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NONUNITARY LIE-ISOTOPIC AND LIE-ADMISSIBLE SCATTERING THEORIES OF HADRONIC MECHANICS:

IRREVERSIBLE DEEP-INELASTIC ELECTRON-POSITRON AND ELECTRON-PROTON SCATTERING

A.O.E. Animalu
Department of Physics & Astronomy
University of Nigeria, Nsukka, Enugu State, Nigeria
nascience@aol.com
R.M. Santilli
The Institute for Basic Research
Palm Harbor, FL, U.S.A.
research@i-b-r.org

Abstract

In this teleconference, we have debated whether high energy scattering regions should be re-inspected under the historical 1935 argument by A. Einstein, B. Podolsky and N. Rosen [10] that "quantum mechanics is not a complete theory" in view of the apparent proofs by R. M. Santilli, especifically, for high density regions suggesting the "completion" the quantum mechanical scattering theory into the iso- and geno-scattering theories of hadronic mechanics. The non-unitary Lie isotopic and Lie-admissible scattering theories of hadronic mechanics, also known as iso- and geno-scattering theories, respectively, were first systematically studied in R. M.Santilli's volumes [11]-[14], particularly in Chapter 12 of Vol. II. In Ref. [27], A. O. E. Animalu and R.M. Santilli continued these studies by developing a generalization of the Feynman graph method for the computation of the S matrix for high density scattering regions that cannot be consistently decomposed into a finite number of isolated point-particles according to various "no-decomposition theorems" [22]. More recently, by using Myung's nonlinear-Riccati differential realization of Santilli's Lie-admissible equation of motion to characterize the generalized structure functions, A.O. E. Animalu [32] has extended the results of Paper [27] to the geno-scattering theory of deep inelastic, thus irreversible electron-positron and electron-proton scattering processes, and obtained a good agreement between the geno-theory and the experimental data, which are presented and discussed in relation to the EPR argument..

Key words: Nonunitary theories, Irreversible deep-inelastic scattering theories,

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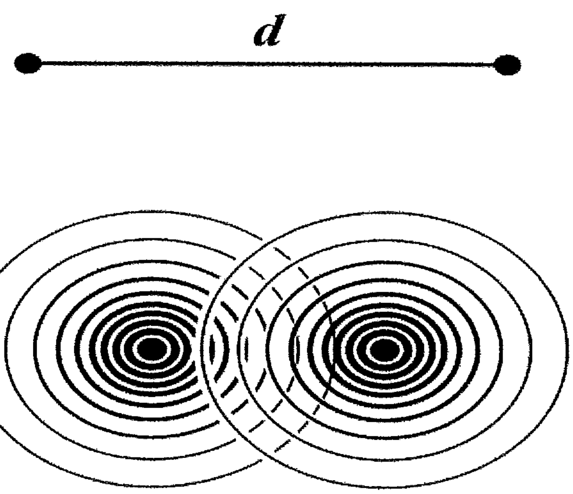
1. IMPLICATIONS OF THE EPR ARGUMENT ON HIGH ENERGY SCATTERING EXPERIMENTS

1.1 The EPR argument.

A most mysterious experimental evidence in nature is the capability of particles to influence each other instantly at a distance. The scientific community of the early 1900 assumed that such an effect is predicted by quantum mechanics, for which reason the effect was called and continue to be called to this day *quantum entanglement as charaterized* in Fig.1 below. By contrast, Albert Einstein noted that quantum mechanics can only represent point-like particles isolated in vacuum, thus being unable to predict their entanglement, in which case the sole possible representation of the entanglement is that via superluminal communications that would violate special relativity. To avoid such a violation, Einstein published in 1935 a historical paper jointly with his graduate students, Boris Podolsky and Nathan Rosen arguing that "quantum mechanics is not a complete theory" (EPR argument) [1], in the sense that quantum mechanics is valid for the atomic structure and other systems, but there may exist more complex systems in nature requiring a suitable "completion" for their consistent treatment, as it is the case for particle entanglement, thus including the possible recovering of classical determinism at least under limit conditions.

Immediately following the appearance of paper [1], Niels Bohr [2] voiced strong opposition to the EPR Argument essentially on grounds that complex systems can be reduced to their elementary constituents, thus being fully treatable with quantum mechanics. Subsequently, J. S. Bell [3] proved a theorem essentially stating that a quantum mechanical system of particles with spin$\frac{1}{2}$ cannot admit a classical counterpart, and therefore prevents the achievement of Einstein's determinism, and *de facto* confirms the view that quantum mechanics is valid for all possible conditions existing in the universe (see, Ref.[4] for a vast bibliography).

Fig.1; Schematic elaboration of quantum entanglement



Since coordinates, potentials, differential calculus and wavefunctions can only be defined at isolated points, quantum mechanics can only represent particles as being isolated in vacuum (top view), thus requiring superluminal speeds for their instantaneous entanglement at a distance, as pointed out by A. Einstein, B. Podolsky and N. Rosen (EPR argument) [1]. Beginning with his Ph. D. studies in the 1960s, R. M. Santilli and a number of scholars [8-31] have: pointed out that the wavepacket of particles fills up the entire universe; the interactions caused by their overlapping/entanglement (bottom view) is not representable via a Hamiltonian; constructed an axiom-preserving completion of quantum mechanics into hadronic mechanics which represents particles as extended in permanent and continuous overlapping by therefore avoiding any need for superluminal speeds; provided a number of proofs, applications and experimental verifications of the EPR argument

1.2 Earlier EPR verifications.

Despite authoritative opposing views, the search for a completion of quantum mechanics was continued by several authors, among whom we recall:

1) A completion of quantum mechanics by W. Heisenberg [5], hereon referred to as *Heisenberg's non-linear theory*, includes interactions non-linear in the wave functions as expected, for instance, in nuclear structures. Heisenberg's non-linear theory is based on the only possible quantum mechanical representation, in the Hamiltonian form $H(r,p)\psi(r)=E\psi(r)$ that, unfortunately, violates the superposition principle $\psi(r)=\sum_{k=1,2,\dots,N}\psi_k(r)$, and, therefore, prevents a quantitative representation of the individual constituents $\psi_k(r)$.

2) A non-local completion of quantum mechanics by D. Bohm and L. de Broglie [6], hereon referred to as the *Bohm-de Broglie non-local theory*, includes non-local interactions that are also expected in the nuclear and other structures, which completion appears to be the first attempt of achieving Einstein's determinism in scientific records. Unfortunately, the Bohm-de Broglie non-local theory is semi-classical and, as such, cannot be considered a completion of quantum mechanics according to the ERP argument [1]. Additionally, interactions occurring over a volume cannot be reduced to a finite number of isolated points and, therefore, they cannot be represented with a Hamiltonian.

3) D. Bohm hypothesis of *hidden variables* [6] has realization expected to void Bell's theorem[3], with ensuing genuine broadening of quantum mechanics. Unfortunately, under the use of 20th century applied mathematics, Bell's inequality [3] prevents hidden variables to have concrete and explicit realizations because the said inequality applies for the infinite family of unitary equivalence of quantum mechanics.

1.3. EPR verification with energy-releasing processes.

An international teleconference was held on September 1 to 5, 2020 [7], to discuss available studies on the completion of quantum mechanics according to the EPR argument, including the studies of the preceding section and the lifetime research by R. M. Santilli on the proof of the EPR argument [8-27], and others.

In his Ph. D. thesis in the mid-1960s [8,9] at the University of Torino, Italy, , R. M. Santilli noted that Heisenberg's equations for the time evolution of an observable A in the infinitesimal and finite forms,

$$i\frac{dA}{dt}=[A,H]=AH-HA \qquad (1)$$

$$A(t)=e^{Hti}A(0)e^{-itH} \qquad (2)$$

are invariant under anti-Hermiticity, thus are unable to represent time-irreversible processes such as combustion or nuclear fusions. Additionally, Santilli proved a theorem establishing that a macroscopic time-irreversible system cannot be consistently decomposed into a finite number of quantum mechanical particles, thus establishing that macroscopic irreversibility originates at the level of elementary particles, as confirmed experimentally by the clear irreversibility of high energy scattering experiments at CERN,. FER- MILAB and other particle physics laboratories. Hence, Santilli concluded that *the inability to provide a consistent representation of irreversible processes is a first clear evidence of its lack of completion of quantum mechanics according to the EPR argument [1].*

To initiate the identification of an appropriate completion for irreversible processes, Santilli proposed in 1967 a completion of quantum mechanics for time-irreversible processes characterized by brackets (A,H) that are no longer invariant under anti-Hermiticity. Following an extensive search in mathematical libraries, Santilli selected brackets (A,H) that are Lie-admissible according to A. A. Albert [10] in the sense that the attached anti-symmetric brackets $[A,B]^* \equiv (A,B)-(B,A)$ are Lie. Therefore, Santilli proposed in 1967 the completion of Heisenberg's time evolution into a Lie-admissible form for the consistent representation of irreversible processes [8,9], which he subsequently finalized in the infinitesimal and finite form [11,12]

$$i\frac{dA}{dt} = (A,H) \equiv ARH - HSA = A < H - H > A \tag{3}$$

$$A(t) = e^{HSti} A(0) e^{-itRH} \tag{4}$$

with corresponding completion of Schrödinger equation [13-15]

$$\langle\hat{\psi}| < H \equiv \langle\hat{\psi}|RH = \langle\hat{\psi}|ER \tag{5}$$

$$H > |\hat{\psi}\rangle \equiv HS|\hat{\psi}\rangle = ES|\hat{\psi}\rangle \tag{6}$$

where: R is generally assumed to have unit value, $R=1; S=1-F/H$ with ensuing Eq. (3) $idA/dt = AH - HA + AF$, is an operator representing the external terms F in Lagrange's and Hamilton's analytic equations; and $\hat{\psi}$ is the completed wavefunction under irreversibility. Note that irreversibility is assured because $R \neq S$.

The mathematics underlying the Lie-admissible dynamical equations, known as *Lie-admissible mathematics* or *genomathematics* for short, was subsequently developed in collaboration with a number of mathematicians (see the general review [13-15]) and Tutoring Lecture IV of Ref. [7]). Applications of the Lie-admissible treatment of irreversible processes, including nuclear fusions, are reviewed in Ref. [15].

1.4. EPR verification with the neutron synthesis.

late 1977, when at Harvard University under DOE support, Santilli [16] was requested to study the synthesis of the neutron from the hydrogen atom in the core of stars. He discovered in this way that quantum mechanics is completely inapplicable to the moist fundamental synthesis in nature because of various technical reasons, including the fact that the mass of the neutron is 0.782 MeV *bigger* than the sum of the masses of the proton and of the electron. Under these conditions the Schrödinger equation of quantum mechanics would require a *positive potential energy* resulting in a *mass excess* that are outside scientific boundaries. Santilli noted that, despite the clear evidence of the synthesis of the neutron in the core of stars, quantum mechanics admits no bound state at short distance between a pro- ton and an electron despite the fact that, at 1 fm mutual distance, they experience the *Coulomb attraction*

$$F = k\frac{e^2}{r^2} = (8.99\times10^9)\frac{(1.60\times10^{-19})^2}{(10^{-15})^2} = 230N \tag{7}$$

which is astronomically big for particle standards. Consequently, *the synthesis of the* *neutron in the core of stars is a second clear evidence of the lack of completion of quantum mechanics according to the EPR argument [1].* In Table V of Ref. [16] of 1978, Santilli noted that, when mesons were assumed to be bound states of elementary particles produced free in their spontaneous decays, there was a systematic appearance of a 'mass excess' similar to that of the neutron.

To initiate these studies, Santilli suggested the completion of quantum mechanics into the covering *hadronic mechanics* [11-12] with general Lie-admissible dynamical equations (3)-(5) and their particularization for $R = S = \hat{T}$ in the infinitesimal form and finite form

$$i\frac{dA}{dt} = [A,H]^* \equiv ATH - HTA = A*H - H*A \qquad (8)$$

and finite form $$A(t) = e^{H\hat{T}ti}A(0)e^{-it\hat{T}H} \qquad (9)$$

with completion of the Schrödinger equation

$$H*|\hat{\psi}\rangle \equiv HT|\hat{\psi}\rangle = ET|\hat{\psi}\rangle \qquad (10)$$

The mathematics and mechanics underlying the above equations are known as *Lie-isotopic mathematics and mechanics,* and characterize branches of hadronic mechanics also known as *isomathematics and isomechanics,* (see Refs. [13-15] and Tutoring Lecture I of Refs. [7]).

It should be noted that isoeigenvalue equation (10) verifies the superposition principle, by therefore admits the decomposition of the isowavefunction into those of the constituents of the hadronic bound state $\hat{\psi} = \sum_{k=1,2,...,N}\hat{\psi}_k$, Hence, isomathematics resolves the limitation of Heisenberg's non-linear theory [5] indicated in Section 1.2 via the embedding of all non-linear terms in the isotopic element $\hat{T} = \hat{T}(\hat{\psi}_1, \hat{\psi}_2, ..., \hat{\psi}_N)$ and reconstructing linearity on isospaces.

Note that Eqs. (8)-(10) are also invariant under anti-Hermiticity as it is the case for Eqs.(1), (2), thus solely able to represent time reversal invariant systems. However, the representation of systems requires the knowledge of *two* operators, namely the conventional Hamiltonian H and the isotopic element $\hat{T}$ representing extended, thus deformable and hyperdense constituents in conditions of mutual penetration/entanglement with ensuing non-linear, non-local and non-potential interactions. Hence, Eqs.(8)-(10)) are particularly suitable for the representation of the entanglement of particles (see Figure 1 for details) and its application to stable nuclei as established by experimental evidence, namely, composed of extended nucleons in partial mutual penetration.

Under the approximation that the considered two-body hadronic bound states are stable, the neutron synthesis was studied with realizations of the isotopic operator $\hat{T}$ of the type [15]

$$\hat{T} = \prod_{k=1,2} Diag.(1/n_{1k}^2, 1/n_{2k}^2, 1/n_{3k}^2, 1/n_{4k}^2)\exp[-\Gamma(\psi, \hat{\psi}, ...)] > 0, \qquad (11)$$

normalized to the values $n_{k\alpha}^2 = 1$, for the sphere; $n_{k\alpha}^2$ represents the density of the exponent is a positive-normalized to the value $n_{4\alpha}^2 = 1$ for the vacuum; and the exponent is a positive definite function representing non-linear, non-local and non-potential interactions caused by mutual overlapping/entanglement of the particles considered with realizations of the type

$$\exp[-\Gamma(\psi,\hat{\psi},...)] = \exp\left[-|\psi/\hat{\psi}|\int\hat{\psi}_1\hat{\psi}_2^+ d^3r\right] << 1 \quad 2 \tag{12}$$

where ψ is a quantum mechanical wave function, and $\hat{\psi}$ is the completed wave function under isotopy. Note the recovering of quantum mechanics identically and uniquely whenever the overlapping of the wavepackets is ignorable and the integral of realization (12) can be assumed to be null.

It should be note that the representation (11), (12) of non-local interactions characterizes a full operator theory, by therefore resolving the semi-classical limitation of the Bohm-de Broglie non-local theory [6] indicated in Section 1.2.

In order to achieve a consistent representation of the neutron synthesis, Santilli assumed that the integral in Eq. (12) is a constant and introduced the following simple realization of the isotopic element [16]

$$\hat{T} = e^{-\frac{|V_H|}{V_Q}} \approx 1 - \frac{|V_H|}{V_Q} \tag{13}$$

where V_Q is the quantum mechanical Coulomb potential $V_Q = +e^2/r$ for the proton-electron system at short distances, and V_H is the strongly attractive hadronic potential caused by the mutual penetration of the wavepackets which can be represented by the Hulten potential,

$$V_H = W\frac{e^{-kr}}{1-e^{-kr}} \tag{14}$$

But the Hulten potential behaves like the Coulomb potential at short distances. Therefore, Santilli absorbed the latter in the former (except for a renormalization of W which is here ignored), and reached the following eigenvalue equation for the two-body hadronic bound state of a proton and an electron at 1 fm mutual distances (Table V of Ref. [16])

$$\left[-\frac{1}{\overline{m}}\Delta - W\frac{e^{-kr}}{1-e^{-kr}}\right]\hat{\psi}(r) = \mathrm{E}\hat{\psi}(r) \tag{15)i.$$

where $\overline{m}$ is renormalization of the reuced mass caused by wave overlapping identified more details in Section 1.9

Under the above formalism, the Lie-isotopic branch of hadronic mechanics was able to achieve an *exact* representation of all characteristics of the neutron in its synthesis from the hydrogen at the non-relativistic and relativistic levels (see the review in Ref. [15] and Ref. [25]).

In paper [17] of 1995, A. O. E. Animalu and R. M. Santilli noted that hadronic bound state (15) holds with a strongly attractive Hulten potential irrespective of whether the Coulomb potential is attractive or repulsive, thereby reaching in this way the first known *attractive force between the identical electrons of the Cooper pair in superconductivity* and, therefore, confirming *Animalu's isosuperconductivity* [18].

1.5 EPR verification with classical images.

In Ref. [19] of 1998, Santilli confirmed the validity of Bell's inequality for point-like particles with spin 1/2 under potential interactions, but indicated its inapplicability (rather than violation) un- der extended, therefore deformable particles in conditions of mutual entanglement with ensuing non-linear, non-local and non-potential interactions.

By using the Lie-isotopic $SU(2)$-spin symmetry (see the review in Ref. [24]) with the explicit and concrete realization of Bohm's hidden variables

$$\hat{T} = Diag.(\lambda, 1/\lambda), \ Det\hat{T} = 1, \qquad (16)$$

Santilli proved that systems of extended, deformable and hyperdense particles with spin $\frac{1}{2}$ in conditions of deep mutual entanglement do indeed admit classical counterparts, and provided specific examples (see the review in Tutoring Lecture II of Ref. [7]).

Thanks to the deformability of neutrons according to Eq. (11), with consequential alteration of their magnetic moments, the above verification if the EPR argument was then used in Ref. [19] for the first known numerically exact and time invariant representation of nuclear magnetic moments (see the review in Ref.]25]).

1.6. EPR verification with Einstein's determinism.

The recovering of classical images in Ref. [19] evidently established the foundations for the achievement of Einstein's determinism [1].

In fact, in paper [20] of 2019, Santilli proved that, under the standard isonormalization

$$\langle\hat{\psi}| * \hat{T} * |\hat{\psi}\rangle = \hat{T} \qquad (17)$$

Heisenberg's uncertainties are completed into the form

$$\Delta r \Delta p = \frac{1}{2}\left|\langle\hat{\psi}(r)| * [\bar{r}, \bar{p}] * |\hat{\psi}(r)\rangle\right| \approx \frac{1}{2}\hat{T} << 1 \qquad (18)$$

(18) where the very small value of $\hat{T}$ is established by structure (12) as well as by all fits of experimental data to date [15] [23].

Isodeterministic principle (18) establishes the progressive validity of Einstein's determinism in the interior of hadrons, nuclei and stars, and its full achievement in the interior of gravitational collapse. The latter result is due to the fact that the isotopic element admits a realization in terms of the space component of Schwartzchild's metric with

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}$$

in ensuing full achievement of Einstein's determinism,

$$\Delta\bar{r}\Delta\bar{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r \to 0} = \hat{U} \qquad (19)$$

in the interior of a black hole whose center of gravity verifies full classical determinism.

1.7. EPR verification with valence bonds.

One of the biggest insufficiencies of quantum mechanics and chemistry discussed at the teleconference [7] is the lack of a consistent representation of valence electron bonds in molecular structures. In fact, quantum mechanics and chemistry predict that, due to their equal elementary charge, the identical electrons in valence bonds experience a *Coulomb repulsion* that, ac- cording to Eq. (7), has the value of 230 N which is so enormous for particle standards to prevent any possibility that current quantum chemical models if valence boinds may achieve the needed attraction.

The lack if a quantitative model of molecular structure has then implied the inabil- ity by quantum chemistry to achieve an exact representation of molecular binding ener- gies and other experimental data from unadulterated first principles, with deviations for the binding energies of about 2% that, rather than being small, are equivalent to about 950 BTU.

Following the joint work with A. O. E. Animalu for the identical electrons of the Cooper pairs in superconductivity [17], Santilli confronted the above limitations with systematic studies presented in monograph [24] and, via a procedure similar to that of Eqs. (11)-(15), did achieve the first and only known *attractive force between valence electron pairs in molecular structures.* In particular, the hadronic force resulted to be so strong that valence electron pairs bond into a quasi-particle called *isoelectronium* (see Chapter 4 of Ref. [24]). The alteration of the structure of the valence electrons to achieve an attraction when in total mutual overlapping is studied in Ref. [23] via the notion of *isoparticle* as a representation of the applicable symmetry indicated in Section 1.9.

In joint works with the chemist D. D. Shillady, Santilli proved that the notion of strong valence bond of hadronic chemistry achieves exact representations from unadulterated first principles of experimental data of the Hydrogen [25] and water [26] molecules.

1.8. EPR verification with the removal of quantum divergencies.

Recall the necessary condition for the completion of quantum mechan- ics and chemistry into isomechanics and isochemistry, respectively, according to which the isotopic product $A * B = A\hat{T}B$ must be applied to the *totality* of the products, thus including all products appearing in perturbative series. But the isotopic element $\hat{T}$ has very small values in all known applications. Consequently, perturbative series that are generally divergent in quantum mechanics and chemistry are turned under isotopy into strongly convergent series, as illustrated by the strong convergence of the series

$$A(t) = A(0) + (ATH - HTA)/1! + ... = K << \infty, \quad T << 1. \quad (20)$$

Consequently, *the validity of Einstein's determinism per Section 1.6 implies the removal of quantum mechanical divergencies* (see Corollary 3.7.1, page 128, Ref. [22]). The actual verification of the above important property has been provided by D. D. Shillady and R. M. Santilli in papers [25] [26] with the proof that *the perturbative series of hadronic chemistry converge at least one thousand times faster than the corresponding quantum chemical series.*

1.9. Implications for high energy scattering experiments

Note that the validity of relativistic quantum mechanics for particles in the vacuum of accelerators is beyond scientific doubt because particles can be well approximated as being point-like, the sole possible interactions being at a distance, thus derivable from a potential. Equally beyond scientific doubts are the numeric values of quantities actually measured, such as scattering angles, cross sections, etc.

The verifications of the EPR argument outlined in Sections 1.2 to 1.8 imply that *rel- ativistic quantum mechanics is inapplicable (rather than being violated) for the interior of the scattering region due to its density so big to approach that of black holes, with ensuing general lack of validity of the characteristics of intermediate particles predicted by quantum mechanics, such as mass, spin, etc.,.*

In the authors' view, the most important experimental evidence is that *high energy inelastic scattering events are strictly irreversible over time, thus requiring the Lie-admissible branch of hadronic mechanics and its related scattering theory (Section 1.3), while high energy elastic, thus time- reversible scattering events require the use of the Lie-isotopic branch of hadronic mechanics and related scattering theory (Section 1.4)*, see Refs. [15] [27] and more detailed presentation in the subsequent sections.

To illustrate the alteration, originally called *mutation* [16] and now called *isorenormal- ization* [14] of the rest energy and spin of intermediate particles for high energy elastic scattering events, recall that the scattering region is represented by the iso-Minkowskian isospace $\hat{M}(\hat{x},\hat{\eta},\hat{I})$ with isounit $\hat{I}=1/\hat{T}$, isometric $\hat{\eta}=\hat{I}\eta$, were η is the Minkowskian metric, $\hat{x}=x\hat{I}$ where x represents the Minkowskian spacetime coordinates, and isotopic line element [28]

$$(\hat{x}-\hat{y})^2=\left[\frac{(x_1-y_1)^2}{n_1^2}+\frac{(x_2-y_2)^2}{n_2^2}+\frac{(x_3-y_3)^2}{n_3^2}-(t_1-t_2)^2\frac{c^2}{n_4^2}\right]\hat{I}, \qquad (21)$$

with isounit

$$\hat{I}=1/\hat{T}=Diag\left(n_1^2,n_2^2,n_3^2,n_4^2\right)>0 \qquad (22)$$

where exponential (12) is incorporated in the n- characteristic quantities. The universal symmetry of line element (21) is the Lorentz-Poincaré-Santilli isosymmetry $_{I}\wedge(3.1)$ [29]. The isorelativistic equation characterized by the second order iso-Casimir invariant (Eq. (39), page 91, Ref. [22]) is given by

$$\hat{P}^{\alpha}=\hat{\eta}_{\mu\nu}P^{\mu}P^{\nu}=0, \qquad (23)$$

, which, under the approximation of a spherical scattering region $n_1^2=n_2^2=n_3^2=1$ and the time isounit $\hat{I}_t=1$, yields the iso-Klein-Gordon equation (for $\hbar=1$)

$$(\Lambda+\overline{m}c^2)\hat{\psi}=0 \qquad (24)$$

where $\overline{m}=m/n_4^2$ and m is the value predicted by relativistic quantum mechanics. To appreciate the value of the isorenormalization, $m\rightarrow\overline{m}$, we assume that the density of the scattering region is the same as that of the fireball of the Bose-Einstein correlation, in which case the fit of the experimental data yields the value $n_4^2 = 0.429$ [30] [31], resulting in the isorenormalized mass

$$m \rightarrow \overline{m} = \frac{m}{0.429} = 2.331m \qquad (25)$$

Note the general increase of the rest energies of intermediate particles compared to the value predicted by quantum mechanics.

2. OUTLINE OF THE PRESENTATION

The presentation is based on A.O.E. Animalu's_ personal knowledge of P.A.M. Dirac (as a post-graduate student at Cavendish Lab, Cambridge, UK) and R. P. Feynman (briefly at UK and subsequently through interaction with his former student, R.Oakes, as post-doctoral research associate staff at W.W.Hansen Lab, Stanford University, Calif. USA) in the 1960s. Thereafter, Animalu was involved at Chapel Hill, NC, as theoretician in experimental measurement of angular distribution of photons from positron annihilation with valence electrons in metals using [*geno-dual* model] pseudo-potential theory...

The foundation for comparison of numerical values predicted by the iso- and/or geno-scattering theory of HM with those of the conventional QM scattering theory was laid while Animalu was working at M.I.T. Lincoln Lab and later (with R.M.Santilli & others) in a paper entitled[32], *"Iso-Feynman Propagator and Iso-Matrix of Hadronic Mechanics"* , published in the Hadronic Journal Vol.31, p.317-350(2008). with the following partial abstract :

In this paper, we present in the framework of the Feynman space-time picture of quantum electrodynamics (QED) a systemic method, based on non-linear, non-unitary transformation of Feynman's propagator and S-matrix into the corresponding iso-propagator and iso-S-matrix in hadronic mechanics (HM) for eliminating three basis of divergences in contemporary physical theories [of quantum mechanics (QM)]. (1) Arbitrariness of the (external) boundary conditions on the quantization volume for normalizations and associated differences in the quantum [Bose-Einstein(BE) and Fermi-Dirac (FD) statistics;(ii) the singularities in the Lorentz transformation and the theorem of addition of addition of two velocities, and (3) the singularities arising from the assumption that particles are point-like and the interaction between them long-range in character which calls for arbitrary cut-off of divergent integrals..... We wish, in this paper, to investigate the scattering processes of such *non-conservative time-irreversible* systems, under the name *genoscattering* theory, as exemplified by deep-inelastic electron-proton scattering, and electron-positron annihilation into two photons.

At Chapel-Hill, North Carolina, Animalu had reported (see, page 51 of Animalu's 1977 Prentice-Hall published book)[33] that experimental measurement of angular distribution of photons from positron annihilation with valence electrons into two γ -rays in metals, is given, for a single plane wave state, by the distribution function(at $0^0 K$):

$$n_k = \begin{cases} 1 & (k \leq k_F) \\ 0 & (k > k_F) \end{cases}$$

where k_F is the Fermi wave number that is determined, in accordance with Fermi-Dirac statistics, by the electron density (ρ) via the relation, $k_F = (3\pi^2\rho)^{1/3}$. The quantity measured by positron annihilation is

$$N(k_z) = \int\int_{-\infty}^{+\infty} n_k dk_x dk_y = \begin{cases} \pi(k_F^2 - k_z^2) & (\text{for } k_z \le k_F) \\ 0 & (\text{for } k_z > k_F) \end{cases}$$

which represents the area of a cross-section of the Fermi surface at $k = k_z$ as shown in Fig. 2.(a) and $N(k_z)$ versus k_z is an inverted parabola as shown in Fig2(b).

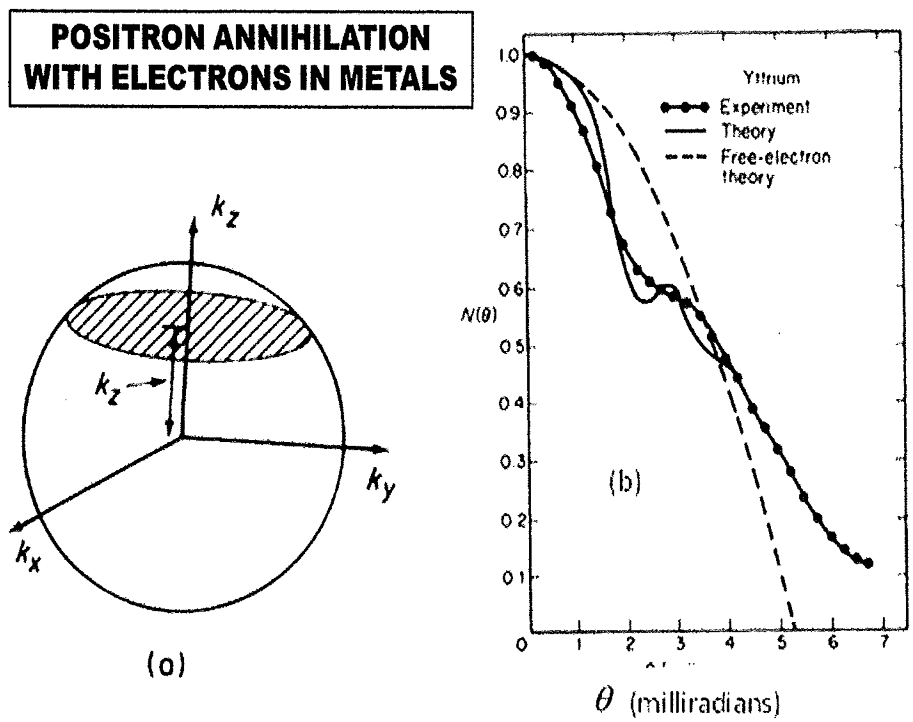


Fig. 2 (a) Cross-sectional area of the Fermi sphere represented by $N(\theta), (\theta = hk_x / mc)$**; (b) Experimental and theoretical ([*geno-dual model*] pseudopotential calculation) angular distribution of photons from positron annihilation in a c-axis Yttrium crystal [From R.W. Williams, T.L. Loucks and R. Mackintosh, Phys. Rev. Letters, 16, p.168 (1966) & sourced as Fig.(2.14) from A.O.E. Animalu, *Intermediate Quantum Theory of Crystalline Solids*, Prentice-Hall, (1977) p.51.**

Bearing the insight from the above results in mind, let us turn next to the use of the Feynman space-time picture of quantum electrodynamics (QED) in terms of the Feynman propagator and the S-matrix method for characterizing the following unitary time-reversible processes:

$$p+\bar{p}\to\gamma,\ e^- + e^+ \to \gamma,$$

$$e^- + e^+ \to \pi^0 \to e^- + e^+$$

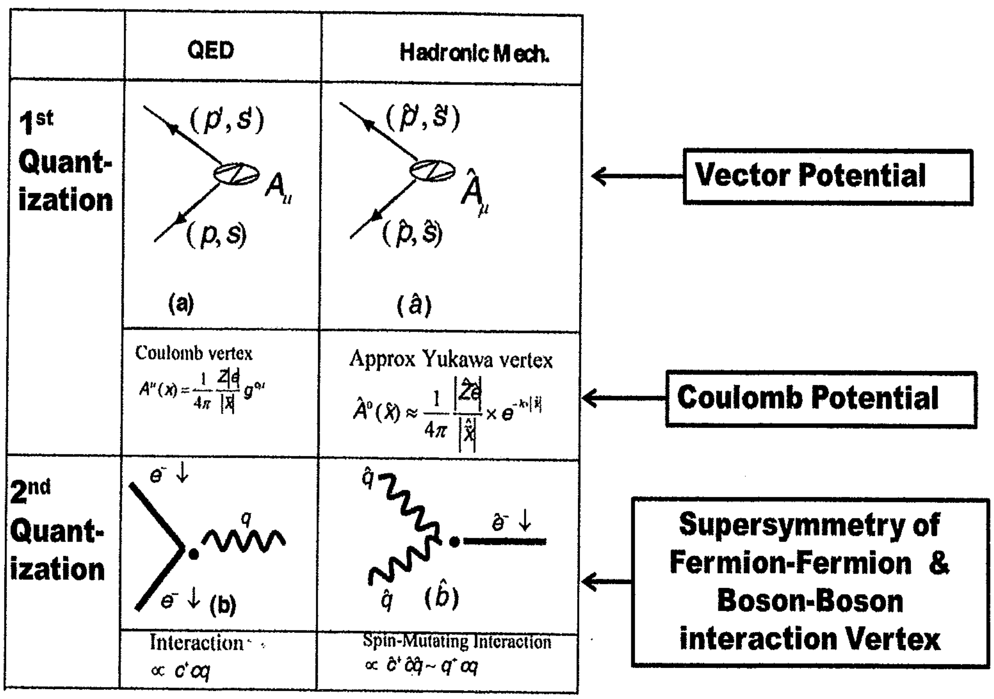
and non-unitary time-irreversible processes:

$$e^- + p \to e^- + p_{resonance} \to e^- + X$$

$$e^- + e^+ \to \gamma + \gamma^d$$

Before going to detailed review in Sec.2, we summarize in Fig.3a the three sources of divergences in contemporary physical theories of quantum mechanics (QM) and how they are eliminated by "lifting" (non-unitary transformation) into the corresponding iso-propagator and iso-S-matrix of hadronic mechanics (HM).

Fig.3a: **ISOTOPIES OF FEYNMAN' DIAGRAMS WITHOUT DIVERGENCES**

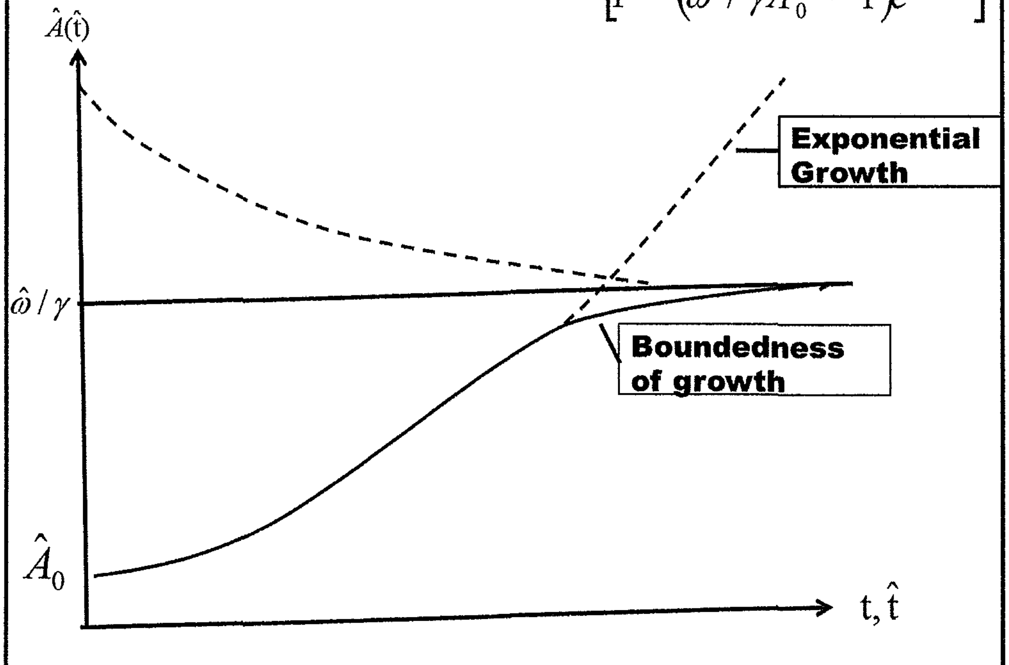


Correspondence Principle between the Feynman graphs for fermion-fermion interaction vertex with boson emission in 1st & 2nd Quantization schemes in QED & the isotopic images for the boson-boson interaction vertex with single fermion spin emission in HM.

The characterization of the transition from the classical (Coulomb vertex) (A_0) to quantum (approximately Yukawa vertex, $\hat{A}_0$) is shown in Fig. 3b. Further generalization involves non-linear processes typified by the existence of current loops as shown in Fig. 4, and the progressive characterization of the Lie-isotopic and Lie-admissible regimes of scattering processes in Fig. 5; the characterization of the internal cube-hexagon (geno-dual) symmetry and existence of current loop in the Lie-Admissible geno-scattering processes in Fig.6; and the hyperspace geometry of current loop(~Kepler vortex) system in Fig. 7.

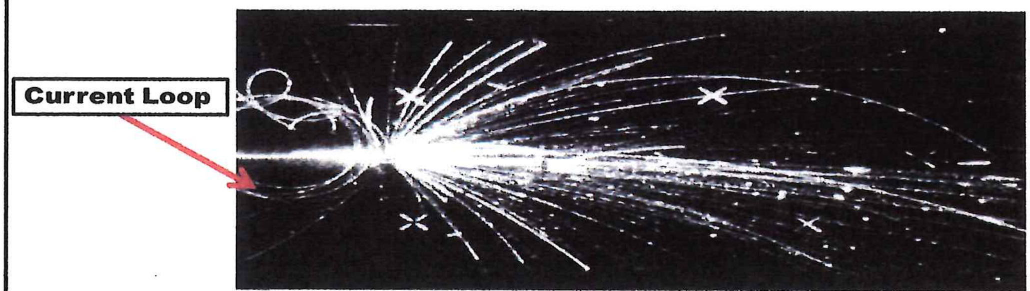
Fig.3b: The classical (Coulomb vertex) (A_0) and its quantum (approximately Yukawa vertex,)

$$A(\mathrm{t}) = A_0 \exp(-\ \omega \mathrm{t}),\ \hat{A}(\hat{\mathrm{t}}) = \frac{\omega_0 / \gamma}{\left[1 - \left(\hat{\omega} / \gamma \hat{A}_0 - 1\right) e^{-\omega \hat{t}}\right]}$$



Two Cases of Eqn for $\hat{A}_0 < \hat{\omega}/\gamma$ & $\hat{A}_0 > \hat{\omega}/\gamma$

Fig. 4: Examples of Typically Irreversible Inelastic Scattering Processes with Current Loops



$$e^- + e^+ \rightarrow 2\gamma\gamma \quad \text{or} \quad e^- + e^d \rightarrow \gamma + \gamma^d$$
$$p + \bar{p} \rightarrow 2\gamma \quad \text{or} \quad p + p^d \rightarrow \gamma + \gamma^d$$
$$e^- + p \rightarrow e^- + p_{resonance} \rightarrow e^- + X$$

Fig. 5: Progressive Characterization of the Lie-Isotopic & Lie-Admissible Regimes of Scattering Processes

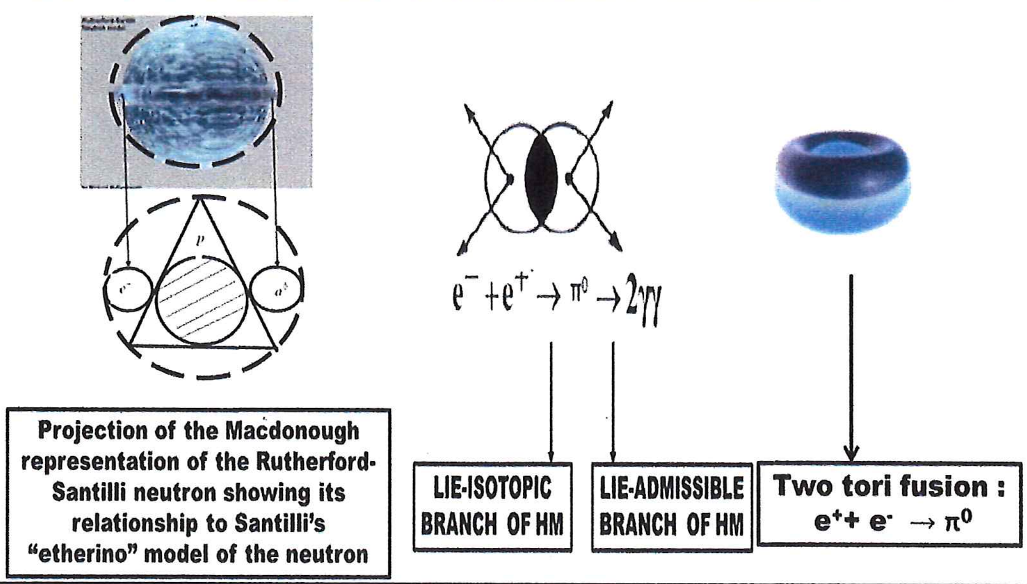


Fig. 6: Characterization of the Internal Cube-Hexagon (geno-dual) Symmetry & Existence of Current loop in Lie-Admissible Geno-scattering Processes

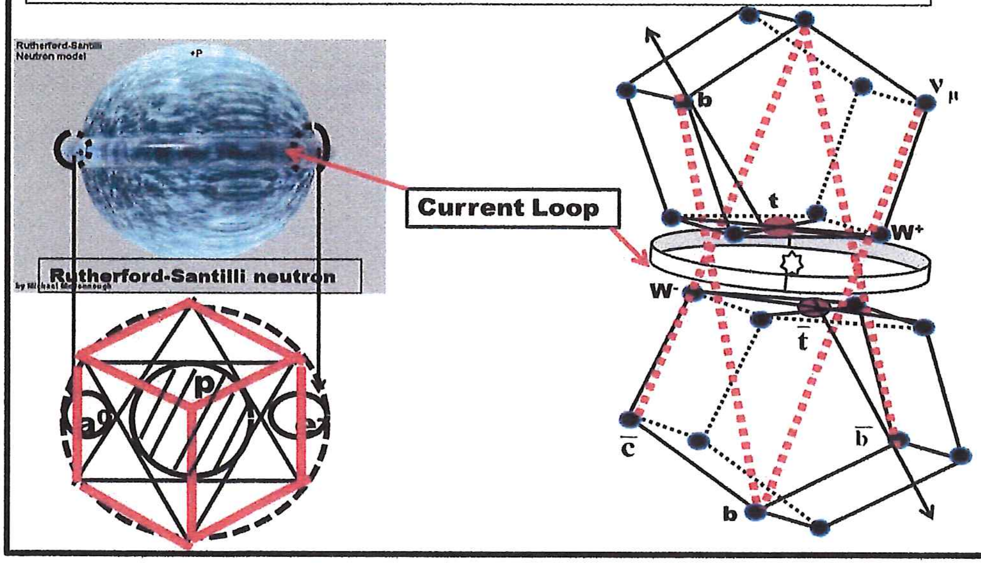
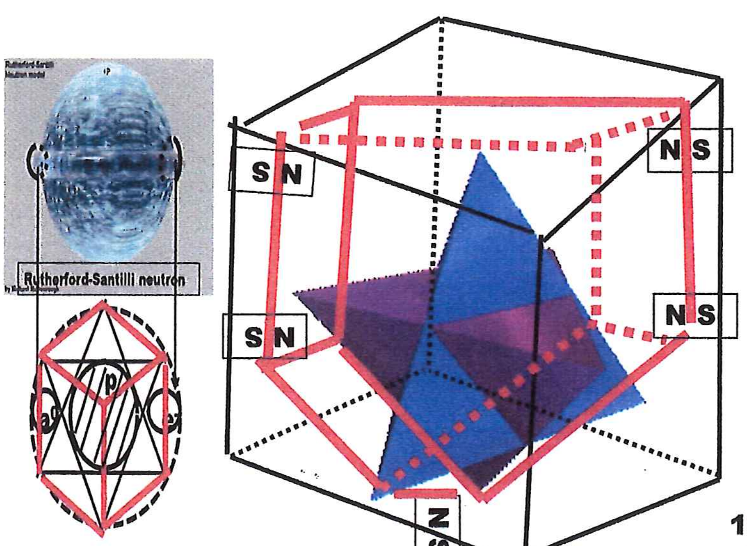


Fig. 7: Hyperspace Geometry of Current Loop (~ Kepler's Vortex) System



Geno-Dual (4x4) Metric Tensor

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	±1

$16+10+7-1 = 32 = 2^5$ =Dim of $2^N x 2^N$ rep of SO(2N)=SO(10) (N=5).

No. of (points+lines+planes) are Cube(26)+(Kepler) Vortex(7x6=42) +Pentagon(32): 26+42+32=100

Let me underscore the question that led the presenter to the above scenario of cube-hexagon (geno-dual) metric tensor principle in three-dimensional projective geometry in the era of search for a unified field theory of gravity and electromagnetism from my interaction in 1968 with P.A.M. Dirac while he was searching for a Maxwell-like **(dual) gauge principle** to relate electric charge and magnetic charge in projective geometric terms. Following treatment of the Lorentz force and the linear Dirac (negative) energy relativistic wave equation for electric charge on the same footing as a corresponding dual of Lorentz force and a positive energy relativistic wave equation for magnetic charge and non-negative mass (represented by a current loop). Dirac's way of thinking led me (as summarized below) to a correspondence principle

Particle ⟷ Point, Field ⟷ Line, Current ⟷ Plane,

Such a correspondence principle implies geometric characterization of conventional electrodynamics and an analogous so-called e-magnetodynamics of current loop and to the Feynman space-time diagrammatic approach based on 10x10 representations of the dynamical group unifying strong interaction with electromagnetism and space-time geometry and violations of the discrete symmetries – parity, charge conjugation, time-reversal and spin-parity:

$$SU^{>}(3)\times SU^{<}(3)\times U^{>}(1)U^{<}(1)\times O^{>}(4,2)\times O^{<}(4,2)$$

This defines the SO(2N)xSO(2N) group-theoretic approach to scattering of N-particle (N=5) systems which we proceed next to review and compare with experiment .in section.3. Discussion of results and conclusions will be presented in Sec.4.

Conventional Electrodynamics	**E-magnetodynamics of Current Loop**
$\|\mathbf{F}_{\mu\nu}\| \equiv \begin{pmatrix} 0 & \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ -\mathbf{E}_1 & 0 & \mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{E}_2 & -\mathbf{B}_3 & 0 & \mathbf{B}_1 \\ -\mathbf{E}_3 & \mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$	$\|G_{\mu\nu}\| \equiv \begin{pmatrix} 0 & \mathbf{J}_1 & \mathbf{J}_2 & \mathbf{J}_3 \\ -\mathbf{J}_1 & 0 & \mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{J}_2 & -\mathbf{B}_3 & 0 & \mathbf{B}_1 \\ -\mathbf{J}_3 & \mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$
Maxwell's eqn & Lorentz force	$(\mathbf{J}^{m} \to \mathbf{B},\ \mathbf{B} \to -\mathbf{J}^{m})$
$\partial^{\nu} F_{\mu\nu} = J^{e}_{\mu};\mathbf{f}^{e} = e\mathbf{E} + \mathbf{J}^{e} \times \mathbf{B}.$	$\mathbf{J}^{m} = -(2e\hbar/mc)\mathbf{A},\quad \mathbf{B} = \mathrm{curl}\mathbf{A} \equiv \mathrm{curl}(\zeta\mathbf{J}^{m}),$
$\mathbf{E} \to \mathbf{B},\ \mathbf{B} \to -\mathbf{E}.\ \mathbf{J}^{e} \to \mathbf{J}^{m}$	$\zeta = (mc/2\|e\|\hbar)$

$$\mathrm{Det}\|F_{\mu\nu} - \lambda\eta_{\mu\nu}\| \equiv \lambda^4 - (R_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma})\lambda^2 + (\epsilon_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma})^2 = 0,$$

$$\mathrm{Det}\|G_{\mu\nu} - \lambda\eta_{\mu\nu}\| \equiv \lambda^4 - (R_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma})\lambda^2 + (\epsilon_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma})^2 = 0,$$

$$\|\eta_{\mu\nu}\| \equiv diag(+1,-1,-1,-1), R_{\mu\nu\rho\sigma} \equiv (\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\sigma\nu}),$$

$$R_{\mu\nu\rho\sigma} F^{\mu\rho} F^{\nu\sigma} - 2\epsilon_{\mu\nu\rho\sigma} F^{\mu\rho} F^{\nu\sigma} = \begin{cases} \mathrm{E}^2 \\ \mathrm{B}^2 \end{cases} \Rightarrow \begin{cases} \mathrm{E}^2 - \mathrm{B}^2 \pm 2\mathrm{B.E} = 0 \\ \mathrm{B}^2 - \mathrm{E}^2 \pm 2\mathrm{E.B} = 0 \end{cases}$$

$$R_{\mu\nu\rho\sigma} G^{\mu\rho} G^{\nu\sigma} - 2\epsilon_{\mu\nu\rho\sigma} G^{\mu\rho} G^{\nu\sigma} = \begin{cases} \mathbf{J}^{m2} \\ B^2 \end{cases} \Rightarrow \begin{cases} \mathbf{J}^{m2} - \mathbf{B}^2 \pm 2\mathbf{B.J}^{m} = 0 \\ \mathbf{B}^2 - \mathbf{J}^{m2} \pm 2\mathbf{J}^{m}.\mathbf{B} = 0 \end{cases}$$

The significance of this algebraic 4x4-matrix unification scheme of Maxwell's equation with Lorentz force on one hand, and e-magnetodynamics with current loop on the other hand, is that we are now able to unify the gravitational field tensor with Maxwell's field tensor in an algebraic relation exhibiting dual symmetry with respect to interchange of electric and magnetic fields, as well as electric and mass/magnetic charges, and hence SO(2N) (N=5) group symmetry.

3. PRESENTATION OF RESULTS

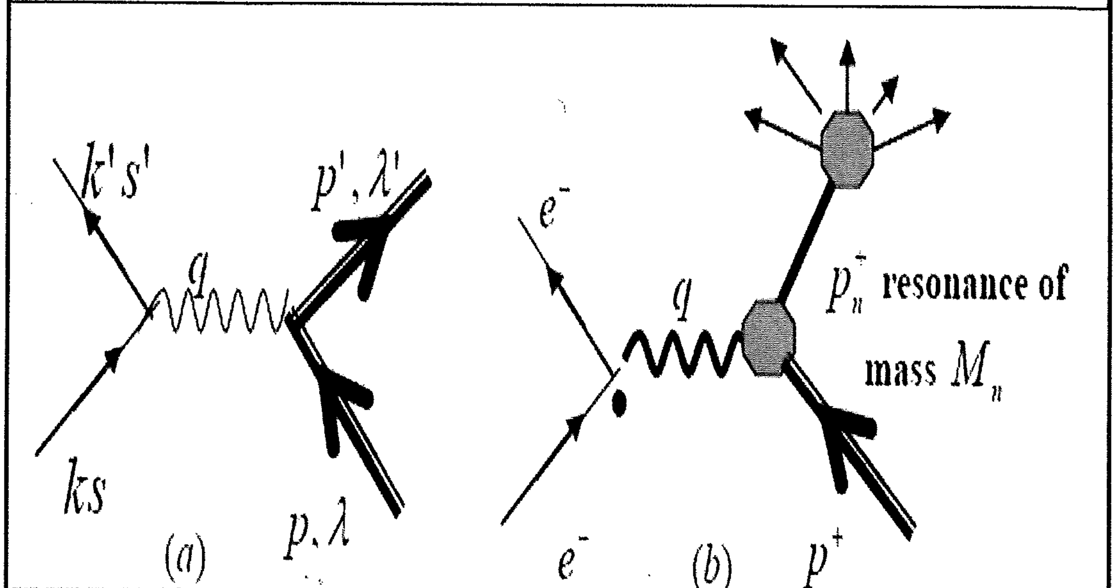
3.1 Feynman Graphs for Scattering Cross-Section

In an article entitled: *Review of Deep-inelastic e-p Scattering: A hadronic mechanics Viewpoint* by A.O.E. Animalu & C.E. Ekuma published in *African Journal of Physics* Vol.1, p.133-153 (2008) cited in 2011 ref. [27]:

www.santilli-foundation.org/docs/Isoscattering-V.pdf

we have exhibited the Feynman graph (shown below in Fig. 8) and summarized the conventional quantum mechanical (QM) and the corresponding hadronic mechanics (HM) expressions for the electron-proton scattering cross-section.

Fig. 8: Feynman graph for (a) electron-point-proton (Mott) scattering & (b) resonance model of deep-inelastic electron-proton scattering



CONVENTIONAL QM EXPRESSIONS FOR THE SCATTRING CROSS-SECTION

A resonance of mass M_n is produced. $M_n^2 = M^2 + 2M\nu + q^2$, with $M^2 \equiv p^2$, $\nu \equiv q.p$,

$$\frac{d\sigma_{e^- - point - p^+}}{d\Omega} = \left.\frac{(Z\alpha)^2 E^2 \left(1 - \beta \sin^2(\theta/2)\right)}{4p^4 \sin^4(\theta/2)}\right|_{\beta=1}$$

$$= \frac{(Z\alpha)^2 \cos^2(\theta/2)}{E^2 \sin^4(\theta/2)} \equiv \frac{d\sigma_{Mott}}{d\Omega}$$

$$\beta = \frac{|\vec{p}|}{E}, \ \tfrac{1}{2}(\vec{p}' - \vec{p})^2 = \vec{q}^2 = 2p^2(1 - \cos\theta) = 4p^2 \sin^2(\theta/2),$$

$$\frac{d^2\sigma}{d\Omega dE} = \frac{d\sigma_{Mott}}{d\Omega}\left[W_2(\nu, q^2) + 2W_1(\nu, q^2)\tan^2\frac{\theta}{2}\right]$$

$$M^2 = -q^2 \to \infty, \nu = q.p \to \infty,$$

$$2MW_2(\nu, q^2) = F_1(\omega), \ \nu W_2(\nu, q^2) = F_2(\omega)$$

$$\int_{q^2/2M}^{\infty} d\nu \left[W_2^p(\nu, q^2) + W_2^n(\nu, q^2)\right] \geq \frac{1}{2} \qquad \int_1^{20} \frac{d\omega}{\omega} \nu W_2^p = 0.78 \pm 0.04$$

R.M. Santilli has presented the predictions for the rest energy of particles within the scattering region

HADRONIC MECHANICS EXPRESSIONS FOR THE SCATTRING CROSS-SECTION

Our characterization of the HM viewpoint is anchored on an association of the two Lorentz scalars, $(q^2 - p^2)$ and $q.p$, involved in the Bjorken scale:

$$-q^2 \to \infty,\ \nu = q.p \to \infty,\ \text{such that } x \equiv (p^2 - q^2)/2p.q \to -q^2/2M\nu \equiv 1/\omega,$$

with the ***self-corresponding points*** of a non-unitary, nonlinear ("lifting") transformation, $(q,p) \to (\hat{q}, \hat{p})$ of a rectangular hyperboloid into a torus,

$$q.p = (1/2)(\hat{q}^2 - \hat{p}^2).$$

$$\frac{d\sigma}{d\Omega} \to \frac{d\hat{\sigma}}{d\hat{\Omega}} \equiv \frac{d\hat{\sigma}_{Mott}}{d\hat{\Omega}}\left[\hat{F}_2(\hat{\omega}) + 2\hat{F}_1(\hat{\omega})\tan^2(\hat{\theta}/2)\right].$$

$$\begin{cases} d\hat{F}_1 / d\hat{\omega} = \gamma\hat{F}_1(\hat{F}_0 - \hat{F}_1) \equiv \hat{x}_0\hat{F}_1 - \gamma\hat{F}_1^2 \\ d\hat{F}_2 / d\hat{\omega} = -\gamma\hat{F}_2(\hat{F}_0 - \hat{F}_2) \equiv -\hat{x}_0\hat{F}_2 + \gamma\hat{F}_2^2 \end{cases}$$

$$\Rightarrow \begin{cases} \hat{F}_1(\hat{\omega}) = \dfrac{\hat{x}_0/\gamma}{\left[1 - \left(\hat{x}_0/\gamma\hat{F}_{10} - 1\right)e^{+\hat{x}_0\hat{\omega}}\right]} \\ \hat{F}_2(\hat{\omega}) = \dfrac{\hat{x}_0/\gamma}{\left[1 - \left(\hat{x}_0/\gamma\hat{F}_{20} - 1\right)e^{-\hat{x}_0\hat{\omega}}\right]} \end{cases}$$

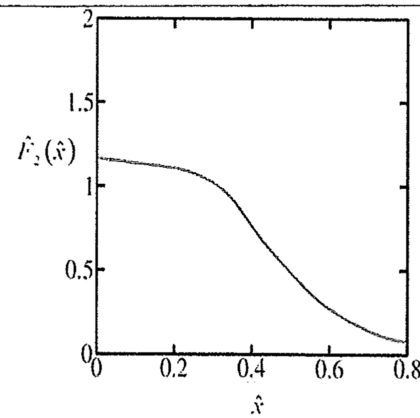
In terms of the reciprocal Bjorken variable, $\hat{x}$ the corresponding curve $\hat{F}_2(\hat{x})$ turns out to be an image of $\hat{F}_2(\hat{\omega})$ and has the form

$$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0/\xi}{\left[1 + \left(\hat{\omega}_0/\xi\hat{F}_{20} - 1\right)e^{+\hat{x}\hat{\omega}_0}\right]}$$

Fig. 9:

$$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0/\xi}{\left[1+\left(\hat{\omega}_0/\xi\hat{F}_{20}-1\right)e^{-\hat{x}\hat{\omega}_0}\right]}$$ is tabulated for $\hat{F}_{20} > \hat{\omega}_0/\xi$ & sketched below

#	
0	1.1672308
0.088888889	1.1424205
0.17777778	1.1164058
0.26666667	1.0621161
0.35555556	0.91052375
0.44444444	0.63087342
0.53333333	0.40818084
0.62222222	0.23984199
0.71111111	0.13305208
0.8	0.080026037



We summarize next in Fig.10 the characterization of isoscattering and genoscattering theories without divergences in hadronic mechanics.

Fig. 10: A schematic view of the main assumptions of (a) Lie-isotopic scattering theory[8] and (b) the Lie-admissible (genoscattering) theory[9], namely the exact validity of conventional quantum mechanics (QM) for the *EXTERIOR* conditions and the validity of the covering hadronic mechanics (HM) for the *INTERIOR* conditions of the scattering region, generally of arbitrary shape, but represented by spherical "extended" particle with a radius of about 10^{-13} cm in isoscattering theory, and by a torus with outer radius of ($r_0 = 10^{-13}cm$) and an inner (hole) radius ($r_i < r_0$) defining two "fundamental lengths" that ensure *ab initio* convergence of possible corrections.

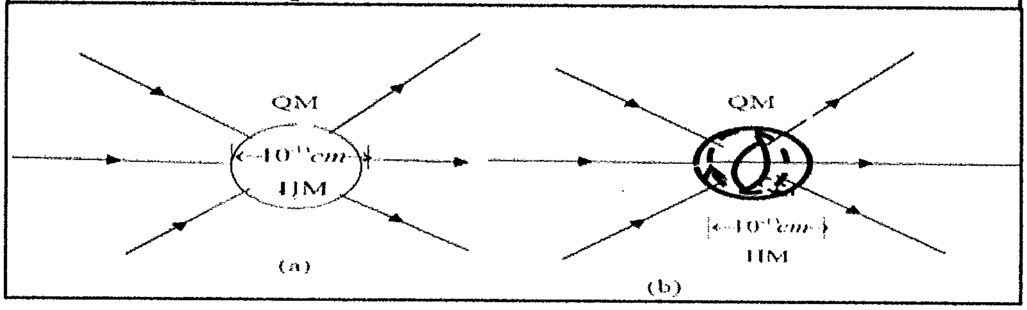


Fig. 11: Four times characterized in Santilli's iso-selfdual symmetry[7] of the theory of ANTI-MATTER

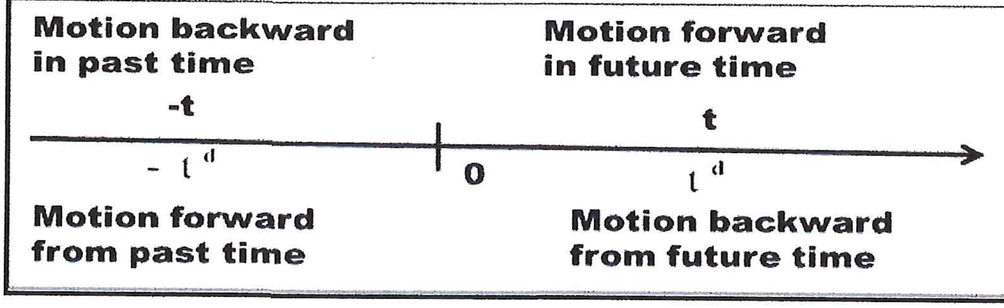


Fig.12: **Corresponding Feynman SPACE-TIME graphs[8] for:**

(a) $e^+ + e^- \to 2\gamma$; **(b)** $e^- + e^+ \to \gamma + \gamma^d$.

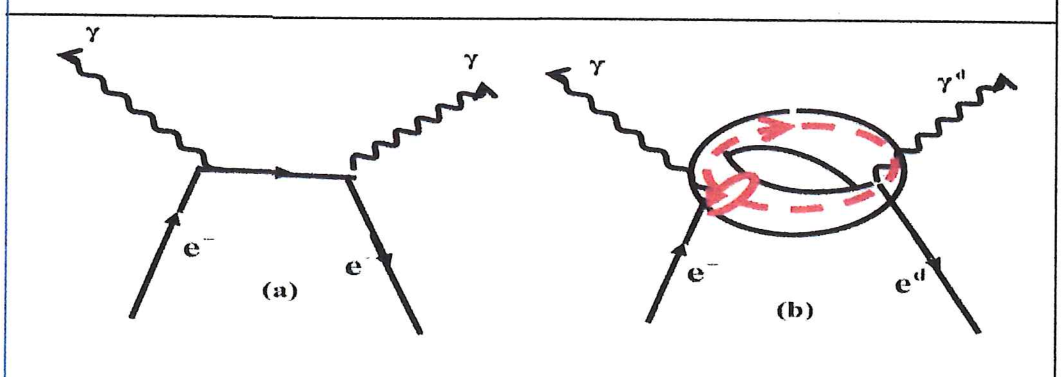


Fig.13. Conventional Feynman Graph/Rule for e-p Coulomb Scattering in :
www.santilli-foundation.org/docs/Isoscattering-IV.pdf

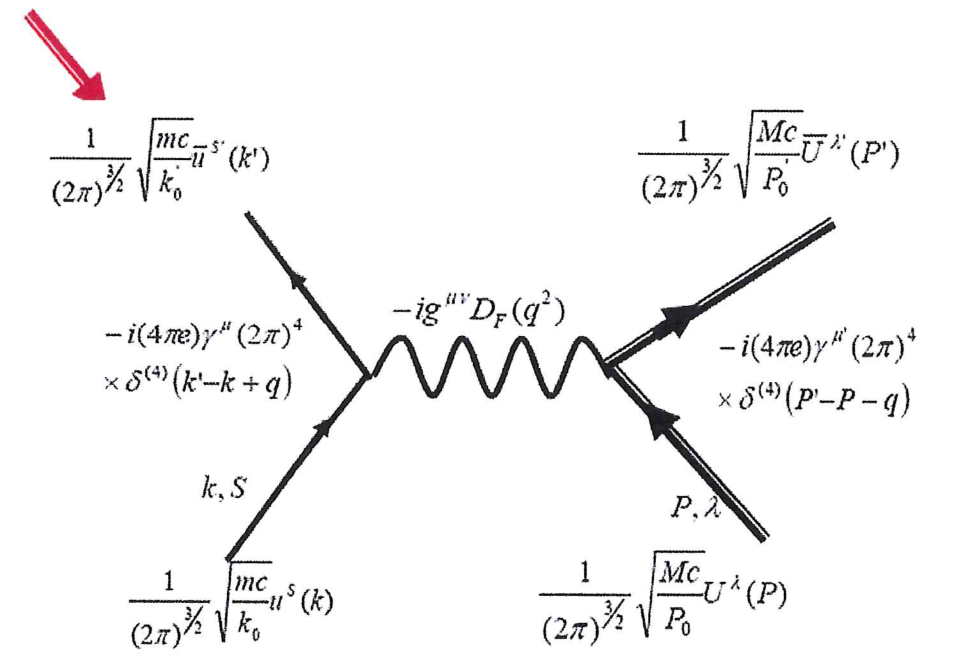
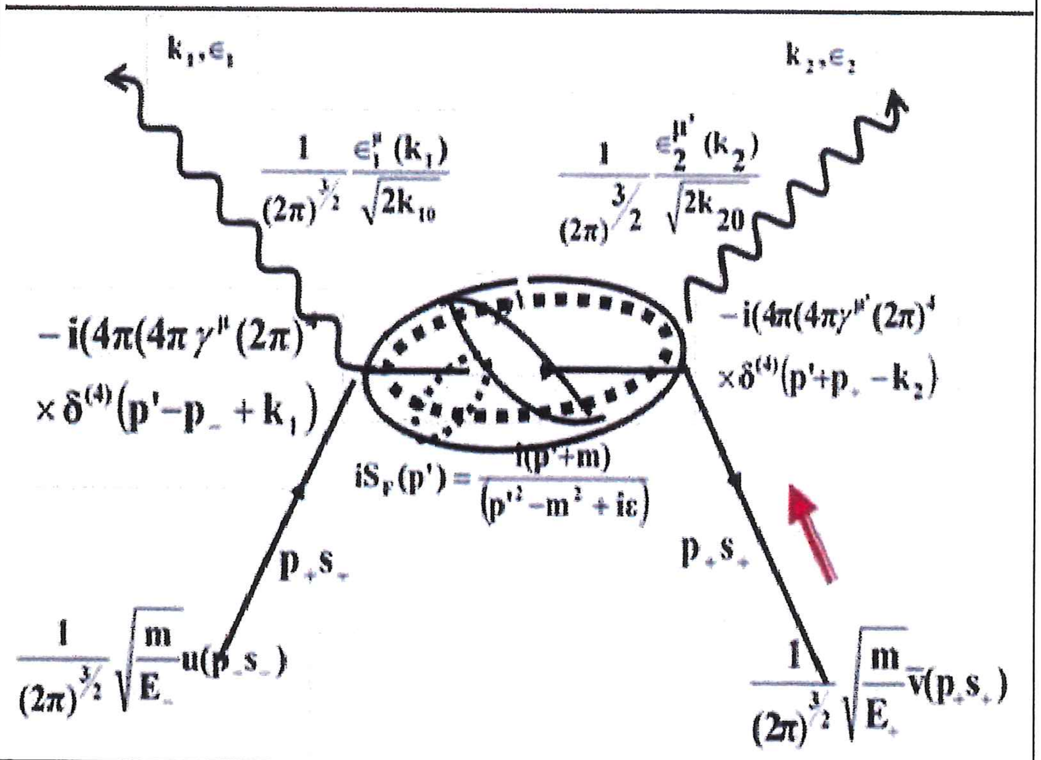
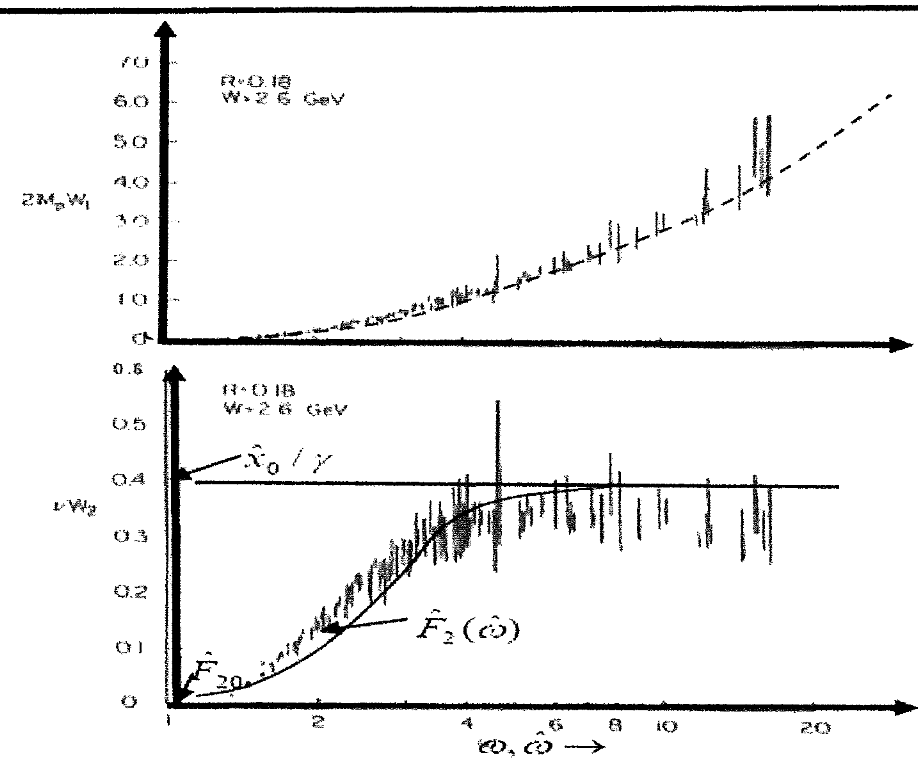


Fig.14: Conventional Feynman Graph/Rules for computation of the S-matrix for the annihilation process $e^+ + e^- \rightarrow 2\gamma$ and its elaboration for the isoselfdual process $e^+ + e^- \rightarrow \gamma + \gamma^d$



3.2 Comparison of Non-Unitary Geno-Sacttering Theory with Experiment (ref.[27])

Fig. 15 : Comparison of $\hat{F}_1(\hat{\omega})$ & $\hat{F}_2(\hat{\omega})$ for $\hat{F}_{20} < \hat{x}_0$ with $2MW_1 = \hat{F}_1(\hat{\omega})$ & $\nu W_2 = F_2(\hat{\omega})$ where $\omega = 2M\nu/q^2$ for proton; $W > 2.6\ GeV$, $q^2 > 1(GeV/c_0^2$ and R=0.18. Data from G.Miller *et al* Phys. Rev. D5, 528(1972).



As stated above, in terms of the reciprocal Bjorken variable, $\hat{x}$ the corresponding curve $\hat{F}_2(\hat{x})$ turns out to be an image of $\hat{F}_2(\hat{\omega})$ and has the form

$$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0 / \xi}{\left[1 + \left(\hat{\omega}_0 / \xi \hat{F}_{20} - 1\right) e^{+\hat{x}\hat{\omega}_0}\right]}$$

whose feature is as shown for $\hat{F}_{20} < \hat{\omega}_0 / \xi$ and $\hat{F}_{20} \geq \hat{\omega}_0 / \xi$ in Figs 16-18

from A.O.E. Animalu and E. C. Ekuma, *Review of Deep-Inelastic e-p Scattering: A Hadronic Mechanics Viewpoint*, in African J. Phys. Vol.1, p.133-153 (2008) which was cited as Ref. [27] in the Proc. ICLATIP-3 Kathmandu Univ. Nepal (2011) as Santilli-Animalu-Isoscattering-V.pdf

Fig. 16 : Comparison of $\hat{F}_1(\hat{\omega})$ & $\hat{F}_2(\hat{\omega})$ for $\hat{F}_{20} < \hat{x}_0$ with $2MW_1 = \hat{F}_1(\hat{\omega})$ & $\nu W_2 = F_2(\hat{\omega})$ where $\omega = 2M\nu / q^2$ for proton; $W > 2.6\ GeV$, $q^2 > 1(GeV / c_0^2$ and R=0.18. Data from G.Miller *et al* Phys. Rev. D5, 528(1972).

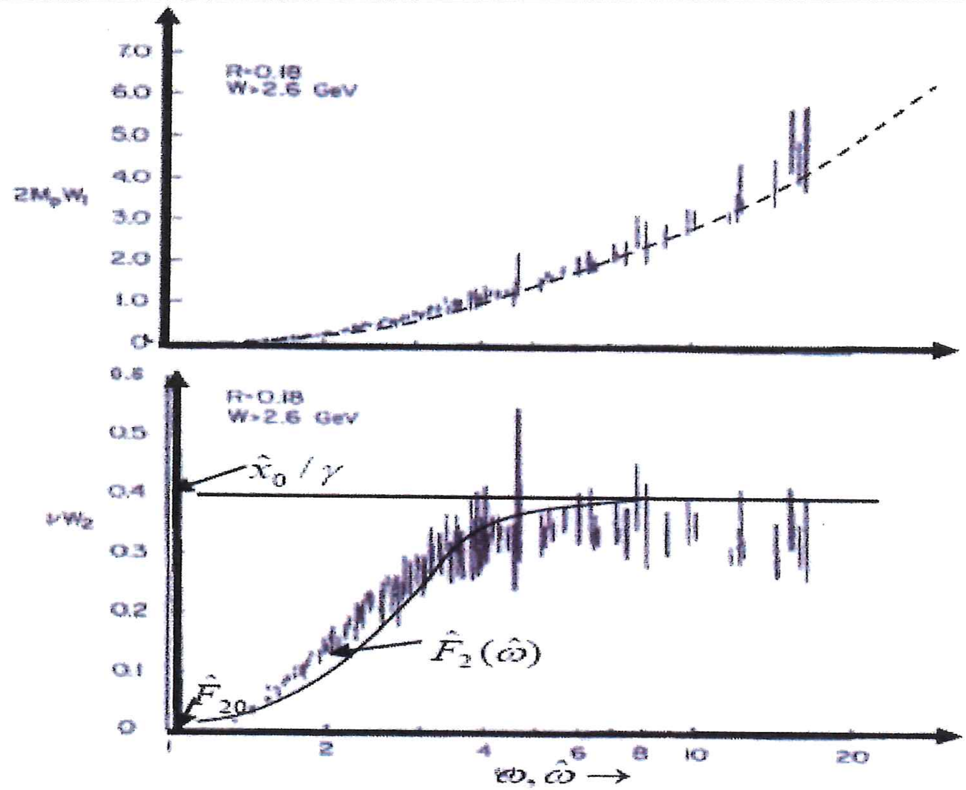


Fig. 17 : Comparison of $\hat{F}_2(\hat{x})$ for $\hat{F}_{20} < \hat{\omega}_0/\xi$ (solid line) with νW_2 versus $x = q^2/2M\nu$ for the proton for $W > 2.0\ GeV$, $q^2 > 2(GeV/c_0^2$ Data from Bodek *et al* Phys. Rev. Lett/ 30, 1087(1973); Phys. Lett. 51B, 417(1974); Phys. Rev. D20, 1471 (1979)

$\hat{x}$	$\hat{F}_2(\hat{x})$
0.0475727	0.334991
0.127295	0.325612
0.204184	0.331821
0.28397	0.310725
0.355572	0.279105
0.449308	0.219128
0.537629	0.144757
0.623138	0.0820683
0.697531	0.0426724
0.782875	0.00992946
0.86254	0.0109668
0.892858	-0.00686764

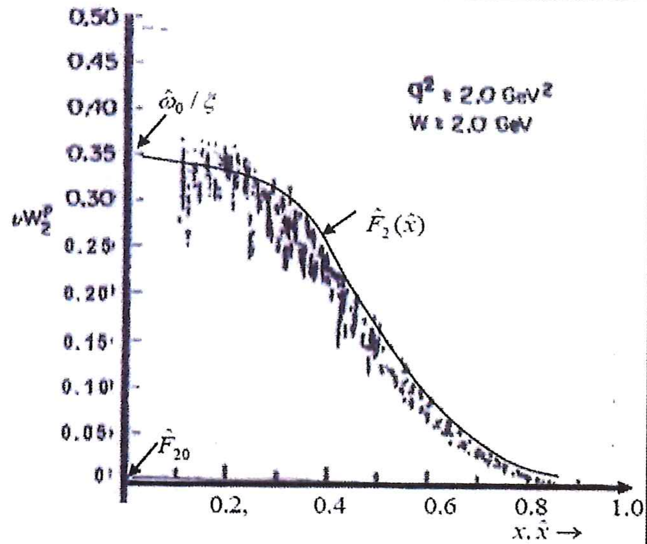
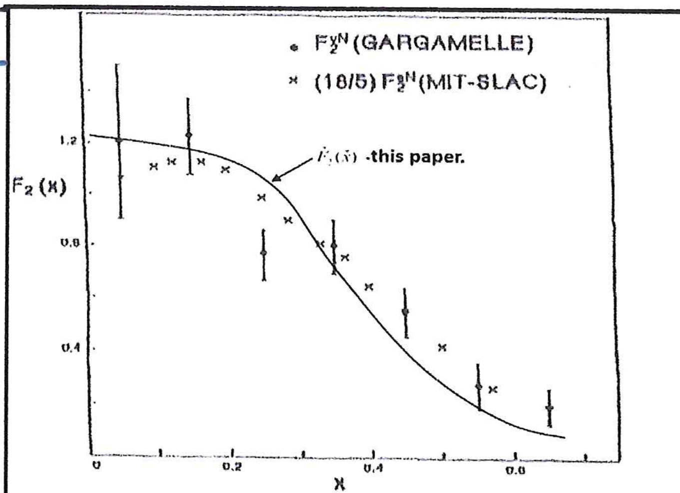


Fig.18:Elaborated from A.O.E. Animalu and E. C. Ekuma, *Review of Deep-Inelastic e-p Scattering: A Hadronic Mechanics Viewpoint*, in African J. Phys. Vol.1, p.133-153 (2008) which was cited as Ref. [27] :
www.santilli-foundation.org/docs/Isoscattering-V.pdf

$\# \hat{x}$	$\hat{F}_2(\hat{x})$
0	1.1672308
0.088888889	1.1424205
0.17777778	1.1164058
0.26666667	1.0621161
0.35555556	0.91052375
0.44444444	0.63087342
0.53333333	0.40818084
0.62222222	0.23984199
0.71111111	0.13305208
0.8	0.080026037



Early Gargamelle measurements of $F_2^{\nu N}$ compared with $(18/5)F_2^{eN}$ calculated from the MIT-SLAC results) and with HM model $\hat{F}_2^{\nu N}(\hat{x})$. (Source of data J.I. Friedman Nobel Lecture 1990, *Physics 1990 p. 715*

A question may be raised as to the fact that the data we have presented are consistent with hadronic mechanics, whereas the proton deep inelastic scattering data are expected to have 2-3 bumps due to high density situations with two bumps for e-p and three bumps for quarks in the standard model and similarly for mesons. An answer is that in the (Bjorken) limit of scale-invariance, the good agreement can only be considered preliminary, in view of the vast amount of available data that need to be analyzed based on the theory.

4 .DISCUSSION AND CONCLUSION

In A PubRelco Interview of R.M. Santilli with Scientific and Industrial implications New York, N.Y., April 15, 2019 In response to the question: "Prof. Santilli, could you please **review in a language** accessible to the general audience **Einstein's 1935 historical prediction that quantum mechanics and, therefore, quantum chemistry are incomplete theories",** the following answer was given: "Einstein did not accept the **uncertainty of quantum mechanics,** including the impossibility to identify the position of a particle with classical precision. For that reason, he made his famous quote **"God does not play dice with the universe."**

Einstein **accepted quantum mechanics for atomic structures, but believed that quantum mechanics is an "incomplete theory,"** in the sense that it could be broadened into such a form to **recover classical determinism** at least under special conditions. The same argument also applies to **quantum chemistry**". It admits interior entanglement/Lie-admissible penetration which is characterized | by progressive generalization of the **Lie-algebraic product** of quantum mechanics (and hence quantum chemistry) in **"hadronic mechanics"** as follows:

$$AB-BA \rightarrow \begin{cases} APB-BP^{-}A,(P^{-}=P=I) \rightarrow \text{Conventional Lie algebraic product} \\ APB-BP^{-}A,(P^{-}=P\neq I) \rightarrow \text{Lie isotopic algebraic product} \\ APB-BP^{-}A,(P^{-}\neq P\neq I) \rightarrow \text{Lie admissible algebraic product,} \end{cases}$$

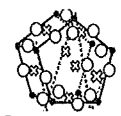
This progressive Lie-admissible algebraic structure has been realized since (2011) in **geometric terms as a deformation of a point sphere into a torus** characterized in the framework of **"Non-unitary scattering theory of hadronic mechanics"** by R.M. Santilli and A.O.E. Animalu

In view of the linguistic challenges of hadronic mechanics, and for ease in translation into other languages, we have done linguistic geometric elaboration of the EPR argument in analogue/digital SO(10) characterization as shown below

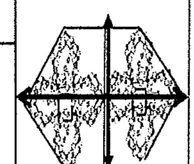
LINGUISTICS GEOMETRIC ELABORATION OF THE EPR ARGUMENT FOR ANALOGUE/DIGITAL SO(10) CHARACTERIZATION OF HADRONIC ENERGY & CORONAVIRUS

Analogue	Digital (GENO-ASCII Code, A ~ 1,B~2,C~3,...,Z~26, so that HADRON ~ H+A+D+R+O+N=8+1+4+18+15+14=60; & (A,B,C,...Z)~(a,b,c,...,z), such that in SO(10),1+2+3+....+36=666.
*Life; Cell; Bit	12+9+6+5= **32 = 2^5**; Cell= 3+5+12+12=**32** ; Bit= 2+9+20=**31**
*Classical Mechanics	(**3+12+1+19+19+9+3+1+12**)+(13+5+3+8+1+14+9+3+19)**=79+75=**155
A Quantum Mechanics Particle Physics A Nuclear Dodecahedron	1+(17+21+1+14+20+21+13=107)+(13+5+3+8+1+14+9+3+19=75) =1+107+75 = 183; (16+1+18+20+9+3+12+5=84)+(16+8+25+19+9+2+19=99) = 84+99=**183.** 1+(14+21+3+12+5+1+18=84)+(4+15+4+5+3+1+8+5+4+18+15+14=**98)=183**
Wave Particle Duality Quantum Chemistry A Dichotomy Exterior	(23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25=92) =227 (17+21+1+14+20+21+13=**107**)+(3+8+5+13+9+19+20+18+25=**120**)=**227** (1+4+9+3+8+15+20+15+13+25=113)+(5+24+20+5+18+9+15+18=**114**)=**227**
Quantum Mechanics Is not Complete Wave-Particle Duality Theory	(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=**75**)+ (9+19)=28)+(14+15+20=49)+(3+15+13+16+12+5+20+5=**89**)+ (23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25=92)+ (20+8+5+15+18+25=91) = **107+75+28+49+89+51+84+92+91 = 666**
God Does Not Play A Dice with the Universe Entanglement Interior	(7+15+4=26)+(4+15+5+19=43)+(14+15+20=49)+(16+12+1+25=54)+ (1+4+9+3+5=22)+(23+9+20+8=60)+(20+8+5=33)+(21+14+9+22+5+18+9+5=113)+ (5+14+20+1+14+7+14+5+13+5+14+20=148)+(9+14+20+5+18+9+15+1=108) = **26+43+49+54+22+60+33+113+148+108 = 666**
A Complete Time Determinism	(1+3+15+13+16+12+5+20+5=80)+**(20+9+13+5=47)+** **4+5+20+5+18+13+9+14+9+19+13=129)= 80+47+129=256**
*Lie-Admissible Penetration	(12+9+5=26)+(1+4+13+9+19+19+9+2+12+5=93) + +(16+5+14+5+20+18+1+20+9+15+14=137) = 26+93+137=256
Hadronic Energy Corona Virus *Corona Virus iso-spin	(8+1+4+18+15+14+9+3=72)+(5+14+5+18+7+25=74) = **72+74 = 146** (3+15+18+15+14+1=66)+(22+9+18+21+19=89); 66+89= 155 **155+** [(9+19+15=43)+(19+16+9+14=58)=155+43+58=155+101 = 256

(*) Remarks :
In SO(10) group representation of ***Life ("cell") by** 32 = 10 points + 15 lines + 7 planes.

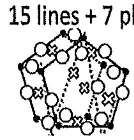
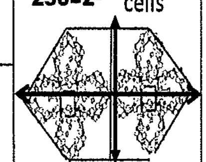
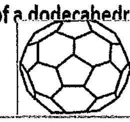


& representation of *Lie-admissible **Penetration by** **$256=2^8$** cells



& *Classical Mechanics is given by 5 Bits (5x31=155) Forming Platonic Icosahedron dual of a dodecahedron.

LINGUISTICS GEOMETRIC ELABORATION OF THE EPR ARGUMENT FOR ANALOGUE/DIGITAL SO(2N) CHARACTERIZATION OF QUANTUM CHEMISTRY AND PARTICLE PHYSICS SCATTERING

Analogue	Digital (GENO-ASCII Code, A ~ 1,B~2,C~3,...,Z~26, so that WORD ~ W+O+R+D=23+15+18+4=60; & (A,B,C,...Z)~(a,b,c,...,z).	(*) Remarks : In SO(10) group
*Life; Cell; Bit	12+9+6+5=**32 = 2^5** ; Cell= 3+5+12+12=**32** ; Bit= 2+9+20=**31**	representation of
*Classical Mechanics	(**3+12+1+19+19+9+3+1+12**)+(13+5+3+8+1+14+9+3+19)=**79+75**=155	***Life ("cell") by**
A Quantum Mechanics **Particle Physics** A Nuclear Dodecahedron	1+(17+21+1+14+20+21+13=107)+(13+5+3+8+1+14+9+3+19=75) =1+107+75 = 183; (16+1+18+20+9+3+12+5=84)+(16+8+25+19+9+2+19=99) = 84+99=**183.** 1+(14+21+3+12+5+1+18=84)+(4+15+4+5+3+1+8+5+4+18+15+14=**98)=183**	32 = 10 points + 15 lines + 7 planes. 
Wave Particle Duality **Quantum Chemistry** **A Dichotomy Exterior**	(23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25=92) =227 (17+21+1+14+20+21+13=**107**)+(3+8+5+13+9+19+20+18+25=**120**)=**227** (1+4+9+3+8+15+20+15+13+25=113)+(5+24+20+5+18+9+15+18=**114**)=**227**	& representation
Quantum Mechanics **Is not Complete** Wave-Particle Duality **Theory**	(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=**75**)+ (9+19)=28)+(14+15+20=49)+(3+15+13+16+12+5+20+5=**89**)+ (23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25= 92) + (20+8+5+15+18+25=91)= **107+75+28+49+89+51+84+92+91 = 666**	of *A Complete **Time Determinism by** **$256=2^8$** cells
God Does Not Play A Dice with the Universe **Entanglement Interior**	(7+15+4=26)+(4+15+5+19=43)+(14+15+20=49)+(16+12+1+25=54)+ (1+4+9+3+5=22)+(23+9+20+8=60)+(20+8+5=33)+(21+14+9+22+5+18+9+5=113)+ (5+14+20+1+14+7+14+5+13+5+14+20=148)+(9+14+20+5+18+9+15+1=108) = **26+43+49+54+22+60+33+113+148+108 = 666**	
***A Complete Time** **Determinism**	(1+3+15+13+16+12+5+20+5=80)+**(20+9+13+5=47)+** **4+5+20+5+18+13+9+14+9+19+13=129) = 80+47+129=256**	& *Classical Mechanics is given by **5 Bits**
Scattering of 2 Fermions into 1 Boson has Incomplete Missing Particle	(19+3+1+20+20+5+18+9+14+7=116)+(15+6=21)+2+(6+5+18+13+9+15+14+19=99)+ (9+14+20+15=58)+1+(2+15+19+15+14=65)+(8+1+19=28)+ (9+14+3+15+13+16+12+5+20+5=102)+ (13+9+19+19+9+14+7=90)+(16+1+18+20+9+3+12+5=84) = 116+21+2+99+58+1+65+28+102 +90+84= **666**	**(5x31=155) Forming Platonic Icosahedron dual of a dodecahedron.** 
Lie-Admissible **Penetration**	(12+9+5=26)+(1+4+13+9+19+19+9+2+12+5)=93) + +(16+5+14+5+20+18+1+20+9+15+14=137) = **26+93+137=256**	

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SIGNIFICANCE FOR THE EPR ARGUMENT OF THE NEUTRON SYNTHESIS FROM HYDROGEN AND OF A NEW CONTROLLED NUCLEAR FUSION WITHOUT COULOMB BARRIER

Simone Beghella Bartoli
Hadronic Technologies Inc.
35246 U.S. Hwy. 19 North, #215
Palm Harbor, FL, 34684, U.S.A.
sibebart@gmail.com

Abstract

In this paper, we present studies on the most fundamental nuclear fusion in nature, the synthesis of the neutron from the hydrogen atom in the core of stars which literally allows stars to "turn on". Such a synthesis is an important confirmation of the EPR argument on the lack of completeness of quantum mechanics (and of its wave-function) due to impossibility by quantum mechanics to describe the synthesis of the neutron, since the mass of the neutron is "bigger" than the sum of the masses of the proton and of the electron, as well as for other technical insufficiencies. For this reason, R. M. Santilli [1] proposed in April 1978 the "completion" of quantum mechanics into an axiom-preserving but non-unitary form which he called hadronic mechanics. Via hadronic mechanics it was possible to achieve a numerically exact representation of "*all*" characteristics of the neutron in its synthesis from the hydrogen at the non-relativistic and relativistic levels. This representation is a fundamental starting point for the description of other syntheses which are again impossible to be correctly represented by quantum mechanics, such as the compression of an electron inside a neutron to synthesize a "pseudo-proton" and of an electron pair in singlet coupling inside a natural deuteron to synthesize a "pseudo-deuteron". The importance of such particles is given by their natural ability to win the Coulomb barrier and be attracted instead of being repelled by positively charged nuclei. Considering that at the mutual distance of 1 fm two deuterons experience a repulsive force of 230 N, turning this repulsive force into an attractive one has enormous implications, such as the realization of the fusion of two deuterons into a Helium nucleus without the need to supply any energy in order to win the Coulomb barrier. Santilli has additionally shown that, as it is the case for the neutron, negatively charged pseudo-proton and pseudo-deuteron are evidently unstable, yet they have mean lives of the order of seconds, thus being sufficient for industrial applications. It is evident that that the synthesis of pseudo-protons and pseudo-deuterons is prohibited by quantum mechanics, while being allowed by its "completion" into hadronic mechanics, thus showing the importance of the original argument by Einstein, Podolsky and Rosen. See [2], [3] and [4] for an extensive treatment of the problem.

1. Historical notes

Stars initiate their lives as being composed of hydrogen. They grow in time via the accretion of interstellar-intergalactic hydrogen, and eventually reach such pressures and temperatures in their core to synthesize the neutron as a "compressed" hydrogen atom according to the model proposed by H. Rutherford.

In Santilli's view, the synthesis of the neutron from the hydrogen in the core of stars is one of the best illustrations of the validity of the EPR argument because said synthesis cannot be represented with quantum mechanics for numerous reasons.

First of all, there is the positive binding energy required by the proton and electron in order to synthesize a neutron, that has an excess mass with respect to the sum of the masses of proton and electron. Second, the contact interaction of the two particles implies a situation of mutual overlapping of their wavepackets, which is deeply non-local and non-Hamiltonian, therefore can NOT be represented by a potential, making Quantum Mechanics inapplicable in these conditions.

Dr. Ernest J. Sternglass was the first known performer of a synthesis of the neutron from hydrogen gas, during his Master's thesis at Cornell University, NY, in 1951 using an electric arc running through an X-ray tube filled with hydrogen. He was in contact with the late Einstein, who showed great interest in this kind of experiments, and encouraged him to pursue his research. The experiment was repeated independently by Edward Trounson, a physicist at the Naval Ordinance Laboratory in 1952 with similar results. Then again the Italian priest physicist don Carlo Borghi repeated similar experiments in Brazil in the 60's. [5]

These early tests on the neutron synthesis were rejected by the scientific community, namely, for the impossibility of deriving strong interactions from a bound state between the electron and the proton. Those researchers experienced indifference and ostracism from the academic community, and none of them ever tried to publish papers on their experiments, we only have private correspondence or diary entries as an historical proof of their existence. The lack of clear neutron detections, due to the lack of proper technical instruments at the time, contributed to the dismissal of the experiments.

2. Hadronic Mechanics

In September 1977 soon after joining Harvard University, R M Santilli was requested by the DOE to study possible new approaches to the controlled nuclear fusion. He accepted under the condition that he could study first the most important fusion in nature, that of the neutron from the hydrogen in the core of stars.

He soon discovered the impossibility of describing this phenomenon via conventional Quantum Mechanics, for the above stated reasons, and he understood that he needed to get back to the bases if he wanted to treat this problem seriously.

He understood that the above insufficiencies originate from the theory at the foundation of Quantum Mechanics, Lie's theory, because said theory solely admits Hamiltonian linear interactions, which cannot possibly happen in conditions of mutual overlapping.

By recalling the fundamental character of Lie's theory, it follows that the mathematics itself underlying quantum mechanics does not allow a consistent representation of nuclear fusions and other physical or chemical energy releasing processes, due to their known irreversibility over time.

Santilli had therefore to conceive a new mathematics, which he called isomathematics, that is able to achieve a "completion" of 20th century mathematical and physical methods for extended, deformable and hyperdense particles in interior dynamical conditions.

This allowed him to propose in April 1978 the "completion" of quantum mechanics into an axiom preserving but non unitary form which he called hadronic mechanics with the first known formulation of the "operator" from the Lie admissible "completion" of Heisenberg's equation [1].

The need to verify Pauli's principle in the interior of hadrons expressed in the title of Harvard's paper [1] originated from the prediction, implied by the EPR argument, that hyperdense hadronic matter alter the conventional spin of particles. In fact, the spin of an electron in the core of stars is expected to be different than its value in vacuum due to the extreme pressures in all directions, provided that the electrons is represented as an extended wavepackets via the isotopic element T of Eqs. (4.15.49) because, when particles are represented as point-like, no resistance can possibly be experienced.
Note the suggestion, also in the title of paper [1] to verify the validity of special relativity in the interior of hadrons because a "completion" of relativistic quantum mechanics for the structure of hadrons implies the "necessary" completion of special relativity into a covering form which were studied in the monographs written at Harvard University [6], [7].

Ref. [1] achieved a representation of all characteristics of mesons as hadronic bound states electrons and positrons, but in 1978 Santilli could not achieve a quantitative representation of the spin of the neutron in its synthesis from the hydrogen due to the need for a detailed study of the isotopies of spin 1/2 which he did in subsequent years.

Following the study of the isotopies of spin and angular momentum in various papers, [8], [9], [10], Santilli was finally able to achieve an exact representation of all characteristics of the neutron in its synthesis from the hydrogen both at the nonrelativistic [11], [12] and at the relativistic level [13], [14].

3. The synthesis of the neutron

Following, and only following a theoretical understanding of the neutron synthesis, Santilli initiated systematic experimental tests, which achieved the first known actual detection of neutrons synthesized from the hydrogen, thanks to the availability of various neutron detectors. [15], [16], [17]

The synthesis of the neutron in a submerged electric arc, according to Hadronic Mechanics, starts with the separation of the hydrogen molecule into H atoms, followed by the ionization of the H atoms and the consequential creation of a plasma composed by protons and electrons. Then said protons and electrons are aligned along the tangent to a local magnetic line with opposite charges, opposing magnetic polarities and opposing spins. This condition is followed by Rutherford's "compression" of protons and electrons, one against the other, caused by the disconnection of the rapid DC discharge that generated the indicated plasma of protons and electrons. If the energy provided by the arc is enough to supply the missing 0.782 MeV, we obtain an actual Neutron. A specific reactor was built by Thunder Energies Corporation (now Hadronic Technologies Inc.) for industrial applications of this process. See [18] for more details.

More systematic tests were then conducted, with the participation of other scientists, that confirmed the possibility of achieving this synthesis in a laboratory, and produced more interesting data [19]. See also lecture [20] for more details.

4. The pseudoproton and its applications

With the same mechanism as the synthesis of the neutron, Hadronic Mechanics predicts the possibility of synthesizing other particles. One of the most interesting is the negatively charged pseudoproton. The pseudoproton is predicted to be generated by the "compression" of an electron, this time, inside the neutron.

The particle so generated has, among its most noticeable characteristics, similar rest energy as the neutron, spin 1/2 and of course negative charge. The industrially significant feature of negatively charged hadrons is that they are attracted by normal nuclei, instead of being repelled. Although it is not possible at the present moment to have a direct experimental verification of the existence of the pseudoproton, mainly because there are no specifically designed detectors available, it is still possible to have a very significative indirect verification via nuclear transmutations of elements. After many tests, using a modified General Motors 14 kW Electric Generator (in order to reduce the neutron interferences and achieve higher pressures in the ignition chambers), Santilli managed to achieve such a verification from two different laboratories that analyzed irradiated and non-irradiated silver samples. [21]

Besides the great scientific interest, these pseudoprotons have also possible industrial applications, for example in the medical field for the treatment of cancer. Currently some tumors are treated with proton irradiation, but the protons are rejected by the atomic nuclei, so this treatment is invasive and with a low efficiency. The irradiation of tumors with pseudoproton rays would have clear advantages because, unlike protons, pseudoprotons are attracted by tumor nuclei, thus requiring low irradiation energies as well as focusing the treatment only in the tumor area reducing collateral damage.

5. The pseudodeuteron and its applications

Another even more interesting product of these syntheses is the pseudo-deuteron. It can be obtained from a deuterium gas, with the "compression" of two electrons inside a normal deuterium nucleus, or deuteron. When the electric discharge happens, a local plasma of deuterium nuclei and electrons is created, especially in singlet coupling form. When the spark is disconnected, some of the deuterons and of the electron valence pairs are compressed one inside the other, forming an isodeuteron nucleus with expected mean life around 1 s. This isodeuteron nucleus is then naturally attracted by the other normal deuterons, with an attractive force that is inversely proportional to the square of the distance, making nuclear fusion simply unavoidable. This process is known as *Santilli Hyperfusion.*

Calculations show that at 1 fm distance two deuterium nuclei experience a repulsive force of about 230 N. This repulsion is called Coulomb barrier, and it's the main reason why, after decades of studies (and billions of dollars of investments) using Quantum Mechanics and the Standard Model to find an efficient way to bring two deuterons one close to the other and make fusion happen, we still have no clear results of a reaction with positive energetic balance, meaning that the energy spent to make the fusion happen has always been bigger than the energy obtained by the fusion itself. The isodeuteron, being negatively charged, is able to turn that repulsive force into an attractive one, with consequences of enormous scientific and industrial importance.

Besides, having opposite magnetic moments with the deuterons, pseudodeuterons are predicted to be able to naturally achieve a singlet coupling configuration, fundamental for the fusion to happen properly, while with normal deuterons this condition happens at random, limiting even more the efficiency of the process. See lecture [22] for more details

6. Conclusions

The synthesis of the neutron achieved by the Directional Neutron Source (DNS) developed by Hadronic Technologies and R. M. Santilli is by itself a big proof of the inability of Quantum Mechanics to provide a complete description of the physical reality, along the objections posed by A. Einstein B. Podolsky and N. Rosen.

The synthesis of negatively charged particles is yet another confirmation of the EPR argument, that has enormous implications for mankind and shows the importance of looking beyond the applicability conditions of QM and investigating reality with basically new perspectives.

Hadronic Mechanics proves to be a valid completion of the quantum mechanical wavefunction into the isowavefunction permitted by the admission of contact non-Hamiltonian interactions due to deep wave overlapping.

This completion, which is only one of the possible completions of commonly accepted Physics, opens the door to applications beyond our present imagination and deserves to be studied for all the possible benefits that it could bring to our society.

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COMPLETENESS IS UNFALSIFIABLE: GODEL AND POPPER FOR THE EPR DEBATE/KUHN AND THE STANDARD MODEL

E.T.D. Boney
boneye@alum.mit.edu

Abstract

We first discuss the relevance of Gödelian incompleteness to the standard of completeness in the EPR argument, following similar arguments as prior work. We note that Einstein was not a dispassionate observer of the developments in Quantum Mechanics when he made his pronouncement about God and dice: he had tiffs with Schrödinger and Hilbert in the development of General Relativity.

We suggest that completeness is unfalsifiable, while incompleteness is falsifiable. Thus while new variables can be considered within varied theoretical frameworks, searching for 'completeness' itself would always suggest additional variables, even when data suggest none. We consider the ramifications for ideas of completeness in Popperian epistemology.

Then turning to Kuhn, we consider the nature of the paradigm shift about to take place in the Standard Model of quantum physics by examining current failings from within the paradigm. We suggest the solution can be found in negative mass antimatter, following prior work. We note that negative mass antimatter (with positive Energy and no symmetry breaking) resolves many of the concerns with the Standard Model, most notably dark matter and dark energy, but including also a reformulation of the CKM model of quarks and an update to the arrow of time for antimatter in Feynman diagrams. We examine several candidates for the crucial experiment that will cause this paradigm shift, primarily at CERN.

I. INTRODUCTION

Completeness is at issue in the EPR paper, indeed from the very title.[1] However, it is unclear the motivation one has for completeness.

One has the historically well-known result about Heisenberg uncertainty,[2] to which most of the completionist arguments have been posed, but we explore in this paper two more important challenges to Einstein's famous concerns: what about Gödelian incompleteness, and is completeness falsifiable?

II. GÖDELIAN INCOMPLETENESS

One imagines along with Einstein a situation where, through additional variables, it is possible to 'complete' quantum mechanics, i.e. to eliminate Heisenberg uncertainty. This still leaves the greater concern of Gödelian incompleteness.[3] If we have used both algebra and Boolean algebra, we have the possibility of statements such as 'this statement is false', which are undecidable, rendering the space of statements incomplete (not all of them are true or false).

It appears this issue is being name-checked by Einstein in the paper, but is never seriously addressed. Even his supposed completion of quantum mechanics would remain incomplete in its logic, unless this concern is addressed.

III. COMPLETENESS IS UNFALSIFIABLE

Falsifiability is the bedrock of scientific epistemology, particularly as understood by Popper.[4] This is the main concern with completeness as a motivation for further examination: it never ends.

There is no amount of evident incompleteness that would satisfy a person ideologically committed to completeness. Every failed attempt at completion only changes the parameters necessary. For instance, the Standard Model currently predicts a neutron electric dipole moment (EDM) too small to measured.[5] However, this was not the case initially, with 90 percent of theoreticans predicting it should have been measured,[6] and then shifting their theories when it was not observed. Expectations shifting past experimental goalposts is the trademark of an unfalsifiable theory, and yet most people persist in the pretense that the Standard Model is falsifiable, despite its history.

One wonders where Einstein has not: is completeness falsifiable? What would such evidence look like, that could not be handled by shifting goalposts, or inventing new ones? Perhaps as much as we as humans enjoy control over every variable, completeness is not a part of nature. It seems this is suggested by the Gödelian nature of logic systems.

But moreover, we must consider the Popperian view of completeness.

Theories can only be part of the scientific universe if they pass this basic muster. As unfalsifiable, completeness is a philosophical longing of humankind, not a scientific hypothesis. A muse for our most creative minds. An inspiration. But not a hypothesis. Not a theory.

IV. INCOMPLETENESS IS FALSIFIABLE

This point is considerably more subtle, and we had not thought of it by the time of the lecture, initially granting that my point of view (incompleteness is ok) was also unfalsifiable, rendering completeness a matter of philosophical choice.

However, we have come to the conclusion that this is not correct. Incompleteness can be disproven by a correct and complete theory.

Take for instance Gödelian Incompleteness. One could imagine building a logic system without such concerns, with a series of rules perhaps. That would be a falsification of Gödel, and a limiting of its domain of application. However such constraints are unknown to the author.

Similarly for uncertainty in position, momentum, energy or time. We have precise lab equipment. We are able to measure these well. If the uncertainty principle has been breached, it would be easy to tell.

That's part of what made uncertainty such a strong hypothesis in the first place: it was not expected.

Incompleteness is both falsifiable and generally where we have been forced because complete theories, with predictions within the realm of experiment, have been falsified, morphing into complete theories with predictions just out of reach of experiment (since the core program of completeness-insistence is beyond question).

We suggest that, because of its falsifiability status, incompleteness is the only hypothesis (between completeness and incompleteness) that meets the criteria of scientific. That this is so because completeness disproves incompleteness seems trivial until one considers that incompleteness does not disprove completeness. Perhaps it seems unfair, like we are holding completeness to a higher standard, but falsifiability is the standard we accept regardless of modern epistemology choice.

V. AFTER FALSIFICATION: PARADIGMS SHIFTING

Another way one can view the falsification of a major theory is through the lens of Kuhn.[7] As the theory is falsified, the paradigm must shift. This proceeds eventually through a crucial experiment, but before that, through the breakdown of the paradigm.

The crucial experiment is meant to be the one that no one can explain and forces the paradigm to shift. But here we are a decade after the Wilkinson Microwave Anisotropy Probe (WMAP) first suggested it,[8] and instead of a new theory, we are just making up dark matter and dark energy to plug our 96 percent empty understanding of the universe. But when a theory does as poorly as ours does, we suggest even naming these things relative to the current theory can be a mistake. We have suggested calling these 'two pieces of evidence of the failure of the Standard Model', since the vacuum catastrophe renders the current model ridiculous for even the dark energy / cosmological constant piece.

Beyond dark matter and dark energy, we have more evidence that the Standard Model has failed, but it has all been bent by human willpower into one standard theory (whose claim to fame is not being falsifiable, but being 'consistent with observation', an unscientific measure). Other evidence of failure:

The lack of symmetry breaking in nearly all of observed re-

ality. With the exception of meson CP violation, we do not observe the preferential creation of matter over antimatter. Symmetry breaking is another unfalsifiable hypothesis designed to patch the Standard Model. And why exactly are we making the entire universe match our local matter antimatter ratio? We have called this the Anthropic Principle, but we would suggest instead calling it the Myopic Principle (or Heliocentric Universe). We do not need to insist the entire universe be like it is here in order for us to exist!

Hyperinflation. Besides dark energy, besides dark matter, we need still another patch to deal with the very early universe. That patch is neat, but unmotivated. Three strikes... and the Standard Model somehow remains at bat!

CP violation / strong CP problem. Even if we wanted to cling to the wild idea that all the matter in the universe were created by the CP violation asymmetry in mesons, we would then be confronted with a huge issue: the lack of quark oscillations in the nucleus. Another falsification requiring another patch to remain intact.

Antimatter moving backwards in time. Feynman had a beautiful mind. Nobody would question this, least of all me. But the idea of antimatter moving backwards in time is wild and does not pass basic Ockam's razor muster. It would be quite surprising to a layman that the Standard Model contains lots of stuff moving backwards in time. No, there's never been an experimental observation of this, nor is there a falsifiable aspect of the proposal, why do you ask? (Remember that the requirement is a result of assumptions about the energy and mass of antimatter, which just may not hold).[9]

Gravitons Not Normallizable. Unlike some, we are not comfortable living with infinites.[10] Wavefunctions must be normallized, including that of the graviton.

VI. NEW PARADIGM: NEGATIVE MASS ANTIMATTER

As a brief aside, we think it's incredible that the Standard Model predicts the exact same energy levels for antimatter molecules as for matter molecules (and this has been experimentally confirmed for anti-Hydrogen). [11] Completely incredible.

However, this suggests the light from far away stars cannot be taken as evidence of the matter or antimatter content of those stars without further corroborating evidence. This suggests we can only take our material adventures as a species for this evidence of matter antimatter ratio, which are needless to say much more limited. To be specific, we've been about 70 or so Astronomical Units from here (69 times the distance from the Earth to the Sun!). And immediately after leaving our solar system... Voyager terminated transmission.

Let's take a step back and marvel at the heft of this assumption! We have only been less than 100 times as far from Earth as the Earth is from the Sun, but what is that in real astronomically relevant units? Well, galaxy clusters are typically 1-10 Mpc diameter (and, given the clustering, might be taken as the largest gravitationally bound units). Otherwise our galaxy is about 30 kpc in diameter, so beyond there we can't really say for sure.

So, how far is the 70 Astronomical Units travelled by Voyager 1. relative to these galactic and cluster diameters? 0.00000034 kpc. And from that experience, we have chosen to extrapolate, not just to the 30 kpc of our galaxy, not just to the Mpc of our cluster, but to the hundreds of Mpc of the entire universe?! (An aside on neutrinos from nearby supernovae, which one could argue extend this knowledge to around 1Mpc,[12] but this would require them to be Dirac not Majorana, in which case they also would be indistinguishable.)

And so we have found the culprit at the core of our assumptions: that symmetry must be broken and antimatter must have positive gravitational mass like matter. It is simply ridiculous to require all of the universe to be like it is so close to here.

This then suggests a pretty obvious candidate for dark matter and dark energy: negative mass antimatter. By this we mean that, relative to our commonly observed matter, antimatter repulses, rather than attracts, matter. For this author, this idea was suggested by combining the above idea about light from antimatter galaxies being indistinguishable with the suggestion from Feynman that this is the behavior expected to be mediated by the Graviton (where its spin-2 nature suggests opposites repel).[13]

It is quite remarkable that such a small change to the Standard Model can account for ALL of the concerns in the prior section! Let us review them one by one:

Dark Energy This is no longer some mysterious cosmological constant or contemptible absolute field, but rather a result of the gravitational repulsion of the rest of the universe which, since it is far away, is roughly constant over galactic distances. Notably, no vacuum catastrophe either.

Dark Matter This is the result of misinterpretation of current data assuming only positive mass allowed. The lumpiness is roughly the projection of the rest of the universe onto our galaxy.

Symmetry Unbroken We no longer have the problem of pair creation and annihilation, because we instead suppose no net Mass to the universe. Symmetry forever unbroken. Every galaxy cluster a hole in the universe compensated by the rest.

Hyperinflation This is the result of protoclusters being closer in the past than they are now. So in the past they were flung apart at first very rapidly (as soon as whatever force holding them together was overwhelmed), since the force goes inversely with distance apart.

CP Violation / strong CP problem This is particularly interesting, since we have a paper from ML Good that actually predicted CP violation in the K meson if its antimatter quark had negative mass![14] Years before it was observed! So why did the hypothesis never receive serious consideration, instead leading to the development of CKM theory? ML Good did not have the astronomical model, and so was forced to introduce an absolute gravitational field, which had not been en vogue since Einstein called it his greatest embarassment. Of course we do not form it as an absolute field, and note that the ultrafine splitting suggested by ML Good would be due instead to the sign of the mass of the rest of the universe outside our cluster (which is the same as the antiquark, but opposite of the quark).

Antimatter moves forward in time Since negative mass changes the relationship between energy and time,[9] Feynman diagrams no longer require antimatter to move backwards in time for a consistent propagator.

Gravitons Normallizable. Now that we have added the assumption that there is no net mass to the universe, we can define for any galaxy or cluster a radius of zero net mass. The integral around this surface then allows the normallization of the graviton, which in turn makes Quantum Field Theory a Grand Unified Theory. Hooray!

Unfortunately so far few are celebrating. But we do have a surprising number of experiments that should be able to distinguish the matter in the next few years,[15–17] and some dozens of theorists that have independently introduced models containing the idea (see my talk slides for a more comprehensive list).[18–23]

Even though WMAP failed to be a crucial experiment despite finding the Standard Model 96 percent wrong, we suggest that antimatter falling up is too far outside the Standard Model to receive the kid glove treatment of dark energy (which has led a renaissance of interest into the cosmological constant, despite Einstein's own embarrassment at the idea). One notes Einstein even originally described his cosmological constant as a uniform background of negative mass density, in response to a concern of Schrödinger's... almost 20 years before the EPR paper and decade before the famous God and dice comment!

Maybe Einstein wasn't mad at God playing dice, so much as who he was playing with.

VII. CONCLUSIONS

We have examined completeness, the idea at the core of the EPR concern, and suggested that as a hypothesis it is unfalsifiable. We have noted that incompleteness and uncertainty seem to be the empirical result as well, in addition to being plainly falsifiable.

We have then considered the ramifications of a paradigm shift in the Standard Model, and why we think negative mass antimatter will be the new paradigm. I am currently placing an open wager on the result of these CERN experiments. When at least one reports a gravitational mass of antimatter to 5 sigmas, I would like to have some nice scotch, or owe somebody a nice scotch. Any takers, just email boneye at alum dot mit dot edu .

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NONLOCALITY, ENTANGLED FIELD AND ITS PREDICTIONS, SUPERLUMINAL COMMUNICATION

Yi-Fang Chang
Department of Physics
Yunnan University
Kunming, 650091, China
yifangch@sina.com

Abstract

From the EPR prediction, the nonlocality and entangled state become the frontline in modern physics. First, we introduce briefly some researches on EPR, which include Santilli's studies. Next, based on the generalized Lorentz transformation (GLT) with superluminal in the complete special relativity, we propose that the entangled states must obey GLT because of they possess the superluminal and some characters as the spacelike vectors. Further, it changes the phase of the entangled field, whose phase particle (phason) has some characters and corresponding equations. It is tachyon, and assume that it is similar to photon and J=1 and m=0 or mass is very small as similar neutrino, and may show the action at a distance. We research that this field as wave has some characters. Third, we discuss the superluminal quantum communication by a pair of entangled states is generated on both positions, or by preparing and transmitting a pair of entangled instruments, so the superluminal quantum communication. Manipulation for one position can pass the same message or information to the other, so we may implement the superluminal communication. Finally, assume that the entangled field has a similar magnetic theory, which may be a quantum cosmic field, or be the extensive quantum theory, or God or the Buddha-fields and so on. These are all macroscopic fields, which correspond to de Broglie-Bohm nonlinear "hidden variable" theory, but it is microscopic. In a word, study and application of nonlocality and entangled field have important scientific and social significance.

Keywords: nonlocality, entangled field, superluminal, special relativity, generalized Lorentz transformation, quantum communication, prediction.

I. Introduction

Based on the Einstein-Podolsky-Rosen (EPR) correlations and Bell inequalities, current new experiments validated that quantum mechanics possesses the nonlocality and entangled state, etc.

First, Aspect, et al., realized EPR experiment by the measure on the linear-polarization correlation of pairs of photons emitted in a radiative cascade of calcium and time-varying analyzers, and it agrees with the quantum mechanical predictions and the greatest violation of generalized Bell inequalities [1,2]. Ghosh and Mandel demonstrated the existence of nonclassical effects in the interference of two photons [3]. Further, the entangled state evolves a great hotspot in physics. Kavassalis and Noolandi discussed a new view of entanglements in dense polymer systems, which predict a geometrical transition from the entangled to the unentangled state in agreement with experimental data [4]. Horne, et al., discussed two-particle interferometry, which employs spatially separated, quantum mechanically entangled two-particle states [5]. Mermin discussed extreme quantum entanglement in a superposition of macroscopically distinct states [6]. Hardy investigated nonlocality for two particles without using inequalities for all entangled states except maximally entangled states such as the singlet state [7]. Goldstein provided a proof on Hardy theorem [8]. Kwiat, et al., reported new high-intensity source of polarization-entangled photon pairs with high momentum definition [9]. Strekalov, et al., reported a two-photon interference experiment that realizes a postselection-free test of Bell inequality based on energy-time entanglement [10].

In 1998 Santilli published a paper showing that the objections against the EPR argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. Now Santilli reviewed, upgraded and specialized the basic mathematical, physical and chemical methods for the study of the EPR prediction that quantum mechanics is not a complete theory. This includes basic methods [11], apparent proof of the EPR argument [12], and examples and applications, in which the validity of the EPR final statement is the effect that the wavefunction of quantum mechanics does not provide a complete description of the physical reality. The axiom-preserving "completion" of the quantum mechanical wavefunction due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature [13]. Moreover, Santill studied the classical determinism of EPR prediction by isomathematics [14].

Pan, et al., reported experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger (GHZ) entanglement, and three specific experiments, involving measurements of polarization correlations between three photons, lead to predictions for a fourth experiment, and found the fourth experiment is agreement with the quantum prediction [15]. Stefanov, et al., investigated the quantum correlations with spacelike separated beam splitters in motion and experimental test of multisimultaneity [16]. Pan, et al., demonstrated experimental entanglement purification for general mixed states of polarization-entangled photons and arbitrary unknown states [17]. Yu, et al., discussed a test of entanglement for two-level systems via the indeterminacy relationship [18]. Zhao, et al., used two entangled photon pairs to generate a four-photon entangled state, which is then combined with a single-photon state, and reported experimental demonstration of five-photon entanglement and open- destination teleportation (for N = 3) [19].

Amico, et al., reviewed the properties of the entanglement in many-body systems [20]. Korbicz, et al., shown structural approximations of positive maps and entanglement-breaking channels [21]. Orus discussed geometric entanglement in a one-dimensional valence-bond solid state [22]. Schmidt, et al., detected entanglement of a mechanical resonator and a qubit in the nanoelectromechanical systems [23]. Thomale, et al., investigated the entanglement gap separating low-energy in the entanglement spectrum of fractional quantum Hall states, and a new principle of adiabatic continuity [24]. Salart, et al., reported the first experiment where single-photon entanglement is purified with a simple linear-optics based protocol [25]. Chavez, et al., observed entangled polymer melt dynamics [26]. Jungnitsch, et al., provided a way to develop entanglement tests with high statistical significance [27]. Huber, et al., detected high-dimensional genuine multipartite entanglement of mixed states [28]. Sponar, et al., discussed the geometric phase in entangled systems for a single-neutron interferometer experiment [29]. Friis, et al., investigated relativistic entanglement of two massive particles [30]. Jack, et al., measured correlations between arbitrary superpositions of orbital angular momentum states generated by spontaneous parametric down-conversion, and quantified the entanglement of modes within two-dimensional orbital angular momentum state spaces [31]. Bussieres, et al., tested nonlocality over 12.4 km of underground fiber with universal time-bin qubit analyzers [32]. Mazzola, et al., investigate the dynamical relations among entanglement, mixedness, and nonlocality in a dynamical context [33]. Miao, et al., discussed universal quantum entanglement between an oscillator and continuous fields [34]. Chitambar, et al., considered multipartite-to-bipartite entanglement transformations and polynomial identity testing [35]. Carmele, et al., discussed the formation dynamics of an entangled photon pair [36].

Many experiments on the quantum entangled state shown some new characteristics: 1). The coherency. 2). The nonlocality [7,13,16,32-34]. 3). The quantum teleportation [17,19]. 4). The superluminal [16]. In this paper, we propose that the entangled states must obey relativity with the superluminal and its quantum theory, and discuss some predictions and the superluminal quantum communication, etc.

2. Complete Special Relativity

Based on the basic principles of the special relativity, according to the constancy of the velocity of light in the vacuum principle, it implies the invariance of the squared interval:

$$s^2 = r_{mn}^2 - c^2 t_{mn}^2 . \qquad (1)$$

From this we derived necessarily two symmetrical types of topological separated structures, i.e., the classification of the timelike and the spacelike intervals, and obtained simultaneously the Lorentz transformation (LT) with smaller velocity v<c and the generalized Lorentz transformation (GLT) with larger velocity $\overline{v}$ >c [37-39].

It is well-known that the Lorentz transformation (LT) is:

$$x_1' = \gamma(x_1 - vt), t' = \gamma(t - vx_1 / c^2), \qquad (2)$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$. GLT is [37-39]:

$$x_1' = \overline{\gamma}(x_1 - c^2 t / \overline{v}), t' = \overline{\gamma}(t - x_1 / \overline{v}), \qquad (3)$$

where $\overline{\gamma} = 1/\sqrt{1-(c/\overline{v})^2}$.

In deriving LT, an additional independent hypothesis has been used, thus the values of velocity are restricted absolutely, and the spacelike interval is excluded. LT and GLT are connected by the de Broglie relation $v\overline{v} = c^2$.

Further, based on the special relativity principle, an invariant speed c_h is necessarily obtained. Therefore, the exact basic principles of the special relativity should be redefined as: I. The special relativity principle, which derives necessarily an invariant speed c_h. II. Suppose that the invariant speed c_h in the theory is the speed of light in the vacuum c. If the second principle does not hold, for example, the superluminal motions exist, the theory will be still the extensive special relativity, in which the formulations are the same, only c is replaced by the invariant speed $c \to c_h$. The fundamental properties of any four-vector and the strange characteristic of these tachyons are described. We discussed various other superluminal transformations and their mistakes.

We think that LT is unsuitable for photon and neutrino, the photon transformation (PT) is unified for space $x'=r+ct$ and time $t'=t+(r/c)$. It may reasonably overcome some existing difficulties, and cannot restrict that the rest mass of photon and neutrino must be zero. LT, GLT and PT together form a complete structure of the Lorentz group. If the invariant speed c_h are various invariant velocities, the diversity of space-time will correspond to many worlds [37,39]. Moreover, it may prove [39] that the local Lorentz transformations for different systems cannot derive the varying speed of light (VSL) theory searched warmly [40-42]. VSL is probably connected only with the general relativity.

3. Entangled Field and Its Predictions

The earliest entangled state originated from the induction and unification between men and nature in the Chinese traditional culture. Now we research some possible theories of the entangled states and corresponding predictions.

3.1. Entangled Relativity and Predictions

Since the quantum entangled state possesses some characters, for example, coherency, nonlocality and superluminal, etc., we propose that it may and must apply the complete special relativity (CSR) and GLT.

From Eq.(1) we derive $s^2=r_{mn}^2-c^2t_{mn}^2$>0 for the spacelike interval, the speed defined as $|v|=|r_{mn}/t_{mn}|$>c is always superluminal. We may choose an inertial frame that $t'_{mn}=0$ (the simultaneity), so that calibration time. In this case $|\overline{v}|=\infty$, i.e., the action at a distance. But $r'_{mn}=0$ (at the same space position) cannot be obtained [37-39]. Its mass should be $m=m_0/\sqrt{1-(c/\overline{v})^2}$.

Further, the quantum entangled states are related with space-time, which form the entangled fields. Their chance are the superluminal phase velocities, and as the spacelike vectors possess some fundamental characteristics in any four-vectors [37,39]:

(P; E/c)	$(j;c\rho)$	(A; φ)	(k; ω/c)	(dk; dω/c)	$(w_\alpha=\frac{\gamma}{c^2}\frac{d(\gamma v)}{dt};w_0=\frac{\gamma}{c}\frac{d\gamma}{dt})$
P=mv>E/c	$j=\rho v>c\rho$	$A>\varphi$	$k>\omega/c$	dk>dω/c	$w_\alpha<w_0$

Here P is momentum, and E is energy, etc. Because $\omega/k=\overline{v}$ is phase velocity, (k; ω/c) is usually a timelike vector with $\overline{v}$>c. While dω/d k=v is group velocity, so (dk; dω/c) is usually a spacelike vector with c>v. Usual the timelike and spacelike intervals are two topological separated parts by the light-cone.

In the timelike vectors, only x, p, j, A, k, dk and w_α can be zero, and then LT is derived. In the spacelike vectors, only t, E, $\rho, \varphi, \omega, d\omega$ and w_0 can be zero, then GLT is derived. In this case φ=0, but A≠0; $\rho = 0$, but $j \neq 0$, etc. These are some predictions based on CSR. Mariwalla [43] let E=0, so GLT of the four-vector (p; E/c) was derived.

For any four-vector ($\vec{A}; A_0$), its LT is

$$A_1' = \gamma(A_1 - vA_0 / c), A_0' = \gamma(A_0 - vA_1 / c), \quad (4)$$

and GLT is

$$A_1' = \bar{\gamma}(A_1 - cA_0 / \bar{v}), A_0' = \bar{\gamma}(A_0 - cA_1 / \bar{v}). \quad (5)$$

Both possess the most perfect symmetrical form. Only A_1, A_0 interchange each other between A_1 and A_0 representations, and LT (4) and GLT (5) also interchange from v/c to c/$\bar{v}$.

The entangled field is in the spacelike interval, so the area is larger and the possibility is more.

3.2. Entangled Quantum Theory and Predictions

The quantum representations on the entangled fields are known, in which two basic spin states are quantized $|\frac{1}{2}>$ and $|-\frac{1}{2}>$, and group of two particles are (s^A, s^B), so there are four eigen-states:

$$|\frac{1}{2}>_A|\frac{1}{2}>_B, \ |\frac{1}{2}>_A|-\frac{1}{2}>_B, \ |-\frac{1}{2}>_A|\frac{1}{2}>_B, \ |-\frac{1}{2}>_A|-\frac{1}{2}>_B. \quad (6)$$

While the coupling represents are:

$$SX_{SM} = MX_{SM} \ (S = s^A + s^B). \quad (7)$$

For S=0 and M=0, $X_{00} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B - |-\frac{1}{2}>_A|\frac{1}{2}>_B]$ is single state.

For S=1 and M=0, ± 1, $X_{10} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B + |-\frac{1}{2}>_A|\frac{1}{2}>_B]$, $X_{11} = |\frac{1}{2}>_A|\frac{1}{2}>_B$ and $X_{1-1} = |-\frac{1}{2}>_A|-\frac{1}{2}>_B$ are threefold states. Here X_{00}, X_{10} are two entangled states. X_{11}, X_{1-1} carry through equal weight superposition, it may compose four entangled states:

$$|\psi^\pm>_{AB} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B \pm |-\frac{1}{2}>_A|\frac{1}{2}>_B], \quad (8)$$

$$|\phi^{\pm}>_{AB}=\frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|\frac{1}{2}>_B \pm|-\frac{1}{2}>_A|-\frac{1}{2}>_B]. \qquad (9)$$

This derives the non-locality, and is similar to the Bell basis in quantum mechanics. It is the entangled state of quantum theory, and may describe quantum teleportation [44].

The entanglement is probably a new field, and exchanges tachyon or phase particle (phason), which corresponds to change of phase. They each other are the phase velocities. Its character is tachyon, and assume that it is similar to photon and J=1 and m=0 or mass is very small as similar neutrino. It shows the action at a distance.

Assume that the entangled field has the wave-particle duality, from which we propose its quantum theory: They are bosons, and based on the same energy-momentum relation:

$$E^2 = p^2c^2 + m^2c^4, \qquad (10)$$

we derive Klein-Gordon equation of quantum mechanics:

$$\frac{\partial^2\psi}{\partial x_\mu^2}-\frac{m^2c^2}{\hbar^2}\psi = 0. \qquad (11)$$

When m_0 =0, it is Maxwell equation. The Schrödinger equation is:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\hbar^2\nabla^2\psi + V\psi. \qquad (12)$$

It agrees with the complete relativity, and is a quantum theory of the complete relativity.

Quantum entanglement corresponds possibly to the nonlinear superposition principle [35,45]. These are also some predictions based on the similar quantum theory of entangled fields.

Maudlin researched the quantum nonlocal entanglement and special relativity in modern physics [46,47].

3.3. Entangled Waves and Predictions

The entangled fields possess the wave property. It is known that two entangled fields can interference each other. They should diffract from each other, or even reflect. Especially combined quantum theory it will have the barrier penetration.

At the same time, wave may further develop to the field. It may combine the mechanical wave theory [48]. Such Schrödinger equation may develop to following nonlinear equations:

$$\frac{\partial\psi}{\partial t} + \alpha\psi\nabla\psi = \frac{i\hbar}{2m}\nabla^2\psi; \qquad (13)$$

and a similar KdV equation:

$$\frac{\partial\psi}{\partial t}+\sigma\psi\frac{\partial\psi}{\partial x}=\frac{\partial^3\psi}{\partial x^3}. \qquad (14)$$

From Boussinesq equation and Klein-Gordon equation we may develop similarly to the nonlinear equation:

$$\phi_{tt}-\phi_{xx}=\sigma(\phi^2)_{xx}+\phi_{xxxx}. \qquad (15)$$

Further, many phase spaces exist probably in our world, for example, body-mind-spirit and they are entangled each other. The extensive quantum entanglement is a real "spooky action at a distance" (Einstein). Entangled states in parapsychology can explain synchronization, telepathy as resonance of the thought field [37,49]., and the unification of men and nature, etc. In quantum mechanics, the participant in the Wheeler interpretation is the unification of men and nature. Combined animism, it can be further explained prediction, premonition and other phenomena of parapsychology. The same frequency of the thought field is easy to synchronize. Quantum entanglement among living things produces their synchronization and magic special functions.

4. Superluminal Quantum Communication

First, Bernett, et al., proposed the quantum teleportion via dual classical information and nonclassical EPR channels [50]. Then Bouwmeester, et al., investigated experimental quantum teleportation [44]. Pan, et al., realized experimentally entangled freely propagating particles that never physically interacted with one another or which have never been dynamically coupled by any other means. It demonstrates that quantum entanglement requires the entangled particles neither to come from a common source nor to have interacted in the past [51].

Raimond, et al., performed manipulating quantum entanglement experiments with Rydberg atoms and microwave photons in a cavity, and investigated entanglement as a resource for the processing of quantum information, and operated a quantum gate and applied it to the generation of a complex three-particle entangled state [52]. Pan, et al., experimentally demonstrated observation of highly pure four-photon GHZ entanglement. Their technique can, in principle, be used to produce entanglement of arbitrarily high order or, equivalently, teleportation and entanglement swapping over multiple stages [53]. Zbinden, et al., reported an experimental test of nonlocal quantum correlation in relativistic configurations, in which entangled photons are sent via an optical fiber network to two villages near Geneva, separated by more than 10 km where they are analyzed by interferometers [54].

It already is widely applied, for example, quantum information [52], quantum swapping [53], quantum non-cloning and so on. Energy-time entangled photon pairs violate Bell inequalities by photons more than 10.9 km

[54] and 12.4 km [51]. At present, some physicists proposed that their entangled distance is infinite, and even is an action at a distance. I think, the quantum entangled state is probably a new fifth interaction [39]. Its strength seems to obey neither the Newtonian long-range gravitational law nor the short-range strong-weak interactions.

Cocciaro, et al., searched superluminal quantum communications on recent experiments and possible improvements. Some physicists (Bell, Eberhard, Bohm and Hiley) suggested that quantum correlations could be due to superluminal communications (tachyons) that propagate isotropically with velocity v>c. For finite values of v, quantum mechanics and superluminal models lead to different predictions. Some years ago a Geneva group and Cocciaro group did experiments on entangled photons to evidence possible discrepancies between experimental results and quantum predictions [55]. But, so far, no evidence for these superluminal communications has been obtained and only lower bounds for the superluminal velocities have been established. Cocciaro, et al., described an improved experiment that increases by about two orders of magnitude the maximum detectable superluminal velocities. No evidence for superluminal communications has been found and a higher lower bound for their velocities has been established [56].

Gao Shan analyzed the relation between quantum collapse, consciousness and superluminal communication. Quantum collapse as result of quantum nonlocality may permit the realization of quantum superluminal communication (QSC). He demonstrated that the combination of quantum collapse and the consciousness of the observer will permit the observer to distinguish nonorthogonal states in principle. This provides a possible way to realize QSC [57]. He introduced a possible mechanism of nonlinear quantum evolution and investigated its implications for quantum communication, so it is shown that the distinguishability of nonorthogonal states can be used to achieve quantum superluminal communication, which must exist based on the quantum nonlocal influence [58]. Reversely, Zhang analyzed the relation and the difference between the quantum correlation of two points in space and the communication between them, and proved the impossibility of the superluminal quantum communication from statistical separability [59].

Walleczek, et al., discussed the apparent conflict between quantum mechanics and the theory of special relativity, and nonlocal quantum information transfer without superluminal signalling and communication [60].

So far, it is generally believed that the entangled states do not transmit information, but affect each other instantly. We suppose that it is similar to electromagnetic field, and may apply to the superluminal communication. In a word, entanglement seems to be a particular synchronism.

A pair of entangled states is generated on both positions, or preparing and transmitting a pair of entangled instruments, so the superluminal quantum communication can be realized. Thus encoding the states (different phases) to form two codes: yes or no, ↑ (positive) or ↓ (opposite), both correspond to 1 or 0. Manipulation for one position can pass the same message or information to the other, so we may implement the superluminal communication. Further, it develops to the corresponding information theory.

Its important basis is that Enrique Galvez, et al., proposed a complete set of instruments to generate entangled photons in the laboratory, and the experimental process into a manual placed on the network [61].

Moreover, the exact quantum communication seems to be inconsistencies with the quantum non-cloning theorem [62]. Further, Barnum, et al., proposed the quantum non-broadcasting theorem [63].

5. Applications, Tests and Other Predictions

Musser searched the spooky action at a distance as the phenomenon that reimagines space and time, and what it means for black holes, the Big Bang and theories of everything [64]. He proposed that the nonlocality exists widely in black holes, the cosmic macrostructure and particle collisions [64]. Jacques, Kaiser and Peruzzo, et al., realized delayed-choice experiments [65-67].

Quantum theory is reversible and localized. Reversible black holes have radiation, so entropy decrease [68-71]. The nonlocal black holes have information overflow. General black holes are only come into and no leave to all v<c material. But, for $\overline{v}$>c black holes may generate information. Both are linked by $v\overline{v}=c^2$. Giddings, et al., discussed black holes, quantum information, unitary evolution and observables in effective gravity [72,73].

We propose that the entangled fields may be developed by a similar magnetic field.

First, the magnetic induction seems to be transient transmission, in which A≠0, but φ =0. It presupposes that there should be a large external field similar to the geomagnetic field. This may be a quantum cosmic field, whose wave function of Universe obeys the Wheeler-de Witt equation:

$$(\hbar^2 G_{ijkl}\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}}+\sqrt{G^3}R)\psi(g)=0\,. \qquad (16)$$

Here $G_{ijkl}=\frac{1}{2\sqrt{G}}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl})$. This may also be the extensive quantum theory [74-76], the mysterious natural field on the induction between men and nature, God or the Buddha-fields [77,78] which correspond to real world is computer simulation or computational universe [79,80], three dimensional truth-goodness-beauty space [81] or body-mind-spirit space [82] and so on. These are all macroscopic fields, which corresponds to the Gaia hypothesis as a intertwined complex whole, and to de Broglie-Bohm nonlinear "hidden variable" theory [83,84], but it is microscopic. In fact, any order society can form a gauge field with customs and law to regulate standards of conduct for all.

If a similar magnetic field exists, it will be the rotation field, whose equations are:

$$\oint_S BdS = 0, \text{ and } \nabla B = 0. \qquad (17)$$

Such it may be used and analogous to communication. Probably, these fields are the origin of the entangled field, the entanglement is only a result.

Further, we should try to find a magnetic monopole. If we find a similar charge, the theory will be developed to the similar electrodynamics, in which Maxwell equations are:

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0, \qquad (18)$$

$$\frac{\partial F_{ik}}{\partial x_k} = \frac{4\pi}{c} j_i. \qquad (19)$$

Lorentz equation is:

$$\frac{dp}{dt} = e(-grad\varphi - \frac{\partial A}{c\partial t} + \frac{v}{c} \times rotA) = e(E + \frac{v}{c} \times H). \qquad (20)$$

Third, we may develop a similar magnetohydrodynamics. Fourth, combining quantum mechanics, it derives a similar quantum electrodynamics (QED). Fifth, combining general relativity, it derives the electromagnetic general relativity [85].

Discover these fields and their corollaries are also our predictions. The general predictions include the telepathy and the induction between men and nature, etc.

In a word, study and application of nonlocality and entangled field have important scientific and social significance.

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COMPARISON OF VARIOUS NUCLEAR FUSION REACTIONS AND ICNF

Indrani B. Das Sarma
Department of Applied Chemistry
Jhulelal Institute of Technology
Lonara, Nagpur-441 111, India
dassarmaindrani@rediffmail.com

Abstract

Modern day demand of clean, cheap and abundant energy gets fulfilled by the novel fuels that have been developed through hadronic mechanics / chemistry. In the present paper, a short review of Hadronic nuclear energy by intermediate controlled nuclear synthesis and comparison with other fusion reactions has been presented.

Introduction

Atomic nucleus and sub-nuclear particles have always been considered an unlimited source of energy. The discovery of nuclear fission by Otto Hahn and Fritz Strassmann paved the way for conventional nuclear energy. However, nuclear fission generates large amount of nuclear waste that risks ecosystem whereas nuclear synthesis is known to create much less pollution, thus is green. It is also comparatively more inexhaustible energy source. Hence, harnessing energy through nuclear synthesis reactions has been so far the Holy Grail. With the discovery of stellar nucleo-synthesis by Hans Bethe paved the way for nuclear synthesis of two or more light nuclei into a heavier nucleus. Of course, the energy released in this process could be harnessed.

The energy conversion from thermonuclear fusion reaction is marred by very low energy gains of the thermonuclear reactions. The energy input was larger than output obtained, hence was not economically feasible. Cold fusion on the other hand does not have sufficient energy to bring about fusion reaction in a sustained way.

With advent of ultra-short pulse laser technology, low temperature initiation of fusion even at the high plasma density can be materialized. This technology has allowed fusion of hydrogen-boron for low-cost fusion energy. However, amongst them ICNF does have an upper-edge as with the Hadronic mechanics, the processes taking place are easier to understand and hence more reproducible. Hadronic mechanics is of paramount importance for understanding nuclear synthesis as in this case nucleus cannot be considered as point mass.

Intermediate Controlled Nuclear Fusion (ICNF)

Intermediate Controlled Nuclear Fusion (ICNF) as proposed by Prof. Santilli are systematic energy releasing nuclear syntheses. The reaction rate is controllable via one or more mechanisms capable of performing the engineering optimization of the applicable laws.

Basic assumptions of Hadronic mechanics as proposed by Prof. Santilli are-

i) Nuclear force: Nuclear force is partly represented by a Hamiltonian and partly by the non-potential type terms that is the latter cannot be represented with a Hamiltonian.
ii) Stable nuclei: According to Heisenberg-Santilli Lie-isotopic equations the sub-nuclear particles are in contact with each other (technically, in conditions of mutual penetration of about 10^{-3} of their charge

distributions). Consequently, the nuclear force is expected to be partially of potential and partially of nonpotential type, with ensuing nonunitary character of the theory, and related applicability of hadronic mechanics.

iii) Unstable nuclei and nuclear fusion: In case of Heisenberg-Santilli Lie-admissible equation (1) for the time evolution of a Hermitean operator A, in their infinitesimal and finite forms

$$i\frac{dA}{dt} = (A \hat{,} H) = ARH - HSA \qquad (1)$$

where Hermitean, H represents non-conserved total energy; the genotopic elements R and S represent non-potential interactions. Thus, irreversibility is assured.

Irreversibility is assured in this case by the different values of the genounit for forward (f) and backward (b) motions in time by equation (2)

$$I^{>} = 1/R \neq {}^{<}I = 1/S \qquad (2)$$

Lie-admissible branch of hadronic mechanics is ideally suited to represent the decay of unstable nuclei and also nuclear synthesis, since both are irreversible over time.

iv) Neutron synthesis: Neutron is assumed to be compressed hydrogen atom (as originally conjectured by Rutherford) as shown by reaction (i).

$$p^{+} + a + e^{-} \rightarrow n \qquad (i)$$

where 'a' is Santilli's etherino (It represents in a conventional Hilbert space transfer of 0.782 MeV and spin ½ missing in the synthesis of neutron from the environment to the neutron structure.)

Don Borghi's experiment and Santilli's hadronic mechanics appropriately explains the Rutherford's conjecture of neutron as a compressed hydrogen atom.

Thus, the CNF is governed by Santilli's laws of controlled nuclear synthesis:

- ➢ The orbitals of peripheral atomic electrons are controlled such that nuclei are systematically exposed.
- ➢ CNF occurs when nuclei spins are either in singlet planar coupling or triplet axial coupling.

- The most probable CNF is those occurring at threshold energies and without the release of massive particles.
- CNF requires trigger, an external mechanism that forces exposed nuclei to come in femto-meter range.

The CNF has been realized using magnecules. The magnecules have systematic and controlled exposure of nuclei which have singlet planar or triplet axial coupling. In case of ICNF, proposed by Prof. Santilli energy supplied is of threshold value just sufficient to expose the atomic nuclei from within the electron cloud. Since the energy is not very high the production of ionizing radiations or sub-nuclear particles are avoided. The reaction is carried out in sealed tanks called hadronic reactors.

HyperCombustion

Hypercombustion is combination of Magnecular Combustion and controlled nuclear fusion. Initially the fossil fuels are ignited with a series of rapid DC discharges, each having at least 100 kV and at least 100 J known as HyperSparks. This convert fossil fuels from their natural molecular to a magnecular form which enhances the combustion temperature, magnecular conversion and the energy output via the fusion of Carbon-12 and Oxygen-16 into Silicon-28. This reduces emission of carbondioxide (green house gas) and also enhances energy output due to additional fusion reaction as compared to only the chemical energy released in case of molecular combustion. Since the fusion is ICNF there is release of no radioactive contaminants either, making the process green.

HyperFusion

Pseudo-protoid, an intermediate state prior to the full synthesis of the pseudo-proton is given by a bound state in singlet coupling of an electron and a neutron under the strongly attractive contact interactions of iso-mechanics, this is given by the equation-

$$n_{\uparrow} + e_{\downarrow}^{-} \rightarrow \tilde{p}$$

HyperFusions are fusions of natural, positively charged nuclei and synthesized negatively charged nuclei.

E.g. pseudo-deuteron and a deuteron into the helium

$$\widehat{D}\ (-1, 2, 1_{\uparrow}) + D\ (1, 2, 1_{\downarrow}) \rightarrow He\ (2, 4, 0) + 2e^{-}$$

The energy released by each hyperfusion is

$\Delta E = E_{He} - (E\,\widehat{D} + E_D) = 23.8 \text{ MeV} = 3.81 \times 10^{-12} \text{ J}$

It is estimated that 10^{18} controlled fusions per hour would yield the significant release of about 10^6 J of clean energy per hour without harmful radiations or waste.

Low Energy Fusion

It was first reported by Fleishmann, Pons and Hawkins in 1989, popularly called as cold fusion as it takes place at room temperature. The major drawback was non-reproducibility by other laboratories. This could be due to insufficient energy required to expose the atomic nuclei from within the covering atomic electron cloud. Difficulty in obtaining required triggering mechanism within the lattice of the metal crystal structure may have been encountered.

High Energy Fusion

It is reported by various laboratories, basically trying to mimic thermonuclear reactions taking place in stars. Hence popularly called as hot fusion. The reactions are not self-sustaining and compound nucleus undergoes fission leading to formation radioactive wastes. Atomic electron clouds are completely stripped off. Kinetic energy of the nuclei is increased to overcome coulombic barrier and the energy attained by the compound nucleus is generally higher than the fission barrier which results in fission reaction or nuclear decay as prominent exit channels.

The advantages of Hadronic fusions are-

1. **Aneutronicity of the reaction:** Aneutronic fusion is a nuclear fusion reaction without formation of neutrons. The majority of the energy released is released in form of charged particles. The charged particles like protons or alpha particles are easy to handle and can be directly used to convert to electricity. This reduces problem related to neutron radiation such as ionizing damage, neutron activation and requirements for biological shielding, remote handling and safety.

   E.g:

   $$C(12,6,0^+,12.0000) + D(2,1,1^+,2.0141) + TR \rightarrow N(14,7,1^+,14.0030) + \text{Heat}$$
   $$\Delta E = 0.0111 \text{ amu} = 0.339 \text{ MeV}$$

   where TR is trigger mechanism (high voltage DC arc).

2. **Hybridization with conventional molecular combustion:** ICNF can be amalgamated with the conventional molecular combustion by using engineering innovations 'Hyper Combustion'. These can be realized by using 'Hyper Furnaces' which is advanced version of Hadronic Reactors. This would drastically decrease the environmental impact as the combustion would be more complete. The global warming is more problematic due to hydrocarbons formed on incomplete combustion of fossil fuels. Also the amount of CO_2 generated per unit energy output would be less. Moreover magnecular combustion by itself is known to have better energy output as compared to conventional molecular combustion.

Conclusions

ICNF and HyperFusions are more promising than hot or cold fusion in terms of reproducibility and energy input to output ratio. The successful achievement of ICNF with industrial relevance depends on the proper selection of the hadronic fuel. The original and final nuclides are light, natural and stable isotope. The nuclear fusion causes no emission of ionizing radiations. The energy produced ΔE is much bigger than the total energy used by the equipment for its production. ICNF relies on magnetic properties of the precursor where as HyperFusion relies on the overcoming the coulombic barrier by the opposite electrical charge. Production of negatively charged nuclei holds promising applications in other fields such as medicines, etc. Thus, it can be concluded that hadronic nuclear energy is truly green.

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INAUGURAL LECTURE

Jeremy Dunning-Davies
Department of Mathematics and Physics (Retd.)
University of Hull, England
and
Institute for Basic Research
Palm Harbor, FL, U.S.A.
masjd@masjd.karoo.co.uk

Abstract

This opening lecture for this very important conference is intended to describe very briefly the background to the major topic to be discussed, as well as both highlighting the possibly changing scientific times in which we find ourselves and raising one or two speculative thoughts which might lead to further work in the not too distant future. There can be little doubt in the minds of most open-minded scientists that science in general and physical science in particular face several problems which are normally hidden from view. Many of these, but by no means all, are involved with the issue of uncertainty and it is this which forms the basis for the larger part of what will follow in these proceedings. The well-known Einstein-Podolsky-Rosen article will be central as will the, until now, little known resolution of the problems raised by that article for the world of science. It will be shown how recent events might indicate a possible change in the attitude towards criticism of some widely accepted results as well as towards some slightly more unconventional explanations for phenomena which have not, in reality, been afforded truly watertight explanations up to the present. As far as this latter point is concerned, the possibly provocative idea of openly reintroducing an aether into the physical description of events will be mooted.

Ladies and Gentlemen,

May I first express my sincere thanks for the generous invitation to present this opening talk for this somewhat unusual conference – unusual because of its format caused by the coronavirus outbreak. However, although the format may be unusual, that cannot detract from the extreme importance of the contents of what is to follow these general opening remarks. There can be little doubt to all open-minded people that science in general, and physical science in particular, faces several oft hidden problems. Many of these, although by no means all, are involved with issues of uncertainty and it is with this issue that the larger part of this conference will be concerned. In a totally unrelated area of physics, changes seem to have occurred recently which might indicate some hope that more open-mindedness has entered into that area and, therefore, the possibility of a change of attitude in other spheres of endeavour might have arisen also.

In 1988, together with a colleague, I published a letter in a well-respected journal in which the validity of the Bekenstein-Hawking expression for the entropy of a black hole was questioned. The follow-up article detailing the entire argument was, however, rejected and subsequently my colleague and I encountered real difficulties in having articles accepted for publication in front line journals. However, thirty years later, shortly after Hawking's death, I was contacted by that same original journal to referee an article. I did so, as much out of curiosity than anything and found it to be a piece of work dealing, amongst other things, with the aforementioned entropy expression. Consequently, I roundly criticised the submitted article in my report – not out of any sense of pique but because I genuinely believed it to be incorrect. The article was rejected for publication. Shortly afterwards I was asked, by the same journal, to referee another paper on a totally different topic. Again I did so and my recommendations were accepted and followed through exactly. The end result was that a few months later I received an award as a Referee of the Year! Was it a coincidence that, after thirty years, but following Hawking's death, I seemed to have been accepted back into the fold? I accept that all this could be an almost unbelievable coincidence but is it just possible that it is an indicator of a change in philosophy of at least some in the hierarchy that appears to control so much in the physics community? If so, now could well be exactly the right time to push for a true open-minded examination of at least some of the major problems facing modern day science and which are the fundamental topics of this conference.

The question of uncertainty affects many areas, including my own special interest of thermodynamics, although, in that case, the affect may be felt indirect. For a moment consider the situation in thermodynamics. In traditional classical thermodynamics there are no uncertainties; all the variables, for example the internal energy and total number of particles, possess definite values. However, when systems composed of a large number of particles are to be considered, the methods of statistical mechanics have to be employed due to our present state of knowledge. As a consequence, when incorporated into thermodynamics, the realm known as statistical thermodynamics is entered. This is, in some crucial ways, totally different from classical thermodynamics because the introduction of statistical techniques has introduced uncertainty into the picture. No longer are there definite values for the internal energy or total number of particles; rather average values are considered. These average values, as with the average values of other thermodynamic variables, can fluctuate in this new regime. Hence, a degree of uncertainty is introduced which leads to the derivation of thermodynamic uncertainty relations. It is important to note, though, that these relations have been introduced via the recourse to statistical methods to describe details of the system under consideration. They have been introduced because, in a system composed of a large number of particles, it is not possible to write down all the equations of motion of the individual particles, let alone solve the resulting set of simultaneous equations. The uncertainty, therefore, has been introduced as a result of our inability to solve the exact problem; there is no inherent uncertainty in the original system. This reasoning follows for all statistical thermodynamic theories and indicates a very real difference between classical and statistical thermodynamics.

Indeed, the same reasoning may be seen to apply to many, if not all, problems considered utilising probability theory. For example, in introducing probability, it is popular to consider the tossing of a coin. If the coin is simply tossed, the outcome when it lands – head or tails – is totally uncertain. However, this is not so if someone is in possession of all the initial conditions pertaining to the toss. If the initial speed is known, the height to which the coin rises may be found, as may the time taken to reach that height. Similarly, the time taken to fall back to a given level may be found. If the rate of rotation is also known, that, together with the total time of flight, should enable the state of the coin on reaching the desired final level to be ascertained. Hence, the uncertainty associated with this problem really arises through a lack of knowledge of the initial conditions in the problem; it is not an inherent property of the actual system.

It may be seen, therefore, that neither statistical thermodynamics nor probability may be termed complete theories in the sense that neither provides exact solutions to problems. In both, uncertainty is introduced as a result of the inability to write down and solve a set of exact equations and/or a lack of knowledge of initial conditions.

Recent rereading of some books on quantum mechanics would seem to indicate a similar situation existing in that branch of physics as well. For example, in Heisenberg's well-known book *The Physical Principles of the Quantum Theory*, the initial derivation of the uncertainty relations relies on an obvious approximation which might raise a few minor queries but the slightly later, more rigorous, derivation draws on notions from probability. Indeed the ideas of probability are closely associated with the wave function as is seen from discussions of Schrodinger's equation and its wave function. Once probability enters any discussion an element of uncertainty must follow in the subsequent theory. Hence, one must wonder if the uncertainty relations of quantum mechanics are a product of the theory rather than a natural property of the systems the theory is purporting to portray? This, of course, is highly reminiscent of the situation already mentioned as occurring in statistical thermodynamics. However, the very fact that probabilistic ideas enter the subject at all must surely indicate that the theory cannot be complete? Here the idea of a theory being complete is intended to indicate that the theory is capable of describing any relevant physical system exactly without any degree, however slight, of uncertainty. That may, or may not, be the precise notion put forward in the famous Einstein-Podolsky-Rosen article but that is the meaning adopted here so far and, in that sense, neither statistical thermodynamics nor quantum theory may be adjudged complete.

Some might well feel that at least some of my comments so far – if not all of them – are a little naïve, even childlike. However, I would remind everyone that two quotations from the Bible might seem appropriate at this juncture. We might be reminded of the quote from St. Paul's Epistle to the Corinthians where he says 'when I became a man, I put away childish things'. However, it seems we might take note also of Jesus's comment that one needs to become like a little child if one wishes to enter the Kingdom of Heaven It seems to me that it's just possible that scientists could learn something from these two quotes if taken together. Over the years, science has become more and more dependent on more and more advanced abstruse mathematics and maybe all scientists should stand back a little and reflect, rather than rushing blindly on using methods and results authenticated by 'conventional wisdom' but not necessarily by common sense. Maybe we should return to some childlike thinking. I would say I don't feel this mild criticism applies to the methods of Hadronic Mechanics, although there is a huge amount of new mathematics to absorb in that field but, when you become used to the new notation, that mathematics is not too difficult to comprehend; - unlike some of the modern additions to accepted conventional theory, several of which seem to be attempting to transport us to some mythical land of make-believe!

As far as the Einstein-Podolsky-Rosen, or EPR, ideas are concerned, it is worth noting that questions about the completeness of quantum mechanics as a physical theory have been discussed at length ever since that famous, some might be tempted to say infamous, paper first appeared. Many experiments were carried out in attempts to both prove and disprove the assertions contained therein and a great deal of thought went into the theoretical investigations of such as Bell. All the references to this work may be found in the collected papers by Bell on quantum philosophy, which may found in *Speakable and Unspeakable in Quantum Mechanics* as published by Cambridge University Press. Less well-known is the resolution of the paradox advanced by Santilli in 1998 and it is the lack of publicity for this work which poses a significant question for the scientific community. Although, when you read even just the abstract for that paper, maybe some answers become apparent. With talk of such concepts as nonlinear, nonlocal, non-canonical, axiom-preserving isotopies and spin-isospin symmetry and iso-spaces, some will be put off by the implied effort to understand properly what follows in the body of the paper, while others will dismiss the work out-of-hand because it depends crucially on concepts unfamiliar to them. This may be a totally improper attitude towards proposed new science but many will have forged impressive curricula vitae based on what they regard as well-established concepts and procedures and will be reluctant to jeopardise their personal positions. Hence, the huge question for the scientific community - when do we agree to examine with a truly open mind, radical new proposals for help in solving age-old problems? It seems there was no difficulty in examining and accepting a wide range of results from Riemannian geometry, as well as the uncertainties introduced by quantum mechanics, into physics and chemistry some one hundred years ago, so why not afford the same respect to Hadronic Mechanics or are the fundamental results of quantum mechanics to remain sacrosanct even when they don't answer all the important questions facing the scientific community?

These are vitally important questions in general but are particularly apposite when considering the so-called EPR paradox and work related to it. Basically, the EPR claims that quantum mechanics is an incomplete theory because its description of physical reality does not include all elements of reality, while every element of physical reality should be precisely represented in a complete theory. Santilli's new approach has important consequences as far as the EPR argument is concerned. Traditionally, commuting quantities are believed to be independent but, in the so-called iso-topic completion of quantum mechanics, iso-commuting quantities can be mutually interacting, although it should be understood that such interactions are structurally different from those of action-at-a-distance/potential type. Fundamentally, quantum mechanics may be considered an incomplete theory in that it does not contain the element of reality given by the nonlocal structure of interactions expected from the mutual wave overlapping. Hadronic mechanics overcomes this problem.

It is important to realise though that, as Santilli himself points out, hadronic mechanics is not intended to represent all elements of reality; it is not meant to be a final theory. Physics is, after all, a discipline which will never admit final theories. Hadronic mechanics simply provides one type of completion of quantum mechanics – that of axiom preserving type. It might also be noted at this point that Santilli has also shown via his new mathematics that von Neumann's theorem on hidden variables is quite simply inapplicable under isotopies – note, not violated, but inapplicable! He has also established that the oft-quoted Bell's inequality is not valid universally but holds for the conventional form of quantum mechanics specifically.

Recently, of course, the matter has resurfaced with the announcement of experimental results supporting the EPR assertions at Basel. This has provoked further contemplation of this whole issue of completeness and just what it really means. The Basel team noted that the phenomenon dated back to thought experiment of 1935 and that it allowed measurement results to be predicted precisely but, of course, it must be remembered always that thought experiments are just that – thought experiments – and such are very difficult to interpret due to the assumptions made not always being totally clear, possibly not even to the originators themselves. In fact, in a purely thought experiment, it is easy to imagine a situation where a fundamental assumption is made with no-one realising that has occurred. Remember that we all indulge in thought experiments – some even when we are asleep – but their true validity only becomes apparent when we have ceased our contemplation and committed our thoughts to paper and resulting concrete scrutiny. Supposedly, the essence of a good practical experiment is that it should be readily repeatable. It is relatively easy to see how this could be true, but could equally well be untrue, of any thought experiment. Hence, in my personal view, important results derived via thought experiments should always be treated with extreme care. Nevertheless, as far as the thought experiment leading to the EPR paradox is concerned, it is one which has been viewed and examined over a large number of years and, seemingly, has always led to a genuine paradox in physics.

Basically, via a thought experiment, Einstein, Podolsky and Rosen showed that precise predictions are possible theoretically in certain circumstances. Briefly, such a notion may be explained as follows:- they considered two systems in an entangled state in which their properties are strongly correlated. In this case, the results of measurements on one system may be used to predict the results of corresponding measurements on the second system with arbitrary precision in principle. It was also the case that the two systems could be separated spatially. The resulting paradox is that an observer may use measurements on the first system to make more precise statements about the second system than an observer who has direct access to that second system but not the first.

The Basel team used lasers to cool atoms to a small fraction of a degree above the absolute zero of temperature. At such low temperatures, the atoms are thought to behave completely according to the rules of quantum mechanics and form a Bose-Einstein condensate. In this ultra-cold cloud, the atoms collide with one another constantly, causing their spins to become entangled. The researchers involved then took measurements of the spin in spatially separated regions of the condensate. By using high-resolution imaging, they were able to measure the spin correlations between the separate regions directly and simultaneously localise the atoms in precisely defined positions. Hence, in this experiment, the researchers seem to have succeeded in using measurements in a given region to predict precisely the results for another region.

Experimental physics is certainly not my forte; in fact, I've not been directly involved in that area since my undergraduate days. Hence, I don't know if any serious objections to this work by the Basel team have surfaced since I read of their claims. If such have emerged, the argument over the validity of the EPR paradox will, no doubt, rumble on. If none has, or does, emerge than it is conceivable that a new era for physics might be opening up since it is surely the case that applications will follow which we all hope will be of benefit to mankind rather than the opposite.

As a follow-up to these comments, it might be worth raising the question of the presumed boundary between classical and quantum mechanics. Precisely when is something small enough to warrant the use of quantum mechanics to describe it? Is this boundary clear cut or does the transition evolve over what might be thought of as a blurred region in which either or both apply? I confess this seemingly simple point is one I have never seen discussed but is one that has preyed on my mind for years with no apparent resolution in the offing. It might be wondered if the reintroduction of an aether could help in the resolution of this and possibly many other difficulties encountered in modern physics. For example, the uncertainty in the position and speed of a very small particle could be accounted for by the presence of a boundary layer between the said small particle and the aether. It is certain that, if the existence of an aether is true, then such a boundary layer must exist and, if the ideas put forward by C. Kenneth Thornhill concerning an aether are valid, then the size of aether particles would be extremely small and small in comparison with the size of recognised elementary particles. Obviously this situation would not apply so obviously to macroscopic bodies because their individual size would far outweigh that of the proposed aether particles. The notion of reintroducing the idea of an aether receives some support these days with the renewed interest in some quarters in the work of Nikola Tesla. His writings, as well as those of the myriad major scientists working on problems of, or at least involving, electromagnetic ideas towards the end of the nineteenth century, contain constant references to this medium. It seems we should all be approaching problems with much more open minds and not be guided too rigidly by conventional wisdom. As the saying goes – think outside the box!

These latter points are all speculative thoughts but, nevertheless, thoughts which have materialised over years and lead to questions, at least, which I feel need carefully considered answers in order to serve the cause of the advancement of scientific knowledge well. At this point, at the very beginning of this conference, it might be remembered also that this event, which could prove vitally important to the future in physical science, has come about due to one person – Ruggero Santilli. Most of us know the enormous contribution he has made, a contribution far too extensive to even begin to summarise here. However, one example which has been totally ignored by those in authority deserves mention and that is his proposal of a method for disposing of nuclear waste safely and on site. This proposal needed to be checked independently by approximately three relatively small experiments. Request after request was made for these to be carried out but to no avail. I myself drew attention to this in 2008 at a conference in Monza which included in the audience the then European Commissioner dealing with energy matters. Again nothing transpired. This is just one example of Ruggero Santilli's work but one which, since it has been ignored, could prove costly to mankind. Before closing, I would like to draw your attention to one other small, but I feelsignificant, point he raised many years ago and which serves to illustrate the point I was making earlier about the assumptions we all make in physics. I drew attention to the difficulty all must really experience in thought experiments in remaining totally aware of any, and all, assumptions made at the outset. As I said also, when one comes to write down thoughts on paper, the assumptions made and their consequences, become somewhat clearer but they may never be forgotten. When Einstein proposed his special theory of relativity many years ago, he made an assumption concerning the constancy of the speed of light. Today it is commonplace both in the media and, crucially, in scientific circles as well to hear people claim that 'Einstein said the speed of light is constant'; this is almost a basic statement of modern physics to some. All have forgotten that, as Ruggero Santilli pointed out so clearly several years ago, Einstein's assumption was that the speed of light remained constant in a vacuum. Here Ruggero stressed, via a popular example, a vitally important scientific truth – when you are using, or quoting, a previously derived result in science, check diligently to see what precise assumptions have been made in deriving the said result. Many errors could be avoided so easily if this simple procedure was adhered to strictly.

May I close by expressing the sincere hope that this proves to be an enormously successful conference and one which leads to a more open-minded approach to the solution of the important problems facing twenty-first century science. Finally, on behalf of all participating in this event, I should like to thank Ruggero Santilli, his wife Carla and all his colleagues, working unnamed and unrecognised behind the scenes, for organising it!

Thank you.

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ZUR THEORIE DER q,ω-LIESCHEN MATRIXGRUPPEN

Thomas Ernst
Department of Mathematics, Uppsala University
P.O. Box 480, SE-751 06 Uppsala, Sweden
thomas@math.uu.se

Abstract

Based on the three papers by Hahn 1949, Annaby et. al. 2012 and Varma et. al. 2018, we introduce the matrix of multiplicative q, ω-polynomials of order $M \in \mathbb{Z}$ with the corresponding q-addition. We prove that this constitutes a so-called q, ω-Lie group with two dual q, ω-multiplications. We also show that the corresponding q, ω-Bernoulli and q, ω-Euler matrices form q, ω-Lie subgroups. In the limit $\omega \to 0$ we obtain corresponding formulas for q-Appell polynomial matrices.

Primary 17B37; Secondary 11B68, 33D15

Keywords— q, ω-Lie group; multiplicative q-Appell polynomial matrix; Hahn–Pascal matrix

ZUSAMMENFASSUNG. Basierend auf den drei Veröffentlichungen von Hahn 1949, Annaby et. al. 2012 und Varma et. al. 2018, führen wir die multiplikative q, ω-Polynommatrix der Ordnung $M \in \mathbb{Z}$ ein, mit der entsprechenden q-Addition. Wir beweisen, dass dies eine sogenannte q, ω-Liesche Gruppe mit zwei dualen q, ω-Multiplikationen darstellt. Wir zeigen auch, dass die entsprechenden q, ω-Bernoulli und q, ω-Euler Matrizen q, ω-Liesche Untergruppen bilden. Im Grenzwert $\omega \to 0$ erhalten wir entsprechende Formeln für q-Appell-Polynommatrizen.

INHALTSVERZEICHNIS

1. Einführung

Wir stellen einige neue Konzepte für q, ω Polynommatrizen vor, von denen einige vorher nur im q-Fall aus den Artikeln des Autors bekannt waren. Durch die logarithmische Methode für q-Analysis erfolgt dieser Übergang fast automatisch, weil die Addition durch die q, ω-Addition ersetzt wird. In dem Artikel [8] wurden Matrixgruppen mit zwei dualen Multiplikationen eingeführt. Später in [10] wurde bewiesen, dass die sogenannte q-Appell-Polynommatrix-Gruppe ein erstes konkretes Beispiel von q-Lie-Gruppen war. Obwohl wir die q, ω-Addition verwenden, werden die q-Binomialkoeffizienten beibehalten. Stattdessen wird die Potenz von x zu den zwei Hauptfolgen geändert.

In diesem Artikel werden die vorherigen Formeln mit q-Pascal-Matrizen einfach zu sogenannten q, ω-Pascal-Matrizen erweitert. Summenformeln mit der neuen q, ω-Addition können dabei in Matrixform umgeschrieben werden.

Dieser Artikel ist wie folgt organisiert: Im Abschnitt 1 werden die Hauptdefinitionen angegeben.

Der Hauptzweck des Abschnitts 2 ist die Einführung der q, ω-Addition und der multiplikativen q, ω-Appellschen Polynome und Zahlen. Das Umbral-Kalkül wird immer implizit angenommen. Im Abschnitt 3 werden die relevanten Matrizen und die Hauptmatrix q, ω-Differenzengleichung eingeführt. Im Unterabschnitt 3.1 werden die multiplikativen q, ω-Polynommatrizen zur Vorbereitung für die q, ω-Liesche Gruppe, den Hauptzweck dieses Artikels, eingeführt.

Im Unterabschnitt 3.2 wiederholen wir zunächst die Matrixformen der q, ω-Bernoulli und q, ω-Eulerschen Polynome aus [9] zur Vorbereitung für die Berechnung ihrer Zahleninverse.

Sei $\omega \in \mathbb{R}$, $\omega > 0$. Man setze $\omega_0 \equiv \frac{\omega}{1-q}$, $0 < q < 1$. Sei I ein Intervall, das ω_0 enthält. Wir gehen davon aus, dass die Variable x zu I gehört.

Definition 1. Der Endomorphismus ϵ im Vektorraum der Polynome wird definiert durch

$$\epsilon f(x) \equiv f(qx + \omega). \tag{1}$$

Dieser Endomorphismus ist eine Verallgemeinerung des Operators mit demselben Namen im q-Kalkül [5]. In [3, S. 136] ist bewiesen, dass

$$\epsilon^k f(x) = f(q^k x + \omega\{k\}_q), \; k \in \mathbb{N}. \tag{2}$$

Definition 2. [11] Sei φ eine stetige reelle Funktion von x. Wir definieren den q, ω-Differenzenoperator $\mathrm{D}_{q,\omega}$ wie folgt:

$$\mathrm{D}_{q,\omega}(\varphi)(x) \equiv \begin{cases} \frac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{3}$$

Eine Funktion $f(x)$ ist n Mal q, ω-differenzierbar, wenn $\mathrm{D}^n_{q,\omega} f(x)$ vorhanden ist. Wenn wir darauf hinweisen möchten, dass dieser Operator auf der Variable x operiert, werden wir $\mathrm{D}_{q,\omega,x}$ für den Operator schreiben. Weiterhin, $\mathrm{D}_{q,\omega}(K) = 0$, wie für die Ableitung.

Dieser Operator interpoliert zwischen zwei bekannten Operatoren, dem Nørlundschen Differenzenoperator

$$\triangle_\omega[f(x)] \equiv \frac{f(x+\omega)-f(x)}{\omega}, \tag{4}$$

und der Jacksonschen q-Ableitung

$$(\mathrm{D}_q\varphi)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\backslash\{1\}, \; x \neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q = 1; \end{cases} \tag{5}$$

Die folgende Definition erscheint zum ersten Mal.

Definition 3. Ein q, ω-Analogon des mathematischen Objekts G ist eine mathematische Funktion $F(q, \omega)$ mit der Eigenschaft $\lim_{\omega \to 0} F(q, \omega) = G_q$, das q-Analogon von G. Sowohl F als auch G können Funktionen von mehreren Variablen sein. Sie können auch Operatoren sein. Die Funktion $F(q, \omega)$ wird ω-Analogon von G_q genannt.

Satz 1.1. [3, (16), S. 137] *Der q, ω-Differenzenoperator für ein Produkt von Funktionen.*

$$\mathrm{D}_{q,\omega}(fg)(x) = \mathrm{D}_{q,\omega}(f)(x)g(x) + f(qx+\omega)\mathrm{D}_{q,\omega}(g)(x). \tag{6}$$

Bemerkung 1. Diese Formel wird zum Nachweis von (28) verwendet.

Wir führen nun zwei Hauptfolgen ein, die die Ciglerschen Polynome in [5, 5.5] verallgemeinern.

Definition 4.

(7) $$[13,\ (15)]\ [x]_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1} (q^m x + \omega\{m\}_q).$$

(8) $$[13,\ (16)]\ (x)_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1} (x - \omega\{m\}_q),$$

wobei $\{m\}_q$ das q-Analogon von m bezeichnet.

Die beiden folgenden Formeln entsprechen der Formel $\mathrm{D}x^n = nx^{n-1}$:

(9) $$[12,\ 2.5],\ [13,\ (17)]\ \mathrm{D}_{q,\omega}(x)_{q,\omega}^{n} = \{n\}_q (x)_{q,\omega}^{n-1}.$$

(10) $$[13,\ (18)]\ \mathrm{D}_{q,\omega}[x]_{q,\omega}^{n} = \{n\}_q [qx+\omega]_{q,\omega}^{n-1}.$$

Als nächstes führen wir zwei q,ω-Analoga der Exponentialfunktion ein:

Definition 5. Die q,ω-Exponentialfunktion $\mathrm{E}_{q,\omega}(z)$ [13, (21)] wird definiert durch

(11) $$\mathrm{E}_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^{k}}{\{k\}_q!},\ |(1-q)z-\omega| < 1.$$

Die komplementäre q,ω-Exponentialfunktion $\mathrm{E}_{\frac{1}{q},\omega}(z)$ [13, (26)] wird definiert durch

(12) $$\mathrm{E}_{\frac{1}{q},\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q,\omega}^{k}}{\{k\}_q!},\ |\omega| < 1.$$

Wir haben den Namen geändert zu $\mathrm{E}_{\frac{1}{q},\omega}(z)$, weil $\mathrm{E}_{\frac{1}{q},0}(z) = \mathrm{E}_{\frac{1}{q}}(z)$ [5].

Satz 1.2. [13, (19)] *Die q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

(13) $$\mathrm{D}_{q,\omega} f(z) = f(z),\ f(0) = 1.$$

[13, (24)] *Die komplementäre q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

(14) $$\mathrm{D}_{q,\omega} f(z) = f(qz+\omega),\ f(0) = 1.$$

Satz 1.3. [13, (21)] *Die meromorphe Fortsetzung der q,ω-Exponentialfunktion $\mathrm{E}_{q,\omega}(z)$ ist gegeben durch*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega;q)_\infty}{((1-q)z-\omega;q)_\infty}. \tag{15}$$

[13, (26)] *Die meromorphe Fortsetzung der komplementären q,ω-Exponentialfunktion $\mathrm{E}_{\frac{1}{q},\omega}(z)$ ist gegeben durch*

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z+\omega;q)_\infty}{(\omega;q)_\infty}. \tag{16}$$

Korollarium 1.4.

$$\mathrm{E}_{q,\omega}(z)\mathrm{E}_{\frac{1}{q},-\omega}(-z) = 1 \tag{17}$$

2. Die q,ω-Liesche Gruppe von q,ω-Appellschen Polynommatrizen

Wir erweitern zunächst einige Definitionen von [8].

Definition 6. Eine q,ω-Liesche Gruppe $(G_{n,q,\omega,\cdot,\cdot_{q,\omega}}, \mathrm{I}_g) \supseteq \mathrm{E}_{q,\omega}(\mathfrak{g}_q)$ ist eine möglicherweise unendliche Menge von Matrizen $\in \mathrm{GL}(n,\mathbb{R})$ und eine Mannigfaltigkeit mit zwei Multiplikationen: $\cdot$, der üblichen Matrixmultiplikation und der verdrehten Matrixmultiplikation $\cdot_{q,\omega}$, die separat definiert wird.

Jede q,ω-Liesche Gruppe hat eine Einheit, die für beide Multiplikationen mit I_g bezeichnet wird. Jedes Element $\Phi \in G_{n,q,\omega}$ hat eine Inverse Φ^{-1} mit der Eigenschaft $\Phi \cdot_{q,\omega} \Phi^{-1} = \mathrm{I}_g$.

Definition 7. Angenommen, $(G_1, \cdot_1, \cdot_{1:q,\omega})$ und $(G_2, \cdot_2, \cdot_{2:q,\omega})$ sind zwei q,ω-Liesche Gruppen, dann ist $(G_1 \times G_2, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Gruppe mit dem Namen Produkt-q,ω-Liesche Gruppe. Diese Gruppe hat Gruppenoperationen definiert durch

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{18}$$

und

$$(g_{11}, g_{21}) \cdot_{q,\omega} (g_{12}, g_{22}) = (g_{11} \cdot_{1:q,\omega} g_{12}, g_{21} \cdot_{2:q,\omega} g_{22}). \tag{19}$$

Definition 8. Wenn $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Gruppe ist und $H_{n,q,\omega}$ eine nichtleere Teilmenge von $G_{n,q,\omega}$ ist, dann wird $(H_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q,ω-Liesche Untergruppe von $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ genannt, falls

(1)

$$\Phi \cdot \Psi \in H_{n,q,\omega} \text{ und } \Phi \cdot_{q,\omega} \Psi \in H_{n,q,\omega} \text{ for all } \Phi, \Psi \in H_{n,q,\omega}. \tag{20}$$

(2)

$$\Phi^{-1} \in H_{n,q,\omega} \text{ for all } \Phi \in H_{n,q,\omega}. \tag{21}$$

(3) $H_{n,q,\omega}$ eine Untermannigfaltigkeit von $G_{n,q,\omega}$ ist.

Um die folgenden Polynome verwenden zu können, müssen wir die q-Addition verallgemeinern. Die gewöhnliche q-Addition ist der Sonderfall $\omega = 0$. Genau wie bei der q-Addition verwenden wir Buchstaben in einem Alphabet für die q, ω-Additionen. Die Gleichheit der Buchstaben wird mit $\sim$ bezeichnet. Man beachte im Folgenden die Tatsache, dass jeweils die Variable x in $(x)_{q,\omega}^{\nu}$ oder in $[x]_{q,\omega}^{\nu}$ durch die Konstante a multipliziert wird, müssen wir auch ω mit a multiplizieren.

Definition 9. Die NWA q, ω-Addition wird wie folgt definiert:

$$(x \oplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (y)_{q,\omega}^{k}. \tag{22}$$

Die NWA q, ω-Subtraktion wird wie folgt definiert:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (-y)_{q,-\omega}^{k}. \tag{23}$$

Die JHC q, ω-Addition wird wie folgt definiert:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [y]_{q,\omega}^{k}. \tag{24}$$

Die JHC q, ω-Subtraktion wird wie folgt definiert:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [-y]_{q,-\omega}^{k}. \tag{25}$$

Korollarium 2.1. *Eine Erweiterung der Formel* [5, 4.29]

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \{n\}_q (x \oplus_{q,\omega} y)^{n-1}, \ \oplus_{q,\omega} \equiv \oplus_{q,\omega} \vee \boxplus_{q,\omega}. \tag{26}$$

Beweis.

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \sum_{k=0}^{n-1} \binom{n}{k}_q \{n-k\}_q (x)_{q,\omega}^{n-k-1} (y)_{q,\omega}^k = \mathrm{RS}. \tag{27}$$

□

Satz 2.2. *Die Kettenregel für den q,ω-Differenzenoperator.*

$$\mathrm{D}_{q,\omega}\left((ax)_{q,a\omega}^n\right) = a\{n\}_q (ax)_{q,a\omega}^{n-1}. \tag{28}$$

$$\mathrm{D}_{q,\omega}\left([ax]_{q,a\omega}^n\right) = a\{n\}_q [aqx + a\omega]_{q,a\omega}^{n-1}. \tag{29}$$

Beweis. Wir beweisen (28) durch Induktion. Die Formel (28) gilt für $n = 1, 2$. Angenommen, sie gilt für $n-1$. Dann haben wir

$$\begin{aligned} &\mathrm{D}_{q,\omega}\left[(ax)_{q,a\omega}^{n-1}(ax - \{n-1\}_q a\omega)\right] \\ &\overset{\text{durch}(6)}{=} a(ax)_{q,a\omega}^{n-1} + a^2\left[qx + \omega - \{n-1\}_q\right]\{n-1\}_q (ax)_{q,a\omega}^{n-2} \\ &= a(ax)_{q,a\omega}^{n-1}\left[1 + q\{n-1\}_q\right] = \mathrm{RS}. \end{aligned} \tag{30}$$

Die Formel (29) wird in ähnlicher Weise bewiesen. □

Korollarium 2.3. *Vier q,ω-Additionen für die q,ω-Exponentialfunktion.*

$$\mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y). \tag{31}$$

$$\mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y). \tag{32}$$

$$\mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y). \tag{33}$$

$$\mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y). \tag{34}$$

Beweis. Man verwende die Formeln (22) und (24). □

Satz 2.4. *Die q,ω-Differenzen für die q,ω-Exponentialfunktionen sind:*

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{q,a\omega}(ax) = a\ \mathrm{E}_{q,a\omega}(ax), \tag{35}$$

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\ \mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega), \tag{36}$$

Beweis. Dies ergibt sich aus der Kettenregel (28) und (29). □

Satz 2.5. *Die NWA q,ω-Addition ist kommutativ und assoziativ.*

Beweis. Ähnlich dem Nachweis für NWA q-Addition. □

Satz 2.6. *Die JHC q,ω-Addition ist assoziativ. Wir nehmen an, dass alle JHC q,ω-Additionen ganz rechts im Funktionsargument stehen.*

Beweis. Dies ergibt sich aus der Assoziativität der Multiplikation. □

Definition 10. Die Ward-ω-Zahl $\overline{n}_{q,\omega}$ wird definiert durch

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \ldots \oplus_{q,\omega} 1, \tag{37}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Definition 11. Die Jacksonsche ω-Zahl $\tilde{n}_{q,\omega}$ wird definiert durch

$$\tilde{n}_{q,\omega} \sim 1 \boxplus_{q,\omega} 1 \boxplus_{q,\omega} \ldots \boxplus_{q,\omega} 1, \tag{38}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Die allgemeinste Form von Polynom in diesem Artikel ist das q,ω-Appell-Polynom, das wir nun definieren werden.

Definition 12. Sei $\mathcal{A}_{q,\omega}$ die reelle Zahlenfolgen $\{u_{\nu,q}\}_{\nu=0}^{\infty}$, so dass

$$\sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^{\nu}}{\{\nu\}_q!} < \infty, \tag{39}$$

für einen q,ω-abhängiger Konvergenzradius $r = r(q) > 0$, wobei $0 < q < 1$.

Die q,ω-Appellsche Zahlenfolge wird mit $\{\Phi_{\nu,q,\omega}^{(n)}\}_{\nu=0}^{\infty}$ bezeichnet.

Definition 13. Sei $h(t,q,\omega)$, $h(t,q,\omega)^{-1} \in \mathbb{R}[[t]]$. Für $f_n(t,q,\omega) = h(t,q,\omega)^n$ werden die multiplikativen q,ω-Appellschen Zahlen von Grad ν und Ordnung n, $\Phi_{\nu,q,\omega} \in \mathcal{A}_{q,\omega}$ durch die folgende erzeugende Funktion gegeben:

$$f_n(t,q,\omega) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi_{\nu,q,\omega}^{(n)}, \quad \Phi_{0,q,\omega}^{(n)} = 1. \tag{40}$$

Der Bequemlichkeit halber fixieren wir den Wert für $n = 0$ und $n = 1$:

$$\Phi_{\nu,q,\omega}^{(0)} \equiv \delta_{0,\nu}, \quad \Phi_{\nu,q,\omega}^{(1)} \equiv \Phi_{\nu,q,\omega}. \tag{41}$$

Definition 14. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die multiplikative q,ω-Appellsche Polynomfolge $\{\Phi_{\nu;q,\omega}^{(n)}(x)\}_{\nu=0}^{\infty}$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion gegeben:

$$f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu;q,\omega}(x). \tag{42}$$

Definition 15. Der Bequemlichkeit halber fixieren wir wieder den Wert für $n=0$ und $n=1$:

$$\Phi^{(0)}_{\nu,q,\omega}(x) = (x)^\nu_{q,\omega},\ \Phi^{(1)}_{\nu,q,\omega}(x) \equiv \Phi_{\nu,q,\omega}(x). \tag{43}$$

Motivation: Die erste Definition folgt, weil die zwei Hauptfolgen die Potenzfunktion ersetzen.

Als nächstes werden Verallgemeinerungen der beiden Formeln [5, 4.107, 4.111] vorgestellt.

Satz 2.7.

$$\mathrm{D}_{q,\omega}\Phi_{\nu;q,\omega}(x) = \{\nu\}_q\Phi_{\nu-1;q,\omega}(x). \tag{44}$$

Die Formel [13, (30)] *in Umbralform:*

$$\Phi_{\nu;q,\omega}(x) \doteq (\Phi_{q,\omega} \oplus_{q,\omega} x)^\nu. \tag{45}$$

Definition 16. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die komplementäre q,ω-Appellsche Polynomfolge $\{\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x)\}_{\nu=0}^\infty$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion definiert:

$$f_n(t,q,\omega)\mathrm{E}_{\frac{1}{q},\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!}\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x). \tag{46}$$

Bemerkung 2. Diese Definition wird in der Formel (49) verwendet.

Satz 2.8. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir ein ω-Analogon von* [10, (43)]*:*

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \tag{47}$$

Beweis. Wir zeigen, dass beide Seiten von (47) dieselbe erzeugende Funktion haben.

$$
\begin{aligned}
&f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x\oplus_{q,\omega}y)t) \overset{\text{durch}(22)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt)\\
&f_K(t,q,\omega)\mathrm{E}_{q,\omega t}(yt) \overset{\text{durch}(42)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;q,\omega}(y)\\
&=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y).
\end{aligned}
\tag{48}
$$

□

Bemerkung 3. Die Formel (47) definiert $\Phi^{(M+K)}_{\nu,q,\omega}(x\oplus_{q,\omega}y)$ als rechte Seite der Formel. Es gibt keine andere Definition von dieser Funktion.

Satz 2.9. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir:*

$$
\Phi^{(M+K)}_{\nu,q,\omega}(x\boxplus_{q,\omega}y)=\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y).
\tag{49}
$$

Beweis. Wir zeigen, dass beide Seiten von (49) dieselbe erzeugende Funktion haben.

(50)

$$
\begin{aligned}
&f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x\boxplus_{q,\omega}y)t) \overset{\text{durch}(24)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt)f_K(t,\tfrac{1}{q},-\omega)\\
&\mathrm{E}_{\frac{1}{q},\omega t}(yt) \overset{\text{durch}(42),(46)}{=} \sum_{k=0}^{\infty}\frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x)\sum_{l=0}^{\infty}\frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;\frac{1}{q},\omega}(y)\\
&=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\{\nu\}_q!}\sum_{k=0}^{\nu}\binom{\nu}{k}_q\Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y).
\end{aligned}
$$

□

3. Der Matrixansatz

In diesem Abschnitt verallgemeinern wir Resultate aus [9] und [10] durch die Einführung der Variable ω.

Definition 17. [6], [9] Ein q-Analogon der Polya-Veinschen Matrix. Die $n \times n$ Matrix $\mathbf{D}_{n,q}$ ist gegeben durch

$$\begin{aligned} \mathbf{D}_{n,q}(i, i-1) &\equiv \{i\}_q, \; i = 1, \ldots, n-1, \\ \mathbf{D}_{n,q}(i,j) &\equiv 0, \; j \neq i-1. \end{aligned} \tag{51}$$

Die folgende Vektorform für q, ω-Appellsche Polynome und Zahlen wird in den Formeln (69), (89), (90), (108) und (109) verwendet.

$$\phi_{n,q,\omega}(x) \equiv (\Phi_{0,q,\omega}(x), \Phi_{1,q,\omega}(x), \ldots, \Phi_{n-1,q,\omega}(x))^T, \tag{52}$$

$$\phi_{n,q,\omega} \equiv \phi_{n,q,\omega}(0). \tag{53}$$

Definition 18. Die folgende Abkürzung wird verwendet.

$$\xi_{n,q,\omega}(x) \equiv ((x)^0_{q,\omega}, (x)^1_{q,\omega}, (x)^2_{q,\omega}, \ldots, (x)^{n-1}_{q,\omega})^T. \tag{54}$$

Definition 19. Man definiere die q, ω-Appellsche Polynommatrix durch

$$\overline{\Phi}_{n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \Phi_{i-j,q,\omega}(x), \; 0 \leq i, j \leq n-1. \tag{55}$$

Definition 20. Die q, ω-Appellsche Zahlenmmatrix wird definiert durch

$$\overline{\Phi}_{n,q,\omega}(i,j) \equiv \overline{\Phi}_{n,q,\omega}(0)(i,j), \; 0 \leq i, j \leq n-1. \tag{56}$$

Satz 3.1. *Die Formel (44) kann in Matrixform geschrieben werden. Vergleiche mit* [6, (83)].

$$\mathrm{D}_{q,\omega}\phi_{n,q,\omega}(x) = \mathbf{D}_{n,q}\phi_{n,q,\omega}(x). \tag{57}$$

Angenommen, $y(t)$ ist ein Vektor der Länge n, ist die folgende q, ω-Differenzengleichung in $\mathbb{R}^n$ von grundlegender Bedeutung:

$$\mathrm{D}_{q,\omega}y(t) = \mathbf{D}_{n,q}y(t), y(0) = y_0, \; -\infty < t < \infty. \tag{58}$$

Gemäß der Formel (57), ist die allgemeine Lösung von (58) der q, ω-Appell-Polynomvektor von Grad ν und Ordnung m. Die Anfangswerte sind dann der Vektor von q, ω-Appellschen Zahlen der Ordnung m u.s.w.. Der Anfangswert kann auch die Vektorfunktion e_0 sein. Die Lösung ist dann die Vektorfunktion $\xi_{n,q,\omega}(x)$.

Definition 21. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}(x)$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j)(x) \equiv \begin{cases} \binom{i}{j}_q (x)_{q,\omega}^{i-j}, i \geq j \\ 0, \text{ sonst} \end{cases} \tag{59}$$

Diese Matrix wird in der Formel (71) verwendet.

Satz 3.2. *Die allgemeine Lösung von (58) kann auch geschrieben werden:* $y(t) = \mathrm{E}_{q,\omega}(\mathbf{D}_{n,q}t)y_0$. *Dies ist eigentlich eine endliche Reihe, die sich in der folgenden Form ausdrücken lässt:*

$$\sum_{k=0}^{n-1} \frac{(\mathbf{D}_{n,q}t)_{q,\omega}^k}{\{k\}_q!} \equiv \mathrm{P}_{n,q,\omega}(t). \tag{60}$$

Der folgende Sonderfall wird häufig verwendet.

Definition 22. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j) \equiv \mathrm{P}_{n,q}(i,j)(1) = \binom{i}{j}_q (1)_{q,\omega}^{i-j}, i,j = 0,\ldots,n-1. \tag{61}$$

Des Weiteren haben wir das folgende q,ω-Analogon von [1, S. 233 (7)], was daraus folgt, dass $\mathrm{P}_{n,q,\omega}(t)$ eine q,ω-Exponentialfunktion ist.

$$\mathrm{P}_{n,q,\omega}(s \oplus_{q,\omega} t) = \mathrm{P}_{n,q,\omega}(s)\mathrm{P}_{n,q,\omega}(t), \quad s,t \in \mathbb{C}. \tag{62}$$

Das impliziert

$$\mathrm{P}_{n,q,\omega}^k = \mathrm{P}_{n,q,\omega}(\overline{k}_{q,\omega}). \tag{63}$$

Durch (62) erhalten wir viele kombinatorische Identitäten. Einige davon sind

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k}[-1]_{q,\omega}^{k-j} = \delta_{i,j} \tag{64}$$

und

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k}(1)_{q,\omega}^{k-j} = (\overline{2_{q,\omega}})^{i-j} \binom{i}{j}_q, \quad i \geq j. \tag{65}$$

3.1. **Multiplikative q,ω-Appellsche Polynommatrizen.**

Definition 23. Ein ω-Analogon von [10, (47)]. Die multiplikativen q,ω-Appell-Polynommatrizen $(\mathcal{M}_{x,q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ der Ordnung $M \in \mathbb{Z}$ sind definiert durch

$$(66) \qquad \overline{\Phi}^{(M)}_{n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \Phi^{(M)}_{i-j,q,\omega}(x), \; 0 \le i,j \le n-1.$$

Definition 24. Die multiplikativen q,ω-Appell Zahlenmatrizen oder die q,ω-Übertragung-Matrizen $(\mathcal{M}_{q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}$ der Order $M \in \mathbb{Z}$ sind definiert durch

$$(67) \qquad \overline{\Phi}^{(M)}_{n,q,\omega}(i,j) \equiv \overline{\Phi}^{(M)}_{n,q,\omega}(0)(i,j), \; 0 \le i,j \le n-1.$$

Satz 3.3. *Eine Formel für die q,ω-Übertragung-Matrix*

$$(68) \qquad \overline{\Phi}_{n,q,\omega} = f_n(t,q,\omega)\mathbf{D}_{n,q},$$

wobei $f_n(t,q,\omega)$ durch (42) definiert wird.

Beweis. Dies ist ähnlich wie bei der Formel [9, (79)]. □

Satz 3.4. *Der q,ω-Appell-Polynomvektor von x kann als Produkt der q,ω-Appellzahlen-Matrix mal $\xi_{n,q,\omega}(x)$ ausgedrückt werden. Ein q,ω-Analogon von* [2, (3.9), S. 432] *und ein ω-Analogon von* [10, (49)].

$$(69) \qquad \phi_{n,q,\omega}(x) = \overline{\Phi}_{n,q,\omega}\xi_{n,q,\omega}(x).$$

Beweis. Dies ist die Formel (45) in Matrixform.

□

Satz 3.5. *Der q,ω-Appell-Polynomvektor von $x \oplus_{q,\omega} y$ kann als Produkt der q,ω-Appellschen Matrix von x mal der q,ω-Appellschen Vektor von y ausgedrückt werden. Ein q,ω-Analogon von* [2, (4.1), S. 436].

$$(70) \qquad \phi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}_{n,q,\omega}(x)\phi_{n,q,\omega}(y).$$

Beweis. Dies ist die Formel (47) in Matrixform. □

Satz 3.6. *Ein q,ω-Analogon von* [2, S. 436].

$$(71) \qquad \xi_{n,q,\omega}(x \oplus_q y) = \mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y).$$

Beweis. Wir haben durch [12, 2.3]

$$\begin{aligned} &\xi_{n,q,\omega}(x \oplus_q y)(i) = (x \oplus_q y)^i_{q,\omega} \\ &= \sum_{k=0}^{i} \binom{i}{k}_q (x)^{i-k}_{q,\omega} (y)^k_{q,\omega} \overset{\text{durch}(59)}{=} (\mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y))(i). \end{aligned} \tag{72}$$

□

Satz 3.7. *Ein q, ω-Analogon von* [2, S. 436].

$$\overline{\Phi}^{(M+K)}_{n,q,\omega} \xi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(K)}_{n,q,\omega} \overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y). \tag{73}$$

Beweis. Die beiden Matrizen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ und $\overline{\Phi}^{(K)}_{n,q,\omega}$ sind Potenzreihen in $\mathbf{D}_{n,q}$ und wir haben

$$\begin{aligned} &\overline{\Phi}^{(M)}_{n,q,\omega}(x)\phi^{(K)}_{n,q,\omega}(y) \\ &\overset{\text{durch}(69)}{=} \overline{\Phi}^{(M)}_{n,q,\omega}(x)\overline{\Phi}^{(K)}_{n,q,\omega}\xi_{n,q,\omega}(y) = \overline{\Phi}^{(K)}_{n,q,\omega}\overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y). \end{aligned} \tag{74}$$

Andererseits gemäß der Formel (47) ist dies gleich

$$\phi^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) \overset{\text{durch}(69)}{=} \overline{\Phi}^{(M+K)}_{n,q,\omega} \xi_{n,q,\omega}(x \oplus_{q,\omega} y). \tag{75}$$

Die Formel (73) folgt, indem die letzten Ausdrücke von (74) und (75) gleichgesetzt werden. □

Bemerkung 4. Die Formel (73) ergibt eine implizite Definition der Funktion $\xi_{n,q,\omega}(x \oplus_{q,\omega} y)$.

Satz 3.8. *Ein ω-Analogon von* [10, (55)]. *Wir gehen davon aus, dass M und K die x-Ordnung bzw. y-Ordnung sind. Die Formel (47) kann in der folgenden Matrixform umgeschrieben werden, wobei $\cdot$ auf der rechten Seite eine Matrixmultiplikation bezeichnet.*

$$\overline{\Phi}^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(M)}_{n,q,\omega}(x) \cdot \overline{\Phi}^{(K)}_{n,q,\omega}(y). \tag{76}$$

Beweis. Wir berechnen das Matrixelement (i,j) der Matrixmultiplikation auf der rechten Seite.

$$
\begin{aligned}
&\sum_{k=j}^{i}\binom{i}{k}_q \Phi^{(M)}_{i-k,q,\omega}(x)\binom{k}{j}_q \Phi^{(K)}_{k-j,q,\omega}(y)\\
&=\binom{i}{j}_q\sum_{k=j}^{i}\binom{i-j}{k-j}_q \Phi^{(M)}_{i-k,q,\omega}(x)\Phi^{(K)}_{k-j,q,\omega}(y)\\
&=\binom{i}{j}_q\sum_{k=0}^{i-j}\binom{i-j}{k}_q \Phi^{(M)}_{i-j-k,q,\omega}(x)\Phi^{(K)}_{k,q,\omega}(y)\\
&\overset{\text{durch}(47)}{=}\binom{i}{j}_q \Phi^{(M+K)}_{i-j,q,\omega}(x\oplus_{q,\omega} y)=\text{LS}.
\end{aligned}
\tag{77}
$$

□

Durch die Formel (66) sind die $\overline{\Phi}^{(M)}_{n,q}(x)$-Matrizen mit Matrixelementen q,ω-Appellsche Polynome multipliziert mit q-Binomial-Koeffizienten, und wir gelangen zur nächsten wichtigen Definition.

Definition 25. Ein ω-Analogon von [10, (57)]. Wir definieren die zweite Matrixmultiplikation $\cdot_{q,\omega}$ durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot_{q,\omega}\overline{\Phi}^{(K)}_{n,q,\omega}(y)\equiv\overline{\Phi}^{(M+K)}_{n,q,\omega}(x\boxplus_{q,\omega} y), \tag{78}$$

wobei $\overline{\Phi}^{(M+K)}_{n,q,\omega}(x\boxplus_q y)$ durch die Formel (49) definiert wird.

Satz 3.9. *Die Menge $(\mathcal{M}_{x,q,\omega},\cdot,\cdot_{q,\omega},\mathrm{I}_n)$ mit Multiplikationen gegeben durch (76) und (78), und Inverse $\overline{\Phi}^{(-M)}_{n,q,\omega}(-x)$ ist eine q,ω-Liesche Gruppe. Das Einheitselement ist die Einheitsmatrix I_n und das assoziative Gesetz gilt wie für Gruppen.*

Wir geben eine vereinfachte Version des entsprechenden Nachweises.

Beweis. $\mathcal{M}_{x,q,\omega}$ ist geschlossen unter den beiden Operationen durch (76) und (78). Durch (78) haben wir

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot_q\overline{\Phi}^{(-M)}_{n,q,\omega}(-x)=\overline{\Phi}^{(0)}_{n,q,\omega}(\theta)=\mathrm{I}_n, \tag{79}$$

das die Existenz eines inversen Elements und einer Einheit beweist.

Zur Vereinfachung der Notation wird der letzte Teil in einem Spezialfall angegeben, das leicht zu verallgemeinern ist. Das assoziative Gesetz lautet:

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x) \cdot \overline{\Phi}_{n,q,\omega}^{(K)}(y)\right) \cdot_q \overline{\Phi}_{n,q,\omega}^{(J)}(z) = \overline{\Phi}_{n,q,\omega}^{(M)}(x) \cdot \left(\overline{\Phi}_{n,q,\omega}^{(K)}(y) \cdot_q \overline{\Phi}_{n,q,\omega}^{(J)}(z)\right), \tag{80}$$

das äquivalent zu

$$\overline{\Phi}_{n,q,\omega}^{(M+K+J)}((x \oplus_{q,\omega} y) \boxplus_{q,\omega} z) = \overline{\Phi}_{n,q,\omega}^{(M+K+J)}(x \oplus_{q,\omega} (y \boxplus_{q,\omega} z)) \tag{81}$$

ist. Die Formel (81) folgt jedoch aus der Assoziativität der beiden q, ω-Additionen. □

Sei

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x)\right)^k \equiv \overline{\Phi}_{n,q,\omega}^{(M)}(x) \cdot \overline{\Phi}_{n,q,\omega}^{(M)}(x) \cdot \cdots \cdot \overline{\Phi}_{n,q,\omega}^{(M)}(x). \tag{82}$$

Dabei steht auf der rechten Seite das Produkt von k gleichen Matrizen $\overline{\Phi}_{n,q}^{(M)}(x)$.

Die Formel (63) kann zu

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x)\right)^k = \overline{\Phi}_{n,q,\omega}^{(kM)}(\overline{k}_{q,\omega} x) \tag{83}$$

verallgemeinert werden.

3.2. **q, ω-Bernoulli und q, ω-Eulersche Polynome.** Wir betrachten auch die besonderen Fälle q, ω-Bernoulli- und q, ω-Eulersche Polynome.

Definition 26. Es gibt zwei q, ω-Bernoulli-Polynome $\mathrm{B}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q, ω-Bernoulli-Polynome, und $\mathrm{B}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q, ω-Bernoulli-Polynome. Sie sind definiert durch die beiden erzeugenden Funktionen

$$\frac{t}{(\mathrm{E}_{q,\omega}(t) - 1)} \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathrm{B}_{\mathrm{NWA},\nu,q}(x)}{\{\nu\}_q!}, \quad |t| < 2\pi. \tag{84}$$

und

$$\frac{t}{(\mathrm{E}_{\frac{1}{q},\omega}(t) - 1)} \mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu \mathrm{B}_{\mathrm{JHC},\nu,q}(x)}{\{\nu\}_q!}, \quad |t| < 2\pi. \tag{85}$$

Definition 27. Die Ward q, ω-Bernoullischen Zahlen sind gegeben durch

$$\mathrm{B}_{\mathrm{NWA},n,q,\omega} \equiv \mathrm{B}_{\mathrm{NWA},n,q,\omega}(0). \tag{86}$$

Die Jackson q, ω-Bernoullischen Zahlen sind gegeben durch

$$\mathrm{B}_{\mathrm{JHC},n,q,\omega} \equiv \mathrm{B}_{\mathrm{JHC},n,q,\omega}(0). \tag{87}$$

Um Platz zu sparen, verwenden wir die folgende Abkürzung in den Gleichungen (89) - (93), (96), (97), (100), (104), (105), (108)-(112), (115)-(116), (119)-(120). Für den JHC-Fall ändern wir gegebenenfalls $\oplus_{q,\omega}$ zu $\boxplus_{q,\omega}$.

$$\mathrm{NWA} = \mathrm{NWA} \vee \mathrm{JHC}. \tag{88}$$

Wir werden die folgenden Vektorformen für die q, ω-Bernoulli-Polynome verwenden, die q, ω-Analoga von [1, S. 239] entsprechen.

$$b_{\mathrm{NWA},n,q,\omega}(x) \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}(x), \mathrm{B}_{\mathrm{NWA},1,q,\omega}(x), \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega}(x))^T. \tag{89}$$

Die entsprechenden Vektorformen für Zahlen sind

$$b_{\mathrm{NWA},n,q,\omega} \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}, \mathrm{B}_{\mathrm{NWA},1,q,\omega}, \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega})^T. \tag{90}$$

Wir stellen die NWA und JHC verschobenen, q, ω-Bernoulli-Matrizen vor.

Definition 28. Ein ω-Analogon von [9, (54)].

$$\begin{aligned} &\mathcal{B}_{\mathrm{NWA},n,q,\omega}(x) \\ \equiv &(b_{\mathrm{NWA},q,\omega}(x) \quad \mathrm{E}(\oplus_{q,\omega}) b_{\mathrm{NWA},q,\omega}(x) \quad \cdots \quad \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}} b_{\mathrm{NWA},q,\omega}(x)), \end{aligned} \tag{91}$$

wobei $\mathrm{E}(\oplus_{q,\omega})^{\overline{k}_{q,\omega}}((x)^n_{q,\omega}) \equiv (x \oplus_{q,\omega} \overline{k}_{q,\omega})^n,\ 0 \leq k \leq n-1$.

Wir benötigen zwei ähnliche Matrizen basierend auf den $\mathrm{B}_{\mathrm{NWA}}$ und $\mathrm{B}_{\mathrm{JHC}}$-Polynomen und Zahlen.

Definition 29. Zwei ω-Analoga von [9, (58),(67)] und zwei q, ω-Analoga von [4, S. 193]. Die NWA und JHC q, ω-Bernoulli-Matrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}(x), 0 \leq i,j \leq n-1. \tag{92}$$

Definition 30. Ein ω-Analogon von [10, 84]. Die NWA und JHC q, ω-Bernoulli-Zahlenmatrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}, 0 \leq i,j \leq n-1. \tag{93}$$

Definition 31. Ein ω-Analogon von [10, 85]. Die Matrix $\mathcal{D}_{\text{NWA},n,q,\omega}$ hat Matrixelemente

$$d_{\text{NWA},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q}\binom{i}{j}_q(1)_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{94}$$

Definition 32. Ein ω-Analogon von [10, 86]. Die Matrix $\mathcal{D}_{\text{JHC},n,q,\omega}$ hat Matrixelemente

$$d_{\text{JHC},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q}\binom{i}{j}_q[1]_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{95}$$

Satz 3.10. *Ein ω-Analogon von* [10, 87]. *Die Inversen der q,ω-Bernoulli-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\text{B}}_{\text{NWA},n,q,\omega}\right)^{-1} = \mathcal{D}_{\text{NWA},n,q,\omega}, \ \left(\overline{\text{B}}_{\text{JHC},n,q,\omega}\right)^{-1} = \mathcal{D}_{\text{JHC},n,q,\omega}. \tag{96}$$

Dies impliziert, dass

$$\overline{\text{B}}_{\text{NWA},n,q,\omega}^{-k} = \mathcal{D}_{\text{NWA},n,q,\omega}^{k}. \tag{97}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\text{B}}_{\text{NWA},n,q,\omega}\mathcal{D}_{\text{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n} \frac{1}{\{k+1\}_q}\binom{n}{k}_q \text{B}_{\text{NWA},n-k,q,\omega}(1)_{q,\omega}^{k+1} = \delta_{n,0}. \tag{98}$$

Dann haben wir

(99)

$$\begin{aligned}
&\sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q}\binom{i}{k}_q \text{B}_{\text{NWA},i-k,q,\omega}\binom{k}{j}_q (1)_{q,\omega}^{k+1-j} \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q}\binom{i-j}{k-j}_q \text{B}_{\text{NWA},i-k,q,\omega}(1)_{q,\omega}^{k+1-j} \\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \frac{1}{\{k+1\}_q}\binom{i-j}{k}_q \text{B}_{\text{NWA},i-j-k,q,\omega}(1)_{q,\omega}^{k+1} \overset{\text{durch}(98)}{=} \binom{i}{j}_q \delta_{i-j,0}.
\end{aligned}$$

□

In [9] haben wir die folgenden Formeln betrachtet.

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x \oplus_q y) = \mathrm{P}_{n,q}(x)\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(y). \tag{100}$$

Diese Formeln können verallgemeinert werden zu

Satz 3.11. *Ein ω-Analogon von* [10, 92].

$$\overline{\Phi}_{n,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}(y). \tag{101}$$

Insbesondere haben wir

$$\overline{\Phi}_{n,q,\omega}(x) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}. \tag{102}$$

Beweis.

$$\begin{aligned}
\mathrm{RS} &= \sum_{k=j}^{i} \binom{i}{k}_q (x)_{q,\omega}^{i-k} \binom{k}{j}_q \Phi_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q (x)_{q,\omega}^{i-k} \Phi_{k-j,q,\omega}(y) \\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q (x)_{q,\omega}^{i-j-k}(x) \Phi_{k,q,\omega}(y) \\
&\overset{\text{durch}(43),(47)}{=} \binom{i}{j}_q \Phi_{i-j,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{LS}.
\end{aligned} \tag{103}$$

□

Satz 3.12. *Ein ω-Analogon von* [10, 95]. *Die Inversen der q,ω-Bernoulli-Polynommatrizen sind gegeben durch*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)\right)^{-1} = \left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} \mathrm{P}_{n,q,\omega}(x)^{-1} = \mathfrak{D}_{\mathrm{NWA},n,q,\omega}\mathrm{P}_{n,q,\omega}(x)^{-1}. \tag{104}$$

Wenn die Ordnung erhöht wird, für $y = 0$ in (76), multiplizieren wir die q,ω-Übertragung-Matrix mit $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$. Wenn die Ordnung konstant ist, in (102), multiplizieren wir die q,ω-Übertragung-Matrix mit der q,ω-Pascal-Matrix.

Satz 3.13. *Die Funktionen der q,ω-Bernoulli Polynommatrizen $(\mathcal{B}_{\mathrm{NWA},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ und $(\mathcal{B}_{\mathrm{JHC},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ mit Elementen*

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x) \tag{105}$$

sind q,ω-Liesche Untergruppen von $\mathcal{M}_{x,q}$.

Beweis. Die Mengen $\mathcal{B}$ sind geschlossen unter den beiden Operationen durch (76) und (78). Die Existenz von Inversen folgt wie für $\mathcal{M}_{x,q}$.

□

Wir wenden uns nun den q,ω-Euler-Polynomen zu.

Definition 33. Es gibt zwei Arten von q,ω-Euler-Polynomen, $\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q,ω-Euler-Polynomen, und $\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q,ω-Euler-Polynomen. Sie sind definiert durch die folgenden zwei erzeugenden Funktionen:

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{q,\omega}(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x),\ |t|<\pi, \tag{106}$$

und

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{\frac{1}{q},\omega}(t)+1}=\sum_{\nu=0}^{\infty}\frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x),\ |t|<\pi. \tag{107}$$

Definition 34. Wir werden die folgenden Vektorformen für diese Polynome benutzen.

$$f_{\mathrm{NWA},n,q,\omega}(x)\equiv(\mathrm{F}_{\mathrm{NWA},0,q,\omega}(x),\mathrm{F}_{\mathrm{NWA},1,q,\omega}(x),\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega}(x))^{T}. \tag{108}$$

Die entsprechenden q,ω-Euler-Zahlvektoren sind

$$f_{\mathrm{NWA},n,q,\omega}\equiv(\mathrm{F}_{\mathrm{NWA},0,q,\omega},\mathrm{F}_{\mathrm{NWA},1,q,\omega},\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega})^{T}. \tag{109}$$

Wir stellen die NWA und JHC verschobenen, q,ω-Eulerschen Matrizen vor.

Definition 35. Ein ω-Analogon von [10, 103].

$$\begin{aligned}&\mathcal{F}_{\mathrm{NWA},n,q,\omega}(x)\\&\equiv(f_{\mathrm{NWA},q,\omega}(x)\ \ \mathrm{E}(\oplus_{q,\omega})f_{\mathrm{NWA},q,\omega}(x)\ \ \cdots\ \ \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}}f_{\mathrm{NWA},q,\omega}(x)).\end{aligned} \tag{110}$$

Wir benötigen zwei ähnliche Matrizen, basierend auf den $\mathrm{F}_{\mathrm{NWA}}$-Polynomen.

Definition 36. Die beiden q, ω-Eulerschen Polynommatrizen sind definiert durch

$$\overline{F}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q f_{\mathrm{NWA},i-j,q,\omega}(x). \tag{111}$$

Definition 37. Ein ω-Analogon von [10, 105]. Die NWA und JHC q, ω-Eulerschen Matrizen sind definiert durch

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{F}_{\mathrm{NWA},i-j,q,\omega},\, 0 \leq i,j \leq n-1. \tag{112}$$

Definition 38. Ein ω-Analogon von [10, 106]. Die Matrix $\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{(1)_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{113}$$

Definition 39. Ein ω-Analogon von [10, 107]. Die Matrix $\mathcal{C}_{\mathrm{JHC},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{[1]_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{114}$$

Satz 3.14. *Ein ω-Analogon von* [10, 108]. *Die Inversen der q, ω-Euler-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}. \tag{115}$$

Dies impliziert, dass

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}^{-k} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}^{k}. \tag{116}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n}(1)_{q,\omega}^{k}\binom{n}{k}_q \mathrm{F}_{\mathrm{NWA},n-k,q} + \mathrm{F}_{\mathrm{NWA},n,q} = 2\delta_{n,0}. \tag{117}$$

Man führe eine Funktion $G(k)$ ein. Dann haben wir

$$\begin{aligned}&\sum_{k=j}^{i}\binom{i}{k}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega}G(k-j)\binom{k}{j}_q\\ (118)\qquad &=\binom{i}{j}_q\sum_{k=j}^{i}\binom{i-j}{k-j}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega}G(k-j)\\ &=\binom{i}{j}_q\sum_{k=0}^{i-j}\binom{i-j}{k}_q \mathrm{F}_{\mathrm{NWA},i-j-k,q,\omega}G(k)\overset{\mathrm{durch}(117)}{=}\binom{i}{j}_q\delta_{i-j,0}.\end{aligned}$$

Es ist jetzt offensichtlich, dass $G(k)=\frac{1}{2}\left[(1)_{q,\omega}^k+\delta_{k,0}\right]$ diese Gleichung für NWA löst, und dass der JHC-Fall ähnlich gelöst wird. □

Die letzten beiden Sätze werden auf ähnliche Weise belegt.

Satz 3.15. *Die Inversen der q,ω-Eulerschen Polynommatrizen sind gegeben durch*

$$(119)\qquad \left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)\right)^{-1}=\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}\right)^{-1}\mathrm{P}_{n,q,\omega}(x)^{-1}=\mathcal{C}_{\mathrm{NWA},n,q}\mathrm{P}_{n,q,\omega}(x)^{-1}.$$

Satz 3.16. *Ein ω-Analogon von* [10, 114]. *Die q,ω-Eulerschen Polynommatrizen $(\mathcal{F}_{\mathrm{NWA},q},\ \cdot,\ \cdot_q,\ \mathrm{I}_n)$ und $(\mathcal{F}_{\mathrm{JHC},q},\ \cdot,\ \cdot_q,\ \mathrm{I}_n)$ mit Elementen*

$$(120)\qquad \overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)$$

sind q,ω-Lieschen Untergruppen von $\mathcal{M}_{x,q}$.

4. Schlussfolgerung

Wir haben Formeln aus Arbeiten von Arponen [4], Aceto et al. [2], Ernst [8], [9], [10], vereinigt und q,ω-deformiert, um eine erste Synthese von q,ω-Appell-Polynommatrizen vorzustellen. Einige Formeln für q-Pascal -Matrizen sowie Formeln für q-Bernoulli- und q-Eulerschen Matrizen werden verallgemeinert. Wir haben die ersten konkreten Beispiele von q,ω-Lieschen Untergruppen angegeben. Wir glauben, dass es keine weitere Analoga gibt, die q,ω-Analoga sind trotzdem sehr interessant.

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Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden,

E-mail address: thomas@math.uu.se

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Bi-α ISO-DIFFERENTIAL CALCULUS

Svetlin Georgiev Georgiev
Sorbonne University, Paris, France
svetlingeorgiev1@gmail.com

Abstract

In this paper we define bi-α(multiplicative) iso-derivative for iso-functions of first, second, third, fourth and fifth kind. They are deducted the main properties of the multiplicative iso-derivative. They are deducted and proved mean value theorems for multiplicative iso-differentiable functions, criteria for increasing and decreasing of multiplicative iso-differentiable functions, criteria for concavity and convexity of multiplicative iso-differentiable functions. In the paper it is introduced the concept for multiplicative iso-integral and they are deducted the main properties. As applications of multiplicative iso-derivative and multiplicative iso-integral we consider some classes multiplicative iso-differential equations.

1 Introduction

As it is well known, Isaac Newton had to develop the differential calculus, (jointly with Gottfried Leibniz), with particular reference to the historical definition of velocities as the time derivative of the coordinates, $v = dr/dt$, in order to write his celebrated equation $ma = F(t, r, v)$, where $a = dv/dt$ is the acceleration and $F(t, r, v)$ is the Newtonian force acting on the mass m. Being local, the differential calculus solely admitted the characterization of massive points. The differential calculus and the notion of massive points were adopted by Galileo Galilei and Albert Einstein for the formulation of their relativities, thus acquiring a fundamental role in 20th century sciences.

In his Ph. D. thesis of 1966 at the University of Turin, Italy, the Italian-American scientist Ruggero Maria Santilli[1] pointed out that Newtonian forces are the most widely known in dynamics, including action-at-a-distance forces derivable derivable from a potential, thus representable with a Hamiltonian, and other forces that are not derivable from a potential or a Hamiltonian, since they are contact dissipative and non-conservative forces caused by the motion of the mass m within a physical medium. Santilli pointed out that, due to their lack of dimensions, massive points can solely experience action-at-a-distance Hamiltonian forces.

On this ground, Santilli initiated a long scientific journey for the generalization of Newton's equation into a form permitting the representation of the actual extended character of massive bodies whenever moving within physical media, as a condition to admit non-Hamiltonian forces. Being a theoretical physicist, Santilli had a number of severe physical conditions for the needed representation. One of them was the need for a representation of extended bodies and their non-Hamiltonian forces to be invariant over time as a condition to predict the same numerical values under the same conditions but at different times.

The resulting new calculus, today known as Santilli IsoDifferential Calculus, or IDC for short, stimulated a further layer of studies that finally signaled the achievement of mathematical and physical maturity. In particular, we note: the isotopies of Euclidean, Minkowskian, Riemannian and symplectic geometries; the isotopies of classical Hamiltonian mechanics, today known as the Hamilton-Santilli isomechanics;and the isotopies of quantum

[1]Prof. Santilli's curriculum is available in from the link http://www.world-lecture-series.org/santilli-cv

mechanics, today known as the isotopic branch of Hadronic mechanics.

In this paper we define bi-α(multiplicative) iso-derivative for iso-functions of first, second, third, fourth and fifth kind. They are deducted the main properties of the multiplicative iso-derivative. They are deducted and proved mean value theorems for multiplicative iso-differentiable functions, criteria for increasing and decreasing of multiplicative iso-differentiable functions, criteria for concavity and convexity of multiplicative iso-differentiable functions. In the paper it is introduced the concept for multiplicative iso-integral and they are deducted the main properties. As applications of multiplicative iso-derivative and multiplicative iso-integral we consider some classes multiplicative iso-differential equations.

2 Definition for Multiplicative Iso-Derivative

Suppose that $A \subset \mathbb{R}$, $f, \hat{T} : A \to (0, \infty)$ be enough times differentiable functions. If it is necessary, we suppose that $\frac{x}{\hat{T}(x)} \in A$ or $x\hat{T}(x) \in A$ for any $x \in A$ so that to be defined the iso-functions of the second, third, fourth and fifth kind. With $\widetilde{f}$ we will denote the corresponding iso-function of the first, second, third, fourth and fifth kind.

Definition 2.1. *Define the multiplicative iso-derivative of $\widetilde{f}$ as*

$$\widetilde{f}^{*\circledast}(x) = e^{\widetilde{f}^{\circledast}(x) \measuredangle \widetilde{f}(x)}, \quad x \in A.$$

1. Iso-functions of the first kind.

$$\hat{f}^{\wedge\wedge*\circledast}(x) = e^{\frac{1}{f(x)(\hat{T}(x))^2} \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

2. Iso-functions of the second kind.

$$\hat{f}^{\wedge*\circledast}(x) = e^{\frac{1}{f(x\hat{T}(x))(\hat{T}(x))^2} \frac{f'(x\hat{T}(x))(\hat{T}^2(x) + x\hat{T}(x)\hat{T}'(x)) - f(x\hat{T}(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

3. Iso-functions of the third kind.

$$\hat{\hat{f}}^{*\circledast}(x) = e^{\frac{1}{f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x))^2} \frac{f'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

4. Iso-functions of the fourth kind.

$$f^{\wedge *\circledast}(x) = e^{\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x)+x\hat{T}'(x))}{\hat{T}(x)f(x\hat{T}(x))\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)}}.$$

5. Iso-functions of the fifth kind.

$$f^{\vee *\circledast}(x) = e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)f\left(\frac{x}{\hat{T}(x)}\right)}}.$$

Example 2.2. *Let* $A = [0, \infty)$, $\hat{T}(x) = 1 + x$, $f(x) = 1 + x^2$, $x \in A$. *Then*

$$\begin{aligned} \hat{f}^{\wedge\wedge}(x) &= \frac{f(x)}{\hat{T}(x)} = \frac{1+x^2}{1+x}, \\ f'(x) &= 2x, \\ \hat{T}'(x) &= 1, \quad x \in A. \end{aligned}$$

Hence, the iso-derivative of the iso-function $\hat{f}^{\wedge\wedge}$ *is given by*

$$\hat{f}^{\wedge\wedge\circledast}(x) = \frac{-1+2x+x^2}{1+x}, \quad x \in A,$$

and its multiplicative iso-derivative is given by

$$\hat{f}^{\wedge\wedge *\circledast}(x) = e^{\frac{-1+2x+x^2}{(1+x)(1+x^2)}}, \quad x \in A.$$

3 Properties of the Multiplicative Iso-Derivative

In this section, we will deduct some of the properties of the multiplicative iso-derivative.

Theorem 3.1. *Let* $\hat{f}, \hat{g} : A \to \mathbb{R}$ *be iso-differentiable functions. Then for any* $a, b \in \mathbb{R}$, *we have*

$$\left(a\hat{f} + b\hat{g}\right)^{*\circledast} = \left(\left(\hat{f}^{*\circledast}\right)^{\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right)\left(\left(\hat{g}^{*\circledast}\right)^{\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right).$$

Proof. We have

$$\begin{aligned}
\left(a\hat{f}+b\hat{g}\right)^{*\circledast} &= e^{(a\hat{f}+b\hat{g})^{*\circledast}\hat{\wedge}(a\hat{f}+b\hat{g})} \\
&= e^{\frac{a\hat{f}^{\circledast}+b\hat{g}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}} \\
&= e^{a\frac{\hat{f}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}+b\frac{\hat{g}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}} \\
&= \left(e^{\left(\hat{f}^{\circledast}\hat{\wedge}\hat{f}\right)\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right) \\
&\quad \left(e^{\left(\hat{g}^{\circledast}\hat{\wedge}\hat{g}\right)\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right) \\
&= \left(\left(\hat{f}^{*\circledast}\right)^{\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right)\left(\left(\hat{g}^{*\circledast}\right)^{\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $\hat{f},\hat{g}:A\to(0,\infty)$ be iso-differentiable functions. Then*

$$\left(\hat{f}\hat{\times}\hat{g}\right)^{*\circledast}=\hat{f}^{*\circledast}\hat{g}^{*\circledast}e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}}.$$

Proof. By the properties of the iso-derivative, we have

$$\begin{aligned}
\left(\hat{f}\hat{\times}\hat{g}\right)^{\circledast} &= \hat{g}\hat{\times}\hat{f}^{\circledast}+\hat{f}\hat{\times}\hat{g}^{\circledast} \\
&\quad +\hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T}-x\hat{T}'}.
\end{aligned}$$

Then

$$\begin{aligned}(\hat{f}\hat{\times}\hat{g})^{\circledast} \curlywedge (\hat{f}\hat{\times}\hat{g}) &= \frac{1}{\hat{f}\hat{g}\hat{T}^2}\Bigg(\hat{g}\hat{T}\hat{f}^{\circledast} + \hat{f}\hat{T}\hat{g}^{\circledast} \\ &\quad + \hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T} - x\hat{T}'}\Bigg) \\ &= \left(\hat{f}^{\circledast} \curlywedge \hat{f}\right) + (\hat{g}^{\circledast} \curlywedge \hat{g}) \\ &\quad + \frac{\hat{T}'}{\hat{T}(\hat{T} - x\hat{T}')}.\end{aligned}$$

Hence,

$$\begin{aligned}\left(\hat{f}\hat{\times}\hat{g}\right)^{*\circledast} &= e^{(\hat{f}\hat{\times}\hat{g})^{\circledast}\curlywedge(\hat{f}\hat{\times}\hat{g})} \\ &= e^{\left(\hat{f}^{\circledast}\curlywedge\hat{f}\right)+\left(\hat{g}^{\circledast}\curlywedge\hat{g}\right)+\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}} \\ &= e^{\hat{f}^{\circledast}\curlywedge\hat{f}}e^{\hat{g}^{\circledast}\curlywedge\hat{g}}e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}} \\ &= \hat{f}^{*\circledast}\hat{g}^{*\circledast}e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}}.\end{aligned}$$

This completes the proof. □

Theorem 3.3. *Let $\hat{f}, \hat{g} : A \to (0, \infty)$ be iso-differentiable functions. Then*

$$(\hat{f}\hat{g})^{*\circledast} = \hat{f}^{*\circledast}\hat{g}^{*\circledast}.$$

Proof. By the properties of the iso-derivative, we have

$$(\hat{f}\hat{g})^{\circledast} = \hat{f}^{\circledast}\hat{g} + \hat{g}\hat{f}^{\circledast}.$$

Hence,

$$\begin{aligned}(\hat{f}\hat{g})^{\circledast} \curlywedge (\hat{f}\hat{g}) &= \frac{1}{\hat{T}\hat{f}\hat{g}}\left(\hat{f}^{\circledast}\hat{g} + \hat{f}\hat{g}^{\circledast}\right) \\ &= \hat{f}^{\circledast} \curlywedge \hat{f} + \hat{g}^{\circledast} \curlywedge \hat{g}.\end{aligned}$$

Therefore

$$\begin{aligned}(\hat{f}\hat{g})^{*\circledast} &= e^{(\hat{f}\hat{g})^{\circledast}\leftthreetimes(\hat{f}\hat{g})}\\ &= e^{\hat{f}^{\circledast}\leftthreetimes\hat{f}+\hat{g}^{\circledast}\leftthreetimes\hat{g}}\\ &= e^{\hat{f}^{\circledast}\leftthreetimes\hat{f}}e^{\hat{g}^{\circledast}\leftthreetimes\hat{g}}\\ &= \hat{f}^{*\circledast}\hat{g}^{*\circledast}.\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $\hat{f},\hat{g} : A \to (0,\infty)$ be iso-differentiable functions and $\hat{g} \neq 0$ on A. Then*

$$\left(\hat{f}\leftthreetimes\hat{g}\right)^{*\circledast} = \frac{\hat{f}^{*\circledast}}{\left(\hat{g}^{*\circledast}\right)^{\frac{1}{\hat{T}^2}}}e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}}.$$

Proof. We have

$$\hat{f}\leftthreetimes\hat{g} = \frac{\hat{f}}{\hat{T}\hat{g}}$$

and by the properties of the iso-derivative, we get

$$\begin{aligned}\left(\hat{f}(x)\leftthreetimes\hat{g}(x)\right)^{\circledast} &= \frac{1}{\left(\hat{T}(x)\right)^2(\hat{g}(x))^2}\Big(\hat{g}(x)\hat{\times}\left(\hat{f}(x)\right)^{\circledast} - \hat{f}(x)\hat{\times}(\hat{g}(x))^{\circledast}\\ &\quad -\hat{f}(x)\hat{g}(x)\frac{\hat{T}(x)\hat{T}'(x)}{\hat{T}(x)-x\hat{T}'(x)}\Big)\\ &= \frac{1}{\hat{T}^2\hat{g}^2}\Big(\hat{T}\hat{g}\hat{f}^{\circledast} - \hat{T}\hat{f}\hat{g}^{\circledast}\\ &\quad -\hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T}-x\hat{T}'}\Big)\\ &= \frac{1}{\hat{g}\hat{T}}\hat{f}^{\circledast} - \frac{1}{\hat{T}\hat{f}\hat{g}}\left(\frac{\hat{g}^{\circledast}}{\hat{g}}\right)\\ &\quad -\frac{\hat{f}}{\hat{T}^2\hat{g}}\cdot\frac{\hat{T}'}{\hat{T}-x\hat{T}'}.\end{aligned}$$

Hence,

$$\left(\hat{f} \curlywedge \hat{g}\right)^{\circledast} \curlywedge (\hat{f} \curlywedge \hat{g}) = \frac{\hat{g}}{\hat{f}}\left(\frac{1}{\hat{g}\hat{T}}\hat{f}^{\circledast} - \frac{1}{\hat{T}\hat{f}\hat{g}}\left(\frac{\hat{g}^{\circledast}}{\hat{g}}\right) - \frac{\hat{f}}{\hat{T}^2\hat{g}} \cdot \frac{\hat{T}'}{\hat{T} - x\hat{T}'}\right)$$

$$= \hat{f}^{\circledast} \curlywedge \hat{f} - \frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast} \curlywedge \hat{g}\right) - \frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T} - x\hat{T}')}$$

and

$$e^{\left(\hat{f} \curlywedge \hat{g}\right)^{\circledast} \curlywedge (\hat{f} \curlywedge \hat{g})} = e^{\hat{f}^{\circledast} \curlywedge \hat{f} - \frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast} \curlywedge \hat{g}\right) - \frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T} - x\hat{T}')}}$$

$$= e^{\hat{f}^{\circledast} \curlywedge \hat{f}} e^{-\frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast} \curlywedge \hat{g}\right)} e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T} - x\hat{T}')}}$$

$$= \frac{\hat{f}^{*\circledast}}{\left(\hat{g}^{*\circledast}\right)^{\frac{1}{\hat{f}^2}}} e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T} - x\hat{T}')}}.$$

This completes the proof. □

4 Monotonicity

Theorem 4.1. *Let $\hat{f} : A \to (0, \infty)$ be iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) > 0, \quad x \in A. \tag{4.1}$$

If $\hat{f}^{\wedge\wedge*\circledast} > 1$ on A, then $f^{\vee}$ is an increasing function.

Proof. Since $\hat{f}^{\wedge\wedge*\circledast} > 1$ on A, we have that

$$\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}(x) - x\hat{T}'(x)} > 0, \quad x \in A.$$

By the last inequality and by the second inequality of (4.1), we get

$$f'(x)\hat{T}(x) - f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Hence, using the first inequality of (4.1), we arrive at

$$f'(x)\hat{T}(x) > f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Therefore $f'(x) > 0$, $x \in A$. Because $\frac{Id}{\hat{T}}$ is an increasing function, we find that $f^{\vee}$ is an increasing function. This completes the proof. □

Theorem 4.2. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) < 0, \quad x \in A. \tag{4.2}$$

If $\hat{f}^{\wedge\wedge*\circledast} < 1$ on A, then $f^{\vee}$ is a decreasing function.

Proof. By the definition of multiplicative iso-derivative and $\hat{f}^{\wedge\wedge*\circledast} < 1$, it follows that

$$\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}(x) - x\hat{T}'(x)} < 0, \quad x \in A.$$

Hence and the second inequality of (4.2), we conclude that

$$f'(x)\hat{T}(x) - f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Now, applying the first inequality of (4.2), we get

$$f'(x)\hat{T}(x) > f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Therefore $f'(x) > 0$ for $x \in A$ and f is an increasing function on A. By the second inequality of (4.2), we find

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x > y.$$

Hence,

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &< f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x > y. \end{aligned}$$

Thus, $f^{\vee}$ is a decreasing function on A. This completes the proof. □

Theorem 4.3. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) > 0, \quad x \in A. \tag{4.3}$$

If $\hat{f}^{\wedge *\circledast} > 1$ on A, then $f^{\wedge}$ is an increasing function on A.

Proof. By the definition of the multiplicative iso-derivative and by the condition $f^{\vee *\circledast} > 1$ on A, we get

$$\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x) + x\hat{T}'(x))}{f(x\hat{T}(x))(\hat{T}(x) - x\hat{T}'(x))} > 0, \quad x \in A,$$

Applying (4.4), we find

$$f'(x\hat{T}(x)) > 0, \quad x \in A.$$

Thus, f is an increasing function on A. Since $\hat{T}'(x) > 0$, $x \in A$, we get that

$$x\hat{T}(x) > y\hat{T}(y), \quad x, y \in A, \quad x > y.$$

Hence, using that f is an increasing function on A, we find

$$\begin{aligned} f^{\wedge}(x) &= f(x\hat{T}(x)) \\ &< f(y\hat{T}(y)) \\ &= f^{\wedge}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Therefore $f^{\wedge}$ is an increasing function on A. This completes the proof. □

Theorem 4.4. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) > 0, \quad x \in A. \tag{4.4}$$

If $\hat{f}^{\wedge *\circledast} < 1$ on A, then $f^{\wedge}$ is a decreasing function on A.

Proof. Applying the multiplicative iso-derivative and the condition $f^{\vee *\circledast} > 1$ on A, we find

$$\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x) + x\hat{T}'(x))}{f(x\hat{T}(x))(\hat{T}(x) - x\hat{T}'(x))} <, \quad x \in A,$$

Employing (??), we arrive at

$$f'(x\hat{T}(x)) < 0, \quad x \in A.$$

Thus, f is a decreasing function on A. Since $\hat{T}'(x) > 0$, $x \in A$, we find that

$$x\hat{T}(x) > y\hat{T}(y), \quad x, y \in A, \quad x > y.$$

From here and from the fact that f is an increasing function on A, we find

$$\begin{aligned} f^{\wedge}(x) &= f(x\hat{T}(x)) \\ &> f(y\hat{T}(y)) \\ &= f^{\wedge}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\wedge}$ is a decreasing function on A. This completes the proof.

□

Theorem 4.5. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function on $A$and $\hat{T}'(x) < 0$, $x \in A$. If $f^{\vee * \circledast}(x) > 1$, $x \in A$, then $f^{\vee}$ is an increasing function on A.*

Proof. By the definition of the multiplicative iso-derivative and by the condition $f^{\vee * \circledast}(x) > 1$, $x \in A$, we find

$$\begin{aligned} f^{\vee * \circledast}(x) &= e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)}} \\ &> 1, \quad x \in A. \end{aligned}$$

Hence,

$$\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)} > 0, \quad x \in A.$$

Therefore

$$f'\left(\frac{x}{\hat{T}(x)}\right) > 0, \quad x \in A,$$

and f is an increasing function on A. Since $\hat{T}'(x) < 0$, $x \in A$, we get

$$\hat{T}(x) > \hat{T}(y), \quad x, y \in A, \quad x < y.$$

Then

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x < y.$$

From here, we arrive at

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &< f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\vee}$ is an increasing function on A. This completes the proof. □

Theorem 4.6. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function on Aand $\hat{T}'(x) < 0$, $x \in A$. If $f^{\vee *\circledast}(x) < 1$, $x \in A$, then $f^{\vee}$ is a decreasing function on A.*

Proof. Applying the definition of the multiplicative iso-derivative and the condition $f^{\vee *\circledast}(x) > 1$, $x \in A$, we arrive at

$$\begin{aligned} f^{\vee *\circledast}(x) &= e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)}} \\ &< 1, \quad x \in A, \end{aligned}$$

whereupon

$$\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)} < 0, \quad x \in A,$$

and

$$f'\left(\frac{x}{\hat{T}(x)}\right) < 0, \quad x \in A.$$

So, f is a decreasing function on A. Because $\hat{T}'(x) < 0$, $x \in A$, we find

$$\hat{T}(x) > \hat{T}(y), \quad x, y \in A, \quad x < y.$$

Thus,

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x < y,$$

and

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &> f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\vee}$ is a decreasing function on A. This completes the proof. □

5 Definition for Multiplicative Iso-Integral. Properties

Definition 5.1. *Suppose that $\hat{T}(x) - x\hat{T}'(x) > 0$, $x \in A$. Define indefinite multiplicative iso-integral for the iso-function of the first kind $\hat{f}^{\wedge\wedge}$ as follows*

$$\hat{\int} \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\int \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x) - x\hat{T}'(x)\right)\hat{\times} dx}. \tag{5.1}$$

By (5.1), it follows

$$\hat{\int} \hat{f}^{\wedge\wedge*\circledast}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\int \log\left(\hat{f}^{\wedge\wedge*\circledast}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\int \frac{1}{f(x)(\hat{T}(x))^2}\frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}}\left(\hat{T}(x)-x\hat{T}'(x)\right)dx}$$

$$= e^{\int \frac{1}{\frac{f(x)}{\hat{T}(x)}}\frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{(\hat{T}(x))^2}dx}$$

$$= e^{\int \frac{\left(\frac{f}{\hat{T}}\right)'(x)}{\frac{f(x)}{\hat{T}(x)}}dx}$$

$$= e^{\log\frac{f(x)}{\hat{T}(x)}}$$

$$= \frac{f(x)}{\hat{T}(x)}$$

$$= \hat{f}^{\wedge\wedge}(x).$$

Suppose that $[a,b] \subset \mathbb{R}$.

Definition 5.2. *Suppose that* $\hat{T}(x) - x\hat{T}'(x) > 0$, $x \in A$. *Define indefinite multiplicative iso-integral for the iso-function of the first kind* $\hat{f}^{\wedge\wedge}$ *as follows*

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}. \tag{5.2}$$

In this case, we say that $\hat{f}^{\wedge\wedge}$ *is multiplicative iso-integrable on* $[a,b]$

Below, we will deduct some of the properties of the multiplicative iso-integral.

Theorem 5.3. *Let* $\hat{f}^{\wedge\wedge}$ *is multiplicative iso-integrable on* $[a,b]$. *Then*

$$\hat{\int}_a^a \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = 1.$$

Proof. We have

$$\hat{\int}_a^a \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^a \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= 1.$$

This completes the proof. □

Theorem 5.4. *Let $c \in [a,b]$ and $\hat{f}^{\wedge\wedge}$ is multiplicative iso-integrable on $[a,b]$. Then*

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = \left(\hat{\int}_a^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_b^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

Proof. We have

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx+\hat{\int}_b^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx} e^{\hat{\int}_b^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= \left(\hat{\int}_a^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_b^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

This completes the proof. □

Theorem 5.5. *Let $\hat{f}^{\wedge\wedge}$, $\hat{g}^{\wedge\wedge}$ be multiplicative iso-integrable on $[a,b]$. Then*

$$\hat{\int}_a^b \left(\hat{f}^{\wedge\wedge}(x)\hat{g}^{\wedge\wedge}\right)^{\hat{\times}\hat{d}\hat{x}} = \left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

Proof. By the definition for multiplicative iso-integral, we get

$$\hat{\int}_a^b \left(\hat{f}^{\wedge\wedge}(x)\hat{g}^{\wedge\wedge}(x)\right)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx+\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}e^{\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= \left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)..$$

This completes the proof. □

Theorem 5.6. *Let* $\hat{f}^{\wedge\wedge}$, $\hat{g}^{\wedge\wedge}$ *be multiplicative iso-integrable on* $[a,b]$ *and*

$$\hat{g}^{\wedge\wedge}(x) \neq 0, \quad \hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} \neq 0.$$

Then

$$\hat{\int}_a^b \left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)^{\hat{\times}\hat{d}\hat{x}} = \frac{\left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}{\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}.$$

Proof. Using the definition for multiplicative iso-integral, we find

$$\hat{\int}_a^b \left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx-\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= \frac{e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}}{e^{\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}}$$

$$= \frac{\left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}{\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}.$$

This completes the proof. □

6 Linear Homogeneous Multiplicative Iso-Differential Equations

Consider the

equation
$$\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x) = a(x)\hat{\times}\hat{f}^{\wedge\wedge}(x).$$

Hence,
$$\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x) \curlywedge \hat{f}^{\wedge\wedge}(x) = a(x)$$

and
$$\begin{aligned}\hat{f}^{\wedge\wedge*\circledast}(x) &= e^{\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x)\curlywedge\hat{f}^{\wedge\wedge}(x)}\\ &= e^{a(x)}.\end{aligned}$$

Then the solution of the considered iso-differential equation can be represented in the form
$$\hat{f}^{\wedge\wedge}(x) = \hat{\int} e^{a(x)}\hat{\times}\hat{d}\hat{x}.$$

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