HYPER-REPRESENTATIONS BY NON SQUARE MATRICES. HELIX-HOPES.

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Abstract

Hyperstructure theory can overcome restrictions which ordinary algebraic structures have. A hyperproduct on non-square ordinary matrices can be defined by using the so called helix-hyperoperations. We define and study the helix-hyperstructures on the representations and we extend our study up to Lie-Santilli theory by using ordinary fields. Therefore the related theory can be faced by defining the hyperproduct on the extended set of non square matrices. The obtained hyperstructure is an H_v-algebra or an H_v-Lie-alebra.

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1. Introduction.

We deal with the largest class of hyperstructures called *H_v-structures* introduced in 1990 [23], [26], which satisfy the *weak axioms* where the non-empty intersection replaces the equality.

Basic definitions:

Definitions 1.1 *In a set H equipped with a* ***hyperoperation*** *(which we abbreviate it by* ***hope****)* $\cdot: H\times H \to \mathcal{P}(H)$, *we abbreviate by*

WASS *the* ***weak associativity****:* $(xy)z\cap x(yz)\neq\emptyset$, $\forall x,y,z\in H$ *and by*

COW *the* ***weak commutativity:*** $xy\cap yx\neq\emptyset$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called ***H_v-semigroup*** *if it is WASS and is called* ***H_v-group*** *if it is reproductive H_v-semigroup:* $xH=Hx=H$, $\forall x\in H$.

$(R,+,\cdot)$ is called ***H_v-ring*** *if $(+)$ and $(\cdot)$ are WASS, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is* ***weak distributive*** *with respect to $(+)$:*

$$x(y+z)\cap(xy+xz)\neq\emptyset, \quad (x+y)z\cap(xz+yz)\neq\emptyset, \ \forall x,y,z\in R.$$

For more definitions, results and applications on H_v-structures, see books [26], [2], [8] and the survey papers [6], [25], [30]. An extreme class is the following [26]: An H_v-structure is *very thin* iff all hopes are operations except one, with all hyperproducts singletons except only one, which is a subset of cardinality more than one. Therefore, in a very thin H_v-structure in a set H there exists a hope $(\cdot)$ and a pair $(a,b)\in H^2$ for which $ab=A$, with $\text{card}A>1$, and all the other products, with respect to any other hopes (so they are operations), are singletons.

The fundamental relations β* and γ* are defined, in H_v-groups and H_v-rings, respectively, as the smallest equivalences so that the quotient would be group and ring, respectively [22], [23], [26], [27], [28], [35]. The way to find the fundamental classes is given by analogous theorems to the following:

Theorem 1.2 *Let $(H,\cdot)$ be an H_v-group and let us denote by U the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then the fundamental relation β^* is the transitive closure of the relation β.*

The main point of the proof of this theorem is that β guaranties that the following is valid: Take two elements x,y such that $\{x,y\}\subset u\in U$ and any hyperproduct where one of these elements is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Thus, if the 'hyperproducts'of the above β-classes are 'products', then, they are fundamental classes. Analogously for the γ in H_v-rings.

An element is called *single* if its fundamental class is a singleton.

Motivation *for H_v-structures:*

We know that the quotient of a group with respect to an invariant subgroup is a group.

Marty states that, the quotient of a group with respect to any subgroup is a hypergroup.

Now, the quotient of a group with respect to any partition is an H_v-group.

Definition 1.3 *Let $(H,\cdot)$, $(H,\otimes)$ be H_v-semigroups defined on the same H. $(\cdot)$ is* ***smaller*** *than $(\otimes)$, and $(\otimes)$* ***greater*** *than $(\cdot)$, iff there exists automorphism*

$$f\in Aut(H,\otimes) \text{ such that } xy\subset f(x\otimes y),\ \forall x\in H.$$

Then $(H,\otimes)$ ***contains*** *$(H,\cdot)$ and write $\cdot\leq\otimes$. If $(H,\cdot)$ is structure, then it is* ***basic*** *and $(H,\otimes)$ is an* ***H_b-structure****.*

The Little Theorem [26]. Greater hopes of the ones which are *WASS* or *COW*, are also *WASS* and *COW*, respectively.

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H_v-field one uses the fundamental relation γ*:

Definition 1.4 [23], [26], [27]. *The H_v-ring $(R,+,\cdot)$ is called* ***H_v-field*** *if the quotient R/γ^* is a field.*

Let ω* be the kernel of the canonical map from R to R/γ*; then we call *reproductive H_v-field* any H_v-field $(R,+,\cdot)$ if the following axiom is valid:

$$x(R-\omega^*) = (R-\omega^*)x = R-\omega^*,\ \forall x\in R-\omega^*.$$

From the above definition we introduced a new class [31],[38]:

Definition 1.5 *The H_v-semigroup $(H,\cdot)$ is called* ***h/v-group*** *if the H/β^* is a group*.

Similarly the *h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces* etc, are defined. The h/v-group is a generalization of the H_v-group since the reproductivity is not necessarily valid. Sometimes a kind of *reproductivity of classes* is valid, i.e. if H is partitioned into equivalence classes σ(x), then the quotient is reproductive $x\sigma(y)=\sigma(xy)=\sigma(x)y$, $\forall x\in H$ [31].

An H_v-group is called *cyclic* [17], [26], if there is an element, called *generator*, which the powers have union the underline set, the minimal power with this property is the *period* of the generator. If there exists an element and a special power, the minimum one, is the underline set, then the H_v-group is called *single-power cyclic*.

To compape classes we can see the small sets. The problem of enumeration and classification of H_v-structures, or of classes of them, is complicate in H_v-structures because we have great numbers. The partial order, introduced in H_v-structures [26], restrict the problem in finding the minimal, *up to isomorphisms*, H_v-structures. In this direction we have results recently by Bayon & Lygeros as the following [1], [13]: In sets with three elements: Up to isomorphism, there are 6.494 minimal H_v-groups. The 137 are abelians; the 6.152 are cyclic. The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. The 7.926 are abelians; 1.013.598 are cyclic. 16 are very thin. Abelian H_v-groups with 4 elements are, 8.028.299.905, the 7.995.884.377.

Definitions 1.6 [25],[26], [38] *Let $(R,+,\cdot)$ be an H_v-ring, $(M,+)$ be COW H_v-group and there exists an external hope $\cdot : \boldsymbol{R}\times M\to P(M): (a,x)\to ax$, such that, $\forall a,b\in R$ and $\forall x,y\in M$ we have*

$$a(x+y)\cap(ax+ay)\neq\varnothing,\quad (a+b)x\cap(ax+bx)\neq\varnothing,\quad (ab)x\cap a(bx)\neq\varnothing,$$

*then M is called an **H_v-module** over R. In the case of an H_v-field F instead of H_v-ring R, then the **H_v-vector space** is defined.*

The fundamental relation ε* is defined to be the smallest equivalence such that the quotient M/ε* is a module (respectively, a vector space) over the fundamental ring R/γ* (resp. the fundamental field F/γ*). The analogous theorem to Theorem 1.2, is:

Theorem *Let $(\boldsymbol{M},+)$ be H_v-module over the H_v-ring **R**. Denote by **U** the set of all expressions consisting of finite hopes either on **R** and **M** or the external hope applied on finite sets of elements of **R** and **M**. We define the relation **ε** in **M** as follows:*

$$x\boldsymbol{\varepsilon}y \text{ iff } \{x,y\}\subset u \text{ where } u\in\boldsymbol{U}.$$

*Then the relation **ε*** is the transitive closure of the relation **ε**.*

Definitions 1.7 [28], [29], [38]. *Let $(H,\cdot)$ be hypergroupoid. We **remove** $h\in H$, if we consider the restriction of $(\cdot)$ in the set $H-\{h\}$. We say that $\underline{h}\in H$ **absorbs** $h\in H$ if we replace h by $\underline{h}$ and h does not appear in the structure. We say that $\underline{h}\in H$ **merges** with $h\in H$, if we take as product of any $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$, therefore the element h does not appeared in the hyperstructure.*

Let $(H,\cdot)$ be an H_v-group, then, if an element $\underline{h}$ absorbs all elements of its own fundamental class then this element becomes a single in the new H_v-group.

Definition 1.8 [35], [37] *Let $(\boldsymbol{L},+)$ be H_v-vector space over the field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\to\boldsymbol{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\boldsymbol{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi': \boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Consider the **bracket** (commutator) hope:*

$$[\ ,\ ] : \boldsymbol{L}\times\boldsymbol{L}\to P(L): (x,y)\to[x,y]$$

*then **L** is an H_v-Lie algebra over **F** if the following axioms are satisfied:*

(L1) *The bracket hope is bilinear, i.e.*

$[\lambda_1 x_1+\lambda_2 x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \varnothing$

$[x,\lambda_1 y_1+\lambda_2 y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2]) \neq \varnothing,\ \forall x,x_1,x_2,y,y_1,y_2 \in L$ *and* $\lambda_1,\lambda_2 \in \mathbf{F}$

(L2) $[x,x]\cap\omega_L \neq \varnothing,\ \forall x \in \mathbf{L}$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L \neq \varnothing,\ \forall x,y \in \mathbf{L}$

A well known and large class of hopes is given as follows [17],[21]:

Definitions 1.9 *Let $(G,\cdot)$ be a groupoid, then for every subset $P\subset G$, $P\neq\varnothing$, we define the following hopes, called* ***P-hopes****:* $\forall x,y \in G$

$$\underline{P}:\ x\underline{P}y = (xP)y\cup x(Py),$$

$$\underline{P}_r:\ x\underline{P}_r y = (xy)P\cup x(yP), \qquad \underline{P}_l:\ x\underline{P}_l y = (Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called ***P-hyperstructures****. The usual case is for semigroup $(G,\cdot)$, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.*

A generalization of P-hopes is the following [9], [10]*: Let $(G,\cdot)$ be abelian group and P a subset of G with more than one elements. We define the hope $\times_P$ as follows:*

$$x\times_P y \ = \begin{cases} x\cdot P\cdot y = \{x\cdot h\cdot y \,/\, h\in P\} & \text{if } x\neq e \text{ and } y\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope, ***P_e-hope****. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.*

A general definition of hopes, is the following [32], [35], [36], [37]:

Definitions 1.20 *Let H be a set with n operations (or hopes) $\otimes_1,\otimes_2,\ldots,\otimes_n$ and one map (or multivalued map) $f: H\to H$, then n hopes $\partial_1,\partial_2,\ldots,\partial_n$ on H are defined, called* ***∂-hopes*** *by putting*

$$x\partial_i y = \{f(x)\otimes_i y,\ x\otimes_i f(y)\},\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

or in case where $\otimes_i$ is hope or f is multivalued map we have

$$x\partial_i y = (f(x)\otimes_i y)\cup(x\otimes_i f(y)),\ \forall x,y\in H,\ i\in\{1,2,\ldots,n\}$$

Let $(G,\cdot)$ groupoid and $f_i:G\to G$, $i\in I$, set of maps on G. Take the map $f_\cup:G\to\mathbf{P}(G)$ such that $f_\cup(x)=\{f_i(x)\,/\,i\in I\}$, call it the union of the $f_i(x)$. We call the ***union ∂-hope*** *(∂), on G if we consider the map $f_\cup(x)$. An important case for a map f, is to take the union of this with the identity id. Thus, we consider the map $\underline{f}\equiv f\cup(id)$, so $\underline{f}(x)=\{x,f(x)\}$, $\forall x\in G$, which is called* ***b-∂-hope****, we denote it by $(\underline{\partial})$, so we have*

$$x\underline{\partial}y = \{xy,\ f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.$$

Remark If $\otimes_i$ is associative then ∂_i is WASS. If $\underline{\partial}$ contains the operation $(\cdot)$, then it is b-operation. Moreover, if $f:G\to P(G)$ is multivalued then the b-∂-hope*s* is defined by using the $\underline{f}(x)=\{x\}\cup f(x)$, $\forall x\in G$.

Motivation for the definition of ∂-hope is the *derivative* where only multiplication of functions can be used. Therefore, for functions s(x), t(x), we have $s\partial t=\{s't,st'\}$, (′) is the derivative.

Example. Consider all polynomials of first degree $g_i(x)=a_ix+b_i$. We have

$$g_1\partial g_2 = \{a_1a_2x+a_1b_2,\ a_1a_2x+b_1a_2\},$$

so it is a hope in the set of first degree polynomials. Moreover all polynomials x+c, where c be a constant, are units.

In hyperstructures there is the *uniting elements* method. This is defined as follows [3], [26], [28]: Let G be a structure and d be a property, which is not valid, and d is described by a set of equations. Consider the partition in G for which it is put together, in the same class, every pair of elements that causes the non-validity of d. The quotient G/d is an H_v-structure. The quotient of G/d by β*, is a stricter structure (G/d)β* for which d is valid.

2. Matrix representations.

H_v-structures are used in Representation (abbreviate by *rep*) Theory. Reps of H_v-groups can be considered either by generalized permutations or by H_v-matrices [18], [20], [24], [25], [26], [38]. The reps by generalized permutations can be achieved by using left or right translations. We present here the hypermatrix rep in H_v-structures and there exist the analogous theory for the h/v-structures.

Definitions 2.1 [20], [26] ***H_v-matrix*** *is called a matrix with entries elements of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices $\boldsymbol{A}=(a_{ij})$ and $\boldsymbol{B}=(b_{ij})$, of type $m\times n$ and $n\times r$ respectively, is defined, in the usual manner,*

$$\boldsymbol{A}\cdot\boldsymbol{B} = (a_{ij})\cdot(b_{ij}) = \{\boldsymbol{C}=(c_{ij})\ /\ c_{ij}\in \boxplus \boldsymbol{\Sigma} a_{ik}\cdot b_{kj}\},$$

and it is a set of $m\times r$ H_v-matrices. The sum of products of elements of the H_v-field is the union of the sets obtained with all possible parentheses put on them, called ***n-ary circle hope*** *on the hyperaddition.*

The hyperproduct of H_v-matrices does not nessesarily satisfy *WASS*.

The problem of the H_v-matrix representations is the following:

Definitions 2.2 *Let $(H,\cdot)$ be an H_v-group. Find an H_v-ring or an H_v-field $(F,+,\cdot)$, a set $M_R=\{(a_{ij})\ /\ a_{ij}\in R\}$ and a map*

$$\boldsymbol{T}: H \to M_R: h \mapsto \boldsymbol{T}(h) \text{ such that } \boldsymbol{T}(h_1h_2)\cap\boldsymbol{T}(h_1)\boldsymbol{T}(h_2) \neq \varnothing,\ \forall h_1,h_2\in H.$$

The map ***T*** *is called* ***H_v -matrix rep****.*

If the $\boldsymbol{T}(h_1h_2)\subset\boldsymbol{T}(h_1)\boldsymbol{T}(h_2)$, $\forall h_1,h_2\in H$ is valid, then ***T*** *is called* ***inclusion rep****.*

If $\boldsymbol{T}(h_1h_2)=\boldsymbol{T}(h_1)\boldsymbol{T}(h_2)=\{\boldsymbol{T}(h)\ /\ h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then ***T*** *is called* ***good rep*** *and then an induced rep* ***T**** *for the hypergroup algebra is obtained. If* ***T*** *is one to one and good then it is a* ***faithful rep****.*

The problem of reps is complicated because the cardinality of the product of H_v-matrices is very big. But it can be simplified in special cases such as the following: The H_v-matrices are over H_v-fields with scalars 0 and 1. The H_v-matrices are over very thin

H_v-fields. On 2×2 H_v-matrices, since the circle hope coincides with the hyperaddition. On H_v-fields which contain singles, then these act as absorbings.

The main theorem of reps is the following [20], [25], [26]:

Theorem 2.3 *A necessary condition in order to have an inclusion rep* $\mathbf{T}$ *of an* H_v*-group* $(H,\cdot)$ *by* $n\times n$ H_v*-matrices over the* H_v*-rind or* H_v*-field* $(F,+,\cdot)$ *is the following:*

For all classes $\beta^*(x)$, $x\in H$ *there must exist elements* $a_{ij}\in H$, $i,j\in\{1,...,n\}$ *such that*

$$\mathbf{T}(\beta^*(a)) \subset \{ A=(a'_{ij}) \,/\, a'_{ij}\in \gamma^*(a_{ij}),\ i,j\in\{1,...,n\}\}$$

So every inclusion rep $\mathbf{T}:H\to M_R$: $a\mapsto \mathbf{T}(a)=(a_{ij})$ *induces a homomorphic rep* $\mathbf{T}^*$ *of the group* H/β^* *over the field* F/γ^* *by setting* $\mathbf{T}^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, *where the* $\gamma^*(a_{ij})\in R/\gamma^*$ *is the ij entry of the matrix* $\mathbf{T}^*(\beta^*(a))$. $\mathbf{T}^*$ *is called* ***fundamental induced rep*** *of* $\mathbf{T}$.

Denote $\mathrm{tr}_\varphi(\mathbf{T}(x)) = \gamma^*(\mathbf{T}(x_{ii}))$ the fundamental trace, then the mapping

$$X_T : H \to R/\gamma^*: x\mapsto X_T(x) = \mathrm{tr}_\varphi(T(x)) = \mathrm{tr}T^*(x)$$

is called *fundamental character*. There are several types of traces.

Using several classes of H_v-structures one can face several reps [26], [29], [30], [38]:

Definition 2.4 *Let* $M=M_{m\times n}$ *be a module of* $m\times n$ *matrices over a ring* R *and take sets*

$$S=\{s_k:k\in K\}\subseteq R,\quad Q=\{Q_i:j\in J\}\subseteq M,\quad P=\{P_i:i\in I\}\subseteq M.$$

Define three hopes as follows

$$\underline{S}:\ R\times M\to P(M):(r,A)\to r\underline{S}A=\{(rs_k)A:k\in K\}\subseteq M$$

$$\underline{Q}_+:\ M\times M\to P(M):(A,B)\to A\underline{Q}_+B=\{A+Q_j+B:j\in J\}\subseteq M$$

$$\underline{P}:\ M\times M\to P(M):(A,B)\to A\underline{P}B=\{AP^t{}_iB:i\in I\}\subseteq M$$

Then $(M,\underline{S},\underline{Q}_+,\underline{P})$ *is a hyperalgebra over* R *called* ***general matrix P-hyperalgebra***.

The hope $\underline{P}$, which is a bilinear map, is a generalization of Rees' operation where, instead of one sandwich matrix, a set of sandwich matrices is used. The hope $\underline{P}$ is strong associative and the inclusion distributivity with respect to addition of matrices

$$A\underline{P}(B+C)\subseteq A\underline{P}B+A\underline{P}C\quad \forall A,B,C\in \boldsymbol{M}$$

is valid. Therefore, $(\boldsymbol{M},+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.

In a similar way a generalization of this hyperalgebra can be defined considering an H_v-ring or an H_v-field instead of a ring and using H_v-matrices instead of matrices.

Definition 2.5 *Let* $A=(a_{ij})$, $B=(b_{ij})\in\boldsymbol{M}_{m\times n}$, *we call* (A,B) *a* ***unitize pair*** *of matrices if* $A^tB=I_n$, *where* I_n *denotes the* $n\times n$ *unit matrix.*

The following theorem can be applied in the classical theory [37], [38].

Theorem 2.6 *If* $m<n$, *then there is no unitize pair.*

Proof. Suppose that n>m and that $A^tB=(c_{ij})$, $c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}$. Denote by A_m the block of the matrix A such that $A_m=(a_{ij})\in \boldsymbol{M}_{m\times m}$, i.e. we take the matrix of the first m columns. Then we suppose that we have $(A_m)^tB_m = I_m$, therefore we must have $\det(A_m)\neq 0$. Now, since n>m, we can consider the homogeneous system with respect to the 'unknowns' $b_{1n},b_{2n},\ldots, b_{mn}$:

$$c_{in}= \sum_{k=1}^{m} a_{ik}b_{kn} = 0 \quad \text{for } i= 1,2,\ldots,m.$$

From which, since $\det(A_m)\neq 0$, we obtain that $b_{1n}=b_{2n}=\ldots=b_{mn}= 0$. Using this fact on the last equation, on the same unknowns, $c_{nn}= \sum_{k=1}^{m} a_{nk}b_{kn} =1$ we have $0=1$, absurd. ■

We recall some definitions from [18], [20], [25].

Definition 2.7 *Let $(G,\cdot)$ hypergroupoid, is called* ***set of fundamental maps*** *on G, a set of onto maps*

$$Q = \{ q: G\times G\to G: (x,y) \xrightarrow{onto} z \mid z\in xy \}.$$

Any subset $Q_s\subset Q$ defines a hope $(\circ_s)$ on G as follows

$$x\circ_s y = \{ z \mid z= q(x,y) \text{ for some } q\in Q \}$$

$\circ_s \leq \cdot$, and $Q_s\subset Q_{os}$, where Q_{os} is the set of fundamental maps with respect to $(\circ_s)$. A $Q_a\subset Q$ for which every $Q_s\subset Q_a$ has $(\circ_s)$ associative (resp. WASS) is called associative (resp. WASS). A hypergroupoid $(G,\cdot)$ is ***q-WASS*** *if there exists an element $q_o\in Q$ which defines an associative operation $(\circ)$ in G. Remark that for H_v-groups we have $Q\neq\varnothing$.*

If G is finite, $cardG=|G|=n$, it is q-WASS with associative $q_o\in Q$. In the set K[G] of all formal linear combinations of elements of G with coefficients from a field K, we define an operation (+):

$$(f_1+f_2)(g)=f_1(g)+f_2(g),\forall g\in G, f_1,f_2\in K[G]$$

and a hope $(*)$, the ***convolution,***

$$f_1*f_2 = \{ f_q: f_q(g) = \sum_{q(x,y)=g} f_1(x) f_2(y), q\in Q \}.$$

$(K[G],+,*)$ is a multiplicative H_v-ring where the inclusion distributivity is valid, which is called ***hypergroupoid H_v-algebra****.*

For all $q\in Q$, $g\in G$, we have

$$|Q| \leq \prod_{(x,y)inGxG} (|xy|), \quad 1 \leq |q^{-1}(g)| \leq n^2-n+1 \quad \text{and} \quad \sum_{ginG} |q^{-1}(g)| = n^2.$$

The zero map $f(x)=0$ is a scalar element in K[G].

In the representation theory several constructions are used, some of them are the following [26], [28], [29], [30]:

Constructions 2.8 Let $(H,\cdot)$ be H_v-group, then for all $(\oplus)$ such that $x\oplus y\supset\{x,y\}$, $\forall x,y\in H$, the $(H,\oplus,\cdot)$ is an H_v-ring. These H_v-rings are called *associated to* $(H,\cdot)$ H_v-rings.

In rep theory of hypergroups, in sense of Marty where the equality is valid, there are three associated hyperrings $(H,\oplus,\cdot)$ to $(H,\cdot)$. The $(\oplus)$ is defined respectively, $\forall x,y\in H$, by:

$$\text{type a: } x\oplus y = \{x,y\}, \quad \text{type b: } x\oplus y = \beta^*(x)\cup\beta^*(y), \quad \text{type c: } x\oplus y = H.$$

In the above types the strong associativity and strong or inclusion distributivity, is valid.

Let $(H,\cdot)$ be H_V-semigroup and $\{v_1,\ldots,v_n\}\cap H=\varnothing$, an ordered set, where if $v_i<v_j$, when $i<j$. Extend $(\cdot)$ in $\underline{H}_n=H\cup\{v_1,v_2,\ldots,v_n\}$ as follows:

$$x\cdot v_i = v_i\cdot x = v_i\,, \quad v_i\cdot v_j = v_j\cdot v_i = v_j\,, \quad \forall i<j \quad \text{and}$$

$$v_i\cdot v_i = H\cup\{v_1,\ldots,v_{i-1}\},\ \forall x\in H,\ i\in\{1,2,\ldots,n\}.$$

Then $(\underline{H}_n,\cdot)$ is an H_V-group (*Attach Elements Construction).* We have $(\underline{H}_n,\cdot)/\beta^*\cong Z_2$ and v_n is single.

Some open problems arising on the topic of rep theory of hypergroups, are:

Open Problems.

a. Find standard H_v-rings or H_v-fields to represent all H_v-groups by H_v-matrices.

b. Find reps by H_v-matrices over standard finite H_v-rings analogous to Z_n.

c. Using matrices find a generalization of the ordinary multiplication of matrices which it could be used in H_v-rep theory (see the helix-hope [40]).

d. Find the 'minimal' hypermatrices corresponding to the minimal hopes.

e. Find reps of special classes of hypergroups and reduce these to minimal dimensions.

3. Helix-hopes and applications

Recall some definitions from [40], [16], [11]:

Definition 3.1 *Let $A=(a_{ij})\in M_{m\times n}$ be an $m\times n$ matrix and $s,t\in N$ be natural numbers such that $1\le s\le m$, $1\le t\le n$. Then we define the characteristic-like map $\underline{cst}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to the matrix A, the matrix $A\underline{cst}=(a_{ij})$ where $1\le i\le s$, $1\le j\le t$. We call this map* ***cut-projection*** *of type $\underline{st}$. In other words $A\underline{cst}$ is a matrix obtained from A by cutting the lines, with index greater than s, and columns, with index greater than t.*

We can use cut-projections on several types of matrices to define sums and products, however, in this case we have ordinary operations, not multivalued.

In the same attitude we define hopes on any type of matrices:

Definition 3.2 *Let $A=(a_{ij})\in M_{m\times n}$ be an $m\times n$ matrix and $s,t\in N$, such that $1\le s\le m$, $1\le t\le n$. We define the mod-like map $\underline{st}$ from $M_{m\times n}$ to $M_{s\times t}$ by corresponding to A the matrix $A\underline{st}=(\underline{a}_{ij})$ which has as entries the sets*

$$\underline{a}_{ij} = \{a_{i+\kappa s,j+\lambda t} \,/\, 1\le i\le s,\ 1\le j\le t. \text{ and } \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n\}.$$

Thus we have the map

$$\underline{st}\colon M_{m\times n}\to M_{s\times t}\colon A\to A\underline{st}=(\underline{a}_{ij}).$$

We call this multivalued map ***helix-projection*** *of type* $\underline{st}$*. Thus* $A\underline{st}$ *is a set of* $s\times t$*-matrices* $X=(x_{ij})$ *such that* $x_{ij}\in\underline{a}_{ij}$, $\forall i,j$*. Obviously* $A\underline{mn}=A$*. We may define helix-projections on 'matrices' of which their entries are sets.*

Let $A=(a_{ij})\in M_{m\times n}$ be a matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then it is clear that we can apply the helix-projection first on the columns and then on the rows, the result is the same if we apply the helix-progection on both, rows and columns. Therefore we have

$$(A\underline{sn})\underline{st} = (A\underline{mt})\underline{st} = A\underline{st}.$$

Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$ such that $1\le s\le m$, $1\le t\le n$. Then if $A\underline{st}$ is not a set of matrices but one single matrix then we call A *cut-helix matrix* of type $s\times t$. In other words the matrix A is a helix matrix of type $s\times t$, if $A\underline{cst}= A\underline{st}$.

Definitions 3.3 *(a) Let* $A=(a_{ij})\in M_{m\times n}$ *and* $B=(b_{ij})\in M_{u\times v}$ *be matrices and* $s=\min(m,u)$, $t=\min(n,u)$*. We define a hope, called* ***helix-addition*** *or* ***helix-sum****, as follows:*

$$\oplus: M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}): (A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t},$$

where

$$(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij})\,/\,a_{ij}\in\underline{a}_{ij}\text{ and }b_{ij}\in\underline{b}_{ij}\}.$$

(b) Let $A=(a_{ij})\in M_{m\times n}$ *and* $B=(b_{ij})\in M_{u\times v}$ *be matrices and* $s=\min(n,u)$*. We define a hope, called* ***helix-multiplication*** *or* ***helix-product****, as follows:*

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v},$$

where

$$(\underline{a}_{ij})\cdot(\underline{b}_{ij})=\{(c_{ij})=(\sum a_{it}b_{tj})\,/\,a_{ij}\in\underline{a}_{ij}\text{ and }b_{ij}\in\underline{b}_{ij}\}.$$

The the helix-addition is an external hope since it is defined on different sets and the result is also in different set. The commutativity is valid in the helix-addition. For the helix-multiplication we remark that we have $A\otimes B=A\underline{ms}\cdot B\underline{sv}$ so we have either $A\underline{ms}=A$ or $B\underline{sv}=B$, that means that the helix-projection was applied only in one matrix and only in the rows or in the columns. If the appropriate matrices in the helix-sum and in the helix-product are cut-helix, then the result is singleton.

Remark. In the set $M_{m\times n}$ the addition of matrices is an ordinary operation, therefore we are interested only in the 'product'. From the fact that the helix-product on non square matrices is defined, the definition of the Lie-bracket is immediate, therefore the ***helix-Lie Algebra*** is defined [36], [37], as well. This algebra is an H_v-Lie Algebra where the fundamental relation ε^* gives, by a quotient, a Lie algebra, from which a classification is obtained.

In the following we restrict ourselves on the matrices $M_{m\times n}$ where $m<n$. Obviously, we have analogous results in the case where $m>n$ and for $m=n$ we have the classical theory. In order to simplify the notation, since we have results on modm, we will use the following notation:

Notation. *For given* $\kappa\in\mathbb{N}-\{0\}$*, we denote by* $\underline{\kappa}$ *the remainder resulting from its division by m if the remainder is non zero, and* $\underline{\kappa}=m$ *if the remainder is zero. Thus a matrix*

$$A=(a_{\kappa\lambda})\in M_{m\times n},\ m<n\ \text{ is a }\textbf{cut-helix matrix}\text{ if we have }\ a_{\kappa\lambda}=a_{\underline{\kappa}\underline{\lambda}},\ \forall\kappa,\lambda\in\mathbb{N}-\{0\}.$$

Moreover let us denote by $I_c=(c_{\kappa\lambda})$ the ***cut-helix unit matrix*** *which the cut matrix is the unit matrix I_m. Therefore, since $I_m=(\delta_{\kappa\lambda})$, where $\delta_{\kappa\lambda}$ is the Kronecker's delta, we obtain that, $\forall\kappa,\lambda$, we have $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$.*

Proposition 3.4 *For $m<n$ in $(M_{m\times n},\otimes)$ the cut-helix unit matrix $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, is a left scalar unit and a right unit. It is the only one left scalar unit.*

Proof. Let $A,B\in M_{m\times n}$ then in the helix-multiplication, since $m<n$, we take helix projection of the matrix A, therefore, the result $A\otimes B$ is singleton if the matrix A is a cut-helix matrix of type $m\times m$. Moreover, in order to have $A\otimes B=A\underline{mm}\cdot B=B$, the matrix $A\underline{mm}$ must be the unit matrix. Consequently, $I_c=(c_{\kappa\lambda})$, where $c_{\kappa\lambda}=\delta_{\kappa\underline{\lambda}}$, $\forall\kappa,\lambda\in\mathbb{N}-\{0\}$, is necessarily the left scalar unit element.

Now we remark that it is not possible to have the same case for the right matrix B, therefore we have only to prove that cut-helix unit matrix I_c is a right unit but it is not a scalar, consequently it is not unique.

Let $A=(a_{uv})\in M_{m\times n}$ and consider the hyperproduct $A\otimes I_c$. In the entry $\kappa\lambda$ of this hyperproduct there are sets, for all $1\le\kappa\le m$, $1\le\lambda\le n$, of the form

$$\Sigma\underline{a}_{\kappa s}c_{s\lambda}=\ \Sigma\underline{a}_{\kappa s}\delta_{s\underline{\lambda}}=\ \underline{a}_{\kappa\underline{\lambda}}\ni a_{\kappa\lambda}.$$

Therefore $A\otimes I_c\ni A$, $\forall A\in M_{m\times n}$. ■

In the following examples of the helix-hope, we denote E_{ij} any type of matrices which have the ij-entry 1 and in all the other entries we have 0.

Example 3.5 [38] Consider the 2×3 matrices of the following form,

$$A_\kappa=E_{11}+\kappa E_{21}+E_{22}+E_{23},\quad B_\kappa=\kappa E_{21}+E_{22}+E_{23},\ \forall\kappa\in\mathbb{N}.$$

Then we obtain $A_\kappa\otimes A_\lambda=\{A_{\kappa+\lambda}, A_{\lambda+1}, B_{\kappa+\lambda}, B_{\lambda+1}\}$

Similarly, we have $B_\kappa\otimes A_\lambda=\{B_{\kappa+\lambda}, B_{\lambda+1}\}$, $A_\kappa\otimes B_\lambda=B_\lambda=B_\kappa\otimes B_\lambda$.

Thus the set $\{A_\kappa,B_\lambda\,|\,\kappa,\lambda\in\mathbb{N}\}$ becomes an H_v-semigroup which is not COW because for $\kappa\neq\lambda$ we have $B_\kappa\otimes B_\lambda=B_\lambda\neq B_\kappa=B_\lambda\otimes B_\kappa$, however

$$(A_\kappa\otimes A_\lambda)\cap(A_\lambda\otimes A_\kappa)=\{A_{\kappa+\lambda}, B_{\kappa+\lambda}\}\neq\varnothing,\ \forall\kappa,\lambda\in\mathbb{N}.$$

All elements B_λ are right absorbing and B_1 is a left scalar element, because $B_1\otimes A_\lambda=B_{\lambda+1}$ and $B_1\otimes B_\lambda=B_\lambda$. The element A_0 is a unit.

Example 3.6 Consider the 2×3 matrices of the forms,

$$A_{\kappa\lambda}=E_{11}+E_{13}+\kappa E_{21}+E_{22}+\lambda E_{23},\ \forall\kappa,\lambda\in\mathbb{Z}.$$

Then we obtain $A_{\kappa\lambda}\otimes A_{st}=\{A_{\kappa+s,\kappa+t}, A_{\kappa+s,\lambda+t}, A_{\lambda+s,\kappa+t}, A_{\lambda+s,\lambda+t}\}$.

Moreover $A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+s},A_{\kappa+s,\lambda+t},A_{\kappa+t,\lambda+s},A_{\kappa+t,\lambda+t}\}$, so

$$A_{\kappa\lambda}\otimes A_{st}\cap A_{st}\otimes A_{\kappa\lambda}=\{A_{\kappa+s,\lambda+t}\},\ \text{thus }(\otimes)\text{ is COW.}$$

The helix multiplication $(\otimes)$ is associative.

Example 3.7 Consider all *traceless* matrices $A=(a_{ij})\in M_{2\times3}$, in the sence that $a_{11}+a_{22}=0$. In this case, the cardinality of the helix-product of any two matrices is 1, or 2^3, or 2^6. These correspond to the cases: $a_{11}=a_{13}$ and $a_{21}=a_{23}$, or only $a_{11}=a_{13}$ either only $a_{21}=a_{23}$,

or if there is no restriction, respectively. For the Lie-bracket of two traceless matrices the corresponding cardinalities are up to 1, or 2^6, or 2^{12}, respectively. We remark that, from the definition of the helix-projection, the initial 2×2, block guaranties that in the result there exists at least one traceless matrix.

From this example it is obvious the following:

Theorem 3.8 *The Lie-bracket of any two traceless matrices $A=(a_{ij})$, $B=(b_{ij})\in M_{m\times n}$, $m<n$, contain at least one traceless matrix.*

During last years hyperstructures there is a variety of applications in mathematics and in many other sciences. The hyperstructures theory can now be widely applicable in industry and production, too. In several books and papers [2], [4], [5], [7], [8], [10], [12], [19], [26], [33], [39] one can find numerous applications.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1996 and they are called *e-hyperfields* [9], [14], [15], [33], [36]. The H_v-fields can give e-hyperfields which can be used in the isotopy theory for applications.

Definitions 3.9 *A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called* ***e-hyperstructure****, where we assume that $\forall x$, there exists an inverse x^{-1}, so $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. A hyperstructure $(\boldsymbol{F},+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called* ***e-hyperfield*** *if the following are valid:*

$(\boldsymbol{F},+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS,

$(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing: $0\cdot x=x\cdot 0=0$, $\forall x\in F$,

exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$,

$\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called ***e-hypernumbers****. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a* ***strong e-hyperfield****.*

A general construction based on the partial ordering of the H_v-structures:

Construction 3.10 [6], [36] *The* ***Main e-Construction****. Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ by extended $(\cdot)$, as follows:*

$$x\otimes y = \{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\}, \text{ and } g_1, g_2,\ldots\in G-\{e\}$$

Then $(G,\otimes)$ becomes an H_v-group, in fact is H_b-group which contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if $\forall x,y$ such that $xy=e$, so we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

An application combining hyperstructures and fuzzy theory, is to replace the scale of Likert in questionnaires by the *bar of Vougiouklis & Vougiouklis* [41]:

Definition 3.11 *In every question substitute the Likert scale with 'the bar' whose poles are defined with '0' on the left end, and '1' on the right end:*

The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point they feel expresses their answer to the question.

The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing [41]. The final suggested length of the bar, according to the *Golden Ratio*, is 6.2cm.

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