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Lie-isotopic representation of stable nuclei II: Exact and time invariant representation of the Deuteron data

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Abstract

In the preceding paper, we have presented apparent insufficiencies of quantum mechanics in nuclear physics To attempt a resolution of the indicated insufficiencies, in this paper we present a systematic and upgraded outline of the axiom-preserving, time reversal invariant, *Lie-isotopic branch of hadronic mechanics* for the characterization of *stable* nuclei via the representation of the dimension, shape and density of protons and neutrons in the experimentally measured conditions of partial mutual penetration in a nuclear structure, with ensuing potential-Hamiltonian and contact non-Hamiltonian terms in nuclear force. We show that the Lie-isotopic methods allow a numerically exact and time invariant representations of all Deuteron data in the ground state without orbital contributions. We finally show that said representations are primarily due to the violation by Lie-isotopic methods of Bell's inequalities with explicit and concrete realizations of Bohm's hidden variables, as well as to the completion of Heisenberg's uncertainty principle for point-like particles in vacuum under electromagnetic interactions into the *isouncertainty principle* of hadronic mechanics for extended nucleons in condiitons of partial mutual penetration allowing a progressive recovering of Einstein's determinism under strong interactions up to its full recovery at the limit of Schwartzschild's horizon.

Keywords: nuclear physics 81V35, EPR argument, hadronic mechanics, nuclear data. [1]

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1 Introduction

1.1. Experimental foundations

Inspired by the 1935 historical argument by A. Einstein, B. Podolsky and N. Rosen (EPR) that *Quantum mechanics is not a complete theory* [1] (see also recent studies [2] [3] [4]), in the preceding paper [5] (herein referred to with the prefix I), we have presented apparent insufficiencies of quantum mechanics in nuclear physics with particular reference to the prohibition by Heisenberg's uncertainty principle for point-like particles in vacuum to achieve a quantitative representation of the neutron synthesis from the Hydrogen in the core of stars, Eq. I-3) and of other nuclear data (Sect. I-2).

In this paper, we attempt a resolution of the indicated insufficiencies based on the rather dramatic physical differences between atomic and nuclear structure. In fact, atomic structures can be well approximated as being composed by point-like constituents in vacuum under sole action at a distance, thus potential interactions and ensuing exact validity of quantum mechanics.

By comparison, experimental evidence [6]-[**?**] establishes that nuclei are composed by *extended* protons and neutrons (collectively called nucleons) in conditions of *partial mutual penetration* when they are members of a nuclear structure. As an example, the charge radius of the Helium-4 $D_{He} = 1.67\ fm$ [9] is $0.07\ fm$ *smaller* then the diameter of the proton $D_N = 1.74\ fm$ [11]. Consequently, the charge distributions of the four nucleons composing the Helium-4 are in conditions of $0.07\ fm$ mutual penetration of their dense structure, with generally increasing values for heavier nuclei.

The latter conditions imply the expectation that nuclei are composed by extended constituents under conventional action at a distance, thus Hamiltonian interactions, plus basically new contact, thus zero range and non-Hamiltonian interactions outside any possible representation by quantum mechanics with ensuing need for its suitable completion.

1.2. Hamiltonian interactions

We are collectively referring to interactions that are linear, local and derivable from a potential, thus being fully representable by the conventional Hamiltonian of quantum mechanics (Sect. 2).

1.3. Non-Hamiltonian interactions

We are collectively referring to interactions that are: *Nonlinear* (in the wave function) as pioneering by Werner Heisenberg [13]; *Nonlocal* (distributed in a volume not reducible to points) as pioneered by Louis de Broglie and David Bohm [14]; *Nonpotential* (of contact zero-range type) as pioneered by R. M. Santilli in the 1978 Harvard University monograph [15] via the *conditions of variational self-*

adjointness according to which Hamiltonian interactions are variationally selfadjoint (SA), while non-Hamiltonian interactions are variationally nonselfadjoint (NSA) (Sect. I-3).

1.4. Hadronic mechanics

In this paper, we shall outline and update the EPR completion of quantum into *hadronic mechanics,* first introduced by R. M. Santilli in the 1983 second volume [16] for the representation of extended nucleons under Hamiltonian/SA and non-Hamiltonian/NSA interactions in such a way to preserve the basic axioms of quantum mechanics at their abstract level, according to the following two primary branches:

TIME REVERSIBLE LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS studied in this paper, which was introduced in Charts 5.2, 5.3 and 5.4, p. 165 on of [16] and comprises the axiom-preserving *isomathematics* (Sect. 3) and *isomechanics* (Sect. 4) for the representation of *stable* (thus, time-reversal invariant) systems of *extended* particles at short mutual distances under Hamiltonian/SA and non-Hamiltonian/NSA interactions. The Lie-isotopic methods are based ion the generalization oi the conventional associative product $AB = A \times B$ of generic quantities into the associativity-preserving form $A\hat{\times}B = A\hat{T}B, \hat{T} > 0$ (first presented in Eq. (5), p. 71 of [16]) where T, called the isotopic element (and at times the Santillian) is positive-definite but possesses otherwise an unrestricted functional dependence on all needed local variables, with ensuing *Lie-isotopic generalization of Heisenberg equation* $idA/dt = A\hat{\times}H - H\hat{\times}A = A\hat{T}H - H\hat{T}A$ (Eq. (18a), p. 153 of [16]) where the Hamiltonian H represents all SA interactions and the Santillian $\hat{T}$ represents the extended character of particles and their NSA interactions.

TIME IRREVERSIBLE LIE-ADMISSIBLE BRANCH OF HADRONIC MECHANICS introduced in Chart 5.1, p. 148 on of [16] which comprises *genomathematics* and *genomechanics.* Lie-admissible formulations are intended to achieve an axiomatization of irreversibility by restricting all associative products $AB = A \times B$ to be ordered to the right $A > B = A\hat{S}B,\ \hat{S} > 0$ (left $A < B = A\hat{R}B,\ \hat{R} > 0$) for motion forward (backward) in time. Irreversibility is then assured for $\hat{R} \neq \hat{S}$ (for technical details, see the 1981 paper [17] from Harvard's Department of Mathematics). The basic dynamical equation is given by the *Lie-admissible generalization of Heisenberg's equations* (Eqs.(19a), p. 153 of [16]) $idA/dt = (A, H) = A < H - H > A = A\hat{R}H - H\hat{S}A$ which admits the Lie-isotopic case for $\hat{R} = \hat{S} = \hat{T} > 0$ and, for the particular case $\hat{R} = 1,\ \hat{S} = 1 - F/H$, assumes the form $idA/df = (AH - HA) + FA$, by therefore providing the first known operator realization of Lagrange's and Hamilton's external terms F for the representation of irreversibility in their celebrated analytic equations. Lie-admissible methods are suggested for the axiomatically consistent representation of controlled nuclear fusions and other irreversible processes to be studied in subsequent papers.

For advances on hadronic mechanics that occurred during the past decades, the interested reader can inspect: the final classification of hadronic mechanics (including the additional *hyperstructural branch* for biological structures and the *isodual branch* for antiparticles) [18]; the collection of papers [19]; the list of early workshops and conferences [20]; independent studies [21]-[29]; and the general presentation [30] [31] [32].

1.5. EPR entanglement

Experimental evidence known since Einstein's times establishes that particles, which are initially bounded together and then separated, can influence each other continuously and instantaneously at arbitrary distances [33]. Albert Einstein strongly objected against the terms "quantum entanglement" on grounds that the sole possible representation of particle entanglements via the Copenhagen interpretation of quantum mechanics would require superluminal communications that violate special relativity.

For the intent of honoring the generally forgotten Einsteins view, R. M. Santilli [34] proved that the sole possible representation of particle entanglement by the Copenhagen interpretation of quantum mechanics is that *the particles are free,* evidently because the sole possible interactions admitted by said interpretation are those derivable from a potential which is identically null for particles at large mutual distances.

By recalling that the wave packet of particles is identically null solely at infinite distance, Ref. [34] then pointed out that the sole interactions that are continuous, instantaneous and at arbitrary distances are given by the mutual penetration of wave packets of particles which, being NSA [15], are beyond any hope of treatment via quantum mechanics.

Thanks to the prior development of isomathematics for the representation of nonlinear, nonlocal and NSA interactions [26] [29] [30], Santilli [34] proposed the axiom-preserving completion of quantum into hadronic entanglement under the suggested name of *EPR entanglement* which does indeed provide a *quantitative*representation of particle entanglements without superluminal speeds via contact, nonlinear, nonlocal and non-Hamiltonian, thus NSA interactions between the wave packets of entangled particles (Fig. 1).

Einstein's entanglement has intriguing implications for the studies presented in this paper, such as:

1.5.A) A new conception of nuclei as hadronic bound states of extended nucleons in condition of deep entanglement that persists following nuclear fission or fusion.

1.5.B) A new conception of computers based on isomathematics, which has been suggested under the name of *EPR computers* [35], with: increased energy efficiency because NSA interactions carry no potential energy; better cybersecurity (Sect. 3.5) and faster computations (Sect. 4.6).

1.5.C) A new conception of continuous and instantaneous interactions that, in view of the recent detection of entanglement at classical distances [36], is here suggested as a possible realization of the *fifth interaction* (see also Ref. [37]. Its lack of identification to date is easily explained by the fact that the EPR entanglement cannot be conceived, let alone treated via quantum mechanics.

1.6. Bell's inequalities

In the 1950's, J. S. Bell [38] proved a number of quantum inequalities, the first one of which essentially states that systems of (point like) particles with spin $1/2$ represented via quantum mechanics do not admit classical counterparts. This view was assumed by mainstream physicists for over half a century to be the final disproof of the EPR argument, and to establish the validity of Heisenberg's uncertainty principle for all possible particle conditions existing in the universe.

Again thanks to the prior development of isomathematics, R. M. Santilli [39] proved in 1998 a number of hadronic inequalities essentially stating that *systems of extended particles with spin* $1/2$ *represented via the Lie-isotopic branch of hadronic mechanics do indeed admit classical counterparts,* while providing explicit examples. Santilli's hadronic inequalities are confirmed by experiments [40] [41] [42] establishing the existence in nature of particle conditions which violate Bell's inequalities (see also experiments [7]-[19] of Paper I).

A deeper understanding of this paper requires the knowledge that the theoretical [39] and experimental works [40] [41] [42] disproving Bell's inequalities imply the expectation that *Heisenberg's uncertainties principle is correspondingly violated by strong interactions between extended nucleons in conditions of mutual penetration.*

1.7. Einstein's isodeterminism

Soon after joining the faculty of Harvard University under DOE support in late 1977, R. M. Santilli expressed doubts on the exact validity for strong interactions of Heisenberg's uncertainty principle and other basic quantum mechanical laws, as one can see from the titles of the 1978 introductory memoir on hadronic mechanics [43] (see also the subsequent papers [44] [45]). The argument underlying such a conviction is that the standard deviations for coordinates Δr, momenta Δp and their product are certainly valid for the conditions of their original conception, i.e., for *point-like charged particles under electromagnetic interactions,* because a point-like particle can move within a star by solely sensing action-at-a-distance interactions due to their dimensionless character.

The situation is conceptually, mathematically, theoretically and experimentally different when considering *extended* nucleons in conditions of *mutual penetration* because, in view of their 'strength', their strong interactions imply the emergence of a *pressure* created on a given nucleon by its surrounding nucleons (Fig. 3) according to a view pioneered by L. de Broglie and D. Bohm with their nonlocal theory [14]. It is then evident that the standard deviations for the indicated nucleon Δr and Δp *cannot* be the same as the corresponding standard deviation for an electron in vacuum, thus implying the need for a suitable completion of Heisenberg's uncertainty principle for strong interactions.

Thanks to works written at Harvard University in the late 1970's [43] [44] and the recent works [45] [34] [39], R. M. Santilli [46] finally achieved the axiom-preserving, EPR completion of Heisenberg's uncertainty principle for point-like particle under electromagnetic interactions into the *isouncertainty principle for extended nucleons under electromagnetic, weak and strong interactions* collectively called *Einstein's isouncertainies* (see Sect. 4.7 for its formulation), which new principle essentially states that Einstein's determinism [1] is progressively recovered with the increase of the density in the interior of hadrons, nuclei and stars and fully recovered at the limit of Schwartzschild's horizon.

Ss known to experts, despite one century of studies under large public funds, nuclear physics has been unable to achieve exact representations of nuclear data, the synthesis of the neutron from the Hydrogen atom in the core of stars, nuclear stability despite the natural instability of the neutron and extremely repulsive protonic Coulomb forces and other open problems.

A main point which is attempted to convey in this and in the next paper is that the indicated open nuclear problems appear to be due to the *theoretical assumption* that Heisenberg's uncertainty principle for point-like particles under electromagnetic interactions is also valid for extended nucleons under strong interactions. However, in view of the lack of any direct experimental verification, said assumption is a personal view by the individual physicist, while the indicated open problems deal with clear physical evidence, by therefore suggesting the need for collegial mathematical, theoretical and experimental studies on the uncertainties of nucleons in a nuclear structure as a necessary premise for advances in the recycling of radioactive nuclear waste, controlled nuclear fusions and other clear societal needs.

1.8. Bohm hidden variables

From the preceding outline one can see that Bohm's hidden variables [48] [49] are *hidden in the associative axiom of quantum mechanics* with generic realization in terms of the Santillian $\lambda = T,\ A\hat{\times}B = A\lambda B,\ A\hat{\times}(B\hat{\times}C) = (A\hat{\times}B)\hat{\times}C.$

It should be noted that, despite its apparent elementary character, the quantitative study of the indicated realization of Boh's hidden variables required nonlocal lifting of 20-th century applied mathematics, including the Newton-Leibnitz differential calculus [50] (see also studies [29]). A preliminary realization of Bohm's hidden variables has been presented in Sect. 4.C.3, p. 170 on and Sect. 6.8, p. 254 on of [31] (see Sect. 5.5 for details). Additional explicit and concrete realizations of Bohm's hidden variables have been presented in Refs. [51] [52].

The deterministic interpretation of Bohm's hidden variables resulted to be progressive [46] because dependent on the density of a composite strongly interacting system represented by the isotopic element $\hat{T}$ which becomes null for gravitational collapse under which nucleons are essentially frozen in their position with identically null standard deviations $\Delta r \equiv 0, \;\; \Delta p \equiv 0$ due to the immense surrounding pressures. (see Sect. 4.7 for details).

1.9. Simple construction of hadronic models

On pedagogical grounds, the main objectives of this paper are:

1.A) Show that quantum mechanical models for point-like nucleons with sole Hamiltonian interactions can be mapped via simple *nonunitary transformations* into covering hadronic models for extended, deformable and hyperdense nucleons under Hamiltonian and non-Hamiltonian interactions.

1.1.B) Show that, when formulated on quantum mechanical spaces over conventional fields, said nonunitary models are afflicted by serious consistency problems, such as the lack of conservation of the Hermiticity/observability, lack of prediction of the same numerical values at different times, loss of causality and others (Sect. 3.11).

1.C) Show that, the indicated inconsistencies are resolved when said quantum mechanical nonunitary models are reformulated into hadronic models on isospaces over isofields (3.12).

In this paper, we shall: review the main axiomatic structure of quantum mechanics; outline and upgrade the foundations of the axiom-preserving isomathematics and isomechanics; and then show that the resulting Lie-isotopic methods, including above all Einstein's isodeterminism presented in Sect. 4.7 [46], allow the first known numerically exact and time invariant representation of the Deuteron experimental data in a form expectedly extendable to other stable nuclei.

2 Axiomatic foundations of quantum mechanics

In this section, we recall a few axiomatic aspects of quantum mechanics that are relevant later on for comparison with their corresponding axiom-preserving completion.

As it is well known (see, e.g., Refs. [53] [54] [55], nuclear physics has been based since its inception on the *universal enveloping associative algebra* $\xi\ :\ \{A, B, ...; AB, =\ A\ \times\ B; 1\}$ of Hermitean operators $A, B, ...$ on a Hilbert space $\mathcal{H}$ with states $|\psi(r)\ >$ over the field of complex numbers $\mathcal{C}$ with conventional associative product

$$AB = A \times B, \tag{1}$$

and unit

$$1 \times A = A \times 1 = A \,\forall\, A\ \in\ \xi. \tag{2}$$

In fact, the above basic axioms are sufficient to characterize, uniquely and unambiguously, all quantum mechanical methods used in nuclear physics including:

2.1) The representation of stable nuclei via a primitive, k-dimensional space-time or internal Lie symmetry with anti-symmetric Lie algebra attached to the enveloping algebra $L\ \equiv\ \xi^-$ with commutation rules for Hermitean generators $X_k = X_k^\dagger,\ k = 1, 2, ..., N$

$$[X_i, X_j] = X_iX_j - X_jX_i = C_{ij}^k X_k, \tag{3}$$

including the rotational symmetry $SO(3)$, the spin symmetry $SU(2)$, the Lorentz symmetry $SO(3.1)$, the Lorentz-Poincaré symmetry $P(3.1)$, the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1)$, and others.

2.2) The infinite class of equivalent nuclear models under the unitary law

$$UU^\dagger = U^\dagger U = I = Diag_{k\times k}(1, 1, ..., 1). \tag{4}$$

2.3) *Heisenberg's representation* of the time evolution of a Hermitean operator A in terms of the Hamiltonian H here expressed in the infinitesimal and its unitary finite form

$$i\tfrac{dA}{dt} = [A, H] = A \times H - H \times A,$$

$$A(t) = e^{H\times t\times i} \times A(0) \times e^{-i\times t\times H} = W(t)A(0)W^\dagger(t), \tag{5}$$

$$WW^\dagger = W^\dagger W = I.$$

2.4) The unitarily equivalent *Schrödinger representation* of the time evolution of A on a Hilbert space $\mathcal{H}$ with states $|\psi(r)\rangle$ over the field of complex numbers $\mathcal{C}$

$$\langle\psi(r)|\psi(r)\rangle = I,$$

$$< \langle\psi(r)H|\psi(r)\rangle =< \langle\psi(r)|H\psi(r)\rangle^*,$$

$$H \times |\psi(r) >= [\tfrac{1}{2m}\delta^{ij}p_i \times p_j + V(r)] \times |\psi(r) >= E \times |\psi(r) >, \tag{6}$$

$$p \times |\psi(r)\rangle = -i\partial_r|\psi(r)\rangle,$$

$$[r_i, p_j] \times |\psi(r)\rangle = -i\delta_{ij},\ \ [r_i, r_j] \times |\psi(r)\rangle = [p_i, p_j] \times |\psi(r)\rangle = 0.$$

2.5) The important prediction of the same numerical values under the same conditions at different times (hereon called time invariance), which is evidently due to the unitary structure of time evolutions (5) (6) with ensuing invariant of numeric values, e.g.,

$$1 \;\rightarrow\; 1' = U1U \equiv 1. \tag{7}$$

2.6) The *perturbation theory* based, e.g., on power series

$$A(t) = A(0) + w \times \frac{[A, H]}{1!} + w^2 \times \frac{[[A, H], H]}{2!} + ... \tag{8}$$

2.7) The remaining well known methods used for a century in the quantum mechanical study of nuclear physics.

3 Elements of isomathematics

3.1. Historical notes

On September 8, 1977, R. M. Santilli joined the Lyman Laboratory of Physics of Harvard University with a research contract from the Department of Energy (DOE) to search for possible new clean forms of nuclear energies. To conduct this task, Santilli requested authorization from the DOE to initiate the research with the study of the most fundamental fusion in nature, that of the neutron from the Hydrogen atom in the core of stars [56], (the inability by quantum mechanics to provide any quantitative representation of the neutron synthesis despite the extremely big $e - p$ Coulomb attraction) was identified as part of this initial study.

The origin of the insufficiency was identified with the need to represent the charge distributions of the proton as being *extended*. The admission of the extended character of particles then implied Insufficiency I.2.1 (the expectation at short mutual distances of non-Hamiltonian interactions 1.3, Fig, I.1), which expectation is essentially implied by Rutherford's "compression" [56] of the electron within the hyperdense proton. The analysis essentially confirmed the EPR argument [1] because a quantitative representation of the neutron synthesis was beyond any hope of achievement via quantum mechanics, thus leaving no other possibility than that of constructing a suitable completion.

Since the task was of primary *mathematical* nature, Santilli was moved in 1979 from the Lyman Laboratory of Physics to the Department of Mathematics of Harvard University (see, e.g., paper [17]) where he proposed the completion of quantum mechancs into *hadronic mechanics* [16] (to indicate its intended use for strong interaction), under the challanging intent of maintaining the abstract axioms of quantum mechanics despite the admission of non-Hamiltonian interactions.

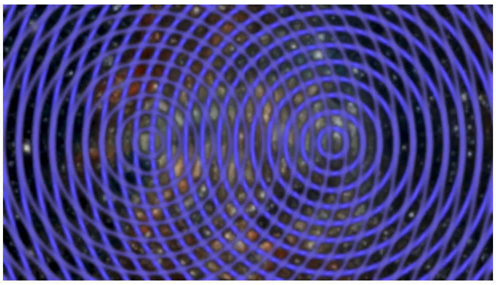

Figure 1: *In this figure we illustrate the notion at the foundation of the studies presented in these papers, which is given by the "Einstein-Podolsky-Rosen (EPR) entanglement of particles" when interpreted as being due to the continuous and instantaneous overlapping of the wave packets of particles at arbitrary mutual distance [34], with resulting nonlinear, nonlocal and nonpotential interactions whose representation required the construction of the new isomathematics, isomechanics and isosymmetries (see also the overview [3]).*

3.2. The birth of isomathematics

As it is well known, Lie's theory (see, e.g., Ref. [57] and works quoted therein) has acquired a justly historical value for the representation of *point-like particles under sole Hamiltonian interactions,* such as the representation of the spin $S = 1/2$ of atomic electrons via Pauli's matrices, and the consequential exact representation of the magnetic moment of the electron via the giromagnetic factor, $\mu = gS$.

By contrast, Lie's theory appears to be excessively restrictive in nuclear physics due to its inability to represent the *extended* character of nucleons and their expected *non-Hamiltonian* interactions, with consequential insufficiencies studied in Paper I. At the same time, the numeric value $1/2$ of the spin of nucleons is beyond scientific doubt. Hence, the problem addressed by Santilli in the late 1970's was that of identifying a completion of Lie's theory for the representation of extended particles under Hamiltonian and non-Hamiltonian interactions while preserving conventional spin values.

To achieve the intended needs, Santilli proposed in Eq. (5) p. 71 of the 1983 Springer-Verlag monograph [16] the completion (also called lifting) of the universal enveloping associative algebra $\xi\{A, B, ...; AB = A \times B; I\}$ (Sect. 2) into the *associativity-preserving,* thus isotopic envelope $\hat{\xi}$ characterized by the new product called the *isoproduct* (first introduced in Eq. (5), p. 71, and then studied in details from Sect. 5.1, p. 148 and Sect. 5.2 of [16])

$$A \times B \ \rightarrow \ A \hat{\times} B = A \hat{T} B, \quad \hat{T} > 0, \tag{9}$$

with related new unit, called the *isounit*

$$\hbar = 1 \ \rightarrow \ \hat{I} = 1/\hat{T}, \quad \hat{I}\hat{\times}A = A\hat{\times}I = A, \, \forall A \ \in \ \hat{\xi}, \tag{10}$$

where the isotopic element $\hat{T}$, at times called the Santillian, is solely restricted by the condition of being positive-definite $\hat{T} \ > \ 0$, while admitting an arbitrary functional dependence on the relative coordinate r, momentum p, acceleration a, energy E, charge radius R, density d, pressure π, temperature τ wave function ψ, and any other needed local variable (Fig. 3),

$$\hat{T} = 1/\hat{I} = \hat{T}(r, p, a, E, R, d, \psi, \pi, \tau, \psi, ...) > 0, \tag{11}$$

which dependence is hereon tacitly assumed.

The reader should be unformed that the generalization of the basic unit of quantum mechanics, $\hbar \ = \ 1$, into a nonlinear, nonlocal and nonpotential operator $\hat{I} \ = \ 1\hat{T}$ intends to represent the possible ack of quantized energy exchanges within a nuclear structure in favor of rather complex integro-differential energy exchanges, as expected for extended nucleons in conditions of mutual penetration (see Sect. 3.12 for more comments, experimental works [47] and papers quoted therein).

3.3. The Lie-isotopic theory

Despite its simplicity, isotopic completion (9) of associative product (1) has required, for consistency, a compatible isotopic completion of the totality of 20th century applied mathematics (with no exception known to the author) into the novel *isomathematics* [15] [16] (see monographs for physicists [28] [30] and those for mathematicians [26] [29]) whose knowledge is essential to avoid structural inconsistencies in isotopic nuclear models that may remain unknown to non-experts in the field.

To begin a brief outline of the isomathematics implied by basic axiomatic assumptions (9) and (10), in Chapter 5, p. 148 on of [16] Santilli constructed the step-by-step isotopies of the various branches of Lie's theory, including the isotopies of the universal enveloping associative algebras (Sect. 5.2), Lie algebras and Lie's Theorems (Sect. 5.3), Lie groups and the Lie transformation theory (Sect. 5.4), resulting in the *Lie-isotopic theory,* today known as the *Lie-Santilli isotheory* [22] [58] [59], with generic *isocommutation rules* for a k-dimensional isoalgebra with iso-Hermitean generators

$$\begin{aligned} [J_i, J_j] = J_i\hat{\times}J_j - J_j\hat{\times}J_i = J_i\hat{T}J_j - J_j\hat{T}J_i = \\ = C_{ijk}(r, p, a, E, R, d, \psi, \pi, \tau, \psi, ...)J_k. \end{aligned} \tag{12}$$

Lie-Santilli isoalgebras are nowadays classified into [60]: *regular isoalgebras* when the structure quantities C_{ij}^k are constant, and *irregular isoalgebras* when said structure quantities are functions of local variables. As we shall see in Sect. 3.11, regular isoalgebras can be easily constructed via *nonunitary transformations* of the original Lie algebras, while irregular isoalgebras cannot, by therefore constituting new realizations of Lie's axioms which remain vastly unexplored to this day.

It should be noted since these introductory words that the above assumptions have permitted various explicit and concrete realizations of Bohm's *hidden variables* λ [48] (Sect. 1.8). Such a realization is known to be impossible for the Copenhagen interpretation of quantum mechanics, but it becomes possible for isomathematics because *Bohm's variable are hidden in the axiom of the associative product of quantum mechanics* [31]

$$\lambda = \hat{T},$$
$$A\hat{\times}(B\hat{\times}C) = A\lambda(B\lambda C) = (A\hat{\times}B)\hat{\times}C = (A\lambda B)\lambda C. \tag{13}$$

The time invariant and numerically exact representation of nuclear data presented in Sect. 5 is a direct consequence of the capability of isomathematics (and, therefore, of isomechanics) to represent Bohm's hidden variables.

It should also be noted that isoproduct (9) assures that hadronic mechanics is outside the infinite class of unitary equivalence of quantum mechanics (because the new product $A\hat{T}B$ cannot be obtained via a unitary transformation of the original product AB). Despite that, isoproduct (9) does preserve the "majestic axiomatic structure of quantum mechanics" indicated in Sect. I.1.

3.4. Representation of extended particles and their non-Hamiltonian interactions

The main objective of the isotoopic element $\hat{T} > 0$ is that of resolving axiomatic insufficiencies I-3.1, I-3.2 and I-3.3 of quantum mechanics in nuclear physics, (namely, its linear, local and potential character) via its representation of nonlinear, nonlocal and nonpotential/NSA interactions among extended nucleons.

The representation was initiated by Newton's historical conception of velocity-dependent resistive, thus nonpotential forces which were then represented by Lagrange and Hamilton via the external terms in their analytic equations.

As indicated in Sect. 1, particle interactions nonlinear in the wave function bur represented with a Hamiltonian were initiated by W. Heisenberg [13]; interactions occurring in volumes not reducible to points but also represented with a Hamiltonian were initiated by L. de Broglie and D. Bohm [14]; nonlinear and nonlocal interactions not representable with a Hamiltonian were initiated by R. M. Santilli in the1978 Harvard University monograph [15] via their technical characterization as being variationally non selfadjoint (NSA).

The first known application of NSA interactions in particle physics occurred in Santilli's 1978 memoir [43] via an isotopic element of the simple type

$$\hat{T} = e^{-\frac{\hat{F}(r)}{F(r)}}, \quad \hat{F}(r) > 0, \quad F(r) > 0, \tag{14}$$

where $\hat{F}(r)$ ($F(r)$) are positive-definite, generally nonlinear and nonlocal functions of relative coordinates r representing NSA (SA) forces in the *Schrödinger-Santilli isoequation* $H\hat{\times}|\hat{\psi}\rangle = H\hat{T}|\hat{\psi}\rangle = E|\hat{\psi}\rangle$ (see Sect. 4.3 for the full representatiion).

Despite its simplicity, isotopic element (14) allowed the first known representation of *all* characteristics of the π^0 meson, the $\mu^{\pm}$ lepton and other particles (including the mechanism for their spontaneous decay and the representation of their mean lives) via hadronic structure equations of physical constituents, electrons and positrons, produced free in the spontaneous decay with the lowest modes (Sect. 5.1, p. 827 on of Ref. [43]).

The isotopic element for the joint representation the extended character of particles and their non-Hamiltonian/NSA interactions adopted in these papers is given by the (contravariant) form [3]

$$\hat{T} = \hat{T}_{4\times 4} = \Pi_{\alpha=1,2,3,4} \begin{pmatrix} \frac{1}{n^2_{1,\alpha}} & 0 & 0 & 0 \\ 0 & \frac{1}{n^2_{2,\alpha}} & 0 & 0 \\ 0 & 0 & \frac{1}{n^2_{3,\alpha}} & 0 \\ 0 & 0 & 0 & \frac{1}{n^2_{4,\alpha}} \end{pmatrix} \times e^{-\Gamma(r,p,a,R,d,\psi,\hat{\psi},\dots)} \; > \; 0,$$

$$n_{\mu,\alpha} > 0, \;\; \Gamma > 0, \tag{15}$$

wqhere

3.4.A) The representation of the dimension and shape of the particle is done via semi-axes $n^2_{k,\alpha}$, $k = 1, 2, 3$ (with n_3 parallel to the spin) with normalization for the vacuum $n^2_{k,\alpha} = 1$ (Fig. 2).

3.4.B) The representation of the density is done via the characteristic quantity $n^2_{4,\alpha}$ per individual nucleons with normalization for the vacuum $n^2_{4,\alpha} = 1$.

3.4.C) The representation of the nonlinear, nonlocal and nonpotential interactions between extended particle is done via the exponential term $e^{\Gamma(\psi,\dots)}$ generally realized in terms of volume integrals, as we shall see in Sects. 3 and 4.

3.4.D) When representing nucleons and their NSA interactions, the dimension of the ellipsoids is restricted not to surpass the range of strong interactions $R = 1\, fm = 10^{-13}\, cm$.

3.4.E) When considering fifth interactions (Sect. 1.5.C), the dimension of the particle is generally assumed to be infinite so that isointegrals [30] [29] yield measurable predictions.

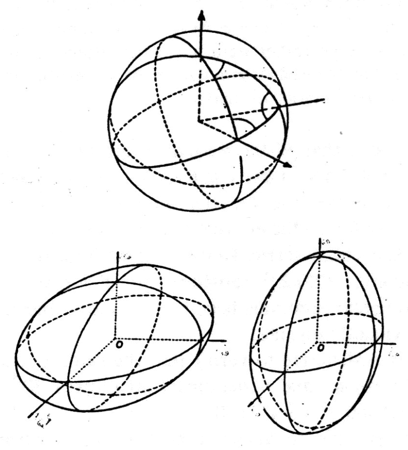

Figure 2: *In this figure, we illustrate the second fundamental notion of these studies, the implementation of the nonlocality advocated by Einstein-Podolsky-Rosen [1] via the representation of protons and neutrons (nucleons) as being extended, thus deformable charge distribution that, as such, can be prolate or oblate ellipsoids [3].*

The reader should keep in mind that the isotopic element for *covariant* nuclear models is the *inverse* of contravariant isotopic element (15), thus have a positive value in the exponent.

3.5. Isonumber theory

The formulation of experimental values of isotopic nuclear models over the conventional field of real numbers $\mathcal{R}(n, \times, 1)$ implies the loss of their invariance over time because the conventional unit 1 of the theory is no longer invariant under the nonunitary transformations of hadronic mechanics, $1 \rightarrow 1' = W 1 W^{\dagger} \neq 1$, with the consequential loss of the base field over time and the invalidation of experimental measurements.

In order to maintain the crucial time invariance of isotopic numeric predictions, it has been necessary to complete the conventional number theory into a form applicable for arbitrary, positive-definite isounits $\hat{I} > 0$. This completion was achieved by Santilli in the 1993 memoir [61] (see independent studies [25] [62]) via the notion of *isofields* $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$ for which the infinite family of rings with elements $\hat{n}$, called *isoreal, isocomplex or isoquaternionic isonumbers,* characterized by the ordinary product of conventional numbers n times a given isounit $\hat{I}$

$$\hat{n} = n \times \hat{I}, \tag{16}$$

when equipped with isoproduct (9)

$$\hat{n} \hat{\times} \hat{m} = (n\hat{I})\hat{T}(m\hat{I}) = (nm)\hat{I} \in \hat{F}, \tag{17}$$

verify all axioms of a numeric field, thus being fully acceptable for experimental values.

Predictably, *all* conventional operations with numbers should be replaced, for consistency, with the compatible operations for isonumnbers, such as (see [30] for details)

$$\hat{n}^{\hat{2}} = \hat{n}\hat{\times}\hat{n} = (n^2)\hat{I}, \;\; \hat{n}\hat{/}\hat{m} = \hat{n}(/\hat{I})\hat{m}, \;\; \hat{n}^{\hat{1/2}} = n^{1/2}\hat{I}, etc. \tag{18}$$

In essence, Santilli discovered that *the abstract axioms of a numeric field do not necessarily require that the multiplicative unit be the millenary unit* 1 since said axioms merely require that the multiplicative unit be invertible and it is here assumed to be positive definite for isotopic theories.

It should be noted that the primary difficulty in using isonumbers in nuclear physics is of *conceptual,* rather than of mathematical nature because from realizations (15), *isonumbers represent volumes,* as expected for the needed representation of extended nucleons at all levels of study, thus including the number theory. For example, the isoreal isonumbers for the nonrelativistic representation of one nucleon are given by the following *volumes* on isospaces over isofields which, when projected into the Euclidean space, acquire the meaning of volumes (Fig. 2)

$$\hat{n} = n \times Diag.(n_1^2, n_2^2, n_3^2), \tag{19}$$

although, on grounds of abstract axions, there is no difference between n and $\hat{n}$ due to the positive-definiteness of the isounit.

The isonumbers for a nucleus with A nucleons are given by the following collection of overlapping volumes

$$\hat{n} = n \times \Pi_{\alpha=1,2,\ldots,A} Diag.(n_{1,\alpha}^2, n_{2,\alpha}^2, n_{3,\alpha}^2). \tag{20}$$

It should also be noted that the implications of Santilli's isonumber theory are not trivial. As an illustration, consider the isoreal isofield $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$ for which $\hat{I} = 3 \in F$. In this case $2\hat{\times}3 = 2$ and

$$\hat{1} = 3, \;\; \hat{2} = 6, \;\; \hat{3} = 9, \;\; \hat{5} = 15, \;\; \hat{7} = 21, \;\; etc., \tag{21}$$

by therefore suggesting that *the numeric values of prime numbers depend on the selected multiplicative unit.*

The above reformulation of prime numbers is not trivial because it has permitted the development and use of *isocryptograms,* (Appendix 2C, p. 84 of Ref. [30]) consisting of conventional cryptograms reformulated via isomathematics with a change of the unit such to require infinite time for the solution by hackers (see also Ref. [63]).

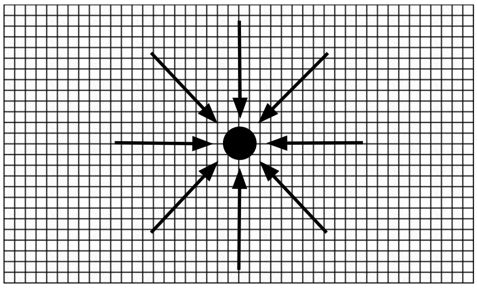

Figure 3: *In this figure, we illustrate a third basic notion of these studies, which is given by the representation in the nuclear force of pressure, density, temperature and other measurable quantities for nuclei conceived as physical media (generally called 'hadronic media') [31].*

3.6. Isofunction analysis

In quantum mechanics, all scalar quantities must have values in an ordinary field F. Corresponding quantities of isotopic theories must be *isoscalar,* that is, they must have values in the isofield $\hat{F}$. This means that all functions used in 20th century physics must be replaced by suitable *isofunctions* whose study was initiated with the *isoexponential isofunction* (Eq. (55), p. 171, Ref.[16])

$$\hat{e}^X = \hat{I} + X + X\hat{\times}X/\hat{1}! + ... = (e^{X\hat{T}})\hat{I} = \hat{I}(e^{\hat{T}X}). \tag{22}$$

Consequently from definition (16), ordinary coordinates $r = (x, r, z)$ representing a point of the 3-dimensional Euclidean space are mapped into *iso-coordinates* representing volumes

$$\hat{r} = r \times \hat{I} = r \times Diag(n_1^2, n_2^2, n_3^2). \tag{23}$$

This implies that Newtonian massive points r are replaced by *massive volumes* $\hat{r}$ (Fig. 2).

Similarly, functions $f(r)$ of local coordinates r must be completed into isofunctions $\hat{f}$ of isocoordinates $\hat{r}$, and in addition, have values in $\hat{F}$, thus suggesting the structure

$$\hat{f}(\hat{r}) = [f(\hat{r})]\hat{I}. \tag{24}$$

Interested readers should be aware that the elaboration of isotopic nuclear models studied in these papers requires the use of *isotrigonometric, isohyperbolic and other isofunctions* that cannot be reviewed here for brevity (see monographs [30] [31] and Refs. [64]-[66]).

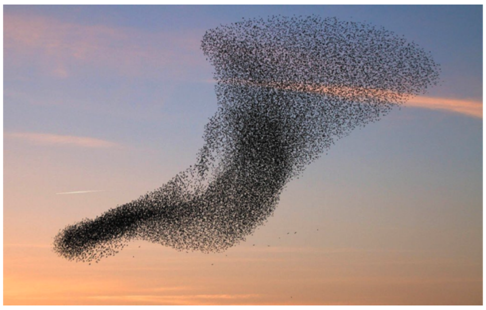

Figure 4: *This picture illustrates the isodifferential calculus [50] with birds flying in close formation without wing interferences. This flying formation can be best understood by assuming that birds conceive themselves as a volume encompassing their wings, rather than a mass concentrated in their center of gravity as it would be requested by the Newton-Leibnitz differential calculus [50]*

3.7. Isodifferential calculus

Following the construction of all possible isotopies of 20th century mathematics reviewed in the preceding sections, in the early 1990's hadronic mechanics was still missing a consistent isotopic formulation of the linear momentum, since the sole available formulation at that time was the familiar form from Eq. (6) $p|\psi\rangle = -i\partial_r|\psi\rangle$. This insufficiency implied the inability, at that time, of formulating the isotopies of the Schrödinger equations in a way compatible with all other aspects of isomathematics, with ensuing inability to formulate meaningful experiments (Fig. 4).

Studies of this impasse revealed that its ultimate origin rested in the conventional differential calculus due to its sole definition at isolated points, while the representation of the extended character of particles was needed for consistency at all levels, including the differential calculus.

The above insufficiencies left no other option than that of constructing the completion of the Newton-Leibnitz differential calculus, from its historical definition at isolated points r, to a formulation defined over *volumes* $\hat{r} = r\hat{I}$, which completion was finally achieved by Santilli in the 1996 special issue of Springer Nature *Rendiconti* [50] via the following:

Isoaxiom 3.7.1: *The isodifferential* $\hat{d}$ *of an isocoordinate* $\hat{r} = r\hat{I}$ *incudes all possible completions* $\hat{d}\hat{r}$ *of the conventional differential* dr *under the sole condition of recovering the latter at the limit when the isounit* $\hat{I}$ *recovers the conventional unit* 1

$$Lim_{\hat{I}=1}\hat{d}\hat{r} = dr, \tag{25}$$

with solution for the isodifferential [50]

$$\hat{d}\hat{r} = \hat{T}d\hat{r} = \hat{T}d[r\hat{I}(r,...)] = dr + r\hat{T}d\hat{I}, \tag{26}$$

and consequential solution for the isoderivative

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{27}$$

The discovery of the isodifferential calculus finally allowed the formulation of the isolinear isomomentum with the consequential consistent isotopies of Schrödinger equations presented in next section (for systematic studies on the isodifferential calculus, see the monumental works by the mathematician, S. Georgiev [29]).

3.8. Isospaces

Let $E(r^k, \delta_{ij}, I),\ i,j = 1,2,3$ be the three-dimensional Euclidean spaces in the (contravariant) coordinates r^k, over the field of real numbers $\mathcal{R}$ with invariant $r^2 = r^i\delta_{ij}r^j = r_1^2 + r_2^2 + r_3^2$. The infinite family of isotopic images of $E(r^k, \delta_{ij}, I)$ are given by the *iso-Euclidean isospaces* [26] [31]) $\hat{E}(\hat{r}^k, \hat{\Delta}_{ij}, \hat{I})$ characterized by the contravariant isotopic element

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}), \tag{28}$$

in which the exponential term of Eq. (15) has been imbedded in the n-characteristic quantities, with isocoordinates $\hat{r}^k = r^k\hat{I}$, *isometrics*

$$\hat{\Delta}^{ij} = \hat{\delta}^{ij}\hat{I}_{cr} = (\hat{T}^i_k\delta^{kj})\hat{I}, \tag{29}$$

and *isoinvariant*

$$\hat{r}^{\hat{2}} = \hat{r}^i\hat{\times}\hat{\Delta}_{ij}\hat{\times}\hat{r}^j = (r^i\hat{\delta}_{ij}r^j)\hat{I}_{cr} = (\frac{r_1^2}{n_1^2} + \frac{r_2^2}{n_2^2} + \frac{r_3^2}{n_3^2})\hat{I}, \tag{30}$$

where one should note that $\hat{\Delta} = \hat{\delta}\hat{I}$ is an *iso-metric,* namely, a metric whose elements are iso-numbers. Note also that the covariant isotopic element is the inverse of the contravariant expression (28).

Let $M(x, \eta, I)$ be the conventional Minkowski space over the field of real numbers $\mathcal{R}$ with contravariant spacetime coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = Diag.(1,1,1,-1)$, unit $I = Diag.(1,1,1,1)$ and invariants $x^2 = x^\mu\eta_{\mu\nu}x^\nu$ and $p^2 = p_\mu\eta^{\mu\nu}p_\nu$. The infinite family of *iso-Minkowskian isospaces* [21] [31] $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ are characterized by the infinite family of isotopic elements

$$\hat{T} = 1/\hat{I} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2}), \tag{31}$$

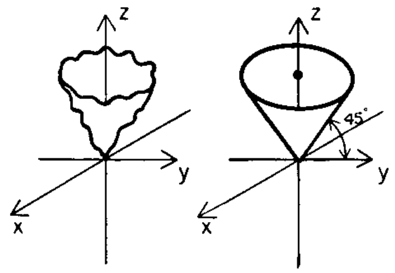

Figure 5: *In this figure, we illustrate a basic notion of the isotopies of the Lorentz symmetry (Sect. 3.10.1), that of providing the invariance of Lorentz's locally varying speed of light $C = c/n$ within physical media with density $n = n_4$ (left view), while reconstructing the universal invariance of the constant speed c on isospaces over isofields (right view) [3].*

isocoordinates $\hat{x} = x\hat{I}$, *isometrics*

$$\hat{\Gamma} = \hat{\eta}\hat{I} = (\hat{T}\eta)\hat{I}, \tag{32}$$

and *isoinvariants*

$$\hat{x}^{\hat{2}} = \hat{x}^{\mu}\hat{\times}\hat{\Gamma}_{\mu\nu}\hat{\times}\hat{x}^{\nu} = (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - \frac{x_4^2}{n_4^2})\hat{I}. \tag{33}$$

3.9. Isogeometries

Euclidean, Minkowskian, Riemannian, symplectic and other 20th century geometries are axiomatically based on the conventional associative product and their values are formulated over numeric fields. It is then evident to see that the lifting of the associative product into its isotopic form (9), with compatible lifting of number theory and functional analysis, implies non-trivial isotopies of 20th century geometries we cannot possibly review here (see Refs. [30] [31], independent studies [24], the recent monoigraph in iso- Euclidean geometry [67] and papers quoted therein).

For future needs, we limit ourselves to recall that the isometric (32) of the Iso-Minkowskian isospoace $\hat{M}$ coincides with the Riemannian metric $\omega(x)$, as a result of which:

3.9.1) The iso-Minkowskian geometry admits the complete machinery of the Riemannian geometry although expressed in terms if the isodifferential calculus [68].

3.9.2) The Riemamnnian metric $\omega(x)$ admits the factorization into the gravitational isotopic element (Santillian) here illustrated with the particular case of the

Schwartzschild's isotopic element [69]

$$\omega(x)_{Riem} = \hat{T}_{Sant} \times \eta_{Mink}, \quad \hat{T}_{Sant} > 0,$$
$$\hat{T}_{Schwar} = Diag.(\frac{1}{(1-\frac{2M}{r})}, \frac{1}{(1-\frac{2M}{r})}, \frac{1}{(1-\frac{2M}{r})}, 1 - \frac{2M}{r}), \tag{34}$$

where the positive definiteness $T_{Sant} > 0$ is guaranteed by the factorization of the Minkowski metric.

3.9.3) Despite its dependence on iso-coordinates, the iso-Minkowskian geometry is *isoflat,* that is, with null curvature when formulated on isospaces over isofields, because the lifting of the Minkowski metric $\eta \to \omega(x) = T(x) \times \eta$ is compensated by the *inverse* lifting of the Riemannian unit $I_{4\times4} \to \hat{I}_{Sant} = 1/\hat{T}_{Sant}$.

3.9.4) Relativistic isomechanics characterized by the isounit $\hat{I} = 1/\hat{T}_{Sant}$ provides an axiomatically consistent unification of Einstein's field equations and quantum mechanical axioms. [70].

3.9.5) In view of the absence of curvature, the above identical reformulation of Einstein's field equations provides an axiomatically consistent unification of gravitation and electroweek interactions [70] studied in detail in the Springer Nature monograph [71].

3.10. Isosymmetries

The main methods used in *nonrelativistic hadronic mechanics* are given by the *isotopies of the Galilean symmetry and relativity* first presented in Chapter 6, p. 199 on of monograph [16] with various applications in pages 253-267. Said isotopies were then studied in the1991 monographs [72] [73] (see also notes [21] from Santilli's 1991 lectures at the ICTP, Trieste, Italy, and review [28]).

The main conceptual and technical difference between the conventional Galileo symmetry and its isotopic image is that the former symmetry is based on the Newtonian notion of *massive points,* as a consequence of which no resistance can be admitted during the free fall in our atmosphere. By contrast, the iso-Galilean symmetry represents the dynamics of *massive volumes* moving within physical media with ensuing resistive and other NSA forces represented via the isounit of the theory in such a way to recover the conventional Galilean symmetry at the limit $\hat{I} \to I$.

Regrettably, we cannot review these Galilean studies to prevent excessive length of this paper and refer the interested reader to monographs [72] [73] and independent reviews [21] [28].

The main methods used for *relativistic hadronic mechanics* are given by the *isotopies of spacetime symmetries* which include: 1) The isotopies $\hat{SO}(3)$ of the rotational symmetry [74]-[76]; 2) The isotopies $\hat{SU}(2)$ of the spin symmetry

$SU(2)$[77]; 3) The isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ in classical [78] and operator [79] formulation; 4) The isotopies $\hat{P}(3.1)$ of the Lorentz-Poincaré symmetry $P(3.1)$ [80]; 5) The isotopies $\hat{\mathcal{P}}(3.1)$ of the spinorial symmetry $\hat{P}(3.1)$ [81]; 6) The isotopies of the various discrete spacetime symmetries [31]; 7) The isotopies $\hat{M}(3.1)$ of the Minkowskian geometry $M(3.1)$ [82].

To render this paper minimally self-sufficient, let us outline the isotopies of the Lorentz and Poincaré symmetries which are needed for the representation of the Deuteron data of Section 5.

3.10.1. Iso-Lorentz symmetry. As it is well known to historians, H. Lorentz attempted in 1904 the construction of the invariance of the speed of light within physical media $C = c/n$ where n is the local density of the medium considered, but failed to do so and had to restrict his study to the invariance of the constant speed of light *in vacuum* c. The latter symmetry was characterized by the unitary irreducible representations of the Lie group $SO(3.1)$, which are the justly celebrated *Lorentz transformations*. It is unfortunate for science that Lorentz's crucial words *in vacuum* were generally ignored in the 20th century physics, resulting in the widespread belief that the speed of light is a universal constant c throughout all possible conditions existing in the universe. A list of inconsistencies for such an assumption is available in Sect. 8.4.4., p. 134 of the overview [3] and a decade of experiments in the U.S.A. and in Europe establishing Lorentz's local value $C = c/n$ of the speed of light in a physical medium, such as our atmosphere, are presented in Ref. [**?**] and papers quoted therein.

Santilli pointed out in the 1983 Springer-Verlag monograph that the restriction of the invariance to the constant speed c was due to the Hamiltonian character of Lie's theory, while Lorentz's problem was highly non-Hamiltonian due to the complex functional dependence (11) of the index of refraction n. For this reason, Santilli constructed in Ref. [16] the Lie-isotopic completion of Lie's theory for non-Hamiltonian systems, and achieved in the 1983 paper [78] the universal invariance of Lorentz's speed of light (34), which is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli isosymmetry $\hat{SO}(3.1)$ of isoinvariant (33), today known as the *Lorentz-Santilli isosymmetry.* It's isoalgebra $\hat{so}(3.1)$ is characterized by the isogenerators (Eq. (10), p. 550 of [78] expressed in terms of the characteristic quantities $b_\mu = 1/n_\mu$)

$$\hat{J}_1 = n_2 n_3 J_1, \quad \hat{J}_2 = n_1 n_3 J_2, \quad \hat{J}_3 = n_1 n_2 J_3, \qquad (35)$$
$$\hat{M}_k = n_k M_k,$$

where J_k, M_k are the conventional generators of $so(3.1)$, with isocommutation rules (Eq. (12), p. 551 of [78])

$$[\hat{J}_i \hat{,} \hat{J}_j] = \epsilon_{ijk}\hat{J}_k, \quad [\hat{M}_i \hat{,} M_j] = C^2\epsilon_{ijk}\hat{J}_k,$$
$$[\hat{J}_i \hat{,} M_j] = -\epsilon_{ijk}\hat{M}_k, \tag{36}$$

and *iso-Casimir isoinvariant* (Eq. (13), p. 551 of Ref. [78])

$$\hat{C}_1 = \hat{J}^{\hat{2}} - \frac{1}{C}\hat{M}^{\hat{2}} = -3\hat{I}, \quad \hat{C}_2 = \hat{\mathbf{J}}\hat{\times}\hat{\mathbf{M}} = 0. \tag{37}$$

A main feature of the Lorentz-Santilli isosymmetry at the abstract level is that of being identical to the conventional Lorentz transformations when formulated on an iso-Minkowskian isospace $\hat{M}(\hat{x}, \hat{\Delta}, \hat{I})$ over the isoreal isofield $\hat{\mathcal{R}}$, in which case the speed of light is indeed the universal constant c (Fig. 5). The invariance of the Lorentz speed $C = c/n$ occurs in the *projection* of the isosymmetry on the conventional Minkowski space $M(x, \eta, I)$ over the field of real numbers $\mathcal{C}$, e.g., for the case of the *isoboosts* (Eq. (15), p. 551 of Ref. [78] for isotransformations in the $(3,4)$-plane, and Chapter 8, p.329 on of Ref. [31] for a general treatment)

$$x^{1'} = x^1, \;\; x^{2'} = x^2,$$
$$x^{3'} = \hat{\gamma}(x^3 - \hat{\beta}\tfrac{n_3}{n_4}x^4), \quad x^{4'} = \hat{\gamma}(x^4 - \hat{\beta}\tfrac{n_4}{n_3}x^3), \tag{38}$$
$$C = \tfrac{c}{n_4} \; = \; invariant,$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \;\; \hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}. \tag{39}$$

It should be indicated that the Lorentz-Santilli isosymmetry leaves invariant the local speed of light $C = c/n_4$, with ensuing wiggly-shaped light cone (left of Fig. 5), only when the isosymmetry is *projected* into the conventional spacetime over the conventional field of real numbers, as it is the case for isotransformations (38). By contrast, the isosymmetry $\hat{SO}(3.1)$ restores the perfect light cone in the isospacetime over the isoreal isofied (right of Fig. 5). We can therefore say that the speed of light c is indeed a *universal constant,* but only on isospacetime over the isoreal isofield, while it is a local variable in the physical realty.

It should be indicated that *isomathematics introduces no restrictions on the density of the medium* besides that of positive-definiteness, $n_4 > 0$ which, as such, can be smaller or greater than one. Consequently, the Lorentz-Santilli isosymmetry confirms Lorentz's local speed of light for transparent media of low density for which $C = c/n4 < c, \;\; n_4 > 1$ and predicts superluminal speeds for physical particles within hyperdense media in the interior of hadronic, nuclei and stars for which $C = c/n_4 > c, \;\; n_4 < 1$ [83] [84] [85], a prediction that has been confirmed by all exact and time invariant representations to date of all characteristics of unstable particles with physical constituents (Sect. 5).

Intriguingly, the Lorentz-Santilli isosymmetry has essentially proved that *the abstract axioms of special relativity do indeed admit arbitrary speeds,* with a consequential considerable increase of its representational capability [86].

Note finally that the Lorentz-Santilli isosymmetry $\hat{SO}(3.1)$ is *irregular* because the structure quantities are isofunctions (Sect. 3.3), thus constituting a new realization of Lie's axioms.

As it is the case for the symmetries, the axioms of special relativity and its isotopic extension, called *isospecial relativity*, coincide at the abstract level, since physical differences solely occur in specific realizations. Regrettably, we are unable to review this additional profile for brevity and have to refer the interested reader to monographs [72] [73] [30] [31], the 2021 upgrade [3] and independent reviews [21] [28].

3.10.2. Iso-Poincaré symmetry. The second basic isosymmetry of relativistic hadronic mechanics is given by the isotopies $\hat{P}(3.1)$ of the lorentz-Poincaré symmetry $P(3.1)$ introduced by Santilli in the 1993 paper [80] [81] (see also Sect. 8.3, p. 342 on of Ref. [16] and independent studies [21]), today called the *Lorentz-Poincaré-Santilli isosymmetry.* It's isoalgebra with Hermitean generators $(J_{\mu\nu} = (J_k, M_k)$ is characterized by the isocommutation rules where

$$\begin{gathered}[J_{\mu\nu}\hat{,}J_{\alpha\beta}] == \imath(\hat{\eta}_{\nu\alpha}J_{\beta\mu} - \hat{\eta}_{\mu\alpha}J_{\beta\nu} - \hat{\eta}_{\nu\beta}J_{\alpha\mu} + \hat{\eta}_{\mu\beta}J_{\alpha\nu}),\\ [J_{\mu\nu}\hat{,}P_{\alpha}] = i(\hat{\eta}_{\mu\alpha}P_{\nu} - \hat{\eta}_{\nu\alpha}P_{\mu}), \quad [P_{\mu}\hat{,}P_{\nu}] = 0,\end{gathered} \tag{40}$$

and *iso-Casimir-isoinvariants*

$$\begin{aligned}\hat{C}_1 &= \hat{I}(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) > 0,\\ \hat{C}_2 &= \hat{P}^{\hat{2}} = \hat{P}_{\mu}\hat{\times}\hat{P}^{\mu} = (\hat{\eta}_{\mu\nu}P^{\mu}\hat{T}P^{\nu})\hat{I},\end{aligned} \tag{41}$$

$$\hat{C}_3 = \hat{W}^{\hat{2}} = \hat{W}_{\mu}\hat{\times}\hat{W}^{\mu}, \;\; \hat{W} = W\hat{I}, \;\; \hat{W}_{\mu} = \hat{\epsilon}_{\mu\alpha\beta\rho} \star J^{\alpha\beta}\hat{\times}P^{\rho}.$$

We should note that the isounit of isoinvariant $\hat{C}_2$ is the inverse as that of the Lorentz-Santilli isosymmetry due to the *covariant* nature of the considers iso-Casimir invariant. We should also note that isomomenta isocommute on isospaces over isofields, but they do not commute when they are projected on conventional spaces over conventional field, thus confirming the nonlinearity of the Lie-Santilli isotheory.

As it is well known, a *particle* can be assumed to be physical according to 20th century physics (that is, existing and experimentally detectable in our spacetime vacuum) if and only if it is represented by a unitary irreducible representation of the Poincaré symmetry $P(3.1)$.

An important feature identified since the 1978 Harvard University memoir [43], and confirmed in subsequent works [3], is that in the transition from motion in vacuum to motion within a hadronic medium (such as a hadron, a nucleus or a star), *an extended particle generally experiences alterations of its intrinsic characteristics* originally called *mutations* and nowadays referred to as the characteristics of *isoparticles* [87].

We shall therefore say that a particle within a hadronic medium is considered to be extended in these papers if and only if it is an isoirreducible isounitary isorepresentation of the Lorentz-Poincaré-Santillii isosymmetry $\hat{P}(3.1)$. As we shall see, the mutations of the intrinsic characteristics of particles appears to be dependent on the density of the hadronic medium, with minimal mutations (if any) in the interior of mesons and maximal mutations in the core of stars or in the interior of gravitational collapse. Intriguingly, the indicated mutations appear ti be deeply linked with, or complementary to the expected deviations from Heisenberg's uncertainty principle for extended particles in the interior of hadronic media [46].

Note that the above isosymmetries are sufficient for the representation of the Deuteron data studied in this paper, but they need a spinorial extension for the studies of the subsequent paper.

3.10.3. Isorelativities. In these papers we gave assumed since the Introduction of Paper I that the Galilean and the special relativity are exactly valid for point-like particles in vacuum under Hamiltonian (thus including electromagnetic and weak) interactions, although they need a suitable completions for extended particles within physical media under Hamiltonian and non-Hamiltonian (thus including strong) interactions. Their study was initiated in 1983 [16] (see Chapter 6), continued in monographs [72] [73] [30] [31] and upgraded in the 2021 work [3] (see independent studies [21] [24] [28]).

By recalling from the preceding sections the isomorphism between conventional space time symmetries and their isotopic formulation, the primary intent for the EPR completion of the Galilean, special and general relativities, collectively known as *isorelativities,* is that of *preserving the original basic axioms and merely introducing broader realizations,* by therefore enlarging their representational capabilities.

3.10.4. Experimental verifications. Let us indicated the generally ignored experimental verifications of isorelativities for:

1) The invariant representation of the the Bose-Einstein correlation [88] [89] with physical characteristics of the proton-antiproton fireball (dimension, shape and density) rather than with unknown "chaoticity parameters".

2) The representation of all characteristics, including their anomalies of muons [35] [90] [91] [92], kaons [93] [94] [95], and unstable baryons [96] via hadronic structure models with physical constituents generally produced free in the spontaneous decays with the lowest modes.

3) The representation of all characteristics of the neutron in its synthesis from the electron and the proton in the core of stars at the non-relativistic and relativistic levels [97] and its experimental verification [98] and generic cases in generic cases in classical mechanics [3] (see the subsequent paper).

3.11. Simple construction of isotopic theories

As indicated in more details in Sect. 4, in practical applications of isotopic methods in nuclear physics, *the Hamiltonian $H(r,p) > 0$ represents all possible linear, local and potential interactions, while the Santillian $\hat{T} = 1/\hat{I} > 0$, represents all possible nonlinear, nonlocal and nonpotential interactions expected from the extended character of nucleons in conditions of partial mutual penetration* (Fig. I.1).

It then follows that, despite their apparent mathematical complexity, all aspects of regular (Sect. 3.3) isotopic formulations, thus including isomathematics, isomechanics and all their applications, can be constructed via the following simple *nonunitary transformation of the quantum mechanical unit $1 = \hbar$ defined on conventional spaces over conventional fields* [99]

$$UU^{\dagger} = \hat{I} > 0, \quad \hat{T} = 1/\hat{I} = (UU^{\dagger})^{-1} > 0, \tag{42}$$

which transformation merely *completes* quantum mechanical models with the representation of the extended character of nucleons and their non-Hamiltonian interactions, provided that said transformation is applied to the *totality* of quantum mechanical quantities and all their operations without any known exception.

In fact, the above procedure maps the quantum mechanical unit, product, Lie algebras, etc., into their isotopic images, according to the simple rules ($1 = \hbar$)

$$\hbar = 1 \ \rightarrow \ U1U^{\dagger} = \hat{I} \neq 1,$$

$$AB \ \rightarrow \ U(AB)U^{\dagger} = (UAU^{\dagger})(UU^{\dagger})^{-1}(UBU^{\dagger}) = \hat{A}\hat{T}\hat{B}, \tag{43}$$

$$[X_i, X_j] \rightarrow U[X_i, X_j]U^{\dagger} = [\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i\hat{T}\hat{X}_j - \hat{X}_j\hat{T}\hat{X}_i = \hat{C}_{ij}^k \hat{\times} \hat{X}_k,$$

where one should note that the transformations are defined on conventional spaces over conventional fields and all products are conventional. Therefore Eqs. (43) characterize *quantum mechanical (rather than hadronic) transformations.*

Note the necessity of abandoning the century-old notion of quantized energy exchanged for point-like electrons in vacuum when dealing with extended nucleons in conditions of mutual penetration (see the experimental works [47] suggesting integro-differential energy exchanges for complex/extended structures and applications in Sect. 5).

Note also that, once quantum mechanical models have been completed via the above simple procedure, when remaininhg within the quantum mechanical formalism, the resulting new models are afflicted by a number of inconsstencies which are typical of all nonunitary theories on conventional spaces over conventional fields, such as:

3.11.1) Lack of invariance of the observables under additional nonunitary transformations $WW^\dagger \neq I$, including the lack of invariance of the isounit, isoproduct, etc.

$$\begin{gathered} \hat{I} \ \rightarrow \ W\hat{I}W^\dagger = \hat{I}' \neq I, \\ \hat{A}\hat{T}\hat{B} \rightarrow W(\hat{A}\hat{T}\hat{B})W^\dagger = \hat{A}'\hat{T}'\hat{B}', \quad \hat{T}' \neq \hat{T}, \end{gathered} \tag{44}$$

with consequential lack of invariance of numeric predictions.

3.11.2) Consequential general lack of conservation of Hermiticity/observability, Eqs. (6), $U(\langle\psi H|\psi\rangle)U^\dagger \neq [U(\langle\psi|H\psi\rangle)U^\dagger]$.

3.11.3) General loss of causality, including the possible admission of solutions in which the effect precedes the cause (see also the analysis by L. Biedenharn [100] on the violation of causality for nonunitarily achieved, half-odd-integer eingenvalues of the conventional $SO(3)$ orbital symmetry). Additional problematic aspects of nonunitary theories formulated over a conventional numeric field are addressed in the so-called *Theorems of catastrophic inconsistencies of nonunitary theories* (see p. 451 on of [101]).

The axiomatic origin of the above inconsistencies is that their basic product is the *conventional* associative product (1), rather than isoproduct (9), with ensuing nonconserved basic unit 1, loss under time evolution of the basic numeric field and other insufficiencies.

3.12. Invariance of isotopic formulations

Problems 3.11.1-3.11.3 were resolved in 1998 by Santilli [99] via the identification of the infinite class of equivalence of the dynamical equations of isotopic theories (Sect. 4) which is given by the completion of unitary law (4) into the *isounitary law*

$$\hat{W}\hat{\times}\hat{W}^\dagger = \hat{W}^\dagger\hat{\times}\hat{W} = \hat{I}, \tag{45}$$

completed by the identical reformulation of transformations (43) into the isounitary form

$$\begin{gathered}
U = \hat{W}T^{1/2}, \\
UU^{\dagger} = \hat{I} \rightarrow \hat{W}\hat{\times}\hat{W}^{\dagger} = \hat{W}^{\dagger}\hat{\times}\hat{W} = \hat{I}, \\
\hat{I} \rightarrow \hat{W}\hat{\times}\hat{I}\hat{\times}\hat{W}^{\dagger} = \hat{I}' \equiv \hat{I}, \\
\hat{A}\hat{\times}\hat{B} \rightarrow \hat{W}\hat{\times}(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{W}^{\dagger} = \\
= (\hat{W}\hat{\times}\hat{A}\hat{\times}\hat{W}^{\dagger})\hat{\times}\hat{W}^{\dagger,-\hat{1}}\hat{\times}\hat{W}^{-\hat{1}}\hat{\times}(\hat{W}^{\dagger}\hat{\times}\hat{B}\hat{\times}\hat{W}^{\dagger}) = \\
= \hat{A}'\hat{\times}\hat{B}' = \hat{A}'\hat{T}\hat{B}', (UU^{\dagger})^{-1},
\end{gathered} \tag{46}$$

(where we have used the isotopoic inverse $W^{-\hat{1}}$ [30]) with consequential invariance of isotopic formulations.

It should be indicated that isoinvariance (46) [99] embodies the very essence of hadronic mechanics as an *axiom-preserving image of quantum mechanics* because:

3.12A) Quantum mechanical *unitary* transformations on $\mathcal{H}$ over $\mathcal{C}$ leave the product $\times$ numerically invariant, i.e., $U \times (A \times B) \times U^{\dagger} = A' \times B',\ U \times U^{\dagger} = 1$.

3.12B) Quantum mechanical *nonunitary* transformations on $\mathcal{H}$ over $\mathcal{C}$ change the numeric value of the product $\times$, Eq. (43), i.e., $W \times (A \times B) \times W^{\dagger} = A'\hat{T}B',\ \times \rightarrow \hat{T} = (W \times W^{\dagger})^{-1},\ W \times W^{\dagger} \neq 1$.

3.12C) Isomechanical *isounitary* transformations on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ leave numerically invariant the issoproduct $\hat{\times}$, Eq. (46), i.e., $\hat{W}\hat{\times}(\hat{A}\hat{\times}\hat{B})\hat{\times}\hat{W}^{\dagger} = \hat{W}\hat{\times}(\hat{A}\hat{T}\hat{B})$-$\hat{\times}\hat{W}^{\dagger} = A'\hat{\times}B' = A'\hat{T}B',\ \hat{W} = W\hat{T}^{-1/2}, \hat{W}\hat{\times}\hat{W}^{\dagger} = \hat{W}^{\dagger}\hat{\times}\hat{W} = \hat{1}$.

Note that the isotopies of spaces and numeric fields. thus of the entire 20th century appleid mathematics, are *necessary* to achieve isoinvariance (46).

The axiomatically and physically important significance of transformations 3.12A and 3.12C is that they coincide at the abstract, realization-free level to such an extent that *quantum mechanics and the Lie-isotopic branch of hadronic mechanics can be expressed via the same equations, only subjected to different realizations of the associative produc,t* to such an extent that the lack of formal identity of quantum and isotopoic expressions is due to the insufficient knowledge and/or use of isomathematics.

It is easy to see that the above reformulation resolves insufficiencies 3.11.1-3.11.3 because: expressions (46) imply the *numerical— invariance of the isounit $\hat{I}$ and of the Santillian $\hat{T}$*, with ensuing invariance of the numerical predictions of the theory; such an invariance implies that all observables of quantum mechanics remain observable in hadronic mechanics, e.g., $H^{\dagger} \equiv H^{\hat{\dagger}}$ and isotopic formulations verify the causality laws of quantum mechanics (see the fully causal isoorbital angular momentum with half-odd-integer isoeigenvalues of Ratherford's electron compressed inside the proton in the core of stars of Sect. 5.3) [30] [31].

On mathematical grounds, the invariance of numeric predictions, the conservation of Hermiticity under unitary transformations and the verification of causality laws are ultimately reducible to the abstract associative enveloping algebra, thus having an assured validity for conventional as well as isotopic realization of said envelope.

On pedagogical grounds, all applications of isotopic formulations in nuclear physics to date (including those of Sect. 5) have been derived via conventional nonunitary transformations (42) (43) and thenir reformulation in terms of isomathematics to prevent the indicated inconsistency problems.

Alternatively we can say that applications of isotopic theories in physics can be initially done via simple rules (42) (43) because they represent the *projection* on conventional spaces over conventional fields of covering formulations (45) (46) on isospaces over isofields.

It should be indicated that this author has received various claims of inconsistency of hadronic mechanics via arguments based on product (1) with basic unit 1, which claims are essentially equivalent to the early 20th century claims of inconsistency of quantum mechanics based on Newtonian mechanics.

3.13. Inconsistencies of deformed Lorentz transformations

It should be indicated that decades following the 1983 isotopies of the Lorentz symmetries [78], iso-Lorentz transformations (38) have been identically used under the name of 'deformed Lorentz transformations' but formulated over a conventional numeric field' (see books [102] [103] and literature quoted therein). As an example, the 2005 deformed Lorentz transformations (4.9), p. 23 of [102] coincide with the 1983 isotransformations (15), p. 351 of [78] and they coincide with Eqs. (38) with $b_\mu = 1/n_\mu$, although without the quotation, specifically, of their origination [78], contrary to admission of said knowledge in preceding works [89] [93] [94] [104]. Similarly, one can see that the 2005 principles of deformed special relativity [102] coincide with the 1991 isoaxioms of special isorelativitry [73].

Readers should be aware that deformed Lorentz transformations and special relativity formulated on conventional numeric fields verify the *Theorems of catastrophic inconsistencies* of Ref. [101]) with particular reference to the lack of invariance over time of the numerical predictions and consequential lack of meaningful application to experiments.

This occurrence illustrates the importance of formulating nonunitary completions of quantum mechanics over isospaces and isofields due to their reconstruction of unitarity, Eq. (45).

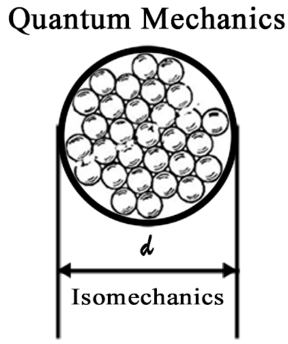

Figure 6: *In this figure, we illustrate that the application of hadronic mechanics (hm) in nuclear physics is restricted to hold in the interior of a sphere of radius $R_{hm} = 10^{-13}$ cm, called the 'nuclear hadronic horizon' in representation of strong nuclear forces, while uniquely and unambiguously recovering quantum mechanics outside said range. The understanding is that, in general, the continuous and instantaneous fifth interactions represented by Santillian (15) hold at arbitrary distances in view of the EPR entanglement [34].*

4 Elements of isomechanics

In order to render this and the following papers minimally self-sufficient, in this section we review, update and specialize to nuclear physics the *Lie-isotopic branch of hadronic mechanics,* also known as *isomechanics,* first proposed in the 1978-1983 Springer-Verlag monographs [15] [16] and subsequently studied by various authors (see the 1995 general treatment [30] [31], monographs [21] [29] and papers quoted therein).

4.1. Iso-Newtonian mechanics

It is important for these papers to indicate that the assumed realization of strong nuclear forces as the most general possible combination of potential/SA (Definition 3.1) and nonpotential/NSA forces (Definition 3.2) originates at the purely Newtonian level, carries over at the operator level and permits the numerically exact and time invariant representation of the experimental data of the Deuteron while preserving the abstract axioms of quantum mechanics (Sect. 2, p. 27 on of Ref. [50]).

Consider the historical Newton's equations for N "massive points" (in Newton's language) on a conventional 3-dimensional Euclidean space $E(r, \delta, I)$

$$m_a \frac{dv_{ak}}{dt} - F_{ak}(t, r, v, ...) = 0, \quad k = 1, 2, 3, \quad a = 1, 2, ..., N, \tag{47}$$

which, under the conditions of variational selfadjointess [15] can be written

$$m_a \frac{dv_{ak}}{dt} - F^{SA}_{ak}(r, v) - F^{NSA}_{ak}(t, r, v, ...) = 0. \tag{48}$$

It was generally believed for centuries that the above equations can solely represent open nonconservative systems generally illustrated via Newton's nonpotential velocity-dependent forces of the type $F^{NSA} = k_1 v + k_2 v^2, ...,$. Eqs. (6.3.6), p. 236 of Ref. [16] introduced the new notion of *closed NS systems,* which are given by systems (39) for $N \geq 2$ verifying all ten conservation laws of Galileo relativity under the conditions

$$\sum\nolimits_{ak} F^{NSA}_{ak} = 0, \ \sum\nolimits_{ak} \mathbf{r} \cdot \mathbf{F}^{NSA} = 0, \ \sum\nolimits_{ak} \mathbf{r} \wedge \mathbf{F}^{NSA} = 0. \tag{49}$$

The main equations of isomechanics are given by the formulation of Newton's equations (39) on iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ (Sect. 3.6) over isoreal isonumbers $\hat{\mathcal{R}}$ (Sect. 3.4) first achieved in Eqs. (2.5), p. 31 of Ref. [50] following the discovery of the isodifferential calculus (Sect. 3.7), and nowadays called *Newton-Santilli isoequations* [21] [28],

$$\hat{m}_a \hat{\times} \frac{\hat{d}\hat{v}_{ak}}{\hat{d}\hat{t}} - F^{SA}_{ak}(\hat{r}, \hat{v}) = 0. \tag{50}$$

The main features of Eqs. (41), hereon considered for the particular case $\hat{t} = t, \ \hat{I}_t = 1$, are the following:

1) By recalling that the isodifferential calculus coincides at the abstract level with the conventional differential calculus, iso-Newton's equation (41) coincide at the abstract level with the Newton's equations (39) with $F^{NSA} = 0$.

2) Eqs. (41) provide the first known, axiomatically consistent representation of the actual shapes and dimensions of the particles considered thanks to their representation via the isodifferential calculus with realization of the isotopic element of type (15).

3) Eqs. (41) are *directly universal* for NSA systems [31] in the sense of representing all possible NSA systems directly in the r-coordinate system of the experimentalist, thus without any needs for the transformation theory. In fact, by writing explicitly the isoderivative $\hat{d}\hat{v}_{ak}/\hat{d}\hat{t}$, we reach the representation of NSA forces in terms of isotopic element (15)

$$F^{NSA}(t, r, v, ...) = mv\hat{T}\frac{d\hat{I}}{dt}, \tag{51}$$

which always admit an isotopic element T for any given (nonsingular) NSA force.

4.2. Iso-Lagrange and iso-Hamilton mechanics

A primary motivation for developing the isodifferential calculus has been the reformulation of NSA Newton's equation in such a way to admit the following *iso-variational principle* (Sect. 2.4, p. 43 on of Ref. [50]), here written for simplicity for the case of one particle with evident extension to many particles

$$\begin{gathered}\hat{\delta}A = \hat{\delta}\hat{\int}(\hat{p}\hat{\times}\hat{d}\hat{r} - \hat{H}\hat{\times}\hat{d}\hat{t}) = \\ = \hat{\delta}\hat{\int}[(p\hat{I}_r)\hat{T}_r\hat{d}\hat{r} - (H\hat{I}_t)T_t\hat{d}\hat{t}] = \hat{\delta}\hat{\int}(pT_r d\hat{r} - HT_t d\hat{t}) = 0,\end{gathered} \tag{52}$$

because important for operator images.

The above principle has permitted the formulation of the following main equations of *classical isomechanics* [50]:

i) The *iso-Hamilton equations*

$$\frac{\hat{d}\hat{r}}{\hat{d}\hat{t}} = \frac{\hat{\partial}H(\hat{r},\hat{p})}{\hat{\partial}\hat{p}}, \quad \frac{\hat{d}\hat{p}}{\hat{d}\hat{t}} = -\frac{\hat{\partial}H(\hat{r},\hat{p})}{\hat{\partial}\hat{r}}, \tag{53}$$

(where we have assumed $\hat{H} = H\hat{I}_t \equiv H$) with time evolution for a quantity Q

$$\frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [Q\hat{,}H] = \frac{\hat{\partial}Q}{\hat{\partial}\hat{r}}\frac{\hat{\partial}H}{\hat{\partial}\hat{p}} - \frac{\hat{\partial}H}{\hat{\partial}\hat{p}}\frac{\hat{\partial}Q}{\hat{\partial}\hat{r}}, \tag{54}$$

where the brackets $[Q\hat{,}H]$ constitute a classical realization of Lie-Santilli isoalgebras [16] [22].

ii) The *iso-Lagrange isoequations*

$$\frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}L(\hat{r},\hat{v})}{\hat{\partial}\hat{v}} - \frac{\hat{\partial}\hat{L}(\hat{r},\hat{v})}{\hat{\partial}\hat{r}} = 0. \tag{55}$$

iii) The *iso-Hamilton-Jacobi equations* also called *Hamilton-Jacobi-Santilli isoequations*

$$\frac{\hat{\partial}A}{\hat{\partial}\hat{t}} + H = 0\ (47a), \quad \frac{\hat{\partial}A}{\hat{\partial}\hat{r}} - \hat{p} = 0\ \ (47b), \quad \frac{\hat{\partial}A}{\hat{\partial}\hat{p}} = 0\ \ (47c). \tag{56}$$

It should be indicated that the above analytic equations represent the *external terms* of the historical Hamilton's and Lagrange's equations via the embedding of said terms in the isodifferential calculus, as one can see for the simple case with $\hat{t} = t,\ \hat{I}_t = 1,\ \hat{r} = r, \hat{I}_r = 1$ and $\hat{I}_v$ representing the external terms via equations of type (50).

4.3. Iso-Schrödinger representation

The isotopic lifting of the conventional naive quantization [105] characterizes the following map of the (classical) iso-Hamiltonian mechanics into the (operator) isomechanics

$$\hat{A} = \hat{\int}(\hat{p}\hat{\times}\hat{d}\hat{r} - \hat{H}\hat{\times}\hat{d}\hat{t}) \rightarrow -\hat{i}\hat{\times}\hat{log}\hat{\psi}(\hat{t},\hat{r}) = -i\hat{I}log|\psi(\hat{t},\hat{r})\rangle, \tag{57}$$

where we use the notion of *isolog* $\hat{log}\hat{\psi} = \hat{I}\log\psi$ [30]. Note that map (57) is restricted in nuclear physics to hold within a radius $R_{hm} = 10^{-13}\ cm$ called the *nuclear hadronic horizon* in representation of strong nuclear forces, while uniquely and unambiguously recovering quantum mechanics outside said range (Fig. 6).

The resulting *iso-Schrödinger representation,* also known as the *Schrödinger-Santilli isorepresentation,* is characterized by:

1) The *iso-Schrödinger equation* here derived from the iso-Hamilton-Jacobi Eq. (47a) (see Chapter 5, p. 182 of Ref. [31] for a general treatment)

$$\begin{aligned} -\hat{i}\hat{\times}\tfrac{\hat{\partial}}{\hat{\partial}\hat{t}}|\hat{\psi}(\hat{t},\hat{r})\rangle = \hat{H}\hat{\times}|\hat{\psi}(\hat{t},\hat{r})\rangle = \\ = [\tfrac{1}{2m}\hat{p}\hat{\times}\hat{p} + \hat{V}(r)]\hat{T}(\hat{\psi},...)|\hat{\psi}(\hat{t},\hat{r})\rangle = \hat{E}\hat{\times}|\hat{\psi}(\hat{t},\hat{r})\rangle = E\times|\hat{\psi}(\hat{t},\hat{r})\rangle, \end{aligned} \tag{58}$$

where we should recall our assumption $\hat{H} = H\hat{I}_t \equiv H$ for our assumption that, in first approximation, $\hat{t} = t,\ \hat{I}_t = 1$, the last identity trivially follows from the value $\hat{E} = E\hat{I}$, and in case the isostates $\hat{|}\hat{\psi}(\hat{t},\hat{r})\rangle$ are k-dimensional, the isotopic element $\hat{t}$ must be a $k\times k$-isomatrix to avoid evident inconsistencies.

2) The *isolinear isomomentum* from Eq. (47b)

$$\hat{p}\hat{\times}\psi(\hat{r}) = -\hat{i}\hat{\times}\hat{\partial}_r\hat{\psi}(\hat{r}) = -i\hat{I}\partial_{\hat{r}}\hat{\psi}(\hat{r}), \tag{59}$$

with ensuing *isocanonical isocommutation rules*

$$[\hat{r}_i\hat{,}\hat{p}_j]\hat{\times}|\hat{\psi}\rangle = -\hat{i}\hat{\times}\hat{\delta}_{i.j}\hat{\times}|\hat{\psi}\rangle = -i\delta_{i.j}|\hat{\psi}\rangle, [\hat{r}_i\hat{,}\hat{r}_j]|\hat{\psi}\rangle = [\hat{p}_i\hat{,}\hat{p}_j]|\hat{\psi}\rangle = 0. \tag{60}$$

3) The independence of the isowavefunction from the isolinear isomomentum from Eq. (47c)

$$\hat{\partial}_{\hat{p}}\hat{\psi} = 0. \tag{61}$$

The above naive isoquantization is formulated on the *Hilbert-Myung-Santilli isospace* [106] on an isocomplex isofield $\hat{\mathcal{C}}$ (Sect. 3.4) with: *isostates* $|\hat{\psi}\rangle$, *isonormalization*

$$\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle = \langle\hat{\psi}|\hat{T}|\hat{\psi}\rangle = \hat{I}, \tag{62}$$

(where different isonormalizations per different applications should be kept in mind); *isoexpectation isovalues* of an iso-Hermitean operator $\hat{A}$,

$$\hat{\langle}\hat{A}\hat{\rangle} = \langle\hat{\psi}|\hat{\times}\hat{A}\hat{\times}|\hat{\psi}\rangle, \tag{63}$$

and *isoidentity*

$$\hat{\langle}\hat{I}\hat{\rangle} = \langle\hat{\psi}|\hat{\times}\hat{I}\hat{\times}|\hat{\psi}\rangle = \hat{I}. \tag{64}$$

It should be recalled from Ref. [106] that the condition of iso-Hermiticity coincides with the conventional Hermiticity. Therefore, *all quantities that are observable in quantum mechanics remain observable in isomechanics* [31].

Note also that the characterization of extended particles via isomechanics requires *two* quantities, the conventional Hamiltonian H for the representation of SA interactions and the isotopic element $\hat{T}$ for the representation of NSA interactions.

We should finally note that the quantum mechanical representations of nonlinear interactions via the Hamiltonian violate the superposition principle, by therefore preventing the study of individual nucleons under nonlinear nuclear forces. This insufficiency has been resolved by isomathematics and isomechanics because the representation of all nonlinear interactions via the isounit restores linearity at the abstract, realization-free level called *isolinearity* (see Sect. 4.2, p. 128 on Ref. [30]), by therefore allowing the decomposition of an isostate $|\Psi\rangle$ (representing a nucleus) into the isostates $|\psi_k\rangle$ (representing the individual nucleons)

$$\Psi(r)\rangle \ = \ \Pi_{k=1,2,...,N}\psi_k(r_k)\rangle, \tag{65}$$

with consequential capability of quantiitative representations of individual nuclear constituents.

4.4. Iso-Heisenberg representation

Nonrelativistic isomechanics is additionally based on the *iso-Heisenberg isoequations,* also called *Heisenberg-Santilli isoequations* (first formulated in Eq. (18a), p. 153, Ref. [16] (see Sect. 3.1, p. 80 on Ref. [31] for a detailed treatment), here written in the isoinfinitesimal form for the time evolution of an iso-Hermitean operator $\hat{Q}$ in terms of the Hamiltonian H representing all linear, local and potential interactions and the Santillian $\hat{T}$ representing all all nonlinear, nonlocal and nonpotential interactions

$$\begin{aligned} \hat{i}\hat{\times}\frac{\hat{d}Q}{\hat{d}\hat{t}} = [Q\hat{,}H] = Q\hat{\times}H - H\hat{\times}Q = \\ = Q\hat{T}(\psi,...)H(r,p) - H(r,p)\hat{T}(\psi,...)Q, \end{aligned} \tag{66}$$

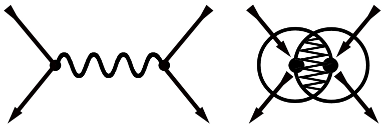

Figure 7: *An illustration on the left of the conventional Feynman's diagrams representing an elastic scattering of two point-like electrons via the exchange of a point-like photon, and an illustration on the right of the elastic scattering of two extended hadrons mediated by their EPR entanglement (Figure 1) without particle exchange due to the density of the scattering region (Ch. 12 of [31]).*

and in their isoexponentiated form

$$Q(t) = \hat{e}^{Htti} \hat{\times} Q(0) \hat{\times} \hat{e}^{-itH} = \hat{U}(t) \hat{\times} Q(0) \hat{\times} U(t)^{\dagger} = \\ = e^{H\hat{T}ti} Q(0) e^{-it\hat{T}H}, \tag{67}$$

where we have used isoexponentiation (22), showing the reconstruction by the Lie-Santilli isogroups of unitarity on isospaces over isofields, called *isounitarity* (see Sect. 4.2, p. 128 on Ref. [30])

$$\hat{U}(t) \hat{\times} \hat{U}(t)^{\dagger} = \hat{U}(t)^{\dagger} \hat{\times} \hat{U}(t) = \hat{I}. \tag{68}$$

Note that *the iso-Heisenberg equations (65) verify the invariance under anti-Hermiticity of the conventional equations (5), by therefore restricting the sole consistent application of the Lie-isotopic formulations to stable nuclei.*

The author contacted Prof. Werner Heisenberg in the early 1970's on the expected nonlinear interactions in nuclear physics, with particular reference to the inability for the Hamiltonian representation to characterize individual nucleons. Prof. Heisenberg responded very nicely to a young physicist (then an Assistant Professor of physics at Boston University), but essentially indicated no knowledge available at the time of including nonlinear interactions while preserving the superposition principle. The isotopic solution of the problem via the incorporation of all nonlinear interactions in the isounit of the theory, Eqs. (49), was the outcome of years of trials and errors by the author stimulated by his correspondence with Prof. Heisenberg.

4.5. Relativistic isoequations

The relativistic isoequations of hadronic mechanics are characterized by the isosymmetry *Poincaré-Santilli isosymmetry* $\hat{\mathcal{P}}(3.1)$ (Sect. 3.8.2) of the covariant formulation of the iso-Minkowski isospacetime $\hat{M}(\hat{p}, \hat{\Omega}, \hat{I})$ over the isoreal isofield $\hat{\mathcal{R}}$, with four-dimensional isomomentum

$$\hat{p}_\mu \hat{\times} |\hat{\psi})(\hat{x})\rangle = -\hat{i} \hat{\times} \hat{\partial}_{\hat{x}_\mu} |\hat{\psi}(\hat{x})\rangle, \;\; \mu = 1, 2, 3, 4, \;\; \hat{x}^4 = \frac{x^4}{n_4}, \tag{69}$$

resulting in the local speed of light within the considered physical medium, Eq. (34), with density $n = n_4$ (Figure 4). Consequently, the isometric $\hat{\Omega} = (\hat{\eta})\hat{I}$ is now the inverse of isometric (29).

The second order relativistic isoequation, called *Klein-Gordon-Santilli isoequation* [107] [108] (see Chapter 10, p.477 on Ref. [31] for a detailed study and review [28]), is characterized by the second order iso-Casimir invariant of $\hat{\mathcal{P}}(3.1)$ and can be written

$$\begin{gathered} \hat{p}^{\hat{2}} |\hat{\psi}(\hat{x})\rangle = \hat{\Omega}^{\mu\nu} \hat{\times} \hat{p}_\mu \hat{\times} \hat{p}_\nu \hat{\times} |\hat{\psi}(\hat{x}) = \\ = \hat{\eta}^{\mu\nu}(-i\hat{I}\partial_\mu)(-i\hat{I}\partial_\nu)|\hat{\psi}(\hat{x}) >= \hat{I}(mC)^2 |\hat{\psi}(\hat{x})\rangle. \end{gathered} \tag{70}$$

The first order isorelativistic equation of isomechanics, called the iso-Dirac equation, will be introduced in Section 5.5.

4.6. Isoconvergence

In all applications to date, the isotopic element (15) resulted to have a numeric value much smaller than one, $||\hat{T}|| \;\ll\; 1$. Since this numeric value has to be sandwiched between *all* products of isomechanics, isomathematics may eliminate quantum mechanical divergencies, besides allowing much faster data elaboration via perturbative and other methods (see Chapter 11, p. 500 on Ref. [31]).

To illustrate this important feature, consider the following divergent quantum mechanical series $A(w) = A(0) + (AH - HA)/1! + ... \to \infty, \;\; w > 1$. But the value of the isotopic element is much smaller than the parameter w. Therefore, the isotopic completion of the above series

$$A(w) = A(0) + (A\hat{T}H - H\hat{T}A)/1! + ... \to N < \infty, \;\; \hat{T} \;\ll\; w, \tag{71}$$

is always strongly convergent.

More specifically, the divergencies of quantum mechanics originate from the singularity existing at the origin of the *Dirac delta distribution* which divergence, in turn, originates from the point-like approximation of particles.

Another important feature of isomechanics is that of avoiding these singularities as illustrated by the isotopic image of Dirac's delta distribution, known as *Dirac-Myung-Santilli isodelta isofunction* first introduced in Ref. [106] (see also Nishioka's studies [109]-[112])

$$\hat{\delta}(\hat{r}) = \hat{\int} \hat{e}^{\hat{k} \star \hat{r}} \star \hat{d}\hat{k} = \int e^{\hat{k}\hat{T}\hat{r}} d\hat{k}, \tag{72}$$

where we have used isoexponentials and isointegrals [30]. In fact, the appearance of the isotopic element in the *exponent* of the integrant may change a sharp singularity at the origin $r = 0$ into a bell-shaped function.

The author visited P. A. M. Dirac, in 1981 at the Department of Physics of the University of Florida, Tallahassee, in connection with his suggestion to remove the divergencies of quantum mechanics so as to avoid the removal of infinities via infinities. During a brief encounter, the author indicated to Prof. Dirac isotopic product (9) with ensuing strongly convergent perturbative series (64) without divergencies. Following a long minute of complete silence, Prof. Dirac simply stated *"Please send me the papers."* Following his return to Harvard University, the author mailed the day after to Prof. Dirac: paper [106] that he was writing with the mathematician H. C. Myung, a copy of monograph [15] and the draft of [16]. It was unfortunate for science that, quite likely, Dirac never saw these works because he became ill and died a few years later.

4.7. Einstein isodeterminism

Recall the physical reality that stars initiate their lives as an aggregate of Hydrogen, synthesize the neutron from the electron and the proton and then synthesize all natural elements. Recall the prohibition by the 1927 Heisenberg's indeterminacy (also called uncertainty) principle to represent the synthesis of the neutron from an electron and a proton in the core of stars despite their extremely big Coulomb attraction at $1\ fm$ mutual distances which can be computed to be of the order of $230\ N$.

The above occurrences suggested the need to generalize Heisenberg's uncertainty principle for *extended* nucleons in conditions of *mutual penetration* whose study was initiated by Santilli in the late 1970s at Harvard University under DOE support (see the title of memoir [43]) which was first formulated in the 1981 paper "Generalization of Heisenberg's uncertainty priciniple for strong interactions" [44], followed by the 1994 paper for gravitational collapse [45], and applied in the 1998 paper [39] to set the foundations for the recovering of Einstein's determinism by showing that a system of *extended* spin $1/2$ particles with potential and non-potential interactions violate Bell's inequalities [38] thus admitting classical counterparts.

The considerable lapse of time between papers [44] - [45] - [39] is due to the need for corresponding prior advances in isomathematics. In fact, starting with the isotopies of Lie's theory [16], the 1981 paper [44] established the need that the formulations had to be defined over numeric isofieds that were discovered in 1993 [61]. The 1994 paper [45] established the need for the extension of the Newton-Leibnitz differential calculus to *volumes*, that was achieved in 1996 [50]. The 1998 paper [45] identified the final needs to predict the same numeric values under the same conditions at different times, such as the invariance under isounitary transformations reviewed in Sect. 3.12.

Santilli's final (isotopic) form of the generalized uncertainty principle for strong interactions was presented in the 2019 publication at *Acta Mathematica* [46] with an extended 2020 presentation also published by Acta Mathematica [113].

Nowadays, the EPR completion of Heisenberg' indeterminacy principle which is needed for the synthesis of the neutron (studied in the next section) and the representation of nuclear stabillity (studied in paper III) can be derived via a truly elementary, *quantum nonunitary* transformation of the conventional principle on conventional spaces and fields (Sect. 3.11) reformulated in terms of *hadronic isounitary transformations* on isospaces over isofields to avoid the inconsistencies of quantum nonunitary theories (Sect. 3.12) and can be reviewed as follows.

For applications in nuclear physics it is sufficient to assume the isonormalization necessary for constant $\hat{T}$

$$\langle\hat{\psi}|\hat{\times}|\hat{\psi}\rangle = \hat{T}, \tag{73}$$

and isocommutation rules (60), under which a simple nonunitary-isounitary transformation of the conventional principle yields the *isoindeterminacy (or isouncertainty) principle of hadronic mechanics* whose projection on our spacetime (as needed for experiments) is given by

$$\begin{aligned}
\Delta r \Delta p \approx \tfrac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r} \hat{,} \hat{p}] \star |\hat{\psi}(\hat{r}) >= \\
= \tfrac{1}{2}| < \hat{\psi}(\hat{r})|\hat{T}\, [\hat{r} \hat{,} \hat{p}]\, \hat{T}|\hat{\psi}(\hat{r}) >= \\
= \tfrac{1}{2}\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T}\hat{\psi}(\hat{r}) \hat{d}\hat{r} \approx \tfrac{1}{2}\bar{h}\hat{T} = \\
\tfrac{1}{2}\bar{h}e^{-\Gamma(r,p,a,R,d,\psi,\hat{\psi},...)} \approx \\
\approx \tfrac{1}{2}\bar{h}[1 - \Gamma(r,p,a,R,d,\psi,\hat{\psi},...) + ...] \ll 1,
\end{aligned} \tag{74}$$

where the Santillian $\hat{T}$ has beed derived from realization (15) to represent the non-potential-NSA component of strong interactions.

It should be noted that isoprinciple (77) contains as particular cases all generalizations of Heisenberg's uncertainty principle known to the author (see, e.g., Refs. [114] [115] [116] and papers quoted therein).

Note in isoprinciple (77) that the quantum expression $\Delta r \Delta p \geq (12)\hbar$ (also denoted $\sigma_r \sigma p \geq (12)\hbar$) is turned into the hadronic form $\Delta r \Delta p \approx (1/2\hat{T}\hbar$ because of the impossibility to maintain quantum mechanical statistical averages of point-like particles in vacuum for extended protons and neutrons under the contact-nonlocal effects cause by the surrounding particles (Fig. 3).

The connection between the above isoindeterminacy principle and Einstein's determinism [1] is intriguing indeed. Recall from Sect. 1:

4.7.1) The prohibition of Einstein's determinism by Bell's inequalities because they prohibit the existence of classical images for quantum mechanical systems of point-like spin $1/2$ particles in vacuum [38];

4.7.2) The violation of Bell's inequalities by extended spin $1/2$ particles in conditions of mutual penetration with the ensuing existence of classical counterparts [39] ;

4.7.3) Bohm's attempt of achieving Einstein's determinism via the hypothesis of hidden variables [48]-[49];

4.7.4) Santilli's explicit and concrete realization of Bohm's variables as being hidden in the axiom of associativity of quantum mechanics [31] [51].

4.7.5) Santilli [46] [113] has shown that isomathematics and isomechanics uniquely and unambiguously characterize the *explicit form of the standard isodeviations* for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$ which, again in their projection on conventional spaces over conventional fields, can be written

$$\begin{aligned}\Delta r &= \sqrt{\hat{T} < \hat{\psi}(\hat{r})|[\,\hat{r} - < \hat{\psi}(\hat{r})| \star \hat{r} \star |\hat{\psi}(\hat{r}) >]^2 |\hat{\psi}(\hat{r}) >},\\ \Delta p &= \sqrt{\hat{T} < \hat{\psi}(\hat{p})|\,[\hat{p} - < \hat{\psi}(\hat{p})| \star \hat{p} \star |\hat{\psi}(\hat{p}) >]^2 |\hat{\psi}(\hat{p}) >}.\end{aligned} \tag{75}$$

In view of the lack of energy by contact zero-range NSA interactions and the ensuing very low value of $\hat{T}$ for all known physical applications, the above isodeviations, *individually* tend to zero with the increase of the density of the hadronic medium up to the limit identity of $\hat{T}$ with Schwartzschild's horizon, Eq. (34),

$$Lim_{\hat{T}=0}\Delta r = 0, \quad Lim_{\hat{T}=0}\Delta p = 0, \tag{76}$$

resulting in the Lemma (p. 127 of [113])

LEMMA 4.7.1 (EINSTEIN'S ISODETERMINISM): The standard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$ progressively and individually tend to be zero for extended particles in the interior of hadrons, nuclei, and stars, and fully achieve Einstein's determinism at the limit of gravitational collapse.

The first significant implication of the above Lemma is a confirmation of the isoconvergence of Sect. 4.6, which can be expressed via the following Corollary (p. 128 of [113]):

COROLLARY 4.7.1. Einstein's isodeterminism according to Lemma 4.7.1 implies the removal of quantum mechanical divergencies in hadronic mechanics.

As we shall see, Einstein's isodeterminism plays a fundamental role for first known, numerically exact and time invariant representation of the Deuteron data

(Sect. 5), as well as of the neutron synthesis from the Hydrogen and, consequently, of nuclear stability (paper III).

4.8. Isoscattering theory without divergencies

As indicated in Section I.1, we have assumed in our study the exact validity of quantum mechanics for point-like particles and electromagnetic waves *in vacuum.* Consequently, we can safely assume that *quantum mechanics is exactly valid for particles in accelerators.*

However, the mathematical, theoretical and experimental evidence presented in these papers imply that *quantum mechanics is not valid for high energy scattering regions,* for numerous reasons, such as the lack of representation of the time irreversibility of nonelastic scattering processes (Insufficiency I-2.7), unavoidable nonlinear, nonlocal and nonpotential effects due to the impact of *extended* protons on *dense* targets, and other reasons (Fig. 7).

Studies on the EPR completion of the conventional scattering theory into a more realistic theory representing the collision of extended particles on dense targets have been initiated by R. Mignani [117]-[118], A. K. Aringazin and D. A. Kirukhin [119], A. O. E. Animalu and R. M. Santilli [120] and others.

These studies have established the foundations of the scattering theories of hadronic mechanics known under the names of *isoscattering and genoscattering theories* and collectively, as *hadronic scattering theories* (Figure 7). Regrettably, we cannot review these studies to avoid a prohibitive length. We limit ourselves to the following comments (see Chapter 12, p. 507 on of Ref. [31] for details):

4.8.1. In view of the time-reversal invariance of the Lie-Santilli theory, the isoscattering theory can solely represent the *elastic* scattering of extended particles/wavepackets on a dense target. The causal and axiomatically consistent representation of *inelastic* scatterings is done via the broader genoscattering theory with a Lie-admissible algebraic structure [120].

4.8.2. Evidently, hadronic scattering theories cannot change experimentally measured values, such as scattering angles, cross sections, etc. However, *hadronic scattering theories do require a revision of the theoretical interpretations derived via the conventional scattering theory.*

4.8.3. Recall that rather vast studies have been conducted on *nonunitary scattering theories* (see, e.g., Ref. [121] and papers quoted therein). These studies had to be abandoned because nonunitary theories violate causality laws (Sect. 3.10). By contrast, hadronic scattering theories reconstruct unitarity on generalized space over generalized fields, thus restoring causality [30].

4.9. Connection between Birkhoffian mechanics and isomechanics

Following the technical characterization of NSA forces in the first volume of the 1978 Springer-Verlag monographs [15], the primary aim of the 1983 second vol-

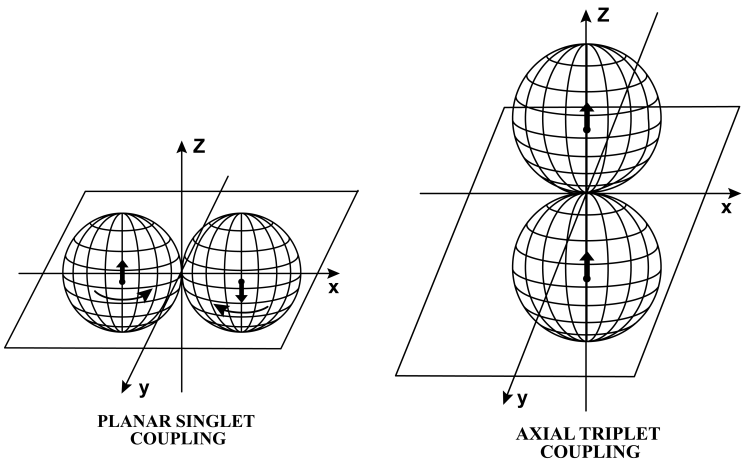

Figure 8: *In this figure, we illustrate the two stable bound states of particles with spin predicted by hadronic mechanics, which are given by the 'planar singlet coupling' on the left and the 'axiom triplet coupling' on the right.*

ume [16] was the achievement, via the use of conventional mathematics, of direct universality for the representation of NSA Newtonian systems in terms of a generalized variational principle so as to allow apparently for the first time the optimization, for instance, of the actual shape of a wing moving in atmosphere (see representative papers [122]-[126]).

This aim was achieved via the formulation of Birkhoffian mechanics in the 6-dimensional phase space with local coordinates $a^{\mu} = (r, p), \quad \mu = 1, 2, ..., 6$, and the assumption of the most general possible action principle in the indicated phase space

$$\delta \mathcal{A} \delta \int [R_{\mu}(t, a) da^{\mu} - B(t, a) dt] = 0, \tag{77}$$

with consequential *Birkhoff's equations* (Sect. 4.2, p. 30 of [16])

$$[\frac{\partial R_{\mu}(t, a)}{\partial a^{\nu}} - \frac{\partial R_{\nu}(t, a)}{\partial a^{\mu}}] \frac{da^{\nu}}{dt} - [\frac{\partial B(t, a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(t, a)}{\partial t}] = 0, \tag{78}$$

where $B(t, a)$ is the *Birkhoffian* (which does not generally represent the energy) with the most general possible realization of the Lie-Santilli isotheory in classical mechanics in terms of brackets between observables while admitting as particular case the conventional Hamilton's equation in phase space (those without external terms) for $R = (0, p)$

$$[A\hat{;}B] = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu},$$

$$\omega_{\mu\nu}\frac{da^\nu}{dt} - \frac{\partial H(t,a)}{\partial a^\mu} = 0, \tag{79}$$

$$[A, B] = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} = \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial r}\frac{\partial A}{\partial p}.$$

Note that the conventional Hamilton's equations can only represent SA Newtonian systems while, by comparison, Birkhoff's equations are directly universal for NSA systems (Sect. 4.5, p. 54 of Ref. [16]).

A crucial point occurred in these studies with the identification of the fact that the *Birkhoffian form of the Hamilton-Jacobi equations* (p. 205 on of Ref. [16]) no longer verifies the crucial property (47c) on the independence of the action from the linear momentum. As a consequence, *the Birkhoffian generalization of Schrödinger equation (p. 211 of Ref. [16]) requires a generalization of quantum mechanical axioms* because wave functions would also depend on momenta $\psi = \psi(t, r, p)$.

In order to maintain the validity of quantum axioms for NSA interactions, R. M. Santilli proposed in Eq. (15), p. 259 of Ref. [16] the decomposition of the Birkhoffian eigenvalue equation into the iso-Hamiltonian

$$B(t, a) \times |\psi(t, a)\rangle = H\hat{\times}|\psi(t, r)\rangle = H\hat{T}|\psi(t, r)\rangle = E|\psi(t, r)\rangle, \tag{80}$$

which is possible thanks to the embedding of all NSA terms in the isounit of the theory.

By looking in retrospect, Birkhoffian mechanics has seen various applications for the classical treatment of NSA systems (see representative papers [122]-[126]) while the isotopic formulation of NSA systems has been preferred over the operator Birkhoffian form in view of its clear preservation of quantum mechanical axioms.

5 Representation of the Deuteron data

5.1. Foreword

Thanks to studies initiated in 1978 [43] and continued thereafter (see Refs. [127]-[131]), in this section we review and upgrade the numerically exact and time invariant representation permitted by the Lie-isotopic mechanics of the following experimental data of the Deuteron [6]-[**?**] in the ground state used for experimental measurements, that with null orbital contributions:

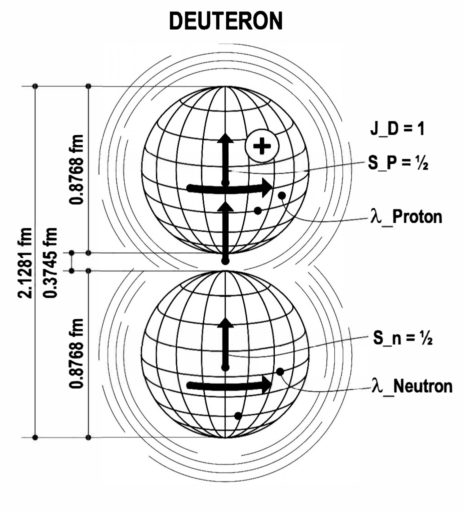

Figure 9: *In this figure, we reproduce known experimental data on the dimensions of the Deuteron [8] and its constituent proton and neutron [11], as well as their interpretation as a hadronic bound state in axial triplet coupling (Fig. 1), thus representing for the first time the spin of the Deuteron $S_D = 1$ in its ground state, that with null angular contributions $L_D = 0$ [51].*

5.1.1. The spin $S_D = 1$.

5.1.2. The magnetic moment $\mu_D^{exp} = 0.85647\ \mu_N$.

5.1.3. The total energy $E_D^{tot} = M_p + M_n - BE = 1875.7\ MeV$.

5.1.4. The charge radius $R_D = \mu_D^{exp} = 0.85647\ \mu_N$.

5.1.5. The mean life $\tau_D = \infty$.

In order to apply isotopic formulations, in this section we assume in first approximation that the Deuteron is a hadronic bound state of a proton and a *permanently stable neutron.* The representation of the stability of the Deuteron despite the natural instability of the neutron and the Deuteron stability despite strongly repulsive protonic forces, are studied in the following paper.

5.2. Representation of the Deuteron spin

Recall that the sole stable bound state predicted by quantum mechanics between two *point-like* particles with spin $S = 1/2$ is the singlet for which $S_D = 0$. The stable bound states predicted by hadronic mechanics for two *extended* particles with spin $S = 1/2$ are the *planar singlet coupling* and the *axial triplet coupling* illustrated in Figure 8. It then follows that, according to the Lie-isotopic branch of hadronic mechanics, *the Deuteron in its true ground state is composed by a proton* *and a neutron in axial triplet coupling* (Fig. 9) [51]

$$D = \begin{pmatrix} p^{\uparrow} \\ n^{\uparrow} \end{pmatrix}, \tag{81}$$

resulting in a structure illustrated in Figure 9 with dimensions from Refs. [8] [11]. In the subsequent Figure 10, we illustrate the consequential structure of the Helium-4 (Fig. 10).

It should be noted that the above representation of the Deuteron spin is highly preliminary because a deeper study of the spin of the Deuteron requires the prior representation of the synthesis of the neutron in the core of stars, which representation is studied in the next paper.

5.3. Hadronic angular momentum

As it is well known, the quantum mechanical angular momentum is given by the familiar expression on an Euclidean space over the field of complex numbers $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}|b\rangle$ acting on a state of the Hilbert space $\mathcal{H}$. The rotational symmetry $so(3)$ represents the conservation of the angular momentum.

The angular momentum for extended nucleons under strong interactions, called *hadronic angular momentum* (first presented in the 1985-1991 papers [76]-[78] and then studied in detail in Chapter 6, p. 209-285 of Ref. [31]) is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli isogroup $\hat{SO}(2)$ characterized by the *isorotations* on the iso-Euclean isospace $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I})$ (Sect. 3.6)

$$\hat{r}' = \hat{R}(\hat{\theta}) \hat{\times} \hat{r}, \;\; \hat{R}(\hat{\theta}) \hat{\times} \hat{R}(\theta)^{-\hat{1}} = \hat{R}(\theta)^{-\hat{1}} \hat{\times} \hat{R}(\theta) = \hat{I},$$

$$\hat{R}(\hat{\theta}) \hat{\times} \hat{R}(\hat{\theta}') = \hat{R}(\hat{\theta}') \hat{\times} \hat{R}(\hat{\theta}) = \hat{R}(\hat{\theta} + \hat{\theta}'), \;\; \hat{R}(\hat{\theta}) \hat{\times} \hat{R}(-\hat{\theta}) = \hat{R}(0) = \hat{I} = 1\hat{T}. \tag{82}$$

The isoalgebra $\hat{so}(2)$ can be realized on an iso-Hilbert isospace over an isofield via a simple isotopy of the conventional angular momentum (Section 3.6)

$$\begin{gathered} \hat{\mathbf{L}} \hat{\times} |\hat{\mathbf{b}}\rangle = (\hat{\mathbf{r}} \hat{\wedge} \hat{\mathbf{p}}) \hat{\times} |\hat{\mathbf{b}}\rangle, \\ \hat{\mathbf{r}} = \mathbf{r}\hat{\mathbf{I}}, \;\; \hat{\mathbf{p}} \hat{\times} |\hat{\mathbf{b}}\rangle = -\mathbf{i}\hat{\partial}_{\hat{\mathbf{r}}} |\hat{\mathbf{b}}\rangle = -\mathbf{i}\hat{\mathbf{I}}\partial_{\hat{\mathbf{r}}} |\hat{\mathbf{b}}\rangle. \end{gathered} \tag{83}$$

As expected, the $\hat{so}(3)$ isosymmetry characterizes the conservation of the hadronic angular momentum, which conservation is evidently verified for stable nuclei.

The simplest possible realization of the isoalgebra $\hat{so}(3)$ can be constructed via the isotransformations of the conventional generators $J_k, \; k = 1, 2, 3$ (Sect. 3.8)

$$\begin{gathered} U J_k U^{\dagger} = \hat{J}_k, \\ UU^{\dagger} = Diag.(\tfrac{1}{n_1^2}, \tfrac{1}{n_2^2}, \tfrac{1}{n_3^2}) = (b_1^2, b_2^2, b_3^2). \end{gathered} \tag{84}$$

It is easy to see that the above isoalgebra preserves the conventional structure constants of $so(3)$

$$[\hat{J}_i \hat{,} \hat{J}_i] \hat{\times} |\hat{b}\rangle = (\hat{J}_i \hat{T} \hat{J}_i - \hat{J}_j \hat{T} \hat{J}_i) \hat{\times} |\hat{b}\rangle = \epsilon_{ijk} \hat{J}_k \hat{\times} |\hat{b}\rangle, \tag{85}$$

as well as conventional quantum mechanical eigenvalues

$$\begin{aligned} &\hat{J}_3 \hat{\times} |\hat{b}\rangle = m|\hat{b}\rangle, \quad m = J, j-1, ..., -J, \;\; j = 0, 1, 2, ... \\ &\hat{J}^2 \hat{\times} |\hat{b}\rangle = (\hat{J}_1 \hat{T} \hat{J}_1 + \hat{J}_2 \hat{T} \hat{J}_2 + \hat{J}_3 \hat{T} \hat{J}_3) \hat{T} |\hat{b}\rangle = j(j+1)|\hat{b}\rangle, \end{aligned} \tag{86}$$

by therefore assuring the isomorphic $\hat{so}(3) \approx SO(3)$.

It should be indicated that the above simplest possible isorealization of $\hat{so}(3)$, called *standard* (p. 212 of Ref. [31]), is sufficient for the representation of the Deuteron magnetic moment done in the next section. Nevertheless, the representation of the conserved angular momentum of the neutron in its synthesis from the Hydrogen atom (studied in the next paper) will require the broader *regular isorepresentation* (p. 213 of Ref. [31]) while the representation of *nonconserved* angular momenta in nuclear fusions requires the more general *irregular isorepresentations* [58] [59] [60].

5.4. Representation of the Deuteron magnetic moment via the hadronic angular momentum

Let us recall the historical view by E. Fermi [53], V. F. Weisskopf [54], and other founders of nuclear physics according to which the inability by quantum mechanics to represent nuclear magnetic moments may be due to a deformation of the charge distribution of nucleons under strong nuclear forces, a prediction hereon referred to as the *Fermi-Weisskopf hypothesis.* In fact, p. 31 of Ref. [54] presents the statement *it is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon.*

We are here referring to the fact that a rotating charge distribution creates a magnetic field along its rotation axis. When said charge distribution is deformed into a prolate (oblate) ellipsoid, the magnetic field decreases (increases). Since the quantum mechanical prediction for the magnetic moment of the Deuteron is in *excess* by about 3% of the experimental value $\mu_D^{exp} = 0.85647\, \mu_N$, the deformation of the charge distribution of the proton and the neutron is expected to be of *prolate* type (bottom right view of Fig. 2).

The representation of the Deuteron magnetic moment via the Fermi-Weisskopf hypothesis achieved for the first time in the 1994 paper [127] (see also monograph [128] and idnependent studies [129]-[131]) is based on the representation of the deformation of nucleons via isotopic element (28) which reconstructs the perfect sphere on the iso-Euclidean isospace over an isofield (top of figure 2). The reformulation (isotopy) via hadronic mechanics (hm) of the conventional connection between magnetic moment and spin of nucleons [55] [7] (here omitted for brevity) implies the following *mutation of the magnetic moments of the protons and the neutron* [127]

$$\mu_D^{hm} = \frac{n_4}{n_3}(g_p S + g_n S) = \frac{n_4}{n_3}\mu_D^(qm). \tag{87}$$

By recalling the experimental value $\mu_D^{exp} = 0.8565\mu_N$, and the quantum mechanical (qm) value $\mu_D^{qm} = 0.8798\mu_N$, we reach the numeric value

$$\frac{\hat{\mu}_D^{exp}}{\mu_D^{qm}} = \frac{n_4}{n_3} = \frac{0.8565}{0.8798} = 0.9735. \tag{88}$$

In the above expressions, n_4 represents the density of individual nucleons whose currently available best value is given by the density $n_4 = 605$ of the $p - \bar{p}$ fireball of the Bose-Einstein correlation [88] [89], resulting in the numeric values

$$n_3 = \frac{0.605}{0.9735} = 0.6215, \;\; n_3^2 = 0.3862. \tag{89}$$

To obtain the value of the remaining characteristic quantities, we assume that the mutation of the magnetic moment does not alter the volume of nucleons. By assuming a normalization of the type

$$n_1^2 + n_2^2 + n_3^2 = 3, \tag{90}$$

we obtain the numeric values of the semiaxes of the spheroids here expressed for the characteristic quantities n_μ of this paper as well as for their inverse $b_\mu = 1/n_\mu$ used in paper [127]

$$\begin{gathered} n_1^2 = n_2^2 = 1.3069, \;\; n_3^2 = 0.3862, \; n_4^2 = 0.3660, \\ b_1^2 = b_2^2 = \tfrac{1}{n_1^2} = \tfrac{1}{n_2^2} = 0.7652, \;\; \tfrac{1}{n_3^2} = 2.5893, \; \tfrac{1}{n_4^2} = 1.7668, \\ n_1 = n_2 = 1.1432, \; n_3 = 0.6215, \; n_4 = 0.605, \\ b_1 = b_2 = \tfrac{1}{n_1} = \tfrac{1}{n_2} = 0.8709, \; b_3 = \tfrac{1}{n_3} = 1.6098, \; b_4 - = \tfrac{1}{n_4} = 1.653, \end{gathered} \tag{91}$$

which confirm their prolate character because $n_3^2 < n_1^2 = n_2^2$.

It should be noted that Ref. [127] studied the isotopic representation of a possible 1% deformation of the neutron under nuclear interactions. By using the same nucleon density $b_4 = 1/n_4 = 1.653$ of Refs. [88] used above [89], Ref.[127] reached the numeric value $b_3 = 1/n_3 = 1.662$ (Eqs. (3.5), p. 124, of Ref.

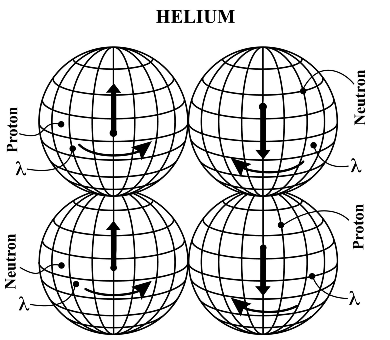

Figure 10: *In this figure, we illustrate the structure of the Helium as a hadronic bound state of two Deuterons composed by protons and neutrons in axial triplet coupling [51].*

[127]) which is rather close to the value $b_3 = 1/n_3 = 1.6098$ of Eq. (90) above confirming the prolate character of the deformation, The $b_1 = b_2$ value of Ref. [127] different than that of Eqs. (90) is due to a normalization of the characteristic quantities different than used above.

The possible application of the above model to a numeric representation of the magnetic moments of light stable nuclei with N nucleons can be done via the following isotopy of standard representation of nuclear magnetic moments [54] [**?**] first presented in Eq. (3.8), p. 125 of Ref. [127]

$$\mu^{hm} = \Sigma_{k=1,2,..,N}(\frac{n_{4k}}{n_{3k}} g_k^L L_{k3} + \frac{n_{4k}}{n_{3k}} g_k^S S_{k3}), \qquad (92)$$

where g_k^L and (g_k^S) are tabulated, orbital (spin) giromagnetic factors of the k-nucleon, n_{4k} is the density of the k-nucleon and n_{3k} is its semiaxis along the spin direction.

By assuming again in first approximation that nuclei are composed by protons and neutrons, it is easy to see that isotopic model (91) provides a numeric representation of the magnetic moment of the Helium and other light stable nuclei with potentially significant information on the deformation of the individual nucleons.

5.5. Hadronic spin

The spin of extended nucleons under strong interactions, called *hadronic spin* (first presented in Ref. [77] is characterized by the isounitary isoirreducible isorepresentations of the Lie-Santilli $\hat{su}(2)$ isoalgebra whose general case is studied in Chapter 6, pages 209-285 of monograph [31], jointly with the hadronic angular momentum of the preceding section and the isotopies of spin-orbit couplings. The hadronic spin was then used for the 1998 verification of the EPR argument [39] and in other applications [32] (Fig. 8).

We here study the particular case of the hadronic spin with conventional quantum value $S = 1/2$, yet with new degrees of freedom representing Bohm's hidden variables [51]. Its formulation requires the isolinearization of the second order isoinvariant $\hat{C}_2$ (41) of the Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ (Sect. 4.8.2) done in Ref. [82] (see [31], Chapter 8, p. 329, particularly Sect. 8.5, p. 353). Said isolinearization required:

1) A four-dimensional real valued isospacetine $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ for the characterization of the orbital motion (orb).

2) A two-dimensional complex valued isospace $E(\hat{z}, \hat{\delta}, \hat{I}_{spin})$ for the representation of the spin (spin).

3) A six-dimensional total (tot) isospace $\hat{M} \hat{\times} \hat{E}$.

From Sect. 3.9, we introduce the corresponding nonunitary transformations

$$U_{orb}U_{orb}^{\dagger} = \hat{I}_{orb} = Diag.(\tfrac{1}{n_1^2}, \tfrac{1}{n_2^2}, \tfrac{1}{n_3^2}, \tfrac{1}{n_4^2}),$$

$$U_{spin}U_{spin}^{\dagger} = Diag.(\lambda^{-1}, \lambda), \tag{93}$$

$$(U_{orb} \times U_{spin}) \times (U_{orb} \times U_{spin})^{\dagger} = \hat{I}_{tot},$$

where λ represents Bohm's hidden variable [48] [49].

We now introduce the 4×4-isomatrices $\hat{\Gamma}$ verifying the conditions

$$\hat{\Gamma}_{\mu} = \hat{\gamma}_{\mu}\hat{I}_{orb}.$$

$$\{\hat{\gamma}_{\mu}\hat{,}\hat{\gamma}_{\nu}\}. = \hat{\gamma}_{\mu}\hat{T}_{orb}\hat{\gamma}_{\nu} + \hat{\gamma}_{\nu}\hat{T}_{orb}\hat{\gamma}_{\mu} = 2\hat{\eta}_{\mu\nu}\hat{I}_{orb}, \tag{94}$$

The second order isoinvariant of $\hat{P}(3.1)$ is then decomposed into the isoproduct (Eq. (6.1a), p. 190, of Ref. [82])

$$\hat{\eta}^{\mu\nu}\hat{p}_{\mu}\hat{T}_{orb}\hat{p}_{\nu} + \bar{m}^2C^2 =$$

$$= (\hat{\eta}^{\mu\nu}\hat{\gamma}_{\mu}\hat{T}_{tot}\hat{p}_{\nu} + i\hat{m}C)\hat{T}_{tot}(\hat{\eta}^{\mu\nu}\hat{\gamma}_{\mu}\hat{T}_{tot}\hat{p}_{\nu} + i\hat{m}C), \;\; C = c/n_4, \tag{95}$$

where the local speed of light $C = c/n_4$ within a hadronic medium with density n_4 originates from the Lorentz-Santilli isosymmetry $\hat{SO}(3.1)$ [78] (Sect. 3.8.1), resulting in the *Dirac-Santilli isoequation* [107] (see also Refs. [50] [31]), here written according to isomathematics and in its projection on the conventional Minkowski space

$$[\hat{\Omega}^{\mu\nu}\hat{\times}\hat{\Gamma}_{\mu}\hat{\times}\hat{P}_{\nu} + \hat{M}\hat{\times}\hat{C}]\hat{\times}|\hat{\psi}(\hat{x}) >=$$

$$= (-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_{\mu}\partial_{\nu} + mC)|\hat{\psi}(\hat{x}) >= 0, \tag{96}$$

where the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\begin{aligned} \hat{\gamma}_k &= \tfrac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 &= \tfrac{i}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \end{aligned} \tag{97}$$

and the *Pauli-Santilli isomatrices* (Eq. (6.8.20), page 254 of Ref. [31] and Ref. [39])

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{98}$$

It is easy to see that the Pauli-Santilli isomatrices verify the iso-commutation rules with the same structure constants of Pauli's matrices

$$\begin{aligned} [\hat{\sigma}_i \hat{,} \hat{\sigma}_j] &= \hat{\sigma}_i \hat{\times} \hat{\sigma}_j - \hat{\sigma}_j \hat{\times} \hat{\sigma}_i = \\ &= \hat{\sigma}_i \hat{T}_{spin} \hat{\sigma}_j - \hat{\sigma}_j \hat{T}_{spin} \hat{\sigma}_i = i2\epsilon_{ijk}\hat{\sigma}_k, \end{aligned} \tag{99}$$

showing the iso-morphism $\hat{su}(2) \approx su(2)$.

The representation of the spin $1/2$ of nucleons is given by the isoeigenvalues on an iso-state $|\hat{b}>$ of the *Hilbert-Myung-Santilli iso-space* $\hat{\mathcal{H}}$[106] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [61]

$$\begin{aligned} \hat{S}_k &= \hat{\tfrac{1}{2}} \hat{\times} \hat{\sigma}_k = \tfrac{1}{2}\hat{\sigma}_k, \\ \hat{\sigma}_3 \hat{\times} |\hat{b}> &= \hat{\sigma}_3 \hat{T} |\hat{b}> = \pm |\hat{b}>, \\ \hat{\sigma}^{\hat{2}} \hat{\times} |\hat{b}> &= (\hat{\sigma}_1 \hat{T} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{T} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}> = 3|\hat{b}> . \end{aligned} \tag{100}$$

It should be noted that the above hadronic spin, called *standard*, is sufficient for our needs for this section, but it is insufficient for the synthesis of the neutron studied in the next paper in favor of the broader notions of *regular and irregular hadronic spins* [31].

5.6. Representations of the Deuteron magnetic moment via the hadronic spin

Let us recall the factorization of the Pauli-Santilli isomatrices in Eqs. (6.8.18), p. 254 of Ref. [31]

$$\hat{\sigma}_k = \sigma_k \hat{I} = \hat{T}\sigma_k, \quad \hat{I} = 1/\hat{T} = Diag.(\lambda^{-1}, \lambda), \quad Det.\hat{I} = Det\hat{T} = 1, \tag{101}$$

and assume the realization of the hidden variable [51] [52]

$$\lambda = e^{\phi} \geq 0,$$

$$\hat{I} = \cosh\phi + \sigma_3 \sinh\phi = e^{\phi\sigma_3}, \tag{102}$$

$$\hat{T} = \cosh\phi - \sigma_3 \sinh\phi = e^{-\phi\sigma_3}.$$

By recalling that σ_3 characterizes the nucleon spin $S = 1/2$, we have

$$\hat{\sigma}_3|\hat{b} >= \sigma_3\hat{I}|\hat{b} >= \sigma_3 e^{\phi\sigma_3}|\hat{b} >, \tag{103}$$

with ensuing characterization of the magnetic moment

$$\mu_{hm}|\hat{b} >= KgS|\hat{b} >, \tag{104}$$

where K is an iso-renormalization constant of the gyromagnetic factor g created by Bohm's hidden variables in the hadronic spin. By using the above property, we reach the relation

$$\mu_{hm}|\hat{b} >= e^{\phi\sigma_3}\mu_{qm}|\hat{b} >= e^{\phi\sigma_3}gS|\hat{b} > . \tag{105}$$

Note that the proton and the neutron have the same spin $1/2$, they have essentially the same mass (thus being characterized by the same λ) and they have the same eigenvalue equation

$$\sigma_3|\hat{b} >= -|\hat{b} > . \tag{106}$$

We can, therefore, write the expression per each nucleon

$$\mu_{hm,k} \approx (1 + \phi\sigma_3)\mu_{qm,k} = (1 - \phi)\mu_{qm.k}, \quad k = p, n, \tag{107}$$

from which we obtain the isorenormalized value of the magnetic moment of the proton and of the neutron

$$\hat{\mu}_p = +(1 - \phi)\ 2.79285\ \mu_N, \quad \hat{\mu}_n = -(1 - \phi)\ 1.91304\ \mu_N, \tag{108}$$

with corresponding value for the magnetic moment of the Deuteron

$$\begin{aligned} \mu_D^{hm} &= (1 - \phi)\ 2.79285 - (1 - \phi)\ 1.91304\ \mu_N = \\ &= (1 - \phi)\ 0.87981\ \mu_N = \mu_D^{ex} = 0.85647\ \mu_N, \end{aligned} \tag{109}$$

from which we obtain the numeric value

$$\phi = 1 - 0.87981/0.85647 = 1 - 0.02334 = 0.97666, \tag{110}$$

with corresponding *numeric value of Bohm's hidden variable for the Deuteron* [51]

$$\lambda = e^{\phi} = e^{0.97666} = 2.65557. \qquad (111)$$

We can, therefore, state that isomathematics and isomechanics do allow a numerically exact representation of the Deuteron magnetic moment in terms of Bohm hidden variable λ which is invariant over time due to the invariance of the Dirac-Santilli isoequation under the isosymmetry $\hat{\mathcal{P}}(3.1)$ [82].

5.7. Representation of the Deuteron mass, stability and charge radius

In accordance with our primary assumptions of Sect. 1, the terms *hadronic structure equations* are referred to iso-Galilean isoinvariant isoequation (57) representing stable bound state of extended nucleons in conditions of partial mutual penetration. The conventional linear, local and potential interactions are represented by the Hamiltonian H, while nonlinear, nonlocal and nonpotential interactions are represented by the isotopic element $\hat{T}$ under the condition of rapidly recovering the conventional unit, $\hat{T} \to I$ at mutual distances bigger than the range of strong interactions $R = b^{-1} = 10^{-13}\ cm$.

The first hadronic structure equation was proposed by Santilli in Sect. 5.1, p. 827 on of the 1978 Harvard University memoir [43] for the representation of *all* characteristics of the pion π^0 as a bound state of an electron and a positron at $R = 10^{-13}\ cm$ mutual distance, so as to have a physical origin of the great instability of the particle as well as of the spontaneous decay with the lowest mode $\pi^0 \to 2\,\gamma$.

The isotopic element was selected to be of type (14), with the function $\hat{F}(r)$ given by the Hulten potential in representation of nonpotential interactions, and the function $F(r)$ representing the Coulomb attraction between the electron and the positron

$$\hat{T} = e^{-\frac{\hat{F}(r)}{F(r)}} \approx 1 - \frac{\hat{F}(r)}{F(r)} = 1 - \frac{\frac{e^{-br}}{1-e^{-br}}}{\frac{e^2}{r}}, \qquad (112)$$

resulting in the iso-Galilean invariant [72] [73] [21] iso-Schrödinger equation

$$[-\frac{1}{\bar{m}}\Delta_r - \frac{e^2}{r} - N\frac{e^{-br}}{1-e^{-br}}]|\psi(r)\rangle = E|\psi\rangle, \qquad (113)$$

where N is a normalization constant, E is the binding energy and the reduced mass

$$m = m_1 m_2/(m_1 + m_2), \qquad (114)$$

is isorenormalized to the form on isospaces over isofields $\hat{m} = m/n_4$ originating from the Lorentz-Santilli isosymmetry (Sect. 3.8.1), with value for the π^0 $\bar{m} = m_e/2n_4$.

The resulting *hadronic structure equations of the* π^0 in their radial form are given by (Eqs. (5.1.14), p. 835 of Ref. [43])

$$\begin{gathered}\left[\tfrac{1}{r^2}\left(\tfrac{d}{dr}r^2\tfrac{d}{dr}\right)+\bar{m}\left(E+\tfrac{e^2}{r}+N\tfrac{e^{-br}}{1-e^{-br}}\right)\right]=0,\\ E_{\pi^0}=E_{e^-}+E_{e^+}-E=135\ MeV,\\ \tau_{\pi^0}=2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_e}{\hbar}=10^{-16}s,\\ R_{\pi^0}=b^{-1}=10^{-13}\ cm=1\ fm.\end{gathered} \tag{115}$$

A detailed study of the analytic solution of the radial *differential* equation with the inclusion of boundary conditions (p. 836-841 of Ref. [43]), reduced the structure model of the π^0 to the solution of the following *algebraic* equations in the parameters $k_1,\ k_2$ (Eqs. (5.1.32a) and (5.1.32b), p. 840 of Ref. [43])

$$\begin{gathered}\tau_{\pi^0}=\tfrac{48\times(137)^2}{4\pi bc}\tfrac{k_1}{(k_2-1)^3}=0,\\ E_{\pi^0}=k_1[1-(k_2-1)2]\tfrac{2\bar{h}c}{b}=135\ MeV,\end{gathered} \tag{116}$$

with energy spectrum

$$E=\frac{1}{4R^2\bar{m}}(k_2\frac{1}{n}-n)^2,\ \ n=1,2,3,...., \tag{117}$$

numeric values for the pi^0

$$k_1=0.34,\quad k_2=1+4.27\times10^{-2}, \tag{118}$$

and consequential approximate value of the Binding Energy $E\approx0$.

Following the above 1978 proposal, the hadronic structure model of the π^0 has been applied to the structure of other particles, in each case providing a numerically exact and time invariant representation of *all* characteristics of the particle considered [91] [35].

The application of the hadronic structure model of the π^0 to the Deuteron as a bound state of a proton and a neutron has been studied extensively in the 1998 monograph [128] resulting in the iso-Galilean isoinvariant *hadronic structure equations of the Deuteron* (we cannot review to prevent a prohibitive length)

$$\begin{gathered}\left[\tfrac{1}{r^2}\left(\tfrac{d}{dr}r^2\tfrac{d}{dr}\right)+\bar{m}\left(E+N\tfrac{e^{-br}}{1-e^{-br}}\right)\right]=0,\\ E_D=E_p+E_n-E=1875.7\ MeV,\\ \tau_D=2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_e}{\hbar}=\infty,\\ R_D=b^{-1}=10^{-13}\ cm=1\ fm,\end{gathered} \tag{119}$$

with values for the k-parameters

$$k_1 = 2.5 \;\; k_2 = 1. \tag{120}$$

The following comments may be helpful for identifying some essential aspects of structure model (119):

5.7.1. As expected, *non-Hamiltonain interactions generate no binding energy,* because for $k_2 = 1$ the finite spectrum of the Hulten potential (116) solely admits the solution $E = 0$. Consequently, *the binding energy and related missing mass in nuclear structures is created by potential interactions.*

5.7.2. Model (114) contains the clearly identified Coulomb potential between the electron and the positron with ensuing well identified binding energy E. By contrast, the author could identify no physical, independently verifiable potential between the proton and the neutron. Consequently, *the representation of the total energy of the Deuteron in model (118) occurs on isospaces over isofields via the isorenormalized mass of the constituents*

$$E_{tot} = \bar{m}_1 + \bar{m}_2 = (\frac{m_1 + m_2}{n_4}), \tag{121}$$

where n_4 is a geometrization of the energy density of the Deuteron.

5.7.3. The author has no words to express the highly approximate character of model (118) and therefore, of the Deuteron conceived as a bound state of a proton and a neutron, thus suggesting the reinspection of model (118) with the synthesis of the neutron done in the next paper.

6 Concluding remarks

In accordance with the Einstein-Podolsky-Rosen argument that *quantum mechanics is not a complete theory* [1], the main aim of these papers is to present the Lie-isotopic methods as a recommendable ore-requisite for the broader Lie-admissible methods for the search (to be conducted in future papers) of new radiation-free controlled nuclear fusions of light element, which have been preliminarily presented in report [132] (see also the video on a reactor built with funds from China [133]).

Along the above lines in this paper we have reviewed and upgraded the EPR completion of quantum mechanics into *hadronic mechanics* as originally proposed in 1978 by R. M. Santilli [43] [15] [16] at Harvard University under DOE support comprising:

6.1) The technical characterization of nonpotential interactions via the conditions of variational self-adjointness presented in the 1978 monograph [15] and their classification into Self-Adjoint/Hamiltionian (SA) interactions and Non-Self-Asdjoint/non-Hamiltionian (NSA) interactions.

6.2) The proposal presented in Charts 5.2-5.4, p. 154 on of the 1983 monograph [16] of representing *stable,* thus time reversal invariant systems, via the axiom-preserving time-reversal invariant *Lie-isotopic formulations* representing SA interactions via the conventional Hamiltonian H and representing NSA interactions via the Santillian $\hat{T} > 0$. The representation is based on the completion of the universal enveloping associative algebra of quantum mechanics with generic product $AB = A \times B$ into the associativity-preserving form with product $A \hat{\times} B = A\hat{T}B,\ \hat{T} > 0$.

6.3) We have then reviewed, upgraded and specialized to nuclear physics studies conducted on hadronic mechanics by various scholars since the late 1970's [30]- [32] [19] [18] [21]-[29].

We have shown that isotopic formulations are based on:

6.i) The novel EPR entanglement characterized by the non-Hamiltonian interactions of the wave packets of particles [34].

6.ii) The violation of Bell's inequalities [38] by a system of extended spin $1/2$ particles under NSA/non- Hamiltonian interactions with ensuing existence of classical counterpart [39].

6.iii) The completion of Heisenberg's uncertainty principle for point-like particles under electromagnetic interactions into the *isouncertainrty principle* for extended particles under electromagnetic and strong interactions (Sect. 4.7) [46] which implies a progressive recovering of Einstein's determinism [1] with the increase of the density of hadronic media (decrease of the value of $\hat{T}$) in the interior of hadrons, nuclei and stars and its full recovering at the limit of gravitational collapse due to the identity of the isotopic element with Schwartzschild's horizon, Eq. (34).

6.iv) Explicit and concrete realizations [51] [52] of Bohm's hidden variables λ [48] [49] in terms of the Santillian $\hat{T} = \lambda$, as being hidden in the axiom of isoassociativity of hadronic mechanics s$A \hat{\times} (B \hat{\times} C) = (A \hat{\times} B) \hat{\times} C, \quad A \hat{\times} B = A \lambda B$.

We have shown that the time reversal invariant isotopic formulations per the above outline permit the representation of *stable,* thus time reversal invariant nuclei as a collection of extended protons and neutrons in conditions of partial mutual penetration with Hamiltonian and Santillian interactions, with the consequential numerically exact and time invariant representation of the experimental data of the Deuteron conceived as a hadronic bound state of a proton and a neutron in its ground state as experimentally measured without orbital contributions.

In the hope that the methods may by applied to strong interactions at large, , we have shown that quantum mechanical nuclear models for *point-like nucleons* with sole Hamiltonian interactions and insufficient representation of nuclear data can be uniquely and unambiguously mapped into hadronic models for *extended nucleons* with Hamiltonian and non-Hamiltonian interactions with ensuing exact representation of nuclear data via the simple *quantum nonunitary transformation* on conventional spaces over conventional fields $U \times U^\dagger = \hat{I} = 1/\hat{T}$ (Sect. 3.11), provided that the resulting model is reformulated in terms of the *hadronic isounitary transformations* on isospaces over isofields $\hat{U} \hat{\times} \hat{U}^\dagger = \hat{I} = 1/\hat{T}$ to avoid insidious inconsistencies (Sect. 3.12).

We have finally shown that, despite the indicated advances, the conception of the Deuteron as a hadronic bound state of a proton and a neutron does not allow a quantitative representation of the Deuteron stability in view of the natural instability of the neutron or of the strongly repulsive protonic Coulomb forces. These insufficiencies, combined with unsolved nuclear problems of societal relevance (such as the recycling of radioactive nuclear waste and the controlled nuclear fusion) suggests the conduction of deeper studies on nuclear structures which are presented in the subsequent paper.

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