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Bi-α ISO-DIFFERENTIAL INEQUALITIES AND APPLICATIONS

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Abstract

In this lecture, firstly we deduct some multiplicative iso-differential inequalities for multiplicative iso-functions of first, second, third, fourth and fifth kind. Then they are deducted and proved some bi-α-multiplicative iso-differential inequalities. As applications, in the lecture are deducted some uniqueness results for some classes multiplicative iso-differential equations and bi-α-multiplicative iso-differential equations.

1 Multiplicative iso-Differential Inequalities

Let D is a domain in $\mathbb{R}^2$, $a > 0$, $x_0 \in \mathbb{R}$, $J = [x_0, x_0 + a)$, $\hat{T} \in \mathcal{C}^1(J)$, $\hat{T}(x) > 0$ in J, $f \in \mathcal{C}(D)$.

Definition 1.1. *(solution of multiplicative iso-differential inequality) A function $y(x)$ is said to be a solution of the multiplicative iso-differential inequality*

(1′) $\quad \left(\hat{y}^\wedge(\hat{x})\right)^{\circledast} > \hat{f}^\wedge(\hat{x}, \hat{y}^\wedge(\hat{x}))$

or

(1) $\quad y'(x) > y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$

in J if

1. *$y'(x)$ exists for all $x \in J$,*

2. *for all $x \in J$ the points $(x, y(x)) \in D$,*

3. *$y'(x) > y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$ for all $x \in J$.*

The solutions of the multiplicative iso-differential inequalities

$$y'(x) \geq y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y'(x) < y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y'(x) \leq y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

are defined analogously. Our first result for multiplicative iso-differential inequalities is stated in the following theorem.

Theorem 1.2. *(basic theorem for the multiplicative iso-differential inequalities) Let $\hat{T}(x)-x\hat{T}'(x) \geq 0$ for every $x \in J$, $y_1(x)$ and $y_2(x)$ be the solutions of the multiplicative iso-differential inequalities*

(2) $\quad y_1'(x) \leq y_1(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$

(3) $$y_2'(x) > y_2(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

on J, respectively. Then the inequality

$$y_1(x_0) < y_2(x_0)$$

implies that

(4) $$y_1(x) < y_2(x) \qquad \text{for} \qquad \forall x \in J.$$

Proof. We suppose that (4) is not true. Then we define the set

$$A = \{x : \qquad x \in J, \qquad y_1(x) \geq y_2(x)\}.$$

From our assumption it follows that $A \neq \emptyset$.

Let x^* be the greatest lower bound of the set A. Then $x_0 < x^*$ and

$$y_1(x^*) \geq y_2(x^*).$$

Let us assume that

$$y_1(x^*) > y_2(x^*).$$

Because $y_1(x)$ and $y_2(x)$ are continuous functions in J then there exists a $\epsilon > 0$ such that

$$y_1(x^* - \epsilon) \geq y_2(x^* - \epsilon),$$

which is a contradiction with the definition of x^*. Consequently

$$y_1(x^*) = y_2(x^*).$$

Let $h < 0$. We have

$$y_1(x^* + h) < y_2(x^* + h),$$

and hence

$$y_1'(x^* - 0) = \lim_{h \longrightarrow 0} \frac{y_1(x^*+h) - y_1(x^*)}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{y_1(x^*+h) - y_2(x^*)}{h}$$

$$\geq \lim_{h \longrightarrow 0} \frac{y_2(x^*+h) - y_2(x^*)}{h}$$

$$= y_2'(x^* - 0),$$

i.e.,

(5) $\quad y_1'(x^* - 0) \geq y_2'(x^* - 0).$

From (2) we get

$$y_1'(x^*) \leq y_1(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\frac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)},$$

from where, using (5),

(6) $\quad y_1(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\frac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \geq y_2'(x^* - 0).$

On the other hand, from (3) we have

$$y_2'(x^* - 0) > y_2(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\frac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)},$$

whereupon, using (6),

$$\begin{aligned} & y_2(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\tfrac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \\ & < y_1(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\tfrac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \\ & = y_2(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\tfrac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)}, \end{aligned}$$

and since $\hat{T}(x^*) - x^*\hat{T}'(x^*) \geq 0$ we get the contradiction

$$f(x^*, y_2(x^*)) < f(x^*, y_2(x^*)).$$

Consequently

$$A = \emptyset,$$

from where we conclude that

$$y_1(x) < y_2(x) \qquad \text{in} \qquad J.$$

□

Corollary 1.3. *Let $\hat{T}(x) - x\hat{T}'(x) \geq 0$ in the interval J. Let also,*

(i) *$y(x)$ be a solution of the initial value problem*

$$(6)\qquad \begin{aligned} &y'(x) = -y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \quad \text{in} \quad (x_0, x_0 + a),\\ &y(x_0) = y_0, \end{aligned}$$

(ii) *$y_1(x)$ and $y_2(x)$ be the solutions of the multiplicative iso-differential inequalities*

$$(7)\qquad y_1'(x) < y_1(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$(8)\qquad y_2'(x) > y_2(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

in J, respectively,

(iii) *$y_1(x_0) \leq y_0 \leq y_2(x_0)$.*

Then

$$y_1(x) < y(x) < y_2(x)$$

for all $x \in (x_0, x_0 + a)$.

Proof. We shall prove that

$$y(x) < y_2(x) \qquad \text{for} \qquad \forall x \in (x_0, x_0 + a).$$

1. case $y_0 < y_2(x_0)$. Then from the last theorem we have that

$$y(x) < y_2(x) \qquad \text{in} \qquad (x_0, x_0 + a).$$

2. case $y_0 = y_2(x_0)$.

Let

$$z(x) = y_2(x) - y(x).$$

Then

$$z(x_0) = y_2(x_0) - y(x_0) = 0,$$

$$z'(x) = y_2'(x) - y'(x),$$

$$z'(x_0) = y_2'(x_0) - y'(x_0)$$

$$> y_2(x_0)\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_2(x_0))\frac{\hat{T}(x_0)-x_0\hat{T}'(x_0)}{\hat{T}(x_0)}$$

$$-y(x_0)\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} - f(x_0, y(x_0))\frac{\hat{T}(x_0)-x_0\hat{T}'(x_0)}{\hat{T}(x_0)}$$

$$= 0,$$

therefore the function z is an increasing function to the right of x_0 in a sufficiently small interval $[x_0, x_0+\delta]$. Consequently $y(x) < y_2(x)$ for all $x \in (x_0, x_0+\delta]$, from where

$$y(x_0+\delta) < y_2(x_0+\delta).$$

Now the last theorem gives that

$$y(x) < y_2(x) \qquad \text{in} \qquad [x_0+\delta, x_0+a).$$

Since δ can be chosen sufficiently small, then

$$y(x) < y_2(x) \qquad \text{in} \qquad (x_0, x_0+a).$$

□

Theorem 1.4. *Let $\hat{T}(x) - x\hat{T}'(x) \geq 0$, $\hat{T}'(x) \leq 0$, $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \leq P$ in J for some positive constant P, and for all (x,y), $(x,z) \in D$ such that $x \geq x_0$, $y \geq z$, we have*

$$f(x,y) - f(x,z) \leq L(y-z),$$

for some positive constant L. Let also,

(i) *$y(x)$ be a solution to the initial value problem (6),*

(ii) *$y_1(x)$ and $y_2(x)$ be solutions to the multiplicative iso-differential inequalities (2) and (3) on J, respectively.*

(iii) $y_1(x_0) \leq y_0 \leq y_2(x_0)$.

Then

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \text{for} \qquad \forall x \in J.$$

Proof. Let $\epsilon > 0$, $\lambda > LP$. Let also,

$$z_1(x) = y_1(x) - \epsilon e^{\lambda(x-x_0)}, \qquad x \in J.$$

Then

$$z_1(x_0) = y_1(x_0) - \epsilon < y_1(x_0)$$

and

$$\begin{aligned} z_1'(x) &= y_1'(x) - \epsilon\lambda e^{\lambda(x-x_0)} \\ &\leq y_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}. \end{aligned} \tag{9}$$

On the other hand, from the definition of the function $z_1(x)$ we have

$$z_1(x) \leq y_1(x) \qquad \text{in} \qquad J.$$

Then

$$f(x, y_1(x)) - f(x, z_1(x)) \leq L(y_1(x) - z_1(x))$$

or

$$f(x, y_1(x)) \leq f(x, z_1(x)) + L(y_1(x) - z_1(x)) \qquad \text{in} \qquad J.$$

From the last inequality and (9) we become

$$z_1'(x) \leq z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, z_1(x)) + L(y_1(x) - z_1(x)))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$= z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + L\epsilon e^{\lambda(x-x_0)}\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$\leq z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + LPe^{\epsilon(x-x_0)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$< z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$$

i.e.,

$$(10) \qquad \begin{aligned} & z_1'(x) < z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \qquad \text{in} \qquad J, \\ & z_1(x_0) < y(x_0). \end{aligned}$$

Let now

$$z_2(x) = y_2(x) + \epsilon e^{\lambda(x-x_0)}, \qquad x \in J.$$

Then

$$z_2(x) > y_2(x) \qquad \text{in} \qquad J.$$

Therefore

$$f(x, z_2(x)) - f(x, y_2(x)) \leq L(z_2(x) - y_2(x)) \qquad \text{in} \qquad J,$$

from where

$$f(x, y_2(x)) \geq f(x, z_2(x)) + L(y_2(x) - z_2(x)) \qquad \text{in} \qquad J.$$

Also, using the last inequality,

$$\begin{aligned} & z_2'(x) = y_2'(x) + \epsilon\lambda e^{\lambda(x-x_0)} \\ & > y_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \geq y_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, z_2(x)) + L(y_2(x) - z_2(x)))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \geq z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - L\epsilon e^{\lambda(x-x_0)}\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \geq z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - L\epsilon P e^{\lambda(x-x_0)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & > z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}, \end{aligned}$$

i.e.,

$$(11) \qquad \begin{aligned} & z_2'(x) > z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \qquad \text{in} \qquad J, \\ & z_2(x_0) > y_2(x_0). \end{aligned}$$

From (10) and (11) it follows that the functions $z_1(x)$ and $z_2(x)$ satisfy all conditions of the basic theorem for the multiplicative iso-differential inequalities. Therefore

$$z_1(x) < y(x) < z_2(x) \qquad \text{in} \qquad (x_0, x_0 + a),$$

i.e.

$$y_1(x) - \epsilon e^{\lambda(x-x_0)} < y(x) < y_2(x) + \epsilon e^{\lambda(x-x_0)} \qquad \text{in} \qquad (x_0, x_0 + a),$$

from here, when $\epsilon \longrightarrow 0$,

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \text{in} \qquad J.$$

□

Corollary 1.5. *Let for every points* (x, y), $(x, z) \in D$ *such that* $x \geq x_0$, *we have*

(12) $\quad |f(x,y) - f(x,z)| \leq L|y - z|$

for some positive constant L, $-P \leq \frac{\hat{T}'(x)}{\hat{T}(x)} \leq 0$, $0 \leq \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \leq P$ *in* J *for some positive constant* P.

Let also,

(i) y *be a solution to the initial value problem* (6),

(ii) $y_1(x)$ *and* $y_2(x)$ *be solutions to the multiplicative iso-differential inequalities* (2) *and* (3) *on* J, *respectively,*

(iii) $y_1(x_0) = y_0 = y_2(x_0)$.

Then for every $x_1 \in J$, $x_1 > x_0$, *either* $y_1(x_1) < y(x_1)$ $(y(x_1) < y_2(x_1))$ *or* $y_1(x) = y(x)$ $(y_2(x) = y(x))$ *for* $\forall x \in [x_0, x_1]$.

Proof. From (12) we have that if $y \geq z$ then

$$-L(y - z) \leq f(x, y) - f(x, z) \leq L(y - z).$$

Therefore all conditions of the last theorem are fulfilled. Consequently

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \text{for} \qquad \forall x \in J.$$

Also, we have

$$y'(x) - y_1'(x) = y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - y_1'(x)$$

$$\geq y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$-y_1'(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$= (y(x) - y_1(x))\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, y(x)) - f(x, y_1(x)))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$\geq -(y(x) - y_1(x))P - LP(y(x) - y_1(x))$$

$$= -P(1 + L)(y(x) - y_1(x)),$$

from where

$$(y(x) - y_1(x))' + P(1 + L)(y(x) - y_1(x)) \geq 0,$$

and

$$\left(e^{P(1+L)x}(y(x)\right)' \geq 0,$$

From the last inequality, when $x \leq x_1$, we get

$$\int_{x_1}^{x} \left(e^{P(1+L)x}(y(x)\right)' dx \leq 0,$$

or

$$e^{P(1+L)x}(y(x) - y_1(x)) \leq e^{P(1+L)x_1}(y(x_1) - y_1(x_1)). \tag{13}$$

Then, if $y(x_1) = y_1(x_1)$, using (13), we have that for every $x \in [x_0, x_1]$

$$y(x) \leq y_1(x),$$

whereupon

$$y(x) = y_1(x) \qquad \text{for} \qquad \forall x \in [x_0, x_1].$$

□

Definition 1.6. *A solution $r(x)$ $(\rho(x))$ of the initial value problem (6) which exists in $J=[x_0,x_0+a)$ is said to be maximal(minimal) if for an arbitrary solution $y(x)$ of (6) existing in J, the inequality $y(x)\le r(x)$ $(\rho(x)\le y(x))$ holds for all $x\in J$.*

Theorem 1.7. *Let $f(x,y)$ be continuous in $\overline{S}_+ = \{(x,y):\quad x_0\le x\le x_0+a,\quad |y-y_0|\le b\}$ and hence there exists a $M>0$ such that $|f(x,y)|\le M$ for all $(x,y)\in\overline{S}_+$. Let also, $\hat{T}(x)-x\hat{T}'(x)\ge 0$ in $[x_0,x_0+a)$, $\frac{|\hat{T}'(x)|}{\hat{T}(x)}\le P$, $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\le P$ in $[x_0,x_0+a)$. Then there exists a maximal solution $r(x)$ and a minimal solution $\rho(x)$ of the initial value problem (6) in the interval $[x_0,x_0+\alpha)$, where*

$$\alpha=\left\{a,\frac{2b}{2P(b+|y_0|+M)+b}\right\}.$$

Proof. We will prove the existence of a maximal solution.

Let

$$0<\epsilon\le\frac{b}{2}.$$

Let us consider the initial value problem

$$(14)\qquad \begin{aligned} &y'(x)=y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)}+f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}+\epsilon \qquad \text{in} \qquad [x_0,x_0+a),\\ &y(x_0)=y_0.\end{aligned}$$

We define

$$\overline{S}_\epsilon=\{(x,y)\in\mathbb{R}^2:\quad x_0\le x\le x_0+a,\quad |y-(y_0+\epsilon)|\le\frac{b}{2}\}.$$

We have that

$$\overline{S}_\epsilon\subset\overline{S}_+,$$

because

$$\begin{aligned}\tfrac{b}{2}&\ge|y-(y_0+\epsilon)|\\ &=|y-y_0-\epsilon|\\ &\ge|y-y_0|-\epsilon,\end{aligned}$$

or

$$|y - y_0| \leq \frac{b}{2} + \epsilon$$

$$\leq \frac{b}{2} + \frac{b}{2}$$

$$= b.$$

Also, for every $(x, y) \in \overline{S}_+$ we have

$$\left|y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\right| \leq |y(x)|\frac{|\hat{T}'(x)|}{\hat{T}(x)} + |f(x,y)|\frac{|\hat{T}(x)-x\hat{T}'(x)|}{\hat{T}(x)} + \epsilon$$

$$\leq P(b + |y_0|) + MP + \epsilon$$

$$\leq P(b + |y_0| + M) + \frac{b}{2}.$$

From here and from the multiplicative iso-Cauchy-Peano's existence theorem it follows that the problem (14) has a solution $y(x, \epsilon)$ which is defined in $[x_0, x_0 + \alpha)$.

Let now

$$0 < \epsilon_2 < \epsilon_1 < \epsilon.$$

We have

$$y(x_0, \epsilon_2) = y_0 + \epsilon_2 < y_0 + \epsilon_1 = y(x_0, \epsilon_1),$$

$$y'(x, \epsilon_2) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_2,$$

$$y'(x, \epsilon_1) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_1$$

$$> y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_2.$$

From here and from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x, \epsilon_2) < y(x, \epsilon_1) \qquad \text{for} \qquad \forall x \in [x_0, x_0 + \alpha).$$

Using the proof of the multiplicative iso-Cauchy-Peano's existence theorem we have that the sequence $\{y(x, \epsilon)\}_{\epsilon > 0}$ is equip-continuous and uniformly bounded.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that

$$\lim_{n\longrightarrow\infty} \epsilon_n = 0$$

and the corresponding sequence $\{y(x,\epsilon_n)\}_{n=1}^{\infty}$ of solutions of (14) is defined in $[x_0, x_0+\alpha)$.

We have

$$y(x,\epsilon_n) = y_0 + \epsilon_n + \int_{x_0}^{x} \left(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t,y(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt,$$

$$y_0 = y(x_0, 0) < y_0 + \epsilon_n,$$

$$y'(x,\epsilon_n) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_n$$

$$> y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}.$$

From here and from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x) < y(x,\epsilon_n) \qquad \text{in} \qquad [x_0, x_0+\alpha).$$

Consequently

$$y(x) \leq \lim_{n\longrightarrow\infty} y(x,\epsilon_n) := r(x) \qquad \text{for} \qquad \forall x \in [x_0, x_0+\alpha).$$

□

Theorem 1.8. *Let $r(x)$ be a maximal solution to the initial value problem (6) in J, $J = [x_0, x_0+a)$. Let also, $y(x)$ be a solution to the multiplicative iso-differential inequality (2) in J. If*

$$y(x_0) \leq y_0$$

then

$$y(x) \leq r(x) \qquad \text{in} \qquad J.$$

Proof. Let $x_1 \in [x_0, x_0+a)$. Let also $\epsilon > 0$ be chosen enough small. We consider the problem

$$(15)\qquad \begin{aligned} &y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x)) + \epsilon \qquad \text{in} \qquad J,\\ &y(x_0) = y_0. \end{aligned}$$

Let $r(x,\epsilon)$ be a maximal solution of the problem (15) in the interval J. We have that

$$\lim_{\epsilon \longrightarrow 0} r(x,\epsilon) = r(x)$$

uniformly in $[x_0, x_1]$.

Since

$$y(x_0) \le y_0 < y_0 + \epsilon,$$

$$y'(x) \le y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

$$< y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} + \epsilon$$

and

$$r'(x,\epsilon) = r(x,\epsilon)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, r(x,\epsilon))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

then from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x) < r(x,\epsilon) \qquad \text{in} \qquad [x_0, x_1],$$

whereupon

$$y(x) \le \lim_{\epsilon \longrightarrow 0} r(x,\epsilon) = r(x).$$

□

2 Existence and Uniqueness of Solutions

In this chapter $(x_0, y_0) \in \mathbb{R}^2$, D is a domain in $\mathbb{R}^2$ containing the point (x_0, y_0), J is an interval in $\mathbb{R}$ containing x_0, $\hat{T}(x) \in \mathcal{C}^1(J)$, $\hat{T}(x) > 0$ for every $x \in J$.

We begin to develop the theory of existence and uniqueness of solutions of the initial value problem

$$(1') \qquad \left(\hat{y}^\wedge(\hat{x})\right)^{\circledast} = \hat{f}^\wedge(\hat{x}, \hat{y}^\wedge(\hat{x})), \qquad x \in J,$$

$$(2) \qquad y(x_0) = y_0,$$

where f will be assumed to be continuous in the domain D.

The equation (1′) can be rewritten in the following form

$$\frac{y'(x)\hat{T}(x)-y(x)\hat{T}'(x)}{\hat{T}(x)\Big(\hat{T}(x)-x\hat{T}'(x)\Big)}=\frac{f(x,y(x))}{\hat{T}(x)}, \qquad x\in J,$$

or

$$y'(x)\hat{T}(x)-y(x)\hat{T}'(x)=f(x,y(x))\Big(\hat{T}(x)-x\hat{T}'(x)\Big), \qquad x\in J,$$

or

$$(1) \qquad y'(x)=y(x)\frac{\hat{T}'(x)}{\hat{T}(x)}+f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}, \qquad x\in J.$$

Definition 2.1. *We will say that a function $y(x)$ is a solution to the initial value problem* (1), (2) *if*

1. $y(x_0)=y_0$,

2. $y'(x)$ *exists for all* $x\in J$,

3. *for all* $x\in J$ *the points* $(x,y(x))\in D$,

4. $y'(x)=y(x)\frac{\hat{T}'(x)}{\hat{T}(x)}+f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$ *for all* $x\in J$.

If $f(x,y(x))$ is not continuous, then the nature of the solutions of (1) is quite arbitrary. For example, let

$$f(x,y(x))=\frac{4(y(x)-2)}{x(1-x)}-\frac{y(x)}{1-x}, \qquad \hat{T}(x)=e^x,$$

and $(x_0,y(x_0))=(0;0)$. Then the equation (1) admits the representation

$$\begin{aligned} y'(x)&=y(x)+\Big(\tfrac{4(y(x)-2)}{x(1-x)}-\tfrac{y(x)}{1-x}\Big)(1-x)\\ &=y(x)+4\tfrac{y(x)-2}{x}-y(x)\\ &=\tfrac{4}{x}(y(x)-2),\end{aligned}$$

its general solution is

(3) $\quad y(x) = 2 + Cx^4,$

where C is a constant. From here, we conclude that

$$y(0) = 2 \neq 0,$$

therefore the considered initial value problem has no any solution. If we take $(x_0, y(x_0)) = (0, 2)$, then every function (2) will be a solution of the considered initial value problem.

We shall need the following result to prove existence, uniqueness, and several other properties of the solutions of the initial value problem (1), (2).

Theorem 2.2. *Let $f(x, y(x))$ be continuous function in the domain D, then any solution of the initial value problem* (1), (2) *is also a solution of the integral equation*

(4) $\quad y(x) = y_0 + \int_{x_0}^{x} \left(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt$

and conversely.

Proof. An integration of the equation (1) yields

$$y(x) - y(x_0) = \int_{x_0}^{x} \left(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt.$$

Conversely, if $y(x)$ is any solution of (4), then

$$y(x_0) = y_0,$$

and since $f(x, y(x))$ is a continuous function in D and $\hat{T}$ is a continuous function in J, then $y(x)$ is a continuous function in J and we can differentiate (4), from where we find

$$y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}.$$

□

We shall solve the integral equation (4) by using the method of successive approximations due to Picard. For this reason, let $y_0(x)$ be any continuous function, we often take $y_0(x) \equiv y_0$, which we will suppose to be initial approximation of the unknown solution of (4), then we define $y_1(x)$ as follows

$$y_1(x) = y_0 + \int_{x_0}^{x} \left(y_0(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt.$$

We pick this $y_1(x)$ as our next approximation and substitute this for $y(x)$ in the right side of (4) and call it $y_2(x)$,

$$y_2(x) = y_0 + \int_{x_0}^{x} \left(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt.$$

Continuing in this way, the $(m+1)$st approximation $y_{m+1}(x)$ is obtained from $y_m(x)$ by means of the relation

(5) $$y_{m+1}(x) = y_0 + \int_{x_0}^{x} \left(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt, \qquad m = 0, 1, 2, \ldots.$$

If the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly to a continuous function $y(x)$ in the interval J and for all $x \in J$ the points $(x, y_m(x)) \in D$, then we may pass to the limit in both sides of (5), to obtain

$$\begin{aligned} y(x) &= \lim_{m \longrightarrow \infty} y_{m+1}(x) \\ &= y_0 + \lim_{m \longrightarrow \infty} \int_{x_0}^{x} \left(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt \\ &= y_0 + \int_{x_0}^{x} \left(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt, \end{aligned}$$

so that $y(x)$ is the desired solution. Below we will suppose that a and b are positive real numbers. Let P be positive real number such that

$$\frac{|\hat{T}'(x)|}{\hat{T}(x)} \leq P, \qquad \frac{|\hat{T}(x) - x\hat{T}'(x)|}{\hat{T}(x)} \leq P \qquad \text{for} \qquad \forall x \in [x_0 - a, x_0 + a].$$

Theorem 2.3. *Let the following conditions be satisfied*

(i) *$f(x,y)$ is continuous in the closed rectangle $\overline{S}: |x-x_0| \le a,\ |y-y_0| \le b$ and hence there exists a $M > 0$ such that $|f(x,y)| \le M$ for all $(x,y) \in \overline{S}$,*

(ii) *$f(x,y)$ satisfies a uniform Lipschitz condition*

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$$

for all (x,y_1), (x,y_2) in the closed rectangle $\overline{S}$,

(iii) *$y_0(x)$ is continuous in $|x-x_0| \le a$, and $|y_0(x) - y_0| \le b$.*

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ generated by Picard iterative scheme (5) converges to the unique solution $y(x)$ of the initial value problem (1), (2). This solution is valid in the interval $J_h : |x-x_0| \le h$, where $h = \min\left\{a, \frac{b}{P(b+|y_0|+M)}\right\}$. Further, for all $x \in J_h$ the following error estimate holds

$$(6) \qquad |y(x) - y_m(x)| \le Ne^{(P+PL)h} \min\left\{1, \frac{((P+PL)h)^m}{m!}\right\}, \qquad m = 0,1,2,\ldots,$$

where

$$\max_{x \in J_h} |y_1(x) - y_0(x)| \le N.$$

Remark 2.4. *This Theorem is called a local existence theorem since it guarantees a solution only in the neighborhood of the point (x_0, y_0).*

Proof. We will show that the successive approximations $y_m(x)$ defined by (5) exist as continuous function in J_h and $(x, y_m(x)) \in \overline{S}$ for all $x \in J_h$. Since $y_0(x)$ is a continuous function for all x such that $|x - x_0| \le a$, the function $F_0(x) = f(x, y_0(x))$ is continuous function in J_h, and hence $y_1(x)$ is continuous in J_h.

Also,

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x} \left(y_0(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left(|y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_0(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left((b + |y_0|)P + MP\right) dt\right|$$

$$= P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Assuming that the assertion is true for $y_m(x)$, $m \geq 1$, then it is sufficient to prove that it is also true for $y_{m+1}(x)$. For this, since $y_m(x)$ is continuous in J_h, the function $F_m(x) = f(x, y_m(x))$ is also continuous function in J_h. Moreover,

$$|y_{m+1}(x) - y_0| = \left|\int_{x_0}^{x} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left(|y_m(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left((b + |y_0|)P + MP\right) dt\right|$$

$$\leq P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Now we will prove that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly in J_h. Since $y_1(x)$ and $y_0(x)$ are continuous in J_h, there exists a constant $N > 0$ such that

$$|y_1(x) - y_0(x)| \leq N \qquad \text{for} \qquad \forall x \in J_h.$$

Also, for every $x \in J_h$, we have

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x} \left((y_1(t) - y_0(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_0(t)))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_0(t))|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} \left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_1(t) - y_0(t)|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right) dt\right|$$

$$\leq \left|\int_{x_0}^{x} (NP + LNP) dt\right|$$

$$= NP(1 + L)|x - x_0|.$$

Supposing that

$$(7) \qquad |y_m(x) - y_{m-1}(x)| \leq N\frac{\left((P+LP)|x-x_0|\right)^{m-1}}{(m-1)!}, \qquad x \in J_h,$$

for some $m \in \mathbb{N}$.

We will prove that

$$|y_{m+1}(x) - y_m(x)| \leq N\frac{\left((P + LP)|x - x_0|\right)^{m}}{m!}, \qquad x \in J_h.$$

Really,

$$|y_{m+1}(x) - y_m(x)|$$

$$= \left|\int_{x_0}^{x}\left((y_m(t) - y_{m-1}(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_m(t)) - f(t, y_{m-1}(t)))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t)) - f(t, y_{m-1}(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}(P + PL)|y_m(t) - y_{m-1}(t)|dt\right|$$

$$\leq N(P + PL)^{m+1}\left|\int_{x_0}^{x}\frac{(t-x_0)^m}{m!}dt\right|$$

$$= N(P + PL)^{m+1}\frac{|x-x_0|^{m+1}}{(m+1)!}.$$

Thus inequality (7) is true for all $m \in \mathbb{N}$.

Next, since

$$N\sum_{m=1}^{\infty}\frac{\left((P+PL)|x-x_0|\right)^{m-1}}{(m-1)!} \leq N\sum_{m=0}^{\infty}\frac{\left((P+PL)h\right)^{m}}{m!}$$

$$= Ne^{(P+PL)h} < \infty,$$

we have that the series

$$y_0(x) + \sum_{m=1}^{\infty}(y_m(x) - y_{m-1}(x))$$

converges absolutely and uniformly in the interval J_h, and hence its partial sums

$$y_1(x), y_2(x), \ldots, y_m(x), \ldots$$

converge to a continuous function in this interval, i.e.,

$$y(x) = \lim_{m\longrightarrow\infty} y_m(x).$$

As we have seen above we have that $y(x)$ is a solution to the problem (1), (2).

To prove that $y(x)$ is the only solution, we assume that $z(x)$ is also a solution to the initial value problem (1), (2) which exists in the interval J_h and $(x, z(x)) \in \overline{S}$ for all $x \in J_h$. The hypothesis (ii) is applicable and we have

$$|y(x) - z(x)| \leq \left| \int_{x_0}^{x} \left(|y(t) - z(t)| \frac{\hat{T}'(t)}{\hat{T}(t)} + |f(t, y(t)) - f(t, z(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left(P|y(t) - z(t)| + LP|y(t) - z(t)| \right) dt \right|$$

$$= (P + LP) \left| \int_{x_0}^{x} |y(t) - z(t)| dt \right|, \qquad x \in J_h.$$

Consequently

$$|y(x) - z(x)| = 0$$

for all $x \in J_h$.

Finally, we will obtain the error bound (6).

For $n > m$ the inequality (7) gives

$$|y_n(x) - y_m(x)| = |y_n(x) - y_{n-1}(x) + y_{n-1}(x) - y_{n-2}(x) + \cdots + y_{m+1}(x) - y_m(x)|$$

$$\leq \sum_{k=m}^{n-1} |y_{k+1}(x) - y_k(x)|$$

$$\leq N \sum_{k=m}^{n-1} \frac{\Big((P+LP)|x-x_0|\Big)^k}{k!}$$

(8) $$\leq N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$= N\Big((P+PL)h\Big)^m \sum_{k=0}^{n--m-1} \frac{\Big((P+PL)h\Big)^k}{(m+k)!} \qquad \left(\tfrac{1}{(m+k)!} \leq \tfrac{1}{m!k!}\right)$$

$$\leq N \frac{\Big((P+PL)h\Big)^m}{m!} \sum_{k=0}^{n-m-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\leq N \frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h},$$

and hence as $n \longrightarrow \infty$, we get

$$|y(x) - y_m(x)| \leq N \frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h}$$

in J_h.

The inequality (8) provides

$$|y_n(x) - y_m(x)| \leq N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\leq N \cdot \sum_{k=0}^{\infty} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$= N e^{(P+PL)h},$$

and as $n \longrightarrow \infty$, we find

$$|y(x) - y_m(x)| \leq N e^{(P+PL)h}$$

in J_h. □

Definition 2.5. *If the solution of the initial value problem* (1), (2) *exists in the entire interval* $|x - x_0| \leq a$, *we say that the solution exists globally.*

The next result is called a global existence theorem.

Theorem 2.6. *Let the following conditions be satisfied*

(i) $f(x, y)$ *is continuous in the strip* $T : |x - x_0| \leq a, |y| < \infty$,

(ii) $f(x, y)$ *satisfies a uniform Lipschitz condition in* T,

(iii) $y_0(x)$ *is continuous in* $|x - x_0| \leq a$.

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ *generated by Picard iterative scheme exists in the entire interval* $|x - x_0| \leq a$, *and converges to the unique solution* $y(x)$ *of the initial value problem* (1), (2).

Proof. For any continuous function $y_0(x)$ in $|x - x_0| \leq a$, as in the proof of the local existence Theorem, can be established the existence of each $y_m(x)$ in $|x - x_0| \leq a$ satisfying $|y_m(x)| < \infty$. Also, as in the proof of the previous Theorem we have that the sequence $\{y_m(x)(\}_{m=1}^{\infty}$ converges to $y(x)$ in $|x - x_0| \leq a$, replacing h by a throughout the proof and recalling that the function $f(x, y)$ satisfies the Lipschitz condition in the strip T. □

Corollary 2.7. *Let* $f(x, y)$ *be continuous in* $\mathbb{R}^2$ *and satisfies a uniform Lipschitz condition in each strip* $T_a : |x| \leq a, |y| < \infty$, *with the Lipschitz constant* L_a. *Then the initial value problem* (1), (2) *has a unique solution which exists for all* x.

Proof. For any x there exists an $a > 0$ such that $|x - x_0| \leq a$. From here and from $T \subset T_{a+|x_0|}$, it follows that the function $f(x, y)$ satisfies the conditions of the previous Theorem in the strip T. Hence, the result follows for any x. □

We will note that there exist positive constants M_1 and M_2 such that

$$\left|\frac{\hat{T}'(x)}{\hat{T}(x)}\right| \leq M_1, \quad \left|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right| \leq M_2 \quad \text{for} \quad x \in [x_0 - a, x_0 + a].$$

Theorem 2.8. *(multiplicative iso-Peano's existence theorem) Let f is defined, continuous and bounded function on the strip $T = \{(x,y) \in \mathbb{R}^2 : |x - x_0| \leq a, |y| < \infty\}$. Then the Cauchy problem (1), (2) has a bounded solution $y(x)$ which is defined on $|x - x_0| \leq a$ and*

$$|y(x)| \leq \left(1 + e^{aM_1}\right)(|y_0| + \sup_{(x,y)\in V} |f(x,y)|M_2 a) \quad \text{for} \quad \forall x \in [x_0 - a, x_0 + a].$$

Remark 2.9. *We can consider our main result as a continuation of the well - known Peano's Theorem.*

If we put

$$g(x,y) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\left(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\right),$$

then g is unbounded function on the strip T. Therefore we can not apply the classical Peano's Theorem for the Cauchy problem (1), (2), because g has to be bounded on T.

Proof. Since f is a bounded function on T then there exists a positive constant M such that

$$|f(x,y)| \leq M \quad \text{for} \quad (x,y) \in T.$$

We will prove our main result for $x \in [x_0, x_0 + a]$. In the same way one can prove the main result for $x \in [x_0 - a, x_0]$.

For $x \in [x_0, x_0 + a]$ we define the sequence $\{y_m(x)\}_{m=1}^{\infty}$ as follows

$$y_m(x) = y_0 \quad \text{for} \quad x \in \left[x_0, x_0 + \tfrac{a}{m}\right],$$

$$y_m(x) = y_0 + \textstyle\int_{x_0}^{x-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt \quad \text{for}$$

$$x \in \left[x_0 + k\tfrac{a}{m}, x_0 + (k+1)\tfrac{a}{m}\right], \quad k = 1, 2, \ldots, m-1.$$

For this sequence we have

1. Let $m \in \mathbb{N}$ is arbitrary chosen.

If $x \in \left[x_0, x_0 + \frac{a}{m}\right]$ then

$$|y_m(x)| = |y_0|.$$

If $x \notin \left[x_0, x_0 + \frac{a}{m}\right]$ and $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ for some $k = 1, 2, \ldots, m-1$, then

$$|y_m(x)| = \left|y_0 + \int_{x_0}^{x-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} (M_1|y_m(t)| + MM_2)dt$$

$$= |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x_0 + (k+1)\frac{a}{m} - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt,$$

i.e. for $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ we have

$$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x} |y_m(t)|dt.$$

From here and the Gronwall's inequality we get

$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x}(|y_0| + MM_2a)e^{M_1(x-t)}dt$

$= |y_0| + MM_2a + e^{M_1x}M_1(|y_0| + MM_2a)\int_{x_0}^{x} e^{-M_1t}dt$

$= |y_0| + MM_2a + e^{M_1x}(|y_0| + MM_2a)\left(e^{-M_1x_0} - e^{-M_1x}\right)$

$\leq |y_0| + MM_2a + e^{M_1(x-x_0)}(|y_0| + MM_2a)$

$\leq |y_0| + MM_2a + e^{aM_1}(|y_0| + MM_2a)$

$= (1 + e^{aM_1})(|y_0| + MM_2a) =: M_3 \quad \text{for} \quad x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right],$

for some $k = 1, 2, \ldots, m-1$.

Consequently for every $x \in [x_0, x_0 + a]$ we have

(9) $\qquad |y_m(x)| \leq M_3$

for every $m \in \mathbb{N}$.

Therefore the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded on $[x_0, x_0 + a]$.

2. Let $x_1, x_2 \in [x_0, x_0 + a]$ and $m \in \mathbb{N}$ is arbitrarily chosen. Then

1. case. $x_1, x_2 \in \left[x_0, x_0 + \frac{a}{m}\right]$. Then

$$y_m(x_1) = y_m(x_2) = y_0,$$

and therefore

$$|y_m(x_2) - y_m(x_1)| = 0.$$

2. case. Let $x_1 \in \left[x_0, x_0 + \frac{a}{m}\right]$, $x_2 \notin \left[x_0, x_0 + \frac{a}{m}\right]$. Then there exists $k \in \{1, 2, \ldots,$
$m-1\}$, such that $x_2 \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ and

$y_m(x_1) = y_0,$

$y_m(x_2) = y_0 + \int_{x_0}^{x_2 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt,$

from here,

$$|y_m(x_2) - y_m(x_1)| = \left|\int_{x_0}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right) dt\right|$$

$$\leq \int_{x_0}^{x_2-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right) dt$$

$$\leq MM_2\left(x_2 - \frac{a}{m} - x_0\right) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)| dt$$

now we use that $x_1 \in \left[x_0, x_0 + \frac{a}{m}\right]$

$$\leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)| dt,$$

i.e.

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2-\frac{a}{m}} |y_m(t)| dt.$$

From here and (9) we obtain

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1M_3 \int_{x_0}^{x_2-\frac{a}{m}} dt$$

$$= MM_2(x_2 - x_1) + M_1M_3\left(x_2 - \frac{a}{m} - x_0\right)$$

$$\leq (MM_2 + M_1M_3)(x_2 - x_1).$$

3. case. Let $x_1, x_2 \notin \left[x_0, x_0 + \frac{a}{m}\right]$. Without loss of generality we can suppose that $x_1 \leq x_2$. Let

$$x_1 \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right], \quad x_2 \in \left[x_0 + i\frac{a}{m}, x_0 + (i+1)\frac{a}{m}\right], \quad k \leq i,$$

$k, i \in \{1, 2, \ldots, m-1\}$.

Then

$$y_m(x_2) = y_0 + \int_{x_0}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt,$$

$$y_m(x_1) = y_0 + \int_{x_0}^{x_1-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt,$$

$$y_m(x_2) - y_m(x_1) = \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt,$$

$$|y_m(x_2) - y_m(x_1)| = \left|\int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq M_1 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} |y_m(t)|dt + MM_2 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} dt$$

$$= M_1 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} |y_m(t)|dt + MM_2(x_2 - x_1)$$

now we apply (9)

$$\leq M_1 M_3 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} dt + MM_2(x_2 - x_1)$$

$$= (M_1 M_3 + MM_2)(x_2 - x_1).$$

From 1, 2, and 3 cases follows that for every $x_1, x_2 \in [x_0, x_0 + a]$ we have

$$(10) \qquad |y_m(x_2) - y_m(x_1)| \leq (M_1 M_3 + MM_2)|x_2 - x_1| \quad \text{for} \quad \forall m \in \mathbb{N}.$$

Let $\epsilon > 0$ is arbitrary chosen and fixed. Let $\delta = \frac{\epsilon}{MM_2 + M_1 M_3}$. Then if $x_1, x_2 \in [x_0, x_0 + a]$, $|x_1 - x_2| < \delta$, using (10), we get

$$|y_m(x_2) - y_m(x_1)| \leq (M_1 M_3 + MM_2)|x_2 - x_1|$$

$$< (M_1 M_3 + MM_2)\delta = \epsilon.$$

Consequently $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous family on $[x_0, x_0 + a]$.

Therefore there exists a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ of the sequence $\{y_m(x)\}_{m=1}^{\infty}$ which is uniformly convergent to $y(x)$ on $[x_0, x_0 + a]$.

For $y_{m_p}(x)$, $x \in [x_0, x_0 + a]$, we have

$$y_{m_p}(x) = y_0 + \int_{x_0}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$(11) = y_0 + \int_{x_0}^{x} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$+ \int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt.$$

Since f is a continuous and bounded function on T we have

$$\lim_{p\longrightarrow\infty} \int_{x_0}^{x} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

(12)

$$= \int_{x_0}^{x} \Big(y(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt.$$

Also,

$$\Big|\int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt\Big|$$

$$\leq \int_{x-\frac{a}{m_p}}^{x} \Big(|y_{m_p}(t)|\Big|\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big| + |f(t, y_{m_p}(t))|\Big|1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big|\Big)dt$$

$$\leq M_1 \int_{x-\frac{a}{m_p}}^{x} |y_{m_p}(t)|dt + MM_2\tfrac{a}{m_p}$$

now we use (9)

$$\leq (M_1 M_3 + MM_2)\frac{a}{m_p} \longrightarrow_{p\longrightarrow\infty} 0.$$

From here and (11), (12), when $p \longrightarrow \infty$, we get

$$y(x) = y_0 + \int_{x_0}^{x} \Big(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

for every $x \in [x_0, x_0 + a]$. Therefore y is a solution of the Cauchy problem (1), (2) which is defined on $[x_0, x_0 + a]$. From (9) follows that $|y(x)| \leq M_3$ for every $x \in [x_0, x_0 + a]$. □

Corollary 2.10. *Let $f(x, y)$ be continuous in $\overline{S}$, and hence there exists a $M > 0$ such that $|f(x, y)| \leq M$ for all $(x, y) \in \overline{S}$. Then the initial value problem* (1), (2) *has at least one solution in J_h.*

Proof. The proof is the same as that of the proof of multiplicative iso-Peano's existence theorem with some obvious changes. □

Definition 2.11. *(ϵ-approximate solution) A function $y(x)$ defined in J is said to be an ϵ-approximate solution of the multiplicative iso-differential equation (1) if*

1. *$y(x)$ is continuous for all $x \in J$,*

2. *for all $x \in J$ the points $(x, y(x)) \in D$,*

3. *$y(x)$ has piecewise continuous derivative in J which may fail to be defined only for a finite number of points, say $x_1, x_2, \ldots, x_k$,*

4. $\left|y'(x) - y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\right| \leq \epsilon$ *for all $x \in J$, $x \neq x_i$, $i = 1, 2, \ldots, k$.*

The existence of an ϵ-approximate solution is provided in the following theorem.

Theorem 2.12. *Let $f(x,y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for every $(x,y) \in \overline{S}$. Then for all $\epsilon > 0$, there exists an ϵ-approximate solution $y(x)$ of the multiplicative iso-differential equation (1) in the interval J_h such that $y(x_0) = y_0$.*

Proof. Because the function $f(x,y)$ is a continuous function in the closed rectangle $\overline{S}$, it is uniformly continuous in this rectangle. Thus, for a given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ so that

$$|f(x,y) - f(x_1,y_1)| \leq \epsilon,$$

$$\left|y\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - y_1\frac{\hat{T}'(x_1)}{\hat{T}(x_1)} - f(x_1,y_1)\frac{\hat{T}(x_1)-x_1\hat{T}'(x_1)}{\hat{T}(x_1)}\right| \leq \epsilon$$

for all (x,y), $(x_1,y_1) \in \overline{S}$ such that

$$|x - x_1| \leq \delta \qquad \text{and} \qquad |y - y_1| \leq \delta.$$

We shall construct an ϵ-approximate solution in the interval $[x_0, x_0 + h]$. A similar process will define it in the interval $[x_0 - h, x_0]$.

For this aim, we divide the interval $[x_0, x_0 + h]$ into m parts

$$x_0 < x_1 < x_2 \ldots < x_m = x_0 + h$$

such that

$$x_i - x_{i-1} \leq \min\left\{\delta, \frac{\delta}{P(|y_0|+b+M)}\right\}, \qquad i = 1, 2, \ldots, m. \tag{13}$$

Now we define a function $y(x)$ in the interval $[x_0, x_0 + h]$ in the following manner

$$\begin{aligned} &y(x) = y(x_{i-1}) + (x - x_{i-1})\Big(y(x_{i-1})\tfrac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1}))\tfrac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\Big), \\ &x_{i-1} \leq x \leq x_i, \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{14}$$

Obviously, this function $y(x)$ is continuous and has a piecewise continuous derivative

$$y'(x) = y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1})\frac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})},$$

$x_{i-1} < x < x_i$, $i = 1, 2, \ldots, m$, which fails to be defined only at the points x_i, $i = 1, 2, \ldots, m - 1$. Since in each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, the function $y(x)$ is a straight line, to prove that $(x, y(x)) \in \overline{S}$ it suffices to show that

$$|y(x_i) - y_0| \leq b$$

for all $i = 1, 2, \ldots, m$.

For this reason, in (14) let $i = 1$ and $x = x_1$ to obtain

$$y(x_1) = y_0 + (x - x_1)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big),$$

$$|y(x_1) - y_0| = \Big|(x - x_1)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big)\Big|$$

$$\leq (x_1 - x_0)\Big(|y_0|\tfrac{|\hat{T}'(x_0)|}{\hat{T}(x_0)} + |f(x_0, y_0)|\tfrac{|\hat{T}(x_0) - x_0\hat{T}'(x_0)|}{\hat{T}(x_0)}\Big)$$

$$\leq (x_1 - x_0)(P|y_0| + MP)$$

$$\leq hP(M + |y_0|)$$

$$\leq hP(b + |y_0| + M)$$

$$\leq \tfrac{b}{P(b+|y_0|+M)} P(b + |y_0| + M)$$

$$= b.$$

Now let the assertion be true for $i = 1, 2, \dots, k-1 < m-1$, then from (14)

$$y(x_1) - y_0 = (x_1 - x_0)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big),$$

$$y(x_2) - y(x_1) = (x_2 - x_1)\Big(y(x_1)\tfrac{\hat{T}'(x_1)}{\hat{T}(x_1)} + f(x_1, y(x_1))\tfrac{\hat{T}(x_1) - x_1\hat{T}'(x_1)}{\hat{T}(x_1)},$$

$$\dots$$

$$y(x_k) - y(x_{k-1}) = (x_k - x_{k-1})\Big(y(x_{k-1})\tfrac{\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})} + f(x_{k-1}, y(x_{k-1}))\tfrac{\hat{T}(x_{k-1}) - x_{k-1}\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})}\Big).$$

From here,

$$y(x_k) - y_0 = \textstyle\sum_{l=1}^{k}(x_l - x_{l-1})\Big(y(x_{l-1})\tfrac{\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})} + f(x_{l-1}, y(x_{l-1}))\tfrac{\hat{T}(x_{l-1}) - x_{l-1}\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})}\Big),$$

which gives

$$|y(x_k) - y_0| \le \sum_{l=1}^{k}(x_l - x_{l-1})\Big(|y(x_{l-1})|\tfrac{|\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})} + |f(x_{l-1}, y(x_{l-1}))|\tfrac{|\hat{T}(x_{l-1}) - x_{l-1}\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})}\Big)$$

$$\le \sum_{l=1}^{k}(x_l - x_{l-1})\Big((b + |y_0|)P + MP\Big)$$

$$= P(M + b + |y_0|)\sum_{l=1}^{k}(x_l - x_{l-1})$$

$$= P(M + b + |y_0|)(x_k - x_0)$$

$$\le P(M + b + |y_0|)h$$

$$\le P(M + b + |y_0|)\tfrac{b}{P(M+b+|y_0|)}$$

$$= b.$$

Finally, if $x_{i-1} < x < x_i$, then from (13) and (14)

$$|y(x) - y(x_{i-1})| = (x - x_i)\Big|y(x_{i-1})\tfrac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}), y(x_{i-1}))\tfrac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\Big|$$

$$\le (x - x_i)\Big(|y(x_{i-1})|\tfrac{|\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})} + |f(x_{i-1}), y(x_{i-1}))|\tfrac{|\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})}\Big)$$

$$\le (x_i - x_{i-1})\Big((|y_0| + b)P + MP\Big)$$

$$\le \tfrac{\delta}{P(|y_0|+b+M)}P(M + |y_0| + b)$$

$$= \delta.$$

Therefore

$$\Big|y'(x) - y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}\Big|$$

$$= \Big|y(x_{i-1})\tfrac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1}))\tfrac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} - y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}\Big|$$

$$\le \epsilon$$

for all $x \in J_h$, $x \neq x_i$, $i = 1, 2, \ldots, m-1$. This completes the proof that $y(x)$ is an ϵ-approximate solution of the multiplicative iso-differential equation (1).

This method of constructing an approximate solution is said to be multiplicative iso-Cauchy-Euler method. □

Theorem 2.13. *(multiplicative iso-Cauchy-Peano's existence theorem) Let $f(x, y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x, y)| \leq M$ for every $(x, y) \in \overline{S}$. Then the initial value problem (1), (2) has at least one solution in J_h.*

Proof. We shall prove the assertion for the interval $[x_0, x_0 + h]$.

Let $\{\epsilon_m\}_{m=1}^{\infty}$ be a monotonically decreasing sequence of positive numbers such that

$$\lim_{m \longrightarrow \infty} \epsilon_m = 0.$$

For each ϵ_m we construct an ϵ_m-approximate solution $y_m(x)$.

As in the proof of the theorem for existence of ϵ-approximate solutions we have

$$|y_m(x)| \leq b + |y_0|$$

for every $m \in \mathbb{N}$ and for every $x \in J_h$. In other words, the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded in J_h.

Let x, $x^* \in [x_0, x_0 + h]$. Then

$$|y_m(x) - y_m(x^*)| \leq \left| \int_x^{x^*} \left(|y_m(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_x^{x^*} \Big(P(|y_0| + b) + MP \Big) dt \right|$$

$$\leq P(M + b + |y_0|)|x - x^*|$$

and from this it follows that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous.

Consequently the sequence $\{y_m(x)\}_{m=1}^{\infty}$ contains a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ which converges uniformly in $[x_0, x_0 + h]$ to a continuous function $y(x)$. We define

$$e_m(x) = \begin{cases} y_m'(x) - y_m(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y_m(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \\ \text{at the points where } y_m'(x) \text{ exists} \\ 0 \quad \text{otherwise.} \end{cases}$$

Then

$$(15)\qquad y_m(x) = y_0 + \int_{x_0}^{x} \left(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) + e_m(t) \right) dt$$

and

$$|e_m(x)| \leq \epsilon_m.$$

Since $f(x,y)$ is continuous in $\overline{S}$ and $y_{m_p}(x)$ converges to $y(x)$ uniformly in $[x_0, x_0 + h]$, the function $f(x, y_{m_p}(x))$ converges to $f(x, y(x))$ uniformly in $[x_0, x_0 + h]$. Thus, by replacing m by m_p in (15) and letting $p \longrightarrow \infty$, we become that $y(x)$ is a solution to the integral equation (4). □

Remark 2.14. *We suppose that all conditions of the multiplicative iso-Cauchy-Peano's existence theorem are satisfied. Further, let the initial value problem* (1), (2) *has a solution* $y(x)$ *in an interval* $J = (\alpha, \beta)$. *We have*

$$|y(x_2) - y(x_1)| \leq P(M + |y_0| + b)|x_2 - x_1|$$

for every x_1, $x_2 \in J$. *Therefore*

$$y(x_2) - y(x_1) \longrightarrow 0$$

as $x_1, x_2 \longrightarrow \alpha^+$. *Thus, by the Cauchy criterion of convergence we have that*

$$\lim_{x \longrightarrow \alpha^+} y(x)$$

exists.

A similar argument holds for

$$\lim_{x \longrightarrow \beta^-} y(x).$$

Theorem 2.15. *Let all conditions of the multiplicative iso-Cauchy-Peano's existence theorem be satisfied. Let also,* $y(x)$ *be a solution of the initial value problem* (1), (2) *in the interval* $J = (\alpha, \beta)$. *Then* $y(x)$ *can be extended over the interval* $(\alpha, \beta + \gamma]$ $([\alpha - \gamma, \beta))$ *for some* $\gamma > 0$.

Proof. We define the function $y_1(x)$ as follows.

$$y_1(x) = y(x) \qquad \text{for} \qquad x \in (\alpha, \beta),$$

$$y_1(\beta) = y(\beta - 0).$$

We observe that for all $x \in (\alpha, \beta]$ we have

$$y_1(x) = y(\beta - 0) + \int_{\beta}^{x} \left(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \right) dt$$

$$= y(x_0) + \int_{x_0}^{\beta} \left(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \right) dt$$

$$+ \int_{\beta}^{x} \left(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \right) dt$$

$$= y(x_0) + \int_{x_0}^{x} \left(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \right) dt.$$

Therefore the left-hand derivative $y_1'(\beta - 0)$ exists and

$$y_1'(\beta - 0) = y_1(\beta)\frac{\hat{T}'(\beta)}{\hat{T}(\beta)} + f(\beta, y_1(\beta))\frac{\hat{T}(\beta) - \beta\hat{T}'(\beta)}{\hat{T}(\beta)}.$$

Thus, $y_1(x)$ is a continuation of $y(x)$ in the interval $(\alpha, \beta]$.

Let $y_2(x)$ be a solution to the problem

$$y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$$

$$y(\beta) = y_1(\beta),$$

existing in the interval $[\beta, \beta + \gamma]$ for some $\gamma > 0$.

We define the function

$$y_3(x) = \begin{cases} y_1(x) & x \in (\alpha, \beta], \\ y_2(x) & x \in [\beta, \beta + \gamma], \end{cases}$$

which is a continuation of $y(x)$ in the interval $(\alpha, \beta + \gamma]$.

Also,

$$y_3(x) = y_0 + \int_{x_0}^{x} \left(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt$$

for every $x \in (\alpha, \beta+\gamma]$, because for all $x \in [\beta, \beta+\gamma]$ we have

$$y_3(x) = y(\beta - 0) + \int_\beta^x \left(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt$$

$$= y_0 + \int_{x_0}^\beta \left(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}x(t)}\right)dt$$

$$+ \int_\beta^x \left(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt$$

$$= y_0 + \int_{x_0}^x \left(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt.$$

□

Theorem 2.16. *(multiplicative iso-Lipschitz uniqueness theorem) Let $f(x,y)$ be continuous and satisfies a uniform Lipschitz condition in $\overline{S}$ with a Lipschitz constant L. Then the problem (1), (2) has at most one solution in $|x - x_0| \le a$.*

Proof. We suppose that the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$, $x \in [x_0 - a, x_0 + a]$. Then

$$y_1(x) = y_0 + \int_{x_0}^x \left(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt,$$

$$y_2(x) = y_0 + \int_{x_0}^x \left(y_2(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_2(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt,$$

whereupon

$$y_1(x)-y_2(x) = \int_{x_0}^x \left((y_1(t)-y_2(t))\frac{\hat{T}'(t)}{\hat{T}(t)}+(f(t,y_1(t))-f(t,y_2(t)))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt,$$

and

$$|y_1(x) - y_2(x)| \le \left|\int_{x_0}^x \left(|y_1(t) - y_2(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\le \left|\int_{x_0}^x \Big(P|y_1(t) - y_2(t)| + LP|y_1(t) - y_2(t)|\Big)dt\right|$$

$$= P(1+L)\left|\int_{x_0}^x |y_1(t) - y_2(t)|dt\right|.$$

From the last inequality and Gronwall's type inequality we conclude that

$$|y_1(x) - y_2(x)| = 0 \qquad \text{in} \qquad [x_0 - a, x_0 + a].$$

□

Theorem 2.17. *(multiplicative iso-Peano's uniqueness theorem) Let $f(x, y)$ be continuous in*

$$\overline{S}_+ = \{(x, y) \in \mathbb{R}^2 : x_0 \le x \le x_0 + a, |y - y_0| \le b\}$$

and nonincreasing in y for all $[x_0, x_0 + a]$. Let also,

$$\hat{T}'(x) \le 0, \qquad \hat{T}(x) - x\hat{T}'(x) \ge 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) has at most one solution in $[x_0, x_0 + a]$.

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0 + a]$ which differ in $[x_0, x_0 + a]$. We assume that

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

while $y_1(x) = y_2(x)$ for $x \in [x_0, x_1]$, i.e., x_1 is the greatest lower bound of the set A consisting of those x for which $y_2(x) > y_1(x)$. This greatest lower bound of the set A exists because the set A is bounded below by x_0 at least. Thus for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_1(x)) \ge f(x, y_2(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \ge f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

$$\ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

for all $x \in (x_1, x_1 + \epsilon)$, and from here

$$y_1'(x) \ge y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon).$$

Hence the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$.

Because

$$z(x_1) = y_2(x_1) - y_1(x_1) = 0$$

we obtain

$$z(x) \leq z(x_1) = 0 \qquad \text{in} \qquad (x_1, x_1 + \epsilon)$$

or

$$y_2(x) \leq y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon).$$

This contradiction proves that

$$y_1(x) = y_2(x) \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

□

Theorem 2.18. *(multiplicative iso-Peano's uniqueness theorem) Let $f(x, y)$ be continuous in $\overline{S}_+$ and nondecreasing in y for every $x \in [x_0, x_0 + a]$. Let also,*

$$\hat{T}'(x) \leq 0, \qquad \hat{T}(x) - x\hat{T}'(x) \leq 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) *has at most one solution in* $[x_0, x_0 + a]$.

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0 + a]$ which differ in $[x_0, x_0 + a]$. Let

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

and

$$y_2(x) = y_1(x) \qquad \text{for} \qquad \forall x \in [x_0, x_1].$$

Therefore for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_2(x)) \geq f(x, y_1(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \geq f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \geq y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$\geq y_2(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

for every $x \in (x_1, x_1 + \epsilon)$. Consequently

$$y_1'(x) \geq y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon)$$

and then the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$, therefore

$$y_2(x) - y_1(x) \leq y_2(x_1) - y_1(x_1) = 0 \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon),$$

which is a contradiction. From here we conclude that $y_1(x) = y_2(x)$ for every $x \in [x_0, x_0 + a]$. □

Lemma 2.19. *(multiplicative iso-Osgood's lemma) Let $w(z)$ be continuous function in $[0, \infty)$, $w(0) = 0$, $z + w(z) > 0$ in $(0, \infty)$, $z + w(z)$ be increasing function in $[0, \infty)$, and*

(16) $\quad \lim_{\epsilon \longrightarrow 0^+} \int_\epsilon^a \frac{dz}{z+w(z)} = \infty.$

Let $u(x)$ be a nonnegative continuous function in $[0, a]$. Then the inequality

(17) $\quad u(x) \leq P \int_0^x (u(t) + w(u(t)))dt, \qquad 0 < x \leq a,$

implies that $u(x) \equiv 0$ in $[0, a]$.

Proof. We define the function

$$v(x) = \max_{0 \leq t \leq x} u(t)$$

and assume that $v(x) > 0$ for $0 < x \leq a$. Then

$$u(t) \leq v(x) \qquad \text{for} \qquad \forall t \in [0, x].$$

Because $u(x)$ is a continuous function in $[0, a]$ then there exists $x_1 \in [0, x]$ such that

$$v(x) = u(x_1).$$

Therefore, using that $z+w(z)$ is an increasing function in $[0,\infty)$,

$$v(x)=u(x_1)\le P\int_0^{x_1}(u(t)+w(u(t)))dt$$

$$\le P\int_0^{x_1}(v(t)+w(v(t)))dt$$

$$\le P\int_0^{x}(v(t)+w(v(t)))dt.$$

Let

$$\overline{v}(x)=P\int_0^x(v(t)+w(v(t)))dt.$$

We have

$$\overline{v}(x)\ge 0,\qquad v(x)\le\overline{v}(x),$$

and

$$\overline{v}'(x)=P(v(x)+w(v(x)))$$

$$\le P(\overline{v}(x)+w(\overline{v}(x))),$$

and since

$$\overline{v}(x)+w(\overline{v}(x))\ge 0,$$

then

$$\frac{\overline{v}'(x)}{P(\overline{v}(x)+w(\overline{v}(x)))}.$$

Consequently for $0<\delta<a$ we have

$$\int_\delta^a\frac{d\overline{v}(x)}{P(\overline{v}(x)+w(\overline{v}(x)))}\le\int_\delta^a dx,$$

whereupon

$$\lim_{\delta\longrightarrow 0^+}\int_\delta^a\frac{d\overline{v}(x)}{P(\overline{v}(x)+w(\overline{v}(x)))}=\lim_{\delta\longrightarrow 0^+}\int_{\overline{v}(\delta)}^{\overline{v}(a)}\frac{dy}{P(y+w(y))}$$

$$\le a,$$

which contradicts with (16). Consequently $u(x)\equiv 0$ in $[0,a]$. □

Theorem 2.20. *(multiplicative iso-Osgood's uniqueness theorem) Let $f(x,y)$ be continuous in $\overline{S}_+$ and for all (x,y_1), $(x,y_2)\in\overline{S}_+$ it satisfies*

$$|f(x,y_1)-f(x,y_2)|\le w(|y_1-y_2|),$$

where $w(z)$ satisfies all conditions of the multiplicative iso-Osgood's lemma. Then the problem (1), (2) has at most one solution in $[x_0,x_0+a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0, x_0 + a]$. Then, if

$$z(x) = |y_1(x) - y_2(x)|, \qquad x \in [x_0, x_0 + a],$$

we have

$$z(x) = \left| \int_{x_0}^{x} \left((y_1(t) - y_2(t)) \frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_2(t))) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \right) dt \right|$$

$$\leq \int_{x_0}^{x} \left(|y_1(t) - y_2(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt$$

$$\leq \int_{x_0} (P|y_1(t) - y_2(t)| + Pw(|y_1(t) - y_2(t)|)) dt$$

$$= P \int_{x_0}^{x} (z(t) + w(z(t))) dt.$$

Let

$$u(x) = z(x_0 + x).$$

Therefore

$$u(x) \leq P \int_{x_0}^{x_0 + x} (z(t) + w(z(t))) dt$$

$$= P \int_0^x (z(x_0 + t) + w(z(x_0 + t))) dt$$

$$= P \int_0^x (u(t) + w(u(t))) dt.$$

Consequently $u(x)$ satisfies the multiplicative iso-Osgood's lemma, from where $u(x) \equiv 0$ in $[0, a]$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

Lemma 2.21. *(multiplicative iso-Nagumo's lemma) Let $u(x)$ be nonnegative continuous function in $[x_0, x_0 + a]$ and $u(x_0) = 0$, and let $u(x)$ be differentiable at $x = x_0$ with $u'(x_0) = 0$. Then*

$$\int_{x_0}^{x} u(t) dt \leq a \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a],$$

and the inequality

$$u(x) \leq \int_{x_0}^{x} \frac{u(t)}{t - x_0} dt, \qquad x \in [x_0, x_0 + a],$$

implies that $u(x) = 0$ in $[x_0, x_0 + a]$.

Proof. Let

$$g(x) = \int_{x_0}^{x} u(t)dt - a\int_{x_0}^{x} \frac{u(t)}{t-x_0}dt, \qquad x \in [x_0, x_0+a].$$

Since

$$\lim_{x \longrightarrow x_0} \frac{u(x)}{x-x_0} = u'(x_0) = 0,$$

then the integral

$$\int_{x_0}^{x} \frac{u(t)}{t-x_0}dt$$

exists for $x \in [x_0, x_0+a]$.

Also,

$$g'(x) = u(x) - a\frac{u(x)}{x-x_0} = u(x)\frac{x-x_0-a}{x-x_0} \leq 0$$

for every $x \in [x_0, x_0+a]$. Therefore g is a nonincreasing function in $[x_0, x_0+a]$, whereupon

$$g(x) \leq g(x_0) \qquad \text{for} \qquad \forall x \in [x_0, x_0+a],$$

or

$$\int_{x_0}^{x} u(t)dt \leq a\int_{x_)}^{x} \frac{u(t)}{t-x_0}dt$$

for every $x \in [x_0, x_0+a]$.

Let now

$$v(x) = \int_{x_0}^{x} \frac{u(t)}{t-x_0}dt, \qquad x \in [x_0, x_0+a].$$

Then

$$u(x) \leq v(x), \qquad x \in [x_0, x_0+a],$$

and

$$v'(x) = \tfrac{u(x)}{x-x_0}$$

$$\leq \tfrac{v(x)}{x-x_0}, \qquad x \in [x_0, x_0+a].$$

Consequently

$$\tfrac{d}{dx}\left(\tfrac{v(x)}{x-x_0}\right) = \tfrac{v'(x)(x-x_0)-v(x)}{(x-x_0)^2}$$

$$\leq 0$$

or the function

$$l(x) = \frac{v(x)}{x - x_0}$$

is a nonincreasing function in $[x_0, x_0 + a]$ and since $l(x_0) = 0$, we have that

$$v(x) \leq 0 \qquad \text{in} \qquad [x_0, x_0 + a],$$

from where

$$v(x) = 0 \qquad \text{in} \qquad [x_0, x_0 + a].$$

Consequently $u(x) = 0$ in $[x_0, x_0 + a]$. □

Theorem 2.22. *(multiplicative iso-Nagumo's theorem) Let $P(a+1) \leq 1$, $f(x, y)$ be continuous in $\overline{S}_+$ and for all (x, y_1), $(x, y_2) \in \overline{S}_+$ it satisfies*

$$|f(x, y_1) - f(x, y_2)| \leq k|x - x_0|^{-1}|y_1 - y_2|, \qquad x \neq x_0. \qquad k \leq 1.$$

Then the problem (1), (2) *has at most one solution in* $[x_0, x_0 + a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0, x_0 + a]$. Then for $x \in [x_0, x_0 + a]$ we have

$$|y_1(x) - y_2(x)| \leq \int_{x_0}^{x} \left(|y_1(t) - y_2(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right) dt$$

$$\leq \int_{x_0}^{x} \left(P|y_1(t) - y_2(t)| + k(t - y_0)^{-1}|y_1(t) - y_2(t)|P\right) dt$$

$$\leq P \int_{x_0}^{x} |y_1(t) - y_2(t)| dt + P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq aP \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt + P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$= (a+1)P \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq \int_{x_0}^{x} \frac{|y_1(t) - y_2(t)|}{t - x_0} dt.$$

Let

$$u(x) = |y_1(x) - y_2(x)|, \qquad x \in [x_0, x_0 + a].$$

Then $u(x_0) = 0$ and from the mean value theorem we have

$$u'(x_0) = \lim_{h \longrightarrow 0} \frac{u(x_0+h)-u(x_0)}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{|y_1(x_0)+hy_1'(x_0+\theta_1 h)+y_2(x_0)-hy_2'(x_0+\theta_2 h)|}{h}$$

$$= (\mathrm{sgn} h) \lim_{h \longrightarrow 0} |y_1'(x_0+\theta_1 h) - y_2'(x_0+\theta_2 h)|$$

$$= 0, \qquad 0 < \theta_1, \theta_2 < 1.$$

Then the conditions of multiplicative iso-Nagumo's lemma are satisfied and $u(x) = 0$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

References

[1] R. M. Santilli, "Embedding of Lie-algebras into Lie-admissible algebras," *Nuovo Cimento* 51, 570 (1967), http://www.santilli-foundation.org/docs/Santilli-54.pdf

[2] R. M. Santilli, "An introduction to Lie-admissible algebras," *Suppl. Nuovo Cimento*, 6, 1225 (1968).

[3] R. M. Santilli, "Lie-admissible mechanics for irreversible systems." *Meccanica*, 1, 3 (1969).

[4] R. M. Santilli, "On a possible Lie-admissible covering of Galilei's relativity in *Newtonian mechanics for nonconservative and Galilei form-noninvariant systems*,". 1, 223-423 (1978), available in free pdf download from
http://www.santilli-foundation.org/docs/Santilli-58.pdf

[5] R. M. Santilli, "Need of subjecting to an experimental verification the validity within a hadron of Einstein special relativity and Pauli exclusion principle," *Hadronic J.* 1, 574-901 (1978), available in free pdf download from
http://www.santilli-foundation.org/docs/Santilli-73.pdf

[6] R. M. Santilli, *Lie-admissible Approach to the Hadronic Structure,* Vols. I and II, Hadronic Press (1978)

http://www.santilli-foundation.org/docs/santilli-71.pdf
http://www.santilli-foundation.org/docs/santilli-72.pdf

[7] R. M. Santilli, *Foundation of Theoretical Mechanics,* Springer Verlag. Heidelberg, Germany, Volume I (1978), *The Inverse Problem in newtonian mechanics,*
http://www.santilli-foundation.org/docs/Santilli-209.pdf
Volume II, *Birkhoffian generalization lof hamiltonian mechanics,* (1982), http://www.santilli-foundation.org/docs/santilli-69.pdf

[8] R. M. Santilli, Nonlocal-Integral Isotopies of Differential Calculus, Mechanics and Geometries, in Isotopies of Contemporary Mathematical Structures, Rendinconti Circolo Matematico Palermo, Suppl. Vol. 42, pp. 7-82, 1996."

[9] S. Georgiev, Foundations of Iso-Diffferential Calculus, Vol.I-VI, 2014-2016, Nova Science Publisher.

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PRINCIPLE(S) OF CAUSALITY AS *DE FACTO* FUNDAMENTAL IN MATHEMATICAL PHYSICS, INCLUDING FOR CHANCE CAUSALITY APPLIED IN QUANTUM MECHANICS

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Abstract

According to the causality theory presented in the differential ontology and epistemology of Johansen (2008), concepts of randomness and probability are (i) *built on fundamental* types of causality, and (ii) represent themselves *particular, elaborated* types of causality. Hence, it is argued that Einstein vas basically correct in insisting on preserving the causality principle against the Copenhagen interpretation of quantum mechanics which he considered to be an incomplete theory. Adequate philosophical interpretations of quantum mechanics and of further developments into hadronic mechanics require concise differentiations and combinations of various types of causality, including chance causality. This is argued to be in agreement with some crucial results from the mathematical physics of David Bohm as well as of Ruggero Maria Santilli in relation to upgraded discussions of the Einstein-Podolsky-Rosen paradox.

Keywords: Einstein; Bohr; Bohm; Santilli; Popper: Einstein-Podolsky-Rosen; differential ontology; differential epistemology; causality; probability; randomness; chance; hadronic mechanics; Fibonacci algorithm; qualitative informatics.

I have mentioned Santilli, and I should like to say that he – one who belongs to a new generation – seems to me to move on a different path. Far be it from me to belittle the giants who founded quantum mechanics under the leadership of Planck, Einstein, Bohr, Born, Heisenberg, de Broglie, Schrodinger, and Dirac. Santilli too makes it very clear how greatly he appreciates the work of these men. But in his approach he distinguishes the region of the 'arena of incontrovertible applicability' of quantum mechanics (he calls it 'atomic mechanics') from nuclear mechanics and hadronics, and his most fascinating arguments in support of the view that quantum mechanics should not, without new tests, be regarded as valid in nuclear and Hadronic mechanics, seem to me to augur a return to sanity: to that realism and objectivism for which Einstein stood, and which had been abandoned by those two very great physicists, Heisenberg and Bohr. (Karl Popper 1982:14)

Die Quantenmechanik ist sehr achtung-gebietend. Aber eine innere Stimme sagt mir, daß das doch nicht der wahre Jakob ist. Die Theorie liefert viel, aber dem Geheimnis des Alten bringt sie uns kaum näher. Jedenfalls bin ich überzeugt, daß der *nicht würfelt.*

(Einstein 1926: Letter to Max Born)

> *(Quantum mechanics is very imposing. But an inner voice tells me that this is still not the true Jacob. The theory delivers much, but it barely brings us closer to the secret of The Old One. In any case, I am convinced that He does not throw dice.)*

What is in the notion of "throwing a dice" ?

Let us inspect and unfold what logical operations that reside *enfolded* in the notion of throwing dices as an exemplar of randomness and probability distributions. There are six classes of possible results for every event, the top face of the cube ending up as 1, 2, 3, 4, 5 or 6. When we consider the total result of many such events, we can group these results into six classes of results E_j where index j varies from 1 to 6.

In each particular event the individual effect is uniquely determined by physical laws and the initial conditions for the particular throw of the dice, such as gravitation, the force and direction of the throw, inertia and texture of the table, shape and texture of the dice etc. Thus, each individual effect is de facto uniquely determined by its corresponding and preceding individual cause, with the effect resulting from the cause by *physical causality*. If we specify and compare the individual causes in sufficient detail to pinpoint the decisive physical differences between causes that result in the physical differences between effects, the six *classes* of effects are also to be regarded as the result of six corresponding and preceding *classes of causes* C_i, where index i varies from 1 to 6. Thus, also the six classes of effects would result from six *classes* of causes by physical causality.

In the case of throwing dice it is difficult to specify and compare the individual causes in sufficient detail to establish the six classes of causes. It is hard to see that any easy attempt to specify significant variation in attributes between individual causes would favor one class of effects towards the other classes of effects. This consideration becomes reinforced when our empirical experience indicates that the six classes of effects occur almost equally often, and the more equal the more events of throwing the dice we consider. Hence, we find it adequate to regard the result of throwing dice *as if* it was *random* which class of effect the dice ended up into. This does not imply that we *really* mean that each individual effect is *not* uniquely determined from each individual cause, or that the six classes of effects are not uniquely determined from six imagined classes of causes, or that we will deny that both these determinations happen by *physical* causality *if* we investigated the events in sufficient microscopic physical detail. It only means that such an investigation is *not worth the effort and trouble* for our purpose at hand. It represents a huge advantage in *thought economy* for description and explanation when we radically simplify the whole constellation of physical events by applying the *simile* category of randomness instead of remaining (solely) at the physical level for description and explanation.

When it is considered *random* which class of effect the throw of the dice ends up into, this implies that the result can be considered random in *relation* to both the *individual* physical cause and to the *classes* of physical causes. Compared to description and explanation by merely physical causality, this represents a radical simplification at the cause side of the logical expression.

The concept of randomness ignores and deletes all *internal* distinctions at the side of the cause, regarding them *as if* they were irrelevant for the effect. However, this does not imply any *annihilation* of cause. Now the form of the logical expression becomes:

> ***Exp1***: IF [cause] a dice is thrown as an individual event, with a resulting individual effect belonging to six possible classes of E_j; THEN [effect] it is considered *random* which class of E_j that will be the effect of the individual event

Thus, introduction of the concept of randomness does no way contradict causality as such, but makes possible a *novel* type of causality which we denote *randomness causality*. By *adding* this novel type of causality, including the according type of simile, the universe of causal relations becomes *expanded*, not *restricted*.

We realize that randomness causality represents a certain, novel type of causality *built on* the preexisting causality type of physical causality, and – further – that the adequacy of randomness causality is *underpinned* by *certain* relations of physical causality. Thus, it represents a philosophical mistake of category, i.e. a mistake in consistent meta-thinking, to consider randomness causality to *contradict* or undermine physical causality.

This is with respect to the very *category* of randomness causality as regarded from conceptual logic. In order for this novel causality type to become *adequately* mobilized and applied, in partial substitution of underpinning relations of physical causality, certain requirements *from* said underpinning physical relations must be met. These requirements fall into two classes, depending on whether the physical requirements for randomness are considered *ad negativo* vs. *ad positivo*:

Ad negativo: The totality of *external* physical relations existing together with the cause in *Exp1* and with the physical underpinnings of individual causes, or during the time span from cause to effect, is considered *irrelevant* for the relation between cause and effect at the level of physical causality. When already having established the concept of randomness, this means that these external physical relations are regarded as *random* and as *cancelling out* in relation to the cause in *Exp1* and to the physical underpinnings of individual causes. Thus, the exclusion of these external physical relations expresses the relevance of the concept of randomness for excluding purposes. We denote this as *negative* randomness.

Ad positivo: *Exp1* expresses *positive* randomness, i.e. the application of randomness causality *after* irrelevant externalities have become excluded by negative randomness. In order for this (positive) causality type to become adequate, it is required, as already stated, that the differences in individual physical causes really do result in a an (approximately) equal – i.e. random – distribution between the six possible classes of physical *effects*. In order for this to happen, it must also be the case that the individual cases of physical causes are distributed (approximately) equally between six imagined classes of physical *causes* C_i preceding and corresponding to their respective six classes of E_j.

That these said (approximately) equal distributions *really* are the case we *discover* by solely investigating the distribution of individual effects among the six classes of E_j. It is this discovery from inspecting the distribution of physical effects from *physical* causality that makes it adequate to *ignore* any inspection of the differences between individual *physical* causes, and thus also to ignore the make-up and internal differences between the imagined six *classes* of physical causes C_i.

When we from our scientific *thinking* do not see any obvious reason for one E_j to occur more often than another one, it is adequate to consider it random which E_j an individual physical effect will show to belong to. In the next step, though, it makes a big difference whether our hypotheses from thinking becomes *supported* or not, through experimental evidence. In the case of throwing dices it *does* become supported from our observation of the distribution of individual physical effects. It is our discovery from observation of the individual *physical* effects that makes it adequate to regard any difference between physical causes *as if* they were random.

Thus, the positive randomness causality of *Exp1* does not contradict physical causality, but presupposes physical causality by observing a *certain* pattern in the distribution of physical effects, i.e. of *physical* effects by *physical* causality from *physical* causes.

Thus, *before* establishing *Exp1*, and especially after observation from experiment, there is another cognitive operator in place, stating that *because* it is regarded as random what E_j the dice ends up into, the application of *Exp1* is regarded as an adequate consequence.

Obviously, observations of a *random* pattern in the distribution of individual physical effects between the classes E_j will *not* happen if the physical system is characterized by *deviations* from an idealized situation of throwing a dice, e.g. if the dice is thrown a very short distance, if the dice is not a regular cube, if the eyes of the dice are magnetic and the dice is thrown at a magnetic table, etc. Thus, there exist obvious constraints for which physical systems that can be adequately described or explained with good approximation by randomness causality. Only *certain* physical systems can be adequately approached by means of randomness causality.

When it is regarded as *random* which E_j a throw of a dice ends up with as physical effect, this means that the six different classes of effects are regarded as *non-differentiated* with respect to *probability*. However, in next steps of thought the considered randomness between each six classes, rather automatically leads to various cases of *non*-randomness and *differentiations* with respect to probability. As trivial examples, the probability of effect E_1 OR E_2 from one throw of the dice will be twice the probability of effect E_3, and the probability of effect E_1 to occur twice from throwing the dice two times will be 1/6 x 1/6 = 1/36. The laws of probability distributions and mathematical probability theory as a whole emerges from systematic unfoldments of what resides enfolded in the very *concepts* of randomness and probability.

Since the concept of probability both presupposes and follows rather directly from the concept of randomness, we realize that *probability causality* also presupposes and follows rather directly from randomness causality as a novel and somewhat more elaborated *type* of causality than randomness causality. *Because* classes of physical effects are regarded as random compared to each other, more elaborated regroupings and sequences of these classes of effects must be *non*-random and *differ in probability* by exact mathematical laws. Thus, explanation of a physical system by means of probability causality will essentially relate to physical causality the same way as explanation of a physical system by means of randomness causality relates to physical causality in manners we already have clarified. We apply the broader term *chance causality* to cover both randomness causality and probability causality.

To sum up we realize that:

(i) randomness does not *contradict* causality, but implies a certain *type* of causality as expressed by *Exp1*;

(ii) randomness causality does not *eliminate physical* causality, but represents a more *elaborated* type of causality (by adding a certain simile) which *presupposes* physical causality;
(iii) randomness causality represents an *adequate wrapping* with (formally regarded) *partial* substitution of physical causality only when being *underpinned* by *certain* cases of physical systems already characterized by *physical* causality.

(From the above it should be clear that what here is stated with respect to randomness and randomness causality, also holds for probability causality and thus for chance causality as a whole.)

It follows from (i) that the very question of whether the universe is based on random events or causality does not make any good sense, and even less to claim the first alternative. It follows from (ii) that the very question of whether the universe is based on chance causality or physical causality does not make any good sense, and even less to claim the first alternative. These conclusions follow from strict philosophical reasons without respect to the scientific theory under consideration. For theories in mathematical physics to reach adequate and fully mature expressions, they should be consistent with points (i), (ii) and (iii), and theoretical developments might benefit from deeper and more detailed reflections on their scientific material in relation to these points.

In theories of mathematical physics interpretations and discussions with respect to the role of causality, tend to consider causality only in the sense of *physical* causality. In our philosophical treatise *Outline of Differential Epistemology* (Johansen 2008), we pretended to have presented a rather complete systematic development and exhibition of the whole *nexus of causality types* (cf. Johansen 2008: ch.3, 113-194; 248-9). It was disclosed and explained that the nexus included several types of causality, classified into ten fundamental types and ten elaborated types. *Chance causality* was presented as one among the ten elaborated types (cf. Johansen 2008: 165-175), while *physical causality* was presented as the *least* fundamental among the ten fundamental type (cf. Johansen 2008: 155-157). Above we have sought to clarify that chance causality for strict reasons of conceptual logic can not constitute any fundamental causality type on an equal footing with physical causality. However, even physical causality can not adequately be considered *that* fundamental as usually regarded in theoretical physics. In the following we will present *some* clarification of why this must be the case, in order to contribute to some according clarification when contemplating the role of causality in theoretical physics. (For further discussion of various aspects, see Johansen 2008a, 2008b, 2008c, 2013, 2017.)

Abstract causality vs. formal-logical causality

When throwing a dice is considered a physical event implying physical causality between individual cause and individual effect, there already reside more fundamental types of causality *enfolded* in the very notion of physical causality.

Before differentiating between *types* of causality there already must exist a *universal* and *abstract* concept of causality *as such* in cognition, namely the concept of the relation between two relata where a logically proceeding relatum denoted 'effect', *with logical necessity* follows from a logically preceding relatum denoted 'cause'. In conventional formal logic causality is approached by the notion *material implication* where binary *truth values* of cause and effect in a logical expression *first* are assumed (or determined) *independently* of each other, *whereafter* material implication is defined as a certain *truth function* of the four pairs of said truth values, more specifically that the material implication is decided as true iff the pair (cause is true; effect is false) does not show up. While highly useful for many purposes, e.g. computer electronics, this approach to define causality is too shallow to hit the mark of that which it attempts to target and catch the essence of.

From the definition of material implication the following expressions will be decided as true:

(1) $p \Rightarrow (q \Rightarrow p)$
$-p \Rightarrow (p \Rightarrow q)$
$(p \Rightarrow q) \vee (q \Rightarrow p)$

However, the claimed truths of such expressions are rather contra-intuitive and not aligned with the concept of the relation IF…THEN… that *tacitly* is *de facto operative* in our ordinary cognition. The modus operandi of our innate, subconscious (or rather *supra*conscious) category of 'causality' does not start out with first separating and establishing candidates to cause and effect, for next comparing pairs of their truth values, and thereafter deciding one pair as a causal relation in distinction to the three other pairs. Our innate category starts out with a logical entity of truth (cause) which *unfolds organically* into another logical entity of truth (effect) as its logically *necessary* result or fruit in a relation which we are not able to reflect upon before *after* the unfoldment has taken place. (And if performing such a reflection we will remobilize the *same* innate category of causality at a meta-level.)

To hit the mark of the implicate, innate cognitive category of causality an adequate approach has to be much more abstract, deeper and accurate than what was the case for establishing the concept of material implication in conventional formal logic. In our treatise (Johansen 2008) we presented a novel – and tentatively completed – theory in order to cover and solve core issues of philosophy. We denoted our philosophical theory *differential ontology,* including *differential epistemology* as the more sophisticated "head" unfolded from and (next) into the ontological "body". Our philosophy presented a systematic unfoldment of categories residing enfolded inside *information as such,* i.e. inside *something being,* whatever it might be, conceived in its most elementary, abstract and universal sense.

The starting point for our systematic philosophical exhibition and successive unfoldments of categories, was *information as such.* The concept of information in the most abstracted *qualitative* sense, was established as close to Gregory Bateson's famous definition of information as a *difference that makes a difference* for something/someone. (Our relatively minor and subtle deviations from the definition by Bateson do not matter much for the present text.) This definition can be reformulated as an input-difference *making* an output-difference for a *subject.* (If the subject is not a human, or not even a living being, when e.g. one billiard ball receives – and reacts to – an input-difference from being hit by another billiard ball, the category of subject is operative by a minimum of anthropomorphic projection applied as an adequate *simile.* Already our grammar, with classes of subjects, verbs and objects, applies a minimum of such a simile.) Thus, the category of subject, whether in the emphatic sense or in the simile sense, is with necessity implied in the very definition of information in the most abstract sense. One striking illustration of the *tacit* de facto inclusion of the subject may be the notion of "rock hard reality", not possible to deny for anyone with their senses intact, applying as exemplar the situation of a heavy stone falling on the toes of a human. Here we notice that a preferred exemplar of "rock hard reality" *depends* on the inclusion of the subject, i.e. of one emphatic subject showing (strong) *emotion.* In the most abstract concept of information the input-difference makes an output-difference for a *third* entity which is considered a *subject* by having *emotion* (in the most abstract sense), and – as already stated – with pseudo-subjects *not* considered as having emotions treated with a minimum of anthropomorphic projection from the analyzing human subject.

When starting out with this most abstract definition of information as such, the very act of the input-difference *making* (or better: *unfolding into*) the output-difference, can be adequately conceived as the *causal* relation between input-difference and output-difference, when this act of making, or unfoldment into, is cognitively regarded as *logically* necessary. In order to conceive this relation as *solely* logically necessary, we have to conceive the input-difference unfolding into an output-difference as *abstracted into* an imagined *pure and free-standing thought universe* of solely logical relations *without* regard to any connection to a physical input-difference and a physical output-difference.

It is *inside* this imagined pure and free-standing universe of thought, *without* regard to correlations to *physical* input-differences and output-differences, we can conceive causality, in its most abstract sense, as the relation from an input-difference unfolding into an output-difference. The cognitive category of causality must be grasped in its purest and most abstract sense before we can study *how* the category is universally *implied* in various *types* of causality such as e.g. physical causality.

In our philosophical treatise we explained how the most abstract notion of causality as organically unfolding the input-difference into an output-difference, could be adequately *back-reflected* by a certain *formal* representation achieved by means of set theory when placing and relating elements and classes at concisely differentiated *ontological levels of thought* inside a freestanding thought universe. The differentiations in ontological levels was presented as unfolding with necessity from consistent reflection from and upon the thought of information as such, in some distinction to modal logic which have tried to overcome – with some success – shortcomings in the notion of material implication by adding various logical operators while still not acknowledging the necessity of developing differentiations in ontological levels of thought in an organic, strict and systematic way quite different from "freely" playing around with voluntaristic constructions inside a logical toy universe.

In the present context it would take too much space to try to represent our formal expressions and according philosophical reasoning. The main point is that we found it possible and adequate to express the most abstract notion of causality by means of formal logic, so that abstract causality could be renamed *formal-logical causality* presupposing that this refers to the *particular* formal expression presented in our treatise.

Projective causality

Let us take a look at sensory perception. Neuroscience has shown that perception has a stepwise constitution, so that there is a lot going on from the subject receives an initial *recept,* defined as the first and most elementary kind of sensory information objectively possible to register for a subject, say a human, until the subject sense a *percept* available for its consciousness. The subject will consider its percept, say a flash of light, as residing *outside* the boundary represented by the skin of the subject, while in reality the percept occurs *inside* the boundary of the subject, with the preceding recept occurring at the *immediate* inside. Thus the subject performs an *outward and backward projection* of the real location of its perceived input-difference. Further, *what* sensory input the subject perceives, both in quality and in quantities of the quality, depends on the algorithms (including their semantics) constituting the sensory apparatus of the subject. During the perception these algorithms are hidden for the subject who de facto applies a projection outwards and backwards also of these subject-internal algorithms and their related subject-*internal* differences. Such projection of one causal relation between input-difference and output-difference to *another* causal relation between input-difference and output-difference, we denote as *projective causality.*

By applying technical instruments more sensitive than our (direct) perception, *another* subject can research the exact relations between input- and output-differences constituting sensory perception, e.g. when studying how inputs of volumes and frequencies of sound from an external source become perceived by a human subject as corresponding but different volumes and frequencies. The researcher will tend to find that such relations between input- and output-differences follow the Weber-Fechner law for sensory perception. The Weber-Fechner relation implies that:

(i) Input-differences are represented *logarithmically* as the output-differences registered by the subject. (If the volume of a sound increases with a factor of 8, the human ear will perceive this as increase with a factor of 3 due to $2^3=8$.)

(ii) Input-differences below the lowest threshold and above the highest threshold for reception by the subject will *not* be represented as output-differences at all. (Frequencies too high or too low will not be heard by the human ear.) (iii) Since the *resolution* of input-differences is higher than the resolution of output-differences, any output-difference will cover *plural* preceding input-differences which hence becomes conflated into the *same* output-difference. (If the difference between two frequencies are too small, the human ear will not perceive any difference.)

With regard to perception we thus see that many input-differences do not unfold into any output-difference, and therefore they do not constitute any information for the perceiving subject (only for the external researcher). Also, we see that when the subject projects his percept (with implied subject-internal algorithms) outwards, there is implied a *quantitative* (logarithmic) transformation between the "real" external input-difference (as measured by the researcher) and the input-difference as perceived by the subject. The *qualitative* incongruence is even more radical since the perceiving subject does not have any *access* to the quality of any *external* difference. The first input-difference that constitutes information is the *recept* located at the immediate *inside* of the subject. Thus, the external input-difference is better considered as a *pre-input-*difference.

When we move from perception to proceeding information processing by the subject, projective causality must still be involved in every step of thought, although the implied incongruences (at least the qualitative ones) will be less radical in most cases. The tacit continued presence of projective causality is due to the fact that the subject can not process or reflect upon its distinctions *before* it has manifested them, and during the act of manifestation the distinction is *hidden* for the subject. Thus, the subject is always processing information *one step ahead* (when observed by the meta-subject of an external researcher) of what itself can be able to conceive.

If, say, you make a distinction between yellow and green in an observed rainbow, there is not inserted any *physical* border between yellow and green as when children draw a line between objects by a black pencil. The border between the two colors is invisible at the perceived physical level, while the border still has a real existence as a mental category in the inherent make-up of the subject. It is the tacit *projection* of the border category, residing in the mental domain, onto the conceived physical domain, that constitutes the difference between yellow and green in the object perceived by the subject. The subject perceives the difference in color *after* the category of border has become projected outwards. What the subject perceives as an input-difference from the physical domain is, when regarded from an external subject with imagined access to the inside of the subject, to be regarded as an output-difference where the inside category of border becomes projected. In general, since the criterion for classification always is hidden (while at the same time expressed) *in* the classification, the subject will always consider the *level* of being/reality it operates on (at least) *one step lower* than what is the case if regarded from an advanced external subject. Thus, projective causality, with this *objectively* implied *simile* of unidirectional level substitution, is with necessity tacitly present in all information processing and thus also tacitly enfolded in all (other) types of causality.

In sensory perception the internal classifications that are projected remain hidden for the subject itself. We denote such as *traceless classifications* which yields trackless representations and processing of information. In more conscious information processing the projected internal classifications can become preserved and reflected upon, and we denote such as *reflexive classifications*.

We may consider, as an example from more refined thought, a logician wrapping his head around whether the expression "I am always lying" (E) is true or false. This seems tricky to decide when he has only one out of two binary truth values to his disposition and the assumption of each of them leads to a contradiction. The contradictions arise when expression E is applied self-referentially to also include itself as something to be true vs. false *about*, as indicated by the term 'always' interpreted as expanded without contextual limitation. Thus, the logician realizes that his trouble originates from that E conflates two different *levels* or logical types of expressions. In order to seek clarification he has to add a meta-level where E can be regarded to also be about itself. The logician unfolds a differentiation already residing enfolded in expression E as soon as he experienced some trouble, so that his reflexive classification into two levels arrives after the more immediate manifestation of expression E in the mind of the logician.

The logician's adequate differentiation into two *levels* of ontological being residing inside a considered freestanding logical universe of logical categories of thought, must be considered to have *physical* correlates in his corresponding brain chemistry, and the logician will seek to express the two levels and the relation between them by physical means as drawings of classification schemes or written logical operators. In general, zero information can exist without being manifested by a *physical* expression and carrier ("physical" as regarded *relatively* to the ontological level of information) as the lower side of the "coin" of the information "atom", whatever minimal (as software or net bank money expressed by pixels at a computer screen). Here, information represents the *upper* side of the coin and the physical manifestation/carrier the *lower* side, as far as the physical manifestation is regarded as *expressing* the information. The information of the same amount of money can be expressed by *plural* alternative physical carriers (coins, bills, net bank pixels), as well as the same universal Turing machine can be expressed by *plural* kinds of computer hardware. Thus, it can not be a 1:1 relation between information and its physical manifestation and carrier, and in this relation information represents the upper and most significant side of the coin vis-à-vis the lower side of the coin that represents the *substance* which incarnate the information. There is no information without substance, and thus without a differentiation (and relation) between two different ontological levels; and the substance is significant only insofar it manifests and carries the information.

The triad of Truth, truth and false

Back to our logician contemplating expression E. *Before* starting out to decide the truth value of the expression, he has first to receive and get the *immediate meaning* of the expression from possessing ordinary skills of language. *Thereafter* he starts out to reflect upon the truth vs. not of the expression from applying formal logic. Thus, he can *not* question the truth of the verbal expression E as being exposed to and received by him in the *first* place, i.e. the reality (inside a thought universe) of his *initial thought object* which only *later on* becomes reflected upon by his logical contemplations. To judge by logic *whether* an expression is true or not, it is presupposed (somewhat pre-meta) that the logician in the first place *did* receive and conceive the expression itself, i.e. the very thought object for his logical reflections, as taken for true. This can be denoted the *prior* truth of expression E without which the following logical investigation of *whether* E is true vs. false can not happen or have any meaning.

Thus, conventional formal logics is not restricted only be some shortcomings shortly indicated previously in this text, but also by *ignoring* the arrival of the initial expressions of the thought *objects* for exercises of formal logics; where these initial expressions are *a priori* to be regarded as prior truths *qua stated.* If we denote the prior truth with capital letter as *Truth,* the truth values of true and false become assigned afterwards as statements *about* the Truth expression. Without Truth there would not *exist* any thought object to think about as true vs. false, so that Truth must have logical priority to both true and false. It is not possible to start out with False as category, since the truth value false only can be *about* Truth and in this sense must be logically secondary and parasitic on the category of Truth. Conventional formal logic is constructed as if truth and non-truth are existing at (only) an equal footing, while from an extended contemplation there is always a *triad* involved in logical reflection, where Truth becomes differentiated into being true or false when Truth becomes reflected upon. In much human thinking Truth is *not* reflected upon, but *unfolds organically* into *another* Truth by mostly unconscious types of causality. It is due to ignoring this circumstance that conventional formal logic includes as true various expressions where claimed truthfulness appear contra-intuitively inadequate, as the examples we gave in (1).

When a newborn baby opens its eyes, it will be confused and perhaps start wondering whether it is true or false that it is not dreaming. But in the first place, before the baby starts wondering about truth values and strives to place its novel visual experience ontologically, the baby can not question the fact of its visual experience as such. Thus, the triad of Truth and truth vs. false is operative in real life phenomena also outside the free-standing thought universe of formal logics.

By analogy, it is mistaken to conceive the categories of *creation and destruction* as existing (only) at an equal ontological footing. It is not possible to destroy anything that is not *already* created; thus the prior category is *Creation* that next differentiates into being treated by (further) *creation* vs. *destruction.* It may be reason to question the adequacy of considering the category *entropy* that fundamental as in most theoretical physics. From a consistent triadic approach it may seem most reasonable to consider *Syntropy* as the prior category which becomes differentiated into *negentropy vs. entropy.* In the present context, however, it will lead way too far to attempt to lift and reinterpret the laws of thermodynamics from a triadic approach.

(As an aside with respect to moral philosophy: From analogous triadic reflections we might find that nor do the categories of *good and evil* exist (only) at an equal ontological footing. The category *evil* (vs. *good)* has to be *about* something, and if this prior something was not *Good* it is hard to see how the category of evil can have any *meaning* as negation of anything.)

The profound ontological significance of the Fibonacci algorithm

Already from our reflections above concerning abstract causality and projective causality it is indicated that when information is constituted as an input-difference unfolding into an output-difference for a subject, this happens by (i) the subject *projecting* subject-internal difference onto an ontological level residing *below* the ontological level where the subject-internal differences reside themselves, so that this projection is implied in the constitution of the input-difference; and by (ii) the subject unfolding the input-difference by means of its inherent *causal* operator(s) into the output-difference. When regarded formally as abstract, universal and elementary as possible, this has the form of the *Fibonacci algorithm* where the subject processes its preceding state by tacitly combining it with a projection of its present state, whereafter the next, proceeding state of the same subject unfolds by causal necessity. This means that the Fibonacci algorithm is tacitly residing inside information as such, when information is considered in its most abstract *qualitative* sense. Consequently, the Fibonacci algorithm must constitute the fundamental bridge *between the qualitative and quantitative* aspects of nature (cf. also Johansen 2006, 2008a, 2014b).

A radical implication from this apparent philosophical result was that even the field of natural numbers, as a distinguished part of (cognitively conceived) nature, should be possible to *unfold* from systematic reflection on the Fibonacci algorithm. In our treatise *Fibonacci generation of natural numbers and prime numbers* (Johansen 2011) the field of natural numbers became established as a supra-structure generated uniquely from the Fibonacci algorithm by successive alternations between ordinal and cardinal aspects of Fibonacci entities/numbers. Thus, while the Fibonacci series trivially is a subset of natural numbers, from this deeper contemplation, representing some Copernican turn, the natural numbers themselves emerged as generated from the Fibonacci algorithm (cf. also Johansen 2014a).

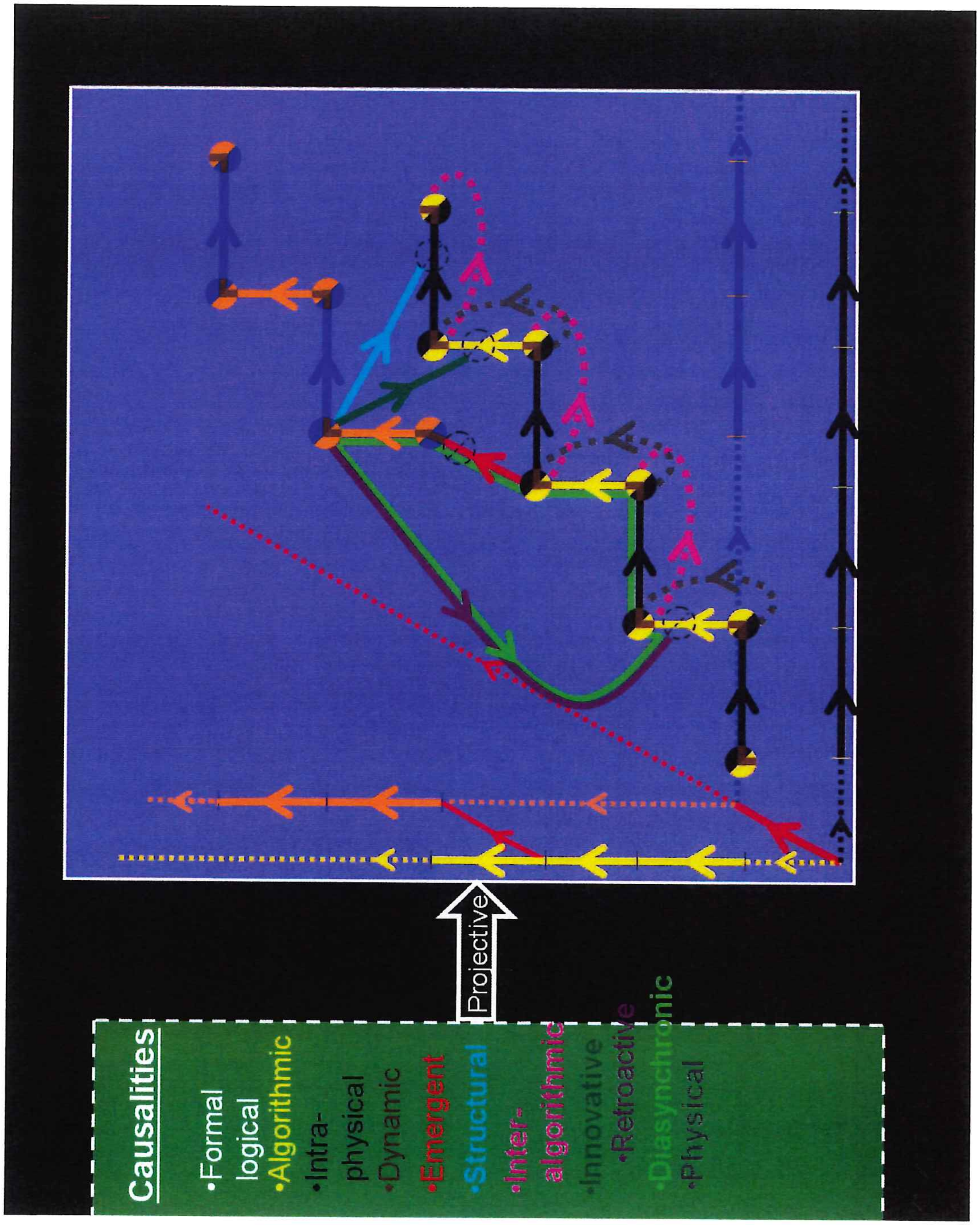
Our mathematical results connected to this refoundation of number theory may suggest that profound and concise reflections on information and causality categories hold a potential for catalyzing clarification and progress also in topics of theoretical physics.

The nexus of causality types

In our philosophical treatise different ontological levels and dimensions are systematically developed from successive reflections on categories residing tacitly enfolded in the very concept of information as such. Our causality theory does not hold any *autonomous* position towards ontology in general, but are *anchored* in *this* differential ontology and epistemology. Our development and differentiations between various causality types are, more accurately expressed, presented as integral and crucial aspects, unfolding more organically, *inside* the development of our differential ontology.

As a whole this causality theory is too extensive and complex to become much presented in this text, but at least we can provide a condensed – and by necessity rather cryptical – description in order to give *some* idea about the nature of the *fundamental* causality types and of the relations between them: (cf. Johansen 2017)

Fig. I: Illustration of the causality nexus anchored in the three dimensions physical (horizontal in black; 3 + 1D compressed as 1D time), algorithmic (vertical in yellow), and transalgorithmic (depth in red). Description of first-order alternates between process (black) and transfiguration (yellow), second-order between blue and orange. Higher orders activate from emergence (red) and unfold as structural change in process (light blue) or innovative change in transfiguration (dark green), with the possibility of the last being retroactive (purple). Whatever degree of order and systemic complexity, the illustrated conglomerate of causality types and arrows constitutes a completed nexus of information flows.


Causalities
•Formal logical
•Algorithmic
•Intra-physical
•Dynamic
•Emergent
•Structural
•Inter-algorithmic
•Innovative
•Retroactive
•Diasynchronic
•Physical
Projective

Formal logical causality: this category is universal for all thinkable information, i.e., for *any* information flow in *any* described information matrix, i.e., in the imagination of a pure and free-standing logical universe. Formal logical causality is deduced in its precise form from specified classification logic between the thinkable classes and elements from ontology differentiated vertically. All other causality types are subtypes and "clothes" of this abstract one, which is what qualify them as causality types. They unfold from specified additions of different *similes, necessary* in any dynamic system description, explicitly stated or not.

Algorithmic causality: this is the causal relation from an input-value to an output-value inside the algorithm.

Intra-physical causality: this is the causal relation from start point to end point of a process.

Dynamic causality: this is the causal relation with the two subclasses: a) from end point of a process to start point in an algorithm; b) from end point of an algorithm to start point in a process.

Projective causality: this is the causal relation from the meta-subject to the thought object as a whole; the potential inner classifications and causal relations being actualized in this projection (including formal logical causality). In fig. 1, the arrow of projective causality originates from the field (in green) of an enfolded nexus of causality types, denoting a segment *inside* the thinking meta-subject that makes the description, and manifests as the field (in indigo) of an unfolded nexus of causality types. The frame of the originating field is marked with broken white lines in order to distinguish its ontological status from the nexus projected into the derived field.

Structural causality: this is the meta-algorithmic causality relation directing the process-output from an algorithm to the process-input for another algorithm and hence positioning all algorithms in a structure.

Inter-algorithmic causality: this is the causal relation from an algorithmic output to the algorithmic input for another algorithm, hence ignoring the intermediary physical process by a projection to the vertical algorithmic axis.

Emergent causality: this is the causal relation from an algorithm to a meta-algorithm.

Innovative causality: this is the causal relation from a meta-algorithm to a first-order algorithm. An important subtype of innovative causality is the *retroactive* causal relation from a meta-algorithm to a first-order algorithm earlier connected to the meta-algorithm by emergent causality.

Diasynchronic causality: this is the causal relation made up by a *circuit* of algorithmic, physical, intraphysical, dynamic, projective, emergent, structural, and retroactive innovative causality.

Physical causality: this is the physical relation from a process output to the process input of the next process; hence, ignoring all intermediary algorithmic and transalgorithmic transfigurations by a projection from the vertical axis or the depth axis to the horizontal axis.

It follows from the illustration of the causality nexus in fig.1, that, e.g., the conventional notion of *physical* causality is far from constituting the *most* fundamental causality type. It is also far from any *trivial* causality types, due to its condensation of many involved causality paths through plural shortcuts and similes. Thus, it follows from strict and consistent philosophical-ontological reflection on the nexus of causality types which *constitutes* the reality of information in the cosmos, that ideas about cosmos as *fundamentally* physical or—even worse—*only* physical, are basically *radically amputated and illusionary* as judged by strict standards of scientifically informed and informing philosophy/meta-science.

From these fundamental causality types, various *elaborated* causality types constituted by combinations of fundamental causality types were exhibited by Johansen (2008: ch. 3.2); among these are: randomness causality, probability causality, stochastic causality, intentional causality, selective causality, and imagined causality. Thus, more elaborated and epistemologically refined causality types, crucial in human and social systems, were understood *inside* the causality nexus anchored in the three ontological dimensions (see Johansen 2008a and 2008c for specified applications of this causality theory).

It follows from our philosophical work that without a sufficiently differentiated and concise ontology, it becomes difficult and in part impossible to discover, differentiate and adequately place and relate several types of causality. Far most of theoretical physics is not much sophisticated in ontological differentiations, which leads – more or less – to corresponding restrictions in reflections on causality in general and on various causality types. Still, the most common "folk ontology" among physicists is limited to the simple binary distinction between the physical world and the mental world, for next to consider the physical world as the primary world or even as the only "real" world. We may take even Einstein as an example expressing a rather naïve ontology subscribing to philosophical materialism, although without the conventional notion of 'matter', expressing support to the tradition from Hume and Mach (cf. Einstein 2000 [1954]:81).

For many – not to say far most – purposes of physics, say, engineering by Newtonian mechanics, shortcomings in more or less subtle ontological differentiations do not matter much – if at all –, nor do shortcomings with regard to understanding the rather intricate relations between more fundamental causality types that reside enfolded in the conflated notion of physical causality. Extensive philosophical meta-reflections will in most cases show contra-productive with respect to solving the task at hand, and the required implied relations between cognitive categories are best delegated to the wisdom, speed and precision performed by unconscious algorithms.

However, in order to adequately approach and treat more *fundamental* issues in theoretical physics, which present crucial *paradigmatic* challenges, more abstract differentiations and meta-reflections may make a constructive difference. With respect to quantum physics more tricky philosophical issues became actualized as soon as the role of the *observer* had to be included into a broader perspective in order to understand *what* real entity that manifests through quantum measurements targeting the wave function.

Approaching the Einstein-Podolsky-Rosen paradox

When approaching the *Einstein-Podolsky-Rosen paradox* (Einstein et al. 1935) in theoretical physics it is not adequate to consider the (mathematical) *chance* distribution as an attribute by the (physical) wave function, as *opposed* to causality (which tacitly is considered as *physical* causality). We have clarified that as located inside our theory of causality, anchored in our ontological framework, this can *not* with logical consistency be considered as an *absolute* opposition, in the sense of representing two opposing categories in their ontological *basis.* Firstly, *physical causality* does not represent the *most* fundamental type of causality, but *enfolds* and is *internally built from* more fundamental types of causality. Secondly, the concept of *chance* (including the concepts of randomness and probability) does itself represent a certain *type* of causality. Thirdly, chance causality does not represent any *fundamental* causality type, but one among several *elaborated* types of causality. Fourthly, chance causality, as when applied in explanation of a physical system, *presupposes physical* causality as *one* among the causality types chance causality is made-up by and from by addition and inclusion of certain simile operators.

When not being consistent with the four points above, theoretical discussion of EPR will contain some categorical conflations and inaccuracies. This does not implicate, however, that the discussion is without intellectual merit or importance, but it does implicate that more basic and consistent categorical differentiations and relations might catalyze further clarification of the issues discussed.

Our differential philosophy might be characterized as a systematic *qualitative* informatics, i.e. that unfoldment into qualitative differentiations and categories precedes related quantifications *of* the unfolded qualities, as already indicated by the significance of the Fibonacci *algorithm* for refoundation of number theory as a whole (and in next steps catalyzing certain novel mathematical results more technically regarded), or by the significance of the Weber-Fechner logarithm in constitution of recepts.

We may contrast this to the *quantitative* informatics presented by Shannon&Weaver (1949) where the concept of information was *defined* (1949: 103f) from the concept of probability when contemplating signal to noise ratios and applying entropy formulas from theoretical physics. Their approach was technically sophisticated and showed highly fruitful, e.g. for developments of telecommunications. However, their quantitative "definition" of information appears as a second-hand pseudo-definition, since it already *tacitly presupposed* the very quality of 'information' to have become established (and thereafter becoming differentiated qualitatively into the concept of 'signal' as input-difference at the sender side and an output-difference at the receiving side) before it became quantified for practical purposes.

Later on both Chaitin and Kolmogoroff presented theories of quantitative informatics which were basically complementary to Shannon&Weaver, and Zurek presented a reasonable synthesis of Shannon vs. Chaitin/Kolmogoroff. In any case these potent developments of quantitative informatics, *based* on i.a. the notion of chance causality, has to be regarded as second-hand as compared to the qualitative informatics represented by our differential philosophy where the very category of chance causality does not occur before rather late in our systematic unfoldment of causality types. Thus, from a more profound and basically qualitative approach it is not adequate to refer to the quantitative second-hand definition of information applied in the information theory of Shannon (or others), with the according fundamental role played by chance causality, in order to approach the deeper issues of theoretical physics as e.g. addressed by the EPR paradox.

The abstract, while concise, definition by Turing (1935) of information as computation, connected to his astonishing invention of the Universal Turing Machine, is definitely more qualitative (in the first place) and independent of probability reflections than the definitions of information referred above. Still though, our definition is more qualitative, abstract and universal than Turing's, with according possible robustness towards more fundamental progresses in informatics (e.g. Deutsch, Diaz/Rowlands, Bohm; cf. Johansen 2008:260f for a short discussion).

Einstein, Podolsky and Rosen (1935) argued quantum mechanics to not represent a *complete (physical) theory* because the *description* of (physical) reality by the wave function in quantum mechanics was judged as not being complete.

They stated as a *necessary* condition for a complete theory that "every element of the physical reality must have a counterpart in the physical theory" (ibid.: 777). (We apply symbol E_R to denote the first kind of element, and symbol E_T to denote the second kind of element.) They stated as a *sufficient* condition for the occurrence of E_R that the "value of a physical quantity" can be predicted with certainty, i.e. with probability=1 (ibid.). Next, from performing a certain *thought experiment*, consistent with quantum mechanics and its mathematical transformation theorems, they argued the occurrences of *certain* E_Rs that were *not* possible to describe by a corresponding counterpart of E_T. Consequently, quantum mechanics could *not* be considered a *complete* theory.

It was concluded as an open question *whether* a complete theory, overcoming their argued limitations of quantum mechanics, could be achieved, but the authors stated their *belief* in such a more advanced and general theory to be possible.

Their thought experiment (ibid.: 779f) considered two (physical) systems interacting for some time, after which system I and system II are separated. The initial states of the two systems when they start to interact, are assumed as known. Then, from the Schrödinger equation with the wave function we can calculate the state of the *combined* system I&II at any time, including *after* the two systems separate. However, we can not calculate the state of *each* system after their interaction has become terminated. According to quantum mechanics, such calculation is only possible from a *measurement* executing *reduction* of the wave packet. It is assumed that measurement only takes place in system I, *either* of *coordinate(s)* (position), thus being considered as E_R, *or* of *momentum* considered as E_R. Depending on which of the two binary alternatives for measurement that is chosen, the inferred wave function for system II, *after* the two systems have been separated, will look different. It does not appear consistent that the *same* system II, after separation, can be assigned with two *different* wave functions.

Next, their thought experiment assumed two E_R candidates, namely two particles P and Q with corresponding eigenfunctions of two *non-commuting* operators with respective eigenvalues. They presented a technical proof, concluded in their eq. (18), that the two *different* eigenfunctions, depending on starting out with measurement of the *momentum* of P vs. with of the *coordinate(s)* of P, represent alternative expressions of the *same* reality, and thus that *both* of the two non-commuting P and Q should be considered *simultaneously* as E_R.

Since the measurement process in system I is considered to not have *any* possible influence on the state of system II *after* the two systems have become separated, so that the measurement process in system I is *irrelevant* for the state of system II, it does not seem to make sense that the objective *E_R status* of something residing in system II *after* the separation should depend on whether the measurement procedure in system I targeted position vs. momentum of a particle residing merely in system I, i.e. *excluded* from system II.

The authors concluded that quantum mechanics offers a non-complete description of objectively existing E_Rs residing in system II since a calculation of *both* E_Rs residing simultaneously in system II is *impossible* to achieve (predict) from a definite measurement in system I, because such a measurement *has to* exclude one calculation on behalf of the other.

Niels Bohr (1935) replied to this critique by pointing out that the very access to *receive any* experimental data about what was going on at the quantum level *required* experimental apparatus and procedures at the *classical* level. From the freely chosen *specifics* of the experimental design at the classical level it would be uniquely *determined whether* the position or the momentum of an elementary particle became targeted and measured. Consequently, it was not empirically *possible* to avoid that measurement of position vs. momentum of an elementary particle *had* to be binary. Thus, it did not make sense to *criticize* quantum mechanics for not being "complete" in its description since the very *access* to the quantum level *presupposed* such incompleteness. Further, *without* such access to the quantum level, with said according narrowing constraints, *zero* predictions or possible assignments of *values* of E_R, or indicating any *existence* of E_R at all, would not be possible. Thus, Bohr argued that you could not criticize quantum mechanics for shortcomings with respect to not achieving *complete* descriptions at the quantum level when these shortcomings with necessity was entailed in the classical apparatus and procedures to get any access to the quantum level *at all.*

Bohr presented the principle of *complementarity,* with respect to the quantum level, in order to account for the fact that even if it was not possible by experimental apparatus to measure (and calculate from) position and momentum of an elementary particle *simultaneously*, *both* approaches, corresponding to according measuring devices, should at the quantum level be regarded as contributing to physical knowledge at an equal footing. Although not referred to by Bohr, this appears basically similar to the *gestalt switch* in psychology of perception, where, say, the alternation between rabbit and duck gives more complete information about the whole object for perception than each of the two perspectives. With respect to natural philosophy, regarded more in general, Bohr's consideration implied that the complementarity principle had to be applied with more strict and basic necessity at the quantum level than at the classical level.

Further, Bohr addressed some possible self-referential inconsistencies in the argument by Einstein et al. by clarifying that their critical conclusion was *based on* applying the transformation theorems developed *inside* the mathematics of quantum mechanics – and thus, at least to some extent, subscribing to the paradigmatic framework of quantum theory. Bohr also pointed out that quantum mechanics involved exchange of energy at the quantum level so that *time and energy* variables should be regarded as conjugates, rather analogous to position vs. momentum, and that this conjugation had an interesting similarity to a certain paradox in Einstein's relativity theory. In order to perform experiments to *test* predictions from relativity theory, highly accurate assignments of time and space coordinates are required, as determined at the *classical* level, despite that relativity theory, especially the *general* theory of relativity, implies a novel theory about the very relation between the coordinates of space and time where these coordinates can *not* be determined independently of each other. Thus, one crucial point from Bohr's anti-critique is that we can not require from an advanced theory of physics that such paradoxes can be completely *avoided*; thus the question is how they are next treated and attempted reconciled from firstly acknowledging the *necessity* of the involved theoretical paradox.

In the proceeding discussions in theoretical physics about the EPR paradox, Bohr's anti-critique tended to be judged as satisfactory. There should be no doubt that Bohr clarified some key issues in a rather concise as well as in a creatively interesting manner. On the other hand, there might be that the arguments by Einstein et al. addressing possible *limitations* by quantum mechanics, compared to an imagined *more* advanced theory of physics, enfolded some rather deep and relevant points, despite that the philosophical clarity in presenting the argument was not that impressing and that the mathematical dressing of the argument did not support the basic argument *that* much, as much clarified by Bohr's anti-critique. The more subtle and challenging request might be to attempt to access the possibly brilliant *intuition* by Einstein, never mind shortcomings in the published presentation of 1935 *from* the intuition.

David Bohm (1951) supported the anti-critique by Bohr (cf. ibid.: 611), while on the other hand Bohm followed Einstein by arguing that quantum theory should not imply denial or downplaying of causal laws. Bohm's rather profound and constructive discussion of the EPR paradox might thus be said to represent some complementary *superposition* of both Bohr and Einstein.

With respect to causality Bohm pointed out "the role of causal laws in making possible the identification of an object, *whether it changes or not*" (ibid.: 163). In general "an object is identified by the way it *reacts* to forces of various kinds...Since the statement than an object reacts in a *definite* way to forces implies that it obeys causal laws, we conclude that no object can even be *identified* as such unless it obeys causal laws" (ibid.:163f; italics by me).

Elementary particles as protons and electrons do not represent any exception in this regard: "It is from the *reaction* to electric and magnetic forces, and from the ionization of other atoms by the electric forces produced by a charged particle, that an electron or proton is *identified*" (ibid.:163; italics by me).

Bohm noted "that this criterion also includes *seeing* the object with the aid of light" (ibid.: 163; italics by me). Since causal laws are involved in all perception (cf. our previous discussion of projective causality and constitution of percepts), and there are no observations without perception, all observations must obey causal laws.

According to Bohm the role of causal laws in order to identify an object was "certainly no less important" (ibid.: 163) in quantum physics than at the classical level. At the same time Bohm acknowledged Bohr's complementarity principle for the quantum level, connected to Heisenberg's uncertainty relation, and did not find it possible to interpret the non-commuting variables of momentum and position as separately and simultaneously existing and precisely defined elements of reality (cf. ibid.: 622f). And "*exact* causal laws would be meaningless in a context in which there were no *precisely* defined variables to which they could apply" (ibid.: 625; italics by me). Bohm interpreted the wave function as describing "the propagation of *correlated potentialities*" (ibid.: 621; italics by me), so that the quantum concept of a potentiality became more fundamental than the notions of momentum and position.

In his general ontology Bohm (1987, 1993) regarded borders distinguishing physical objects not as totally absolute, but more – or less – as dotted lines. While the ontological assumption of complete separation between independent physical objects obviously represented an adequate approximation for theoretical physics at the classical level, Bohm argued that this assumption had to be relaxed with respect to certain phenomena occurring from the quantum level.

Bohm (1951: 624-628) sought to clarify the *interrelation* between the classical and quantum level, as well as between their respective theoretical concepts. Rather than viewing the classical level as some special case from a generalized quantum theory, Bohm argued that the quantum world and the classical world should be understood *complementary* as mutually dependent. One of his points was that quantum theory *presupposes* the classical level because "the last stages of a measuring apparatus are always classically describable" (ibid.: 625). Without such measurements, quantum theory can hardly be said to have any meaning at all. And, if we look at the uncertainty relation between position and momentum, and the related complementarity between wave and particle, this relation does not manifest before in *interaction* with a *classical* system of measuring devices (cf. ibid.: 625, 627).

If we reflect a bit on the very concept of a *physical wave*, it is implied that the form of a wave as a mathematical concept must have a physical manifestation and carrier, as e.g. sound waves carried by molecules in the air or ocean waves carried by water molecules. Thus, a physical wave must be carried by a huge number of physical particles (or at least by *something* physical as in contrast to the information of the wave pattern itself). Here, the form of the wave appears at a much *larger scale* than each of the physical particles that make up the physical wave. *If* we consider a physical wave to have a complementary state as a particle at the *same* scale as itself, the concept of a physical wave will then tacitly (by transitivity) imply a relation *between* the large-scale particle and the small-scale particles. The wave-particle duality at the quantum level is well known, but transformations between the physical state and the wave state have also been documented as possible at the molecular level where (more or less) the same *information* of the molecule is preserved during the transformation to its specific frequency constellation (cf. i.a. Gariaev et al. 2000, 2011; Montagnier et al. 2011, 2014; Marvi&Ghadiri 2020; Brand et al. 2020). The philosophical point here, of some possible relevance for theoretical physics, is that the conceptual contrast between a physical wave and a physical particle is not an *absolute* opposition, but *relative to the scales* considered adequate for description of the involved phenomena.

Then one may ask: When describing a quantum phenomenon as a wave, and consider the wave to be a *physical* wave and not only a pure *mathematical* notion, what are the physical *sub-entities* that make up the wave? *If* such sub-entities are imagined to exist, then the application of chance causality in describing measurement probabilities in quantum mechanics might not be that *completely* different, after all, from applying chance causality when throwing dices.

In his work *Causality and Chance in Modern Physics* (Bohm 1984; first ed. 1957), Bohm presented a sophisticated reflection on the philosophical categories of causality vs. chance. There are significant overlaps between some key points in Bohm's treatment and our own treatment of chance vs. causality (originally presented in a publication from 1991 and developed without knowledge about Bohm's work), especially where he discusses "chance and necessary causal interconnections" (ibid.: 139-146). In his last work, *The Undivided Universe. An Ontological Interpretation of Quantum Theory* (Bohm

1993), Bohm stated: "To sum up then…in no sense is probability being regarded as a fundamental concept. Rather the properties of the individual system are taken as primary, and probabilities are interpreted in terms of these".

In his works Bohm presented various *causal* interpretations of physical phenomena and theories often opinioned *not* to be causal, with related discussions of causality in different aspects. In the present context we have focused mostly on his basic points about the role of causality as expressed in more direct relation to his discussion of the EPR paradox.

Bohm followed Einstein in demanding that principle(s) of causality should prevail also in interpretation of and further development of quantum theory. However, Bohm found Einstein's requirement of one-to-one correspondence between any conceivable E_R with a counterpart E_T to be too strong, and he did not share Einstein's optimistic belief that a *complete* (physical) theory should be possible to achieve. Bohm wrote:

> *A complete theory will always require concepts that are more general than that of analysis into precisely defined elements. We may probably expect that even the more general types of concepts provided by the present quantum theory will also ultimately be found to provide only a partial reflection of the infinitely complex and subtle structure of the world. As science develops, we may therefore look forward to the appearance of still never concepts, which are only faintly foreshadowed at present* (Bohm 1951: 622)

The last words by Bohm may be taken as rather prophetic when reflecting upon the immense contributions to progress in theories of physics, as well as to related progress in mathematics, chemistry, biology and technology, achieved by **Ruggero Maria Santilli**.

Some basics about the hadronic sciences initiated by Santilli

Santilli initiated the establishment of vast new fields of scientific theory and discovery denoted by the umbrella term *hadronic science(s)* covering hadronic mathematics, hadronic mechanics, hadronic chemistry, hadronic biology, and hadronic technology. The main reason for the choice of the term 'hadronic' was that Santilli initially approached the hadrons in the nucleus by regarding elementary particles as *extended* particles, in distinction to conventional quantum mechanics treating elementary particles more simplistically *as if* they were point particles, and found it necessary to develop novel mathematics in order to adequately analyze extended particles. Next, this mathematics and related development of novel physical theory showed both potent and rather necessary when also addressing plural issues of physics *outside* nuclear physics. Hadronic mathematics was structured by a layered architecture where a novel layer of *isonumbers* emerged as a "second floor" above natural numbers where the "elevator" (these metaphors are mine, not Santilli's) between the two floors was constituted by the *isotopic element* which indicated the transform of the conventional unit, represented by the natural number 1, to *another* (arbitrary) unit whereby a whole field of novel numbers emerged from the basic *relation* between the two units. A further layer of *genonumbers* emerged as a "third floor" of non-commuting numbers accounting more directly for irreversibility as category. An even further layer of *hyperstructural numbers* emerged as a "fourth floor" of numbers *themselves* having an intrinsic layered structuring, somewhat similar to one hand possessing plural fingers and thus being multi-valued.

In the architecture of hadronic number theory the numbers residing at each level were included as a *sub-set* of the numbers residing at the level above (that is, when taking the "elevator" down again and performing downwards *degeneration* as the opposite transform of upwards *lifting*). Further, in the architecture of hadronic number theory each level in the number landscape had a "mirrored twin landscape" of numbers, denoted its *isodual*.

In hadronic mathematics *hadronic geometry* corresponds isomorphically to the architecture of hadronic number theory. Although Santilli by far has been the most innovative and important scientist contributing to the development of the hadronic sciences, by a rough estimate some 2-300 scientists world wide have also published contributions in more or less specialized fields inside the hadronic sciences. Some obviously important contributions have been *T. Vougiouklis* creating much of the sufficiently abstract hyperstructures to inspire Santilli's mathematical inventions; *S. Georgiev* lifting the ordinary calculus to the more complex *isocalculus* published in voluminous detail; *J. Dunning-Davis* lifting the laws of thermodynamics to a more general formulation by isomathematics; *A. Animalu* pioneering the field of iso-superconductivity (along with Santilli himself); and *C. Illert* pioneering hadronic biology (along with Santilli himself).

With regard to hadronic geometry the achievements by Illert (1995; cf. also Johansen 2008a, 2008b, 2008c) hold extraordinary significance. Illert wanted to find a universal formula to describe the growth pattern of sea shells, with as few variables as possible, compared with a data base covering some 100 000 empirical cases of sea shell growth. This showed not to be possible by applying Euclidian geometry, nor with the geometries of Minkowski (applied in Einstein SR) or of Riemann (applied in Einstein GR), while it *did* show possible by applying hadronic geometry. Further, formulated by the mathematical concepts of hadronic geometry, the universal formula showed to be surprisingly simple, entailing only two basic variables, while at the same time, for certain *particular* species of sea shells, the growth pattern, as described by hadronic geometry, included non-trivial information flows jumping forwards and backwards as perceived from our ordinary experience of Euclidian time. This circumstance could be interpreted as *further* support for the adequacy and potency of hadronic geometry, since such non-trivial time flows were included as possible at the genotypic level of hadronic geometry, and also – before Illert's discovery – had been *predicted* by Santilli to later become discovered in empirical systems!

Santilli himself presented results in nuclear physics providing further support to the relevance and potency of hadronic geometry. The discovery by Illert stands out as rather spectacular since it provided crucial support to hadronic geometry from an extensive study at the *classical* level involving much more complex entities (sea shells) than elementary particles. More generally regarded, this was not that much of a surprise from hadronic mathematics, since higher and more complex levels of hadronic geometry, in this case: the level of *genotopic* geometry, were assumed to become more relevant for analysis the more complex the targeted empirical system was assumed to be.

When it showed not possible to find a universal formula for sea shell growth at the *classical* level by means of the Minkowski geometry of Einstein SR, nor by means of the Riemann geometry of Einstein GR, while it *did* show possible to find by *hadronic* geometry, it ought to suffice to give Santilli, the main *inventor* of hadronic mathematics and geometry, a very strong voice with regard to an adequate *hadronic* reconsideration of the EPR paradox and the implied relations between the classical and the quantum level.

(For an introductory overview of the hadronic sciences, see Gandzha and Kadeisvili (2011). For a general bibliography per 2008, see Institute for Basic Research (2008). For some key publications, see Santilli 1994, 2001, 2003, 2006, 2008.)

Santilli's reconsideration of the EPR discussion from achievements in hadronic mechanics

From the very onset of developing hadronic mechanics the whole body of conventional quantum mechanics, addressing elementary particles as idealized point particles instead of as extended particles, had to be considered a *sub-set* of, and explained *from,* the lifted and broader theory of hadronic mechanics, due to being based on simplified assumptions and thereby scientifically limited. Quite recently, Santilli (2019, 2020) has directly addressed and provided a rather extensive reconsideration of the EPR discussion, based on achievements by hadronic mechanics. Although he also previously has given substantial comments to the EPR discussion (cf. Santilli 1998), the recent publications of Santilli offer much more and sharpened foods for thought.

In his publications Santilli has often displayed a humble attitude with respect to (anyone) ever achieving a complete or *final* theory about physical/empirical systems, much aligned with the attitude displayed by Bohm in the quotation we referred above before as a transition to introducing Santilli. Santilli (2019) states that " 'completion of quantum mechanics' is used in Einstein's sense for the intent of honoring his memory", and Santilli (2020) claims "there is no doubt that the 'completion' of quantum mechanics is, by far, Einstein's most important legacy". Taken together, we may interpret this as Santilli regarding development *towards* a complete theory in the sense of Einstein as adequate and highly important, and that various achievements of hadronic mechanics as a matter of fact *have* provided important results along that line.

Let us shortly address at least a few key points in Santilli's reconsideration of the EPR discussion from achievements by hadronic mechanics.

Reversible vs. irreversible time. Some of the objections against the EPR argument had as necessary condition the conventional axiom of quantum theory where *time* was considered invariant with respect to time-reversal, i.e. that time at the quantum level could flow backwards with the same probability as forwards as conceived in Euclidian time. The obvious incompatibility between this axiom and the arrow of thermodynamics at the classical level, became resolved by hadronic *mechanics* lifting *both* classical descriptions and quantum descriptions to a *genotypic* level of description which basically accounted for irreversibility of time *across* the distinction between the classical and quantum level (while at the same time allowing three novel and non-trivial categories of time as necessarily "attached", categorically more secondary, to this irreversibility). It may be of some interest to note this theoretical achievements being somewhat foreshadowed by Bohm's closing note in his thick book *Quantum Theory*:

> *We propose also that irreversible processes taking place in the large scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level.*
>
> (Bohm 1951: 628)

Radical shrinking of the span of the insecurity relation. The *isotopic elements* required for adequate descriptions by iso-mechanics (or geno-mechanics) of coordinates and momenta for particles within *hyperdense* media (as the interior of hadrons, nuclei, or stars), have showed to always be *very small*. This reduces rather radically, and proportional to the density of the non-empty medium, the *span* of the insecurity relation between position and momentum when an adequate description by means of hadronic mechanics are being applied. This shrinking could not be discovered by treating elementary particles as point particles instead of as extended particles. From this discovery Santilli provided a mathematical formulation of the so-called *iso-deterministic iso-principle*, implying that the product of (iso)standard (iso)deviations for (iso)coordinates and (iso)momenta progressively *approaches* a classical description for extended particles with the increase in density of the medium.

Generalized lifting and revision of the conventional wave function. By lifting the description of the conventional wave function to a more generalized description by iso-mechanics, Santilli argued that it was possible to include a representation of the *attractive force* between identical electron pairs in valence coupling (the so-called fifth force, or contact force, connected to the notion of the *iso-electron*, with related orbit and magnetic polarization, in hadronic chemistry). This more advanced description of the wave function in quantum mechanics gave support to Einstein's suspicion that the wave function as described in conventional quantum mechanics did not represent a final or complete description.

It should be indicated already from these few key points that a scientifically competent discussion of EPR today, both philosophically and more directly related to theoretical physics, needs to be upgraded to the present state of *de facto* forefront theoretical physics.

References

Bohm, D. (1951): *Quantum Theory*. New York: Prentice Hall

Bohm, D. (1957): *Causality and Chance in Modern Physics.* London: Routledge

Bohm, D. and F.D. Peat (1987): *Science, Order, and Creativity.* Toronto: Bantam

Bohm, D. and B.J. Hiley (1993): *The Undivided Universe. An ontological interpretation of quantum theory.* London: Routledge

Bohm, D. (1994) [1992]: *Thought as a System*. London: Routledge

Bohr, N. (1935): Can quantum mechanical description of physical reality be considered complete? *Phys. Rev.* **48**: 696-702
http://www.informationphilosopher.com/solutions/scientists/bohr\-/EPRBohr.pdf

Brand, C. et al. (2020): Bragg Diffraction of Large Organic Molecules. *Phys. Rev. Lett.* **125**, 033604
https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.125.033604

Einstein, A. (1926): Letter to Max Born 04.12.1926. In: H. Born and M. Born (1972): *Briefwechsel 1916–1955,* 97f. Reinbek bei Hamburg: Rowohlt

Einstein, A., B. Podolsky, and N. Rosen (1935): Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.* **47**: 777-780
http://www.eprdebates.org/docs/epr-argument.pdf

Einstein, A. (2000): *Relativitetsteorien.* Oslo: Bokklubben Dagens Bøker. (Norwegian translation of 6th ed. of *Über die spezielle und die allgemeine Relativitätstheorie, gemeinverständlich* [1954]

Gandzha, I. and J. Kadeisvili (2011): *New Sciences for a New Era: Mathematical, Physical and Chemical Discoveries of Ruggero Maria Santilli.* Kathmandu: Sankata Printing Press.
http://www.santilli-foundation.org/santilli-scientific-discoveries.html

Gariaev, P. et al. (2000): The DNA-wave biocomputer. *Fourth Int. Conf. Comput. Anticip. Syst.* **10**, 290–310
http://www.centar-zdravlje.hr/PDF/The%20DNA-wave%20Biocomputer%20-%20Full%20Paper.pdf

Gariaev, P. et al. (2011): DNA as Basis for Quantum Biocomputer. *DNA Decipher* **1**(1), 25–46

Illert, C. (1995): *Foundations of Theoretical Conchology* (2. ed.). Part I: Mathematical representations of sea shells from self-similarity in non-conservative mechanics. Palm Harbor, USA: Hadronic Press
http://www.santilli-foundation.org/docs/Santilli-109.pdf

Institute for Basic Research (2008): *General Bibliography*: 121-161 in http://www.i-b-r.org/docs/HMMC-1-02-26-08.pdf

Johansen, Stein E. (2006): Initiation of 'Hadronic Philosophy', the Philosophy Underlying Hadronic Mechanics and Chemistry. *Hadronic Journal* **29**(2): 111-135

Johansen, S.E. (2008): *Outline of Differential Epistemology* (in Norwegian). Oslo: Abstrakt

Johansen, S.E. (2008a): Basic considerations about Kozyrev's theory of Time from recent advances in specialist biology, mathematical physics and philosophical informatics. L.S. Shikhobalov (ed.) (Russian and English): *Time and Stars: The centenary of N.A. Kozyrev*, 652-703. St. Petersburg: Nestor-History

Johansen, S.E. (2008b): Spacetimes of Santilli Hypermechanics: From Hadronic Mechanics to Hadronic Biology and Hadronic Psychology. *Hadronic Journal* **31** (5): 451-512

Johansen, S.E. (2008c): Non-trivial Time Flows in Anticipation and Action Revealed by Recent Advances in Natural Science, Framed in the Causality Network of Differential Ontology. *International Journal of Computing Anticipatory Systems* **22**: *Logic and Semantics in Front of Nanoscale Physics:* 141-58

Johansen, S.E. (2011): Fibonacci Generation of Natural Numbers and Prime Numbers. C. Corda (ed.): *Proceedings of the Third International Conference on Lie-admissable Treatment of Irreversible Processes*: 305-410. Kathmandu: Kathmandu University / Sankata Press.
http://www.santilli-foundation.org/docs/Nepal-2011.pdf

Johansen, S.E. (2013): Some Ontological Aspects of Physics for Matter and Anti-matter. *Journal of Computational Methods in Sciences and Engineering* **13**: 135-162

Johansen, S.E. (2014a): Positive approach: Implications for the relation between number theory and geometry, including connection to Santilli mathematics, from Fibonacci reconstitution of natural numbers and of prime numbers. 10th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences. *AIP Conference Proceedings* **1637**:457-468

Johansen, S.E. (2014b): Entering Hi-Time Cybernetics – From recent liftings of science, philosophy, meaning theory and robotics onto the human condition. Mikhail Ignatyev: *Cybernetic Picture of World. Complex cyber-physical systems.* 438-469. St. Petersburg: St. Petersburg State University of Aerospace Instrumentation

Johansen, S.E. (2017) Systematic Unfoldment of Differential Ontology from Qualitative Concept of Information. In C. Thomas (ed.): *Ontology in Information Science*, 225-253 IntechOpen
https://www.intechopen.com/books/ontology-in-information-science/systematic-unfoldment-of-differential-ontology-from-qualitative-concept-of-information

Marvi, M. and M. Ghadiri (2020): M. A Mathematical Model for Vibration Behavior Analysis of DNA and Using a Resonant Frequency of DNA for Genome Engineering. *Sci Rep* **10,** 3439
https://www.nature.com/articles/s41598-020-60105-3

Montagnier, L. et al. (2011): DNA waves and water. *J. Phys. Conf. Ser.* **306**(1), 012007
https://arxiv.org/PS_cache/arxiv/pdf/1012/1012.5166v1.pdf

Montagnier, L. et al. (2014): Transduction of DNA information through water and electromagnetic waves. *Electromagn. Biol. Med.* **8378**(2), 10

Popper, K. (1982): *Quantum Theory and the Schism in Physics.* London: Hutchinson

Santilli, R.M. (1994) [1991]: *Isotopic Generalizations of the Galilei and Einstein Relativities.* Second ed.: Kiev: Ukraine Academy of Science

Santilli, R.M. (1998): Isorepresentation of the Lie-isotopic SU(2) Algebra with Application to Nuclear Physics and Local Realism. *Acta Applicandae Mathematicae* **50**: 177-190
http://www.eprdebates.org/docs/epr-paper-i.pdf

Santilli, R.M. (2001): *Foundations of Hadronic Chemistry With Applications to New Clean Energies and Fuels.* Dordrecht, Netherlands: Kluwer Academic Publishers

Santilli, R.M. (2003): Iso-, geno-, hyper-mechanics for matter, their isoduals for antimatter, and their novel applications in physics, chemistry and biology. *Journal of Dynamical Systems and Geometric Theories,* **1**: 121-19.

Santilli, R.M. (2006): *Isodual Theory of Antimatter with applications to Antigravity, Grand Unification and Cosmology.* Dordrecht, Netherlands: Springer.

Santilli, R.M. (2008): *Hadronic Mathematics, Mechanics and Chemistry. Vol. I-V.* US-Europe-Asia: International Academic Press
http://www.i-b-r.org/Hadronic-Mechanics.htm

Santilli, R.M. (2019): Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen." *Ratio Mathematica* **37**: 5-23
http://www.eprdebates.org/docs/epr-paper-ii.pdf

Santilli, R.M. (2020): Studies on A. Einstein, B. Podolsky, and N. Rosen argument that 'quantum mechanics is not a complete theory'. *Ratio Mathematica* **38**: 5-222
I: Basic methods: 5-69
http://eprdebates.org/docs/epr-review-i.pdf
II: Apparent confirmation of the EPR argument: 71-138
http://eprdebates.org/docs/epr-review-ii.pdf
III: Illustrative examples and applications: 139-222
http://eprdebates.org/docs/epr-review-iii.pdf

Shannon, C.E. and W.Weaver (1949): *The Mathematical Theory of Communication.* Urbana: University of Illinois Press

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A GENERAL RELATIVISTIC THEORY OF ELECTROMAGNETIC FIELD AND ITS CONNECTION WITH PLANCK'S CONSTANT

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Abstract

A general relativistic theory of electromagnetic (EM) field is developed by constructing an EM tensor which is an outer product of EM vector potentials. The Einstein's equations are modified using this EM tensor and the coupling constant is found to be inversely proportional to Planck's constant. Maxwell's equations, in their current form, do not provide equations of motion; equations of motions are provided by Lorentz force equations which do not follow from Maxwell's equations. However, with the proposed theory of EM field, the modified Maxwell's equations lead to Lorentz force equations. The derived wavefunction for photons may be interpreted deterministically as the slowly varying envelope of the EM potential or statistically as the absolute square of the wavefunction is the probability density of photons.

1. Introduction

The problems of classical electrodynamics can be divided into two classes. (i) The charge and current distributions are known and the resulting electromagnetic (EM) fields are calculated, and (ii) the external EM fields are specified and the motion of charged particles under the influence of EM fields are calculated [1]. When these two problems are combined as in the case of bremsstrahlung, the classical treatment is a two-step process: (i) the motion of charged particle in the external field is determined ignoring the emission of radiation by the charged particles, and then (ii) Maxwell's equations are solved to find the EM fields taking into account the trajectory of the moving charges. As pointed by Jackson [1], this way of handling problems in electrodynamics is of approximate validity since the emitted radiation due to accelerating charges carries off energy and momentum, and so must influence the subsequent motion of charged particles. A correct treatment must include the reaction of radiation on the motion of sources. A classical treatment of reactive effects of the radiation does not exist [1]. However, a semiclassical theory in which the field is treated classically and the charged matter is treated quantum mechanically, contain the back-action of the radiation field on the charge [2]. In Maxwell's theory (classical theory), the field equations do not provide the equations of motion for charged particles; equations of motion are given separately by Lorentz force equations. In contrast, in the theory of gravitation, the equation of motion of mass points follow from Einstein's field equations [3]. Bergmann [4] attributed this to the fact that the field equations of gravitation satisfy four identities, while Maxwell's equations satisfy only one. Another important difference is that Maxwell's equations, in the current form, are linear for vacuum. If solutions are obtained by the linear combinations, charged particles will not interact with each other. In contrast, Einstein's field equations are nonlinear; even the classical interaction of mass points is brought about by the nonlinear terms in the field equations [5,6]. Bergmann pointed out that a field theory can lead to laws of motion only if the (i) field equations satisfy at least four identities, and (ii) they are nonlinear [4]. In this paper, four identities that are satisfied by the EM vector potentials are first derived and then a general relativistic theory of EM field is developed. The resulting EM field equations are nonlinear and the equations of motion that resemble the Lorentz force equations follow from these nonlinear EM field equations.

Tolman, Ehrenfest, and Podolsky [7] investigated the gravitational interaction between two electromagnetic waves in vacuum and showed that the "test rays of light" in the neighborhood of an intense electromagnetic pulse are not deflected when the test ray is propagating parallel with the intense pulse. Later, Scully [8] showed that when a probe pulse and an intense laser pulse are propagating parallel with their velocities less than the speed of light in vacuum, interesting gravitational interaction between them can occur. For example, when an intense laser pulse propagates in a dielectric waveguide and the probe pulse propagates in the bulk dielectric (i.e. outside the waveguide) in the same direction as that of intense pulse, the probe pulse undergoes a small shift (towards the intense pulse) due to gravitational interaction between them. There have been many attempts to combine the theory of gravitation and electromagnetesim, which is summarized by Santilli [9]. Kaluza combined electromagnetism with gravitation in 5 D [10] and Klein applied this idea to quantum theory [11], laying a basis for various versions of string theory [12]. As in the Kaluza-Klein theory, the tensor formed by the outer product of electromagnetic vector potentials plays an important role in our approach.

Although Einstein's general theory of relativity (GR) is well accepted, alternative theories and modified GR theories have drawn significant interest [10-14]. The GR has been verified for masses on length scales of the solar system, but it faces challenges on quantum and cosmological scales [14]. In this paper, we retain the structure of GR theory, but introduce a novel electromagnetic tensor as an additional source term in Einstein's equations. Without this term, the modified Maxwell-Einstein equations do not lead to Lorentz Force equations. The relation between matter and EM field can be interpreted from two different standpoints [13]. The first is the *unitarian* standpoint which assumes only one entity, the EM field. The particles of matter are considered as singularities of EM field and mass is a derived notion to be expressed by EM field energy (or EM mass). The second is the *dualistic* standpoint which takes particles and fields as two different entities. The particles are the sources of the field, but are not a part of the field [13]. In classical electrodynamics, charged particles (the cause) are distinguished from EM field (the effect). The charged particles are considered as the sources for the EM field corresponding to dualistic standpoint. In this paper, it is postulated that the cause and effect are inseparable and the charge is embedded in the field itself. Using this idea, an electromagnetic tensor which is an outer product of EM vector potentials is constructed and the divergence of this tensor satisfies four identities. In the theory of gravitation, the constant appearing in Einstein's field equations is connected to the gravitational constant. Similarly, in the proposed theory, the coupling constant (κ) is connected to Planck's constant. Under the slowly varying envelope approximations, it is shown that the nonlinear EM field equations reduce to Schrodinger equation with one of the potential terms being the self-trapping potential. It is also shown that the rate of change of mean momentum of the EM wave packet is given by Newton's laws with one of the forces being Lorentz force. Using a weak field approximation for the metric tensor, modified Manakov equations are derived for orthogonal polarization components of electromagnetic fields. Manakov equations have been used to describe the evolution of orthogonal polarization components in a nonlinear fiber [15].

The modified Maxwell-Einstein equations lead to a wavefunction $\tilde{\phi}$ for photons, which can be interpreted in two different ways: (i) deterministic interpretation - $\tilde{\phi}$ is the slowly varying envelope of EM potential and momentum density of the EM wave is $\omega \mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]/2$, where ω is the mean frequency of the EM field, (ii) statistical interpretation - $|\tilde{\phi}(x,y,z)|^2$dxdydz represents the probability that a photon is present in the region between (x,y,z) and $(x+dx, y+dy, z+dz)$, and the probability current density is $\mathbf{j}(r,t) = \frac{\hbar}{2m}\mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]$. In the classical limit, Schrodinger equation leads to Lorentz force equations.

Next, by treating the nonlinear effects (i.e. spacetime curvature) as a small perturbation on the linear fundamental mode of a rectangular cavity resonator, a dispersion relation is derived. It is found that the resonant frequency of the cavity is shifted by an amount proportional to the square of the EM energy stored in the cavity, due to spacetime curvature. The dispersion relation is expressed as a special relativistic equation describing the relation between the EM energy, the EM momentum and rest mass, from which it is found that the coupling constant is inversely proportional to Planck's constant.

This paper is organized as follows. In Section 2, an EM tensor which is an outer product of EM vector potentials is constructed and the EM energy-momentum tensor appearing in Einstein's equations is modified using this tensor. The field equations are solved under the weak field approximations in Section 3 and it is shown that the modified Maxwell's equations reduce to modified nonlinear Schrodinger equation or modified Manakov equations, which lead to Lorentz force equations. Section 4 deals with the analysis of rectangular cavity resonator under the weak field approximation and the impact of spacetime curvature on the resonant frequency of the cavity is investigated.

2. Electromagnetic Tensor and Field Equations

The electromagnetic potential A^μ may be written as

$$A^\mu = \eta \frac{dx^\mu}{d\lambda} = \eta U^\mu, \tag{1}$$

where η is a scalar similar to charge (although of different dimension), λ is a parameter along the world line, and U^{μ} are the components of 4-velocity. Consider a locally inertial frame O. We follow the notations of [16]. Let

$$\vec{\mathbf{U}} \underset{O}{\rightarrow} (U^0, U^1, U^2, U^3). \qquad (2)$$

In the frame O, the components of $\vec{\mathbf{U}}$ are constants along the worldline at a point P, i.e.,

$$\frac{d\vec{\mathbf{U}}}{d\lambda} = 0,$$

or $\qquad (3)$

$$\frac{dU^{\alpha}}{d\lambda} = U^{\alpha}{}_{,\beta} \frac{dx^{\beta}}{d\lambda} = 0.$$

In the frame O, the Lorentz gauge condition is

$$A^{\mu}{}_{,\mu} = 0. \qquad (4)$$

With the definition of Eq. (1), the Lorentz condition is nothing but the conservation of η. Using Eq. (1) in Eq. (4) and using $U^{\mu}{}_{,\mu} = 0$, we find

$$\frac{d\eta}{d\lambda} = 0. \qquad (5)$$

Using Eqs. (5) and (3), we find

$$\frac{dA^{\alpha}}{d\lambda} = \frac{d\eta}{d\lambda} U^{\alpha} + \eta \frac{dU^{\alpha}}{d\lambda} = 0. \qquad (6)$$

From Eq. (6), we have

$$\frac{dA^{\alpha}}{d\lambda} = A^{\alpha}{}_{,\beta} U^{\beta} = 0,$$

or $\qquad (7)$

$$A^{\alpha}{}_{,\beta} A^{\beta} = 0.$$

Using Eqs. (7) and (4), we find the conservation relation

$$(A^{\alpha} A^{\beta})_{,\beta} = A^{\alpha} A^{\beta}{}_{,\beta} + A^{\alpha}{}_{,\beta} A^{\beta} = 0. \qquad (8)$$

Since (8) is a tensor equation, it is valid in any coordinate frame. So, we have

$$(A^{\alpha}A^{\beta})_{;\beta} = 0, \tag{9}$$

where semicolon denotes the covariant derivative. We define an electromagnetic tensor

$$T^{\alpha\beta} = A^{\alpha}A^{\beta}, \tag{10}$$

with the conservation relation

$$T^{\alpha\beta}_{\ ;\beta} = 0. \tag{11}$$

To the best of our knowledge, the conservation relation (8) is not known in the literature. We choose the unit of A^{α} as $\sqrt{J/m}$ so that the unit of energy density, $(E^2 + H^2)/2$ is J/m^3 and the dimension of $T^{\alpha\beta}$ is J/m. Hence, $T^{\alpha\beta}$ may be termed as power-force tensor.

In fact, the tensor $T^{\alpha\beta}$ is similar to stress-energy tensor for 'dust', which is given by [4,15,16]

$$T^{\alpha\beta}_{dust} = \rho U^{\alpha}U^{\beta}, \tag{12}$$

where ρ is the energy density. Eq. (10) may also be written as

$$T^{\alpha\beta} = \eta^2 U^{\alpha}U^{\beta}, \tag{13}$$

with η^2 playing the role of ρ, although their dimensions are different.

To verify the validity of Eq. (8), consider a forward propagating plane wave

$$A^j = D^j \exp[i(k_{\nu}x^{\nu})], \quad j = 1,2. \tag{14}$$

Using the Lorentz gauge conditions, we find

$$k_1 D^1 = -k_2 D^2. \tag{15}$$

Using Eq. (15), we find that Eq. (8) is automatically satisfied for a forward propagating plane wave. To verify if Eq. (8) is satisfied when the EM field is confined, we solved Maxwell's equations using the finite difference time domain (FDTD) technique for a rectangular cavity resonator, which is a rectangular metallic waveguide that is closed off at both ends by metallic walls (see Fig. 1).

The length of the cavity is L_j in x^j direction and for simplicity, we assumed that L_j=L. The walls of the cavity are assumed to be a perfect conductor so that the tangential component of the electric field is zero at the conducting walls. We excited this cavity on the left side with a propagating plane wave given by Eq. (14) and the constants D^j satisfy Eq. (15) (for example, there is an antenna on the left wall which emits the EM field of the form given by Eq. (14)). Numerical solution of the Maxwell's equations showed that the Lorentz gauge condition, Eq. (4) and conservation relations, Eq. (8) are satisfied at each point in the cavity for $t >= 0$.

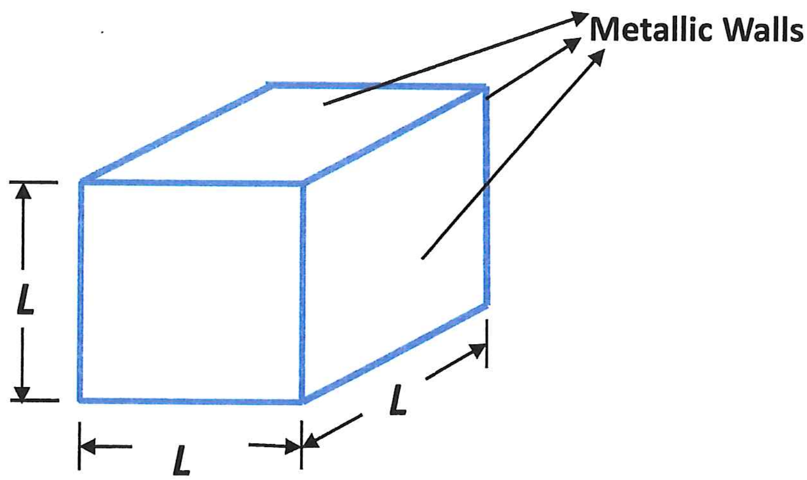

Figure 1. A rectangular cavity resonator.

2.1. Einstein's Field Equations

Einstein's field equations are given by [4,9,17]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}[T^{mat}_{\mu\nu} + T^{em}_{\mu\nu}], \quad (16)$$

where $T^{mat}_{\mu\nu}$ and $T^{em}_{\mu\nu}$ are the energy-momentum tensors of matter and electromagnetic field, respectively. For "dust", we have

$$T^{mat}_{\mu\nu} = \rho V_\mu V_\nu, \quad (17)$$

where ρ is the energy density of the matter and V_μ is its four-velocity. Santilli [9] has analyzed the gravitational field of partons under the assumptions that (i) gravitational field of any massive body is partially due to the EM fields of its charged basic constituents (weak assumption) and (ii) gravitation field is entirely due to the EM fields (strong assumption).

When only the electromagnetic field is present (i.e. $\rho = 0$), Einstein's field equations are given by [4,17]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}^{em}, \tag{18}$$

where the EM energy-momentum tensor is given by

$$T_{\mu\nu}^{em} = F_{\mu\gamma} F_{\nu}^{\ \gamma} - \frac{1}{4} g_{\mu\nu} F_{\gamma\delta} F^{\gamma\delta},$$
$$F_{\gamma\delta} = A_{\delta,\gamma} - A_{\gamma,\delta}, \tag{19}$$

and

$$(T^{\mu\nu})_{;\nu}^{em} = 0. \tag{20}$$

According to the GR with the EM tensor given by Eq. (19), there is no gravitational interaction between two EM fields propagating in parallel in vacuum [7]. In this paper, we modify the EM tensor as

$$T'^{em}_{\mu\nu} = \alpha A_\mu A_\nu + T_{\mu\nu}^{em}, \tag{21}$$

where α is a constant. Since $A_\mu A_\nu$ is the power-force tensor, α has a dimension of $1/m^2$.Using Eqs. (11) and (20), we find this new tensor to be divergence-free, i.e.

$$(T'^{\mu\nu})_{;\nu}^{em} = 0. \tag{22}$$

Since $G^{\mu\nu}_{\ \ ;\nu} = 0$, using $T'^{em}_{\mu\nu}$ instead of $T^{em}_{\mu\nu}$ in Eq. (18), we find

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa A_\mu A_\nu - \frac{8\pi G}{c^4} T_{\mu\nu}^{em}, \tag{23}$$

where $\kappa = -8\pi\alpha G / c^4$. Since we chose the unit of A^μ to be $\sqrt{J/m}$, κ has the dimension of $1/(Jm)$, which is the same as that of $1/(hc)$, where h is Planck's constant. Santlli [18] discusses in detail the importance of the forgotten Freud identity [19] of Riemannian geometry that requires a first order source on the right hand side of Einstein's equations, as in Eq. (23).

Equation (23) may also be derived using the following Lagrangian density for the electromagnetic field,

$$\mathcal{L} = \left[2\kappa A^\mu A_\mu - \frac{8\pi G}{c^4} F^{\mu\nu} F_{\mu\nu} \right] \sqrt{|g|}, \tag{24}$$

where g is the determinant of the matrix of metric components. The effect of the last term on the right hand side (RHS) of Eq. (23) on the evolution of electromagnetic field is already studied in Refs. [7-8]. While studying the evolution of electromagnetic field, we expect that the impact of the term with cosmological constant Λ is negligible and hence, we set $\Lambda = 0$. In this paper, we focus only on the impact of the first term on the right hand side of Eq. (23) on the spacetime curvature and the subsequent changes in the evolution of electromagnetic field.

2.2 Maxwell's Equations in Curved Spacetime

In a locally inertial frame, Maxwell's equations are given by

$$\eta^{\alpha\beta}\frac{\partial^2 A^{\nu}}{\partial\xi^{\alpha}\partial\xi^{\beta}}=0, \tag{25}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric. The Lorentz gauge conditions are

$$\frac{\partial A^{\nu}}{\partial\xi^{\nu}}=0. \tag{26}$$

Using the transformation,

$$A'^{\mu}=\frac{\partial x^{\mu}}{\partial\xi^{\nu}}A^{\nu}, \tag{27}$$

where A'^{μ} is the electromagnetic potential in the new coordinates $\{x^{\mu}\}$. Substituting Eq. (27) in Eqs. (25) and (26) and with $A'^{\mu}\to A^{\mu}$, we find

$$\begin{aligned} g^{\mu\nu}[A^{\alpha}_{,\mu,\nu}+\Gamma^{\alpha}_{\mu\sigma,\nu}A^{\sigma}+2\Gamma^{\alpha}_{\mu\sigma}A^{\sigma}_{,\nu} \\ +\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}A^{\rho}-\Gamma^{\sigma}_{\mu\nu}(A^{\alpha}_{,\sigma}+\Gamma^{\alpha}_{\sigma\rho}A^{\rho})]=0, \end{aligned} \tag{28}$$

$$A^{\nu}_{,\nu}+\Gamma^{\mu}_{\mu\sigma}A^{\sigma}=0, \tag{29}$$

$$g^{\mu\nu}=\eta^{\alpha\beta}\frac{\partial x^{\mu}}{\partial\xi^{\alpha}}\frac{\partial x^{\nu}}{\partial\xi^{\beta}}. \tag{30}$$

It may be noted that Eqs. (26) and (27) could as well be obtained using Einstein's principle of equivalence (comma-goes-to-semicolon rule [16]). In fact, Eqs. (28) and (29) describe the Maxwell's equations in curvilinear coordinates whether or not the spacetime is flat. For example, in a flat spacetime with spherical coordinates, we have

$$g_{00}=-1,\ \ g_{rr}=1,\ g_{\theta\theta}=r^2,\ g_{\phi\phi}=r^2\sin^2\theta, \tag{31}$$

and rest of the metric coefficients are zero. If $\vec{\mathbf{A}}\underset{0}{\to}(A^0,0,0,0)$, Eq. (26) reduces to

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial A^0}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial A^0}{\partial\theta}\right)+ \\ \frac{1}{r^2\sin^2\theta}\frac{\partial^2 A^0}{\partial\phi^2}-\frac{1}{c^2}\frac{\partial^2 A^0}{\partial t^2}=0, \end{aligned} \tag{32}$$

which is nothing but Maxwell's equations in spherical coordinates. However, as the magnitude of $\vec{\mathbf{A}}$ increases, metric coefficients deviate from Eq. (31). Now, they are determined by Einstein's equation (23), and Eq. (28) provides the evolution of EM field. Equations (23) and (28) form a coupled system of equations that govern the evolution of metric coefficients and EM field, respectively, with the conservation relations (22) and (29). In the next Section, we solve this system of equations under the weak field approximations.

3. Weak field approximations

We use a first order perturbation theory and assume that

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{33}$$

where $|\kappa h_{\mu\nu}| << 1$, and $\eta_{\mu\nu}$ is the Minkowski metric. We use a harmonic coordinate system, for which

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{34}$$

Using Eq. (33) and Eq. (34) in Eq. (23) and ignoring the terms proportional to κ^2 and higher, we obtain [17]

$$\Box h_{\mu\nu} = 2\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\mu}_{\ \mu}\right), \tag{35}$$

where $\Box$ is the four-dimensional Laplacian operator and $T_{\mu\nu} = A_{\mu}A_{\nu}$.

3.1 A^1 **only:**

We consider the case for which $A^2 = A^3 = A^0 = 0$, corresponding to an electromagnetic wave with the electric field in x^1-direction and the magnetic field in x^2-direction. For this case, Eq. (35) reduces to

$$\begin{gathered} \Box h_{00} = (A_1)^2, \\ h_{11} = h_{00},\ h_{33} = h_{22} = -h_{00}. \end{gathered} \tag{36}$$

Ignoring the terms proportional to κ^2 and higher, Maxwell's equations (26) and Lorentz gauge condition (29) become

$$\Box A^1 = -\kappa\left(\frac{(A^1)^3}{2} + \eta^{\mu\nu}h_{11,\mu}A^1_{\ ,\nu}\right), \tag{37}$$

$$A^1_{\ ,1} - h_{11,1}A^1 = 0. \tag{38}$$

where

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \text{ and } A_{\mu} = \eta_{\mu\alpha}A^{\alpha}. \tag{39}$$

It may be noted that Eqs. (36)-(38) are Lorentz invariant. Using Eqs. (34) and (38), we find

$$A^1_{\ ,1} = 0 \quad \text{and } h_{11,1} = 0. \tag{40}$$

The first term on the right hand side of Eq. (37) has the form of third order nonlinear effect in nonlinear optics, which is responsible for Kerr effect or self-phase modulation (SPM) and four wave mixing (FWM) [20-21]. Hence, the effect of spacetime curvature may be interpreted as the nonlinear change in refractive index. Let

$$A^1 = \frac{1}{2}\left[\tilde{\phi}(\mathbf{r},t)e^{-i\omega t} + c.c.\right]. \tag{41}$$

Using Eq. (1), let the envelope $\tilde{\phi}$ be $\tilde{\eta}\tilde{U}^1$, where $\tilde{\eta}$and $\tilde{U}^1$ are the envelopes of η and U^1,respectively. Using the slowly varying envelope approximation, we ignore the second order derivative of $\tilde{\phi}$ with respect to *t* and now, Eq. (37) reduces to a modified nonlinear Schrodinger equation (NLSE),

$$\frac{i\omega}{c^2}\frac{\partial\tilde{\phi}}{\partial t} + \frac{1}{2}\nabla^2\tilde{\phi} + \frac{3\kappa}{16}|\tilde{\phi}|^2\tilde{\phi} = -\frac{\kappa}{2}\left[\eta^{jk}h_{11,j}\tilde{\phi}_{,k} + i\omega h_{11,0}\tilde{\phi}\right]. \tag{42}$$

In deriving Eq. (42), we have ignored the third harmonic component proportional to $e^{-i3\omega t}$. In nonlinear optics, while deriving the nonlinear Schrodinger equation (NLSE) from the nonlinear wave equation, the third harmonic component is ignored [21]. Unless there is a special phase matching, the growth of third harmonic component is small. The second and third terms in Eq. (42) denote diffraction and Kerr effect, respectively. When the diffraction/dispersion balances the Kerr effect, a spatial/temporal pulse propagates without pulse broadening and such a pulse is called soliton [22,23]. Hence, κ may be interpreted as the nonlinear coefficient of vacuum. In the absence of the terms on the right hand side, Eq. (42) represents the three-dimensional (3-D) NLSE. In the 1-D case, NLSE admits the well-known soliton solutions [21-23]. Interestingly, in the 1-D case, the terms on the right hand side of Eq. (42) have the forms similar to self-steepening and Raman effects in nonlinear fiber optics [21].

3.1.1. *Lorentz Force*

Electric field intensity E_1 and magnetic field intensity H_2 are related to the vector potential A^1 by

$$E_1 = -A^1_{,0} \text{ and } H_2 = A^1_{,3}. \tag{43}$$

Let

$$E_1 = \frac{1}{2}\left[\tilde{E}_1 e^{-i\omega t} + c.c.\right],$$
$$H_2 = \frac{1}{2}\left[\tilde{H}_2 e^{-i\omega t} + c.c.\right]. \tag{44}$$

Using Eqs. (41), (43) and (44), and using the slowly varying envelope approximation, we find

$$\tilde{E}_1 = \frac{i\omega\tilde{\phi}}{c} \text{ and } \tilde{H}_2 = \tilde{\phi}_{,3}. \tag{45}$$

Using Eq. (44), electromagnetic momentum density may be written as [1]

$$p^3 = \frac{1}{2}\mathrm{Re}\left[\tilde{E}_1\tilde{H}_2^{\ *}\right] = \frac{-\omega}{2c}\mathrm{Im}\left[\tilde{\phi}\tilde{\phi}^{*}_{,3}\right], \tag{46}$$

and the energy density is [1]

$$\mathrm{E} = \tfrac{1}{4}\left[|\tilde{E}_1|^2 + |\tilde{H}_2|^2\right]. \tag{47}$$

We multiply Eq. (40) by $\tilde{\phi}^{*}_{,3}$, add its complex conjugate and integrate over the volume $dV = dx^1dx^2dx^3$. First, consider the first term of Eq. (42):

$$\begin{aligned} \frac{i\omega}{c}\int[\tilde{\phi}_{,0}\tilde{\phi}^{*}_{,3} - \tilde{\phi}^{*}_{,0}\tilde{\phi}_{,3}]dV &= \frac{-\omega}{c}\frac{d}{dx^0}\int \mathrm{Im}(\tilde{\phi}\tilde{\phi}^{*}_{,3})dV, \\ &= 2\frac{d\langle p^3\rangle}{dx^0}. \end{aligned} \tag{48}$$

Next consider the third term:

$$\begin{aligned} \frac{-6\kappa}{16}\mathrm{Re}\int|\tilde{\phi}|^2(\tilde{\phi}\tilde{\phi}^{*}_{,3})dV &= \frac{-6\kappa}{16}\mathrm{Re}\int|\tilde{\phi}|^2\,\tilde{\eta}\tilde{U}^1\tilde{H}^{*}_2dV, \\ &= 2\,\mathrm{Re}\int\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3dV, \end{aligned} \tag{49}$$

where

$$\rho = \frac{-3\kappa}{16}|\tilde{\phi}|^2\,\tilde{\eta}, \tag{50}$$

and $\tilde{\eta}$ is the complex envelope of η. ρ may be interpreted as the density of the embedded charge. Now Eq. (42) leads to

$$\frac{d\langle p^3\rangle}{dx^0} = \mathrm{Re}\left\langle\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\right\rangle - \left\langle h_{00,3}\mathrm{E}\right\rangle + \left\langle h_{33,0}p^3\right\rangle, \tag{51}$$

where the subscript 3 on the first term on the RHS refers to the z-component of $\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*$. The first term on the RHS of Eq. (51) represents the Lorentz force on the embedded charge. It can be shown that in this simple case of transverse electromagnetic wave, the expectation of the Lorentz force is zero. Nevertheless, Eq. (51) shows that the equation of motion is built into Einstein-Maxwell's equations. In contrast, the conventional Maxwell's equations in vacuum do not provide the equations of motion for the charged particle; it has to be supplemented with Lorentz force equations to describe the interaction of charge and electromagnetic field. It may be possible that the time-independent solutions of Eqs. (23) and (28) correspond to elementary electric charges and their interactions would have the form similar to Eq. (51). The second and third terms on the right hand side of Eq. (51) are similar to those present in the Einstein's theory of gravitation under the weak field approximations, except that Eq. (51) has expectation operators.

3.1.2 Interpretation of $\tilde{\phi}$:

In Section 4.3, it will be shown that

$$\kappa = \frac{2(2\pi)^{3/2}}{\hbar c} \tag{52}$$

and using Eq. (52), Eq. (42) may be rewritten as

$$i\hbar\frac{\partial\tilde{\phi}}{\partial t} + \frac{\hbar^2}{2m_{eff}}\nabla^2\tilde{\phi} + V\tilde{\phi} = 0, \tag{53}$$

where

$$V = \frac{3}{8}|\tilde{\phi}|^2 + \left[\eta^{jk}h_{11,j}\tilde{\phi}_{,k}/\tilde{\phi} + i\omega h_{11,0}\right], \tag{54}$$

and

$$m_{eff} = \frac{\hbar\omega}{c^2} \tag{55}$$

is the effective mass of the wave packet. The potential V consists of self-trapping potential (the first term on the RHS of Eq. (54)) and the other terms are due to spacetime curvature. $\tilde{\phi}$

could be interpreted in two different ways.

(i)Deterministic interpretation : $\tilde{\phi}$ is the slowly varying envelope of EM potential A^1 and the momentum density is given by the Poynting vector, $\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^* = -\omega\,\mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]/(2c)$.

(ii)Statistical interpretation: $|\tilde{\phi}(x,y,z)|^2$dxdydz represents the probability that a photon is present in the region between (x,y,z) and $(x+dx, y+dy, z+dz)$, and the probability current density is

$$\mathrm{j}(r,t) = \frac{\hbar}{2m}\mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]. \tag{56}$$

Using Ehrenfest's theorem, Eq. (51) can be retrieved, i.e.

$$\begin{aligned}\frac{d\langle p^3\rangle}{dx^0} &= -\frac{(\nabla V)_3}{c} \\ &= \mathrm{Re}\langle \rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\rangle - \langle h_{00,3}\mathrm{E}\rangle + \langle h_{33,0}p^3\rangle.\end{aligned} \tag{57}$$

3.2 A^1 and A^2 only:

In this section, we assume that $A^3 = A^0 = 0$. Using $T_{11} = (A_1)^2, T_{22} = (A_2)^2, T_{12} = A_1A_2$ and the rest of $T_{\mu\nu}$ being zero, Equation (35) becomes

$$\Box h_{00} = (A_1)^2 + (A_2)^2, \tag{58}$$

$$\Box h_{11} = (A_1)^2 - (A_2)^2, \tag{59}$$

$$\Box h_{12} = 2A_1A_2, \tag{60}$$

$$h_{11} = -h_{22} \text{ and } h_{00} = -h_{33}, \tag{61}$$

and the rest of $h_{\mu\nu}$ are zero. The Maxwell's equations (28) take the following form

$$\begin{aligned}\Box A^1 &= \kappa\left[-\frac{(A^1)^2+(A^2)^2}{2}A^1 - \eta^{\mu\nu}h_{11,\mu}A^1{}_{,\nu} + h^{\mu\nu}A^1{}_{,\mu,\nu}\right] \\ &\quad -\kappa\left(h_{11,2,1}A^2 + \frac{1}{2}(h_{12,2,2} - h_{12,1,1})\right)A^2 \\ &\quad -\kappa\left[h_{11,2}A^2{}_{,1} + (h_{11,1} + 2h_{12,2})A^2{}_{,2} + h_{12,3}A^2{}_{,3} - h_{12,0}A^2{}_{,0}\right],\end{aligned} \tag{62}$$

$$\begin{aligned}\Box A^2 &= \kappa\left[-\frac{(A^1)^2+(A^2)^2}{2}A^2 - \eta^{\mu\nu}h_{22,\mu}A^2{}_{,\nu} + h^{\mu\nu}A^2{}_{,\mu,\nu}\right] \\ &\quad -\kappa\left(h_{22,2,1}A^1 + \frac{1}{2}(h_{12,1,1} - h_{12,2,2})\right)A^1 \\ &\quad -\kappa\left[h_{22,1}A^1{}_{,2} + (h_{22,2} + 2h_{12,1})A^1{}_{,1} + h_{12,3}A^1{}_{,3} - h_{12,0}A^1{}_{,0}\right],\end{aligned} \tag{63}$$

with the Lorentz gauge condition

$$A^1{}_{,1} + A^2{}_{,2} - h_{00,1}A^1 - h_{00,2}A^2 = 0. \tag{64}$$

The first term on the right hand side of Eq. (62) or Eq. (63) leads to the phase modulation proportional to the magnitude square of $\vec{\mathbf{A}}$. Eqs. (62) and (64) reduce to Eq. (37) when A^2 is zero. Let

$$A^j = \frac{1}{2}\left[\tilde{\phi}^j(\mathbf{r},t)e^{-i\omega t} + c.c.\right], j = 1,2. \tag{65}$$

Using the slowly varying envelope approximations and in the absence of terms with $h_{\mu\nu}$ and their first order derivatives, Eqs. (62) and (63) reduce to the three-dimensional Manakov equations

$$\frac{i\omega}{c^2}\frac{\partial\tilde{\phi}^j}{\partial t}+\frac{1}{2}\nabla^2\tilde{\phi}^j+\frac{3\kappa}{16}[|\tilde{\phi}^1|^2+|\tilde{\phi}^2|^2]\tilde{\phi}^j=0,\ j=1,2. \tag{66}$$

One-dimensional Manakov equations describe the evolutions of two polarization components in nonlinear optics [15,21]. From the third term of Eq. (66), it follows that the phase of the polarization component A^1 (or A^2) is modulated not only by its intensity, but also by the intensity of A^2 (or A^1), which is known as cross-phase modulation (XPM). Proceeding as in Section 3.1.1, it can be shown that the following Lorentz force equation can be obtained

$$\frac{d\langle p^3\rangle}{dx^0}=\mathrm{Re}\langle\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\rangle, \tag{67}$$

where

$$p^3=\frac{1}{2}\mathrm{Re}\left[\tilde{E}_1\tilde{H}_2^*-\tilde{E}_2\tilde{H}_1^*\right]=\frac{-\omega}{2c}\mathrm{Im}\left[\tilde{\phi}^1(\tilde{\phi}^1)^*_{,3}+\tilde{\phi}^2(\tilde{\phi}^2)^*_{,3}\right], \tag{68}$$

$$\rho=\frac{-3\kappa}{16}[|\tilde{\phi}^1|^2+|\tilde{\phi}^2|^2]\tilde{\eta}, \tag{69}$$

$$E_j=-A^j_{,0},j=1,2, \tag{70}$$

$$H_2=A^1_{,3},\ \text{and}\ H_1=-A^2_{,3}. \tag{71}$$

3.3 Rectangular cavity resonator

We consider a closed cubicle cavity of dimension L^3 with perfectly conducting walls located at planes $\pm(L/2)\vec{j}, j=x,y,z,$

where $\{x^1,x^2,x^3\}\rightarrow\{x,y,z\}$ (See Fig. 1). Without the loss of generality, we assume that z is the direction of propagation. The electromagnetic field in the cavity are divided into two types – (i) transverse electric (TE) for which the electrical field component $E_z = 0$ and (ii) transverse magnetic (TM) for which the magnetic field component $H_z = 0$. In this paper, we focus only the TE modes for which $A^3=A^0=0$. The linear modes (TE_{mn}) (i.e. the right hand sides of Eqs. (62) and (63) are zero) are given by [24,25]

$$A^j = g_j(x,y)\cos(\omega t)[D_j \exp(ik_z z) + c.c.]/2,\ j=1,2,$$
$$g_1(x,y) = \cos[k_x(x+L/2)]\sin(k_y(y+L/2)],$$
$$g_2(x,y) = \sin[k_x(x+L/2)]\cos(k_y(y+L/2)],$$
$$D_1 k_x + D_2 k_y = 0, \tag{72}$$

where

$k_x = m\pi / L,\ k_y = n\pi / L,\ m$ and n are integers, and $g_j(x,y)$ represents the transverse mode distributions and k_z is the propagation constant. The requirement that the tangential component of the electric field intensity should be zero at the planes *z=-L/2* and *L/2* leads to

$$k_z = j\pi / L,\ j \text{ is an odd integer,}$$
$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 / c^2. \tag{73}$$

In this Section, we focus on the fundamental TE_{01} mode,

$$A^1 = D_1 \cos(k_y y)\cos(k_z z)\cos(\omega t),$$
$$A^2 = 0,$$
$$k_y = k_z = \pi / L, k_x = 0. \tag{74}$$

We wish to find the quasi-linear modes of the cavity under the weak field approximations satisfying the boundary condition that the tangential components of the electric field intensity are zero at the conducting walls. The evolution of A^1 and h_{00} in the cavity are given by Eqs. (37) and (36), respectively.

$$\Box A^1 = -\kappa\left(\frac{(A^1)^3}{2} + \eta^{\mu\nu} h_{00,\mu} A^1{}_{,\nu}\right), \tag{75}$$

$$\Box h_{00} = (A^1)^2. \tag{76}$$

Let

$$A^1 = \psi(y,z)\exp(i\omega t) + c.c. \tag{77}$$

Squaring Eq. (77) and substituting it in Eq. (76), we find that the excitation is at frequencies 0 and 2ω. Hence, let the response be

$$h_{00} = [h_1(y,z)\exp(i2\omega t) + c.c.] + h_2(y,z). \tag{78}$$

Substituting Eq. (78) in Eq. (75) and ignoring third harmonic components, we find

$$\psi_{,2,2} + \psi_{,3,3} + k^2\psi = -\kappa\left(\frac{3|\psi|^2\psi}{2} - 2k^2 h_1\psi^*\right) - \kappa(h_{2,2}\psi_{,2} + h_{2,3}\psi_{,3} + h_{1,2}\psi^*_{,2} + h_{1,3}\psi^*_{,3}), \tag{79}$$

$$h_{1,2,2} + h_{1,3,3} + 4k^2 h_1 = \psi^2, \tag{80}$$

$$h_{2,2,2} + h_{2,3,3} = 2|\psi|^2. \tag{81}$$

To further simplify Eqs. (79)-(81), let

$$\psi(y,z) = \cos(k_y y)g(z), \tag{82}$$

We assume that $g(z)$is real, which reduces to $\cos(k_z z)$ as $h_{\mu\nu} \to 0$. Substituting Eq. (82) in Eqs. (80) and (81), and separating components at the spatial frequencies 0 and $2k_y$, we find

$$\begin{aligned} h_1 &= B(z) + C(z)\cos(2k_y y), \\ h_2 &= D(z) + F(z)\cos(2k_y y), \end{aligned} \tag{83}$$

$$\begin{aligned} &4B(z)k^2 + B_{,z,z} = g^2(z)/2, \\ &4(k^2 - k_y^2)C(z) + C_{,z,z} = g^2(z)/2, \\ &D_{,z,z} = g^2(z), \\ &-4k_y^2 F(z) + F_{,z,z} = g^2(z). \end{aligned} \tag{84}$$

Substituting Eqs. (83) and (84) in Eq. (79), we obtain

$$\begin{aligned} &[(\omega^2/c^2 - k_y^2)g(z) + g_{,z,z}]\cos(k_y y) = -\kappa\left(\frac{3\cos^3(k_y y)g^3}{2}\right) \\ &-\frac{2\kappa\omega^2 g}{c^2}[B - C\cos(2k_y y)\cos(k_y y)] \\ &+\kappa g\left[2k_y^2\sin(2k_y y)\sin(k_y y)(C+F)\right] \\ &-\kappa g_{,z}\cos(k_y y)\left[(B+D) - (C+F)\cos(2k_y y)\right]_{,z}. \end{aligned} \tag{85}$$

In this Section, our objective is to find g(z) which becomes zero at $z = \pm L/2$so that the boundary condition is satisfied. In order to accomplish this, we follow the approach typically used to derive the nonlinear Schrodinger equation (NLSE) from the nonlinear Maxwell's equation [26,21]. In Refs. [26,21], fiber nonlinearity is treated as a small perturbation on the fundamental transverse mode (HE_{11}) and the NLSE is derived to describe the evolution of the mode weight of the fundamental transverse mode as a function of the propagation distance, *z*, by multiplying the nonlinear wave equation by the transverse mode distribution and integrating over the transverse dimensions *x* and *y*. Here, we follow the same approach. We assume that the field in the transverse direction is the same as that of a linear mode and the nonlinear terms appearing on the RHS of Eq. (85) cause a small perturbation to this linear mode. Multiplying Eq. (85) by $\cos(k_y y)$ and integrating from $-L/2$ to $L/2$ with respect to y, we obtain

$$g_{,z,z} + \beta^2 g = \kappa\left[-\frac{9}{8}g^3 + k^2(2B+C)\right] - \kappa[k_y^2 g(C+F) + g_{,z}(B+D+(C+F)/2)_{,z}], \tag{86}$$

where $\beta = \sqrt{(\omega/c)^2 - k_y^2}$ is the eigenvalue to be determined under the condition that $g(z)$ becomes zero at $\pm L/2$. For the linear case (i.e. $\kappa = 0$), from Eqs. (86) and (74), we have $\beta = k_z = \pi/L$. In a general case, Eq. (86) provides the evolution of a quasi-linear mode (in z-direction) with the transverse mode distribution being proportional to $\cos(\pi y/L)$. From the right hand side of Eq. (85), we see that there are excitations proportional to $\cos(3\pi y/L)$and one should expect the generation of such higher order modes due to nonlinear effects. However, when Eq. (85) is multiplied by $\cos(\pi y/L)$ and integrated over y, higher order transverse modes do not contribute and Eq. (86) may be interpreted as the equation that provides the weight of the fundamental transverse mode distribution ($\cos(\pi y/L)$). As the amplitude of A^1 becomes larger, there could be a nonlinear coupling between the fundamental transverse mode ($\propto \cos(\pi y/L)$) and the higher order mode ($\propto \cos(3\pi y/L)$) . However, such nonlinear interactions are not captured in Eq. (86).

Equations (86) and (84) form a coupled nonlinear differential system of equations which are solved using an explicit Runge-Kutta method (Matlab built-in function ode45). We look for a solution that is symmetric with respect to z=0. The problem can be formulated in two ways (i) For the given initial condition $g(0) = g_0$, and $g'(0) = 0$where ' denotes differentiation with respect to z, the propagation constant β is found such that the boundary condition $g(L/2) = 0$is satisfied (i.e. the tangential component of the electric field is zero at the walls) (ii) For the given β (or equivalently for the given ω), find $g(0)$such that the boundary condition $g(L/2) = 0$ is satisfied. We follow the latter approach. Note that in the absence of nonlinearity ($\kappa = 0$), $\beta = k_z$, ω=ck and the amplitude $g(0)$ is arbitrary. Let

$$f_{res} = \frac{ck}{2\pi} = \frac{c}{\sqrt{2}L} \tag{87}$$

be the resonant frequency of the cavity when $\kappa = 0$ for the fundamental mode. In the presence of nonlinearity ($\kappa \neq 0$), as the frequency of the EM field deviates from the resonant frequency, the initial amplitude $g(0)$ (equivalently energy of the EM field) should be changed to satisfy the boundary condition at $z=L/2$. Due to nonlinear effects (which is the signature of spacetime curvature), let the frequency detuning be

$$\Delta f = f - f_{res} \tag{88}$$

where f is the frequency of the EM field. Figure 2 shows the evolution of the field $g(z)$ as a function of distance z. As the frequency detuning increases, the amplitude of the field at $z=0$ increases and hence, the EM energy stored in the cavity increases. If the frequency detuning Δf is negative, we found that the boundary condition that $g(L/2)=0$cannot be satisfied. If Eq. (86) is solved with $\kappa=0$, one finds that Δf should be zero so as to satisfy the boundary condition $g(L/2)=0$ (unless Δf is so large that f coincides with the higher order resonant frequencies) and the amplitude of the field at $z=0$ is arbitrary.

To verify the validity of Eqs. (84) and (86), the coupled partial differential equations (73)-(75) are numerically solved using the FDTD technique with the boundary condition that the tangential components of the electric field intensity are zero at the metallic walls. In the numerical solution, the growth of higher order mode ($\propto \cos(3\pi y/L)$) was observed. To be consistent with the semi-analytical approach, the numerical solution of Eq. (79) is multiplied by $\cos(\pi y/L)$ and integrated from $-L/2$ to $L/2$ to obtain the mode weight of the fundamental mode. '+' in Fig. 2 show the numerical solutions obtained by the FDTD technique and as can be seen, the agreement between the semi-analytical approach and numerical approach is quite good.

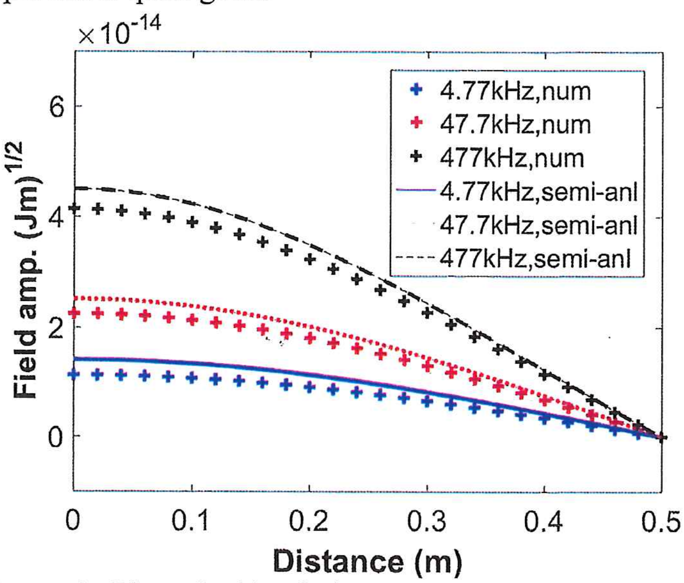

Figure 2. Plot of g(z) of electromagnetic vector potential A^1 vs distance, z for frequency detuning factors, Δf= 4.77 kHz, 47.7 kHz and 477 kHz. L = 1 m,

resonant frequency, f_{res}= 211.98 MHz ; num = numerical, and semi-anl = semi-analytical.

The small discrepancy is attributed to the fact that semi-analytical approach does not take into account the coupling between the fundamental and higher order transverse mode.

We define the rest mass of the EM field confined to the cavity as

$$m = (\mathrm{E}_e + \mathrm{E}_m)/c^2, \tag{89}$$

where E_e and E_m are the mean energy stored in electric and magnetic fields respectively,

$$\mathrm{E}_e = \left\langle \int E_x^2 dV \right\rangle, \mathrm{E}_m = \left\langle \int (H_y^2 + H_z^2) dV \right\rangle. \tag{90}$$

$$E_x = -A^1_{,0},\ H_y = A^1_{,z},\ \text{and}\ H_z = -A^1_{,y}. \tag{91}$$

For the given frequency *f*, the field $g(z)$ is calculated by solving Eqs. (86) and (84) numerically, and using Eq. (91), electric and magnetic field intensities are calculated. Using Eqs. (89) and (90), the rest mass is calculated for the given frequency *f* and plotted in Fig. 3. The line with '+' shows the mass calculated semi-analytically using the above procedure and the solid line shows the curve fitting. A good fit was found by using

$$m = \frac{8\pi^2\sqrt{2\pi(f^2 - f_{res}^2)}}{c^3\kappa}. \tag{92}$$

Equation (92) may be rewritten as the following dispersion relation

$$\omega^2 = (kc)^2 + \frac{m^2c^6\kappa^2}{32\pi^3}, \tag{93}$$

where the wave number at the resonant frequency is

$$k = \sqrt{k_y^2 + k_z^2} = (2\pi f_{res})/c. \tag{94}$$

By setting $E = \hbar\omega$ and $p = \hbar k$, and if

$$\kappa = \frac{2(2\pi)^{3/2}}{\hbar c}, \tag{95}$$

Equation (93) could be rewritten as a special relativistic relation relating the energy, momentum and rest mass of a particle,

$$E^2 = p^2c^2 + m^2c^4. \tag{96}$$

As the amplitude of A^1 goes to zero, $m \to 0$, and hence, $\omega = kc$ is correct only for the EM field with vanishing amplitude. In the absence of spacetime curvature (κ =0), the EM field is governed by the linear Maxwell's equations and in this case, $E = pc$ even if the field is confined to a localized region. The relation between κ

and $\hbar$ given by Eq. (95) may be off by a scaling factor of $O(1)$ due to approximations made in the derivation of Eq. (86). We have considered the impact of spacetime curvature only on the fundamental mode TE_{01} and as the mode order increases, the dependence of the frequency detuning on the field intensity is expected to be given by a formula similar to Eq. (93), but there could be an additional constant in Eq. (93) that may depend on the mode order.

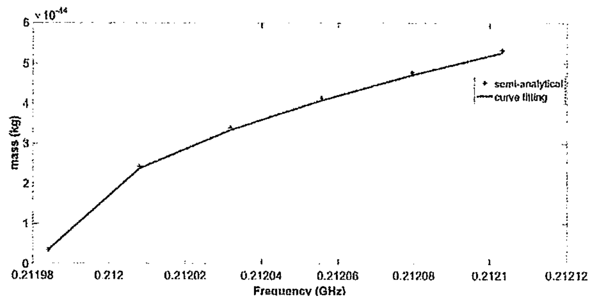

Figure 3. Plot of the rest mass as a function of the frequency of the electromagnetic field. L = 1 m, resonant frequency, f_{res} = 211.98 MHz.

Conclusions

An electromagnetic (EM) tensor which is an outer product of EM vector potential is used to modify Einstein-Maxwell equations. In Einstein's theory of gravitation, the coupling constant connecting Einstein tensor and stress-energy tensor is proportional to gravitational constant. Similarly, we find that the coupling constant connecting Einstein tensor and electromagnetic tensor is inversely proportional to Planck's constant. In classical electrodynamics, Maxwell's equations do not provide the equations of motion for charged particles; they are provided separately by Lorentz force equations. However, Einstein-Maxwell equations with the new EM tensor derived in this paper lead to equations of motion that resemble Lorentz force equations. Using slowly varying envelope approximation, these equations reduce to Schrodinger equation with a self-trapping potential.

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References

[1] J.D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, Inc, New Jersey, USA (1999).

[2] M.O. Scully and M.S. Zubairy, *Quantum Optics*, Cambridge University Press (1997).

[3] A. Einstein, Annalen der Physik **49,** 769 (1916).

[4] P.G. Bergmann, *Introduction to the theory of relativity*, Prentice-Hall Inc, New Jersey,
(1960).

[5] A. Einstein, L. Infeld, B. Hoffmann, Annals of Mathematics **39**, 65 (1938).

[6] A. Einstein, L. Infeld, Annals of Mathematics **41**, 455 (1940).

[7] R.C. Tolman, P. Ehrenfest, and B. Podolsky, Phys. Rev. **37**, 602 (1931).

[8] M.O. Scully, Phys. Rev. D **19**, 3582 (1979).

[9] R.M. Santilli, "Partons and Gravitation: some Puzzling Questions," (MIT) Annals of Physics, **83**, 108 (1974).

[10] T. Kaluza, Sitzungsber Preuss. Akad. Wiss. Berlin (Math. Phys.), 966 (1921).

[11] O. Klein, Nature **118**, 516 (1926).

[12] P.S. Wesson, *Space-time-matter:Modern Kaluza-Klein theory,* World Scientific Publication (1998).

[13] M. Born and L. Infeld, Proc. R. Soc. Lond. **144**, 425 (1934).

[14] A.A. Coley, R.J. van den Hoogen, and D.D. McNutt, J. Math. Phys. **61**, 072503 (2020). See
references therein.

[15] S.V. Manakov, Soviet Physics JETP **38**, 248 (1974).

[16] G. Schutz, *First course in general relativity*, Cambridge University Press, (2011).

[17] S. Weinberg, *Gravitation and cosmology*, John Wiley and Sons, Inc, New Jersey, USA,
(1972).

[18] R. M. Santilli, American Journal of Modern Physics, **4** (5), 59-75 (2015).

[19] P. Freud, Ann. Math. **40**(2), 417 (1939).

[20] Y.R. Shen, *The principles of nonlinear optics*, John Wiley and Sons, Inc, Hoboken, New Jersey, USA (2003).

[21] G.P. Agrawal, *Nonlinear fiber optics*, Academic Press, Oxford, UK (2013).

[22] A. Hasegawa, F. Tappert, App. Phys. Lett. **23**, 142 (1973).

[23] V.E. Zakharov, A.B. Shabat, Soviet Physics JETP **34**, 62 (1972).

[24] A. Ghatak, K. Thyagarajan, *Optical Electronics*, Cambridge University Press, Cambridge, UK, (1998).

[25] M.N.O. Sadiku, *Elements of Electromagnetics*, Oxford University Press, New York, USA, (2006).

[26] S. Kumar and M. J. Deen, *Fiber Optic Communications: Fundamentals and Applications*, John Wiley and Sons, Inc, West Sussex, UK, (2014).

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ROLE OF THE LIE-SANTILLI ISOTHEORY FOR THE PROOF OF THE EPR ARGUMENT

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Abstract

In 1935, A. Einstein stated, in a historical paper with B. Podolsky and N. Rosen [1] that "quantum mechanics is not a complete theory" and that determinism could be recovered at least under limit conditions (EPR argument). In 1964, J. S. Bell [2] proved a theorem according to which a system of quantum mechanical particles with spin 1/2 with $SU(2)$ Lie algebra $[\sigma_i, \sigma_j] = 2\epsilon_{i,j,k}\sigma_k$, where the σs are the Pauli matrices, cannot admit a classical counterpart, thus appearing to disprove the EPR argument. In 1978, R. M. Santilli [3] discovered the axiom-preserving generalization-"completion" of the various branches of Lie's theory (universal enveloping algebras, Lie algebras, and Lie groups) based on the isoassociative product $X_i \star X_j = X_i\hat{T}X_j, \hat{T} > 0$, with Lie-Santilli isoalgebras $[X_i, X_j]^* = X_i \star X_j - X_j \star X_i = C_{i,j,k,}X_k$ classified into *regular* (*irregular*) when the C-quantities are constant (functions). In 1998 [4] Santilli proved that Bell's theorem is valid for point-particles, but it is inapplicable for systems of extended particles with spin 1/2 under deep mutual entanglement, and that said systems do admit classical counterparts when represented with the isotopic $SU(2)$ Lie-Santilli isoalgebars $[\Sigma_i, \Sigma_j]^* = 2\epsilon_{i,j,k}\Sigma_k$, where Σ_k are the new Pauli-Santilli isomatrices, with realization of the isotopic element $\hat{T} = Diag.(1/\lambda, \lambda)$, $\det\hat{T} = 1$ providing a concrete and explicit realization of "hidden variables" under the full validity of quantum axioms. Subsequently, Santilli [5] proved that Einstein's determinism is progressively approached in the structure of hadrons, nuclei and stars and it is fully recovered at the limit of gravitational collapse (see Refs. [6] for a detailed presentation). In this lecture, by following our recent paper [7], we outline the aspects of the Lie-Santilli isotheory which are essential for Santilli's proofs of the EPR argument.

1 Introduction

The well known *EPR argument* was proposed by A. Einstein, B. Podolsky and N.Rosen in 1935 [1] implies that; Quantum mechanics is not a complete theory but should be supplemented by additional variables. i.e. Quantum mechanics has to be deterministic. In this regard Einstein has made a famous statement that "God doesn't play dice with the universe."
In other words, Einstein believed that quantum mechanics is not a complete theory, in the sense that it could be broadened to recover classical determinism at least under limiting conditions.
Numerous objections against EPR argument have been raised by scholars including N.Bohr [8], S.Bell [2, 9], J.Von Neumann [10]. Till date, it is widely believed that Quantum mechanics is the final theory for all conceivable conditions existing in the universe.
Any Physical Theory operates with physical concepts which correspond with the objective reality. Success of a physical theory depends on;

- Correctness
- Completeness

Correctness is judged by the degree of agreement between theoretical conclusions and human experience.
Completeness of a Physical Theory Requires;

- Every element of the physical reality must have a corresponding concept in the physical theory.
- Elements of physical reality must be experiments and measurements.

Scientifically, a reasonable interpretation of physical reality would be; if, without in any way disturbing a system, we can predict with certainty (i.e.with probability unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.
If we start with the assumption that wave function does give a complete description of the physical reality, we arrive at the conclusion that two physical quantities with non commuting operators can have simultaneous reality. This implies that quantum mechanical description of physical reality given by wave function is not complete. i.e. quantum axioms do not admit hidden variables (Local Realism), [2]. So, quantum mechanics can not be described

by local hidden variables. For that matter, assuming the validity of Bell's theorem, any deterministic hidden-variable theory that is consistent with quantum mechanics would be non-local. Hence, dismissal of EPR argument. Following decades of research since 1998, R.M.santilli, assuming the validity of quantum mechanics, with consequential validity of the objections against the EPR argument[11],[12] for point-like particles in empty space under linear, local and potential interactions (exterior dynamical problems); proved the inapplicability (and not their violation) of said objections for the broader class of extended, deformable and hyperdense particles within physical media under the most general known linear and nonlinear, local and non- local and potential as well as non- potential interactions (interior dynamical problems). Santilli's Contribution also provided the apparent proof that interior dynamical systems admit classical counter- parts in full accordance with the EPR argument via the representation of interior systems with of isomathematics, also called isotopic branch of hadronic mathematics, and isomechanics, also called isotopic branch of hadronic mechanics.
The main assumption of apparent proof of EPR argument is; particles can be represented as extended, deformable and hyperdense under the conditions of mutual overlapping/entanglement with ensuing contact at a distance. This eliminates objection 'quantum entanglement' regarding non-locality of quantum mechanics.

In the 2019 paper [5], *Santilli provided the apparent proof (of 'completion' of quantum mechanics* as isotopic/axiom-preserving type, being fully admitted by quantum mechanics merely subjected to a broader realization than that of Copenhegen school) that *Einstein's determinism is progressively approached in the interior of hadrons, nuclei and stars and it is fully achieved in the interior of gravitational collapse.*

Thus, inapplicability of 20th century 'applied mathematics' in general and of Lie's theory in particular to the interior dynamical systems led Santilli to construct a new mathematics, known as *isomathematics* exactly applicable to the interior dynamical systems. In particular, Lie-algebra structure required in quantum mechanics was lifted structurally to show that objections against EPR argument are inapplicable.

Appropriate lifting of conventional Lie theory applicable to exterior dynamical systems to Lie-Santilli isotheory applicable to interior dynamical systems was achieved by Santilli [13].
The Lie-Santilli isoalgebras and isogroups were elaborated with the conven-

tional mathematics of Lie's theory via conventional functional analysis and differential calculus on conventional space such as original Hilbert space H over conventional field $F(n, \times, 1)$.

2 Isomathematics

The basic multiplicative unit 1 is replaced by an arbitrary, positive definite quantity $\hat{I} = \frac{1}{\hat{T}}$ whether or not element of the original field. $\hat{T}$ is called the *isotopic element* and

$$\hat{I} = \frac{1}{\hat{T}} \tag{1}$$

is called the *isounit*, and all possible associative products are lifted via

$$X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j \tag{2}$$

with $\hat{I}$ being the correct left and right multiplicative unit for all the elements of the set considered such that

$$\hat{I} \hat{\times} X = X \hat{\times} \hat{I} = \hat{X} \tag{3}$$

for all X in the resulting new field called as *Santilli Isofield.* The new numbers $\hat{X}$ in the isofield are called as *isonumbers*
This new field is denoted by $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$.

3 Lie-Santilli isotheory

It is well known that **Lie's theory** is at the true structural foundation of quantum mechanics via celebrated product;

$$[A, B] = A \times B - B \times A \tag{4}$$

where $A \times B = AB$ is the conventional associative product.

Today, by **Lie-Santilli isotheory** we mean the infinite family of isotopies of **Lie's theory** formulated on an iso-Hilbert space $\hat{H}$ defined over an isofield $\hat{F}$ generated by iso-Hremitean generators $X_k, k = 1, 2, 3, \ldots.., N$ with all possible products lifted into the isoassociative form (2) and multiplicative isounit $\hat{I} = \frac{1}{\hat{T}}$, [14].

Generalization of Lie's theory by Santilli [15]in 1978 under the name *Lie-isotopic theory* with the basic product;

$$
\begin{aligned}
[A,\hat{}B] &= A\hat{\times}B - B\hat{\times}A \\
&= A \times \hat{T} \times B - B \times \hat{T} \times A \\
&= A\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, \eta, ...)B \\
&- B\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, \eta, ...)A
\end{aligned} \tag{5}
$$

Lie-isotopic theory is also called as *Lie-Santilli isotheory.*

1. *Lie-Santilli isotheory* is based on isotopic product $[A,\hat{}B] = A\hat{T}B - B\hat{T}A$ where $\hat{T}$ is a hermitean matrix or operator with $\hat{T} = \hat{T}^\dagger$.

2. *Lie-admissible theory,* also called as *Lie-Santilli genotheory,* is based on the product $(A, B) = A\hat{T}B - B\hat{T}^\dagger A = A\hat{R}B - B\hat{S}A$ where $\hat{T}$ is a nonhermitean matrix or operator with $\hat{T} = \hat{R} \neq \hat{T}^\dagger = \hat{S}$.

3. *Hypertheory,* the most general formulation of hyperstructural character [16] is based on the product of type $A \otimes B = A\hat{R}B - B\hat{S}A$ where $\hat{R}$ and $\hat{S}$ are sets.

4 Lie Algebra

Let L **be an N-dimensional Lie algebra** over a field $F(n, \times, 1)$ of characteristic zero and associative product $nm = n \times m \in F$ and multiplicative unit 1.
Let the **generators of** L **are the Hermitean operators** $X_k, k = 1, 2, ...n$, on a Hilbert space $\mathcal{H}$ over F.
Let $\xi(L)$ **be the universal enveloping associative algebra** of ordered monomials based on the associative product;

$$X_i \times X_j \tag{6}$$

Let the Lie algebra L be isomorphic to the **anti-symmetric algebra attached to the enveloping algebra** $L \approx [\xi(L)]^-$ with ensuing Lie's theorems and and commutation rules;

$$[X_i, X_j] = X_i \times X_j - X_j \times X_i = C_{ij}^k \times X_k \tag{7}$$

4.1 Isotopies of Lie Algebra

- the ***isotopy of the associative product***

$$X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j \tag{8}$$

  where $\hat{T}$ (the *isotopic element*)is a fixed positive-definite operator with an arbitrary functional dependence on local variables;

- the ***isotopy of the enveloping algebra*** $\hat{\xi}(\hat{L})$ characterized by ordered monomials of the Poincare-Birkhoff-Witt-Santill is a theorem based on isoproduct (2);

- the ***isotopies of Lie algebras***, today called the ***Lie-Santilli isoalgebra*** $\hat{L}$ as the anti-symmetric algebra attached to the isoenvelope

$$\hat{L} \approx [\hat{\xi}(\hat{L})]^- \tag{9}$$

  with *Lie-Santilli isocommutation* rules

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i = \hat{C}_{ij}^k \hat{\times} X_k \tag{10}$$

- the *isotopies of Lie groups* today known as the ***Lie-Santilli isogroups***; and

- the ***isorepresentation theory.***

4.2 Lie-Santilli Isoalgebra

Definition 4.1 *A (finite-dimensional) isospace $\hat{L}$ over an isofield $\hat{F}(\hat{a}, +, \hat{\times})$ of isoreal numbers $\hat{R}(\hat{n}, +, \hat{\times})$, isocomplex numbers $\hat{C}(\hat{c}, +, \hat{\times})$ or isoquaternions $\hat{Q}(\hat{q}, +, \hat{\times})$ with isotopic element $\hat{T}$ and isounit $\hat{I} = \hat{T}^{-1}$ is called a "Lie-Santilli isoalgebra" over $\hat{F}$ when there is a composition $[\hat{A} \hat{,} \hat{B}]$ in $\hat{L}$, called "isocommutator", which verifies the following "isolinear and isodifferential rules" for all $\hat{a}, \hat{b} \in \hat{F}$ and $\hat{A}, \hat{B}, \hat{C} \in \hat{L}$*

$$[\hat{a}\hat{\times}\hat{A}+\hat{b}\hat{\times}\hat{B},\hat{,}\hat{C}]=\hat{a}\hat{\times}[\hat{A},\hat{,}\hat{C}]+\hat{b}\hat{\times}[\hat{B},\hat{C}] \tag{11}$$

$$[\hat{A}\hat{\times}\hat{B},\hat{C}]=\hat{A}\hat{\times}[\hat{B},\hat{C}]+[\hat{A},\hat{C}]\hat{\times}\hat{B} \tag{12}$$

and "Lie-Santilli isoaxioms"

$$[\hat{A},\hat{B}]=-[\hat{B},\hat{A}] \tag{13}$$

$$[\hat{A},\hat{[}\hat{B},\hat{C}]]+[\hat{B},\hat{[}\hat{C},\hat{A}]]+[\hat{C},\hat{[}\hat{A},\hat{B}]]=0 \tag{14}$$

It is important to note that the associative character of the underlying envelope is preserved while using isoreals, isocomplexes and isoquaternions. Consistent isotopic generalization of celebrated Lie's First, Second and Third theorems has been proved by Santilli in [17].

4.3 Isorepresentations of Lie-Santilli isoalgebras

Isorepresentations of Lie-Santilli isoalgebras is classified into;

1. **Regular isorepresentations** which occur due to C's of the rules (10) are constants; and

2. **Irregular isorepresentation** occurring when the C's of the rules (10) are functions of the local variables (an occurrence solely possible for the Lie-Santill isotheory).

5 Construction of Regular Isorepresentations

General Construction: Regular isorepresentation of Lie-Santilli isoalgebras $\hat{L}$ over an isofield of characteristic zero can be constructed via *non-unitary* transformations of the conventional representations of the conventional Lie algebra L.

The general rule for mapping Lie algebras into regular Lie-Santilli isoalgebras were identified for the first time by Santilli in [18] and then studied systematically in monographs [14]. They can be written as follows;

$$U \times U^{\dagger} = \hat{I} \neq I \tag{15}$$

This non-unitary transformation is **applied to the entire mathematics of Lie's theory leading to Santilli's isomathematics.** We get the following important fundamental transformations;

$$I \longrightarrow \hat{I} = U \times I \times U^{\dagger} = \frac{1}{\hat{T}}, \tag{16}$$

$$a \longrightarrow \hat{a} = U \times a \times U^{\dagger} = a \times U \times U^{\dagger} = a \times \hat{I} \in \hat{F}, a \in F \tag{17}$$

$$e^{A} \longrightarrow U \times e^{A} \times U^{\dagger} = \hat{I} \times e^{\hat{T} \times \hat{A}} = (e^{\hat{A} \times \hat{T}}) \times \hat{I} \tag{18}$$

$$\begin{aligned} A \times B \longrightarrow U \times (A \times B) \times U^{\dagger} &= (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) \\ &= \hat{A} \hat{\times} \hat{B} \end{aligned} \tag{19}$$

$$\begin{aligned} [X_i, X_j] \longrightarrow U \times [X_i, X_j] \times U^{\dagger} &= [\hat{X}_i \hat{,} \hat{X}_j] = U \times (C_{ij}^{k} \times X_k) \times U^{\dagger} \\ &= C_{ij}^{k} \times \hat{X}_k \end{aligned} \tag{20}$$

$$\begin{aligned} <\psi \mid \times \mid \psi> \longrightarrow U \times <\psi \mid \times \mid \psi> \times U^{\dagger} &= <\psi \mid \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times \mid \psi> \times (U \times U^{\dagger}) \\ &= <\hat{\psi} \mid \hat{\times} \mid \hat{\psi}> \times \hat{I}, \end{aligned} \tag{21}$$

$$\begin{aligned} H \times \mid \psi> \longrightarrow U \times (H \times \mid \psi>) U^{\dagger} &= (U \times H \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times \mid \psi>) \\ &= \hat{H} \hat{\times} \mid \hat{\psi}> \end{aligned} \tag{22}$$

etc.

6 Classification of regular isounitary isoirreducible isorepresentations of the Lie-Santilli $\widehat{SU}(2)$ isoalgebras over isofields of characteristic zero

Santilli [21, 22] identified and **constructed the following regular isorepresentation of Lie-Santilli isoalgebra** $\widehat{SU}(2)$**, from the conventional**

two-dimensional irreducible representation of the $SU(2)$ Lie algebra defined by the well known Pauli's matrices.

This Classification is merely given by either the nonunitary transform $U - Diag(n_1, n_2), n_k$ real > 0, or by $U - OffDiag(n_1, n_2)$.

Conventional Paulli matrices σ_k [19, 20] satisfy the rules $\sigma_i\sigma_j = i\varepsilon_{ijk}, i, j, k = 1, 2, 3$. We present the identification and classification ref.[21, 22] of these matrices due to isoalgebra $\hat{SU}_Q(2)$.

In general Lie-isotopic algebras are the image of Lie algebras under nonunitary transformations [23, 24]. **Under the transformation $UU^\dagger = \hat{I} \neq I$ a Lie commutator among the matrices acquires the Lie-isotopic form**

$$U(AB - BA)U^\dagger = A'QB' - B'QA',$$
$$A' = UAU^\dagger, B' = UBU^\dagger, Q = (UU^\dagger)^{-1} = Q^\dagger \tag{23}$$

As a result, a first class of fundamental (adjoint) isorepresentations called as regular adjoint isorepresentations are characterized by the maps $J_k = \frac{1}{2}\sigma_k \rightarrow \hat{J}_k = UJ_kU^\dagger, UU^\dagger\hat{I} \neq I$ with isotopic contributions that are factorizable in the spectra, $\pm\frac{1}{2} \rightarrow +\frac{1}{2}f(\Delta), \frac{3}{4} \rightarrow (\frac{3}{4}f^2(\Delta))$ where $\Delta = detQ$ and $f(\Delta)$ is a smooth nowhere-null function such that $f(1) = 1$.

7 Iso-Pauli matrices

Santilli constructed the following example of regular iso-Pauli matrices.

$$\hat{\sigma_1} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \hat{\sigma_2} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & -ig_{11} \\ ig_{22} & 0 \end{pmatrix}$$
$$\hat{\sigma_3} = \Delta^{-\frac{1}{2}} \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} \tag{24}$$

where $\Delta = detQ = g_{11}g_{22} > 0$.

These representations verify the isotopic rules $\hat{\sigma}_i Q \hat{\sigma}_j = i\Delta^{\frac{1}{2}}\epsilon_{ijk}\hat{\sigma}_k$, and consequently the following isocommutator rules and generalized isoeigenvalues for $f(\Delta) = \Delta^{\frac{1}{2}}$ and

$$[\hat{\sigma}_1, \hat{\sigma}_j] = \hat{\sigma}_i Q \hat{\sigma}_j - \hat{\sigma}_j Q \hat{\sigma}_i = 2i\Delta^{\frac{1}{2}}_{ijk}\hat{\sigma}_k \tag{25}$$

$$\hat{\sigma}_3 * \mid \hat{b}^2_i\rangle = \pm\Delta^{\frac{1}{2}} \mid \hat{b}^2_i\rangle \tag{26}$$

$$\hat{\sigma}^{\hat{2}} * \mid \hat{b}^2_i\rangle = 3\Delta \mid \hat{b}^2_i\rangle, i = 1, 2 \tag{27}$$

This confirms the 'regular' character of the generalization considered here. The isonormalized isobasis is then given by a trivial extension of the conventional basis $\mid \hat{b}\rangle = Q^{-\frac{1}{2}} \mid b\rangle$.
In fact, regular iso-Pauli matrices (24) admit the conventional eigenvalues $\frac{1}{2}$ and $\frac{3}{4}$ for $\Delta = 1$ which can be verified by putting $g_{11} = g_{22}^{-1} = \lambda$.
It is important to emphasize the condition of isounitarity, i.e. $UU^\dagger = \hat{I} \neq I$ for which $n_1{}^2 = 1/n_2{}^2 = \lambda > 0$. Thus, realization of isotopic element $\hat{T} = Diag.(1/\lambda, \lambda)$ with $detQ$ *provides a concrete and explicit realization of "hidden variable" under full validity of quantum axioms.*

Remarks:- This degree of freedom has major fundamental implications presented in [22] as well as for the spin component of the first known representation of nuclear magnetic moments presented in the papers [25, 26].

References

[1] A.Einstein, B.Podolsky and N.Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete ?*, Physical Review Vol.47, 1935.

[2] J.S.Bell *ON THE EINSTEIN PODOLSKY ROSEN PARADOX*, Physiscs Publishing Co. , Physics Vol. 1, No. 3, pp.195-200, 1964.

[3] R. M. Santilli, *Foundation of Theoretical Mechanics,* Springer-Verlag, Heidelberg, Germany, Volumes I and II (1978)

http://www.santilli-foundation.org/docs/Santilli-209.pdf
http://www.santilli-foundation.org/docs/santilli-69.pdf

[4] R. M. Santilli, Acta Applicandae Mathematicae Vol. 50, 177 (1998),
http://www.eprdebates.org/docs/epr-paper-i.pdf

[5] R. M. Santilli, Ratio Mathematica Volume 37, pages 5-23 (2019),
http://www.eprdebates.org/docs/epr-paper-ii.pdf

[6] R.M. Santilli, *"Studies on A. Einstein, B. Podolsky, and N. Rosen prediction that quantum mechanics is not a complete theory*, Papers I, II and III
http://eprdebates.org/docs/epr-review-i.pdf
http://eprdebates.org/docs/epr-review-ii.pdf
http://eprdebates.org/docs/epr-review-iii.pdf

[7] A. S. Muktibodh and R. M. Santilli, *"Studies of the Regular and Irregular Isorepresentations of the Lie-Santilli Isotheory,"* Journal of Generalized Lie Theories Vol. 11, p. 1-7 (2017),
http://www.santilli-foundation.org/docs/isorep-Lie-Santilli-2017.pdf

[8] N.Bohr *"Can Quantum mechanical description of physical reality be considered complete?"* Phys. Review, Vo. 48, p.696(1935)
informationphilosopher.com/solutions/scientists/bohr/EPRBohr.pdf.

[9] "On the problem of hidden variables in quantum mechanics", Reviews of Modern Physics, Vol.38, No.3,447, (July 1966).

[10] J.Von Neumann, Mathematische Grundalgen der Quantenmechanik, Springer, Berlin (1951).

[11] R.M.Santilli, *Studies on A. Einstein, B. Podolsky, and N. Rosen argument that quantum mechanics is not a complete theory,* I: Basic methods, submitted for publication,

[12] R.M.Santilli, *Studies on A. Einstein, B. Podolsky, and N. Rosen argument that quantum mechanics is not a complete theory II: Apparent confirmation of the EPR argument,* Submitted for publication

[13] R.M.Santilli, *Foundations of Theoretical Mechanics, Vol.II: Birkhoffian Generalization of Hamiltonian Mechanics,* Springer-Verlag, Heidelberg, New York (1982).

[14] R.M.Santilli, *Elements of Hadronic Mechanics,* Vol. I and II, Ukraine Academy of Sciences, Kiev, second edition 1995. http://www.santilli-foundation.org/docs/Santilli-300.pdf http://www.santilli-foundation.org/docs/Santilli-301.pdf.

[15] R.M.Santilli, Hadronic Journal. 1.,233, and Addendum, Hadronic J. 1. 1279 (1978).

[16] T. Vogiouklis, *Hyperstructures and their representations,* Hadronic Press, Palm Harbor, FL(1994).

[17] R.M.Santilli, Hadronic J., 574 (1978).

[18] R.M.Santilli, *"Non-local Integral isotopies of differential Calculus, Mechanics and Geometries" in isotopies of contemporary mathematical structures* P. Vetro Editor, Rendiconti Circolo Matematico plaermo, suppl. Vol.42, 7-82. (1996)http://www.santilli-foundation.org/docs/Santilli-37.pdf

[19] Blatt, J.M. and Weiskopf, V.F., *Theoretical Nuclear Physics,* Wiley, New York, 1963.

[20] Edger G. *Nuclear Forces,* MIT press, Cambridge, MA, 1968.

[21] *R.M.Santilli, Isotopic Lifting of the SU(2) Symmetr y with Applications to Nuclear Physics* , JINR rapid Comm. Vol. 6, 24-38 (1993), http://www.santilli-foundation.org/docs/Santilli-19.pdf

[22] R.M.Santilli, *Isorepresentation of the Lie-isotopic SU(2)Algebra with Applications to Nuclear Physics and*

Local Realism Acta Applicandae Mathematicae, Vol 50, 177(1998)
http://www.santilli-foundation.org/docs/Santilli-27.pdf

[23] R.M.Santilli, *Isotopic Generalization of Galilei's and Einstein's Relativities,* Vol. I, *Mathematical Foundations*, 1-st edition Hadronic Press, Palm Harbor, FL (1991): 2-nd edition Ukrain Academy of Sciences, Kiev.

[24] Kadeisvili, J.V., *Foundations of Lie-Santilli isotheory, Isotopies of Contemporary Mathematical Structures*, P.Vetro Editor, Rendiconti Circolo Matematico Palermo, Suppl. Vol. 42, 7-82 (1996).

[25] R.M.Santilli, *A quantitative isotopic representation of the dueteron magnetic moment*, Proceedings of International Symposium, Dubna Deuteron-93, Joint Institute of Nuclear Research, Dubna, Russia (1994)
http://www.santilli-foundation.org/docs/Santilli-134.pdf

[26] R.M.Santilli, *Nuclear Realization of hadronic mechanics and the exact representation of nuclear magnetic moments*, R.M.Santilli, Intern. J. Physics. Vol. 4, 1-70 (1998).

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ISODUAL MATHEMATICS FOR ANTIMATTER

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Abstract

Since the discovery of antimatter it has only been treated at the level of second quantization, where as, matter is treated at all levels of study, from Newtonian mechanics to quantum field theory. To resolve this scientific imbalance of 20th century, Santilli in 1993 [1],[2],[3] took up to study antimatter at all levels. In this paper we present the classical representation of antimatter at Newtonian level and emerging images at subsequent levels. The most appropriate theory of antimatter as proposed by Santilli [4] is based on a new map called isoduality which is applicable at the Newtonian level and all the subsequent levels of study of antimatter. Santilli also formulated the new anti-isomorphic isodual images of the Galilean, special and general relativities compatible with the experimental knowledge on electromagnetic interactions. Antigravity for antimatter [6] (and vice versa) is a natural consequence of this study and awaits validity due to lack of sufficient experimental evidence.

1 Introduction

Scientific studies have come a long way from Newtons equations, Galilios relativity and Einsteins special and general relativities. Existence of antimatter asteroids and cosmic rays in the universe has already been suggested by phenomena like; 1) Catastrophic explosion in Tunguska in Siberia in 1908 of the power of thousand Hiroshima bombs with devastating effect and total absence of any debris and crater. Surprisingly, entire Earths atmosphere was charged for some days so much so that people in Sydney could read news papers without any artificial light. Such a large excitation of the atmosphere can only be explained by annihilation of matter by antimatter. 2) NASA has recently reported explosions in our upper atmosphere which can be caused only by small antimatter asteroids annihilating the upper portion of our atmosphere while coming in contact with it and 3) Astronauts and cosmonauts have observed flashes of light in the upper atmosphere which can only be interpreted as being due to antimatter cosmic rays coming in contact with our atmosphere. In short, the evidences of existence of antimatter asteroids hitting our earth has become a major threat to humanity and hence warrants a serious study of antimatter in general and antimatter asteroids, comets and galaxies in particular. We know that matter is described at all levels of study from Newtonian mechanics to Quantum field theory but antimatter is solely treated at the level of second quantization; as antimatter particles with negative-energy do not behave in a physical way. Thus, Newton, Galileo and Einstein's theories were solely describing matter and not antimatter. A. Schuster in 1898 conjectured existence of antimatter. It was discovered by Dirac [5] in 1920, fourteen years following the formulation of general relativity. He even submitted hole theory for the study of antimatter at the level of second quantization. Today, the stand adopted in general is that; As Einsteins special and general relativity do not provide a proper description of antimatter, it does not exist in the universe in appreciable amount; the sole exception being that of a man made antiparticles created in the laboratory. The above scientific imbalance was for the first time identified by Italian- American scientist Ruggero Maria Santilli who decided to ascertain whether a far away star or galaxy is made up of matter or antimatter. Santilli soon discovered the entire body of mathematical, theoretical and experimental formulation [6, 1, 7, 8] applicable to his aim as his previous knowledge at the graduate studies was insufficient. Santilli first took up to formulate the mathematics needed for classical and operator representation either neutral or charged antimatter. Secondly, a reformulation of Newton's, Galileo's and Einstein's theories suitable for the study of neutral or charged antimatter at all possible levels and thirdly, the formulation of experiments to ascertain, in due time, whether far away stars or galaxies are made up of matter or antimatter. Antimatter asteroids must be treated as as they are isolated in space. Also, they are too large for the treatment via operator theories. Hence, scientific studies in the detection of antimatter requires physical theories for classical treatment of antimatter. Santillis mathematical and theoretical studies in antimatter are unique in a way being capable of classical representation of neutral antimatter. In his writings, Santilli has specifically mentioned that A protracted lack of solution of physical problems is generally due to the use of insufficient or inadequate mathematics [6]. Moreover, he says that There can not exist a really new physical theory without a new mathematics, and there can not exist a really new mathematics without new numbers. Santilli spent decades of exhaustive research in developing new numbers; subsequently new mathemetics sufficient to treat neutral or charged antimatter. Santilli introduced new numbers called isodual numbers [9] where the prefix iso was introduced in the Greek sense meaning preserving the conventional axioms used for matter. The term dual indicates the map from matter to antimatter. Santilli's entire theory for of antimatter is called Isodual due to predominant role and importance of Santillis isodual numbers. Subsequently, in 1993 [1] Santilli constructed the isodualities of Euclidean and Minkowskian spaces, evidently needed for possible physical applications. He then proceeded to construct the isodual image of Lies theory [2] needed for the construction of basic symmetries for antimatter viz. isodual images of the Galileo and Lorentz symmetries. Second landmark discovery was a new formulation of differential calculus which was crucial for the achievement of the first ever known formulation of Newtons equation for neutral or charged antiparticles. Readers can find the complete formulation of isodual mathematics in the monograph [11] of 1994 and with more updation in [8]. Thereafter, Santill initiated his physical studies in the paper [12] of 1993 written on his original aim of possible detection of antimatter stars and galaxies. Subsequently, Santilli wrote an important paper [16] on all important classical representation of neutral antimatter; the submission of experiment in paper [7] of 1994 to test the gravity of positrons and paper [14] of 1994 on the causal space-time machine i.e. the capability of moving as desired in space and time without violation of causality which is an invertible consequence of gravitational repulsion between matter and antimatter. A.P.Mills, an experimentalist established that Santillis gravity experiment [15] is resolutory because displacement due to gravity of very low energy positrons on a scintillator at the end of a flight in a super-cooled, supervacuum tube is visible to the naked eye. Paper [16] of 1997 included the first isodualities of Galileo and Einstein's relativities; another basic physical discovery has been discussed in paper [17] of 1997 via the prediction that light emitted by antimatter is repelled by the matter gravitational field. This prediction is an invertible consequence of the main feature of the studies reviewed here, namely, the classical conjugation from neutral matter to neutral antimatter that evidently also applied to light. The prediction can mainly be used to ascertain whether a faraway galaxy is made up of matter or antimatter. One of the best papers of Santilli [1] in 1998 achieves the first ever known representation of the gravitational field of antimatter which serves in and sets the foundation for the first known grand unification of electroweak and gravitational integration including antigravity, developed in details in [6] of 2001. The first quantitative study of thermodynamics of antimatter is available in the paper [18] of 1999 written by J. Dunning Davis. Also, the treatment of matter and antimatter under the general conditions of irreversibility over time over classical operator level has been discussed in [19] of 2006 by Santilli. In his paper [20] in 2011, Santilli finally acquires the position to address his main objective namely To identify experimentally the existence of stars and galaxies, and detections of asteroids. The conformation that Santillis experiment in [7] on gravity of positrons in horizontal flight on earth is resolutory came via paper [21] of 2011 by the experimentalist V. de. Haan by confirming Mills results [15]. In this direction, the paper by Santilli with the mathematicians B. Davaas and T. Vougiouklis is completely the most advanced paper because it establishes that the universe is multi-valued and not multi-dimensional, as matter and antimatter co-exist in physically distinct space-times implies multivaluedness. Subsequently, Santillis studies on antimatter at three successive levels of study, including ; 1) Single-valued reversible 2) Single-valued irreversible and 3) Multi-valued irreversible conditions, are provided in monograph [22] of 2011 by theoretical physicists I. Gandzha and J.V. Kadieisvilli. Reference [23] in 2013 by A.A.Bhalekar is an excellent account of the basic mathematics behind the above subject matter.

2 Santillis Isodual mathematics

Inapplicability of 20th century mathematics for consistent representation of antimatter led to decades of rigorous studies of suitable formulation for quantitative representation of matter-antimatter annihilation. As such, application of the same (existing) mathematics for matter and antimatter proved to be incompatible due to matter-antimatter annihilation. Santilli found that the matter-antimatter annihilation could only be represented by the use of mathematics that is anti-homomorphic to each at all their levels. In fact, the mathematics anti-homomorphic to 20th century mathematics did not exist in 1980s. Physical theories describing antimatter at all the levels asked for construction of entirely new mathematics that would allow classical treatment of neutral or charged antimatter. While at Department of mathematics Harvard University in 1980s, under DOE support Santilli constructed the required new mathematics for the exact representation of antimatter, today known as Santilli isodual mathematics [2, 16, 10](monograph [6] for comprehensive presentation). This new mathematics is anti-homomorphic to the conventional mathematics. We outline the main branches of Santillis isodual mathematics;

2.1 Isodual Map

Note that the *term isodual denotes a conjugation* characterized by the word dual under the preservation of the axioms of conventional mathematics denoted by the Greek prefix "iso".

2.2 Isodual numbers

Isodual numbers are characterized via new basic isodual unit 1^d defined as;

$$1^d = (-1)^\dagger = -1 \tag{1}$$

with resulting isodual real, complex and quaternion numbers;

$$n^d = n1^d = (-n)^\dagger \tag{2}$$

and isodual multiplication defined as;

$$n^{dd}m^d = -n^d m^d \tag{3}$$

with ensuing isodual operations of division power, square root e.t.c. under which 1^d is the basic unit of new theory. Also,

$$1^{dd}n^d = n^{dd}1^d = n^d \tag{4}$$

As numeric field does not necessarily require that the basic unit be positive, it can indeed be taken as negative, and all the operations can be reformulated accordingly. This fact is the ultimate basis of the new theory of antimatter and the resulting new era of cosmology.

Lemma 2.1 *All quantities which are positive-definite when referred to fields(such as mass, energy, angular momentum, density, temperature, time etc.) became negative-definite when referred to isodual fields.*

Positive-definite quantities referred to positive-definite units characterize matter, and negative-definite quantities referred to negative definite units, characterize antimatter.

These characterizations lead to subsequent levels of representation of matter and antimatter.

Definition 2.1 *A quantity is called isoselfdual when it is invariant under isoduality.*

2.3 Isodual functional Analysis:

Functional analysis at large was subjected to isoduality with consistent applications of isodual theories resulting in a simple, unique and significant isodual functional analysis by Kadeisvili [24].

Isodual functions are defined as;

$$f^d(r^d) = -f^\dagger(-r)^\dagger \tag{5}$$

is called the isodual image of the conventional function,

2.4 Isodual differential calculus

This is the isodual image of the conventional differential calculus and related isodual derivative. *Isodual differential* coincides with the conventional differential by Santilli conception,

$$d^d r^d \equiv dr \tag{6}$$

Actually, because of this the new isodual calculus was not discovered since Newton's time till 1996.

2.5 Isodual Lie theory

Let L be an *n-dimensional Lie algebra* with universal enveloping associative algebra $\xi(L), [\xi(L)]^{-} \approx L$ n-dimensional unit $I = diag(1, 1, ...1)$ for the regular representation, *ordered set of Hermitean generators* $X = X^{\dagger} = \{X_k\}$, conventional associative product $X_i \times X_j$, and familiar Lie's Theorems over a field $F(a, +, \times)$.

The *isodual universal associative algebra* $[\xi(L)]^d$ is characterized by the isodual unit I^d, isodual generators $X^d = -X$ and isodual associative product ;

$$X_i{}^d \times^d X_j{}^d = -X_i \times X_j \tag{7}$$

with corresponding infinite-dimensional basis(isodual version of conventional Poincare- Birkhoff-Witt theorem) characterizing the *isodual exponentiation of a generic quantity A*

$$e^{dA} = I^d + A^d/^d 1!^d + A^d \times^d A^d/^d 2!^d + \cdots = -e^{A^{\dagger}} \tag{8}$$

where e is the conventional exponentiation.

The attached *isodual Lie algebra* $L^d \approx (\xi^d)$ over the isodual field $F^d(a^d, +^d, \times^d)$ is characterized by the *isodual commutators*;

$$[X_i^d, X_j^d]^d = -[X_i, X_j] = C_{ij}^{kd} \times^d X_k^d \tag{9}$$

with a classical realization.

Let G be the conventional, connected, n-dimensional Lie transformation group on $S(x, g, F)$ admitting L as the Lie algebra in the neighbourhood of the identity, with generators X_k and parameters $\omega = \{\omega_k\}$. The *isodual Lie group* G^d admitting the isodual Lie algebra L^d in the neighborhood of the isodual identity I^d is the n-dimensional group with generators $X^d = \{-X_k\}$ and the parameters $\omega^d = \{\omega_k\}$ over the isodual field F^d with generic element

$$U^d(\omega^d) = e^{i^d \times^d \omega^d \times^d X^d} = -e^{i(-\omega)X} = -U(-\omega) \tag{10}$$

The *isodual symmetries* are then defined accordingly via the use of the isodual groups and they are ant-isomorphic to the corresponding conventional symmetries, as desired ref.[26] for additional details. *Conventional Lie symmetries are used for the characterization of matter where as Isodual Lie symmetries are used for the characterization of antimatter.*

2.6 Isodual Euclidean Geometry

Let $S = S(x, g, R)$ be a conventional N-dimensional *metric space* with local coordinates $x = \{x_k\}, k = 1, 2, ...N$, nowhere degenerate, sufficiently smooth, real valued and symmetric metric $g(x, ...)$,and *related invariant*

$$x^2 = x^i g_{ij} x^j \tag{11}$$

over the reals R. The *isodual spaces* are the spaces $S^d(x^d, g^d, R^d)$ of $S(x, g, R)$ with isodual coordinates $x^d = x \times I^d$, *isodual metric*

$$g^d(x^d, ...) = -g^\dagger(-x, ...) = -g(-x, ...) \tag{12}$$

and *isodual interval*

$$(x-y)^{d2d} = [(x-y)^{id} \times^d g^d_{ij} \times^d (x-y)^{jd}]^d = [(x-y)^i \times g^d_{ij} \times (x-y)^{jd}] \times I^d \tag{13}$$

defined over the isodual field $R^d = R^d(n^d, +^d, \times^d)$ with the same isodual isounit I^d. The *three dimensional isodual Euclidean space* is defined as;

$$E^d(r^d, \delta^d, R^d) : r^d = \{r^{kd}\} = \{-r^k\} = \{-x, -y, -z\} \tag{14}$$

with

$$\delta^d = -\delta = diag(-1, -1, -1), I^d = -I = diag(-1, -1, -1) \tag{15}$$

Thus, the *isodual Euclidean geometry* is the geometry of the isodual space E^d over R^d which is given by step-by-step isoduality of the various aspects of conventional geometry.

Lemma 2.2 *The isoeuclidean geometry on E^d over R^d is anti-isomorphic to the conventional geometry on E over R.*

Isodual sphere is the perfect sphere in E^d over R^d with negative radius;

$$R^{d2d} = [x^{d2d} + y^{d2d} + z^{d2d}] \times I^d \tag{16}$$

2.7 Isodual Minkowski space

This new space $M^d(x^d, \eta^d, I^d)$ is characterized by the isodual image of the conventional *Minkowski space for matter* $M(x; \eta, I)$ where x denotes space-time coordinates, $\eta = Diag(1, 1, 1, 1)$ denotes the Minkowski metric [25], and

$I = Diag(1,1,1,1)$ is the basic unit. Now the *isodual line element* is given by

$$x^{d2d} = (x^d \times^d \eta^d \times^d x^d)1^d \equiv x^2 \tag{17}$$

where the multiplication by 1^d is necessary for the isodual line element to have values in the isodual field. Note that the above isodual line element coincides with the conventional line element also by Santilli conception
It is important to note that Santilli's studies on antimatter requires a knowledge of the fact that **representation space of antimatter coexists with that of matter while being totally different from the latter**.

2.8 Isodual Riemannian geometry

Let $R(x, g, R)$ be a $3+1$ dimensional *Riemannian space* with basic unit $I = diag(1,1,1,1)$ and *related Riemannian geometry* in local formulation. Then the *isodual Riemannian spaces* are given by

$$R^d(x^d, g^d, R^d) : \begin{array}{rcl} x^d &=& \{-\widehat{x}^\mu\} \\ g^d &=& -g = \{x\}, g \in R(x, g, R) \\ I^d &=& diag(-1,-1,-1,-1) \end{array} \tag{18}$$

with interval $x^{2d} = [x^{dt} \times^d g^d(x^d) \times^d x^d] \times I^d = [x^t \times g^d(x^d) \times x] \times I^d$ on R^d, where t stands for transposed.
The *isodual Riemannian geometry is the geometry of spaces R^d over R^d, and is also obtained by taking step-by-step isodualities of the conventional geometry, including, most importantly, the isoduality of the differential and exterior calculus.*

2.9 Isodual Lie theory and symmetries

These are *characterized by Hermitean generators* $X^\dagger = X$ verifying *Lie-Santilli isodual product*

$$[X^d, Y^d]^d = Y^d \times^d X^d - X^d \times^d Y^d \equiv [X, Y] \tag{19}$$

and related Lie-Santilli isodual theory formulated on isodual spaces over an isodual numeric field and elaborated via the isodual functional analysis and isodual differential calculus.

It is important to note that the above isodual product coincides with the conventional Lie bracket also by Santillis conception; this identifies the deep meaning of the term isoduality. This new symmetry called as IsoSelfDuality (ISD) [3, 6, 1]is simply given by the invariance under isoduality. It can be verified that $P^d(3.1)$ does not verify ISD where as $P(3.1) \times P^d(3.1)$ does verify ISD as each symmetry is transformed in to other, resulting in a total invariance.

2.10 Isodual Lorentz-Poincare - Santilli symmetry

Santilli constructed *isoduality of the Lie theory.* He achieved this by way of the *isodual rotational symmetry* $SO^d(3)$, the isodual symmetry $SU^d(2)$, the *isodual Lorentz symmetry* $SO^d(3:1)$ and finally, the *isodual Lorentz-Poincare symmetry* $P^d(3:1)$ which is the fundamental symmetry of the new theory of antimatter. Here it is important to note that isodual mathematics is solely applicable to point-like abstraction of antimatter masses or particles.

Here it is important to note that *Isodual mathematics is solely applicable to point-like abstraction of antimatter masses or particles. Covering isodual isomathematics* is required for the representation of time reversal invariant systems of extended antimatter particles. Also, representation of their counterparts requires *isodual genomathematics*

The most general conceivable mathematics for antimatter is given by *Santilli isodual hypermathematics* which is particularly suited for multi-valued (rather than multi dimensional) formulations [27, 28].

2.11 Representation of antimatter at Newtonian level

As we know, *Newtonian treatment of antimatter consisting of N point-like particles is based on* a 7-dimensional representation space which is a Kronecker products of the Euclidean spaces of time t, coordinates r, and velocities v as;

$$S(t,r,v) = E(t,R_t) \times E(r,\delta,R_r)E(v,\delta,R_v) \tag{20}$$

where

$$r = (r_a^k) = (r_a 1, r_a^2, r_a^3) = (x_a, y_a, z_a) \tag{21}$$

$$v = (v_{ka}) = (v_{1a}, v_{2a}, v_{3a}) = (v_{xa}, v_{ya}, v_{za}) = \frac{dr}{dt} \tag{22}$$

$$\delta = Diag : (1,1,1), k = 1,2,3, a = 1,2,3, ...N \tag{23}$$

where the base fields R_t, R_r and R_v are trivially identical all having trivial units $+1$, resulting in the trivial total unit

$$I_{Tot} = I_t \times I_r \times I_v = 1 \times 1 \times 1 = 1 \tag{24}$$

Newtons celebrated equations of motion for point-like particles are;

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t,r,v), k = 1,2,3, a = 1,2,3, ...N \tag{25}$$

For the isodual treatment of antiparticles basic space is 7N-dimensional isodual space given by,

$$S^d(t^d, r^d, v^d) = E^d(t^d, R^d_{t,}) \times E^d(r^d; \delta^d, R^d) \times E^d(v^d, \delta^d, R^d) \tag{26}$$

with isodual unit and isodual metric

$$I^d_{Tot} = I^d_t \times I^d_r \times I^d_v \tag{27}$$

$$I^d_t = -1, I^d_r = I^d_v = Diag : (-1,-1,-1) \tag{28}$$

$$\delta^d = Diag(1^d, 1^d, 1^d) = Diag(-1,-1,-1) \tag{29}$$

This transformation results into celebrated Newton-Santilli isodual equations for point-like antiparticles first introduced by Santilli [6] as,

$$m^d_a \times^d dv_k a^d /^d d^d t^d = F^d_{ka}(t^d r^d v^d), k = x, y, z, a = 1,2,3, ...n \tag{30}$$

which has been experimentally verified. It is *important to note that the above isodual equations are anti-isomorphic to the conventional forms.*

2.12 Implications of Newton-Santilli isodual equations

Antimatter exists in a spacetime, co-existing, yet different than our own. As such, isodual Euclidean space $E^d(r^d, \delta^d, R^d)$ co-exist within, but is physically distinct from Euclidean space $E(r, \delta, R)$ and same occurs for full representation spaces $S^d(t^d, r^d, v^d)$ and $S(t, r, v)$.
Antimatter moves backward in time in a way as causal as the motion of matter forward. In fact, Newton-Santillis isodual equations provide the only known causal description of particles moving backward in time.
Antimatter is characterized by negative mass, negative energy and negative magnitudes of other physical quantities.

3 Isodual Relativities

3.1 Isodual Galilean Relativity

First we introduce *isodual Galilean symmetry* $G^d(3.1)$ as *isodual image of the conventional symmetry* $G(3.1)$. For the Galilean symmetry of a system of N particles with non-null masses

$m_a, a = 1, 2, ...N, G^d(3.1)$ has i*sodual parameters and generators defined as;*

$$w^d = (\theta_k^d, r_0^{kd}, v_0^{kd}, t_0^{kd}) = -w\, j_k^d = \sum_{aijk} r_{ja}^d \times^d p_{ja}^k = -J_k, P_k^d = -P_k \quad (31)$$

and

$$G_k^d = \sum_a (m_a^d \times^d r_{ak}^d - t^d \times p_{ak}^d), H^d = \frac{1}{2}^d \times^d \sum_a p_{ak}^d \times^d p_a^{kd} + V^d(r^d) = -H \quad (32)$$

with *isodual commutator defined as*

$$[A^d, B^d]^d = \sum_{a,k} [(\partial^d A^d /^d \partial^d r_a^{kd}) \times^d (\partial^d B^d /^d \partial^d p_{ak}^d) - (\partial^d B^d /^d \partial^d r_a^{kd}) \times^d (\partial^d A^d /^d \partial^d p_{ak}^d)$$

$$= -[A, B] \quad (33)$$

The structure constants and *Casimir invariants of the isodual Lie algebra* $G^d(3.1)$ *are negative-definite.* If $g(w)$ **is an element of the connected component** of the Galilei group $G(3.1)$ then *its isodual* is defined as

$$g^d(w^d) = e^{-i^d \times^d w^d \times^d X^d} = e^{i \times (-w) X} = -g(-w) \in G^d(3,1) \quad (34)$$

The *isodual Galilean transformations* are then given by the following;

$$t^d \to t'^d = t^d + t_0^d = -t, r^d \to r'^d = r^d + r_0^d = -r' \quad (35)$$

$$r^d \to r'^d = r^d + v_0^d \times^d t_0^d = -r', r^d \to r'^d = R^d(\theta^d) \times^d r^d = -R(-\theta) \quad (36)$$

where $R^d(\theta^d)$ is an element of the *isodual rotational symmetry.* The above isodual representation of antimatter is truly consistent with available classical experimental knowledge for matter [6]. *The situation in isodual space is described by the following Lemma.*

Lemma 3.1 *The trajectories under the same magnetic field of a charged particle in Euclidean space and the corresponding antiparticle in the isodual Euclidean space coincide.*

Proof 3.1 *Consider a particle with charge $-e$ in the Euclidean space $E(r, \delta, R)$ i.e. the value $-e$ with respect to positive unit $+1$ of the underlying field of real numbers $R(n, +, \times)$. Suppose the particle is under the influence of the magnetic field B.*
The corresponding antiparticle via isoduality changes the sign (reversal of sign) of all physical quantities resulting in the charge $(-e)^d = +e$ in the corresponding isodual Euclidean space$E^d(r^d, \delta^d, R^d)$ simultaneously reversing the magnetic field $B^d = -B$ defined with respect to the negative unit $(+1)^d = -1$. This establishes the fact that the trajectory of the particle with charge $-e$ in the field B defined with respect to the unit $+1$ in the Euclidean space and that for the antiparticle in the field $-B$ defined with respect to the unit -1in the isodual Euclidean space coincide.

Corollary 3.1 *Antiparticles reverse their trajectories when projected from their own isodual space into our own space.*

3.2 Isodual Special Relativity

Classical relativistic treatment of point-like antiparticles can best be done via isodual special relativity . *Conventional special relativity [29] is constructed with the 4-dimensional unit of the Minkowski space, $I = Diag.(1,1,1,1)$* which represents dimensionless units of space $\{+1, +1, +1\}$ and the dimensionless unit of time $+1$, and is the unit of Poincare symmetry $P(3.1)$. *The isodual special relativity is defined by the map*

$$I = Diag(\{1,1,1\},1) > 0 \to I^d = -Diag(\{1,1,1\},1) < 0 \qquad (37)$$

It is based on negative units of space and time.

The isodual special relativity is expressed by the isodual image of all mathematical and physical aspects of conventional relativity in such a way as to admit the negative definite unit I^d as the correct left and right unit,including: *the isodual Minkowski spacetime* $M^d(x^d, \eta^d, R^d)$ with isodual coordinates $x^d = x \times I^d$, isodual metric $\eta^d = -\eta$ and basic invariant over R^d

$$(x-y)^{d2d} = [(x^\mu - y^\mu) \times \eta^d_{\mu\nu} \times (x^\nu - y^\nu)] \times I^d \in R^d \qquad (38)$$

and *fundamental isodual Poincare symmetry* [12]

$$P^d(3.1) = L^d(3.1) \times^d T^d(3.1) \qquad (39)$$

where $L^d(3.1)$ is the Lorentz-Santilli symmetry, $\times^d$ is the isodual direct product and $T^d(3,1)$ represents isodual translations. *The algebra of the connected component* $P_+^{\uparrow d}$ *of* $P^d(3,1)$ *can be constructed in terms of the isodual parameters* $w^d = \{-w_k\} = \{-\theta, -v, -a\}$ and isodual generators $X^d = -X = \{-M_{\mu\nu}, -P_\mu\}$, where the factorization by the 4-dimensional unit I is understood.

Also, the *isodual commutator rules* are given by;

$$[M^d_{\mu\nu}, M^d_{\alpha\beta}]^d = i^d \times^d (\eta^d_{\nu\alpha} \times^d M^d_{\mu\beta} - \eta^d_{\mu\alpha} \times^d M^d_{\nu\beta} - \eta^d_{\nu\beta} \times^d M^d_{\mu\alpha} + \eta^d_{\mu\beta}) \times^d \hat{M}^d_{\alpha\nu} \quad (40)$$

$$[M^d_{\mu\nu}, p^d_\alpha]^d = i^d \times^d (\eta^d_{\mu\alpha} \times^d p^d_\nu - \eta^d_{\nu\alpha} \times^d p^d_\mu)[p^d_\alpha, p^d_\beta] = 0 \quad (41)$$

The basic postulates of isodual special relativity are simple isodual image of the conventional postulates.
Isodual inversions and spacetime inversions are equivalent.

3.3 Isodual General Relativity

The most effective gravitational characterization of antimatter is isodual general relativity obtained by isodual map of all the aspects of conventional relativity. This is defined on the isodual Riemannian spaces $R^d(x^d, g^d, R^d)$. Isodual Riemannian geometry is defined on the isodual field of real numbers $R^d(n^d, +^d, \times^d)$ for which the norm is negative-definite. As a result, all quantities which are positive in Riemannian geometry become negative under isoduality, including the energy-momentum tensor. *Explicitly, the electromagnetic field follows the simple rule under isoduality*

$$F^d_{\mu\nu} = \partial^d A^d_\mu /^d \partial^d x^{\nu d} - \partial^d A^d_\nu /^d \partial^d x^{\mu d} = -F\mu\nu \quad (42)$$

and for the *energy-momentum tensor* we have

$$T^d_{\mu\nu} = (4m)^{-1d} \times^d (F^d_{\mu\alpha} \times^d F^d_{\alpha\nu} + (1/4)^{-1d} \times^d g^d_{\mu\nu} \times^d F^d_{\alpha\beta} \times^d F^{d\alpha\beta}) = -T_{\mu\nu} \quad (43)$$

In fact, isodual Riemannian geometry has negative-definite energy-momentum tensor and other physical quantities which open up new possibilities for attempting a grand unified theory.
Reader should note that the universal symmetry of the isodual general relativity, the isodual isoPoincare symmetry $\hat{P}^d(3.1)P^d(3.1)$ has been introduced at the operator level in [10].

4 Antigravity

In the words of Santilli "*Isodual theory of antimatter predicts the existence of antigravity (here defined as the reversal of the sign of the curvature tensor in our space-time) for antimatter in the field of matter or vice-versa*" As such, the isodual theory of antimatter predicts in a consistent and systematic way at all levels of study, from Newtonian mechanics to Riemannian geometry, that matter and antimatter must experience gravitational repulsions ref [7, 30] and monograph [6]

We may summarize above results as; ***classical representation of antiparticles via isoduality renders gravitational interactions equivalent to the electromagnetic ones, in the sense that the Newtonian gravitational law becomes equivalent to the Coulombs law.***

These results could not have been achieved without isoduality.

References

[1] R. M. Santilli, *Isominkowskian Geometry for the Gravitational Treatment of Matter and its Isodual for Antimatter*, Intern. J. Modern Phys. D, Vol. 7, pp 351-407, (1998), http://www.santillifoundation.org/docs/Santilli-35.pdf.

[2] R. M. Santilli, *Isotopies, genotopies and isodualities of Lies Theory*, Talk delivered at the International Congress of Mathematicians, Zurich, August 3-11, (1994).

[3] R. M. Santilli, *Elements of Hadronic Mechanics*, Volumes I and II Ukraine Academy of Sciences, Kiev, second edition 1995, http://www.santilli-foundation.org/docs/Santilli-300.pdf http://www.santilli-foundation.org/docs/Santilli-301.pdf.

[4] R.M.Santilli, *Rudiments of Isogravitation for Matter and its Isodual for Antimatter*, American Journal of Modern Physics, 4(5), 59-75, (2015).

[5] P. A. M. Dirac, *The Principles of Quantum mechanics*, Clarendon Press, Oxford, fourth edition (1958).

[6] R. M. Santilli, *Isodual Theory of Antimatter with Application to Antigravity, Grand Unification and the Spacetime Machine*, Springer, New York, (2001).

[7] R. M. Santilli, *Antigravity*, Hadronic J., Vol. 17, pp 257-284, (1994).

[8] R. M. Santilli, *Classical isodual theory of antimatter and its prediction of antigravity*, Intern. J. Modern Phys. A, Vol. 14, pp 2205-2238, (1999). http://www.santilli-foundation.org/docs/Santilli-09.pdf

[9] R. M. Santilli, *Isonumber and genonumbers of dimension 1, 2, 4, 8, their isoduals and pseudoduals, and hidden numbers of dimension 3, 5, 6, 7*, Algebras, Groups and Geometries, Vol. 10, pp 273-321, (1993) http://www.santilli-foundation.org/docs/Santilli-34.pdf.

[10] R. M. Santilli, *Isodual Theory of Antimatter with Applictions to Antigravity and Cosmology.* Springer, (2006).

[11] R. M. Santilli, *Elements of Hadronic Mechanics*, Volumes I and II, Ukraine Academy of Sciences, Naukoa Dumka Publishers, Kiev, 1st edition (1994), 2nd edition (1995). http://www.santillifoundation.org/docs/Santilli-300.pdf http://www.santillifoundation.org/docs/Santilli-301.pdf.

[12] R. M. Santilli, *A new cosmological conception of the universe based on the isominkowskian geometry and its isodual*, Part I and Part II, Contributed paper in Analysis, Geometry and Groups, A Riemann Legacy Volume II, H.M. Srivastava, Editor.

[13] R. M. Santilli, *Representation of antiparticles via isodual numbers, spaces and geometries*, Comm. Theor. Phys., Vol. 3, pp 153-181, (1994). http://www.santilli-foundation.org/docs/Santilli-112.pdf

[14] R. M. Santilli, *Spacetime machine*, Hadronic J., Vol. 17, pp 285-310,(1994). http://www.santilli-foundation.org/docs/Santilli-10.pdf An Introduction to Santillis Isodual Theory of Antimatter and the Open Problem of Detecting Antimatter Asteroids.

[15] A. P. Mills, *Possibilities of measuring the gravitational mass of electrons and positrons in free horizontal ight*, a contributed paper for the proceedings of the International Conference on Antimatter, held in Sepino,

Italy, May 1996, published in the Hadronic J., vol. 19, pp 77-96, (1996). http://www.santilli-foundation.org/docs/Santilli-11.pdf

[16] R.M. Santilli, *Isotopic relativity for matter and its isodual for antimatter, Gravitation*, Vol. 3, no. 2, p 212, (1997).

[17] R. M.Santilli, *Does antimatter emit a new light?*, Invited paper for the proceedings of the Internationalk Conference on Antimatter, held in Sepino, Italy, on May 1996, published in Hyperfine Interactions, Vol. 109, pp 63-81, (1997). http://www.santilli-foundation.org/docs/Santilli-28.pdf

[18] J. Dunning-Davies, *Thermodynamics of antimatter via Santillis isodualities.* Found. Phys., Vol. 12, pp 593-599, (1999) http://www.santillifoundation. org/docs/Isodual-therm.pdf

[19] R. M. Santilli, *Lie-admissible invariant representation of irreversibility for matter and antimatter at the classical and operator levels*, Nuovo Cimento B, Vol. 121, pp 443-486, (2006) http://www.santillifoundation. org/docs/Lie-admiss-NCB-I.pd.

[20] R. M. Santilli, *Can antimatter asteroids, stars and galaxies be detected with current means?* Proceedings of the Third International Conference on the Lie-Admissible Treatment of Irreversible Processes, C. Corda, Editor, Kathmandu University, Nepal, pp 25-36, (2011).

[21] V. de Haan, *Proposal for the realization of Santilli comparative test on the gravity of electrons and positrons via a horizontal supercooled vacuum tube,* Proceedings of the Third International Conference on the Lie-Admissible Treatment of Irreversible Processes, C. Corda, Editor, Kathmandu University, Nepal, pp 57-67 (2011). http://www.santillifoundation. org/docs/deHaan-Arxiv.pdf

[22] I. Gandzha and J. Kadeisvili, *New Sciences for a New Era: Mathematical, Physical and Chemical Discoveries of Ruggero Maria Santilli*, Sankata Printing Press, Nepal (2011) http://www.santillifoundation. org/docs/RMS.pdf.

[23] A. A. Bhalekar, *Santillis New Mathematics for Chemists and Biologists.* An Introductory Account, Hadronic J., (2013) (In press) http://www.santilli-foundation.org/docs/Bhalekar-Math-2013.pdf

[24] J. V. Kadeisvili, Santillis Isotopies of Contemporary Algebras, Geometries and Relativities, Ukraine Academy of Sciences, Second edition (1997), http://www.santilli-foundation.org/docs/Santilli-60.pdf.

[25] R. M.Santilli, *Nonlocal Integral axiom-preserving isotopies and isodualities of the Minkowskian geometry,* in The Mathematical Legacy of Hanno Rund, J.V. Kadeisvili, Editor, Hadronic Press, Palm Harbor, pp. 383-430, (1993).

[26] J. V. Kadeisvili, *Foundations of the Lie-Santilli isotheory and its isodual,* Rendiconti Circolo Matematico Palermo, Suppl., Vol. 42, pp 83-185, (1996) http://www.santilli-foundation.org/docs/Santilli-37.pdf.

[27] B. Davvaz, R. M. Santilli, and T. Vougiouklis , *Studies of Multi- Valued Hyperstructures for the Characterization of Matter-Antimatter Systems and their Extension,* in Proceedings of the 2011 International Conference on Lie-admissible Formulations for Irreversible Processes, C. Corda, editor, Kathmandu University, Nepal, pp 45-57, (2011) http://www.santilli-foundation.org/Hyperstructures.pdf.

[28] R. M.Santilli, I*sotopic Genotopic and Hyperstructural Methods in Theoretcal Biology,* Ukranian Academy of Sciences, Kiev (1996).

[29] W. Pauli, *Theory of Reelativity,*PergamonPress, London(1958).

[30] Richard Anderson, Anil A. Bhalekar, Bijan Davvaz, Pradeep S. Muktibodh, Vijay M. Tangde, Arun S. Muktibodh, and Thomas Vougiouklis *An Introduction to Santillis Isodual Theory of Antimatter and the Open problem of Detecting Antimatter Asteroids* NUMTA BULLETIN, 6(2012-13), 1-33. 22