Ratio Mathematica Volume 52, 2024

Lie-Santilli admissible hyper-structures, from numbers to H_v-numbers

Kostaq Hila*
Ruggero M. Santilli†
Thomas Vougiouklis‡

Abstract

The class of H_v-structures defined on a set is very big and admits a partial order. For this reason, it has a numerous of applications in mathematics and other sciences as physics, biology, linguistics, to mention but a few. Here, we focus on the Lie-Santilli's admissible case, where the hyper-numbers, called H_v-numbers, are used. In order to verify all needed axioms for Lie-Santilli's admissibility, as the irreversibility and uniqueness of living organisms and time, on the one side and small results on the other side, we use the very-thin H_v-fields. Therefore, we take rings and we enlarge only one result by adding only one element in order to obtain an H_v-field. This means that, we use only the associativity on the product and we transfer this to the weak-associativity on the hyper-product. Thus, from a semigroup on the product, we construct an H_v-group on the hyper-product.

Keywords: Lie-Santilli iso-theory, weak axioms, H_v-fields.

2010 AMS subject classification: 20N20, 16Y99§.

* Polytechnic University of Tirana, Tirana P.O. Box 1001, Albania, kostaq_hila@yahoo.com
† Institute for Basic Research, P. O. Box 1577 Palm Harbor, FL 34682, USA, ibr@gte.net
‡ Democritus University of Thrace, Xanthi, Greece, tvougiou@eled.duth.gr

§ Received on April 1, 2024. Accepted on May 30, 2024. Published on June 30, 2024. DOI:10.23755/rm.v52i0.1610. ISSN: 1592-7415. eISSN: 2282-8214.

1. Weak Hyper-structures

The largest class of hyper-structures is the class of *H_v-structures* which was introduced in 1990 [16]. These satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Definitions 1.1 *Hyper-structure* is any set H equipped with, at least one, *hyper-operation:*

$$\cdot: H\times H \to P(H)-\{\varnothing\}.$$

A hyper-operation satisfy the *weak associativity*, if $(xy)z\cap x(yz) \neq \varnothing$, $\forall x,y,z\in H$ and satisfy the *weak commutativity*, if $xy\cap yx \neq \varnothing$, $\forall x,y\in H$.

A hyper-structure $(H,\cdot)$ is an *H_v-semigroup* if it is weak associative and it is an *H_v-group* if it is reproductive H_v-semigroup: $xH = Hx = H$, $\forall x\in H$.

In the classical theory, the quotient of a group with respect to an invariant subgroup is a group. In hyper-structures, the quotient of a group with respect to any subgroup is a hypergroup. Finally, Vougiouklis introduced and proved in 1990, that the quotient of a group with respect to any partition is an H_v-group.

Definitions 1.2 The $(R,+,\cdot)$ is an *H_v-ring* if both (+) and (·) are weak associative, the reproduction axiom is valid for (+) and (·) is *weak distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing, \quad (x+y)z\cap(xz+yz)\neq\varnothing, \quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ be H_v-ring, $(M,+)$ be a weak commuttive H_v-group and there exists an external hyper-operation

$$\cdot : R\times M \to P(M): (a,x) \to ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing, \quad (a+b)x\cap(ax+bx)\neq\varnothing, \quad (ab)x\cap a(bx)\neq\varnothing,$$

then M is an *H_v-module* over F. In the case of an H_v-field, which is defined later, F instead of an H_v-ring R, then the *H_v-vector space* is defined.

For more definitions and applications on H_v-structures one can see the books and papers as the: [1], [2], [5], [7], [16], [17], [18], [28], [29].

Definition 1.3 The *fundamental relations* β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [5], [16], [19], [23], [27].

The way to find the fundamental classes is given by analogous to the following theorems:

Let $(H,\cdot)$ be H_v-group, U the set of all finite products of elements of H. Define the relation β in H by setting $x\beta y$ if, and only if, $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.

Let $(R,+,\cdot)$ be H_v-ring, U the set of all finite polynomials of elements of R. Define the relation γ in R by: $x\gamma y$ if, and only if, $\{x,y\}\subset u$ where $u\in U$. Then γ^* is the transitive closure of γ.

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions:

Definition 1.4 An H_v-ring $(R,+,\cdot)$ is called *H_v-field* if R/γ^* is a field. The elements of an H_v-field are called *hyper-numbers* or *H_v-numbers.*

Definition 1.5 Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, if, and only if, there exists an automorphism

$$f\in Aut(H,*) \quad \text{such that} \quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,*)$ contains a group, then is called *H_b-group*.

The Little Theorem. *Greater hyper-operations than the ones which are weak associative or weak commutative are, as well, weak associative or weak commutative, respectively.*

The Little Theorem leads to a partial order, *posets*, on H_v-structures and introduce several new classes of hyper-structures [16], [27].

Definition 1.6 An H_v-structure is called *very-thin* if all hyper-operations are operations except one, which has all hyper-products singletons except only one, which is a subset of cardinality more than one.

Definition 1.7 Let $(G,\cdot)$ be groupoid, then for all $P\subset G$, $P\neq\emptyset$ we define the following *P-hyper-operation*:

$$\underline{P}:\ x\underline{P}y=(xP)y\cup x(Py),\quad \forall x,y\in G.$$

If $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$, so $(G,\underline{P})$ is a semi-hypergroup.

A generalization of P-hyper-operations is the following [4], [5], [24]:

Let $(G,\cdot)$ be abelian group and P, subset of G. We define the hyper-operation $\times_P$, by

$$\begin{cases} x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\}, & \text{if } x\neq e \text{ and } y\neq e \\ x\cdot y, & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this *P_e-hyper-operation* and the hyper-structure $(G,\times_P)$ is an abelian H_v-group.

The general definition of an H_v-Lie algebra was given as follows [5], [15]:

Definition 1.8 Let $(L,+)$ be an H_v-vector space over $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$ the canonical map, $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is zero of F/γ^*. Let ω_L the core of $\varphi':L\to L/\varepsilon^*$ and denote 0 the zero of L/ε^*. Consider the *bracket hyper-operation*

$$[\ ,\ ]:\ L\times L\to P(L):\ (x,y)\to[x,y]$$

then L is an *H_v-Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hyperoperation is bilinear:

$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \emptyset$

$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2]) \neq \emptyset$, $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\forall\lambda_1,\lambda_2\in F$

(L2) $[x,x]\cap\omega_L \neq \emptyset$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L \neq \emptyset$, $\forall x,y\in L$.

2. H_v-fields and hyper-matrix representations

The enlargements and the reductions of hyper-structures are examined in the sense that extra elements appear in results or we take out elements. In both directions most interesting cases in the research are those H_v-structures when the fundamental structures remain the same [16], [19], [23], [27].

Theorem 2.1 Let $(G,\cdot)$ be semigroup and $v\notin G$ be an extra element appearing in a product ab, where $a,b\in G$, thus the result becomes a hyper-product $a\otimes b=\{ab,v\}$. Then the minimal hyper-opration $(\otimes)$ extended in $G'=G\cup\{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G',\otimes)$ is a minimal H_V-semigroup which has fundamental structure isomorphic to $(G,\cdot)$, is defined by

$$a\otimes b=\{ab,v\},\ \ x\otimes y=xy,\ \ \forall(x,y)\in G^2-\{(a,b)\}$$

$$v\otimes v=abab,\ \ x\otimes v=xab \ \text{ and } \ v\otimes x=abx,\ \ \forall x\in G.$$

$(G',\otimes)$ is very-thin H_v-semigroup. If $(G,\cdot)$ is commutative then we obtain that $(G',\otimes)$ is strongly commutative.

Now we present some 'small' H_v-fields obtained mainly by finite ordinary rings, by enlarging only one result by adding only one element from the underline ring. These H_v-fields are useful since the results are as small as possible. Moreover, they obey in axioms needed in applications in hadronic mechanics [15], [23], [26].

Constructions 2.2 On the rings $(Z_4,+,\cdot)$ and $(Z_6,+,\cdot)$ we will define all the multiplicative H_v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very-thin minimal, (b) weak commutative, (c) have 0 and 1, scalars.

(I) On $(Z_4,+,\cdot)$ we have the isomorphic cases: $2\otimes3=\{0,2\}$ or $3\otimes2=\{0,2\}$. The fundamental classes are $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(Z_4,+,\otimes)/\gamma^*\cong(Z_2,+,\cdot)$. In this H_v-group there is only one unit and every element have a unique double inverse.

(II) On $(Z_6,+,\cdot)$, we have the only one hyperproduct,

(i) $2\otimes3=\{0,3\}$, $2\otimes4=\{2,5\}$, $3\otimes4=\{0,3\}$, $3\otimes5=\{0,3\}$, $4\otimes5=\{2,5\}$. Fundamental classes: $[0]=\{0,3\}$, $[1]=\{1,4\}$, $[2]=\{2,5\}$ and we have $(Z_6,+,\otimes)/\gamma^*\cong(Z_3,+,\cdot)$.

(ii) $2\otimes3=\{0,2\}$ or $2\otimes3=\{0,4\}$, $2\otimes4=\{0,2\}$ or $\{2,4\}$, $2\otimes5=\{0,4\}$ or $2\otimes5=\{2,4\}$, $3\otimes4=\{0,2\}$ or $\{0,4\}$, $3\otimes5=\{3,5\}$, $4\otimes5=\{0,2\}$ or $\{2,4\}$. Fundamental classes: $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have $(Z_6,+,\otimes)/\gamma^*\cong(Z_2,+,\cdot)$.

Representations of H_v-groups, can be considered either by H_v-matrices [5], [14], [16], [17], [18] or by generalized permutations [16]. Here we focus on representations by H_v-matrices because they are used in Santilli's admissibility.

Definition 2.3 *H_v-matrix* is called a matrix with entries from an H_v-ring or H_v-field. The hyper-product of H_v-matrices $A=(a_{ij})$ and $B=(b_{ij})$, of type m×n and n×r, respectively, is a set of m×r H_v-matrices, defined in a usual manner:

$$A\cdot B=(a_{ij})\cdot(b_{ij})=\{C=(c_{ij})\mid c_{ij}\in\oplus\Sigma a_{ik}\cdot b_{kj}\},$$

where ($\oplus$) is the *n-ary circle hyper-operation*, which means the union of all possible patterns of parentheses put for elements, on the hyper-addition.

The representation problem by H_v-matrices is the following:

Definition 2.4 Let $(H,\cdot)$ be an H_v-group, $(R,+,\cdot)$ be an H_v-ring and take a set of H_v-matrices $M_R=\{(a_{ij})\mid a_{ij}\in R\}$, then any

$$T:H\to M_R:\ h\to T(h)\ \text{ with }\ T(h_1h_2)\cap T(h_1)T(h_2)\neq\varnothing,\ \forall h_1,h_2\in H,$$

is called *H_v-matrix representation.* If $T(h_1h_2)\subset T(h_1)T(h_2)$, then T is called *inclusion representation,* if $T(h_1h_2)=T(h_1)T(h_2)$, then T is called *good representation,* in this case an induced representation T* for the H_v-group algebra, is obtained. If T is one to one and good, then it is a *faithful representation*.

Theorem 2.5 A necessary condition in order to have an inclusion representation T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:

For all $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a))\subset\{A=(a'_{ij})\mid a'_{ij}\in\gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}.$$

Thus, every inclusion representation $T:H\to M_R$: $a\mapsto T(a)=(a_{ij})$ induces a homomorphic representation T* of H/β^* over R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

Several classes of H_v-structures can give special representations. Some of those classes are [16], [20], [26]:

Definition 2.6 Let $M=M_{m\times n}$, the set of m×n matrices on R and take $P=\{P_i:i\in I\}\subseteq M$. We define, a kind of, P-hyper-operation $\underline{P}$ on M as follows

$$\underline{P}:\ M\times M\to P(M):\ (A,B)\ A\underline{P}B=\{AP^t_iB:\ i\in I\}\subseteq M$$

where P^t denotes the transpose of P. $\underline{P}$ is bilinear Rees' like operation where, instead of one sandwich matrix, a set is used. $\underline{P}$ is strong associative and the inclusion distributive to addition is valid:

$$A\underline{P}(B+C) \subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M$$

So (M,+,$\underline{P}$) defines a multiplicative hyperring on non-square matrices.

Let $M=M_{m\times n}$ be module of m×n matrices on R and take the sets

$$S=\{s_k:k\in K\}\subseteq R,\quad Q=\{Q_i:j\in J\}\subseteq M,\quad P=\{P_i:i\in I\}\subseteq M.$$

Define three hyperoperations as follows

$$\underline{S}: R\times M\to P(M): (r,A)\to r\underline{S}A= \{(rs_k)A:k\in K\}\subseteq M$$

$$\underline{Q}_+: M\times M\to P(M): (A,B)\to A\underline{Q}_+B= \{A+Q_j+B: j\in J\}\subseteq M$$

$$\underline{P}: M\times M\to P(M): (A,B)\to A\underline{P}B= \{AP^t_iB:i\in I\}\subseteq M$$

Then, (M,$\underline{S}$,$\underline{Q}_+$,$\underline{P}$) is a hyperalgebra on R the *general matrix P-hyperalgebra*.

3. The Lie-Santilli H_v-admissibility

Hyper-structures have applications in other sciences, which range from hadronic physics, leptons, Santilli's iso-theory, bio-mathematics to mention but a few. The hyper-structure theory is related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too [1], [2], [3], [5], [6], [7], [10], [11], [12], [13], [16], [28], [29].

In [21], with '*The Santilli's theory 'invasion' in hyperstructures',* there is a description how Santilli's theories effect in hyper-structures and how new theories in Mathematics appeared by Santilli's pioneer research. In 1996 Santilli & Vougiouklis [14], [21], point out that in physics the interesting hyper-structures are the e-hyperstructures. These hyper-structures contain a unique left ant right scalar unit, which is an important tool in Lie-Santilli theory. See books and related papers for definitions and results related topics: [3], [4], [5], [6], [8], [9], [10], [11], [12], [13], [14], [20], [21], [22], [24], [25].

Definition 3.1 A hyper-structure (F,+,·), where (+) is operation and (·) hyper-operation, is called *e-hyperfield* if the following are valid: (F,+) is abelian group with additive unit 0, (·) is weak associative, (·) is weak distributive with respect to (+), 0 is absorbing: $0\cdot x=x\cdot 0=0,\ \forall x\in F$, there exist a product scalar unit 1: $1\cdot x=x\cdot 1=x,\ \forall x\in F$, and $\forall x\in F$ there is a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If $1=x\cdot x^{-1}=x^{-1}\cdot x$, then we say that we have a *strong e-hyperfield.*

Definition 3.2 *Main e-Construction.* Given a group (G,·), where e the unit, we define in G, a huge number of hyper-operations (⊗) by

$$x\otimes y=\{xy,g_1,g_2,\ldots\},\ \forall x,y\in G-\{e\},\ \text{where } g_1,g_2,\ldots\in G-\{e\}.$$

(G,⊗) is e-hypergroup, in fact is an H_b-group because contains (G,·). Moreover, if ∀x,y such that xy=e, so x⊗y=xy, then (G,⊗) becomes strong e-hypergroup.

The proof is immediate since we enlarge results of the group by putting elements from G applying the Little Theorem. Moreover, e is unique scalar and $\forall x\in G$, there is a unique inverse x^{-1}, such that $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. Finally, if the last condition of this difinition is valid, then $e=x\cdot x^{1}=x^{-1}\cdot x$, so $(G,\otimes)$ is strong e-hypergroup.

The useful e-hypergroups are if only one product is enlarged by one element.

An important topic in hyper-mathematics comes from Santilli's admissibility. We may transfer this theory to representations in two ways: using either, ordinary matrices with a hyper-operation on them, or hyper-matrices with ordinary operations [5], [9], [26].

Definition 3.3 Let L be H_v-vector space over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of F/γ^*. Let ω_L be the core of $\varphi':L\to L/\varepsilon^*$ and denote again by 0 the zero of L/ε^*. Take two subsets R, $S\subseteq L$ then a *Lie-Santilli admissible hyperalgebra* is obtained by taking the Lie bracket hyper-operation

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}= xRy-ySx= \{xry-ysx \mid r\in R, s\in S\}$$

We can define admissibility on non square matrices, as well:

Definition 3.4 Let $(L=M_{m\times n},+)$ be the H_v-vector space of m×n hyper-matrices on the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, canonical map, $\omega_F=\{x\in F:\varphi(x)=0\}$. Similarly, let ω_L the core of $\varphi':L\to L/\varepsilon^*$. Take any two subsets $R,S\subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hyper-operation

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}=xR^ty-yS^tx.$$

Notice that $[x,y]_{RS}=xR^ty-yS^tx=\{xr^ty-ys^tx \mid r\in R \text{ and } s\in S\}$

Definition 3.5 According to Santilli's iso-theory, on a field F=(F,+,·), *a general isofield* $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$, is defined, to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new product

$$\hat{a}\,\hat{\times}\,\hat{b}:=\hat{a}\times\hat{T}\times\hat{b},\quad \text{with}\ \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(i)}$$

called *iso-product*, for which $\hat{1}$ is the left and right unit of F⎾,

$$\hat{1}\,\hat{\times}\,\hat{a}=\hat{a}\times\hat{1}=\hat{a},\ \forall\hat{a}\in\hat{F} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory to hyper-structures, we generalize only the new product $\hat{\times}$ from (i), by replacing with a hyper-operation including the old one. There are two general constructions on this direction as follows:

Construction 3.6 *General enlargement*. On a field $F=(F,+,\cdot)$ and isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b}=\hat{a}\times\hat{T}\times\hat{b},\quad \text{with}\ \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, containing $\hat{T}$, for all $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore, the new iso-hyper-operation is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b} = \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ is *iso-H_v-field.* The elements of F are called *iso-H_v-numbers* or *iso-numbers*.

Important hopes, of this construction, are those where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra element $\hat{x}_i$, is exactly one. Thus, there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{F}$$

and the rest are ordinary, then we have a *very-thin iso-H_v-field*.

The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, where $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the iso-H_v-field has scalar absorbing $\hat{0}$, scalar $\hat{1}$, and every $\hat{a}\in\hat{F}$, has one inverse.

Construction 3.7 *The P-hyper-operation*. Consider an isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, isonumbers, where $a\in F$, and $\hat{1}$ is positive-defined outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{F}$$

Take a $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\hat{F}-\{\hat{0},\hat{1}\}$, define the *isoP-$H_v$-field*, $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times}_P)$, where the hyper-operation $\hat{\times}_P$ is defined as follows:

$$\hat{a}\,\hat{\times}_P\,\hat{b} := \begin{cases} \hat{a}\times\hat{P}\times\hat{b} = \{\hat{a}\times\hat{h}\times\hat{b}\,|\,\hat{h}\in\hat{P}\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{T}\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1} \end{cases} \qquad \text{(iv)}$$

The elements of $\hat{F}$ are called *isoP-H_v-numbers*.

If $\hat{P}=\{\hat{T},\hat{p}\}$, the inverses in isoP-$H_v$-fields, are not necessarily unique.

Example 3.8 The generalized P-construction can be applied on rings to obtain H_v-fields. Thus for, $\hat{\boldsymbol{Z}}_{10}=\boldsymbol{Z}_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, and if we take $\hat{P}=\{\underline{\hat{2}},\underline{\hat{7}}\}$, then we have the table

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{\mathbf{9}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{3}},\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2}},\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{4}},\underline{\hat{9}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0}},\underline{\hat{5}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$
$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2}},\underline{\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{4}}$	$\underline{\hat{3}},\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{1}},\underline{\hat{6}}$
$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\hat{\mathbf{9}}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{4}},\underline{\hat{9}}$	$\underline{\hat{2}}$	$\underline{\hat{0}},\underline{\hat{5}}$	$\underline{\hat{8}}$	$\underline{\hat{1}},\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}},\underline{\hat{7}}$

Then the fundamental classes are

$(0)=\{\underline{\hat{0}},\underline{\hat{5}}\}$, $(1)=\{\underline{\hat{1}},\underline{\hat{6}}\}$, $(2)=\{\underline{\hat{2}},\underline{\hat{7}}\}$, $(3)=\{\underline{\hat{3}},\underline{\hat{8}}\}$, $(4)=\{\underline{\hat{4}},\underline{\hat{9}}\}$,

and the multiplicative table is the following

×	**(0)**	**(1)**	**(2)**	**(3)**	**(4)**
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{\boldsymbol{Z}}_{10} = \boldsymbol{Z}_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, is an H_v-field.

Example 3.9 Consider the $\hat{\boldsymbol{Z}}_{14} = \boldsymbol{Z}_{14}(\underline{\hat{a}},\hat{+},\hat{\times})$, and take $\hat{P}=\{\underline{\hat{2}},\underline{\hat{9}}\}$, then we have the table

$\hat{\times}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$	$\underline{\widehat{10}}$	$\underline{\widehat{11}}$	$\underline{\widehat{12}}$	$\underline{\widehat{13}}$
$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{1}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$	$\underline{\widehat{10}}$	$\underline{\widehat{11}}$	$\underline{\widehat{12}}$	$\underline{\widehat{13}}$
$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{2}}$	$\underline{\widehat{12}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\widehat{10}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\widehat{12}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\widehat{10}}$
$\underline{\hat{3}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\widehat{12}}$	$\underline{\hat{4},\widehat{11}}$	$\underline{\widehat{10}}$	$\underline{\hat{2},\hat{9}}$	$\underline{\hat{8}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{5},\widehat{12}}$	$\underline{\hat{4}}$	$\underline{\hat{3},\widehat{10}}$	$\underline{\hat{2}}$	$\underline{\hat{1},\hat{8}}$
$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\widehat{10}}$	$\underline{\hat{4}}$	$\underline{\widehat{12}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\widehat{10}}$	$\underline{\hat{4}}$	$\underline{\widehat{12}}$	$\underline{\hat{6}}$
$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2},\hat{9}}$	$\underline{\widehat{12}}$	$\underline{\hat{1},\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\widehat{10}}$	$\underline{\hat{6},\widehat{13}}$	$\underline{\hat{2}}$	$\underline{\hat{5},\widehat{12}}$	$\underline{\hat{8}}$	$\underline{\hat{4},\widehat{11}}$
$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\widehat{10}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\widehat{12}}$	$\underline{\widehat{10}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{7}}$
$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\widehat{10}}$	$\underline{\widehat{12}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\widehat{10}}$	$\underline{\widehat{12}}$
$\underline{\hat{9}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{8}}$	$\underline{\hat{5},\widehat{12}}$	$\underline{\hat{2}}$	$\underline{\hat{6},\widehat{13}}$	$\underline{\widehat{10}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{4}}$	$\underline{\hat{1},\hat{8}}$	$\underline{\widehat{12}}$	$\underline{\hat{2},\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{3},\widehat{10}}$
$\underline{\widehat{10}}$	$\underline{\hat{0}}$	$\underline{\widehat{10}}$	$\underline{\widehat{12}}$	$\underline{\hat{4}}$	$\underline{\widehat{10}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\widehat{12}}$	$\underline{\hat{4}}$	$\underline{\widehat{10}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$
$\underline{\widehat{11}}$	$\underline{\hat{0}}$	$\underline{\widehat{11}}$	$\underline{\hat{2}}$	$\underline{\hat{3},\widehat{10}}$	$\underline{\hat{4}}$	$\underline{\hat{5},\widehat{12}}$	$\underline{\hat{6}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2},\hat{9}}$	$\underline{\widehat{10}}$	$\underline{\hat{4},\widehat{11}}$	$\underline{\hat{6}}$	$\underline{\hat{3},\widehat{10}}$
$\underline{\widehat{12}}$	$\underline{\hat{0}}$	$\underline{\widehat{12}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\widehat{12}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\widehat{10}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\widehat{13}}$	$\underline{\hat{0}}$	$\underline{\widehat{13}}$	$\underline{\widehat{10}}$	$\underline{\hat{1},\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4},\widehat{11}}$	$\underline{\hat{2}}$	$\underline{\hat{0},\hat{7}}$	$\underline{\widehat{12}}$	$\underline{\hat{3},\widehat{10}}$	$\underline{\hat{8}}$	$\underline{\hat{3},\widehat{10}}$	$\underline{\hat{4}}$	$\underline{\hat{2},\hat{9}}$

Then the fundamental classes are

(0)=$\{\underline{\hat{0}},\underline{\hat{7}}\}$, (1)=$\{\underline{\hat{1}},\underline{\hat{8}}\}$, (2)=$\{\underline{\hat{2}},\underline{\hat{9}}\}$, (3)=$\{\underline{\hat{3}},\underline{\widehat{10}}\}$, (4)=$\{\underline{\hat{4}},\underline{\widehat{11}}\}$, (5)=$\{\underline{\hat{5}},\underline{\widehat{12}}\}$, (6)=$\{\underline{\hat{6}},\underline{\widehat{13}}\}$,

and the multiplicative table is the following

×	**(0)**	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
(0)	(0)	(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(6)	(1),(4)	(3),(5)	(5),(6)
(2)	(0)	(2),(4)	(1),(2)	(5)	(2)	(6)	(3)
(3)	(0)	(3),(6)	(5)	(4)	(3)	(2)	(1)
(4)	(0)	(1),(4)	(2)	(3)	(4)	(5),(6)	(3),(6)
(5)	(0)	(3),(5)	(6)	(2)	(5),(6)	(1)	(4)
(6)	(0)	(5),(6)	(3)	(1)	(3),(6)	(4)	(2)

Consequently, $\hat{\boldsymbol{Z}}_{14} = \boldsymbol{Z}_{14}(\underline{\hat{a}},\hat{+},\hat{\times})$, is an H_v-field.

4 Conclusions

The very big number of hyper-structures, especially of H_v-structures, defined on a set gives the opportunity to work in applications in an opposite direction. This means that we ask from applied sciences to give more axioms in order to reduce the number of possible H_v-structures to express the problem by a mathematical model. This is the problem of Lie-Santilli's admissibility which can be faced by defining the appropriate hyper-numbers or H_v-numbers, needed to express the irreversibility and uniqueness of living organisms and time.

References

[1] P. Corsini, V. Leoreanu, Application of Hyperstructure Theory, Klower, 2003.

[2] B. Davvaz, V. Leoreanu-Fotea, Hyperring Theory and Applications, Int. Acad. Press, USA, 2007.

[3] B. Davvaz, R.M. Santilli, T. Vougiouklis, Multi-valued Hypermathematics for characterization of matter and antimatter systems, JCMSE, 13, 2013, 37-50.

[4] B. Davvaz, R.M. Santilli, T. Vougiouklis Algebra, Hyperalgebra and Lie-Santilli Theory, J. Generalized Lie Theory Appl., 2015, 9:2, 1-5.

[5] B. Davvaz, T. Vougiouklis, A Walk Through Weak Hyperstructures, H_v-Structures, World Scientific, 2018.

[6] S. Georgiev, Foundations of Iso-Differential Carlculus, Nova Sc.Publ.,V.1-6, 2016.

[7] S. Ostadhadi-Dehkordi, T. Vougiouklis, K. Hila. 2022. H-sets and applications on H_v-groups, J. Algebraic Systems, V.10, N.1, 79-93.

[8] R.M. Santilli, Embedding of Lie-algebras into Lie-admissible algebras, Nuovo Cimento 51, 570, 1967.

[9] R.M. Santilli, Dissipativity and Lie-admissible algebras, Meccanica 1, 3, l969.

[10] R.M. Santilli, Foundations of Hadronic Chemistry, with Applications to New Clean Energies and Fuels, Kluwer Academic Publishers, 2001.

[11] R.M. Santilli, Hadronic Mathematics, Mechanics and Chemistry, Volumes I, II, III, IV and V, International Academic Press, USA, 2007.

[12] R.M. Santilli, Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen, Ratio Mathematica, V.37, 2019, 5-23.

[13] R.M. Santilli, Studies on A. Einstein, B. Podolsky and N. Rosen argument that 'quantum mechanics is not a complete theory,' Ratio Mathematica V. 38, 2020, I: 5-69. II: 71-138, III: 139-222.

[14] R.M. Santilli, T. Vougiouklis, Isotopies, Genotopies, Hyperstructures and their Applications, New frontiers Hyperstr., Hadronic, 1996, 1-48.

[15] R.M. Santilli, T. Vougiouklis, Lie-admissible hyperalgebras, Italian J. Pure Appl. Math., N.31, 2013, 239-254.

[16] T. Vougiouklis, Hyperstructures and their Representations, Monographs in Math., Hadronic, 1994.

[17] T. Vougiouklis, Some remarks on hyperstructures, Contemporary Math., Amer. Math. Society, 184, 1995, 427-431.

[18] T. Vougiouklis, Convolutions on WASS hyperstructures, Discrete Math., 174, 1997, 347-355.

[19] T. Vougiouklis, Enlarging H_v-structures, Algebras Comb., ICAC'97, Hong Kong, Springer, 1999, 455-463.

[20] T. Vougiouklis, Hyperstructures in isotopies and genotopies, Advances in equations and Inequalities, Hadronic Press, 1999, 275-291.

[21] T. Vougiouklis, The Santilli's theory 'invasion' in hyperstructures, AGG 28(1), 2011, 83-103.

[22] T. Vougiouklis, The e-hyperstructures, J. Mahani Math. Research Center, V.1, N.1, 2012, 13-28.

[23] T. Vougiouklis, From H_v-rings to H_v-fields, Int. J. AHA. V.1, N.1, 2014, 1-13.

[24] T. Vougiouklis, On the isoH_v-numbers, Hadronic J., Dec.5, 2014, 1-18.

[25] T. Vougiouklis, Iso-hypernumbers, Iso-H_v-numbers, Proc. ICNAAM 2014, AIP Conf. Proc. 1648, 510019, 2015.

[26] T. Vougiouklis, Hypermathematics, H_v-structures, hypernumbers, hypermatrices and Lie-Santilli admissibility, American J. Modern Physics, 4(5), 2015, 34-46.

[27] T. Vougiouklis, Minimal H_v-fields, RatioMathematica V.38, 2020, 313-328.

[28] T. Vougiouklis, S. Vougiouklis, The helix hyperoperations, Italian J. Pure Appl. Math., N.18, 2005, 197-206.

[29] S. Yavuz, S. Onar, B.A. Ersoy, B. Davvaz, K. Hila, Intuitionistic Complex Fuzzy H_v-subgroups, J. multiple-valued logic & soft comp., V. 41, N. 1-2, 2023, 141-167.