(\ell+1,F)$ for some $\ell \geq 2$.

<u>Proof</u>. This follows from Theorem 7.3.3 and Corollary 6.4.4. //

Note that if, in addition, S is power-associative then S is a Lie algebra under the multiplication in A (Corollary 6.4.5).

In the discussion above, when the classical subalgebra S of A^- is complemented by an ideal R of A^- (i.e., $A^- = S^- \oplus R$ as Lie algebra) then it is possible to determine the multiplication between S and R. Note that, by Levi's Theorem 3.7.6, if A is finite-dimensional of char 0 then A^- contains a semisimple subalgebra which is complemented by the solvable radical Rad A^-.

Let $S_1, S_2, \ldots, S_n$ be Lie algebras over F of char $\neq 2$ (each S_i with multiplication written as xy) and let R be a flexible Lie-admissible algebra over F. Let $f_1, \ldots, f_n$ be linear functionals on R which vanish on $[R,R]$. Let S be the direct sum of Lie algebras $S_1, \ldots, S_n$. Form the vector space direct sum $A = S + R$ and define a multiplication in A by

$$(*) \quad \left(\sum_{i=1}^{n} x_i + r\right)\left(\sum_{i=1}^{n} y_i + s\right) = \sum_{i=1}^{n} \left[x_i y_i + f_i(r) y_i + f_i(s) x_i\right] +$$

$$x_i, y_i \in S_i, \quad r, s \in R, \quad i = 1, 2, \ldots, n .$$

Then one sees that $[x + r, y + s] = 2xy + [r,s]$ for $x, y \in$ and $r, s \in R$. Thus A is Lie-admissible. We also compute

$$[(x + r)(y + s)](x + r)$$

$$= \sum_i \left[(x_i y_i) x_i + f_i(r) y_i x_i + f_i(s) x_i^2 + f_i(rs) x_i + f_i(r) x_i y\right.$$

$$\left. + f_i(r) f_i(r) y_i + f_i(r) f_i(s) x_i\right] + (rs) r$$

$$= \sum_i \left[(x_i y_i) x_i + f_i(rs) x_i + f_i(r)^2 y_i + f_i(r) f_i(s) x_i\right] + (rs) r$$

since each S_i is anticomutative. Likewise we have

$$(x + r)[(y + s)(x + r)]$$

$$= \sum_i [x_i(y_i x_i) + f_i(sr)x_i + f_i(r)^2 y_i + f_i(r)f_i(s)x_i] + r(sr) .$$

Thus $[(x + r)(y + s)](x + r) - (x + r)[(y + s)(x + r)]$ $= \sum_i f_i([r,s])x_i = 0$ since each S_i and R are flexible and each f_i vanishes on $[R,R]$. Therefore, A is flexible Lie-admissible.

In fact we can give a condition that a classical Lie algebra S is embedded into A in such a way that the multiplication between S and R is given by the rule (*).

Lemma 7.3.6. Let M be a subset of a flexible Lie-admissible algebra A. Then the centralizer $C(M)$ $= \{x \in A \mid [x,M] = 0\}$ of M in A^- is a subalgebra of A.

Proof. Let $x,y \in C(M)$. Then $[xy,M] \subset x[y,M]$ $+ [x,M]y = 0$. //

Theorem 7.3.7. Let A be a flexible Lie-admissible algebra over F of char $\neq 2,3$ (not necessarily finite-dimensional). Let S be a finite-dimensional classical subalgebra of A^- which is complemented by an ideal R of A^-. Then S is a Lie algebra under the multiplication in A and is an ideal of A if and only if S contains a classical CSA H which is nil in A and such that $HH \subset S$ and $[H,R] = 0$. In this case, R is a subalgebra of A and the multiplication in A is given by the rule (*) where $f_1,\ldots,f_n$ are linear functionals on R which vanish on $[R,R]$, and n is the number of simple summands in S.

Remark. By Theorem 7.1.4 S is a direct sum of simple classical Lie algebras. //

Proof. One direction is clear. Suppose that S contains a classical CSA H satisfying the conditions. By Lemma 7.3.2 $[H,R] = 0$ implies $[H,S] = 0$, while it follows from $HH \subset S$ that S is a subalgebra of A (Theorem 7.3.3). Therefore, by Corollary 7.2.6 S is a Lie algebra under the multiplication in A. Thus we have $[x,y] = xy - yx = 2xy$, $x,y \in S$. Since $[S,R] = 0$ and S has center 0 (by definition), we have $R = C(S)$. Thus by Lemma 7.3.6 R is a subalgebra of A.

Let $x,y \in S$ and let $r \in R$. Write $yr = ry = z + s$ for some $z \in S$ and $s \in R$. Then the flexible law $(xy)r - x(yr) + (ry)x - r(yx) = 0$ implies $(xy)r = xz$ since S is anticommutative and $[R,S] = 0$. Thus $(SS)R \subset S$ and since S is a direct sum of simple algebras $SS = S$. This implies that S is an ideal of A. Since

$C_S(H) = H$ and $HH = [H,H] = 0$, from $[h',hr] = [h',h]r + h[h',r] = 0$ for all $h,h' \in H$ and $r \in R$ we have

$$HR = RH \subset H . \qquad (2.38)$$

Let $S = H + \Sigma_{\alpha \neq 0} S_\alpha$ be the Cartan decomposition of S relative to H. Recall that $\dim S_\alpha = 1$ for $\alpha \neq 0$ (Theorem 7.1.1), and $xh = \alpha(h)x$, $x \in S_\alpha$, $h \in H$. For a root $\alpha \neq 0$, choose an $h \in H$ such that $\alpha(h) \neq 0$. If $x \in S$ and $r \in R$ then from $(xh)r - x(hr) + (rh)x - r(hx) = 0$ and (2.38) one gets $xr = \alpha(h)^{-1}\alpha(hr)x$. Thus we can put

$$xr = rx = \lambda x , \quad x \in S_\alpha , \quad \alpha \neq 0 \qquad (2.39)$$

where $\lambda \in F$ depends on $r \in R$ and $\alpha \neq 0$. Since S is classical, $S_\alpha S_{-\alpha}$ is one-dimensional for $\alpha \neq 0$. Thus one can choose nonzero elements $x \in S_\alpha$, $y \in S_{-\alpha}$, $h \in H$ such that

$$xh = x , \quad yh = -y , \quad xy = h . \qquad (2.40)$$

Then by (2.39) $xr = rx = \lambda x$ and $yr = ry = \mu y$, $\mu \in F$, $r \in R$. If we let $H = Fh + B$ be a vector space direct sum then by (2.38) $hr = rh = \nu h + b$, $\nu \in F$, $b \in B$. By the flexible law $(xy)r - x(yr) + (ry)x - r(yx) = 0$, it follows from (2.40) that $2(\nu h + b) = 2\mu h$, so $b = 0$ and $\nu = \mu$. Similarly, from $(yx)r - y(xr) + (rx)y - r(xy) = 0$

one gets $\nu = \lambda$. Therefore,

$$xr = rx = f_\alpha(r)x, \; yr = ry = f_\alpha(r)y, \; hr = rh = f_\alpha(r)h \quad (2.41$$

for $\alpha \neq 0$ and $r \varepsilon R$, where f_α is a linear functional on R and $\{x,y,h\}$ is the canonical basis as in (2.40). In particular, we have $f_\alpha = f_{-\alpha}$.

Since S is a direct sum of simple classical Lie algebras, for the remainder of the proof we may assume that S is simple. Let $\Pi = \{\alpha_1,\ldots,\alpha_m\}$ be a fundamental system of roots for S. Then Π is connected (Theorem 7.1. that is, for any roots α , $\beta \varepsilon \Pi$, there are roots $\mu_1,\ldots,\mu_r \varepsilon \Pi$ such that $\alpha = \mu_1$, $\beta = \mu_r$ and $A_{\mu_i,\mu_{i+1}} \neq$ $1 \leq i \leq r$, where $A_{\alpha,\beta}$ is the Cartan integer. For brevity, denote $A_{\alpha_i,\alpha_j} = A_{ij}$ and $f_{\alpha_i} = f_i$. We contend that if $A_{ij} \neq 0$ then $f_i = f_j$. If $A_{ij} < 0$ then $\alpha_i + \alpha_j$ is a root, so $S_{\alpha_i}S_{\alpha_j} \neq 0$ (Theorem 7.1.1(3)) and hence one chooses elements $x \varepsilon S_{\alpha_i}$, $y \varepsilon S_{\alpha_j}$ with $xy \neq 0$. Then by the flexible law $(xy)r - x(yr) + (ry)x - r(yx) = 0$, it follows from (2.41) that $(xy)r = f_j(r)(x$ for all $r \varepsilon R$. Similarly, $(yx)r - y(xr) + (rx)y - r(xy) = 0$ gives $(xy)r = f_i(r)(xy)$. Thus $f_i = f_j$. If $A_{ij} > 0$ then, using $A_{\alpha,-\beta} = -A_{\alpha,\beta}$ (Theorem 7.1.2(3)) and $f_\alpha = f_{-\alpha}$, we argue that $f_i = f_j$. Since Π is connected, this prove that $f_i = f_j$ for all $i,j = 1,2,\ldots,m$. Denote one of these f_i by f.

Let x_i, y_i, h_i be the canonical basis corresponding to the root $\alpha_i \in \Pi$. We have then shown

$$x_i r = rx_i = f(r)x_i,\ y_i r = ry_i = f(r)y_i,\ h_i r = rh_i = f(r)h_i, \quad r \in R,\ i = 1,2,\ldots,m. \tag{2.42}$$

Then S has a basis consisting of elements of the form

$$h_i,\ (\ldots(x_{i_1}x_{i_2})\ldots x_{i_k}),\ (\ldots(y_{i_1}y_{i_2})\ldots y_{i_k}),$$

$$i = 1,2,\ldots,m,\quad \{i_1,\ldots,i_k\} \subset \{1,2,\ldots,m\}$$

(Theorem 7.1.3). The flexible law $(x_i x_j)r - x_i(x_j r) + (rx_j)x_i - r(x_j x_i) = 0$ gives $(x_i x_j)r = f(r)(x_i x_j)$ by (2.42). Therefore by induction we have

$$xr = rx = f(r)x,\quad x \in S,\quad r \in R.$$

Finally, let $r,s \in R$ and let x be a nonzero element of S. Then $(rs)x - r(sx) + (xs)r - x(sr) = 0$ gives $f([r,s]) = 0$, so f vanishes on $[R,R]$. Thus the multiplication in A is given by $(*)$. //

In view of Corollary 6.4.5, the following is immediate from Theorem 7.3.7.

<u>Theorem 7.3.8</u>. Let A be a finite-dimensional

flexible Lie-admissible over F of char 0 . Let R be the solvable radical of A^- and S be a Levi-factor of A^- which is power-associative in A . Then S is a Lie algebra under the multiplication in A and is an ideal of A if and only if S contains a split CSA H such that $HH \subset S$ and $[H,R] = 0$. In this case, the multiplication between S and R is given by (*) . //

Corollary 7.3.9. Let A, S, R be the same as in Theorem 7.3.7 and let S be embedded into A as in Theorem 7.3.7. If $[R,R] = R$ then R is an ideal of A . In particular, if R^- is simple then R is an ideal of A .

Proof. The linear functionals f_i vanish on $[R,R] = R$, so $f_i = 0$, $i = 1,2,\ldots,n$. If R^- is simple, $[R,R] = R$. //

The materials in Sections 7.2 and 7.3 are drawn from Myung :

1. "Some classes of flexible Lie-admissible algebras' Trans. Amer. Math. Soc. 167(1972), 79-88.

2. "A subalgebra condition in Lie-admissible algebras", Proc. Amer. Math. Soc. 59(1976), 6-8.

3. "Embedding of a Lie algebra into Lie-damissible algebras", Proc. Amer. Math. Soc. 73(1979), 303-307.

7.4. Lie-admissible algebras associated with generalized Witt algebras

We discuss in this section an extension of Theorem 7.2.5 which has recently been proved by G. M. Benkart ("The construction of examples of Lie-admissible algebras", Hadronic J. 5(1982), 431-493). The idea is to replace the restriction of $\dim A_\alpha = 1$ for $\alpha \neq 0$ by the weaker condition that the Cartan subalgebra H acts diagonally on each root space A_α. In this way, the conclusion of Theorem 7.2.5 applies to a broader class of flexible Lie-admissible algebras, such as algebras where the attached Lie algebra A^- is a generalized Witt algebra or a Virasoro algebra. The identity (2.29), which can also be expressed as

$$[x \circ y, z] = x \circ [y,z] + [x,z] \circ y \,, \tag{2.42}$$

plays a main role.

Theorem 7.4.1. Let A be a flexible Lie-admissible algebra with product xy (not necessarily finite-dimensional) over a field F of characteristic $\neq 2$ and let H be an abelian Cartan subalgebra of A^-. Assume that A^- has a Cartan decomposition relative to H such that each $\operatorname{ad} h$ $(h \in H)$ diagonally acts on the root space A_α for all α; i.e., $[h,x] = \alpha(h)x$ for all $x \in A_\alpha$ and $h \in H$.

(1) If $h \in H$ and $x \in A_\alpha$ for $\alpha \neq 0$ then hx and xh are multiples of x.

(2) If $x \in A_\alpha$, $y \in A_\beta$ and $\alpha \neq -\beta$ for $\alpha, \beta \neq 0$ then xy is a multiple of $[x,y]$.

(3) If $HH = 0$ then $xy = \frac{1}{2}[x,y]$ for all $x, y \in A$ and hence A is a Lie algebra.

(4) If the center of A^- is zero and H is power-associative under the product xy with property that there exists a positive integer n such that $h^n = 0$ for all $h \in H$, then $HH = 0$ and A is a Lie algebra.

Proof. We first note that, under the assumption, each root α is a linear functional on H. Let α be a nonzero root of H. For $h, h' \in H$ and $x \in A_\alpha$, by (2.42) we have

$$\alpha(h \circ h')x = [h \circ h', x] = [h,x] \circ h' + h \circ [h',x]$$

$$= \alpha(h)x \circ h' + \alpha(h')h \circ x \quad . \tag{2.43}$$

First, we consider the special case where $h = h'$ and $h \notin \ker \alpha$, the kernel of α. Then $\alpha(h)^2 x = \alpha(h)[h,x] = \alpha(h)(hx - xh)$, while $\alpha(h^2)x = \alpha(h)(hx + xh)$ by (2.43). This implies that hx is a multiple of $[h,x]$. If $h \circ h = 0$ then $x \circ h = 0$ by (2.43) and hence

$$hx = \frac{1}{2}\alpha(h)x = \frac{1}{2}[h,x] \quad . \tag{2.44}$$

Assume that $h \notin \ker \alpha$ and $h' \in \ker \alpha$. Thus, by (2.43), $\alpha(h \circ h')x = \alpha(h)x \circ h'$ and so $x \circ h'$ is a multiple of x.

This proves part (1) . In particular, if $h \circ h' = 0$ then by (2.43) $x \circ h' = 0$ and so $h' \circ x = 0 = \frac{1}{2}\alpha(h')x = \frac{1}{2}[h',x]$. By the linearity of α , we conclude that if $HH = 0$ then (2.44) holds for all $h \in H$, $x \in A_\alpha$.

Suppose now that $x \in A_\alpha$ and $y \in A_\beta$ for $\alpha \neq 0$, $\beta \neq 0$ with $\alpha \neq -\beta$. It follows from (2.42) that

$$[h \circ x,y] = h \circ [x,y] + [h,y] \circ x = \beta(h)y \circ x + h \circ [x,y] \qquad (2.45)$$

for $h \in H$. By part (1), the left side and the second term of right side of (2.45) are multiples of $[x,y]$. Since $\beta \neq 0$, we see that $x \circ y$ is a multiple of $[x,y]$ and so is xy . This verifies part (2).

Assume $HH = 0$. For nonzero roots α, β, let $x \in A_\alpha$, $y \in A_\beta$. In light of (2.44), $h \circ x = 0$ for all $h \in H$, $x \in A_\alpha$. This together with (2.45) and Lemma 7.2.3 implies $\beta(h)\, x \circ y = 0$ for all $h \in H$. Since $\beta \neq 0$, we have $x \circ y = 0$ and hence

$$xy = - yx = \frac{1}{2}[x,y] .$$

Therefore, it follows from this and (2.44) that A is a Lie algebra.

For the proof of part (4), it suffices to show that $HH = 0$ under the hypotheses. The proof of this is based on the identity (2.42) and hence is the same as the first part of the proof of Theorem 7.2.5 (also see Lemmas 7.2.1, 7.2.2 and 7.2.4). //

An interesting example of the Lie algebra A^- satisfying the hypotheses in Theorem 7.4.1 is a generalized Witt algebra. Let G be any additive subgroup of the field F and let $G^{(m)} = G \times \cdots \times G$

(m copies) be the direct product. Assume that W is the vector space over F with basis $\{ e^i \}$ where $i = 1,2,\cdots, m$ and $\alpha \varepsilon G^{(m)}$, and define multiplication in W by

$$[e^i_\alpha, e^j_\beta] = \beta_i \, e^j_{\alpha+\beta} - \alpha_j \, e^i_{\alpha+\beta} \qquad (2.46)$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m)$ and $\beta = (\beta_1, \beta_2, \cdots, \beta_m)$.

Then W becomes a Lie algebra over F under the product (2.46) and is called a generalized Witt algebra (see Seligman's book and R. Ree, "On generalized Witt algebras", Trans. Amer. Math. Soc. 83(1956), 510-546). Note that the elements $\{ e^i_0 \}$ for $i = 1,2,\cdots, m$ span an abelian Cartan subalgebra H and the span of $e^1_\alpha, \cdots, e^m_\alpha$ determines a root space for H. Thus W is not classical for $m > 1$. Since

$$[e^i_0, e^j_\alpha] = \alpha_i \, e^j_\alpha$$

by (2.46), H acts diagonally on each root space.

Two special cases of the construction (2.46) are noteworthy. First when F has characteristic zero and $G = \{\cdots, -1, 0, 1, \cdots\}$, the generalized Witt algebra obtained by taking $m = 1$ has the specialized multiplication

$$[e_j, e_k] = (k - j) \, e_{j+k} \qquad (2.47)$$

where we have set $e^1_j = e_j$. The resulting algebra is called the Virasoro algebra which arises in relativistic string dual model theory (J. Scherk, "An introduction to the theory of dual models and strings", Rev. Mod. Phys. 47(1975), 123-164). For the second special case we take F to be a field of characteristic $p > 0$ and G to be the integers modulo p. Then the generalized Witt algebra obtained by (2.46) is the Jacobson - Witt algebra (N. Jacobson, "Classes of restricted Lie algebras of characteristic p, II," Duke Math. J. 10 (1943), 107-121).

Theorem 7.4.2. Let A be a flexible Lie-admissible algebra with product xy over a field F of characteristic $\neq 2$ such that A^- is isomorphic to a generalized Witt algebra. Then $xy = \frac{1}{2}[x,y]$ for all $x, y \in A$, so that A is a Lie algebra.

Proof. In light of Theorem 7.4.1. and the foregoing remarks, it suffices to show that $e_0^i e_0^j = 0$ for the basis $e_0^1, \cdots, e_0^m$ of the Cartan subalgebra H. We first prove that

$$e_0^i e_0^j = \delta_{ij} c_i e_0^i \tag{2.48}$$

where $c_i \in F$ and δ_{ij} is the Kronecker delta. Since ad x $(x \in A)$ is a derivation of A, we have the equation

$$[e_0^i e_\alpha^j, e_{-\alpha}^k] = e_0^i [e_\alpha^j, e_{-\alpha}^k] + [e_0^i, e_{-\alpha}^k] e_\alpha^j . \tag{2.49}$$

If we let $j = k$ and choose α so that $\alpha_i = 0$ but $\alpha_j \neq 0$, then the right side of (2.49) is simply $-2\alpha_j e_0^i e_0^j$. By Theorem 7.4.1 (1), the left side of (2.49) is a multiple of $[e_\alpha^j, e_{-\alpha}^j] = -2\alpha_j e_0^j$. Hence if $i \neq j$ then $e_0^i e_0^j$ is a multiple of e_0^j. But since e_0^i and e_0^j commute, $e_0^i e_0^j = e_0^j e_0^i$ and the latter is a multiple of e_0^i. Thus, for $i \neq j$, $e_0^i e_0^j = 0$. Assume that $i = k$ in (2.49) and that α is chosen so that $\alpha_i = 0$, but $\alpha_j \neq 0$. The second term on the right side of (2.49) vanishes and the first term is $-\alpha_j e_0^i e_0^i$ whereas the left side of (2.49) is a multiple of $[e_\alpha^j, e_\alpha^i] = -\alpha_j e_0^i$ by Theorem 7.4.1. Therefore, $e_0^i e_0^i$ is a multiple of e_0^i, giving the desired relation (2.48).

Consider the relation

$$[e_0^i e_\beta^i, e_\alpha^i] = [e_0^i, e_\alpha^i] e_\beta^i + e_0^i [e_\beta^i, e_\alpha^i] \quad . \tag{2.50}$$

If $\beta = 0$ then we have from (2.50) that

$$c_i \alpha_i e_\alpha^i = \alpha_i (e_\alpha^i e_0^i + e_0^i e_\alpha^i)$$

which, together with $[e_0^i, e_\alpha^i] = \alpha_i e_\alpha^i$, implies that $e_0^i e_\alpha^i = \frac{1}{2}(c_i + \alpha_i) e_\alpha^i$ and $e_\alpha^i e_0^i = \frac{1}{2}(c_i - \alpha_i) e_\alpha^i$ for all $\alpha \neq 0$. If $\beta = -\alpha$ in (2.50) then we see that

$$\frac{1}{2}(c_i - \alpha_i)(2\alpha_i) e_0^i = \alpha_i e_\alpha^i e_{-\alpha}^i + 2\alpha_i c_i e_0^i$$

or

$$e_\alpha^i e_{-\alpha}^i = -(c_i + \alpha_i) e_0^i \quad .$$

Using this, we compute

$$\begin{aligned} -(c_i + \alpha_i) \alpha_i e_\alpha^i &= -(c_i + \alpha_i) [e_0^i, e_\alpha^i] \\ &= [e_\alpha^i e_{-\alpha}^i, e_\alpha^i] \\ &= e_\alpha^i [e_{-\alpha}^i, e_\alpha^i] \\ &= e_\alpha^i (2\alpha_i e_0^i) = 2\alpha_i \frac{1}{2}(c_i - \alpha_i) e_\alpha^i \quad . \end{aligned}$$

This gives $c_i = 0$ for all i, and it follows from (2.48) that $HH = 0$. By Theorem 7.4.1 (3), $xy = \frac{1}{2}[x,y]$ for all $x, y \varepsilon A$ and hence A is a Lie algebra. //

The special case of Theorem 7.4.2 where A^- is a Jacobson - Witt algebra of characteristic $p > 0$ has been proved by a direct computation in Tomber's paper ("Jacobson - Witt algebras and Lie-admissible algebras, Hadronic J. 4(1981), 183-198). For the construction of new flexible Lie-admissible algebras which are not Lie algebras, Theorem 7.4.2 does not seem useful. In this regard we recall that a flexible Lie-admissible algebra A with A^- simple of type A_n $(n \geq 2)$ alone yields a non-Lie, flexible Lie-admissible algebra (Theorem 6.4.2).

Assume that A is a flexible Lie-admissible algebra with product xy and with a prescribed attached Lie algebra A^-. Then

$$xy = \frac{1}{2}[x,y] + x \circ y, \quad x, y \varepsilon A \tag{2.51}$$

where $x \circ y$ is a commutative product defined on the vector space A. Conversely, if there is defined a commutative product $x \circ y$ on A then the product xy given by (2.51) defines a Lie-admissible product on A with the prescribed structure on A^-. What has been discussed in Chapters 6 and 7 shows that the flexible law imposes constraints on the product $x \circ y$ and specifies relations with the Lie product $[x,y]$. In fact, $x \circ y$ is identically zero, except when A^- is a simple Lie algebra of type A_n $(n \geq 2)$ (of char 0). However, there is a well known commutative product that has arisen in an earlier work of L. M. Weiner ("Algebras based on linear functionals", Math. Mag. 28(1954), 9-12) and in Benkart's paper quoted in the beginning of this section. Let A be a vector space over F with a Lie algebra product $[x,y]$. Suppose that there is defined a linear functional τ on A. We define a multiplication xy on A by

$$xy = \frac{1}{2}[x,y] + \tau(x)\, y + \tau(y)\, x\,. \tag{2.252}$$

Then A is a Lie-admissible algebra such that A^- has the product $[x,y]$. Furthermore, A is power-associative, since it is readily seen that $x^m = 2^{m-1}\tau(x)^{m-1}\, x,\ x \in A$. But the product xy is not in general flexible, as we show in

Lemma 7.4.3. Let A be an algebra over F with an anticommutative product xy and let τ be a linear functional on A. Define a product $x * y$ on A by

$$x * y = xy + \tau(x)\, y + \tau(y)\, x\,.$$

Then the algebra $(A, *)$ is flexible if and only if $\tau([A,A]) = 0$.

Proof. Note that $x \circ y = \frac{1}{2}(x*y + y*x) = \tau(x)\, y + \tau(y)\, x$. It suffices to show that $[z, x \circ y] = x \circ [z,y] + [z,x] \circ y$ holds for all $x, y, z \in A$ if and only if $\tau([A,A]) = 0$. For $x, y, z \in A$, we have

$$[z, x \circ y] = \tau(x)[z,y] + \tau(y)[z,x]\,,$$

$$x \circ [z,y] + [z,x] \circ y = \tau([z,x])\, y + \tau(y)[z,x] + \tau(x)[z,y] + \tau([z,y])\, x$$

Thus, if $(A, *)$ is flexible then $\tau([z,x])\, y + \tau([z,y])\, x = 0$ for all $x, y, z \in A$ and this gives $\tau([A,A]) = 0$. //

Let V be the Virasoro algebra defined by (2.47) and let L be the subalgebra of V generated by the basis $\{e_i \mid i = 0,1,2,\cdots\}$. As an application of Theorem 7.4.1 and the remarks above, we have

Theorem 7.4.4. Let A be a flexible Lie-admissible algebra with product xy over a field of characteristic 0 such A^- is the subalgebra L with basis $\{e_i \mid i = 0,1,2,\cdots\}$ of the Virasoro algebra. Then there is a linear functional τ on A with $\tau(e_i) = 0$ for all $i > 0$ such that the multiplication xy in A is given by (2.52). Conversely, any such product determines a flexible Lie-admissible algebra on L .

Proof. Let A be a flexible Lie admissible algebra such that A^- is L and let $xy = \frac{1}{2}[x,y] + x \circ y$, as in (2.51). The element e_0 spans a 1-dimensional Cartan subalgebra of L , whereas each e_k $(k > 0)$ determines a 1-dimensional root space. Thus by Theorem 7.4.1 $e_j\, e_k$ is a multiple of $[e_j,e_k] = (k-j)\, e_{j+k}$ for all j, k , and so is $e_j \circ e_k$. Hence we can write $e_j \circ e_k = \gamma_{jk}\, e_{j+k}$, where $\gamma_{jk} = \gamma_{kj}$. Since A is flexible, we can use (2.42) to compute

$$\begin{aligned}\gamma_{0k}\,(k-j)\, e_{j+k} &= [e_j, e_0 \circ e_k] \\ &= [e_j,e_0] \circ e_k + e_0 \circ [e_j,e_k] \\ &= -j\, \gamma_{jk}\, e_{j+k} + (k-j)\, \gamma_{0,j+k}\, e_{j+k} . \end{aligned} \tag{2.53}$$

If $k = 0$ then equation (2.53) implies $\gamma_{0j} = \frac{1}{2}\gamma_{00}$ for all $j \neq 0$, and, substituting this into (2.53), we have $\gamma_{jk} = 0$ for all $j, k \neq 0$. We define the linear functional τ on A by $\tau(e_0) = \frac{1}{2}\gamma_{00}$ and $\tau(e_i) = 0$ for all $i > 0$. It is clear that $x \circ y = \tau(x)\, y + \tau(y)\, x$ for all $x, y \in A$ and $\tau([A,A]) = 0$. In light of Lemma 7.4.3, this proves the Theorem. //

7.5. Lie-admissible mutations of an associative algebra

We close this chapter with a mutation type of Lie-admissible algebras constructed from an associative algebra. Let A be an associative algebra over a field F of char $\neq 2$ with product xy. Let r, s be fixed elements in A and define a product $x * y$ on the same vector space as A by

$$x * y = xry - ysx \quad . \tag{2.54}$$

Denote the resulting algebra by $A(r,s)$, called the <u>(r,s) - mutation</u> of A. The (r,s) - mutation was originally introducted by Santilli for a formulation of a Lie-admissible time evolution law which generalizes the conventional Heisenberg equation. The structure of $A(r,s)$ has been studied in a number of physical and mathematical publications. In this section, we discuss some basic structure of this algebra. The material is largely drawn from the work:

(1) R. H. Oehmke, "Some elementary structure theorems for a class of Lie-admissible algebras", Hadronic J. 3(1979), 293-319.

(2) J. M. Osborn, "The Lie-admissible mutation $A(r,s)$ of an associative algebra A", Hadronic J. 5(1982), 904-930.

(3) H. C. Myung, "The exponentation and deformations of Lie-admissible algebras", Hadronic J. 5(1982), 771-903.

Related to the (r,s) - mutation is the homotope and isotope of an associative algebra. Let a be a fixed element of an associative algebra A . The algebra, denoted by $A^{(a)}$, with multiplication $x \underset{a}{\cdot} y$ defined by

$$x \underset{a}{\cdot} y = xay$$

is called the <u>a-homotope</u> of A . If a is invertible in A , then $A^{(a)}$ is called the <u>a-isotope</u> of A . Clearly, $A^{(a)}$ is associative.

Denote by $[x,y]^* = x * y - y * x$ the Lie product in $A(r,s)$. Since

$$[x,y]^* = xry - ysx - yrx + xsy = x(r + s) -y(r + s)x \quad ,$$

we see that $A(r,s)^-$ is isomorphic to $A^{(r+s)^-}$. Since the $(r + s)$ - homotope $A^{(r+s)}$ is associative, this implies that $A(r,s)$ is Lie-admissible. Similarly, we compute the Jordan product in

$A(r,s)$ as

$$\frac{1}{2}(x * y + y * x) = \frac{1}{2}(xry - ysx + yrx - xsy)$$

$$= \frac{1}{2}[x(r - s)y + y(r - s)x] .$$

Thus $A(r,s)^+ \simeq A^{(r-s)+}$ and hence $A(r,s)$ is Jordan-admissible also (see p. 12). The (r,s) - mutation is not in general flexible. We can show that flexibility of $A(r,s)$ is equivalent to many of the well known identities when r and s are invertible in A (in the original introduction of $A(r,s)$ in physics, r and s are assumed to be invertible operators in a Hilbert space). For this, we need some definitions.

Assume that A is an associative algebra with unit element 1 over a field F of char $\neq 2,3$. A nonassociative algebra is called <u>third power-associative</u> if it satisfies the identity

$$(x * x) * x = x * (x * x) \quad , \tag{2.55}$$

which is satisfied by the flexible identity $(x * y) * x = x * (y * x)$. Also, any power-associative algebra (see p. 47) clearly satisfies the third power-associative identity (2.55). With the exception of Lie-admissibility and Jordan-admissibility, virtually all the identities that are considered in nonassociative algebras imply (2.55). A non-associative algebra B is called <u>generalized quasi-associative</u> if, up to isomorphism, it arises from an associative algebra A under the new product

$$x * y = \alpha xy + \beta yx$$

for some fixed α,β in the center $Z(A)$ of A, where $Z(A)$ is defined as the set $\{x \mid xy = yx \text{ for all } y \in A\}$. Thus if B is generalized quasi-associative, then $B \simeq A(\alpha,\beta)$. If α and β are just scalars then B is called quasi-associative. Since it can be easily seen that if $\alpha \neq \beta$ then the mapping $x \to (\alpha-\beta)^{-1} x$ is an isomorphism of $A(r,1-r)$ to $A(\alpha,\beta)$ where $r = \alpha(\alpha-\beta)^{-1}$, the present definition agrees with one given in Section 6.6 (see p. 261). It is easy to verify that every generalized quasi-associative algebra is both flexible and power-associative.

<u>Theorem 7.5.1</u>. Let r,s be invertible elements in A. The following properties for the algebra $A(r,s)$ are equivalent:

(i) $A(r,s)$ is third power-associative,

(ii) $A(r,s)$ is flexible,

(iii) $A(r,s)$ is power-associative,

(iv) $A(r,s)$ is generalized quasi-associative,

(v) $s = \alpha r$ for some invertible α in the center of A,

(vi) $A(r,s) \simeq A(1,\beta)$ for some invertible β in the center of A.

<u>Proof</u>. We have already noted that the implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) hold. Also, the implication (vi) $\Rightarrow$ (iv) is obvious. Assume (v) holds. Then $x * y = xry - \alpha yrx$, and since r is invertible, the mapping $x \to xr$ is an isomorphism of the isotope $A^{(r)}$ to A which in turn induces the isomorphism $A(r,s) \simeq A^{(r)}(1,-\alpha) \simeq A(1,\alpha)$. Setting

$\beta = -\alpha$, we have established (vi) . It remains to verify that (i) implies (v) .

If (i) holds then, using (2.54), the relation (2.55) gives

$$xrxrx - xsxrx - xsxrx + xsxsx = xrxrx - xrxsx - xrxsx + xsxsx \quad .$$

This reduces to

$$xsxrx = xrxsx$$

and replacing x by $x + 1$, the terms linear in x are

$$xsr + sxr + srx = xrs + rxs + rsx \quad . \tag{2.56}$$

The special case $x = 1$ in (2.56) gives $sr = rs$ or $r^{-1}s = sr^{-1}$. Using this, (2.56) reduces to $sxr = rxs$ for all $x \in A$. Since r and s are invertible, from this we have $r^{-1}sx = xsr^{-1} = xr^{-1}s$ for all $x \in A$. Thus $r^{-1}s$ is invertible and is in the center of A . Letting $\alpha = r^{-1}s = sr^{-1}$, we have established (v) . //

In most cases of interest, the algebra A has the property that the center of A is scalar multiples of the unit element. This is the case when A is central simple over F . In this case, a generalized quasi-associative algebra derived from A is quasi-associative. It can be shown that if A is simple then the conditions (i) - (vi) in Theorem 7.5.1 are equivalent, under the assumption that $r + s$ or $r - s$ is invertible in A .

<u>Theorem 7.5.2</u>. Let $r - s$ or $r + s$ be invertible in A .

Assume that A is simple. Then the conditions (i) - (vi) in Theorem 7.5.1 are equivalent. In this case r and s are invertible also.

Proof. Assume that $r + s$ is invertible. Let $p = (r + s)^{-1}r$. Then the mapping $f : x \to x(r + s)^{-1}$ is an isomorphism of $A(p,1 - p)$ to $A(r,s)$. In fact,

$$\begin{aligned} f(x) * f(y) &= f(x)rf(y) - f(y)sf(x) \\ &= x(r + s)^{-1}ry(r + s)^{-1} - y(r + s)^{-1}sx(r + s)^{-1} \\ &= [x(r + s)^{-1}ry - y(r + s)^{-1}sx] (r + s)^{-1} \\ &= (xpy - y(1 - p)x) (r + s)^{-1} \\ &= f(x * y) , \end{aligned}$$

since $1 - p = (r + s)^{-1}s$. Since each of the conditions (ii) - (vi) implies third power-associativity, in light of Theorem 7.5.1 it suffices to show that the condition (i) implies that r and s are invertible. Thus, assume that $A(r,s) \simeq A(p,1 - p)$ is third power-associative. The identity (2.55) in $A(p,1 - p)$ implies $xpx(1 - p)x - x(1 - p)xpx = xpx^2 - x^2px = 0$. Replacing x by $1 + \lambda x(\lambda \in F)$ in this, we get $(1 + \lambda x)^2p(1 + \lambda x) = (1 + \lambda x)p(1 + \lambda x)^2$ and

$$0 = \lambda(xp - px) + \lambda^2(x^2p - px^2) + \lambda^3(x^2px - xpx^2) .$$

for all $\lambda \in F$. This gives $xp = px$ for all $x \in A$ and hence

p is in the center $Z(A)$ of A . Since A is simple, $Z(A)$ is a field and so $p = \alpha$ for some invertible α in $Z(A)$. Thus $(r + s)^{-1}r = \alpha$ and $s = (1 - \alpha)\alpha^{-1}r$, hence r and s are invertible. In light of Theorem 7.5.1, this establishes the equivalences.

If $r - s$ is invertible then we set $q = (r - s)^{-1}s$ and, as above, the mapping $x \to x(r - s)^{-1}$ is an isomorphism of $A(q,q - 1)$ to $A(r,s)$. The identity (2.55) then implies that q is in the center of A . This establishes the same conclusion. //

An algebra is called prime if the product of any two nonzero ideals is nonzero. Thus any simple algebra is prime. We note that if the (r,s)-mutation $A(r,s)$ is simple then so is A , since any ideal of A is an ideal of $A(r,s)$. Here, we investigate the converse of this as well as the relation between the primeness of A and $A(r,s)$. The following lemma is useful.

Lemma 7.5.3. Let A be an associative algebra with unit element over F of char $\neq 2,3$. Let r and s be invertible in A .

(1) The subspace C of A spanned by all elements of the form $xry + ysx$ for $x, y \in A$ contains the ideal $A(r + s)A$ of A .

(2) If B is an ideal of $A(r,s)$, then B contains the subspaces $B(r - s)C$, $C(r - s)B$, and $C(r - s)B(r - s)C$.

(3) If B is an ideal of $A(r,s)$, then $A(r + s)A(r - s)B(r - s)A(r + s)A$ is an ideal of A which is contained in B .

Proof. (1) The subspace C contains the elements

$$xr(zsy) + (zsy)sx \quad , \quad (ysx)rz + zs(ysx)$$

for $x, y, z \in A$, and hence their difference which is $xrzsy - ysxrz$. Setting $z = r^{-1}$ gives $xsy - ysx \in C$, and adding this to $xry + ysx \in C$ yields $xry + xsy = x(r + s)y \in C$. Hence $A(r + s)A \subseteq C$.

(2) Assume that B is an ideal of $A(r,s)$ and $u \in B$ and $x, z \in A$. Then

$$(x * u) * z - x * (u * z) = urzsx - usxrz + xrzsu - zsxru \in B ,$$

$$(zsx)ru - us(zsx) \in B ,$$

$$ur(xrz) - (xrz)su \in B .$$

Adding these together, we have

$$urzsx - usxrz + urxrz - uszsx = u(r - s)(xrz + zsx) \in B ,$$

which implies that $B(r - s)C \subseteq B$. For $w \in C$, we get $w(r - s)u = u * w + w * u - u(r - s)w \in B$ and hence

$$C(r - s)B(r - s)C \subseteq \{C(r - s)B\}(r - s)C \subseteq B(r - s)C \subseteq B .$$

This proves part (2).

(3) Clearly, $A(r + s)A(r - s)B(r - s)A(r + s)A$ is an ideal of A and is contained in B since

$$A(r + s)A(r - s)B(r - s)A(r + s)A \subseteq C(r - s)B(r - s)C \subseteq B ,$$

using parts (1) and (2). //

Lemma 7.5.4. Let r,s be invertible elements in A such that $r \neq s$. If A is prime then every nonzero ideal of $A(r,s)$ contains a nonzero ideal of A .

Proof. Let B be a nonzero ideal of $A(r,s)$. First, assume $r + s = 0$. Then $x * y = xry + yrx$ and $A(r,s) \simeq (A^{(r)})^+ \simeq A^+$, since the mapping $x \to rx$ is an isomorphism of $A^{(r)}$ to A . Thus it suffices to show in this case that every nonzero ideal B of A^+ contains a nonzero ideal of A . If $b \in B$ and $x \in A$ then

$$b(bx - xb) + (bx - xb)b = b^2x - xb^2 \in B ,$$

and $b^2 \in B$, $b^2x + xb^2 \in B$, showing that $b^2x \in B$ and $xb^2 \in B$. Hence Ab^2A is an ideal of A which is contained in B , since $xb^2y = [(xb^2)y + y(xb^2)] - (yx)b^2 \in B$ for $x, y \in A$. If $Ab^2A = 0$ then $b^2 = 0$, and $b(bx + xb) + (bx + xb)b = 2bxb \in B$, so $bAb = 0$. Therefore, AbA is an ideal of A that squares to zero. Since A is prime, $bAb = 0$, giving $b = 0$. Thus the proof is complete when $r + s = 0$.

Assume $r + s \neq 0$. Thus $A(r + s)A$ is a nonzero ideal of A . By Lemma 5.7.3(3), the ideal $A(r + s)A(r - s)B(r - s)A(r + s)A$ is an ideal of A contained in B . Thus if this ideal is nonzero, we are done. Suppose that $A(r + s)A(r - s)B(r - s)A(r + s)A = 0$. From this, we see that $A(r + s)A$ and $A(r - s)B(r - s)A(r + s)A$ are ideals of A whose product is zero. Since $A(r + s)A \neq 0$, we have $A(r - s)B(r - s)A(r + s)A = 0$. But then $A(r + s)A$ and $A(r - s)B(r - s)A$ are two ideals of A whose product is zero, hence $A(r - s)B(r - s)A = 0$ or $(r - s)B(r - s) = 0$. Let $b \varepsilon B$ and $y \varepsilon A(r + s)A$. By Lemma 7.5.3 (1) and (2) we obtain $b(r - s)y \varepsilon B$ and this gives

$$(xr^{-1})r(b(r - s)y) - (b(r - s)y)s(xr^{-1}) \varepsilon B$$

for all $x \varepsilon A$. But the second term is in $B(r - s)A(r + s)A \subseteq B$, since $y \varepsilon A(r + s)A$ (see Lemma 7.5.3 (1)). Thus $xb(r - s)y \varepsilon B$ or $AB(r - s)A(r + s)A = 0$. Since A is prime, from this we have $AB(r - s)A = 0$, yielding $B(r - s) = 0$. Thus for $b \varepsilon B$ and $x \varepsilon A$,

$$br(r^{-1}x) - (r^{-1}x)sb = bx - r^{-1}xsb \varepsilon B ,$$

implying that

$$0 = (hx - r^{-1}xsb)(r - s) = bx(r - s)$$

or $BA(r - s) = 0$. Hence the product of ideals ABA and $A(r - s)A$ is zero. Since $B \neq 0$ and $r - s \neq 0$, we have a contradiction. Therefore, $A(r + s)A(r - s)B(r - s)A(r + s)A$ is a nonzero ideal of A contained in B . //

Theorem 7.5.5. Let r and s be invertible elements of A such that $r \neq s$. Then,

(1) $A(r,s)$ is prime if and only if A is prime;

(2) $A(r,s)$ is simple if and only if A is simple;

(3) if $A(r + s)A = A$ and if $r - s$ is invertible, then a subspace of A is an ideal of $A(r,s)$ if and only if it is

an ideal of A .

Proof. (1) It suffices to show that if A is prime then $A(r,s)$ is prime. Suppose that A is prime. Let B_1, B_2 be nonzero ideals of $A(r,s)$. By Lemma 7.5.4, we can choose nonzero ideals D_1, D_2 of A which are contained respectively in B_1, B_2 . Hence $D = D_1 D_2$ is a nonzero ideal of A contained in both B_1 and B_2 . Suppose that $B_1 * B_2 = 0$. Then, for $u, v \in D$, we have $u * v = urv - vsu = 0$, and

$$urvr = (vsu)r = ((vsr^{-1})ru)r = (us(vsr^{-1}))r = usvs ,$$

since $vsr^{-1} \in D$. Hence $D(rvr - svs) = 0$ for $v \in D$, showing that D annihilates the ideal generated by $rvr - svs$. Since A is prime, it follows from this that $rvr = svs$ or $vrs^{-1} = r^{-1}sv$ for all $v \in D$. If $x \in A$, then $xr^{-1}sv = xvrs^{-1} = r^{-1}sxv$ for all $v \in D$, or $(xr^{-1}s - r^{-1}sx)D = 0$. Since A is prime, this gives $xr^{-1}s = r^{-1}sx$ for all $x \in A$ and hence $\alpha = r^{-1}s$ is in the center of A . Thus, $rvr = svs = \alpha^2 rvr$, implying $0 = r^{-1}(1 - \alpha^2)(rvr)r^{-1} = (1 - \alpha^2)v$ or $(1 - \alpha^2)D = 0$. Hence $\alpha^2 = 1$ and since $r \neq s$ by hypothesis, we get $\alpha = -1$ or $r + s = 0$. As in the proof of Lemma 7.5.4 we are in the situation where $A(r,s) \simeq A^+$. Thus we can think of B_1 and B_2 as ideals of A^+ . But then, for $u, v, w \in D$ we have $uv + vu = 0$ and $uvw = - vuw = vwu = - uvw$, giving $D^3 = 0$. This is impossible since A is prime. We have proved that for any nonzero ideals B_1, B_2 of $A(r,s)$, the product $B_1 * B_2$ is nonzero. Hence $A(r,s)$ is prime and this proves part (1).

(2) Suppose that A is simple. Then A is prime and so, by Lemma 7.5.4, any nonzero ideal B of $A(r,s)$ contains a nonzero ideal D of A. Since A is simple, $D = A$ and hence $B = A$. Thus $A(r,s)$ is simple.

(3) Assume that B is an ideal of $A(r,s)$ and that $A(r + s)A = A$. Let C be the subspace of A spanned by $xry + ysx$ for $x, y \in A$. Then by Lemma 7.5.3 (1) $C = A$ and $B(r - s)A \subseteq B$ by Lemma 7.5.3 (2). Since $r - s$ is invertible, $(r - s)A = A$, and hence $BA \subseteq B$. Similarly, $AB \subseteq B$ and B is an ideal of A. //

LIST OF SYMBOLS

Symbol	Meaning	Page
Φ^+	Set of positive roots	113
Φ^-	Set of negative roots	113
Π	Simple system of roots	114
$\|\|\alpha\|\|$	Length of α	116
A_{ji}	$\langle\alpha_i,\alpha_j\rangle$; Cartan integer	115
(A_{ji})	Cartan matrix of Π	115
$\|\beta\|$	Level of β	115
$s\ell(\ell+1,F)$	Special linear algebra	138
$O(2\ell+1,F)$	Orthogonal algebra	139
$sp(2\ell,F)$	Symplectic algebra	142
M_3^8	Exceptional Jordan algebra	149
$T(V)$	Tensor algebra on V	160
$S(V)$	Symmetric algebra on V	160
$U(L)$	Universal enveloping algebra	162
PBW	Poincaré-Birkhoff-Witt	169
P_α	Reflecting hyperplane	176
σ_α	Reflection by α	176
WG	Weyl group	177
$\Phi^+(\gamma)$	$\{\alpha \in \Phi \mid (\gamma,\alpha) > 0\}$	179
$C(\gamma)$	Weyl chamber	179
$C(\Pi)$	Fundamental Weyl chamber	181
$\ell(\sigma)$	Length of σ	186
$n(\sigma)$	Number of $\sigma \in \Phi^+$ with $\sigma(\alpha) < 0$	186
$\overline{C(\Pi)}$	Fundamental domain	189

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SELECTED COMMENTS

A comment by MARVIN L. TOMBER, Professor of Mathematics,
Michigan State University, East Lansing, Michigan 48824
"This book presents a very readable account of the structure, classification, and representation theories of Lie algebras of characteristic zero. This information is used to give a good treatment of the best understood class of flexible Lie–admissible algebras."

A comment by RUGGERO M. SANTILLI, President,
The Institute for Basic Research, Cambridge, Massachusetts 02138
"The material presented in this monograph originated from a graduate course delivered by Professor Myung at the Seoul National University in 1979–1980. The presentation begins with certain foundational aspects of Lie algebras, and then passes to their generalization beyond the graded–supersymmetric level, and known under the name of flexible Lie–admissible algebras. Particular attention is given to the classification of flexible Lie–admissible algebras with attached Lie algebras of reductive, classical, or generalized Witt type. Much of the material is motivated by recent physical applications of the flexible Lie–admissible algebras for the representation of hadrons as extended particles, and the possible interpretation of recent experimental information suggesting a time–asymmetry in nuclear reactions, small deformations of the charge distribution of neutrons under sufficiently intense fields, and other data. The reading of this monograph, which is authored by one of the mathematical leaders in the field, is invaluable for all mathematicians and physicists interested in fundamental advances."