VOLUME 15, NUMBER 4, DECEMBER 1998

ALGEBRAS, GROUPS AND GEOMETRIES

Special issue
dedicated to
Marius Sophus Lie

EDITORIAL OFFICES:

Inst. for Basic Res.,
Palm Harbor, FL, USA

Estonia Academy of
Sciences, Tartu

HADRONIC PRESS, INC.
35246 US 19 N.# 115, PALM HARBOR, FL 34684, U.S.A.

ALGEBRAS, GROUPS AND GEOMETRIES

VOLUME 15, NUMBER 4, DECEMBER 1998

CONTENTS

PUBLISHER'S FOREWORD

Marius Sophus Lie

(December 17, 1842 - February 18, 1899)

is, undoubtedly, the most famous name in the mathematics and physics of this century and, also unquestionably, is the most famous Norwegian scientist of all times.

Therefore, it has been a privilege for the Hadronic Press to publish in this special issue of *Algebras, Groups and Geometries* the only known English translation of Lie's celebrated Doctoral Dissertation. The appearance of this publication in the occasion of the 100-th anniversary of Lie's death is particularly significant.

We have no words to express our appreciation to: Professor **Erik Trell** of the University of Linköping, Sweden, for conducting the translation and illustrating in a subsequent article the importance of Lie's original conception in contemporary physics; Professor **Jeremy Dunning-Davies** of the University of Hull, England, for assisting in the linguistic control of the translation; Professor **Ruggero Maria Santilli** for illustrating in a separate article the importance of preserving the abstract Lie axioms in the most advanced possible ongoing broadenings of Lie's theory as a necessary condition for consistency in physical applications; various Editors and Authors of *Algebras, Groups and Geometries* for their invaluable contribution to this important project.

G. C. Gandiglio
President
Hadronic Press, Inc.
Tarpon Springs
Florida, USA

January 15, 1999

MARIUS SOPHUS LIE'S DOCTORAL THESIS
OVER EN CLASSE GEOMETRISKE TRANSFORMATIONER

English Translation by Erik Trell[1]
Commentary by Erik Trell[1] **and Ruggero Maria Santilli**[2]
1. Faculty of Health and Sciences
University of Linköping
S-581 85 Linköping, Sweden
eritr@all.liu.se
2. Institute for Basic Research
P. O. Box 1577, Palm Harbor, FL 34682, USA
ibr@gte.net

Received April 3, 1998; revised December 15, 1998

Abstract

Marius Sophus Lie discovered and developed the whole mathematical chapter of the continuous transformation groups and related algebras. Their quintessential importance for modern mathematics and physics is well known, but would be even greater had not, instead of more faithful surface geodetical representations, a closed central orbit/standing wave model been chosen, in which case the theory becomes "mystically fit to describe mathematically" the ultimate layers of the universe. The origin and relation of Lie's groups and algebras to spherical geometry were outlined in Lie's celebrated Doctoral Thesis *Over en Classe Geometriske Transformationer* at the University of Christiania (now Oslo) in 1871. This memoir presents the English translation of Lie's Doctoral Thesis by one of us (E.T.) together with a few biographical notes and epistemological comments.

Introduction

Marius Sophus Lie (pronounced Lee) (Fig. 1) was born on the 17th of December, 1842, in Nordfjordeid in the Bergen episcopate on the west coast of Norway. His first graduation was as a public school teacher in 1865; not until a couple of years later was his mathematical creativity awakened under the influence of, above all, Poncelet and Plücker; and from there "his genius was developed at an astonishing exuberance and vigour".[1]

After only a few years work, "he had for ever tied his name to a string of new ideas and theories of the most far-reaching importance"[1], initially mostly within geometry and the principles of differential equations. In the latter field, already during his first eruptive period he enriched mathematics with a wealth of new material, that required all his remaining life to develop and deepen.

Starting from a series of early outlines he advanced an entirely new mathematical subject, the theory of the transformation groups: "a heroic feat that makes an epoch in the history of mathematics. Already his initial works reveal that he is a born mathematician".[1] In 1869 he delivered the first draft of his "imaginärteori" *(Christiania videnskabs-selskabs forhandlinger)*, where the germ of many of his later discoveries was laid down. The work drew attention and acclaim, and the same year he received a scholarship for studies in Berlin, Göttingen and Paris (where he was imprisoned for a month in Fontainebleau and barely escaped execution on suspicion of being a German spy in the then ongoing war).

While in France he published *(Comptes rendus de l'Academie des sciences, vol. LXX, 1870)* his disclosure of the spherical geometry ("one of the most beautiful discoveries in modern geometry", according to Darboux). After his return home in 1871, he received a university stipend and later the same year obtained the Ph. D. degree for the here translated thesis, *Over en Classe geometriske Transformationer,* where he "shed light upon a hitherto unforeseen connection between the metric and projective properties of space".[1] In many respects this thesis is transitional both in an historical and epistemological sense in Lie's *oeuvre* and accordingly should be the suitable reintroduction in relation to the classical roots and direct physical applications at which its translation here aims.

In 1872, when Lie applied for a professorship in Lund, Sweden (then in union with Norway), the Norwegian parliament created a post as extra

---

1 K.V. Hammer, whose compendium on Lie (1912) is followed here.

ordinary professor at the Christiania (now Oslo) University. In 1886 he was offered and accepted a chair at the university of Leipzig. In the meantime he had grown to be one of the most prominent mathematicians of his time. He developed his ideas in a whole sequence of dissertations, published partly in *Göttingische gelehrte Anzeigen (1870-74), Matematische Annalen, Leipzig (1871 and onwards)* and other international journals, and partly in *Christiania videnskabs-selskabs forhandlinger* and, mainly, *Archiv for mathematik og naturvidenskab*, which became his domestic forum par preference. Its first issue (January, 1876) was opened by his pioneering treatise, *Theorie der Transformationsgruppen*, the prelude of a long succession of works in which the new subject is evolved to an entire system, revolutionising a vast area of mathematics and which in origin, exposition and completion in every detail, is exclusively Lie's own achievement.

Fig. 1. Photo of Marius Sophus Lie

In 1883, Picard drew the French mathematicians' attention to Lie's new theories, and each year, while Lie was in Leipzig, École normale supérieure sent its best students to him. Parallel with his elaboration of the transformation group theory, he developed their applications and other of his basic conceptions, inter alia the important study of minimal surfaces in *Classification der Flächen nach der Transformationsgruppe ihrer geodätischen Curven (Univ-*

progr. Christiania, 1879). Gradually, however, the transformation groups absorbed his thoughts and he therefore set out to give them an altogether new rendition, in which all problems that had emerged in the meantime were to be included. In collaboration with his student, Friedrich Engel, he undertook the edition of his gigantic main opus, *Theorie der Transformationsgruppen*, printed in three volumes 1888-93 in Leipzig, and also (with H. Scheffer) his lectures: *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen (1891)* and *Vorlesungen über continuerliche Gruppen mit geometrischen und anderen Anwendungen (1893).*

From 1895, increasing ill-health disturbed his plentiful flow of production, and contributed to his decision to return to his homeland, where a professorial honorarium of 100,000 Norwegian Crowns (and release of lecture duties) had been granted to him by the parliament. However, when he was finally able to leave Lepizig, in 1898, he was terminally ill (with pernicious anaemia, now perfectly curable by vitamin B_{12} supply, but then ascribed to "restless mental exhaustion" by his titanic labour burden). He died on the 18th of February, 1899, in Christiania. No biography is written, but there is a bibliography of his printed works *(Fr. Engel, Bibliotheca Mathematica, ser. 3, vol. 1, Leipzig, 1900)* and there is also a register of his posthumous manuscripts *(C. Störmer, Christiania videnskabs-selskabs forhandlinger, 1904).*

Some remarks on the translation and transcription of his thesis into English may be warranted. First a linguistic one; three words in the Norwegian language for sphere: "Klot", Kugle" and "Sphære", he used interchangeably and may at places have slightly different connotations such as reference to earlier, mainly German works. This also applies to the word "Complex" which often relate to a set or collection or array of quantities. Other examples are "Avbildning" which may mean representation, projection, mapping, as well as transformation. "Krumnings-Curver" ("Krumning" ≈ curve) may appear to be a tautology but is a specification that mainly causes the translational problem: curved curves, or bent, or rounded (as here chosen) curves? Abbreviations like F and f ("flade" = surface) stem from the respective Norwegian word.

The terminology might at places have a slightly archaic ring, and the general epistemological perspective to bear in mind is of course the temporal one. His work was performed and his thesis written well before the days of centrally oriented Rutherford and Bohr orbital models, relationally one-dimensional Schrödinger equations and other present elementary particle theory and practice (one cannot help musing that had the experimental discoveries and projections taken another sequence it might have been natural to apply the Lie groups and algebras in their direct form). The mathematical foundations were likewise primordial, their *chef d'oeuvres* being Cartesius' co-ordinate geometry and Plücker's straight line algebra. These underpinnings, of the time as well as of space and conceptions, are important to consider (and also throw some light upon the philosophical tenor of his work).

One may say that the gist and meaning are in the ideal, the logical; undistracted by later representation theory largely founded upon initial experimental records and their pioneer, yet permanented interpretations - which are concentric, relating to planetary orbits, one-dimensionally extended waves, quantum wells etc. Lie´s own primary realisations were clearly associated with surfaces, of which there are many references in his thesis. For instance: "When x y z and X Y Z are perceived as point co-ordinates for r and R, one can say, that by a transformation of this kind is defined *a correspondence between two spaces' surface-elements and nota bene the most general*"...."*always* defines a reciprocal correspondence between the two spaces' surface elements"...."always determines a transformation that turns surfaces that touch each other in like surfaces" (§ 6, n. 16).

We have, from two complementary approaches, arrived at the critical need of reintroducing the primary Lie groups and algebras; and both approaches have mathematical and biophysical compartments. One is tempted to allude: "for the one, who has immersed himself in the spirit of Lie´s work, there is a fundamental defect in the idea to import precipitations and trajectories on the virtual germ membrane of the space to a concentric confinement in the inside of the film". Whereas on the surface of an arbitrary manifold there are arbitrary positive and negative regions, which are fully equivalent and interchangeable, the singular introjection of them to the origin of the co-ordinate system also annuls this commensurability. This unfortunate anomaly is the root of the isotopic theory which re-establishes full solvability and is thus Lie-admissible in a double sense.

The bio-physical terminology applied above was not accidental. Only recently, by the advent and advancement of refined nano-technology, sweep-electron microscopes etc., has it become evident that physics appears and behaves quite classically and Renaissance-like even at the finest level of resolution, and that events, arrays, particles, filaments generate, align, emerge on surfaces, monolayers, growth strata....and that phenomena like folding of proteins, racemic forms, parity and many others cannot readily be explained in a centre-of-mass reference, or indeed in a space whose co-ordinate axes do not assume the properties and structures by and of the reciprocal elements (one may compare, for instance, foot-note 1 of § 9, n. 24 :....."the sphere (X_0, Y_0, Z_0, H_0) is the image of the axes of the linear complex at hand."). That is, a "shape space"[1] is like shaped itself, which is another of the true Lie group and algebra implications. All of these actual physical facts can only be

1 See L. Holm and C. Sander. Science **273**, 595 (1996).

consistent with and enabled by external, "complex-cones" or "cone-intersection-complexes" as defined by Lie (e.g. § 5, n. 15), and which are genuinely differential, geodetic front functions, and not centrally closed orbital ones. Such is the complementary re-entry to Lie's original works and concepts, as further leading to exact reproduction of elementary particle symmetries and a novel, iso-unit expansion proof of Fermat's last theorem.

One also has to submit to the fact that in the ideal, "hyperbolic" space of the mind, albeit a co-ordinate, a point is an infinitesimally, actually unimaginably small locus, it can yet harbour an infinity of projections, tangents, crossings onto, into and through itself. Hence, in the "philosophical reflections on the nature of Cartesian geometry", that were natural for Lie, it is indeed so that "as element for the geometry of the space can be used what-so-ever a curve that is dependent on three parameters", that is, x, y, and z in the most basic of cases. There remains a congruence, a reciprocity, equivalence, even identity between the elements as well as the spaces. But as mentioned, the latter condition has been overlooked in later representation theory, where even sectored neighbourhoods thus tend to be centred, i.e., mapped "onto a neighbourhood of the origin of R_N".[1]

Lie´s exploration took place in the familiar ordinary Cartesian space, the altogether robust primary appearance of which we have no reason to doubt up to this very day, and whose ground representation, also in the Lie groups and algebras, is the well-known rectilinear one - the x,y and z co-ordinates that span our real world. Their own static mathematical projection and elements would be the straight, Plücker line complexes, from which the Lie extension virtually submerges to the surface of the sphere - or vice versa. It is in that sense, too, a differential mathematics, a transition between space forms, from the straight to the round, and what characterises and monitors such an all-pervasive process.

Marius Sophus Lie presented his thesis on the 12th of June, 1871, at the philosophical faculty of the Royal Norwegian Frederiks University in Christiania (Fig. 2). The ensuing translation may illuminate some of the aforesaid, and place his work in its proper frame and position. As mentioned it reflects a transitional stage between his early ideas and later expansion and refinement of them. It is also a basic work, straightforward to comprehend and trace in its bearings. Therefore, it may serve as a source document for the universal scientist to contemplate in a meta-analysis way.

---

1 See R. Gilmore. Lie Groups, Lie Algebras and Some of Their Applications, pp. 35-36. (John Wiley & Sons, New York, London, Sydney, Toronto, 1974)

Över

en Classe geometriske Transformationer

af

Marius Sophus Lie.

Afhandling

for den philosophiske Doctorgrad ved det Kgl. Norske Frederiks Universitet.

Christiania 1871.

Trykt hos Brøgger & Christie.

☞ Disputations-Actus foregaaer den 12te Juni 1871 i Universitetets Festsal.

Fig. 2. Title page of Marius Sophus Lie´s Ph.D. thesis.

Over a Class of geometrical Transformations.[1]

The rapid development of geometry in our century stands, as is well known, in an intimate dependence on philosophical reflections upon the nature of Cartesian Geometry - reflections, which are expounded in their most universal form by *Plücker* in his oldest works.

For one, who has immersed himself in the spirit of Plücker´s works, there is nothing fundamentally new in the idea, that as element for the geometry of the space can be used any curve that is dependent on three parameters. When none-the-less no one, as far as I know, has realised this thought, the ground must probably be sought in that no advantage that might result from this was seen.

I have been brought to a general study of the said theory by my finding that, through a particularly remarkable transformation, the theory of main tangential curves can be brought back to that of rounded curves.

Following Plücker's trail I discuss the equation system:

$$[F_1\,(x\;y\;z\;X\;Y\;Z) = 0, \quad F_2(x\;y\;z\;X\;Y\;Z) = 0],$$

which in one meaning, later to be explained, defines a general reciprocity between two spaces. When in particular the two equations are linear in relation to each system's variables, a projection is obtained by which to each space's points correspond in the other space the lines of a Plücker Line-complex. The simplest among the class of transformations I obtain in this way is the well-known Ampèreish, which hereby is shown in a new light. In particular I study the aforementioned projection, upon which I found a - as it appears to me - *fundamental relation between the Plücker line geometry and a spatial geometry whose element is the sphere.*

While I was occupied with the present thesis I have been standing in a vivid exchange of thoughts with Plücker's pupil, Dr. Felix Klein, to whom I owe many ideas, more, no doubt, than what by quotation I am able to indicate.

I will also notify that this work has many points of contact with my works over

1 The most important aspects of the present thesis I reported to Christiania Science Association in July and October 1870. One may also compare a note of Mr Klein and me in the Berlin Academy's "Monatsbericht" 15 December, 1870.

the imaginaries of plane geometry. When I am not letting these relations be exposed in my narrative here, it is partly because I consider it incidental, and partly that I don't wish to deviate from the customary language of mathematics.[1]

First Section

Over a new Reciprocity of Space

§ 1.

Reciprocity between two planes or two spaces.

1. The Poncelet-Gergonne reciprocity theory can, as is well known in respect of plane geometry, be derived from the equation:

$$X(a_1x + b_1y + c_1) + Y(a_2x + b_2y + c_2) + (a_3x + b_3y + c_3) = 0 \qquad (1)$$

or by the equivalent:

$$x(a_1X + a_2Y + a_3) + y(b_1X + b_2Y + b_3) + (c_1X + c_2Y + c_3) = 0$$

provided that one interprets (x,y) and (X,Y) as Cartesian point co-ordinates for two planes.

If, namely, one uses the term, *conjugate*, of two points (x,y) and (X,Y), whose co-ordinate values satisfy equation (1), one can say that, to a given point (x,y), conjugate points (X,Y) form a straight line that can be perceived as *corresponding* to the given point.

When all points of a given straight line have a mutual conjugated point in the other plane, their corresponding straight lines go through this common point.

The two planes are thus mapped into each other by equation (1) in such a way that to the points of the one plane correspond the straight lines of the other

1 Guided by the theories expounded in the present thesis, Mr Klein in a recently published note (Gesellschaft d. Wissensch. zu Göttingen, 4 March 1870) brought the Plücker ideas one step forward in that he showed, that the Plücker line geometry - or by my transformation the corresponding sphere geometry - in a remarkable way manifests itself as an illustration of the metric geometry between four variables.

plane. To points of a given line λ correspond the straight lines that go through λ's image point.

But herein lies just the principle of the Poncelet-Gergonne reciprocity theory.

One considers now in the one plane a polygon whose corners are: $(p_1, p_2 \ldots p_n)$, and in the other plane the polygon, whose sides: $(S_1, S_2 \ldots S_n)$ correspond to these points. From what we have said follows also that the last-mentioned polygon's corners: $(S_1\ S_2)\ (S_2\ S_3) \ldots (S_{n-1}\ S_n)$ are projection points of the given sides: $(p_1\ p_2)\ (p_2\ p_3) \ldots (p_{n-1}\ p_n)$, that thus the two polygons stand in a reciprocal relation.

By a limit transition one is brought from here to a consideration of two curves c and C, that correspond to each other in such a way that the tangents of the one project themselves as the points of the other. Two such curves are said to be *reciprocal* relative to equation (1).

2. Plücker[1] has based a generalisation of the above presented theory on the interpretation of the general equation:

$$F[x,y\ \ X,Y] = 0 \qquad (2)$$

Those to a given point (x,y) [or (X,Y)] conjugated points (X,Y) [or (x,y)] now form a curve C [or c], which is produced by equation (2), when in the same (x,y) [or (X,Y)] are regarded as parameters, (X,Y) [or (x,y)] on the contrary as running co-ordinates.

By equation (2) the two planes are thus projected into each other in such a way, that to the points of the one plane unambiguously correspond the curves of a certain curve-net in the other.

Quite as before it is understood, that to points of a given curve c [or C] correspond the curves C (or c) that go through the given image point.

To a polygon of curves c $(c_1\ c_2 \ldots c_n)$ correspond n points: $(p_1\ p_2 \ldots p_n)$ which pairwise lie upon those curves C: $(p_1\ p_2)\ (p_2\ p_3) \ldots (p_{n-1}\ p_n)$, whose image points are corners of the given curvilinear polygon. Eventually one is here also brought to a consideration of curves σ and Σ in the two planes, that stand in such a mutual relation to each other, that to the points of the one correspond the

1 Analytisch geometrische Entwickelungen. T.I. Zweite Abth.

curves c [or C] that envelope the other. In general this reciprocity relation is not complete, though, inasmuch as adjunct forms usually appear.

3. *Plücker*[1] bases the general reciprocity between two spaces on the interpretation of the general equation:

$$F\,(x\ y\ z\ \ X\ Y\ Z) = 0.$$

When F is linear in respect of each system's variables, the Poncelet-Gergonne reciprocity between the two spaces' points and planes is obtained.

In the present thesis and especially in the first section of the same, I aim at studying a new reciprocity of space, which is to be considered side-ordered to Plücker's, and that is defined by the equation system:

$$F_1\,(x\ y\ z\ X\ Y\ Z) = 0$$
$$F_2\,(x\ y\ z\ X\ Y\ Z) = 0,$$

where (x y z) and (X Y Z) are perceived as point co-ordinates of two spaces r and R.

§ 2.

A space curve, that depends upon three parameters can be chosen as the element of the geometry of the space.

4. The transformation of geometric postulates that is founded upon the Poncelet-Gergonne or the Plücker reciprocity can - as Gergonne and Plücker have emphasised - be seen from a higher point of view, which we here want to state, because the same applies to our new reciprocity.

The Cartesian geometry, namely, translates any geometric theorem into an algebraic one and thus of the geometry of the plane renders a faithful representation of the algebra of two variables and likewise of the geometry of space a representation of the algebra of three variable quantities.

Now Plücker in particular has directed attention to the circumstance that to Cartesian analytic geometry is attributed a double conditionality.

Descartes produces a system of values of the variables x and y at a *point* in the plane; he has, as one uses to express it, *chosen the point as the element of the*

---

1 Although I am unable to provide a quotation, I believe that it is correct to attribute this reciprocity to Plücker.

geometry of the plane, while with the same justification one could use the straight line or any curve at all depending upon two parameters. Now - as regards the plane - the geometrical transformation that is founded upon the Poncelet-Gergonne reciprocity can be perceived as consisting of a transition from a point to a straight line as element, and likewise the Plücker plane reciprocity in the same sense rests upon the introduction of a curve depending upon two parameters as the element of the geometry of the plane.

Further, Descartes produces a quantity-system (x,y) by the point in the plane whose distance from two given axes equals x and y; *he has among the unlimited manifold of possible co-ordinate systems chosen a definite one.*

The progress that geometry has made in the 19th century depends to a large part upon the fact that these two conditions in Cartesian analytic geometry have been clearly recognised as such, and it is accordingly close at hand to exploit these important facts even more.

5. The in the following presented new theories are founded upon the fact, *that one can choose any space-curve which depends upon three parameters as the element of the geometry of the space.* If, for instance, one remembers that the equations of the straight line in space contain four essential co-ordinates, one realises that the straight lines that meet a given condition may be used as the element of a geometry of the space, which - like the ordinary one - gives a faithful representation of the algebra of three variables.

Hereby, however, a certain line-system - *the Plücker line-complex* - is distinguished, and it is as a consequence of this seen that a certain representation of this kind can have only a limited utility. If, however, it concerns a *study of the space relative to a given line-complex*, it may be particularly suitable to choose the straight lines of this complex as space-element. As is well known, in the metric geometry, the infinitely distanced imaginary circle and as consequence hereof the straight lines that intersect the same are marked out, *and therefore there might a priori be some grouns to suppose that, as regards the treatment of certain metric problems, it might be advantageous to introduce these straight lines as element.*

It is to be emphasised that when we, for instance, have just said that it is possible to choose the straight lines of a line-complex as space-elements, this is something different, something more particular if one so likes, than those ideas that lie as a ground for Plücker's last work: *"Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raum-Element"*. Plücker had already drawn attention to the fact that it is possible to create a representation of an algebra that embraces an arbitrary number of variables in that one namely introduces a figure that depends upon the necessary number of parameters as element. Especially he emphasised[1], that the space-line has four co-ordinates, that by choosing the same as space element one thus obtains a geometry for which the space has *four* dimensions.

§ 3.

Curve complex. New geometric representation of partial differential equations of the first order. The main tangent-curves of a line-complex.

6. *Plücker* has used the expression *line-complex* to denote the collection of the straight lines, that satisfy a given condition, and which thus depend upon three unspecified (undetermined) parameters. By analogy herewith, in the following, by *curve-complex* I understand an arbitrary system of space-curves c, whose equations:

$$f_1 (x\ y\ z\ \ a\ b\ c) = 0, \quad f_2 (x\ y\ z\ a\ b\ c) = 0 \qquad (3)$$

contain *three essential constants.*

On differentiation of (3) with respect to x y z and elimination of a, b, c between the two new and the initial equations a result is obtained of the form:

$$f(x\ y\ z\ \ dx\ dy\ dz) = 0 \qquad (4)$$

If here x, y, z are perceived as parameters, dx, dy, dz on the other hand as direction-cosines, each point in the space defined by (4) is associated with a cone, namely, the collection of tangents to those complex-curves c, that go through the point in question. These cones I call *elementary complex-cones*; further I use the designation: *elementary complex-direction* to denote an arbitrary line-element (dx dy dz) that belongs to a complex-curve c. *The collection of the to a point corresponding elementary complex-directions generate the to the point associated elementary complex-cone.*

To a given system (3) or - as one may also say - to a given curve-complex correspond a definite equation: [f = 0]; *on the other hand [f = 0], through the mentioned operations, can be derived from an unlimited manifold of systems* (3).

1 Geometrie des Raumes. n. 258. (1846).

If namely one chooses an arbitrary relation of the form:

$$\psi[x\ y\ z\ dx\ dy\ dz\ \alpha] = 0,$$

where α denotes a constant, and represents

$$\varphi_1(x\ y\ z\ \alpha\ \beta\ \gamma) = 0, \quad \varphi_2(x\ y\ z\ \alpha\ \beta\ \gamma) = 0$$

the integral of the simultaneous system:

$$f = 0, \quad \psi = 0,$$

it is evident, that also $[\varphi_1 = 0,\ \varphi_2 = 0]$ by differentiation relative to x, y, z and elimination of α, β, γ leads to: (f=0).

Any curve of this new complex: $[\varphi_1 = 0,\ \varphi_2 = 0]$ is enveloped by curves c, inasmuch as its elements are all complex-directions.

7. A partial differential equation of the first order between x, y, z is, according to *Monge*, equivalent to the following problem: to find the general surface which in each of its points touches a cone associated with the point in question and whose general equation in plane co-ordinates is produced by just the given partial differential equation.

Lagrange and *Monge* have led this problem back to the determination of a definite curve-complex - the so called *characteristic curves* - inasmuch as they have shown, that one always gets an integral surface by adjoining to a surface a collection of characteristic curves each of which intersects the nearest preceding one.

One may note that the equation:

$$f(x\ y\ z\ \ dx\ \ dy\ \ dz) = 0,$$

which the characteristic curves, according to the aforesaid determine, is to be considered as equivalent to the partial differential equation itself, inasmuch as both these equations are the analytic definition of the same three-fold infinity of cones.

8. *A general geometric interpretation of partial differential equations of the first order between x y z is obtained by showing that the task: finding the* *general surface which at all its points has a three-point contact with a curve of a given curve-complex - whereby it is implied, however, that the said curve is not in its whole extension residing upon the surface - finds its analytic expression in a partial differential equation of the first order. When, further:*

$$f(x\ y\ z\ \ dx\ dy\ dz) = 0$$

is the equation, that the characteristic Curves determine, any curve-complex whose equations satisfy (f = 0) will stand in the said geometrical relation to the given partial differential equation.

One considers that a complex of curves c is given, which satisfies (f = 0) and analytically expresses the requirement that a surface [z= F(x y)] at each of its points has a three-point contact with a curve c, *without, however, excluding the possibility of an even more intimate contact.* It is easy to see that to determine z a partial differential equation of the second order ($\delta_2 = 0$) is obtained.[1] But any surface which is generated by infinitely many c, apparently satisfies ($\delta_2 = 0$), and hence its general integral with two arbitrary functions is known. By analytical deliberations of great simplicity - albeit formally of some breadth - I intend to show that the partial differential equation of the first order ($\delta_1 = 0$), that corresponds to (f = 0), satisfies ($\delta_2 = 0$). When now apparently ($\delta_1 = 0$) in general is not included in the aforementioned integral, ($\delta_1 = 0$) is a *singular* integral of ($\delta_2 = 0$).

The equation: [f(x y z dx dy dz) = 0] gives by differentiation:

$$f'_x dx + f'_y dy + f'_z dz + f'_{dx} d^2x + f'_{dy} d^2y + f'_{dz} d^2z = 0, \qquad (6)$$

whereby ($dx\ dy\ dz\ \ d^2x\ d^2y\ \ d^2z$) are to be regarded as belonging to an arbitrary curve, that satisfies: (f = 0). In particular (6) is valid for [$\delta_1 = 0$]'s characteristic curves, and in that we denote these by an index, we obtain:

$$f'_{x_1} dx_1 + \dots f'_{dx_1} d^2x_1 + \dots = 0.$$

Now remarking that any curve that touches one of ($\delta_1 = 0$)'s integral

1 ($\delta_2 = 0$) has the form: [$A(rt - s^2) + Br + Cs + Dt + E = 0$]. One may compare with a dissertation by Boole in Crelles Journal. Bd. 61.

surfaces: (U = 0) satisfies the equation:

$$\frac{dU}{dx}\,dx + \frac{dU}{dy}\,dy + \frac{dU}{dz} = 0, \qquad (7)$$

that further any curve, that by (U = 0) has a three-point contact, in addition fulfills the relation:

$$\frac{d^2U}{dx^2}(dx)^2 + \ldots \left(\frac{dU}{dx}\right) d^2x \ldots = 0, \qquad (8)$$

it is seen, that any characteristic curve, that lies upon (U = 0), satisfies (7) as well as (8).

But at each of its points (U = 0) touches the associated cone of the system: (f = 0), and thus apply the equations:

$$f'_{dx} = \rho \frac{dU}{dx'}, \quad f'_{dy} = \rho \frac{dU}{dy'}, \quad f'_{dz} = \rho \frac{dU}{dz'},$$

in which ρ denotes an undetermined proportionality factor. Thus the accentuated equation (8) transforms into the following

$$\rho\left[\frac{d^2U}{dx_1^2}(dx_1)^2 + \ldots\right] + \left[f'_{dx_1}\, d^2x_1 + \ldots\right] = 0.$$

But we know, that:

$$f'_{x_1}\, dx_1 + \ldots + f'_{x_1}\, d^2x_1 + \ldots = 0$$

and hence is:

$$\rho\left[\frac{d^2U}{dx_1^2} + \ldots\right] = f'_{x_1}\left[dx_1 + \ldots\right]$$

or by exclusion of the now unneccesary index:

$$\rho\left[\frac{d^2U}{dx^2}\,d\,x^2 \;+\ldots\right] = \; f_x\,dx + \ldots$$

Now, however,

$$\rho\,\left[\frac{dU}{dx}\,d^2x + \frac{dU}{dy}\,d^2y \;+ \frac{dU}{dz}\,d^2z\,\right] = \left[f_{dx}\,d^2x + \ldots\right].$$

and thus the equation:

$$\rho\,\left[\frac{dU}{dx}\,d^2x + \frac{dU}{dy}\,d^2y \;+ \frac{dU}{dz}\,d^2z \;+ \frac{d^2U}{dx^2}(dx)^2 + \ldots\;\right] =$$

$$= f'_x\,dx + f'_y\,dy + f'_z\,dz + f'_{dx}\,d^2x + f'_{dy}\,d^2y + f'_{dz}\,d^2z,$$

whose right and left parts thus simultaneously vanish.

Our expansions show, that any curve, that satisfies ($f = 0$), and that touches one upon ($U = 0$) lying characteristic curve with the said surface has a three-point contact; ($\delta_1 = 0$) is thus a singular integral of ($\delta_2 = 0$).

Finally we show that ($\delta_2 = 0$) does not allow any other singular integral.

On an integral surface I of ($\delta_2 = 0$), every point is namely associated with a direction - the respective, three-point-contacting c's tangent. If it is now implied, that I is not generated by a manifold c, so goes through each point of I two converging c, that both touch upon the surface in the point in question. But in consequence, I in each of its points is contacted by the corresponding elementary complex-cone; I satisfies the equation: ($\delta_1 = 0$).

9. *Corollary. The determination of the most general surface that in each of its points has an - not upon the surface lying - main tangent, belonging to a given line-complex, depends upon the solution of a partial differential equation of the first order, whose characteristic curves are enveloped by the complex's lines. The said curves appear in this case as main tangent-curves on the integral surfaces.*

We present an independent geometrical proof of this corollary.

The partial differential equation, whose characteristics are enveloped by a given line-complex's lines, is according to Monge's theory the analytic expression of the following problem: to find the most general surface which at each of its points touches the complex-cone corresponding to the point. But when a curve's tangents belong to a line-complex, the same osculation-plane is the tangent plane to the corresponding complex-cone, and thus our characteristic curves' osculation plane is the tangent plane for all integral surfaces which contain the curve at hand. Here a couple of further remarks are required, which, however, may be a repetition of what we have said before.

Any line-complex determines, according to the above-mentioned, a complex of curves that are enveloped by the line-complex's lines, and which have the property to be main tangent-curves on any surface that is generated by a system of these curves, each of which intersects the preceding. *This complex of curves we in the following designate the line-complex's main tangent-curves.*

I owe Mr. Klein the acknowledgement, that the congruence of straight lines, that Plücker calls a line-complexe's *singular lines*, belong to the said curve-complex. If the given complex is formed by a surface's tangents [or by the straight lines that intersect a curve], then all the lines of the line-complex are singular lines and hence also main tangent-curves.

§ 4

The equation system : $F_1(x\ y\ z\ \ X\ Y\ Z) = 0$, $F_2(x\ y\ z\ \ X\ Y\ Z) = 0$, determines a reciprocity between two spaces.[1]

10. We begin a study of the spatial reciprocity determined by the equations:

$$\left.\begin{aligned} F_1(x\ y\ z\ \ X\ Y\ Z) &= 0 \\ F_2(x\ y\ z\ \ X\ Y\ Z) &= 0 \end{aligned}\right\} \qquad (9)$$

when in the same (x y z) and (X Y Z) are perceived as point co-ordinates of two spaces r and R.[2]

1 One compares this paragraph with § 1.

2 Things, that belong to the space r, we as a rule denote by small letters; on the other hand versals are used for everything belonging to R.

When the term conjugated is used of two points, whose co-ordinate values (x y z) and (X Y Z) satisfy the relations (9), one can say, that the to a given point (x y z) conjugated points (X Y Z) generate a curve C, which is formulated by (9), when in the same (x y z) are regarded as parameters, (X Y Z) on the other hand as running co-ordinates.

To the curves of the space r, thus unambiguously correspond the curves C of a definite curve-complex in R, and likewise appears in r a complex of curves c, which stand in the same relation to R's points.

A curve c's points have a mutual conjugated point in R, and in consequence their corresponding cuirves C go through this mutual point.

The two spaces are thus mapped by the equation system (9) into each other in such a way, that to each space's points unambiguously correspond in the other the curves of a definite complex. When a point describes a complex-curve, the complex-curve corresponding to the point turns round[1] *the image-point of the intersected one.*

11. It is now possible to show, that the equations (9) determine a general reciprocity between figures in the two spaces and especially between curves that are enveloped by complex-curves c and C.

When two curves of the one complex have a mutual-point - which is obviously not so in general - their imagepoints lie upon a complex-curve. Note in particular, that two endlessly close-lying complex-curves, which intersect each other, project themselves as two points whose infinitesimal connection-line is an elementary complex-direction.

One now considers a curve σ in r, that is enveloped by curves c, and all curves C that correspond to σ's points. Two consecutive of these C would, after what we have just said, intersect each other, and thus their collection determines an envelope-curve Σ.

It is further apparent, that when a point runs through Σ, the corresponding c will envelop a curve σ^1, and it can be shown that σ^1 is precisely the originally given curve σ.

One may namely consider on the one hand a curved polygon formed by

---

1 The term, "turns round" is unfortunate inasmuch as, of course, a turn associated by a change of form is meant.

complex-curves ($c_1\ c_2\ c_3 \ldots c_n$), whose corners are ($c_1\ c_2$) ($c_2\ c_3$)...($c_{n-1}\ c_n$) - and on the other hand the imagepoints of the curves c: ($P_1\ P_2 \ldots P_n$), which obviously pairwise: ($P_1\ P_2$) ($P_2\ P_3$)...($P_{n-1}\ P_n$) lie upon complex-curves c, those, namely, which correspond to the corners of the given polygon. The new polygon in R and the given one thus stand in a fully reciprocal relation to each other.

By a limit transition one obtains in the two spaces curves, which are enveloped by complex-curves c and C, and which stand in such a mutual relation to each other, that to the points of the one correspond the complex-curves that envelope the other.

A curve enveloped by complex-curves is thus projected in a double sense as another, likewise by complex-curves enveloped curve, which we say is the rendered *reciprocal* relative to the equation system (9).

One may also notice that elementary complex-directions (dx dy dz) (dX dY dZ) arrange themselves pairwise as reciprocals, and that thus two rounded lines enveloped by complex-curves, that touch upon each other, are transformed in the other space as curves that stand in the same mutual relation.

12. Also between other space-forms equations (9) determine a correspondence, which, however, in general is not a complete reciprocity.

A given surface f's points are, namely, projected in R as a double infinity of curves C; as a curve-congruence, whose focus-surface[1] *is F. Likewise correspond to F's points a congruence of curves c, whose focus-surface, as we will later see, contains f as reducible part.*

The elementary complex-cones whose apex-points lie upon the surface f, intersect the corresponding tangent-planes of this in n straight lines - by n is understood the said complex-cones' order - and thus in each point of f determine n elementary complex-directions. The continuous succession of these directions forms an f n-fold enveloping curve set, which is all enveloped by complex-curves c. *The geometric locus for this curve-collection's reciprocal curves, or, as we may also say, collection of image-points of the c, which touch upon f, forms the focus-surface F.*

---

1 In analogy with the terminology used for line-congruences, I understand by this curve-congruence's focus-surface: the geometric locus for intersection-points between infinitesimally close-lying curves C. If the curve-congruence is thought of as defined by a partial linear differential equation, its focus-surface is just what in general one calls the differential equation's singular integral.

In order to prove this, one remembers, that two infinitesimally close-lying, each other intersecting curves C project themselves as two points, whose infinitesimal connection-line is an elementary complex-direction. Now there go from a point p_0 on f, n complex-directions, and thus p_0's image-point C_0 is intersected at n points of the adjacent C, which belong to our curve-congruence earlier considered - in those n points, namely, which correspond to the n complex-curves x, which touch the surface f in the point p_0. F's points are thus the image of the c that touch f.

When now f has a general location in the space r, a c, that touches f in a point, will in general not have more contacts with the same. But all these c form a congruence in which each c touches the focus system in N points - by N is understood the order of the elementary complex-cones in R -, and thus, as said above, our congruence's focus-system decomposes in f and a surface φ, which is touched by each c in (N-1) points.

If thus the correspondence determined by equations (9) between surfaces in r and R is to be a complete reciprocity, it is necessary and sufficient that n and N both equal 1. *In general, the reciprocity-relation is incomplete inasmuch as analogous operations on the one hand transform f in F, and on the other, F in the collection of f and φ.*

The above deliberations are also valid, when f, and as a consequence hereof, F are surface-elements; if f is infinitesimal in one direction alone, the same is the case with F.

One considers finally a curve k, which is not enveloped by complex-curves c, together with the surface F, that is generated by all C, which correspond to k's points. The points of a c are transformed to the through C's image-point going curves c, and thus correspond to F's points the collection of curves c, that intersect k. *The interrelation between k and F is thus a double one.*

Equations (9) which map the two spaces into each other, transform the according to the above-mentione given spaceforms to new ones that stand in a reciprocal relation to the given ones, and can thus serve to transform geometrical theorems and problems. For a special form of the equations (9) we will make important uses of this transformation-principle later on.

§ 5

13. Legendre has given a general method to - in the language of modern

1 One compares also: Plücker, Geometrie des Raumes. § 2. (1846).

geometry - transform a partial differential equation between point co-ordinates x y z into a differential equation between plane-co-ordinates t, u, v, or - as one may also say - between point-co-ordinates t, u, v, of a space, that stands in a reciprocal relation to the given one.

When the curves c are introduced as elements of the space r, it is in a similar way possible to transform a partial differential equation between x, y, z into a differential equation between the new space-element's co-ordinates X Y Z, whereby one may also interpret X, Y, Z as point-co-ordinates of the space R, - a notion that will prevail in our exposition.

Hence, given an arbitrary partial differential equation of the first order between x, y, z and all surfaces ψ that generate a so called *"integral complet"* of the same, one should bear in mind, that any other integral surface f can be represented as envelope of single-infinitely many ψ.

One considers further in the space R all surfaces Ψ and Φ which correspond to the surfaces ψ and f. We will soon show, that any F is the envelope-surface of single-infinitely many Ψ, that thus the surfaces F satisfy a partial differential equation of the first order, for which all Ψ form an "integral complet".

Two given surfaces in r, which possess a mutual surface-element, namely project themselves in R as surfaces which touch each other, and likewise surfaces that possess infinitely many mutual surface-elements are transformed in surfaces, which like the given one touch each other along a curve.

This provided, one considers an integral surface f_0 and all single-infinitely many ψ_0, that touch the same along a characteristic curve, and finally the corresponding F_0 and Ψ_0. It is clear that F_0 is touched by each Ψ_0 along a curve, and F_0 thus is the envelope-surface of all Ψ_0.

14. A particular interest is offered by the fact that the *partial differential equation,* that is transformed, is precisely that, *which is determined by the complex-curves c* (compare § 3); in that case it can be shown, that the corresponding differential equation between X, Y, Z is decomposed into two equations, of which one *is just that which corresponds to the complex-curves C.*

Consider an integral surface by the given differential equation between x y z, and all to the surface f's points corresponding complex-cones. These cones after § 4 in each point of f determine n complex-directions, of which *in casu* two coincide; thus the in § 4 on the surface f considered collection of curves, which are enveloped by complex-curves c, decompose into f's characteristic curves and a curve-system, that covers f (n-2)fold.

The curve-congruence in the space R corresponding to f's points thus has a focus-system, which is decomposed into two surfaces, of which the one - that we are calling Φ - is touched by each c in two coinciding points, while (n-2) contact-points fall upon the other. *The surfaces Φ thus satisfy the partial differential equation that, after the theorem in § 3, are determined by the complex-curves C.*

Now noting, that Φ is the geometric locus for the reciprocal curves of f's characteristic curves, it is seen that the two integral surfaces f_1 and f_2, which touch each other along a characteristic curve k, are transformed into two surfaces Φ_1 and Φ_2,which touch each other along k's *reciprocal* curve; k is namely enveloped by complex-curves c.

The characteristic curves for the two partial differential equations which, after § 3, are determined by the curve-complexes c and C, are reciprocal curves relative to the equation-system (9).

15. The theorem just stated gives the following general method for the transformation of partial differential equations of the first order.

One determines after the customary method the equation:

$$f(x\ y\ z\ dx\ dy\ dz) = 0,$$

which the given partial differential equation's characteristics satisfy, and choose an arbitrary relation of the form:

$$\psi(x\ y\ z\ dx\ dy\ dzX) = 0,$$

where X denotes a constant. The simultaneous system:

$$f = 0,\ \psi = 0$$

be integrated in the form:

$$F_1(x\ y\ z\ X\ Y\ Z) = 0.\ \ F_2(x\ y\ z\ X\ Y\ Z),$$

where Y and Z are the constants introduced by the integration.

By differentiation and elimination one obtains a relation of the form:

$$F_3(X\ Y\ Z\ dX\ dY\ dZ) = 0,$$

which we regard as an equation for the characteristic curves of a definite partial differential equation:

$$F_4\left(X\ Y\ Z \frac{dZ}{dX}\ \frac{dZ}{dY}\right) = 0.$$

Our earlier expansions show that ($F_4 = 0$), that is derived of ($F_3 = 0$) according to the ordinary rules, and the given partial differential equation stand in such a mutual interrelation, that if the one can be integrated, so is the other also open for treatment.

One may draw from this general conclusions on the reduction in degree of partial differential equations of the first order, defined by a complex of curves, the order of which is given.

Thus, for example, any partial differential equation of the first order, that is defined by a line-complex (§ 3), may be transformed into a partial differential equation of the second degree.[1]

Likewise, any partial differential equation, defined by a cone-intersection-complex, can be transformed into a differential equation of the 30th degree.[2]

§ 6.

Over the most general transformation that turns surfaces that touch each other into similar surfaces.

16. In the study of partial differential equations, an important role is played by transformations that can be expressed in the form:

$$X = F_1(x\ y\ z\ p\ q),\ Y = F_2(x\ y\ z\ p\ q),\ Z = F_3(x\ y\ z\ p\ q).$$

---

1 This reduction is due to the fact that each line of a line-congruence touches the focus-system in 2 points (§ 4, 12).

2 The number 30 comes up as product of 6 and (6-1); 6 is the number of points, in which the focus-system of a focus-intersection-congruence is touched by each focus-intersection.

By p and q one as usual understands the partially derived: dz/dx, dz/dy; likewise P and Q denote dZ/dX and dZ/dY.

In the following we would consider the instance where the functions F_1, F_2 and F_3 are chosen in such a way that P and Q also only depend of (x y z p q):

$$P = F_4(x\ y\ z\ p\ q); \quad Q = F_5(x\ y\ z\ p\ q).$$

In that we imply that, from the above 5 equations, a relation between (X Y Z P Q) cannot be derived, note also that each separate quantity (x y z p q) can be expressed as a function of (X Y Z P Q).

When x y z and X Y Z are perceived as point co-ordinates for r and R, one can say, that by a transformation of this kind is defined *a correspondence between the two spaces' surface-elements, and nota bene the most general.* We will show, that *these transformations fall into two distinct, side-ordered classes, of which the one*[1] *corresponds to the Plücker reciprocity, while the other corresponds to the by me propounded reciprocity.*

By elimination of p, q, P and Q between the five equations:

$$X = F_1, \ Y = F_2, \ Z = F_3, \ P = F_4, \ Q = F_5$$

two essentially different situations may occur. Either only an equation between (x y z X Y Z) is obtained, or two relations exist between these quantities. (The existence of *three* mutually independent equations between the two spaces' point-co-ordinates requires the transformation in question to be a *point*-transformation.)

But it is known, that the equation:

$$F(x\ y\ z\ X\ Y\ Z) = 0$$

always defines a reciprocal correspondence between the two spaces' surface-elements; and likewise I have in the foregoing shown, that the equation-system:

$$F_1(x\ y\ z\ X\ Y\ Z) = 0, \ F_2(x\ y\ z\ X\ Y\ Z) = 0$$

always determines a transformation, that turns surfaces, that touch each other in like surfaces.

---

1 Compare: Du Bois-Reymond, Partielle Differential-Gleichungen. 75 - 81.

Hereby my proposition is proved.

At this point I will draw attention to the fact that these transformations possess the remarkable ability to project an arbitrary differential equation of the form: $[A(rt - s^2) + Br + Cs + Dt + E = 0]$, in which A, B, C, D depend only on x, y, z, p, q into an equation of the same form. Inasmuch as the given equation satisfies a general first integral, the same is of course also the case with the new equation. (Compare *Boole's* thesis in Crelle's Journal Bd. 61).

Second Section

The Plücker Line-Geometry can be transformed into a Sphere-Geometry

§ 7.

The two curve-complexes are line-complexes.

17. When we imply that the equations, that map the two spaces into each other, are linear in relation to any system variables:

$$(10)\quad \begin{cases} 0 = X(a_1x+b_1y+c_1z+d_1) + Y(a_2x+b_2y+c_2z+d_2) + Z(a_3x+b_3y+c_3z+d_3) + (a_4 + \ldots) \\ 0 = X(\alpha_1x+\beta_1y+\gamma_1z+\delta_1) + Y(\alpha_2x+\beta_2y+\gamma_2z+\delta_2) + Z(\alpha_3x+\beta_3y+\gamma_3z+\delta_3) + (\alpha_4x+\beta_4y+\gamma_4z+\delta_4), \end{cases}$$

the points conjugate to a given point in the other space obviously generate a straight line. The two curve-Complexes are Plücker line-complexes[1], and in consequence the equations (10) determine a correspondence between r and R, that possesses the following characteristic properties:

a) *To each space's points correspond unambiguously the lines of a line-complex in the other.*

b) *When a point describes a complex-line, the corresponding line in the other space turns around the intersected's image-point.*

c) *Curves that are enveloped by the two complexes' lines, arrange themselves together pairwise as reciprocals in such a way that the tangents*

---

1 Regarding the theory of line-complexes I assume as known: 1) *Plücker*, Neue Geometrie des Raumes, gegründet auf etc....1868-69; 2) *Klein*, Zur Theori der Complexe.....math. Annalen. Bd. II.

of each correspond to the points of the other.

d) To a surface f in the space r is associated a surface F in R for two reasons. On the one hand, F is the focus-surface of the line-congruence whose image is f; on the other F's points correspond to those of f's tangents which belong to the line-complexes in r.

e) On the surfaces f and F just mentioned all curves arrange themselves together pairwise, conjugated in such a way, that to one upon f [or F] lying curve's points correspond in the other space a line-surface, which contains the conjugated curves and after the same touches F [or f].

f) To a curve upon f, which is enveloped by the line-complexes lines, corresponds as conjugated a likewise by complex-lines enveloped curve on F, and these curves are reciprocal curves, in the sense stated under (c).

Any one of the equations (10) determines an an-harmonic correspondence between points and planes in the two spaces, and thus each of our line-complexes may be defined as collections of an-harmonically corresponding planes' intersection-lines - or as an-harmonically corresponding points' intersection-lines. But the complex of the second degree here defined is according to Mr. *Reye* identical to the line-system that initially *Binet* has considered as the collection of a material body's stationary revolution-axes and that later on numerous mathematicians, especially *Chasles* and *Reye*, have studied.

When the constants in equations (10) are particularised, the two complexes can either get a special status - they may for example coincide, the case of which Mr. Reye has treated in his *"Geometrie des Lages, 1868"*, second part, in that he simultaneously set forth the equations obeying (a) and (b) - or they may themselves be particularised. Without entering a discussion of all the possible special-varieties, I wish to stress the two most important degenerations:[1]

Both complexes can be *special, linear.* This case leads to the well-known *Ampere's* transformation, which can thus be regarded as dependant upon the fact that one introduces as space-element, instead of the point, the collection of straight lines which intersect a given line.

The one complex may degenerate into the collection of straight lines, that

1 Lie, "Repräsentation der Imaginæren etc. Christiania Vidensk.-Selskab 1869. Februar og August". The in the mentioned dissertation's §§ 17 and 27-29 treated spatial transformation is identical to the one, I treat in the present Paragraph. In § 25 I explicitly stress the first of the degenerations here reported.

of each correspond to the points of the other.

d) To a surface f in the space r is associated a surface F in R for two reasons. On the one hand, F is the focus-surface of the line-congruence whose image is f; on the other F's points correspond to those of f's tangents which belong to the line-complexes in r.

e) On the surfaces f and F just mentioned all curves arrange themselves together pairwise, conjugated in such a way, that to one upon f [or F] lying curve's points correspond in the other space a line-surface, which contains the conjugated curves and after the same touches F [or f].

f) To a curve upon f, which is enveloped by the line-complexes lines, corresponds as conjugated a likewise by complex-lines enveloped curve on F, and these curves are reciprocal curves, in the sense stated under (c).

Any one of the equations (10) determines an an-harmonic correspondence between points and planes in the two spaces, and thus each of our line-complexes may be defined as collections of an-harmonically corresponding planes' intersection-lines - or as an-harmonically corresponding points' intersection-lines. But the complex of the second degree here defined is according to Mr. *Reye* identical to the line-system that initially *Binet* has considered as the collection of a material body's stationary revolution-axes and that later on numerous mathematicians, especially *Chasles* and *Reye*, have studied.

When the constants in equations (10) are particularised, the two complexes can either get a special status - they may for example coincide, the case of which Mr. Reye has treated in his *"Geometrie des Lages, 1868"*, second part, in that he simultaneously set forth the equations obeying (a) and (b) - or they may themselves be particularised. Without entering a discussion of all the possible special-varieties, I wish to stress the two most important degenerations:[1]

Both complexes can be *special, linear*. This case leads to the well-known *Ampere's* transformation, which can thus be regarded as dependant upon the fact that one introduces as space-element, instead of the point, the collection of straight lines which intersect a given line.

The one complex may degenerate into the collection of straight lines, that

---

1 Lie, "Repräsentation der Imaginæren etc. Christiania Vidensk.-Selskab 1869. Februar og August". The in the mentioned dissertation's §§ 17 and 27-29 treated spatial transformation is identical to the one, I treat in the present Paragraph. In § 25 I explicitly stress the first of the degenerations here reported.

intersect a given cone-section. In that case, the other complex is a general linear complex. I here wish to mention, that Mr *Nöether* (Gӧtting. Nachr. 1869) has just reported a projection of the linear complex in a point-space, which is identical to the one we consider here. The fundamental notion for us: that *each* space contains a complex, whose lines are mapped as the other space's curves, is not expressed in Mr. Nöether's brief note. - It is this degeneration that we wish to study in the following, under the condition that the fundamental cone-intersection is the infinitely distanced imaginary circle.

18. We have found that the two curve-complexes are line-complexes, when the transformation-equations are linear in respect of any variable system, and we are hereby led to investigate whether this sufficient condition is necessary.

When the one complex is a general line-complex, the corresponding curve-complex's elementary complex-cones must be decomposed in cones of the second degree. The proof (§4, 12) of this lies in the fact that a line-congruence's lines touch the focus-surface in *two* points. When the one complex is a special line-complex, the corresponding curve-complex's elementary complex-cones in the other space are decomposed into plane bundles.

Thus, when both complexes would be line-complexes, the elementary complex-cones in both spaces must decompose into cones of the 2nd or 1st degree. But when a line-complex's cones always decompose, the complex itself is reducible,[1] and thus it is shown that, when two line-complexes are transformed into each other in the previously stated way, either both must be of 2nd degree, or the one a special complex of 2nd degree and the other linear, or both special linear complexes. All these three cases are represented by the equation-system (10), and we wish indicate *that (10) defines the most general mutual transformation of two line-complexes.*

When, namely, both complexes are of 2nd degree, it can be shown, that the singularity-surface cannot be a *curved* surface.

From each point of the surface in question emanate two plane bundles, whose lines project in the other space as a straight line's points. It follows that all lines of one bundle correspond to one and the same point in the other space.

But the collection of lines which do not have an independent mapping cannot

1 I don't know of any proof of this proposition, which, however, is reported to me as certain.

form a complex, at the most a congruence or a number of congruences. When, however, the collection of plane ray-bundles which emanate from all points of a *curved* surface necessarily form a complex, our proposition, that the singularity-surface cannot be a *curved* surface, is proved.

When two 2nd degree complexes are transformed into each other - in which case neither of them can be a special complex - in both the singularity-surface must consist only of planes, and in consequence both systems are such as those Binet first examined.

When a 2nd degree complex and a linear complex are transformed into each other, two cases are conceivable: the 2nd degree complex could be formed by lines, which intersect a conesection - this occasion according to the aforesaid actually exists -; the 2nd degree complex could consist of all a 2nd degree surface's tangents. I have through considerations, that have something in common with those I use in § 12, proved that this second case does not exist; because I could in that event, from the fact that a linear complex can be turned into itself by a three-fold infinity of linear transformations permutable between themselves deduce, that the same must be the case with the 2nd degree surface, which, however, is not how the matter stands.

§ 8.

Reciprocity between a linear complex and the colletion of straight lines, which intersect the endlessly distanced imaginary circle.

19. In the following we subject to a closer study the equation system:

$$\left. \begin{aligned} -\frac{\lambda}{2B} Zz &= x - \frac{1}{2A}(x + iY) \\ \frac{1}{2B}(x - iY)z &= y - \frac{1}{2\lambda A} Z, \end{aligned} \right| \; i = \sqrt{-1} \; \Big| \qquad (11)$$

which is linear relative to both variable systems, and which after § 7 determines a correspondence between two line-complexes. First we wish to seek these complexes' equations in the Plücker line co-ordinates.

Plücker writes the straight line's equations in the form:

$$rz = x - \rho, \quad sz = y - \sigma,$$

where he considers the five quantities: r, ρ, s, σ, (rσ - sρ) as line-co-ordinates. The equations (11) thus reproduce, provided that one perceives X, Y, Z as parameters in the same, the system of straight lines, whose co-ordinates satisfy the relations:

$$r= -\frac{\lambda}{2B} Z, \quad \rho = \frac{1}{2A} (x + iY),$$

$$s= \frac{1}{2B} (x - iY), \quad \sigma = \frac{1}{2\lambda A} Z ,$$

which by elimination of X, Y and Z give as our complex's equation:

$$\lambda^2 A\sigma + Br = 0 \qquad (12).$$

The line-complex in the space r is thus a linear complex and it is a general linear complex which - as one may notice - contains the xy-planes endlessly distanced straight line.

To determine the line-complexes in R, one replaces the system (11) by the equivalent:

$$\left(\frac{\lambda A}{2B} z - \frac{B}{2\lambda Az}\right) Z = x - \left(Ax + B\frac{y}{z}\right)$$

$$\frac{1}{i}\left(\frac{\lambda A}{2B} z + \frac{B}{2\lambda Az}\right) Z = Y - \frac{1}{i}\left(Ax - B\frac{y}{z}\right) ,$$

which, by combination with the equations of the straight line in R:

$$RZ = X - P; \; sZ = Y - \Sigma \qquad (13)$$

give:

$$R = \frac{\lambda A}{2B} z - \frac{B}{2\lambda Az} Z , \quad P = Ax + B\frac{y}{z} ,$$

$$S = \frac{1}{i}\left(\frac{\lambda A}{2B} z + \frac{B}{2\lambda Az}\right) , \quad \Sigma = \frac{1}{i}\left(Ax - B\frac{y}{z}\right) ,$$

and thus is found as equation of the line-complexes in R:

$$R^2 + S^2 + 1 = 0 \tag{14}$$

According to (13), however:

$$R = dX/_{dZ}\ ,\ S = dY/_{dZ}\ ,$$

and as a consequence, (14) can also be written in the form:

$$dX^2 + dY^2 + dZ^2 = 0. \tag{15}$$

The line-complexes in R are thus formed by the imaginary straight lines, whose length equals zero, or as one may also say, of those lines that intersect the endlessly distanced imaginary circle.

The equations (11) transform the two spaces into each other in such a way that to r's points correspond in R the imaginary straight lines whose length equals zero, while R's points transform as the lines of the linear complex (12).

One sees that, when a point runs through a line of this linear complex, the corresponding straight line in R describes an infinitesimal sphere - a point-sphere.

20. According to the general theory for reciprocal curves, as expounded in § 4, one can, when a curve is known, by simple operations find the image-curve that is enveloped by the other complex's lines. Now *Lagrange* has engaged himself with the most general determination of space-curves whose length equals zero, whose tangents thus possess the same property. He has found these curves' general equation, and thus it is by the aforesaid *also possible to specify general formulas for the curves whose tangents belong to a linear complex.*

In order not to depart from our aim we will not enter here into a closer consideration of the simple geometrical relations that occur between reciprocal curves in the two spaces.[1]

Our earlier expositions of the correspondence between surfaces in the two spaces are now somewhat modified thereby, in that all congruences of straight

---

1 When the given curve of length equalling zero has an apex, the corresponding curve in the linear complex has a stationary tangent. On the whole, stationary tangents occur as ordinary singularities, when curves are perceived as line-generations, that is, as enveloped by a given line-complex's lines.

lines, which intersect the endlessly distanced circle possess a mutual focus-curve - namely, that circle - and that furthermore a line-congruence's straight lines only touch the focus-surface in two points.

Because if one imagines a surface F given in R, and that f is the geometric locus for the points in r that correspond to F's tangents of length zero, then also, inversely, F is the *complete* geometric locus for the imagepoints of the straight lines in the linear complex (12), touching f.

On the other hand, the case stands as in the general instance, when a surface φ of general location in r is given, inasmuch as the straight lines of the linear complex (12) which touch φ, in addition envelope another surface ψ, φ's so called reciprocal polarity relative to (12).

The above mentioned line-system transforms in R as a surface Φ, that obviously is the focus-surface for two congruences - firstly for the collection of straight lines, of length zero that correspond to φ's points - secondly for the other collection of the lines that stand in the same relation to ψ's points.

Φ's tangents of length equal to zero thus decompose into two systems, or as one can also say: Φ's geodetic curves of length equalling zero form two distinct sets.

En passant we note that the determination of *the curves that are enveloped by the straight lines of a congruence belonging to a linear complex, according to our general theories can be traced back to the searching out on the image-surface F of the geodetic curves, whose length equals zero.* For these curves are reciprocal between each other (17, f) relative to the equation system (11).

21. In the following we will make use of the ensuing theorems a few times:

a. *A surface F of n^{th} order, which contains the endlessly distanced imaginary circle as p-double line is the image of a congruence, whose order, and in consequence also class, equals (n - p).*[1]

An imaginary line of length equal to zero namely intersects F in (n - p) points

1 I will at this occasion express an, as it seems, nowhere explicitly articulated, but none-the-less for any one mathematician, who deals with line-geometry,well-known lemma: *For a congruence that belongs to a linear complex, the order is always equal to the class.*

which lie in the finite space, and thus are always given (n - p) lines of the image-congruence, that run through a given point - or that lie in a given plane in the space r.

b. *A curve C of n^{th} order, which intersects the infinitely distanced circle in p points, is projected in r as a linesurface of the $(2n - p)^{th}$ order.*

A straight line of the linear complex (12) namely intersects the mentioned linesurface in as many points as the number of - not infinitely distanced - mutual-points between the curve C and an infinitesimal sphere.

§ 9.

The Plücker line geometry can be transformed into a sphere-geometry.

22. In this paragraph we establish a *fundamental relation that takes place between the Plücker line-geometry and a geometry whose elements are the space's spheres.*

Because equations (11) transform the space r's straight lines into the space R's spheres, and that for a double rendition (12).

On the one hand the straight lines of the complex l_2, which intersect a given line l_1, and thus likewise the same's reciprocal polarity l_2 relative to (12), transform according to an earlier lemma (21,b) as a sphere's points; on the other hand the lines l_1 and l_2's points are transformed into this sphere's rectilinear generatrices.

By the following analytic expositions one can find the relations, that take place between l_1 and l_2's line-co-ordinates X', Y', Z' and radius H'.

When

$$\rho z = x - r, \quad \sigma z = y - s$$

are the line l_1 [or l_2's] equation, and it is remembered that the linear complex (12)'s straight lines can be expressed by:

$$-\frac{\lambda}{2B} Zz = x - \frac{1}{2A}(X + iY)$$

$$\frac{1}{2B}(X - iY)\, z = y - \frac{1}{2\lambda A} Z,$$

it is seen, that one must eliminate x y z between these four lines in order to subject the just mentioned lines to the condition of intersecting l_1. One hereby finds the following relation:

$$[Z-(A\sigma\lambda-B/_{\lambda}\, r)]^2 + [X-(A\rho+Bs)]^2 + [Y- i(Bs-A\rho)]^2 =$$

$$[A\lambda\sigma+B/_{\lambda}\, r]^2 \qquad (16)$$

between these linear parameters (X, Y, Z) or, as one may also say, between the imagepoints' co-ordinates.

The immediate interpretation is that this equation confirms what we have said above, and in addition yields the following formulas:

$$X' = A\rho + Bs \qquad iY' = A\rho - Bs$$
$$Z' = \lambda A\sigma - B/_{\lambda}\, r \qquad \pm H' = \lambda A\sigma + B/_{\lambda}\, r \qquad (17)$$

or the equivalent

$$\rho = \frac{1}{2A}\,(X' + iY') \qquad s = \frac{1}{2B}\,(X' - iY')$$

$$(18)$$

$$\sigma = \frac{1}{2\lambda A}(Z \pm H') \qquad r = -\frac{\lambda}{2B}(Z \pm H')$$

In which one may without disadvantage exclude the sphere-co-ordinates X'Y'Z'H''s accents, in that for our perception the space R's points are spheres, whose radius equals zero.

The formulas (17) and (18) show, that a straight line in r transforms as an unambiguously determined sphere in R, while to a given sphere correspond two lines in r:

$$(X, Y, Z, +H) \qquad (X, Y, Z, -H),$$

which are each other's reciprocal polarities relative to the linear complex:

$$H = 0 = \lambda A\sigma + B/_{\lambda}\, r\,, \qquad (12)$$

(17) and (18) evidently express, when H is defined as zero, the unambiguous association between the complexe (12)'s straight lines and the space R's point-spheres.

A plane - that is, a sphere, whose radius is infinitely large - projects as two straight lines (l_1 and l_2), which intersect the xy-planes endlessly distanced straight lines, and according to the above are l_1 and l_2's points the projection of those imaginary lines in the given plane, which go to the same's endlessly distanced circle-points.

Note in particular, that to a plane, touching the endlessly distanced imaginary circle, corresponds a line of the complex (H = 0) parallel to the xy plane.

23. Two lines l_1 and λ_1, which intersect each other, transform as spheres, between which contact takes place.

For l_1 and λ_1's polarities relative to (H = 0) also intersect each other, and in consequence the mentioned spheres have two mutual generatrices. But 2nd degree surfaces, whose intersection-curves consist of a conesection and two straight lines, touch each other in three points - the section-curves double-points. l_1 and λ_1's image-spheres thus have three contact points, of which two, however, imaginary and infinitely distanced, in ordinary parlance do not come into question.

Analytically our theorem is proved in the following way:

The condition for the intersection between the two lines:

$$r_1 z = x - l_1 \qquad r_2 z = x - \rho_2$$
$$s_1 z = y - \sigma_1 \qquad s_2 z = y - \sigma_2$$

is known to be expressed by the equation:

$$(r_1 - r_2)(\sigma_1 - \sigma_2) - (\rho_1 - \rho_2)(s_1 - s_2) = 0\,,$$

which by use of (18) gives:

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 + (iH_1 - iH_2)^2 = 0\,,$$

which proves our statement.

Our theorem shows, that the collection of the straight lines, which intersect a given, transforms as all spheres, that touch a given, *and in consequence we know the special linear complex's projection.*

Conversely, corresponding to two spheres, which touch each other, there are two line-pairs, whose mutual relation is such that each line of the one pair intersects a line of the other.

24. The general linear complex's transformation. The general linear complex is produced by the equation:

$$(\rho\sigma - \pi\sigma) + mr + n\sigma + p\rho + qs + t = 0\,, \qquad (19)$$

which by use of (18) is found to be the equation of the corresponding "linear sphere-complex":

$$[X^2 + Y^2 + Z^2 - H^2] + MX + NY + PZ + QH + T = 0.$$ [1]

Here M, N, P, Q, T signify constants that depend upon m, n, p, q, t, *while X, Y, Z, H are to be considered as - non-homogeneous - sphere co-ordinates.*

The last equation determines, as one easily sees, all spheres that intersect the image-sphere of the complexes (19) and (H = 0)'s linear mutual-congruence under constant angle.

If these complexes are simultaneous invariants equalling zero, or the two complexes, as Klein puts it, lie in involution, the constant angle is right.

To spheres, which intersect a given sphere under constant angle, correspond in the space r those straight lines of two linear complexes that are each other's reciprocal polarities relative to (H = 0).

In particular it should be noted that the spheres, which intersect a given one orthogonally, transform as the straight lines of a linear complex, lying in involution with (H = 0).

---

1 This equation can be posed under the form:

$$(X-X_0)^2 + (Y-Y_0)^2 + (Z-Z_0)^2 + (iH-iH_0)^2 = C_0^2$$

in which we perceive X_0, Y_0, Z_0, H_0, C_0 as non-homogeneous co-ordinates of the linear complex. Mr Klein has drew my attention to the fact, that the sphere (X_0, Y_0, Z_0, H_0) is the image of the axes of the linear complex at hand.

Now given a linear complex, whose equation has the form:

$$ar + bs + c\rho + d\sigma + e = 0, \quad (20)$$

the corresponding relation between X, Y, Z, H is also linear, and thus the linear sphere-complex in question is generated by all *spheres, which intersect a given plane under a given angle.*

This one may also conclude from the fact that the complex (20) contains the xy-plane's infinitely distanced straight line, that thus the same's mutual-congruence with (H = 0) possesses directrices, which intersect this line.

When the complexes (20) and (H = 0) lie in involution, (20)'s lines transform as all spheres which intersect a given plane orthogonally, or, equivalently, as the spheres whose centres lie in a given plane.

The following four complexes:

$$X = 0 = A\rho + Bs \qquad Z = 0 = \lambda A\sigma - B/_{\lambda}\, r$$

$$iY = 0 = A\rho - Bs \qquad H = 0 = \lambda A\sigma + B/_{\lambda}\, r$$

lie, as one easily sees, pairwise in involution and furthermore contain, as mutual line, the xy-planes infinitely distanced line.

The special linear complex: (Const = 0), that is generated by all lines parallel with the xy-plane in association with the four general linear complexes (X = 0) (Y = 0) (Z = 0) (H = 0), thus forms a system, that is to be perceived as a degeneration of Mr. Klein's 6 fundamental-complexes. In analogy with the fact that, above we have introduced X, Y, Z, H as non-homogeneous co-ordinates of a geometry with four dimensions, the element of which is the sphere, these quantities can also be used as non-homogeneous line-co-ordinates.

It is of interest to note, that the linear complexes, whose equation is:

$$H = \lambda A\sigma + B/_{\lambda}\, r = \text{Const.},$$

and which according to the equation-form touch each other after a special linear congruence, whose directrices have joined themselves in the xy-plane's endlessly distanced line, transform as a set of sphere-complexes, which are characterised thereby, that all spheres of the same complex have equally large radii.

25. Various projections. A surface f and all its tangents in a given point project themselves as a surface F and all spheres that touch the same in a given point.

A line lying upon f as a sphere, that touches F along a curve.

When f is a linesurface, F is a sphere-envelope - a tube-surface.

If in particular f is a 2nd degree surface and as consequence thereof contains two systems of rectilinear generatrices, in two ways F may be perceived as a sphere-envelope, and it is significant that in this manner we*obtain the most general surface, which possess this property (the cyclide).*

A developable surface transforms itself in the envelope-surface of a set of spheres, of which two consecutive ones always touch each other - that is to say, in an imaginary linesurface, whose generatrices intersect the infinitely distanced imaginary circle. These line-surfaces are, one knows, just those that *Monge* characterises by their possessing only one system of rounded curves.

26. It is known that the immediate consequence of the Plücker understanding, that when ($l_1 = 0$) and ($l_2 = 0$) are the equations for two linear complexes,

$$l_1 + \mu\ l_2 = 0\ ,$$

provided that μ signifies a parameter, represent a set of linear complexes, which contain a mutual linear congruence. Our projection-principle transforms this theorem into the following:

The spheres K, that intersect two given spheres S_1 and S_2 under given angles, V_1 and V_2 stand in the same relation to infinitely many spheres S. There are, corresponding to the said line-congruence's two directrices, two S, which are touched by all K.

The variable line complex: ($l_1 + \mu l_2 = 0$) intersects the complex ($H = 0$) along a linear congruence, whose directrices describe a 2nd degree surface - the average of the three complexes: $l_1 = 0$, $l_2 = 0$, $H = 0$, and in consequence the just mentioned spheres S envelop a cyclide, which, by the way, in this case is degenerated into a circle, after which all S intersect each other.

Here we also wish to draw attention to the fact that our sphere-projection allows the deduction of corresponding sphere-groups from interesting discontinuous line-groups, and vice versa. For example, from the well-known theory for the 3rd degree surface's 27 straight lines we derive the existence of groups of 27 spheres, of which each touches ten of the others.

On the other hand, for example, sphere-columns yield strangely discontinuous arrangements of a linear complex's lines.

§ 10.

Transformation of particulars concerning spheres in line-problems.

27. In this paragraph we wish to solve a few well-known problems concerning spheres, in that we consider the corresponding line-problems by our transformation-principle.

Problem I. How many spheres touch four given spheres?

The four spheres transform in four line-pairs $(l_1\ \lambda_1)(l_2\ \lambda_2)(l_3\ \lambda_3)(l_4\ \lambda_4)$, and the corresponding line-problem is thus to find the lines, which intersect four lines, chosen in such a way among the 8 stated, that one line is taken by each pair.

The lines l and λ can be arranged in 16 distinct groups of four:

$$l_1\, l_2\, l_3\, l_4 \qquad \lambda_1\, \lambda_2\, \lambda_3\, \lambda_4$$
$$l_1\, l_2\, l_3\, \lambda_4 \qquad \lambda_1\, \lambda_2\, \lambda_3\, l_4$$
. . . .

in such a way, that each group only contains one line of each pair. These 16 groups are, however, pairwise generated by lines, that are each other's reciprocal polarities relative to (H = 0), and in consequence also two associated groups' transversal-pairs $(t_1\ t_2)\ (\tau_1\ \tau_2)$ are each other's polarities relative to (H = 0). The four last-mentioned lines are thus projected as *two* spheres, and in consequence there exist 16 spheres, arranged in 8 pairs, that touch the four given.

Problem II. How many spheres intersect four given spheres under four given angles?

The spheres, which intersect a given sphere under the same angle, project themselves as those straight lines of two linear complexes, which are each other's reciprocal polarities relative to (H = 0). One thus has to consider four pairs of complexes $(l_1\ \lambda_1)(l_2\ \lambda_2)(l_3\ \lambda_3)(l_4\ \lambda_4)$, and the problem is to find the lines, that belong to four of these complexes, which are chosen in such a way, that one of each pair is taken.

Four linear complexes have two mutual-lines, and thus as solution one obtains, by following the same method as we used in the preceding problem's treatment, 16 spheres that are arranged in 8 pairs.

Our problem is simplified, when one or more of the given angles are right, insofar as a given sphere's orthogonal-spheres transform as the lines of *one* complex lying in involution with (H = 0) (*n.24*). When all angles are right, the question is how many mutual-lines four with (H = 0) in involution lying complexes have. There are two such lines, which are each other's reciprocal polarities relative to (H = 0), *and in consequence there is only one sphere, that intersects the four given orthogonally.*

Problem III. To construct the spheres, that intersect five given spheres under the same angle.

Our transformation-principle turns this problem into the following: to find the linear complexes, which contain a line of each of five given line pairs $(l_1\ \lambda_1)......(l_5\ \lambda_5)$.

These 10 lines can be arranged in 32 different groups of five, in such a way, that each group contains *one* line of each pair:

$$(l_1\ l_2\ l_3\ l_4\ l_5)\ (\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ \lambda_5)$$

. . . .

by which, however, note that pairwise these groups are each other's reciprocal polarities relative to (H = 0). Each group gives a line-complex and, in all, 32 pairwisely conjugated linear complexes are thus obtained, which transform as 16 linear sphere-complexes. The 16 spheres, each of which is intersected under constant angle by the mentioned system's spheres, are our problem's solutions.

Two line-groups like:

$$l_1\ l_2\ \lambda_3\ \lambda_4\ l_5 \qquad \lambda_1\ \lambda_2\ \lambda_3\ \lambda_4\ l_5$$

contain four mutual lines, and thus the two corresponding linear complexes intersect each other after a linear congruence, whose directrices d_1 and d_2 are the mentioned four lines' transversals.

But the complex (H = 0) intersects that congruence along a 2nd degree surface, which is the image of a circle - the average-circle between two of the sought spheres, but likewise between d_1 and d_2's image-spheres. These last spheres can also be defined by the fact that they touch four of the five given spheres and thus one can, by the just stated construction, determine a number of circles upon an arbitrary one of the spheres searched for.

On each of the 16 spheres that intersect five given under the same angle, five circles can be constructed, provided that one can construct the spheres, which touch the five given.

§ 11.

Relation between rounded curves' and maintangent-curves' theory.

28. The transformation considered in the foregoing gains particular interest due to the following, in my opinion important theorem:

To a surface F's rounded curves given in R correspond in r line-surfaces which touch the image-surface f along maintangent-curves.

The surface f's tangents transform into spheres that touch F, and the idea is thus that, to *f's maintangents*, correspond *F's main-spheres*. This is also the case.

Because f is intersected by a maintangent in three coinciding points, which shows that three consecutive generatrices of the maintangent's image-sphere touch F. But such a sphere intersects F along a curve, which in both's point of contact has an apex, and this is just characteristic of main-spheres.

When it is now further considered that this apex's direction is tangent to a rounded curve, it is seen that two consecutive points of a maintangent-curve on f project as two lines, which touch F in consecutive points of the same rounded curve. To *f's maintangent-curves, perceived as point-creations, thus correspond imaginary linesurfaces which touch F along a rounded curve.*

But curves on f and F arrange themselves pairwise together as conjugated in such a way *(n. 17, e)* that the one's points are images of lines, which touch the other surface in points of the conjugated curve, and *thus our theorem is proved.*

The two ensuing examples can be regarded as a verification of this proposition.

A sphere in R is the image of a linear congruence, as whose focus-surface the two directrices are to be perceived. Now as is well-known any curve on a sphere is a rounded curve, and in reality the directrices also appear as maintangent-curves on any line-surface that belongs to a linear congruence. - A hyperboloid f in the space r gives in R a surface, which in two ways can be perceived as a sphere-envelope. Now the line-surfaces in the complex (H= 0), touching f after its maintangent-curves, that is to say, after its rectilinear generatrices, are themselves 2nd degree surfaces, and in consequence *the cyclide F's rounded curves are circles.*

As an interesting consequence of our theorem the following may be contemplated.

Kummer's surface of the first order and class has algebraic maintangent-curves of the 16th order, which generate the complete contact-average between the respective surface and linesurfaces of the 8th order.

Kummer's surface is namely the focus-surface for the general line-congruence of 2nd order and class, which projects - provided it belongs to (H = 0) - as a fourth degree surface which contains the infinitely distanced circle twice (n. 21, a).

But Mr. *Darboux* and *Moutard* [1] have shown, that the last mentioned surface's curvelines are curves of the 8th order, which intersect the infinitely distanced imaginary circle in 8 points, and thus these line transform as linesurfaces of the 8th order (n. 21, b).

Finally, if it is remembered, that these linesurfaces' generatrices are doubletangents to the Kummer surface, our theorem's correctness is realised.[2]

It is evident, that also the Kummer surface's degenerations, e.g.: *the wavesurface, the Plücker complex-surface, the Steiner surface of the 4th order and 3rd class*[3], a linesurface of 4th degree, the 3rd degree linesurface....have algebraic maintangent-curves.

29. Mr *Darboux* has shown that on an arbitrary surface in general a curveline located in the finite space can be determined - the touching-curve with the imaginary developable, which is circumscribed simultaneously around the given surface and the infinitely distanced imaginary circle.

---

1 Comptes rendus. Year 1864.

2 Klein and Lie. Berliner Monatsbericht. 15 Decbr. 1870.

3 Clebsch has determined the Steiner surface's maintangent-curves.

As a consequence hereof it is in general possible to identify one maintangent-curve on the focussurface of a congruence belonging to a linear complex - the geometric locus of the points, for which the tangentplane likewise is the plane associated with the linear complex.

The infinitely small spheres, which touch F, namely consist of F's points in connection with the above stated imaginary developables, and in consequence the straight lines of the complex (H = 0), which touch the imagesurface f, divide into two systems - one system of doubletangents, and on the other hand the collection of lines that touch f in the points of a definite curve. But this curve is, as the projection of an imaginary linesurface that touches F along a rounded curve, one of f's maintangent-curves.

However, this determination of a maintangent-curve is rendered illusory, when not the congruence, but the focus-surface - or, more correctly, a reducible part of the same - is conditionally stated. For on a surface, as a rule only a finite number of points exist, whose tangentplane moreover is the plane which is associated with the said point by a given linear complex.

It is of interest to note, that a linesurface, whose generatrices belong to a linear complex, contains infinitely many points, for which the tangentplane in addition is the plane assigned by the linear complex. The collection of these points generates, by simple operations - differentiation and elimination -, a determinable maintangent-curve.

But Mr. *Clebsch* has shown, that when a maintangent-curve is known upon a linesurface, the others can be found by squaring.

The determination of maintangent-curves upon a linesurface belonging to a linear complex depends only on squaring.

In that we use our transformation-principle on the mentioned theorem of Mr Clebsch as well as on the deduced consequence, we obtain the following theorems:

When upon a tubesurface (sphere envelope) a rounded curve which is not circular is known, the others can be found by squaring.

Single-infinitely many spheres, that intersect a given sphere S under constant angle, envelope a tubesurface upon which a curveline can be defined, and the others thus determined by squaring.

That one can find a rounded curve upon the tubesurface mentioned in the last theorem, is apparent also from the fact that the tubesurface intersects S under constant angle. But this section-curve must be one of the tubesurface's rounded curves by the known lemma: When two surfaces intersect each other under constant angle, and the section-curve is a rounded line on the one surface, it must also be so on the other; but on a sphere all curves are rounded lines.

§ 12.

Correspondence between transformations of the two spaces.

30. Our projection can, according to *n. 16,* be expressed by five equations, which determine an arbitrary quantity of the two groups:

$$(x\,y\,z\,p\,q) \qquad (X\,Y\,Z\,P\,Q),$$

as function of quantities of the other group. If now the one of the two spaces is subjected to an, e.g., transformation, by which surfaces that touch each other, are turned into similar surfaces, the corresponding transformation of the other space will possess the same property. The mentioned transformation of r can namely be expressed by five equations between x_1, y_1, z_1, p_1, q_1 and x_2, y_2, z_2, p_2, q_2 - the indices 1 and 2 refer to the space r's two states - and these relations are turned by aid of the transformation-equations between (x y z p q) and (X Y Z P Q) to relations between $(X_1, Y_1, Z_1, P_1, Q_1)$ and $(X_2, Y_2, Z_2, P_2, Q_2)$, which proves our proposition.

In that we restrict ourselves to linear transformations of r, we find between the corresponding transformations of R: *all movements (translation-movement, rotation-movement and the helicoidal movement), semblability-transformation, transformation by reciprocal radii, parallel transformation*[1] *- by that is understood transition from a surface to its parallel-surface - a reciprocal transformation studied by Mr. Bonnet*[2] etc., which all, corresponding to linear transformations of r, possess the property of turning rounded curves into rounded curves. We finally prove, *that to the general linear transformation of r correspond the most general transformation of R, by which rounded lines are covariant curves.*

31. When we now firstly consider such linear point-transformations of r, to which correspond linear point-transformations of R, it is evident, that we can only find such transformations of R, by which the endlessly distanced imaginary circle remains unchanged, and inversely it is also true that we obtain all of these.

1 Bonnets "dilatation."

2 Comptes rendus. Many times in the 50-ies.

For as we know, such a linear point transformation of R on the one hand turns straight lines which intersect that circle into similar lines; on the other hand spheres into spheres, and thus the corresponding transformation of r is at the same time a point- and line-transformation, that is: a linear point transformation, which was to be proved.

The general linear transformation of R, that does not distort the infinitely distanced imaginary circle, contains 7 constants and can, as is well known, be composed by translation- and rotation-movements in combination with semblability-transformations. The corresponding transformation of r, that obviously also depends upon 7 constants, can be characterised so that it turns a linear complex (H = 0) and one determined by the same lines - the xy-planes' infinitely distanced line - into itself. One could also define this transformation so that it turns a special linear congruence into itself.

By analytical considerations one can in the following way determine the linear point-transformation of r corresponding to a transformation-movement of R. A translation-movement is expressed by the equations:

$$X_1 = X_2 + A;\quad Y_1 = Y_2 + B;\quad Z_1 = Z_2 + C;\quad H_1 = H_2\,,$$

which by using the formulas (17) give:

$$r_1 = r_2 + a;\quad s_1 = s_2 + b;\quad \rho_1 = \rho_2 + c;\quad \sigma_1 = \sigma_2 + d.$$

On insertion of these expressions in a straight line's equations:

$$r_1 z_1 = x_1 - \rho_1 \quad s_1 z_1 = y_1 - \sigma_1.$$

are obtained as definition of the mentioned transformation of r:

$$z_1 = z_2;\quad x_1 = x_2 + a z_2 + c;\quad y_1 = y_2 + b z_2 + d.$$

Likewise it is easy to determine analytically the transformation of r corresponding the to a *semblability-transformation* of R. For the equations:

$$X_1 = mX_2;\quad Y_1 = mY_2;\quad Z_1 = mZ_2;\quad H_1 = mH_2$$

give, by using (17):

$$r_1 = mr_2;\quad \rho_1 = m\rho_2;\quad s_1 = ms_2;\quad \sigma_1 = m\sigma_2\,,$$

which define a linear transformation of r that can also be expressed by:

$$z_1 = z_2; \quad x_1 = mx_2; \quad y_1 = my_2 .$$

But these last relations define a linear point-transformation that can be defined so that *two straight lines retain their places.*

By geometric consideration we will show, that also rotation-movements of R metamorphose into transformations of the just stated kind. Let A be the rotation-axis and M and N the two points of the imaginary circle not distorted by the rotation. It is evident, that all imaginary lines, that intersect A, and that go through M or N, retain their position under the rotation, and in consequence the same is the case with these lines' imagepoints, which form two straight lines parallel with the xy-plane.

32. Transformation by reciprocal radii of the space R transforms points into points, spheres into spheres and finally straight lines of length equal to zero into similar lines; the corresponding transformation of r is thus a *linear point-*transformation, that turns the complex (H = 0) into itself. When one further notes that transformation by reciprocal radii lets a definite sphere's points and rectilinear generatrices maintain their position, it is realised, that the corresponding point-transformation does not distort two straight lines' points.

Mr *Klein*[1] has drawn our attention to the fact that the just mentioned transformation can be perceived as composed of two transformations relative to two linear complexes lying in involution, of which *in casu* (H = 0) is one, while the other corresponds to the collection of spheres which intersect the fundamental-sphere of the given transformation by reciprocal radii.

According to the above it is evident, that to a surface D, which through a transformation by reciprocal radii is turned into itself, corresponds in the space r one to (H = 0) belonging congruence, which is its own reciprocal polarity relative to a linear complex lying in involution with (H = 0). The focus-surface (f) of the said congruence is thus its own reciprocal polarity relative to both the stated linear complexes, and in consequence the collection of f's doubletangents generally decomposes into three congruences, of which the two relationally belong to (H = 0) and the complex lying in involution with the same.

33. One now considers all line-transformations of r, by which straight

---

1 Zur Theorie . . . math. Annalen, Bd. II.

lines, that intersect each other are turned into similar lines[1], and on the other hand the corresponding transformations of R, which possess the property to turn spheres into spheres, spheres that touch each other in similar spheres.

By the stated line-transformation, the collection of a surface f_1's tangents is turned into all of another surface f_2's tangents, and especially f_1's main-tangents go over into f_2's maintangents - this irrespective of whether the line-transformation is a point-transformation or a point-plane-transformation.

By the corresponding transformation of R, the threefold infinity of spheres, that touch a given surface F_1 is turned into the collection of spheres, standing in the same relation to the other surface F_2, and especially F_1's main-spheres are transformed into F_2's main-spheres. A simple consequence hereof is that F_1's and F_2's arcuate-lines correspond to each other in the sense that when in an arbitrary relation:

$$\Phi(X_1\ Y_1\ Z_1\ P_1\ Q_1) = 0\,,$$

which is valid along one of F's rounded lines, are inserted $X_1\ Y_1\ Z_1\ P_1\ Q_1$'s values at $X_2\ Y_2\ Z_2\ P_2\ Q_2$, an equation is obtained, that is valid for one of F_2's rounded curves.

I will now show, that any transformation of R of the form:

$$X_1 = F_1\left(X_2\ Y_2\ Z_2\ \frac{dZ_2}{dX_2}\ \frac{dZ_2}{dY_2}\ \frac{d^2 Z_2}{dX_2}\ \cdots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

$$Y_1 = F_2\left(X_2\ Y_2\ Z_2\ \cdots\cdots\cdots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

$$Z_1 = F_3\left(X_2\ Y_2\ Z_2\ \cdots\cdots\cdots\ \frac{d^{m+n} Z_2}{dX_2^m \cdot dY_2^n}\right)$$

---

1 Here are, as we know, two cases to be considered, insofar as lines, that go through a point, can either be transformed in similar lines, or in lines that lie in a plane.

which turns an arbitrary surface's rounded lines into rounded lines for the new surface, through my transformation corresponds to a linear transformation of r.

The proof can be straightforwardly reduced to demonstrating that when a transformation of r turns an arbitrary surface's maintangent-curves into maintangent-curves of the transformed surface, straight lines that intersect each other must be turned into similar lines by the same.

Firstly, the transformation in question must turn straight lines into straight lines, follows from that the straight line is the only curve, which is maintangent-curve on any surface that contains the same.

Further, to straight lines that intersect each other, must correspond lines of the same relative mode, can be deduced from the fact that the developable surface is the only linesurface, which possesses the property, that through each of its points runs only one maintangent-curve - that thus our transformation must turn developable surfaces into developable surfaces.

Our proposition is thus proved.

One may note that, corresponding to the two essentially different kinds of linear *transformations, exist two distinct classes of transformations, for which rounded curves are covariant curves.*

When one chooses among the stated transformations of R those which are point-transformations, *the most general point-transformation of R, by which rounded lines are covariant curves, is obtained,* a problem that *Liouville* first solved. That hereunder equivalence in the smallest parts is maintained, follows by the fact that infinitesimal spheres are transformed into infinitesimal spheres.

Parallel-transformation is known to turn rounded lines into rounded lines, and it is in reality easy to verify that the corresponding transformation of r is a linear point-transformation.

For the equations:

$$X_1 = X_2; \quad Y_1 = Y_2; \quad Z_1 = Z_2; \quad H_1 = H_2 + A$$

transform (compare our considerations over translation-movement *n. 31*) into relations of the form:

$$z_1 = z_2; \quad x_1 = x_2 + uz_2 + b; \quad y_1 = y_2 + cz_2 + d.$$

34. Mr. *Bonnet* has many times considered a transformation, which he defines by the equations:

$$Z_2 = i\, Z_2 \sqrt{1+p_2^2+q_2^2} \;;\; x_1 = x_2 + p_2 z_2 \;;\; y_1 = y_2 + q_2 z_2 \,,$$

whereby the two indices refer to the given and the transformed surfaces.

Mr. Bonnet shows that this transformation is reciprocal - in the sense, that twice applied it brings back the given surface, that it transforms curvelines into curvelines, that finally the following two relations:

$$\zeta_1 = iH_2 \,,\;\; H_1 = -i\zeta_2 \qquad (\alpha)$$

find place, provided that H_1 and H_2 signify curve-radii for corresponding points, that further ζ_1 and ζ_2 are z-ordinates for the corresponding curve-centres.

The Bonnet transformation is, as we will soon show, the image of a transformation of r relative to the linear complex:

$$Z + iH = 0.$$

Because, remembered that (X = 0) (Y = 0) (Z = 0) (H = 0) pairwise lie in involution, it is found, that the co-ordinates of the straight lines, that are each other's polarities relative to [Z + iH = 0], fulfil the relations:

$$X_1 = X_2;\;\; Y_1 = Y_2;\;\; Z_1 = iH_2;\;\; H_1 = -iZ_2 \,. \qquad (\beta)$$

But these formulas determine a pairwise correspondence between all spheres of the space when X, Y, Z, H are interpreted as sphere-co-ordinates. This is just the same as the Bonnet transformation.

Because a surface F_1's mainspheres are transformed hereunder into a surface F_2's mainspheres, and hence we recover Bonnet's formulas (α). When one further considers F_1 generated by point-spheres, the equations (β) define F_2 as an envelope of spheres, whose centres lie in the plane (z = 0), in that the equation (H_1 = 0) draws (Z_2 = 0) after itself as a consequence. In reality we are hereby carried to precisely the geometric construction described by Mr. Bonnet.

Concluding remarks

Thus ends Marius Sophus Lie's thesis - and starts his deeper exploration of the new chapter he founded. The aim here has been to present his well-spring opus for a wider scientific community and to retain thereby also the direct and pristine voice of his own formulations. They may at times sound somewhat out of date, but they are not - they are just original. There are numerous examples, which hardly need to be recollected. At large and in detail, we recognise basic concepts and notions also of the modern versions and terminology, which, however, as so often in superspecialising science, have been alienated in efforts of technical rationalisation and diversification - to virtual unintelligibility, not only for the laymen but for fellow scientists as well.

But an even more serious departure is the distancing from the real world, from the robust practicability that signifies Lie's thesis. It deals with the very existence, with authentic geometry both in a logical and substantial sense: in descriptive terms of what is essentially possible and hence absolutely necessary. Mind's dimensionalities and scope are endless, but when it is here a structural matter of, for instance, actual spheres coming into reciprocal physical being and continuous, i.e. space-filling and non-overcrossing, functioning, there has to be a synthesis. It can be seen quite concretely that infinitesimal projections of them, be they imaginary or tangible, distribute into "16 spheres, which are arranged in eight pairs" *(§ 10, n. 27, problem II)*: virtually on the surface of the four northern and four southern both Cartesian and Lie algebra A_2 in R_3 hemispheral space sectors as each of them further bi-sected by the hexagonal symmetry axes therein. Then, the clock works and is free to wind out the ensuing mathematical dimensions by cyclically regenerating expansion over the equally rendered and folded space extensions.[1]

This is a significant aspect of the Lie groups and algebras, otherwise they would not apply at all to manifested physics. In science, returning to the sources has always proven to be a good plan, and in doing so one not exceptionally finds that the message is divergent from the present-day account. When we go back with rational sincerity and interest to the primordial Lie groups and algebras we see, that it may not be they that are "mystically fit to describe mathematically"[2] the elementary particles and their patterns and behaviour. Rather it may be the quite subtle and ingenious but undeniably also disconnected, secondary adoption and application of them that makes the current elementary particle iconography arguably more mystical than fit.

1 Compare E. Trell. Hadronic Journal Supplement 12, 217-40, 1997.

2 M. Carmeli. Group Theory and General Relativity. Representations of the Lorentz Group and their Applications to the Gravitational Field. (NY, McGraw-Hill, 1977)

THE EIGHTFOLD EIGHTFOLD WAY: APPLICATION OF LIE'S TRUE *GEOMETRISKE TRANSFORMATIONER* TO ELEMENTARY PARTICLES

Erik Trell
Faculty of Health and Sciences
University of Linköping
S-581 85 Linköping, Sweden
eritr@all.liu.se

Received June 5, 1998; revised December 12, 1998

Abstract

Despite outstanding achievements, the Gell-Mann/Neeman eightfold model of elementary particle spectroscopy still remains with open problems, such as a direct representation of energy levels/mass numbers, or a direct inclusion of antiparticles. In this paper we show that these insufficiencies can be resolved by formulating the eightfold model via the original theory of *geometriscke Transformationer* of Marius Sophus Lie. By using the canonical orthogonal coset decomposition of SU(3), we obtain a double eightfold representation consisting of two ordinary Cartesian coordinate quadrants, in each of which the Gell-Mann/Neeman eightfold way acts, by directly reproducing the mass spectroscopy of particles and directly including antiparticles. All transitions and channels are clearly identified and, importantly, the masses are directly and amply retrieved from the pertaining ellipsoidical domain transformation as reciprocally proportional to the minor semiaxis contraction in relation to the proton ground state. Not only the baryons, but also the mesons and leptons as well as more exotic elementary particles, such as the *bottom flavour*, and the W and Z gauge vector bosons, are generated and here exemplified. In conclusion, the use of the original Lie's theory provides strong support of the standard SU(3) model via the achievement of more complete data. On the philosophical level the implications are likely far reaching. Taking place between and on surfaces instead of in the center, and being in that sense projective and observer-related, our findings entail a truly relativistic-dualistic, rather than monistic description of reality.

Introduction

Present elementary particle and theoretical physics remain firmly founded upon Marius Sophus Lie´s (pronounced Lee) continuous transformation groups and algebras - but not the real primary ones! This remarkable situation is rooted in the fact that his pertaining Ph.D. thesis, *"Over en Classe geometriske Transformationer"* [1], has remained un-translated from the, *per se* rich and comely Norwegian language, and therefore in a sense both contains and keeps the ordinary geometrical prescriptions by which current physics would have looked different - and quite familiar at the otherwise consolidated nanoscale of natural sciences today.

This has become apparent to me (who by linguistic kinship was able to read his thesis already when as a cardiologist I attempted to reproduce the full three-dimensional electrocardiographic potential distribution by rotational transformations of chestwall and other external recordings) as well as during the English thesis translation[2] to professor Santilli (who by his ingenious, per definition Lie-admissible isotopic mathematics[3] in the meantime kept Lie's flame alive in the flat complex space of the special unitary representation where for the usual irrational reasons it has been kept on reduced light since elementary particle physics and related cosmology[2,4] comparatively recently were established).

The paradox is that the special unitary representations are beyond doubt true in a circumscribed algorithm sense - but their isolated root-space diagram deficient both philosophically and physically: in the former monistic, self-referential, and in the latter hence centred and singular. This is apparent also in the cosmological expansion as manifested by the (etymologically Nordic) Big Bang scenario towards that extremity of the scale.

But the original Lie, basically real-structural geometrical representation is dualistic, double, three-dimensional, and surfacial - and in fact the forbear of the unitary groups and algebras, whose non-compensated spatial flatness and centredness are caused by the bi-dimensionality and polar projection of the complex numbers. This might be illuminated both principally and methodologically by the ensuing series of quotations from his thesis:

"The rapid development of geometry in our century stands, as is well known, in an intimate dependence on philosophical reflections upon the nature of Cartesian Geometry - reflections, which are expounded in their most universal form by *Plücker* in his oldest works. For one, who has immersed himself in the spirit of Plücker´s works, there is nothing fundamentally new in the idea, that as element for the geometry of the space can be used any curve that is dependent on three parameters. When none-the-less no one, as far as I know, has realised this thought, the ground must probably be sought in that no advantage that might result from this was seen."

"I have been brought to a general study of the said theory by my finding that, through a particularly remarkable transformation, the theory of main tangential curves can be brought back to that of rounded curves.

Following Plücker's trail I discuss the equation system:

$$[F_1 (x\ y\ z\ X\ Y\ Z) = 0, \quad F_2(x\ y\ z\ \ X\ Y\ Z) = 0],$$

which in one meaning, later to be explained, defines a general reciprocity between two spaces. When in particular the two equations are linear in relation to each system's variables, a projection is obtained by which to each space's points correspond in the other space the lines of a Plücker Line-complex.....In particular I study the aforementioned projection, upon which I found a - as it appears to me - *fundamental relation between the Plücker line geometry and a spatial geometry whose element is the sphere.*"

Methods

Also the methods are best reintroduced in Lie´s own words:

" The Cartesian geometry translates any geometric theorem into an algebraic one and thus of the geometry of the plane renders a faithful representation of the algebra of two variables and likewise of the geometry of space a representation of the algebra of three variable quantities."

"The in the following presented new theories are founded upon the fact, *that one can choose any space-curve which depends upon three parameters as the element of the geometry of the space*."

"When x y z and X Y Z are perceived as point co-ordinates for r and R, one can say, that by a transformation of this kind is defined *a correspondence between the two spaces' surface-elements, and nota bene the most general.*"

"It is known, that the equation:

$$F(x\ y\ z\ X\ Y\ Z) = 0$$

always defines a reciprocal correspondence between the two spaces' surface-elements; and likewise I have in the foregoing shown, that the equation-system:

$$F_1(x\ y\ z\ X\ Y\ Z) = 0, \ F_2(x\ y\ z\ X\ Y\ Z) = 0$$

always determines a transformation, that turns surfaces, that touch each other in like surfaces".

"We establish a *fundamental relation that takes place between the Plücker line-geometry and a geometry whose elements are the space's spheres."*

" When f is a linesurface, F is a sphere-envelope".

"If now the one of the two spaces is subjected to an, e.g., transformation, by which surfaces that touch each other, are turned into similar surfaces, the corresponding transformation of the other space will possess the same property...In that we restrict ourselves to linear transformations of r, we find between the corresponding transformations of R: *all movements (translation-movement, rotation-movement and the helicoidal movement), semblability-transformation, transformation by reciprocal radii, parallel transformation - by that is understood transition from a surface to its parallel-surface - a reciprocal transformation studied by Mr. Bonnet* etc., which all, corresponding to linear transformations of r, possess the property of turning rounded curves into rounded curves."

Of course this is an extremely pruned selection of the simplest of the quite subtle and profound theorems and lemmas in the thesis. But it may visualise the contours of a "double rendition" [1,2] in ordinary real space where there is mutual reciprocity, both in the "transition" [1,2,4] between rectilinear Plücker and rounded Lie geometries, and in the likewise doubly isomorphic transformations on the "envelope" [1,2] of either of these fulfilling the same analytical (that is, geodetic feasibility) conditions.

Hence, what might be called tangential surface events are generated, analogous to precipitations on germinal membranes, monolayers, films etc. where structure may first be seen to appear in neighbouring branches of the natural sciences today, like microbiology and chemistry.[4,5] This is a categorical difference between the currently applied unitary and the original, real orthogonal Lie algebras relating to the elementary particle spectroscopy, and of which Gilmore has provided the most lucid graphical exposition (Figures 1 - 2).[6]

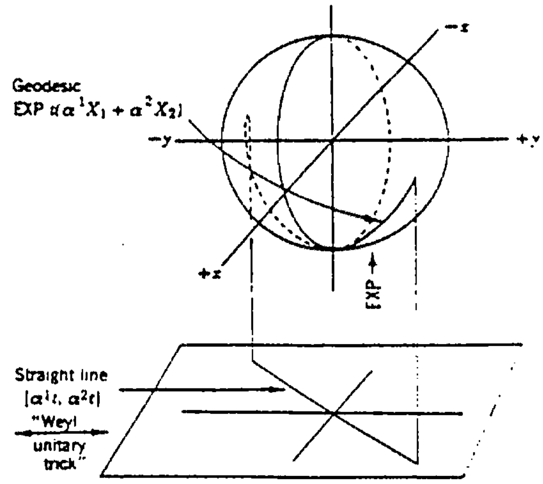

Fig. 1 Graphical exposition of Lie's projection of the surface-to-surface "fundamental relation that takes place between the Plücker line-geometry and a geometry whose elements are the space's spheres." (After Gilmore)

The ground plan has an ordinary geometrical, rotationally symmetric SO(3) x O(5) composition (Fig. 2), which, again documenting that the results are supporting and saluting the established theories, actually is parent to SU(3) although in the latter-day reverse order referred to as the canonical involutive automorphism, or orthogonal coset decomposition of this.[6,7]

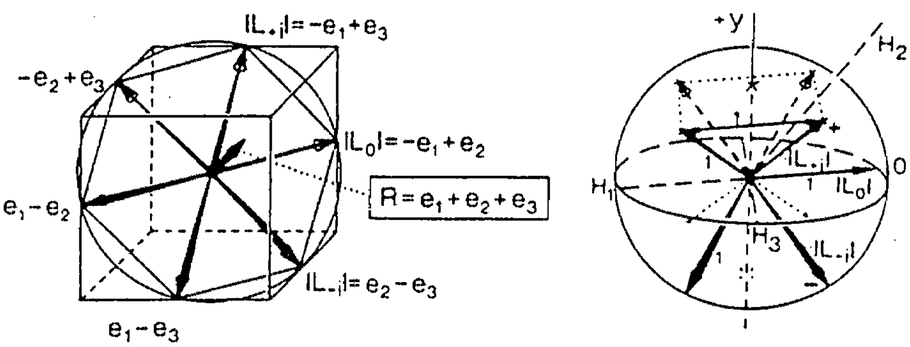

Fig. 2 a-b Full SO(3) x O(5) Lie transformation diagram. The five-dimensional, perpendicular and diagonal Cartesian projective space axes inside the unit "endlessly distanced sphere"[1,2] outline a duplicated a_2 "eightfold way" [8] root space diagram by which the unit t isospin (and sphere radius) transition steps from any of their end-points on the surface conduct the SO(3) volume- and spheroidal symmetry-preserving domain transformations (From Gilmore and Trell)

The figure shows the full SO(3) x O(5) transformation system on either surface, where O(5) corresponds to the perpendicular and diagonal projective axes of the Cartesian reference space. It is seen that their extensions inside the ground unit sphere form a duplicated A_2 root space diagram. Thus, the flat "eightfold way"[8] t isospin unit step ladder, or 'switch-board' of state shifts is expanded to a three-dimensional lattice (Fig. 3) along which by exactly the same procedure the initial transformations are carried out on the domain surface in any of the four northern and four southern, Cartesian neighbourhood hemisphere segments linearly independent of each other.

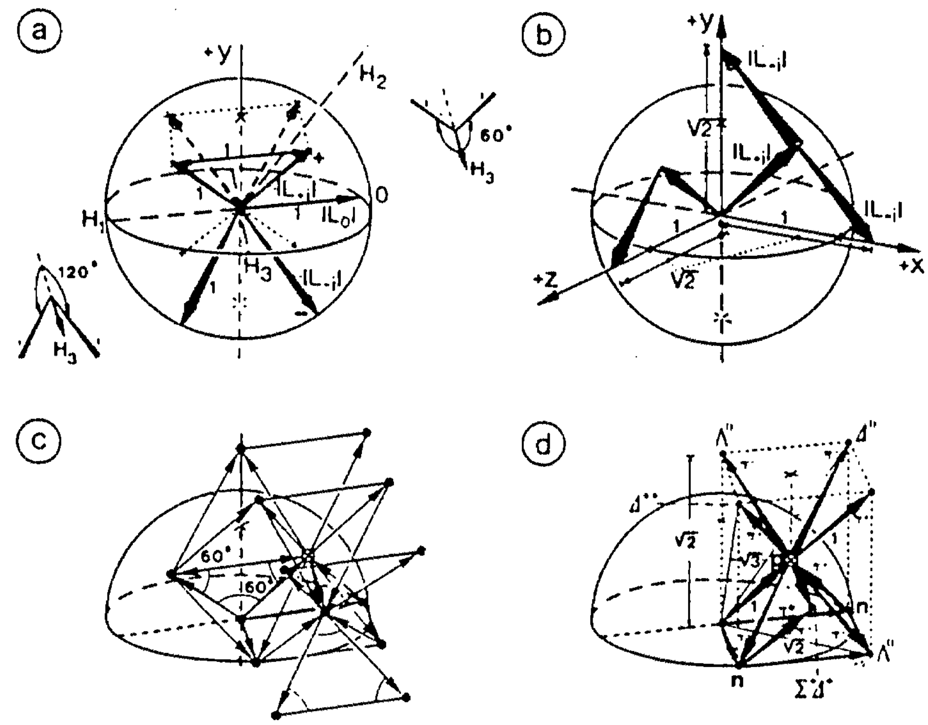

Fig. 3 Graphical representation of (a) SO(3) x O(5) root space, (b-c) three-dimensional transition lattice by unit length pion (and lepton) vector elements there, and (d) some of the basic supermultiplet transitions originating by eightfold way steps by these from the projective surface in one of the four northern perpendicular Cartesian co-ordinate segments (further bisected by diagonal symmetry axis).

A_2 is, of course, the root space diagram of SU(3) in its complex number formulation. However, when transferred 1:1 to the real space and numbers it becomes flat and self-centred because it lodges the duplicity inside itself.

This deficiency therefore afflicts the otherwise perfectly accurate eightfold way method.[8] It is followed here when tracking down channels and modes along the same pion and other unit transferences displayed there as well as in the observations. But by the full Lie representation, the previously combined electromagnetic and gravitational, "hypercharge" axis is decomposed into its two, basically perpendicular semiaxes, of which the *t* isospin vector, just as in the prototype eightfold way[8] constituting also the alternatingly and opposingly positive, negative and neutral (horizontal) lattice (or quark/antiquark) sides, always remains confined to discrete Q levels.

The mass is already implied to be associated with the semiaxis simultaneously liberated "in the plane perpendicular to the lines of electromagnetic flux".[9] The length to the surface of the new by SO(3) determined volume-invariant spheroidal transformation's elongated major semiaxis *(a)* set up by the new *t* isospin vector can always be calculated. The reciprocally contracted minor semiaxis length *(b* and *c* in a symmetric ellipsoid, *c* if *b* is taken over from the previous state) can equally easily be obtained from the unit gauge preservation condition.

And this enables the exact mass calculation as $1/c$ x 938.27 MeV, that is, inversely proportional by both the "quark pressure" formula,

$$\Delta p = h/\Delta x$$

and the domain curvature straightening to the mass carried by the Proton, all semiaxes = 1, perfectly round unit spheroid[4,7,10-12] according to the accepted theoretical understanding that all transformation "properties are attributable to this nonperturbative ground state".[9]

Results

Besides providing an overview of the faithful "eightfold eightfold way" of real form Lie transformations, Fig. 3 also exemplifies basic operations and resulting state reproductions in one arbitrary of the eight Cartesian space segments.

It is seen that the transitions are in the outward direction and hence distinct in each space segment. There is therefore separate room in opposing segments for antiparticle events. That does not hinder full particle domains, i.e. the baryons, to be Weyl-reflected in a free such diametrical sector of their space co-ordination.[7,11,12] But surface differentials as well as the single lattice steps and envelopes between the domain transformations that primarily take place in the separate segments are naturally cut off (but may themselves continue from or be transformed) there.

The Baryons

By these simple rules it is possible to virtually extemporise the elementary particle spectroscopy, and since detailed results of the basic baryon supermultiplets[7,10-12], the Λ[10,11], Σ[7,11], N and Δ[11,12] resonances, the basic mesons and leptons[10-12] as well as the charmed hadrons[4] have already been reported, only a condensed, mainly figure and table rhapsody of them is recapitulated here, giving more attention to the *bottom* mesons and the Z and W gauge vector bosons which, remarkably enough, the direct Lie representations also cover more than hundred years postponed.

The basic baryon supermultiplets are surveyed in Table 1 and Figures 4-5.

The table first summarises the obtained major semiaxis lengths of all the basic baryon supermultiplets as well as the consequential, volume-preserving length of the minor semiaxis changing in the transformation, easily obtained in the unit scale from the equation *(a x b x c)* = 1 and giving the mass number as *1/c x 938.27* MeV. As seen, there is extremely good correspondence with the available observational data.[7,10,13]

Table 1 Survey of the basic baryon supermutiplets

	Major semiaxis	Minor semiaxis	Mass	
			Calculated	Observed
Λ^0	$\sqrt{2}$	$\sqrt[4]{\frac{1}{2}}$	1115.8	1115.6
$\Sigma^{+,0,-}$	1.60804	0.788591	1189.8	1189.4 – 1197
$\Delta^{++,+,0,-}$	$\sqrt{3}$	$\sqrt[4]{\frac{1}{3}}$	1234.8	1230 – 1236
$\Xi^{0,-}$	1.975	0.7116	1318.5	1314.9 – 1321.3
$\Sigma(1385)^{+,0,-}$	$\sqrt{4.71} - \sqrt{4.75}$	0.679 – 0.678	1382.2 – 1385	1383 – 1386
$\Lambda(1405)^0$	$\sqrt{5}$	$\sqrt[4]{\frac{1}{5}}$	1403	1405 ± 5
$\Xi(1530)^{0,-}$	$\sqrt{7.06}$	0.6134778	1529.5	1528 – 1534
Ω^-	2.505 – 2.51	0.561 – 0.560*	1673.5 – 1677	1672 – 1674

* Minor semiaxis changed in the transformation (c).

The figures illustrate how in an arbitrary Cartesian space segement and exactly matching the channel spectrum observed in reality, the eightfold way, unit t isopspin steps are taken, and the root expression of resulting major semiaxis length to the centre or poles (or focal point) of parent states.

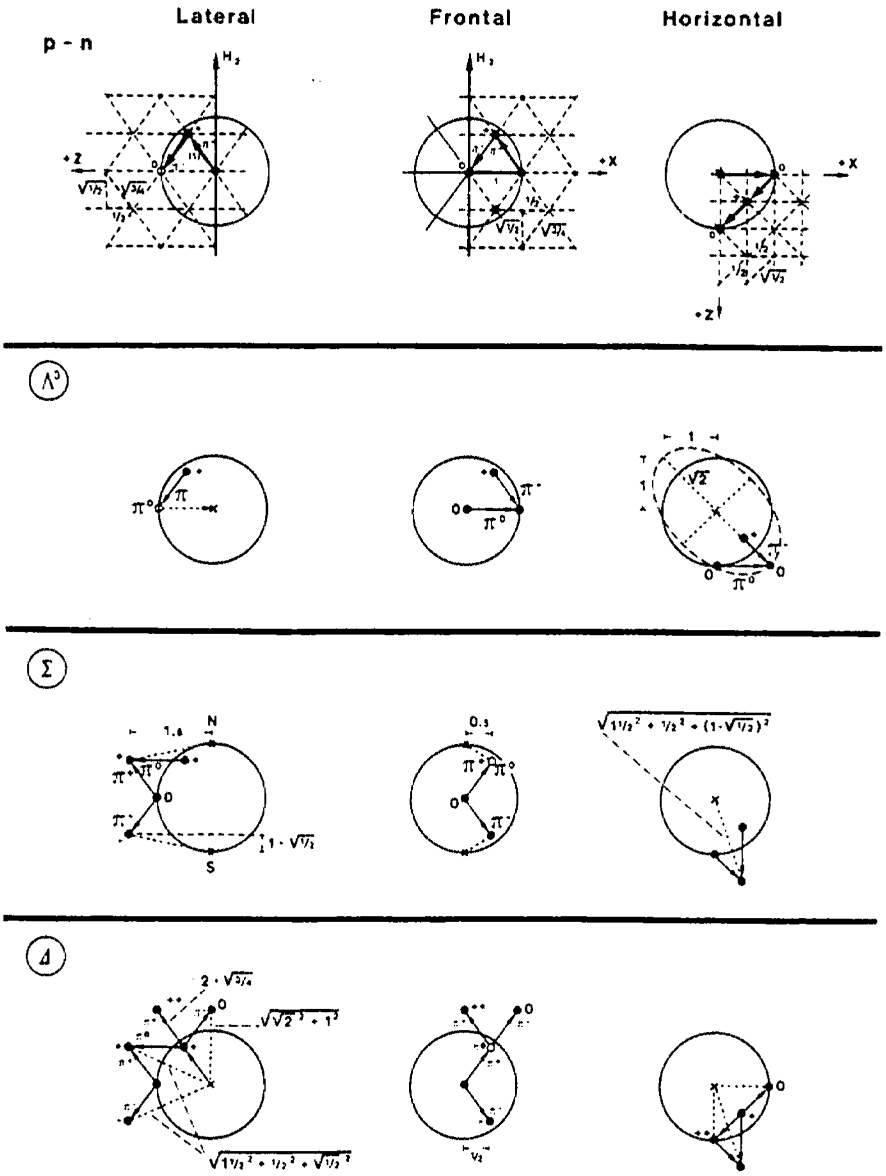

Fig. 4 Graphs of basic, p-n, Λ, Σ and Δ supermultiplet transitions (like in ensuing figures, except 8, 11 and 12, plane projections are shown)

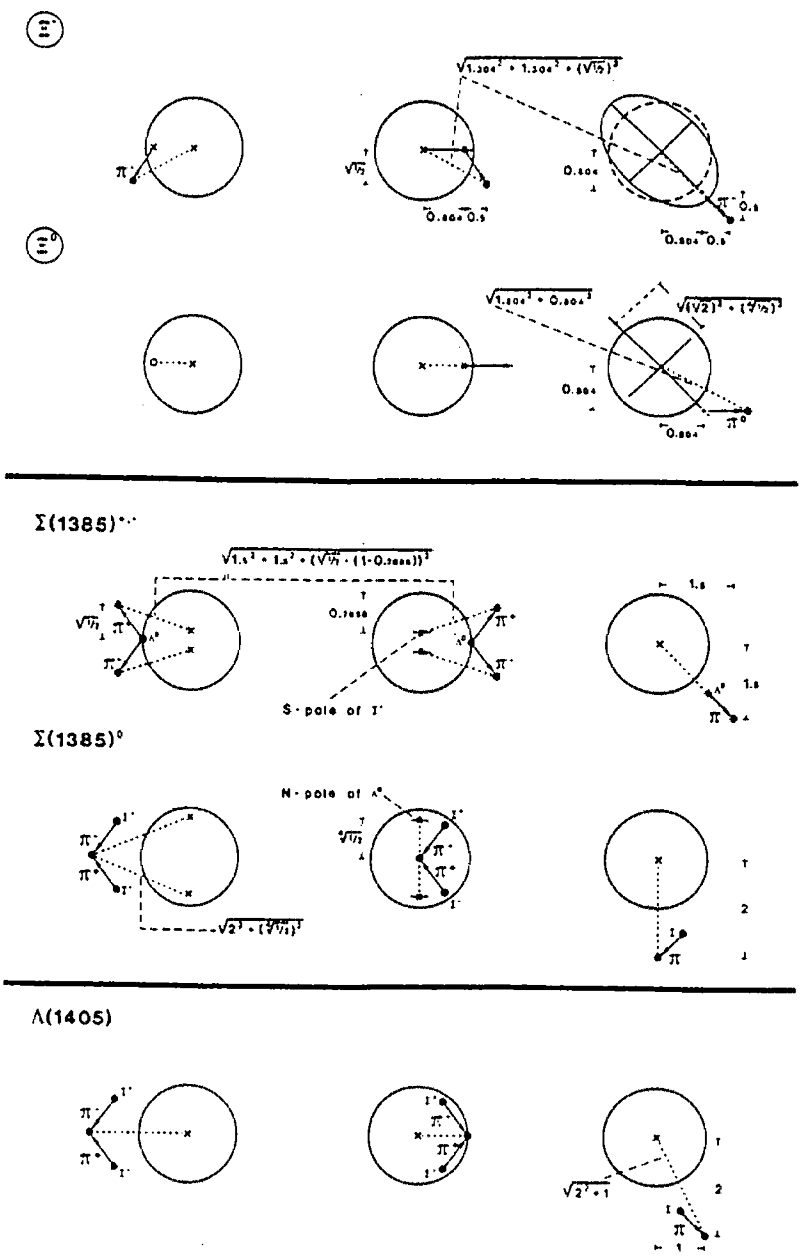

Fig. 5 Graphs of the basic Ξ, Σ(1385) and Λ(1405) supermultiplet transitions (here and elsewhere occasionally too much size-reduced number in root signs is 4)

Table 2 summarises the same findings in the <u>Λ resonances.</u>[10]

Table 2 Survey of the basic Λ resonances

State	Reproducible channels	Major semiaxis (a)		Minor semiaxes		Mass	
		Projection to	Length	b	c	Calculated	Observed
$\Lambda'(1520)$	$\Lambda\pi\pi$ $\Sigma\pi$ $N-\bar{K}$ $\Sigma(1385)\pi$	Λ meridian in $Q=0$ plane	$\sqrt{6.9}$	$\sqrt[4]{1/6.9}$	$\sqrt[4]{1/6.9}$	1520.5	~ 1520
$\Lambda''(1670)$	$N-\bar{K}$ $\Lambda\eta$ $\Sigma\pi$	Nucleon centre	$\sqrt{10}$	$\sqrt[4]{1/10}$	$\sqrt[4]{1/10}$	1668.5	1662 – 1680
$\Lambda''(1690)$	$\Lambda\pi\pi$ $\Sigma\pi$ $N-\bar{K}$	Λ meridian in $Q=0$ plane	2.13	$\sqrt[4]{1/2}$ (Λ minor axis)	0.5583	~ 1680	1685 – 1695
$\Lambda(1815)$	$\Lambda\pi\pi$ $\Sigma\pi$ $\Sigma(1385)\pi$	Σ centre	$\sqrt{6}$	0.7886 (Σ minor axis)	0.5177	~ 1812	~ 1820
$\Lambda(1830)$	$\Sigma\pi$	π from Σ focus; to vertical axis	2.036	0.9624 (to Σ centre)	0.51046	~ 1838	1810 – 1830
$\Lambda(1890)$	$N-\bar{K}$ $\Sigma\pi$ $\Sigma(1385)\pi$	Nucleon centre or periphery	2	1 (N radius)	0.5	~ 1880	1850 – 1910
$\Lambda(2100)\alpha$	$\Sigma\pi$ $\Lambda\omega$ $N-\bar{K}$	Nucleon centre	$\sqrt{5}$	1 (N radius)	$1/\sqrt{5}$	~ 2098	~ 2100
$\Lambda(2100)\beta$	$\Lambda\eta$ $N-\bar{K}$ $N-\bar{K}-\pi$	Nucleon centre	$\sqrt{8}$	0.7886 (Σ)	0.4483	~ 2093	~ 2100
$\Lambda(2350)$	$N-\bar{K}$	Σ centre	$\sqrt{10}$	0.7886 (Σ)	0.4001	~ 2340	~ 2350

No illustration of the Λ series is given here. Instead, the well confirmed <u>Σ resonances</u>[7](****/*** in particle properties review[13]) are outlined (Fig.6).

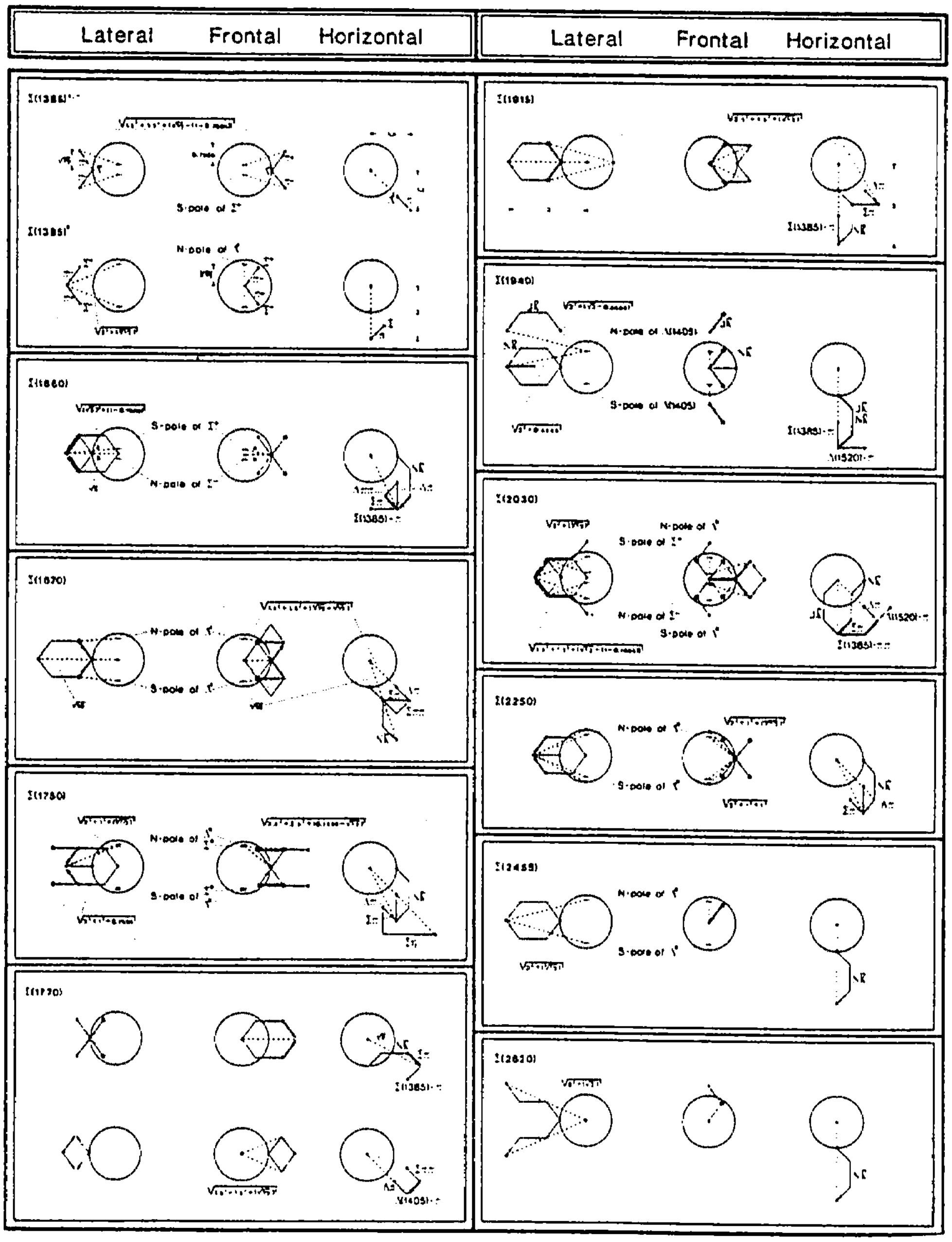

Fig. 6 Graphical reproduction of main members of Σ resonance series.

In fact, all the Σ resonances and their masses and J^P numbers can be accurately reproduced[7], but for matters of brevity only the graph but no table of the channels and trigonometric major semiaxis calculation of the better verified members was shown. In exchange, a tabular account of the N and Δ resonance series[12] is given below (Table 3) without illustration.

Table 3 N and Δ resonances, *a* and *c* semiaxes and computed masses (MeV)

State	*Major semi-axis*	*Minor semi-axis c*	*Mass Expression (MeV)*
$N^{+,0,(-)}$	1	1	938.28
$\Delta(1232)^{++,+,0,-}$	$\sqrt{3}$	$\sqrt{1/3}$	1234.8
N(1440)	$\sqrt{6}$	$\sqrt{1/6}$	1468.5
N(1520) *a*	2.6131	0.6186	1517
b	$\sqrt{7}$	$\sqrt{1/7}$	1526
N(1535)	$\sqrt{7}$	$\sqrt{1/7}$	1526
$\Delta(1620)$	$\sqrt{3}$	0.5774	1625
N(1650) *a*	$\sqrt{9.5}$	$\sqrt{1/9.5}$	1647
b	$\sqrt{10}$	$\sqrt{1/10}$	1669
N(1675)	$\sqrt{10}$	$\sqrt{1/10}$	1669
N(1680) *a*	$\sqrt{10.25}$	$\sqrt{1/10.25}$	1679
b	2.5495	0.5547	1691
N(1700)	$\sqrt{10.8284}$	$\sqrt{1/10.8248}$	1702
$\Delta(1700)$ *a*	$\sqrt{10}$	$\sqrt{1/10}$	1669
b	$\sqrt{10.8284}$	$\sqrt{1/10.8248}$	1702
N(1710)	$\sqrt{3} - \sqrt{4}$	0.5774 – 0.5	1630 – 1876
N(1720)	$\sqrt{13}$	$\sqrt{1/13}$	1781
$\Delta(1900)$	2 or $\sqrt{2}$	0.5	1876
$\Delta(1905)$	$\sqrt{16}$	$\sqrt{1/16}$	1878
$\Delta(1910)$	$\sqrt{17}$	$\sqrt{1/17}$	1905
$\Delta(1920)$	$\sqrt{18}$	$\sqrt{1/18}$	1932
$\Delta(1930)$	$\sqrt{18.66}$	$\sqrt{1/18.66}$	1950
$\Delta(1950)$	2.7321	0.4816	1948
N(2190)	3	0.4387	2140
N(2220), N(2250)	$\sqrt{6}$	0.4082	2298
$\Delta(2420)$	$\sqrt{7}$	0.3780	2482
N(2600)	2	0.3536	2654
$\Delta(2750)$, $\Delta(2950)$	$\sqrt{3}$ or $\sqrt{9}$	1/3	2815
N(3030)	$\sqrt{5}$ or $\sqrt{10}$	0.3162	2967
$\Delta(3230)$	2	0.2887	3250

Even the *Charm* baryons and their extra degree of projection freedom, their channels and masses and other properties can be adequately reproduced.[4] Only the graphs will be recapitulated here (Fig. 7).

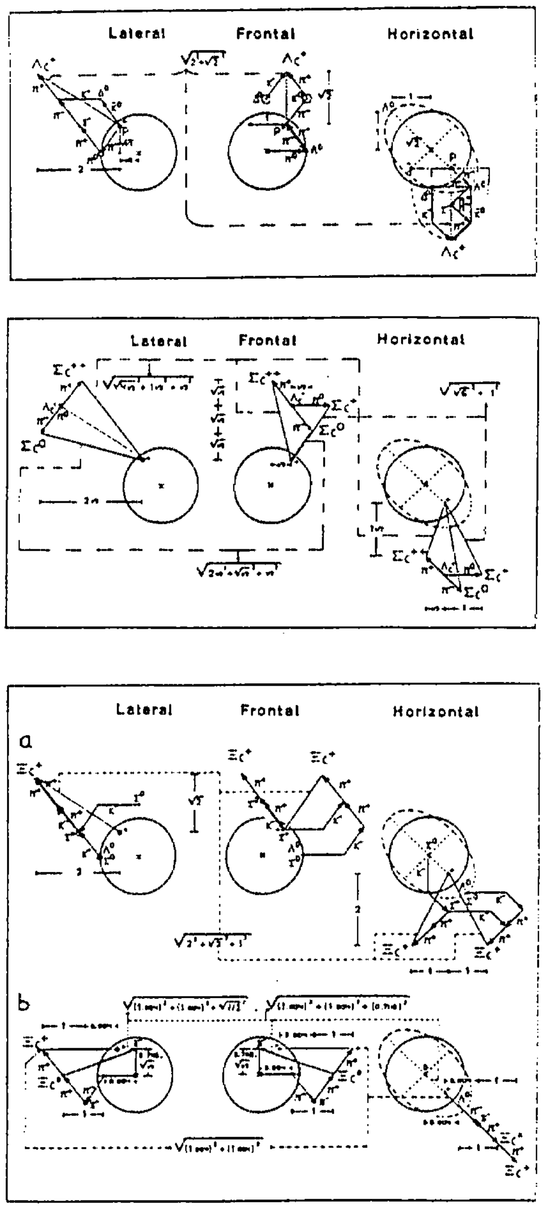

Fig. 7 Graphical reproduction of *Charm* baryons

The Mesons

Also the mesons manifest "the very nice distinction between quarks and antiquarks via Lie´s original approach" in a quite concrete way.[14]

Here the pions and the ensuing, most basic *(u,d,s)* mesons, i.e. $K^{\pm}$, K^0_S, K^0_L, η, $\rho(770)$ and $\omega(783)$[7,10] will be described (fig. 8).

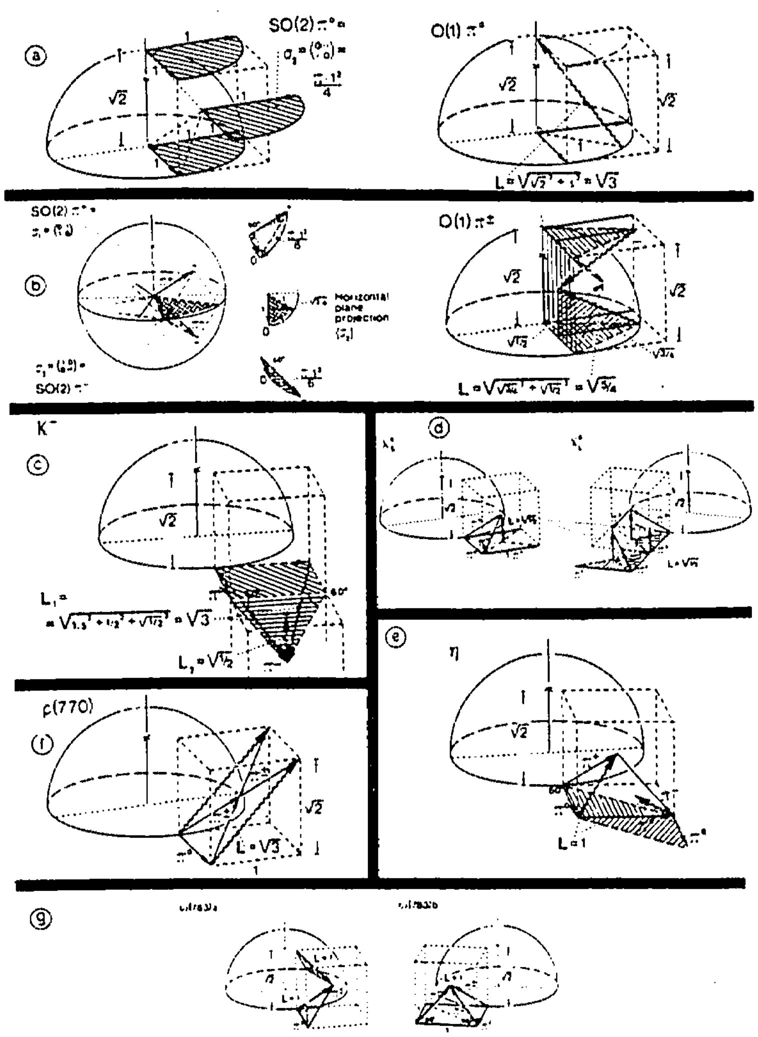

Fig. 8 Graphical reproduction of basic mesons

The respective real forms of the SU(2) x U(1) product group assigned to the mesons[15-18] appear as differential elements on and from the surface between transforming baryons, thus definitely within separate Cartesian co-ordinate segments. Their gauge theory product group is SU(2) x U(1), whose real form projections are summarised in the figure.

The SU(2) plane counterpart is spanned by opposingly alternate t isospin vectors which thus exhibit the prescribed quark-antiquark relation to each other. Being the symmetric element the mass contribution of the plane is proportional to its fraction of the proton equatorial plane area, whereas the counterpart of the antisymmetric U(1) translation vector, being the distance to the next same projection from the parent surface, contributes to the mass inversely to its length. Computed and actual masses are shown in Table 4.

Table 4 Mass calculations of pions and other basic (u,d,s) mesons (and leptons)

π^0	$1/4 \times 938.27 \times 1/\sqrt{3}$	135.4	135.0
$\pi^{\pm}$	$1/6 \times 938.27 \times 1/\sqrt{5/4}$	139.87	139.57
$K^{\pm}$	$(938.27/4 \times 1/\sqrt{3}) + (938.27/4 \times 1/\sqrt{3}) + (938.27/6 \times 1/\sqrt{1/2})$	492.0	493.65
K_S^0	$938.27/(4 \times \sqrt{1/2}) + 938.27/(8 \times \sqrt{1/2})$	497.6	497.67
K_L^0	$938.27/(8 \times \sqrt{1/2}) + 938.27/(8 \times \sqrt{1/2}) + 938.27/(8 \times \sqrt{1/2})$	497.6	497.67
η	$(938.27/6) + (938.27/6) + (938.27/4)$	547.33	548.8±0.6
$\rho(770)$	$(938.27/\sqrt{2})/\sqrt{3/4}$ or $(938.27 \times \sqrt{2})/\sqrt{3}$	766.1	768.3±0.5
$\omega(783)$	$938.27/4 + 938.27/4 + 938.27/6 + 938.27/6$	781.9	781.95±0.14
$\mu^{\pm}$	$1/(2\pi \times \sqrt{2}) \times 938.27$ or $1/(2\pi \times 2\sqrt{1/2}) \times 938.27$	105.59	105.66
$e^{\pm}$	$1/(137.035966 \times 6\pi \times \sqrt{1/2}) \times 938.27$	0.514	0.511
ν	$1/\infty \times 938.27$	0	0 (< 17-35)
γ	$1/\infty \times 938.27$	0	0 (< $3 \cdot 10^{-33}$)

It is evident that they are differential surface events spanning own volumes which can be inflated by further transformations but always departed further and further out in the periphery of the lattice segment. A plethora of permutations are possible which remarkably enough correspond to the equally abundant, observed spectroscopy.[7,11]

Nowhere is this more evident than in the *Charm* and *Bottom* mesons, which may appear far-fetched, bordering to fantastic, in the Lie representation; but not more so than - and matching - the factually observed modes and channels and their gauge field theory explanations including for each of them a corresponding extra "degree of freedom."[18] Since the charmed mesons have been reported separately[4,19], only two, somewhat diminished graphs of $\underline{D}^+$, $\underline{D}^0$ and $\underline{\bar{D}}^0$ are shown here (Fig. 9 a-b).

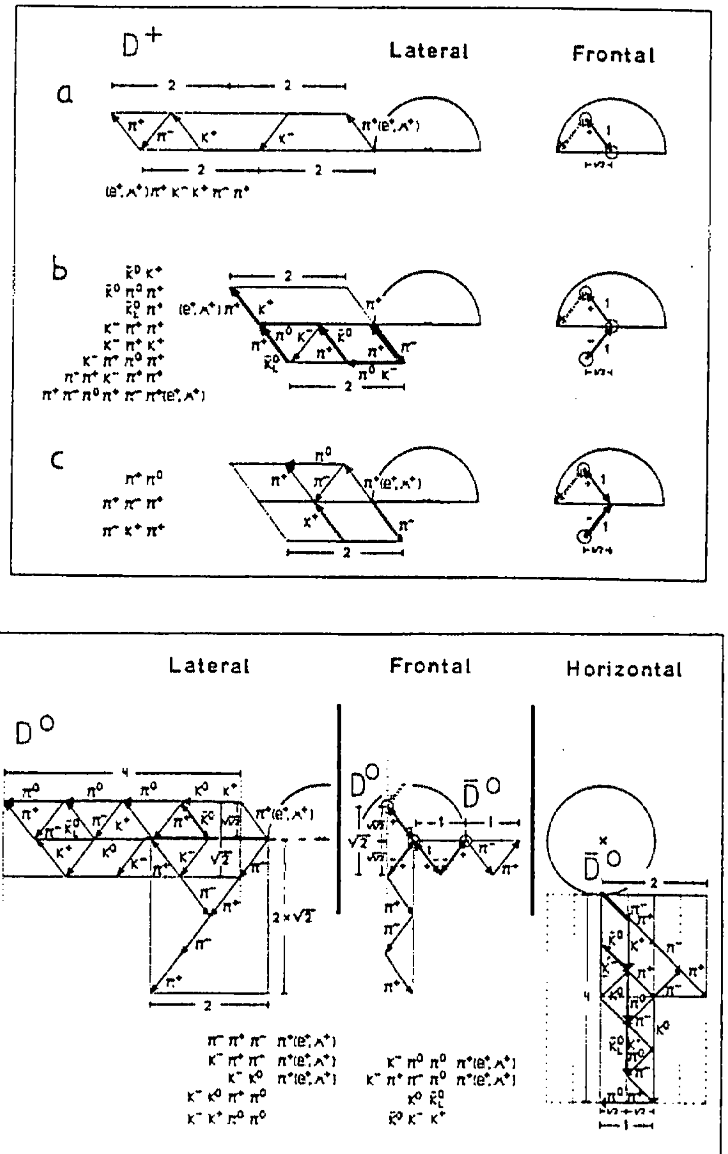

Fig. 9 a-b Real Lie image of D^+, D^0 and $\bar{D}^0$

The further dimensionality of the <u>*Bottom*</u> flavour deserves a somewhat more extensive consideration. Something extra obviously happens with the first *Bottom* mesons that can bring about their great jump in energy so immediately or closely upon the D mesons.[13] Much speaks in favour of that it is a reflection of one or both of the weak, quark-antiquark isospin root vectors to genuine semiaxes in elliptic or circular SU(2) planes. It is clearly a new degree of freedom and the large mass increase of the *Bottom* mesons becomes comprehensible, and also that they can evolve from the D isomultiplet almost directly or by single pion steps.

The latter, and a recorded $\Lambda_c^+ \leftrightarrow$ p-B^0 reaction are sketched in Fig. 10 a-b.

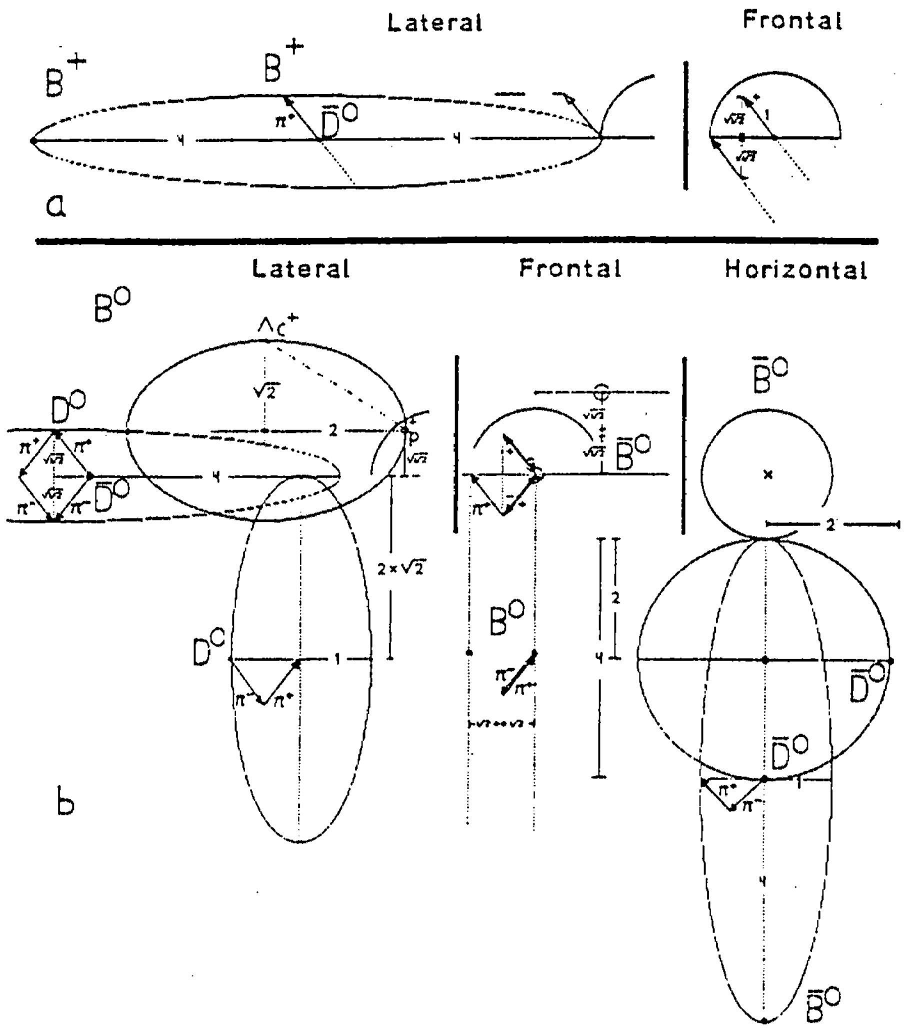

Fig. 10 a-b Real Lie image of the B *Bottom* mesons, modes and channels

B^+ (B^- obtained by charge conjugation)[13] is visualized by the D^0-π^+ mode. It produces a t isospin vector of length = 1 at a distance = 4 from the end-point of the other weak isospin vector (Fig. 10 a). When both are reflected to elliptic semiaxes, these are aligned in the electromagnetic charge plane and span a surface area of $(\pi)\cdot 1\cdot 4$. The distance to surrounding B^+ states is $(0.5)^{1/2}$. The mass is then 4 x $1/(0.5)^{1/2}$ x 0.93827 GeV = $4(2)^{1/2}$ x 0.93827 GeV ≈ 5.30 GeV, or just the actual mass of ≈ 5.28 GeV.[13]

In B^0, three varieties are shown (Fig. 10 b). In the $D^0/\bar{D}^0$ - $\pi^+\pi^-$ modes the elliptic semiaxes are $(0.5)^{1/2}$ and 4, and 1 and $2(2)^{1/2}$, and the distance to the neighbours = 0.5. The mass calculations are $(0.5)^{1/2}$ x 4 x 1/0.5 = $4(0.5)^{1/2}$ x 2 = $2(8)^{1/2}$ x 0.93827 GeV, and 1 x $2(2)^{1/2}$ x 1/0.5 = $2(8)^{1/2}$ x 0.93827 GeV ≈ 5.30 GeV. In the $\Lambda_C{}^+$ channel it is likewise 2 x $(2)^{1/2}$ x 1/0.5 = $4(2)^{1/2}$ x 0.93827 GeV ≈ 5.30 GeV.

The $\bar{B}^0$ antiparticle is recaptured as a degeneracy in the horizontal plane and caused by one semiaxis = 4 and the other = 1, or both semiaxes = 2, and distance to the next SU(2) plane of the same kind = $(0.5)^{1/2}$ (Fig. 10 b). Both give masses = 4 x 1 x $(2)^{1/2}$ x 0.93827 or 2 x 2 x $(2)^{1/2}$ x 0.93827 GeV ≈ 5.30 GeV in comparison with the ≈ 5.28 GeV in reality.[13]

The leptons

The leptons are contrived as one-dimensional wave sections, packets, and orbits. Their symmetries are governed by the simple U(1) group[15-17], the real forms of which in R_3 can only be straight or curved lines; put together, and thus not restricted to unit length, by iterations of the t isospin vectors or the difference vectors between them in the geometrical representation of the SU(3) root space.[7,11]

Interesting similarities between the leptons and the quarks have been noted since long.[20] Both may be truly elementary and genuinely point-like so that there is only one dimension left for their geodesics. The lepton masses are already considered to be inversely proportional to these, for example, the orbital wavelength or path distance in the electron.[21]

Fig. 11 shows the direct counterparts of the root vector sequences of the <u>muon</u> and the <u>electron/positron</u> geodesics in the transformation lattice. Especially "the muon has remained a mystery" and "the question 'why does the muon weigh?' has remained unanswered".[20]

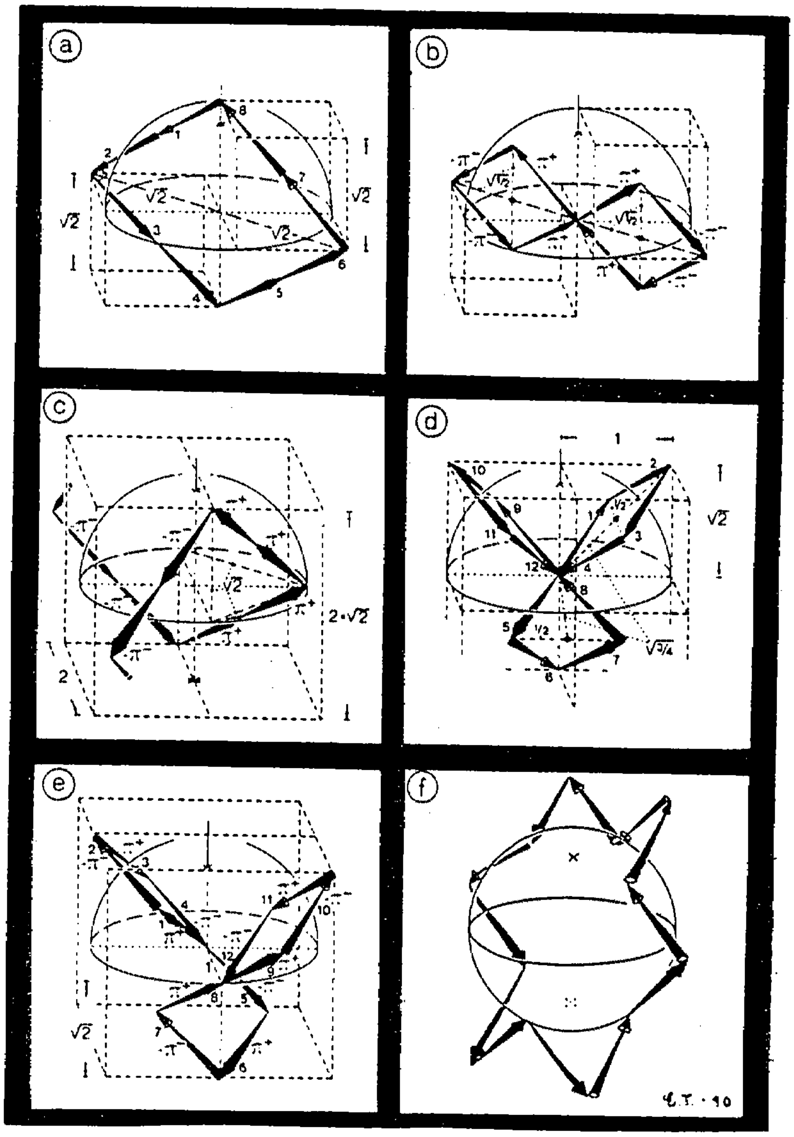

Fig. 11 a-f Vector trains of $e^{\pm}$ and $\mu^{\pm}$ geodesics in real Lie representation space

Here it is seen that one type of orbits in and around the nucleon domain can be established by quite straightforwardly repeating adjoining *t* isospin root vectors of equal charge sign and inclinated 90 - 180° to each other (Fig. 11

a-c). All have a length of $2\pi \times (2)^{1/2}$ and consequential mass number = $1/(2\pi \times (2)^{1/2}) \times 938.27$ MeV = 105.59 MeV, so that indeed in every respect the muon with actual mass 105.66 MeV[13] is reproduced.[7,11]

Further it is seen that the difference between an electron and muon trajectory is that of turning into a 60° or 120° adjoining t isospin root vector sequence instead of a perpendicular one (Fig. 11 d-f). The difference can be projected as straight angular momentum-like vectors in the same electromagnetic charge plane; one set up by the new, electron/positron root vector alignment, and one corresponding to the abandoned muon (not shown). It is also how the muon decays into the electron/positron by "a typically weak interaction"[20], involving both the muon-associated neutrino and the electron-associated anti-neutrino.[13] Their continuations result in straight lines within unchanged electromagnetic charge planes. Such rectilinear neutral current trajectories would be very hard to catch in a recording device and would be of infinite length with consequential mass number = $1/\infty \times 38.27$ MeV = 0. The γ counterparts, e.g. in the π^0 decay, are analogous but assume a sinusoidal form. Their iterations would accordingly be easier to record but still of infinite length with mass $1/\infty \times 938.27$ MeV = 0.

As mentioned, the electron/positron is retrieved in the 60°-120° t isospin root vector sequences (Fig. 11 d-f). This is the most "natural" orientation in the A_2 root space and, like in the muon, various types of closed or helical orbits can be generated; either "three-jet propeller" or "Mercedes" configurations (Fig. 11 d-e) as reported in high energy electron-positron annihilations[22], or a more likely six-pointed course around the nucleon region (Fig. 11 f). The length of all trajectories can be calculated as $3 \times 2\pi \times (1/2)^{1/2}$. However, since the electron circulates at a distance outside the nucleus expressed by multiples of the fine structure constant[21], the actual path length of the ground orbit is $137.035986 \times 3 \times 2\pi \times (1/2)^{1/2}$, with mass equivalent of $1/(137.035986 \times 3 \times 2\pi \times (1/2)^{1/2}) \times 938.27 \approx 0.5137$ MeV, which is again precisely as the real electron/positron with mass ≈ 0.511 MeV.[13]

The W and Z gauge vector bosons

Moreover, the τ heavy lepton and its neutrino adjuncts (not shown), and the W and Z gauge vector bosons can be identified in the faithful structure diagrams. The latter will be illustrated, also to demonstrate the far-reaching identities between the complete real Lie representation, and reality and theory alike (fig. 12 a,b).

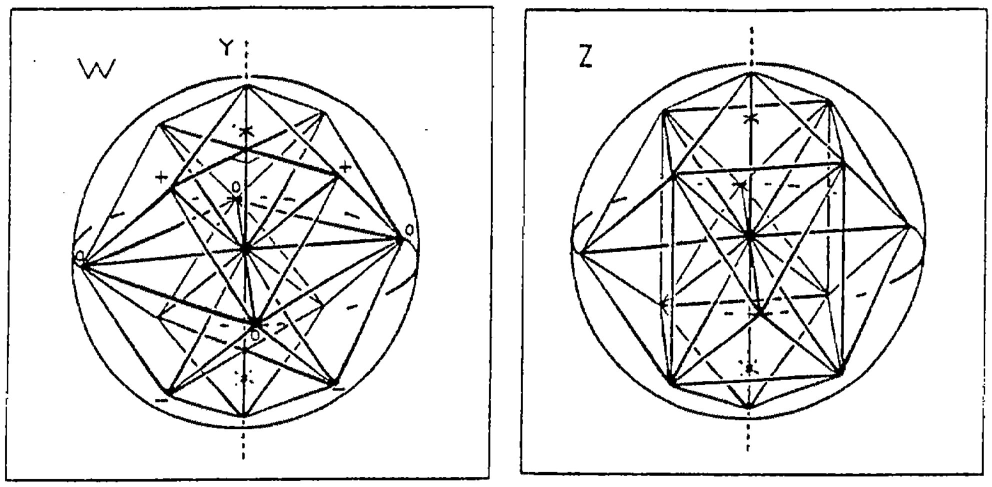

Fig. 12 a-b. W and Z gauge vector collection in full Lie SO(3) x O(1) domain

There is agreement that the vector bosons represent the whole gauge field, including particles and antiparticles (from the collision of which they may reciprocally be created). Hence, recapitulating the full SO(3) x O(5) embodiment and the gauge vector elements in the same (Fig. 3 a-d), two crystal-like collections and arrangements of all of these come up in the spherical domain, one with main diagonal co-ordination (Fig. 12 a), and one with a somewhat more probable-looking, orthogonal alignment (Fig. 12 b).

When fully split up along the diagonal symmetry axes of the orthogonal complementary subspace in the SU(3) geometrical automorphism (Fig. 7a), there would be a predominant release of charged t isospin root vectors and associated neutrinos, and also some γ. The partition is in the horizontal planes so that the decays would assume a charge isodoublet distribution.

Counting and summing up the generated vector quantities in this type of polyhedral division of colliding proton and anti-proton domains yield in each of them 32 charged *t* isospin vectors of length = 1; 4 neutral *t* isospin vectors of length = 1; 2 y axes of length = $(2)^{1/2}$, and 8 diagonal symmetry axes of the complementary subspace of length = $(2)^{1/2}$. The binding/splitting energy of the unit vectors should equal the proton mass of 0.93827 GeV, whereas in the $(2)^{1/2}$ axes it should be $1/(2)^{1/2}$ x 0.93827 GeV.

This gives 36 x 0.93827 + (10 x $1/(2)^{1/2}$) x 0.93827 GeV = 43.071 x 0.93827 = 40.4123 GeV. Adding the same amount from the other moiety in the reaction gives 2 x 40.4123 GeV ≈ 80.82 GeV, which perfectly suits with the recorded 80.6 ± 0.4 GeV in $W^{\pm}$.[13]

The Z^0 boson is more common. The mass is currently determined at ≈ 91.16 GeV from the peak energy at which proton-antiproton annihilation collisions may result in swarms of charged lepton pairs (e^+e^- 3.2%, $\mu^+\mu^-$ 3.4 %, $\tau^+\tau^-$ 3.3%, $e^{\pm}\mu^{\pm}$ 2 %), their associated neutrinos (19 %), γ's (2.8-3.7 %) and hadrons/antihadrons of all known flavours (70.9 %).[13]

Fig. 12 b shows that a cleavage along the horizontal and vertical Cartesian symmetry planes in a proton/anti-proton gauge vector lattice pair releases 2 x 32 charged *t* isospin axes of unit length and corresponding to weak isospin vector or lepton/antilepton pairs, 2 x 4 neutral t isospin axes of unit length; and 2 x 8 horizontally and, including the y co-ordinates, 2 x 6 vertically inclined symmetry axes of length 1 and $(2)^{1/2}$, respectively.

They either come out by themselves as neutrinos or γ or combine with *t* isospin root vectors to meson states. The total mass summation amounts to 2(44 + 6 x$(2)^{1/2}$) x 0.93827 GeV ≈ 90.53 GeV, which fits within 0.7 % with the decided value of Z^0 at 91.16 GeV.[13]

Discussion

Can also the *Top*, or in alternative terminology, *Truth* be reached by the real Lie algebras? Preliminary data is that it can, but this is beyond the scope of the present paper, which aims at recapitulating some of the straightforward but entirely faithful graphical executions of the original Lie transformations and how extensive and exact a correspondence with reality and theories alike they exhibit.

As noted, even the "distinction between quarks and antiquarks...comes out very nicely"[14] and "does not appear that clearly in the conventional treatment of SU(3)".[14,23] In the present geometrical reproductions both the alternate and opposing quark/antiquark root vector extensions in and of separate Cartesian space segments and adjoining quark/antiquark t isospin root vectors of the meson SU(2) plane counterparts are quite evident and complementary.

Furthermore, the "formulation is deeply linked to Dirac's equation in which particle and antiparticles do appear jointly. Finally, the connection appears to be squarely representable via isoduality"[4], which is in remarkable agreement with observations of both Clebsh and Lie.[1,2]

And it incorporates and preserves both gauge - unit volume - and flavour - spheroidal symmetry - and is in consequence in full accord also with quantum chromodynamics. When expressed as probability of chance coincidence (in which terms, for instance, new particle events are often judged to be confirmed at the odds ratio of one to a few thousands), the present findings, in extra regard of the spectroscopic correspondences, would therefore seem to be in effect infinitely well verified.

And they are fully and immediately reproducible, and should accordingly be of utility and value in exploratory structural research of both individual and collective properties, preferably by further development and application of computer techniques.[7,11,12] Mathematically, though, I have to agree that the exercises are rather simplistic.

But on the philosophical plane, where I am a bit better oriented and which Lie and his contemporaries considered essential to Mathematics[1,2], they might be of significance in indicating a way out of the monistic cul-de-sac which current elementary particle and theoretical physics seem to have entered during this century. As may have appeared in the foregoing, Lie's original continuous transformations are principally dualistic, a transition between and onto category spaces. This reminds very much of archaic ideas and notions like, for instance, the Yin-Yang whirl and later Aristotle's prime mover[4] - and remains the perhaps only imaginable realistic system where the perpetually rephrased cosmological dilemma "how a sort of spark in the primordial nothingness could have set off the Big Bang" [24] has a natural resolution.

References

1. M. S. Lie, Over en Classe geometriske Transformationer, Ph.D. Thesis, Kristiania (now Oslo), 1871.
2. E. Trell and R.M. Santilli, Algebras, Groups and Geometries **15** (1998).
3. R.M. Santilli, Isominkowskian Gravitation and Cosmology (with further references). Physical Interpretations of Relativity Theory, Imperial College, London: 11-14/9, 1998.
4. E. Trell, Real Charm of Form - real Form of Charm. Duality in Transition. Fundamental open problems in Science, Chinese Academy of Sciences, Beijing, August 28, 1997 and Hadronic J **21** (1998).
5. E. Trell. Hadronic J **21** (1998).
6. R. Gilmore, *Lie Groups, Lie Algebras and Some of their Applications* (John Wiley, NY, London, Toronto, 1974).
7. E. Trell, Phys. Essays **5**, 362 (1992).
8. M. Gell-Mann and Y. Neeman, *The eight-fold Way* (Benjamin, NY, 1964).
9. R.L. Jaffe, Nature **268**, 201 (1977).
10. E. Trell, Acta Phys. Austriaca **55**, 97 (1983).
11. E. Trell, Hadronic J. **13**, 277 (1990).
12. E. Trell, Phys. Essays **4**, 272 (1991).
13. Particle Data Group, Phys. Lett. B **239**, 1 (1990).
14. R.M. Santilli, Personal Communication. Physical Interpretations of Relativity Theory, Imperial College, London: 11-14/9, 1998.
15. R. P. Feynman and M. Gell-Mann, Phys. Rev **109**, 193 (1958).
16. S.L. Glashow, Nucl. Phys. **22**, 579 (1961).
17. A. Salam and J.C. Ward, Phys. Lett. **13**, 168 (1964).
18. F.E. Close, *An Introduction to Quarks and Partons* (Academic Press, London, NY and San Francisco, 1979).
19. E. Trell, The Eightfold Eightfold Way. A lateral view of the standard model. Physical Interpretations of Relativity Theory, Imperial College, London: 11-14/9, 1998. *
20. E. Close, Nature **266**, 679 (1977).
21. J. L. T. Waugh, *The Constitution of Inorganic Compounds* (Wiley Interscience, NY, London, Sydney, Toronto, 1972).
22. F. E. Close, Nature **281**, 104 (1979).
23. L.P. Horwitz, Personal Communication. Physical Interpretations of Relativity Theory, Imperial College, London: 11-14/9, 1998.
24. J. Glanz, Science **280**, 1008 (1998).

* Of which presentation this article is an elaboration

Dedicated to the centennial of the death of
Marius Sophus Lie
February 18, 1899

ISOTOPIC, GENOTOPIC AND HYPERSTRUCTURAL LIFTINGS OF LIE'S THEORY AND THEIR ISODUALS

Ruggero Maria Santilli
Institute for Basic Research
P.O.Box 1577, Palm Harbor, FL 34682, U.S.A.
ibr@gte.net

Received August 3, 1998, Revised December 21, 1998

Abstract

After reviewing the basic role of Lie's theory for the mathematics and physics of this century, we identify its limitations for the treatment of systems beyond the local-differential, Hamiltonian and canonical-unitary conditions of the original conception. We therefore outline three sequential generalized mathematics introduced by the author under the name of *iso-, geno- and hyper-mathematics* which are based on generalized, nonsingular, Hermitean, non-Hermitean and multi-valued units, respectively. The resulting *iso-, geno- and hyper-Lie theories*, for which the new mathematics were submitted, have been extensively used for the description of nonlocal-integral systems with action-at-a-distance potential and contact nonpotential interactions in reversible, irreversible and multi-valued conditions, respectively. We then point out that conventional, iso-, geno- and hyper-Lie theories are unable to provide a consistent *classical* representation of *antimatter* which yields the correct charge conjugate states at the operator counterpart. We therefore outline yet novel mathematics proposed by the author under the names of *isodual conventional, iso-, geno- and hyper-mathematics*, which constitute anti-isomorphic images of the original mathematics characterized by *negative-definite units and norms.* The emerging isodual generalizations of Lie's theory have permitted a novel consistent characterization of antimatter at all levels of study, from Newton to second quantization. The main message emerging after three decades of investigations is that the sole generalized theories as invariant as the original theory, the sole admitting physical applications, are those preserving the original abstract Lie axioms, and merely realizing them in broader forms.

1. Majestic Consistency of Lie's Theory.

Since I was first exposed to the theory of *Marius Sophus Lie* [1] during my graduate studies in physics at the University of Torino, Italy, in the 1960's, I understood that Lie's theory has a fundamental character for the virtual entire contemporary mathematics and physics.

I therefore dedicated my research life to identify the *limitations* of Lie's theory and construct possible *generalizations* for physical conditions broader than those of the original conception. In this paper I outline the most salient aspects of this scientific journey (as representative references, see my original papers in the field [3], mathematical studies [4], physical studies [5], monographs [6], applications and experimental verifications [7-10]).

Let $F = F(a, +, \times)$ be a field of conventional numbers a (real, complex or quaternionic numbers) with conventional sum +, (associative) product ×) additive unit 0 and multiplicative unit I. When formulated on a Hilbert space $\mathcal{H}$ over F, the physically most important formulation of Lie's theory is that via connected transformations of an operator A on $\mathcal{H}$ over F in the following finite and infinitesimal forms and interconnecting conjugation

$$A(w) = U \times A(0) \times U^{\dagger} = e^{iX \times w} \times A(0) \times e^{-iw \times X}, \tag{1.1a}$$

$$idA/dw = A \times X - X \times A = [A, X]_{operator}, \tag{1.1b}$$

$$e^{iX \times w} = [e^{-iw \times X}]^{\dagger}, X = X^{\dagger}, w \epsilon F, \tag{1.1c}$$

with classical counterpart in terms of vector-fields on the cotangent bundle (phase space) with local chart (r^k, p_k), k = 1, 2, 3, over F

$$A(w) = U \times A(0) \times U^{t} = e^{-w \times (\partial X/\partial r^k) \times (\partial/\partial p_k)} \times A(0) \times e^{w (\partial/\partial r^k) \times (\partial X/\partial p_k)}, \tag{1.2a}$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k} = [A, X]_{classical}, \tag{1.2b}$$

and unique interconnecting map given by the conventional or symplectic quantization.

As it is well known, Lie's theory is at the foundation of the mathematics of this century, including topology, vector and metric spaces, functional analysis, differential equations, algebras and groups, geometries, etc.

As it is also well know, Lie's theory is at the foundation of all physical theories of this century, including classical and quantum mechanics, particle physics, nuclear physics, superconductivity, chemistry, astrophysics, etc. In fact, whenever the parameter w represents time t, Eqs. (1.1) are the celebrated Heisenberg equations of motion in finite and infinitesimal form, while Eq.s (1.2) are the classical Hamilton equations, also in their finite and infinitesimal forms. Characterization via Lie's theory of all classical and operator branches of physics then follows.

A reason for the majestic consistency of Lie's theory most important for physical applications is that of being *form invariant* under the transformations of its own class. In fact, connected Lie groups (1.1a) constitute *unitary transforms* on $\mathcal{H}$ over F,

$$U \times U^{\dagger} = U^{\dagger} \times U = I, \tag{1.3}$$

under which we have the following invariance laws for units, products and eigenvalue equations

$$I \rightarrow U \times I \times U^{\dagger} = I' = I, \tag{1.4a}$$

$$A \times B \rightarrow U \times (A \times B) \times U^{\dagger} = (U \times A \times U^{\dagger}) \times (U \times B \times U^{\dagger}) = A' \times B', \tag{1.4b}$$

$$H \times |\psi >= E \times |\psi > \rightarrow U \times H \times |\psi >= (U \times H \times U^{\dagger}) \times (U \times |\psi >) = H' \times |\psi' >=$$
$$U \times E \times |\psi >= E' \times |\psi' >, E' = E. \tag{1.4c}$$

with corresponding invariances at the classical level here omitted for brevity.

It then follows that *Lie's theory possesses numerically invariant units, products and eigenvalues,* thus verifying the necessary condition for physically consistent applications.

2. Initial Proposals of Generalized Theories.

Despite the above majestic mathematical and physical consistency, by no means Lie's theory can represent the totality of systems existing in the universe. In fact, inspection of structures (1.1) and (1.2) reveals that, in its conventional formulation, *Lie's theory can only represent isolated-conservative-reversible systems of point-like particles with only potential-Hamiltonian internal interactions.* In fact, the point-like structure is demanded by the local-differential character of the underlying topology; the isolated-conservative character of the systems is established by the fact that the brackets [A, B] of the time evolution are totally antisymmetric, thus implying conservation laws of total quantities; the sole potential character is established by the representation of systems solely via a Hamiltonian; and the reversibility is established by the fact that all known action-at-a-distance interactions are reversible in time (i.e., their time reversal image is as physical as the original one, as it is the case for the orbit of a planet). All admissible interactions are represented via time-independent potentials V in the Hamiltonian H $= p^2/2m + V$, resulting in manifestly reversible systems.

I therefore initiated a long term research program aiming at generalizations (I called *liftings*) of Lie's theory suitable for the representation of broader systems.

The first lifting I proposed as part of my Ph.D. thesis [3a,3b] back in 1967 is that for the representation of *open-nonconservative systems*, that is, systems whose total energy H is not conserves in time, $idH/dt \neq 0$, because of interactions with the rest of the universe. This called for the formulation of the theory in such a way that its brackets are not totally antisymmetric. In this way I proposed, apparently for the first time in 1967, the broader (p-q)-parametric deformations (known in more recent times as the q-deformations),

$$A(w) = U \times A(0) \times U^{\dagger} = e^{iw \times p \times X} \times A(0) \times e^{-iw \times q \times X}, X = X^{\dagger}, \quad (2.1a)$$

$$idA/dw = p \times A \times X - q \times X \times A = (A, X)_{operator}, \quad (2.1b)$$

where p, q and $p + / - q$ are non-null parameters, with classical counterpart [3c]

$$A(w) = U \times A(0) \times U^{t} =^{-w \times q \times (\partial X/\partial r^k) \times (\partial/\partial p_k)} \times A(0) \times e^{w \times p \times (\partial/\partial r^k) \times (\partial X/\partial p_k)}, \quad (2.2a)$$

$$\frac{dA}{dw} = p \times \frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - q \times \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k} = (A, X)_{classical}. \quad (2.2b)$$

Prior to releasing papers [3a] for publication, I spent about one year in European mathematical libraries to identify the algebras characterized by brackets (A, B) which resulted to be *Lie-admissible* according to Albert [2] (a generally nonassociative algebra with product (A, B) is said to be Lie-admissible when the attached algebra with antisymmetric product [A, B] = (A, B) - (B, A) is Lie). At the time of proposal [3a] only *three* papers had appeared in Lie-admissible algebras and only in the mathematical literature (see Ref. [3a]).

The (p, q)-parametric deformations (2.1), (2.2) did indeed achieve the desired objective. In fact, the total energy and other physical quantities *are not* conserved by assumption, because idH/dt = (p - q)$\times H \times H \neq 0$.

In 1968 I emigrated with my family to the U. S. A, where I soon discovered that Lie-admissible theories were excessively ahead of their time because unknown in *mathematical*, let alone physical circles. Therefore, for a number of years I had to dedicated myself to more mundane research along the preferred lines of the time.

When I passed to Harvard University in 1978 I resumed research on Lie-admissibility and proposed the most general possible (P, Q)-operator Lie-admissible theory according to the operator structures [3d]

$$A(w) = U \times A(0) \times U^{\dagger} = e^{iw \times X \times Q} \times A(0) \times e^{-iw \times P \times X}, X = X^{\dagger}, P = Q^{\dagger}, \tag{2.3a}$$

$$idA/dw = A \times P \times X - X \times Q \times A = (A \hat{,} X)_{operator}, \tag{2.3b}$$

where $P, Q, and P+/-Q$ are non-singular matrices (or operators) such that P-Q characterizes Lie brackets, with classical counterpart [3d]

$$A(w) = U \times A(0) \times U^{t} = e^{-w \times \times (\partial X/\partial r^{i}) \times Q^{i}_{j} \times (\partial/\partial p_{k})} \times A(0) \times e^{w \times (\partial/\partial r^{i}) \times P^{i}_{j} (\partial X/\partial p_{j})}, \tag{2.4a}$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^{i}} \times P^{i}_{j} \times \frac{\partial X}{\partial p_{j}} - \frac{\partial X}{\partial r^{i}} \times Q^{i}_{j} \times \frac{\partial A}{\partial p_{j}} = (A, X)_{classical}. \tag{2.4b}$$

A primary motivation for generalizations (2.3) and (2.4) over (2.1) and (2.2) is that the latter constitute nonunitary-noncanonical transforms. The application of a nonunitary transform to Eqs. (2.1) then yields precisely Eqs. (2.3) with $P = p \times (U \times U^{\dagger})^{-1}$ and $Q = q \times (U \times U^{\dagger})^{-1}$, as we shall see better below. The application of any additional nonunitary transform then preserves the Lie-admissible structure. A similar case occurs for the classical counterpart.

Additional studies established that structures (2.3) constitute the most general possible transformations admitting an algebra in the infinitesimal form. In particular, the product $(A\hat{,}B)$ results to be jointly *Lie- and Jordan admissible*, although the attached Lie and Jordan algebras are more general than the conversional forms.

The latter generalized character permitted me to propose a particularization of the above Lie-admissible theory I called *Lie-isotopic* [3d,3r] , in which the brackets did verify the Lie axioms, but are more general then the conventional versions, with operator formulation

$$A(w) = U \times A(0) \times U^{\dagger} = e^{iX \times T \times w} \times A(0) \times e^{-iw \times T \times X}, \hat{T} = hatT^{\dagger}, \quad (2.5a)$$

$$idA/dw = A \times T \times X - X \times T \times A = [A\hat{,}X]_{operator}, \quad (2.5b)$$

and classical counterpart [3d,3r]]

$$A(w) = e^{-w \times (\partial X/\partial r^i) \times T^i_j \times (\partial/\partial p_j)}) > A(0) < e^{-w(\partial/\partial p_j) \times T^i_j \times (\partial X/\partial r^i)}, \quad (2.6a)$$

$$\frac{dA}{dw} = \frac{\partial A}{\partial r^i} \times T^i_j \times \frac{\partial X}{\partial p_j} - \frac{\partial X}{\partial r^i} \times T^i_j \times \frac{\partial A}{\partial p_j} = [A\hat{,}X]_{classical}. \quad (2.6b)$$

As one can see, the latter theories too are nonunitary-noncanonical, and the application of additional nonunitary-noncanonical transforms preserves the Lie-isotopic character. This establishes that transformations (2.5), (2.6) are the most general possible ones admitting a Lie algebra in the brackets of their infinitesimal versions.

3. Inconsistencies of Initial Generalizations.

Following the proposals of theories (2.1)-(2.6), I discovered that, even though *mathematically* intriguing and significant, the above Lie-isotopic and Lie-admissible theories had *no physical applications.* This is due to the fact that all the broader theories considered have a *nonunitary structure* at the operator level with a *noncanonical structure* st the classical counterpart.

In the transition from unitary to nonunitary theories, invariances (1.4) are turned into the following noninvariances,

$$U \times U^{\dagger} = U^{\dagger} \times U \neq I, \quad (3.1a)$$

$$I \to U \times I \times U^\dagger = I' \neq I, \tag{3.1b}$$

$$A \times B \to U \times (A \times B) \times U^\dagger =$$

$$(U \times A \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times B \times U^\dagger) = A' \times T \times B', T = (U \times U^\dagger)^{-1}, \tag{3.1c}$$

$$H \times |\psi> = E \times |\psi> \to U \times H \times |\psi> = (U \times H \times U^\dagger) \times (U \times U^\dagger)^{-1} \times (U \times |\psi>) =$$

$$H' \times T \times |\psi'> = U \times E \times |\psi> = E' \times |\psi'>, E' \neq E, \tag{3.1d}$$

It then follows that all theories with a nonunitary structure have the following physical inconsistencies studied in detail in Refs. [12]: 1) nonunitary theories do not have invariant units of time, space, energy, etc., thus lacking any physically meaningful applications to measurements (for which the invariance of the basic units is a necessary pre-requisite); 2) nonunitary theories do not preserve in time the original Hermiticity of operators, thus having no physically acceptable observables; 3) nonunitary theories do not have invariant conventional and special functions and transforms, thus lacking unique and invariant numerical predictions; nonunitary theories violate probability and causality laws; nonunitary theories are incompatible with Galilei's and Einstein's relativities; and suffer from other serious shortcomings. Similar inconsistencies exist at the classical level.

Corresponding mathematical inconsistencies also occur [12f,12g]. In fact, nonunitary theories are generally formulated on a conventional metric or Hilbert space defined over a given field which, in turn, is based on a given unit I. But the fundamental unit is not left invariant by nonunitary transforms by conception. It them follows that the entire mathematical structure of nonunitary theories becomes inapplicable for any value of the parameters different than the initial values.

It should be noted that the above catastrophic inconsistencies also hold for any other theory departing from Lie's theory, yet formulated via conventional mathematics, such as deformations, Kac-Moody algebras, superalgebras, etc. [12].

After systematic studies I realized that the *only* possibility to reach invariant formulations of generalized Lie theories was that of *constructing new mathematics* specifically conceived for the task at hand.

Since no other mathematics was available for the representation of the broader theories here considered, as a physics I had to initiate long and laborious mathematical studies in constructing the new mathematics, as a pre-requisite for conducting physical research.

Predictably, the task resulted to be more difficult than I suspected. In fact, after having lifted all the essential aspects of conventional mathematics (such as numbers and fields, vector and metric spaces, algebras and groups, geometries, etc.) [3s] into the needed broader form, I continued to miss the crucial invariance.

Insidiously, the problem resulted to exist where I was expect it the least, in the *ordinary differential calculus.* It was only in memoir [3i] of 1966 that I finally achieved invariance following suitable liftings of the ordinary differential calculus. The reader is therefore warned that *all papers on Lie-isotopic and Lie-admissible theories prior to memoir [3i] have no consistent physical applications because they lack invariance.*

The invariant liftings of Lie's theory which resulted from these efforts can be summarized as follows.

4. Lie-Santilli Isotheory.

The main idea [3d] is the lifting the conventional, trivial, n-dimensional unit I = diag. (1, 1, ..., 1) of Lie's theory into a real-values, nowhere singular and positive-definite $n \times n$-dimensional matrix $\hat{I}$, called *isounit* (where the prefix "iso-" means "axiom-preserving"), with an unrestricted functional dependence on time t, coordinates $r = (r^k)$, momenta p = (p_k), k = 1, 2, 3, wavefunctions ψ, and any other needed variable,

$$I = diag.(1, 1, ..., 1) \rightarrow \hat{I}(t, r, p, \psi, ...) = 1/\hat{T} \neq I. \qquad (4.1)$$

The applicable mathematics, called *isomathematics*, is the lifting of the *totality* of conventional mathematics (with a well defined unit), without any exception known to me, into a new form admitting $\hat{I}$, rather than I, as the correct left and right unit. This calls for:

1) The lifting of the associative product $A \times B$ amon g generic quantities A, B (such as numbers, vector-fields, operators, etc.) into the form, called *isoassociative product*, for which $\hat{I}$ is indeed the left and right unit,

$$A \times B \rightarrow A \hat{\times} B = A \times \hat{T} \times B, \hat{I} \hat{\times} A = A \times \hat{\times} \hat{I} = A; \qquad (4.2)$$

2) The lifting of fields $F = F(a, +, \times)$ into the *isofields* $\hat{F} = \hat{F}(\hat{a}, \hat{+}. \hat{\times})$ of *isonumbers* $\hat{a} = a \times \hat{I}$ (isoreal, isocomplex and isooctonionic numbers)

with *isosum* $\hat{a}\hat{+}\hat{b} = (a+b)\times\hat{I}$, *isoproduct* $\hat{a}\hat{\times}\hat{b} = (a\times b)\times\hat{I}$, *isoquotient* $\hat{a}\hat{/}\hat{b} = (\hat{a}/\hat{b})\times hatI$, etc. (see [3h] for details) ;

3) The lifting of functions f(r) on F into *isofunctions* $\hat{f}(\hat{r})$ on $\hat{F}$, such as the *isoexponentiation* $\hat{e}^{\hat{A}} = \hat{I}+\hat{A}/1!+\hat{A}\hat{\times}\hat{A}/2!+\ldots = (e^{\hat{A}\times\hat{T})}\times\hat{I} = \hat{I}\times(e^{\hat{T}\times\hat{A}}$, and related lifting of transforms into *isotransforms* (see [3i,3s] for details);

4) The lifting of the ordinary differential calculus into the *isodifferential calculus*, with basic rules $\hat{d}\hat{r}^k = \hat{I}^k_i\times d\hat{r}^i, \hat{d}\hat{p}_k = \hat{T}^i_k\times d\hat{p}_i$ (because r^k and p_k are defined on isospaces with isometrics inverse of each other), *isoderivatives* $\hat{\partial}/\hat{\partial}\hat{r}^i = \hat{T}^j_i\times\partial/\partial\hat{r}^j$, $\hat{\partial}/\hat{\partial}\hat{p}_k = \hat{I}^i_k\times\partial/\partial\hat{p}_i$, $\hat{\partial}\hat{r}^i/\hat{\partial}\hat{r}^j = \hat{\delta}^i_j = \delta^i_j\times\hat{I}$, etc.(see [3i] for details);

5) The lifting of conventional vector, metric and Hilbert spaces into their isotopic images, e.g., the lifting of the Euclidean space $E(r,\delta,R)$ with local coordinates $r = (r^k)$ and metric $\delta = Diag.(1,1,1)$ into the *isoeuclidean spaces* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ with isocoordinates $\hat{r} = r\times\hat{I}$ and isometric $\hat{\delta} = \hat{T}\times\delta$ over the isoreals $\hat{R}$, or the lifting of the Hilbert space $\mathcal{H}$ with inner product $<|\times|>\times I$ over the complex field C into the *isohilbert space* $\hat{\mathcal{H}}$ with *isoinner product* $<|\hat{\times}|>\times\hat{I}$ over the isocomplex field $\hat{C}$; etc. (see [3s] for details).

6) The lifting of geometries and topologies into their corresponding isotopic images (see [3n] for details);

7) The isotopic lifting of all various branches of Lie's theory, such as the liftings of: universal enveloping associative algebras (including the Poincaré-Birkhoff-Witt theorem), Lie's algebras (including Lie's first, second and third theorems); Lie's groups, transformation and representation theory, etc.

The main operator formulation of the Lie-Santilli isotheory can be written

$$\hat{A}(\hat{w}) = \hat{e}^{i\hat{X}\hat{\times}\hat{w}}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{e}^{-i\hat{w}\hat{\times}\hat{X}} =$$

$$[e^{iw\times\hat{X}\times\hat{T}}\times A(0)\times e^{-iw\times\hat{T}\times hatX}]\times\hat{I}, X = X^\dagger, \hat{T} = \hat{T}^\dagger, \quad (4.3a)$$

$$i\hat{d}\hat{A}\hat{/}\hat{d}\hat{w} = \hat{A}\hat{\times}\hat{X} - \hat{X}\hat{\times}\hat{A} = \hat{A}\times\hat{T}\times\hat{X} - \hat{X}\times\hat{T}\times\hat{A} = [A\hat{,}X]_{operator}, \quad (4.3b)$$

$$hatei\hat{X}\times\hat{w} = (\hat{e}^{-i\hat{w}\hat{\times}\hat{X}})^{\hat{d}agger}, \quad (4.3c)$$

with classical counterpart

$$\hat{A}(\hat{w}) = e^{-w\times(\hat{\partial}\hat{X}\hat{/}\hat{\partial}\hat{r}^k)\hat{\times}(\hat{\partial}\hat{/}\hat{\partial}\hat{p}_k)}\hat{\times}\hat{A}(\hat{0})\hat{\times}\hat{e}^{w(\hat{\partial}\hat{/}\hat{\partial}\hat{r}^k)\hat{\times}(\hat{\partial}\hat{X}\hat{/}\hat{\partial}\hat{p}_k)}, \quad (4.4a)$$

$$\frac{\hat{d}\hat{A}}{\hat{d}\hat{w}} = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}X}{\hat{\partial}\hat{p}_k} - \frac{\hat{\partial}\hat{X}}{\hat{\partial}\hat{r}^k} \hat{\times} \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_k} =$$

$$[\frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k} - \frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k}] \times \hat{I} = [A\hat{,}X]_{classical}, \tag{4.4b}$$

and unique interconnecting map called *isosymplectic quantization* [3s].

A most salient feature of the Lie-Santilli isotheory is that it is *form invariant under all possible nonunitary transforms*, thus achieving the fundamental physical objective indicated earlier. In fact, an arbitrary nonunitary transform on $\mathcal{H}$ over F can always be uniquely written as the *isounitary transform* on $\hat{\mathcal{H}}$ over $\hat{F}$,

$$V \times V^\dagger = \hat{I} \neq I, V = \hat{V} \times \hat{T}^{1/2}, V \times V^\dagger = \hat{V} \hat{\times} \hat{V}^\dagger = \hat{V}^\dagger \hat{\times} \hat{V} = \hat{I}, \tag{4.5}$$

under which we have the isoinvariance laws

$$\hat{I} \rightarrow \hat{V} \times \hat{I} \times \hat{V}^\dagger = \hat{I}' = \hat{I}, \tag{4.6a}$$

$$\hat{A} \hat{\times} \hat{B} \rightarrow \hat{V} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{V}^\dagger = (\hat{V} \hat{\times} \hat{A} \hat{\times} V^\dagger) \hat{\times} (\hat{V} \hat{\times} \hat{B} \hat{\times} \hat{V}^\dagger) = \hat{A}' \hat{\times} \hat{B}', \tag{4.6b}$$

$$\hat{H} \hat{\times} |\hat{\psi}> = \hat{E} \hat{\times} |\hat{\psi}> \rightarrow \hat{V} \hat{\times} \hat{H} \hat{\times} |\hat{\psi}> = \hat{V} \hat{\times} \hat{H} \hat{\times} \hat{V}^\dagger \hat{\times} \hat{V} \hat{\times} |\hat{\psi}> = \hat{H}' \hat{\times} |\hat{\psi}'> =$$

$$\hat{V} \hat{\times} \hat{E} \hat{\times} |\hat{\psi}> = \hat{E}' \hat{\times} |\hat{\psi}'>, \hat{E}' = \hat{E}, \tag{4.6c}$$

with corresponding isoinvariances for the classical counterpart.

As one can see, isomathematics achieves the *invariance of the numerical values of the isounit, isoproduct and isoeigenvalues*, thus regaining the necessary conditions for physical applications.

It is easy to prove that *isohermiticity coincides with the conventional Hermiticity.* As a result, all conventional observables of unitary theories remain observables under isotopies. The preservation of Hermiticity-observability in time is then ensured by the above isoinvariances. Detailed studies conducted in Ref. [3l] established the resolution of all inconsistencies of nonunitary theories.

By comparing Eqs. (1.1)-(1.2) and (4.3)-(4.4) it is evident that the Lie theory and the Lie-Santilli isotheory coincide at the abstract level by conception and construction [3d,3i]. In fact, the latter can be characterized by "putting a hat" to the totality of quantities and operations of Lie's theory with no exception known to me (otherwise the invariance is lost).

Despite this mathematical similarity, the physical implications of the Lie-Santilli isotheory are far reaching. By recalling that Lie's theory is at the foundation of all of physics, Eqs. (4.3) and (4.4) have permitted a structural generalization of the fundamental dynamical equations of classical and quantum mechanics, superconductivity and chemistry into new disciplines called *isomechanics* [3] *isosuperconductivity* [7] and *isochemistry* [8]. These new disciplines essentially preserve the physical content of the old theories, including the preservation identically of the total conserved quantities, but add internal nonhamiltonian effects represented by the isounit that are outside any hope of representation via Lie's theory.

In turn, these novel effects have permitted momentous advances in various scientific fields, such as the first axiomatically consistent unification of electroweak and gravitational interactions [3k,3q].

An illustrative classical application of the Lie-Santilli isotheory is the representation of the structure of Jupiter when considered isolated from the rest of the Solar system, with action-at-a-distance gravitational and other interactions represented with the potential V in the Hamiltonian H and additional, internal contact-non-Hamiltonian interactions represented via the isounit $\hat{I}$.

An illustrative operator application is given by novel structure models of the strongly interacting particles (called *hadrons*) for which the theory was constructed [3j]. In turn, the latter application has far reaching implications, including the prediction of novel, clean *subnuclear* energies.

5. Lie-Santilli Genotheory.

The main insufficiency of the Lie-Santilli isotheory is that it preserves the totally antisymmetric character of the classical and operators Lie brackets, thus being unsuited for a representation of open-nonconservative systems. In particular, despite the broadening of unitary-canonical theories into nonunitary-noncanonical extensions, the fundamental problem of the *origin of the irreversibility* of our macroscopic reality does not admit quantitative treatment via the Lie-Santilli isotheory because the latter theory is also is structurally reversible (that is, the theory coincides with its time reversal image for reversible Hamiltonians and isounits).

The resolution of this insufficiency required the broadening of the Lie-Santilli isotheory into a form whose brackets are neither totally antisymmetric nor totally symmetric. In turn, the achievement of an invariant formulation of the latter theory requires the construction of a new mathematics I suggested back in 1978 [3d] under the name of *genomathematics* (where the prefix "geno" now stands for "axiom-inducing").

The main idea of genomathematics is the selection of *two different generalized units* called *genounits*, the first $\hat{I}^>$ for the *ordered multiplication to the right* $A > B$, called *forward genoproduct*, and the second $^<\hat{I}$ for the *ordered multiplication to the left* $A < B$, called *backward genoproduct*, according to the general rules [3d,3i,3l]

$$\hat{I}^> = 1/\hat{S}, A > B = A \times \hat{S} \times B, \hat{I}^> > A = A > \hat{I}^> = A, \tag{5.1a}$$

$$^<\hat{I} = 1/\hat{R}, A < B = A \times \hat{R} \times B, ^<\hat{I} < A = A <^< \hat{I} = A, \tag{5.1b}$$

$$A = A^\dagger, B = B^\dagger, \hat{R} = \hat{S}^\dagger \tag{5.1c}$$

The broader genomathematics is then given by:

1) The lifting of isofields $\hat{F}(\hat{a}, \hat{+}, \hat{\times})$ into the *forward and backward genofields* $\hat{F}^>(\hat{a}^>, \hat{+}^>, >)$ and $^<\hat{F}(^<\hat{c}, ^<\hat{+}, <)$ with *forward and backward genonumbers* $\hat{a}^> = a \times \hat{I}^>$ and $^<\hat{a} = ^<\hat{I} \times a$, and related operations [3h];

2) The lifting of isofunctions $\hat{f}(\hat{r})$ on $\hat{F}$ into the *forward and backward genofunctions* $\hat{f}^>(\hat{r}^>)$ and $^<\hat{f}(^<\hat{r})$ on $\hat{F}^>$ and $^<F$, respectively, such as $\hat{e}_>^{\hat{X}>} = (e^{\hat{X}>\times\hat{R}}) \times \hat{I}^>$ and $\hat{e}_<^{<\hat{X}} = ^<\hat{I} \times e^{\hat{S}\times<\hat{X}}$,with consequential genotopies of transforms and functional analysis at large [3i,3s];

3) The lifting of the isodifferential calculus into the *forward and backward genodifferential calculus* with main forward rules $\hat{d}^>\hat{r}^{>k} = \hat{I}_i^{>k} \times d\hat{r}^{>i}, \hat{d}^>\hat{p}_k^> = \hat{T}_k^{>i} \times d\hat{p}_i^>, \hat{\partial}^>/\hat{\partial}^>\hat{r}^{>i} = \hat{S}_i^{>j} \times \partial/\partial\hat{r}^{>j}, \hat{\partial}^>/\hat{\partial}^>\hat{p}_k^> = \hat{S}_k^{>i} \times \partial/\partial\hat{p}_i^>, \hat{\partial}^>\hat{r}^{>i}/\hat{\partial}^>\hat{r}^{>j} = \hat{\delta}_j^{>i} = \delta_j^i \times \hat{I}^>$, etc., and corresponding backward rules easily obtainable via conjugation (see [3i] for details);

4) The lifting of isotopologies, isogeometries,etc. into the dual forward and backward genotopic forms; and

5) The lifting of the Lie-Santilli isotheory into the genotheory, including the genotopies of the various aspects, such as universal enveloping associative algebras for ordered product to the right and to the left, etc. [3i,3r,3s].

The explicit realization of the Lie-Santilli genotheory can be expressed via the following finite and infinitesimal forms with related interconnection (at a fixed value of the parameter w, thus without its ordering) [3i,3l]

$$\hat{A}(\hat{w}) = e_{>}^{i\hat{X}>>\hat{w}} > \hat{A}(\hat{0}) < e_{<}^{-i\hat{w}<<\hat{X}} =$$

$$[e^{i\hat{X}\times\hat{S}\times w} \times \hat{I}^{>}] \times \hat{S} \times \hat{A}(\hat{0}) \times \hat{R} \times [{}^{<}\hat{I} \times e_{<}^{-iw\times\hat{R}\times\hat{X}}], \tag{5.2a}$$

$$i\hat{d}\hat{A}/\hat{d}\hat{w} = \hat{A} < \hat{X} - \hat{X} > \hat{A} =$$

$$\hat{A} \times \hat{R} \times \hat{X} - \hat{X} \times \hat{S} \times \hat{A} = (\hat{A},\hat{X})_{operator}, \tag{5.2b}$$

$${}^{<}\hat{X} = (\hat{X}^{>})^{\dagger}, \hat{R} = \hat{S}^{\dagger} \tag{5.2c}$$

classical counterpart [3i]

$$\hat{A}(\hat{w}) = \hat{e}_{>}^{-\hat{X}>>\hat{w}} > \hat{A}(\hat{0}) < \hat{e}_{<}^{\hat{w}<<\hat{X}} =$$

$$e^{-w\times(\hat{\partial}^{>}\hat{X}^{>}/\hat{\partial}^{>}\hat{r}^{>k})>(\hat{\partial}^{>}/\hat{\partial}^{>}\hat{p}_k^{>})} > \hat{A}(\hat{0}) < \hat{e}^{w({}^{<}\partial/{}^{<}\hat{\partial}^{<k}\hat{r})<({}^{<}\hat{\partial}^{<}\hat{X}/{}^{<}\hat{\partial}_k^{<}\hat{p})}, \tag{5.3a}$$

$$\frac{\hat{d}\hat{A}}{\hat{d}\hat{w}} =^{<} \frac{\hat{\partial}^{<}\hat{A}}{{}^{<}\hat{\partial}^{<k}\hat{r}} < \frac{{}^{<}\hat{\partial}X}{{}^{<}\hat{\partial}_k^{<}\hat{p}} - \frac{\hat{\partial}^{>}\hat{X}^{>}}{\hat{\partial}^{>}\hat{r}^{>k}} > \frac{\hat{\partial}^{>}\hat{A}^{>}}{\hat{\partial}^{>}\hat{p}_k^{>}} =$$

$${}^{<}\hat{I} \times [\frac{\partial A}{\partial r^k} \times \frac{\partial X}{\partial p_k}] - [\frac{\partial X}{\partial r^k} \times \frac{\partial A}{\partial p_k}] \times \hat{I}^{>} = (A\hat{,}X)_{classical} \tag{5.3b}$$

with unique interconnecting map called *genosymplectic quantization* [3s].

A most important feature of the Lie-Santilli genotheory is its *form invariance.* This can be seen by noting that a *pair* of nonunitary transforms on $\mathcal{H}$ over $\hat{C}$ can always be identically rewritten as the *genounitary transforms* on genohilbert spaces over genocomplex fields,

$$V\times V^{\dagger} \neq 1, V =^{<} \hat{V}\times\hat{R}^{1/2}, V\times V^{\dagger} =^{<} \hat{V} <^{<} \hat{V}^{\dagger} =^{<} \hat{V}^{\dagger} <^{<} \hat{V} =^{<} \hat{I}, \tag{5.4a}$$

$$W \times W^{\dagger} \neq 1, W = \hat{W}^{>} \times \hat{S}^{1/2}, W \times W^{\dagger} = \hat{W}^{>} > \hat{W}^{>\dagger} = \hat{W}^{>\dagger} > \hat{W}^{>} = \hat{I}^{>}, \tag{5.4b}$$

under which we have indeed the following forward genoinvariance laws [3j]

$$\hat{I}^{>} \rightarrow \hat{I}'^{>} = \hat{W}^{>} > \hat{I}^{>} > \hat{W}^{>\dagger} = \hat{I}^{>}, \tag{5.5a}$$

$$\hat{A} > \hat{B} \to \hat{W}^{>} > (\hat{A} > \hat{B}) > \hat{W}^{>\dagger} = \hat{A}' > \hat{B}', \tag{5.5b}$$

$$\hat{H}^{>} > | >= \hat{E}^{>} > | >= E \times | > \to \hat{W}^{>} > \hat{H}^{>} > | >= \hat{H}'^{>} > | >' = \hat{W}^{>} > \hat{E}^{>} > | >= E \times | >', \tag{5.5c}$$

with corresponding rules for the backward and classical counterparts.

The above rules confirm the achievement of the *invariance of the numerical values of genounits, genoproducts and genoeigenvalues*, thus permitting physically consistent applications.

By recalling again that Lie's theory is at the foundation of all of contemporary science, the Lie-Santilli genotheory has permitted an additional structural generalization of classical and quantum isomechanics, isosuperconductivity and isochemistry into their genotopic coverings.

Intriguingly, the product $\hat{A} < \hat{B} - \hat{B} > \hat{A} = \hat{A} \times \hat{R} \times \hat{B} - \hat{B} \times \hat{S} \times \hat{A}$, $\hat{R} \neq \hat{S}$, is manifestly non-Lie on conventional spaces over conventional fields, yet it becomes fully antisymmetry and Lie when formulates on the bimodule of the respective envelopes to the left and to the right, $\{{}^{<}\hat{\mathcal{A}}, \hat{\mathcal{A}}^{>}\}$ (explicitly, the numerical values of $\hat{A} < \hat{B} = \hat{A} \times \hat{R} \times \hat{B}$ computed with respect to ${}^{<}\hat{I} = 1/\hat{R}$ is the same as that of $\hat{A} > \hat{B} = \hat{A} \times \hat{S} \times \hat{B}$ when computed with respect to $\hat{I}^{>} = 1/\hat{S}$) [3i,3l].

A primary feature of the broader classical and operator genotheories is that it represents open-nonconservative systems, as desired, because now the total energy H is not conserved in our spacetime, $idH/dt = H \times (\hat{R} - \hat{S}) \times H \neq 0$. Yet, the notion of *genohermiticity* on $\hat{\mathcal{H}}^{>}$ over $\hat{C}^{>}$ coincides with conventional Hermiticity. Therefore, the Lie-admissible theory provides the only operator representation of open systems known to this author in which the *nonconserved Hamiltonian and other quantities are Hermitean, thus observable.* In other treatments of nonconservative systems the Hamiltonian is generally *nonhermitean* and, therefore, *not observable.*

More importantly, genotheories have permitted a resolution of the historical problem of the *origin of irreversibility* via its reduction to the ultimate possible layers of nature, such as particles in the core of a star. The interested reader can find the invariant genotopic formulations of: Newton's equations in Ref. [3i]; Hamilton's equations with external terms in Ref. [3i]; quantization for open-irreversible systems in Ref. [3i,3l]; operator theory of open-irreversible systems in Ref. [3l].

6. Lie-Santilli Hypertheory.

By no means genotheories are sufficient to represent the entirely of nature, e.g., because they are unable to represent *biological structures* such as a cell or a sea shell. The latter systems are indeed open-nonconservative-irreversible, yet they possess a structure dramatically more complex than that of a nonconservative Newtonian system. A study of the issue has revealed that the limitation of genotheories is due to their *single-valued character.*

As an illustration, mathematical treatments complemented with computer visualization [10] have established that the *shape* of sea shells can be well described via the conventional single-valued three-dimensional Euclidean space and geometry according to the empirical perception of our three Eustachian tubes. However, the same space and geometry are basically insufficient to represent *the growth in time* of sea shells. In fact, computer visualization shows that, under the exact imposition of the Euclidean axioms, sea shells first grow in time in a distorted way and then crack.

Illert [10] showed that a minimally consistent representation of the sea shells growth in time requires *six dimensions.* But sea shells exist in our environment and can be observed via our *three-dimensional* perception. The solution of this apparent dichotomy I proposed [10] is that via *multi-valued hypermathematics* essentially characterized by the relaxation of the single-valued nature of the genounits while preserving their nonsymmetric character (as a necessary condition to represent irreversible events), according to the rules [3i,3t]

$$\hat{I}^{>} = \{\hat{I}_1^{>}, \hat{I}_2^{>}, \hat{I}_3^{>}, ...\} = 1/\hat{S}, \qquad (6.1a)$$

$$A > B = \{A\times\hat{S}_1\times B, A\times\hat{S}_2\times B, A\times\hat{S}_3\times B, ...\}, \hat{I}^{>} > A = A > \hat{I}^{>} = A\times I, \qquad (6.1b)$$

$$^{<}\hat{I} = \{^{<}\hat{I}_1, ^{<}\hat{I}_2, ^{<}\hat{I}_3, ...\} = 1/\hat{R}, \qquad (6.1c)$$

$$A < B = \{A\times\hat{R}_1\times B, A\times hat R_2\times B, A\times\hat{R}_3\times B, ...\}^{<}\hat{I} < A = A <^{<}\hat{I} = I\times A, \qquad (6.1d)$$

$$A = A^{\dagger}, B = B^{\dagger}, \hat{R} = \hat{S}^{\dagger}. \qquad (6.1e)$$

All aspects of the bimodular genotheories admit a unique, and significant extension to the above hyperstructures and their explicit form is here omitted for brevity [3i,3t]. The expression of the theory via *weak equalities and operations* was first studied by Santilli and Vougiouklis in Ref. [11].

7. Isodual theories.

Mathematicians appear to be unaware of the fact that, contrary to popular beliefs, *the totality of contemporary mathematics, including its isotopic, genotopic and hyperstructural liftings, cannot provide a consistent classical representation of antimatter.* In fact, all these mathematics admit *only one quantization channel.* As a result, the operator image of any classical treatment of antimatter via these mathematics simply cannot yield the correct charge conjugate state, but it merely yields a particle with the wrong sign of the charge.

The occurrence should not be surprising because the study of antimatter constitutes one of the biggest scientific unbalances of this century. In fact, matter is treated at all possible mathematical and physical levels, from Newton's equations and underlying topology, all the way to second quantization and quantum field theories, while antimatter is solely treated at the level of *second quantization.* However, astrophysical evidence suggests quite strongly the existence of macroscopic amounts of antimatter in the universe, to the point that even entire galaxies and quasars could eventually result to be made up entirely of antimatter.

The only possible resolution of this historical unbalance is that via the construction of a *yet new mathematics*, specifically conceived for a consistent *classical* representation of antimatter whose operator counterpart yields indeed the correct charge conjugate states.

Recall that charge conjugation is anti-homomorphic, although solely applies at the operator level. It then follows that the new mathematics for antimatter should be, more generally, anti-isomorphic and applicable at all levels of study.

After a laborious research, I proposed back in 1985 [3g] the *isodual mathematics*, namely, mathematics constructed via the *isodual map* of numbers, fields, spaces, algebras, geometries, etc..

The *isodual conventional mathematics* is characterized by the simplest conceivable anti-isomorphic map of the unit into its *negative-definite form*,

$$I > 0 \rightarrow -I = I^d < 0, \tag{7.1}$$

under which we have the transformation law of a generic, scalar, real-valued quantity

$$A(w) \to A^d(w^d) = -A(-w), \tag{7.2}$$

with reconstruction of numbers, fields, spaces, algebras, geometries, quantization, etc. in such a way to admit I^d, rather than I, as the correct left and right unit.

The isodual map characterizing the broader *isodual iso-, geno- and hyper-mathematics* is instead given by

$$\hat{I}(\hat{t}, \hat{r}, \hat{p}, \hat{\psi}, ...) \to -\hat{I}^{\dagger}(-\hat{t}^{\dagger}, -\hat{r}^{\dagger}, -\hat{p}^{\dagger}, -\hat{\psi}^{\dagger}, ...) = \hat{I}^d(\hat{t}^d, \hat{r}^d, \hat{p}^d, \hat{\psi}^d, ...), \tag{7.3}$$

and consequential reconstruction of the entire formalism to admit $\hat{I}^d$ as the correct left and right new unit.

The above map is not trivial, e.g., because it implies the reversal of the sign of *all* physical characteristics of matter (and not only of the charge). As such, isodual theories provide a novel intriguing representation of antimatter which *begins* at the primitive classical Newtonian level, as desired, and then persists at all subsequent levels, including that of second quantization, in which case isoduality becomes equivalent to charge conjugation [3m].

The most general mathematics presented in this paper is the *isoselfdual hypermathematics* [3i], namely, a hypermathematics that coincides with its isodual, and is evidently given by hypermathematics multiplied by its isodual. The latter mathematics has been used for one of the most general known cosmologies [3p] inclusive of antimatter as well as of biological structures (as any cosmology should be), in which the universe: has a multivalued structure perceived by our Eustachian tubes as a single-valued three-dimensional structure; admits equal amounts of matter and antimatter (in its limit formulation verifying Lies conjugation (1.1c)); removes any need for the "missing mass"; reduces considerably the currently believed dimension of the measured universe; possesses all *identically null* total characteristics of time, energy, linear and angular momentum, etc.; eliminates any singularity at the time of creation.

8. Simple Construction of Generalized Theories.

Unpredictably, the need for new mathematics has been a major obstacle for the propagation of the generalized Lie theories outlined in this paper in both mathematical and physical circles.

I would like to indicate here that *all* generalized Lie theories, *all* their underlying new mathematics and *all* their applications can be uniquely and unambiguously constructed via the following elementary means accessible to undergraduate students.

First, isotheories can be constructed via the systematic application of the following nonunitary transform

$$U \times U^{\dagger} = \hat{I}, (U \times U^{\dagger})^{-1} = \hat{T}, \tag{8.1}$$

to the *totality* of the original formalism with no exceptions.

In fact, transform (8.1) yields the isonumbers $U \times n \times U^{\dagger} = n \times \hat{I}$, the isoproduct, $U \times (A \times B) \times U^{\dagger} = (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) = A' \times \hat{T} \times B' = A' \hat{\times} B'$; the correct isofunctions, such as $U \times e^{X} \times U^{\dagger} = \hat{e}^{\hat{X}}$; and the correct expression of all other aspects, including the Lie-Santilli isotheory and its underlying basic theorems.

Once the isotopic structure has been achieved in this way, its invariance is proved via the reformulation of nonunitary transforms in the isounitary form (4.5), with consequential invariance of the isotheory as in Eqs. (4.6).

The construction of the Lie-Santilli genotheory is equally elementary, and requires the use, this time, of *two* nonunitary transforms

$$U \times U^{\dagger} \neq I, W \times W^{\dagger} \neq I, U \times W^{\dagger} = \hat{I}^{>}, W \times U^{\dagger} =^{<} \hat{I}, \tag{8.2}$$

to the *totality* of the original formalism, again, without any exceptions.

In fact, transforms (8.2) yields the correct form of forward and backward genonumbers, e.g., $U \times n \times W^{\dagger} = n \times \hat{I}^{>}$, the correct form of the forward and backward genoproduct, genofunctions and genotransforms, including the correct structure and representation of the Lie-Santilli genotheory. Once reached in this way, the invariance is proved by rewriting the nonunitary transforms in their genounitary version (5.4). Genoinvariant laws (5.5) then follow.

The Lie-Santilli hypertheory can be constructed and proved to be invariant via the mere relaxation of the single-valued character of the genounits. The explicit construction is is here omitted for brevity [3t]).

Finally, the isodual Lie theory can be easily constructed via the systematic application of the anti-isomorphic transform

$$U \times U^{\dagger} = -I = I^{d}, \tag{8.3}$$

to the totality of the original formalism with no exceptions.

This yields isodual numbers, fields, products, functions, etc. The isodualities of isotopic, genotopic and hyperstructural theories can be similarly constructed via the anti-isomorphic images of the preceding transforms.

Note that the above methods is useful on both mathematical and physical grounds. On mathematical grounds one can start from one given structure, e.g., the representation of the conventional Poincaré symmetry and construct explicitly all infinitely possible irreps of the Poincaré-Santilli iso-, geno- and hyper-symmetries as well as their isoduals [3,4].

The methods is also useful for the ongoing efforts to unify all simple Lie groups of the same dimension in Cartan's classification (over a field of characteristic zero) into one single isogroup, whose study has been initiated by Gr. Tsagas and his group [4].

On physical grounds, the method presented in this section is also particularly valuable to generalize existing applications of Lie's theory via the appropriate selection of the nonunitary transform representing the missing characteristics or properties, e.g., the representation of a locally varying speed of light.

9. Ultimate Significance of Lie's Axioms.

A unitary Lie group has the structure of a *bi-module* in both its finite and infinitesimal forms with an action from the left $U^{>} = e^{iX \times w}$ and an action from the the right ${}^{<}U = e^{-iw \times X}$ interconnected by Hermitean conjugation (1.1c) [3e]. Eqs. (1.1) can then be written

$$A(w) = U^{>} > A(0) <^{<} U = e^{iX \times w} > A(0) < e^{-iw \times W}, \tag{9.1a}$$

$$idA/dt = A < X - X > A, \tag{9.1b}$$

$${}^{<}U = (U^{>})^{\dagger}, X = X^{\dagger}. \tag{9.1c}$$

In the Lie case both products $A < B$ and $A > B$ are evidently conventional associative products, $A < B = A > B = A \times B$, resulting in Lie's bimodule. However, axiomatic structure (9.1) *does not* require that such products have necessarily to be conventionally associative, because they can also be isoassociative, thus yielding the Lie-Santilli isotheory. Moreover, axioms (9.1) *do not* require that the forward and backward isoassociative products have to be necessarily the same, because they can also be different, provided that conjugation (9.1c) is met. In the latter case the axioms yield the Lie-Santilli genotheory with an easy extension to the hypertheory via multi-valued realizations. Isodual theories emerge along similar lines because axioms (9.1) do not necessarily demand that the underlying unit be positive-definite.

It then follows that the axiomatic consistency and invariance of the generalized theories studied in this paper can be inferred from the original invariance of Lie's theory itself, of course, when treated with the the mathematics leaving invariant the basic units. The only applicable mathematics are then the iso-, geno-, and hyper-mathematics and their isoduals.

In conclusion, by looking in retrospect some three decades of studies on the topics outlined in this paper, the emerging most important message is that *the sole invariant classical and operator theories are those preserving the abstract Lie axioms*, Eq.s (1.1) and (1.2), *and merely providing their broder realizations treated with the appropriate mathematics.*

References

[1] M. S. Lie, Doctoral Thesis, English translation by E. Trell, *Algebras, Groups and Geometries* 15, 395 (1998).

[2] A.A. Albert, Trans.Amer. Math.Soc. 64, 552 (1948).

[3] R. M. Santilli, Nuovo Cimento 51 570 (1967) [3a]; Supl.Nuovo Cimento 6, 1225 (1968) [3b]; Meccanica 1, 3 (196 [3c]; Hadronic J. 1,224, 574 and 1267 (1978) [3d]; Hadronic J. 3, 440 (1979) [3e]; Lett. Nuovo Cimento 37 509, (1983) [3f]; Hadronic J. 8, 25 and 36 (1985) [3g]; Algebras, Groups and Geometries 10, 273 (1993) [3h]; Rendiconti Circolo Matematico Palermo, Suppl. 42, 7 (1996) [3i]; Found. Phys. 27, 625 (1997) [3j]; Found. Phys. Letters 4, 307 (1997) [3k]; Found. Phys. 27,

1159 (1997) [3l]; Hyperfine Interactions, **109**, 63 (1997) [3m]; Intern. J. Modern Phys. D **7**, 351 (1998) [3n]; Acta Appl. Math. **50**, 177 (1998) [3o]; in *Modern Modified Theories of Gravitation and Cosmology*, E. I. Guendelman, editor, Hadronic J. **21**, 113 (1998) [3p]; in *Proceedings of the VIII M. Grossmann Meeting on General Relativity*, Jerusalem, June 1997, World Scientific, in press (1999) [3q] *Foundations of Theoretical Mechanics*, Vol. I (1978) and II (1983), Springer–Verlag, Heidelberg–New York [3r] *Elements of Hadronic Mechanics*, Vol. I (1995) and II (1995) and III (in preparation), Ukraine Academy of Sciences, Kiev (1995) [3s]; *Isotopic, Genotopic and Hyperstructural Method in Theoretical Biology*, Ukraine Academy of Sciences, Kiev (1996) [3t].

[4] Gr. T. Tsagas and D. S. Sourlas, Algebras, Groups and Geometries **12**, 1 and 67 (1995) [4a] and Algebras, Groups and Geometries 13, 129 (1996) [4b]. D. Rapoport-Campodonico, in *New Frontiers in Physics*, T. L. Gill, editor, Hadronic Press (1996). [4c]. R. Aslaner and S. Keles, Algebras, Groups and Geometries bf 14, 211 (1997) [4d]. S. Vacaru, Algebras, Groups and Geometries **14**, 225 (1997) [4e]. E. B. Lin, Hadronic J. **11**, 81 (1988) [4f]. C. X. Jiang, Algebras, Groups and Geometries **15**, 351 (1998) [4g]. E. Trell, Algebras, Groups and Geometries **15**, 299 (1998) [4h]. N. Kamiya et al, Hadronic J. **16**. 168 (1993) [4i]. D.S.Sourlas and G.T.Tsagas, J. V. Kadeisvili, Algebras, Groups and Geometries **9**, 283 and 319 (1992) [4j]; in *Symmetry Methods in Physics* (Dedicated to Smorodinsky), G. Pogosyan et al., editors, JINR, Dubna (1994) [4k]; Indian J. Appl. of Math. **25**, 7 (1995) [4l]; Math. Methods in Applied Sciences, **19**, 362 (1996) [4m]; Algebras, Groups and Geometries **13**, 171 (1996) [4n]; Acta Appl. Math. **50**, 131 (1998) [4o]. IBR Web Site http://home1.gte.net/ibr, page 18 [4p].

[5] S.L. Adler, Phys.Rev. **17,** 3212 (1978). Cl.George, F. Henin, F.Mayene and I.Prigogine, Hadronic J. **1,** 520 (1978). C.N.Ktorides, H.C.Myung and R.M.Santilli, Phys.Rev. **D 22,** 892 (1980). G.Eder, Hadronic J. **4,** (1981) and **5,** 750 (1982). C.Myung and R.M.Santilli, Hadronic J. **5,** 1277 (1982). Mignani, Hadronic J. **5,** 1120 (1982). R.Mignani, Nuovo Cimento Lett. **39,** 413 (1984). M.Nishioka, Nuovo Cimento **82A,** 351 (1984). M.Gasperini, Hadronic J. **7,** 971 (1984). A.K.Aringazin, A.

Jannussis, D.F.Lopez, M.Nishioka and B.Veljanosky, *Santilli's Lie-Isotopic Generalization of Galilei's and Einstein's Relativities* (1990), Kostarakis Publisher, Athens, Greece. A.Jannussis, R.Mignani and D.Skaltsas Physics A **187,** 575 (1992). D.Rapoport–Campodonico, Algebras, Groups and Geometries 8, 1 (1991). T.Gill, J.Lindesay and W.W.Zachary, Hadronic J. 17, 449 (1994). J.Ellis, N.E. Mavromatos and D.V.Napoulos in *Proceedings of the Erice Summer School, 31st Course: From Superstrings to the Origin of Space–Time*, World Scientific (1996).

[6] A.K.Aringazin,A.Jannussis, D.F.Lopez, M.Nishioka and B.Veljanosky, *Santilli's Lie-Isotopic Generalization of Galilei's and Einstein's Relativities*, Kostarakis Publisher, Athens, Greece (1990). D.S.Sourlas and G.T.Tsagas, *Mathematical Foundations of the Lie-Santilli Theory*, Ukraine Academy of Sciences, Kiev (1993). J.Lôhmus, E.Paal and L.Sorgsepp, *Nonassociative Algebras in Physics*, Hadronic Press, Palm Harbor, USA (1994). J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, Second Edition, Ukraine Academy of Sciences, Kiev, (1996). J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, Second Edition, Ukraine Academy of Sciences, Kiev, (1996). S. Vacaru, *Interactions, Strings and Isotopies in Higher Order Anisotropic Superspaces* , Hadronic Press (1999). C. X. Jiang, *An Introduction to Santilli's Isonumber Theory*, Hadronic Press, to appear in 1999.

[7] A. O. E. Animalu, Hadronic J. 17 , 379 91994). A.O.E. Animalu and R.M.Santilli, Int. J.Quantum Chemistry 29, 175 (1995).

[8] R. M. Santilli and D. D. Shillady, Intern. J. Hydr. Energy, in press (1999); and in *Fundamental Open Problems in Science at the End of the Millennium"*, T. L. Gill et al., editors, to appear (1999).

[9] Yu. Arestov et al. Found. Phys. Letters, in press [9a]. R. M. Santilli, in *Collection of Papers on the Lorentz Symmetry*, V. V. Dvoeglazov,, Editor, Nova Publisher, in press [9b]. F.Cardone et al., Phys. **C18,** L61 and L141 (1992) [9c]. R. M. Santilli, Hadronic J. 15, 1 (1992) [9d]. R. Mignani, Lettere Nuovo Cimento **39**, 413 (1982) [9e]. A.J.Kalnay,

Hadronic J. 6, 1 (1983) [9f]. R. M. Santilli, Comm. Theor. Phys. 4, 123 (1995) [9g]. R. M. Santilli, Comm. Theor. Math. Phys. , in press [9h]. R. M. Santilli,. Hadronic J. 17, 311 (1994) [9i]. R.Mignani, Physics Essays 5, 531 (1992) [9j]. R. M. Santilli, in *Frontiers of Fundamental Physics*, M. Barone and F. Selleri, editors, Plenum (1994) [9k]. A. O. E. Animalu and R. M. Santilli, Intern. J. Quantum Chem. 29, 175 (1995) [9l]. E. Trell et al., in press (1999) [9m]. Toups Technologies Web Site http://www.toupstech.com/aquafuel/aquaII-pIII.html [9n].

[10] C. Illert and R. M. Santilli, *Foundations of Conchology*, Hadronic Press (1996).

[11] R. M. Santilli and T. Vougiouklis, *New Frontiers in Hyperstructures*, T. Vougiouklis, editor, Hadronic Press (1996).

[12] S.Okubo, *Hadronic J.* 5, 1667 (1982) [12a]. D.F.Lopez, in *Symmetry Methods in Physics* (Smorodinski Memorial), A.N.Sissakian, G.S.Pogosyan and X.I.Vinitsky, Editors (JINR, Dubna, Russia, 1994), and *Hadronic J.* 16, 429 (1993) [12b]. A.Jannussis and D.Skaltsas,*Ann. Fond. L.de Broglie* 18, 137 (1993) [12c]. A Jannussis, R.Mignani and R.M.Santilli, *Ann.Fond. L.de Broglie* 18, 371 (1993) [12d]. D.Schuch, *Phys. Rev. A* 55, 955 (1997) [12e]. R. M. Santilli, *Modern Phys. Letters* 13, 327 (1998) [12f]. T. M. Santilli, Problematic aspects and invariant formulation of classical and quantum deformations, Intern. J. Modern Phys. A (1999), in press [12g].

ALGEBRAS GROUPS AND GEOMETRIES 15, 497–508 (1998)

COMMENTS ON A RECENT PAPER BY MOROSI AND PIZZOCCHERO ON THE LIE–SANTILLI ISOTHEORY

J. V. Kadeisvili
Institute for Basic Research
P. O. Box. 1577, Palm Harbor, FL 34682, U. S. A.
ibr@gte.net; http://home1.gte.net/ibr/

Received March 31, 1997

Abstract.

In a recent paper [28] published in *Letters in Mathematical Physics,* C. Morosi and L. Pizzocchero study certain implications of Lie algebras equipped with the product they call "twisted Lie product" [L, N] = LAN - NAL. In these comments we point out that the above product was first introduced by R. M. Santilli in 1978 [30,31,50] as the foundation of his systematic, axiom-preserving isotopies of Lie's theory, today known as Lie-Santilli isotopic theory [3,20,23,63]. We then point out that, as formulated in [28] via conventional spaces and fields, the emerging theory is generally nonlinear, nonlocal and noninvariant, thus possessing a number of physical drawbacks. We finally point out that, when formulated on certain generalized spaces over generalized fields, the results of paper [28] can be identically written in a form verifying the conditions of linearity, locality and invariance, thus resolving said problematic aspects by therefore permitting rather intriguing mathematical and physical advances. These Comments were submitted to *Letters in Mathematical Physics* to establish a record of the correct paternity of the algebra studied in the paper by Morosi and Pizzocchero [28] in the journal of its publication, but they were rejected, by the editors even though admitted in as being "certainly correct".

In the recent paper [28] published in *Letters in Mathematical Physics*, C. Morosi and L. Pizzocchero study a Lie algebra of skew-symmetric n×n-dimensional matrices L, N, . . ., equipped with the ' twisted ' Lie product (Eq. (2.1), p. 119)

$$[L, N]_A = L A N - N A L. \qquad (1)$$

where A is a symmetric n×n matrix which remains fixed for all commutators. By preserving the assumptions and symbols of ref. [28] for brevity, we present hereon the following comments:

1) Product (1) was first introduced by R. M. Santilli in 1978 [30,31] as part of his systematic studies (see the representative papers [33–48] and monographs [49–56]) of certain axiom-preserving liftings of the various branches of Lie's theory (enveloping algebras, Lie algebras, Lie groups, transformation and representation theories, etc.) today known as *Lie-Santilli isotopic theory* or *isotheory* for short (see ref.s [1–29] and [57–63], particularly independent monographs [3,20,23,63] and large bodies of references quoted therein).

2) Even though *mathematically* impeccable, when formulated via conventional spaces and fields, the studies of ref. [28] are generally *nonlinear, nonlocal and noninvariant* (as specified below), thus having a number of *physical* drawbacks studied by Santilli [31,48,54], Lopez [24], Jannussis *et al.* [12,13], Schuch [57,58], Kadeisvili [14,15], Tsagas and Sourlas [59,60], and others.

The loss of linearity originates from the fact that, besides the conditions of being nowhere degenerate and symmetric with well behaved elements, the lifting of the product $[L, N] \rightarrow [L, N]_A$ implies no restrictions on the functional dependence of the fixed quantity A which can therefore depend on the local coordinates r, wavefunctions, their derivatives and other local variables, $A = A(r, \dot{r}, \psi, \partial\psi, ...)$. Brackets $[L, N]_A$ therefore characterize a theory which is nonlinearity non only on the coordinates, but also in the wavefunctions and their derivatives [54]. In turn, such nonlinearity implies the loss of the traditional *linearity* of Lie's theory with physical implications, such as the lack of consistent applications to composite systems in view of the loss of the superposition principle studied by Santilli [48,54], and other problematic aspects studied by Schuch [57,58].

The loss of locality originates from the fact that the quantity A is generally outside the original algebra and, as such, its element can also admit nonlocal realizations, e.g., those of *integral* type expressing in classical mechanics the dependence of resistive forces from the surface σ of the body via a kernel $\mathfrak{F}$ representing the local medium in which motion occurs,, $A = A_0 \exp(n\int_\sigma d\sigma \mathfrak{F}(\sigma, ...))$, or operator realizations characterized by volume integrals on the region of wave-overlappings, e.g., $A = A_0 \exp(n\int d^3r \psi^\dagger(r)\psi(r))$. In turn, the latter occurrence implies the loss of the traditional local-differential *topology* of Lie's theory as studied by Tsagas and Sourlas [59,60], with physical implications such as problematic aspects such as general loss of causality as studied by Santilli [48,54].

The loss of invariance was studied in details by Santilli since the original proposal [31] (see [46] for a recent mathematical study). It can be first studied for operator realizations of the theory, the classical counterpart via vector-fields on manifolds being an evident consequence. Let $\mathcal{H}$ be a conventional Hilbert space with states $|\psi>, |\phi>, \ldots$, and inner product $<\phi|\psi>$ over a conventional field $C(c,+,\times)$ of complex numbers c with the usual sum + and product ×. It is then easy to see that the group structure admitting of product (1) in the neighborhood of the unit is *nonunitary* because of the general lack of commutativity of A with the generator of the algebra i.e. [31,54]

$$U = e^{iLAw}, \qquad UU^{\dagger} \neq I. \tag{2}$$

and, as such, it implies the following problematic aspects:

2.A) The unit of ref. [28] (here referred to that of the enveloping associative algebra) is not invariant under the group action,

$$I \rightarrow I' = UIU^{\dagger} \neq I, \tag{3}$$

with consequential lack of unambiguous applications to measurements evidently due to the possible variation in time of stationary units of measures during the measurements process itself;

2.B) The 'twisted' associative product of ref. [28] is also not invariant,

$$LAN \rightarrow ULANU^{\dagger} = l'\tilde{A}n', \qquad A \rightarrow \tilde{A} = U^{\dagger -1}AU^{-1}, \tag{4}$$

with consequential lack of invariance of basic mathematical operations defined on it, such as exponentiation, and loss of acceptable physical laws derived via the exponentiation, e.g., uncertainties;

2.C) The 'twisted' Lie product of ref. [28] is also not invariant,

$$[L,N]_A = LAN - NAL \rightarrow U[L,N]_A U^{\dagger} = L'\tilde{A}N' - N'\tilde{A}L', \tag{5}$$

with consequential lack of invariant dynamical equations;

2D) When formulated on $\mathcal{H}$ over C, structure (1) does not admit an invariant Hermiticity, i.e., an operator Q which is Hermitean at w = 0, $Q = Q^{\dagger}$, is no longer Hermitean under g,

$$(<\phi|AQ^{\hat{\dagger}})|\psi> = <\phi|(QA|\psi>), \qquad Q^{\dagger} \rightarrow Q^{\hat{\dagger}} = A^{-1}Q^{\dagger}A \neq Q^{\dagger}, \tag{6}$$

thus lacking physically acceptable observables (a property also known as *Lopez's lemma* [24]);

2.E) The numerical predictions of structure (1) on conventional Hilbert spaces over conventional fields are not unique and not invariant. For additional problematic aspects one may consult ref.s [12,13,24,48,54,57,58].

3) Santilli [30–56] constructed the isotopies of Lie's theory for the primary purpose of bypassing the above problematic aspects. His fundamental assumption is the *isotopy of the basic, n-dimensional unit* I = Diag. (1, 1, ..., 1) of the enveloping algebra $\xi(L)$ of a Lie algebra L into an n×n-dimensional, nowhere singular and Hermitean matrix with an arbitrary, generally nonlinear and nonlocal functional dependence of its elements, $\hat{1} = \hat{1}(r, t, \psi, \partial\psi, \ldots) = \hat{1}^\dagger$. The lifting $I \rightarrow \hat{1}$ is constructed in such a way to be the the *inverse* of the lifting of the associative product [30,31,50], i.e.,

$$I \quad \rightarrow \quad \hat{1}(r, t, \psi, \partial\psi, \ldots) = \hat{1}^\dagger, \quad L\,N \;\rightarrow\; L \hat{\times} N = L\,A\,N, \quad \hat{1} = A^{-1}, \qquad (7)$$

under which conditions $\hat{1}$ is indeed the correct left and writ unit of the new envelope $\hat{\xi}$, called *isounit*, while A is called the *isotopic element*.The above fundamental assumption assure *ab initio* the local isomorphism $\xi \approx \hat{\xi}$ for all positive-definite A (although not so for realization of A with different signature [53]). The above assumption also ensure that the new unit is in the center of the new enveloping algebra, thus being invariant as we shall see. By comparison, in ref. [28] only the associative product is lifted, LN → LAN, while the original unit I is preserved, thus being no longer in the center of the new envelope.

The joint liftings of the unit and of the associative product then require, for consistency, the lifting of the totality of mathematical formulation of Lie's theory, with no exception known to this author. As an illustration we here indicate:

3a) The lifting of the base fields of (real, complex of quaternionic) numbers $F(n,+,\times)$ into *Santilli's isofields* $\hat{F} = \hat{F}(\hat{n},\hat{+},\hat{\times})$ [39] with *isonumbers*, i.e. *numbers with an arbitrary unit*, $\hat{n} = n\hat{1}$ equipped with the *isosum* and *isoproduct*

$$\hat{n} \hat{+} \hat{m} = (n + m)\hat{1}, \qquad \hat{n} \hat{\times} \hat{m} = (n \times m)\hat{1}, \qquad (8)$$

and related operations, additive unit $\hat{0} \equiv 0$ and multiplicative unit $\hat{1} \neq I$. Despite its simplicity, the lifting $F \rightarrow \hat{F}$ is mathematically nontrivial, as illustrated by the fact that, e.g., the number 4 becomes prime under the isounit $\hat{1} = 3$ (see ref.s [39,53] for some of the revisions of number theory required by arbitrary units).

3b) The lifting of metric spaces, e.g., the Euclidean space $E(r,\delta,R)$, $r = \{r^k\}$, k = 1, 2, 3, δ = Diag. (1, 1, 1,) over the reals $R(n,+,\times)$ into *Santilli's isoeuclidean spaces* [34] $\hat{E} = \hat{E}(\hat{r},\hat{\Delta},\hat{R})$, $\hat{r} = r\hat{1}$, $\hat{\Delta} = A\delta\hat{1} = \hat{\delta}\hat{1}$, over the isoreals $\hat{R} = \hat{R}(\hat{n},\hat{+},\hat{\times})$ with *isotopic interval*

$$(\hat{r}_1 - \hat{r}_2)^{\hat{2}} = (\hat{r}_1 - \hat{r}_2)^i \hat{\times} \Delta_{ij} \hat{\times} (\hat{r}_1 - \hat{r}_2)^j = [\,(r_1 - r_2)^i\, \hat{\delta}_{ij}\, (r_1 - r_2)^j\,]\,\hat{1} \in \hat{R}\,. \qquad (9)$$

The nontriviality of the isotopy is illustrated by the arbitrary functional dependence of the isometric $\hat{\delta}(r, \dot{r}, ...)$ admitting as a particular case all possible 3-dimensional nowhere singular metrics, including evidently the Riemannian metric, yet the space is *isoflat*, that is, it verifies the condition of flatness on the isospace $\hat{E}$ over the isofield $\hat{R}$ (because of the evident local isomorphism $E \approx \hat{E}$) [53]. In fact, the component of a Riemannian metric $g(r)$ truly representing curvature, the term $A(r)$ in the factorization $g(r) = A(r)\delta = \hat{\delta}$, is now referred to a unit which is its *inverse*, $\hat{1} = [A(r)]^{-1}$, thus eliminating curvature itself while preserving the Riemannian metric unchanged, an occurrence with rather intriguing novel possibilities in classical and quantum indicated later on.

3c) The lifting of continuity (studied by Kadeisvili [14,15]), topology (studied by Tsagas and Sourlas [59,60]), differential calculus (studied by Santilli [46]), and of functional analysis at large [53], here referred to for brevity to the quoted literature.

3d) The lifting of the Hilbert space $\mathcal{H}$ over C into the *Myung-Santilli isohilbert space* $\hat{\mathcal{H}}$ [29] with isostates $|\hat{\psi}>, |\hat{\phi}>, \ldots,$ *isoinner product* and *normalization* [53]

$$<\hat{\phi}| A |\hat{\psi}> \hat{1} \in \hat{C}, \quad <\hat{\psi}| A |\hat{\psi}> = 1, \qquad (10)$$

and related *isounitarity conditions*

$$\hat{U} \hat{\times} \hat{U}^{\dagger} = \hat{U}^{\dagger} \hat{\times} \hat{U} = \hat{1}; \qquad (11)$$

3e) The lifting of conventional transforms on E over R into the *isotransforms* on $\hat{E}$ over $\hat{R}$ [30,53]

$$\hat{r} \rightarrow \hat{r}' = \hat{U} \hat{\times} \hat{r}; \qquad (12)$$

and the lifting of all remaining aspects, e.g., conventional and special functions and transforms (see [53] for brevity).

The *Lie-Santilli isotheory* can now be outlined as being characterized by:

3-I) The *universal enveloping isoassociative algebra* $\hat{\xi}(L)$ with infinite-dimensional basis expressed in terms of the finite-dimensional ordered basis $X = (X_k)$ of Hermitean generators X_k of L (*isotopic Poincare'-Birkhoff-Witt theorem* [30,50]

$$\hat{\xi}: \ \hat{1}, \quad X_i, \quad X_i \hat{\times} \hat{X}_j, \ i \leq j, \ X_i \hat{\times} X_j \hat{\times} X_k, \ i \leq j \leq k, \ldots, \qquad (13)$$

with unique and unambiguous *isoexponentiation*

$$\hat{e}^{iXw} := \hat{1} + X/1! + (Xw)\hat{\times}(Xw)/2! + \ldots = (e^{iXAw})\hat{1}. \qquad (14)$$

The nontriviality of the lifting is illustrated by the *appearance of a matrix A with nonlinear-integral terms in the exponent.*

3-II) The *isoalgebra* $\hat{L} \approx [\hat{\xi}(L)]^{-}$ characterized by the same basis of L but now re-interpreted as an isovector space over isofields with *Lie-Santilli second theorem* [loc.

cit.]

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i = \hat{C}_{ij}{}^k \hat{\times} X_k, \qquad (15)$$

where the $\hat{C}$'s are now *structure functions.*

3-III) The (connected) *isogroup* $\hat{g}$ characterized by the isounitary transforms on $\hat{\mathcal{H}}$ over $\hat{C}$ and related isogroup laws [loc. cit.]

$$\hat{g}: \ \hat{U} = \hat{e}^{iXw} = \hat{e}^{\hat{i} \hat{\times} X \hat{\times} \hat{w}} = (e^{iXAw})\hat{1}, \qquad (16a)$$

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \ \ \hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{-}\hat{w}) = \hat{U}(\hat{0}) = \hat{1}. \qquad (16b)$$

3-IV) The *isotransformation theory*

$$\hat{r} \rightarrow \hat{r}' = \hat{U} \hat{\times} \hat{r} = (e^{iXAw})\hat{r}, \qquad (17)$$

3-V) The *isorepresentation theory*; etc.

It is easy to see that the Lie-Santilli isotheory is *isolinear*, i.e., it verifies the conditions of linearity in isospace over isofields, while its projection on conventional spaces over conventional fields is generally nonlinear as in ref. [28]. In particular, all nonlinear terms are embedded in the *unit* of the theory, thus admitting a fully valid superposition principle, with consequential axiomatically correct treatment of composite systems (see [54] for operator details).

Similarly, the isotheory is *isolocal* in the sense of verification the condition of locality in isospace over isofields, while again its projection on conventional spaces is generally nonlocal-integral. Again, the result is reached by embedding all nonlocal terms in the *unit*. The theory therefore admit a fully causal description of nonlocal interactions, as one can see from the isomorphism of the isotopic and conventional one-dimensional groups of time evolutions (see also [54] for brevity).

Finally, the Lie-Santilli isotheory can be readily shown to be invariant under the action of its own isogroups. In fact, any *nonunitary* operator W admits the *identical* isounitary realization

$$W = \hat{W} \hat{A}^{1/2}, \ \ W W^\dagger = \hat{W} \hat{\times} \hat{W}^\dagger = \hat{W}^\dagger \hat{\times} \hat{W} = \hat{1}, \qquad (18)$$

under which: the fundamental isounit is invariant, $\hat{1} \rightarrow \hat{1}' = \hat{W} \times \hat{\times} \hat{1} \hat{\times} \hat{W}^\dagger \equiv \hat{1}$ (namely, its matrix elements are *numerically* unchanged in the chart considered); the isoassociative product is invariant $L \hat{\times} N \rightarrow \hat{W} \hat{\times} (L \hat{\times} N) \hat{\times} \hat{W}^\dagger = L' \hat{\times} N'$ (i.e., again, the quantity A is *numerically* unchanged); the Lie-Santilli products is consequentially invariant; the notion of *isohermiticity* on $\hat{\mathcal{H}}$ over $\hat{C}$ *coincides* with the conventional one, $X^{\hat{\dagger}} \equiv X^\dagger$, and it is indeed invariant; the numerical predictions are unique and invariant; etc. (see ref.s [54] for brevity).

The above properties imply: invariant units of measurements as necessary for physically effective measurements; uniqueness and invariant physical laws; Hermiticity-observability at all times; uniqueness and invariance of numerical predictions; etc. [54]. In other words, the Lie-Santilli theory preserves all the physical characteristics of the conventional theory, without any known exclusion (again, for positive-definite A).

We should also recall [54] that isotopic formulations can be uniquely and unambiguously constructed from the conventional ones via nonunitary transforms $WW^{\dagger} = \hat{1} \neq I$ *provided* that they are systematically applied to the *totality* of the original mathematical structure. On the contrary, any *partial* application of the nonunitary map leads to the problematic aspects identified earlier.

In fact, under a nonunitary map $WW^{\dagger} = \hat{1}$, $A = (WW^{\dagger})^{-1}$ the original unit I becomes Santilli's isounit $\hat{1}$ with the correct Hermiticity property, $I \rightarrow \hat{1} = WIW^{\dagger} = \hat{1}^{\dagger}$; the conventional numbers n become isonumbers $n \rightarrow WnW^{\dagger} = \hat{n} = n\hat{1}$ with the correct isooperations; metric spaces acquires the structure of Santilli's isospaces with the correct isocoordinates $r \rightarrow \hat{r} = WrW^{\dagger} = r\hat{1}$, and isoinvariant $r^2 \in R \rightarrow \hat{r}^2 = Wr^2W^{\dagger} = (Wr^tW^{\dagger})(WW^{\dagger})^{-1}\delta(WrW^{\dagger}) = (r^i\delta_{ij}r^j)\hat{1} \in \hat{R}$; the original associative product acquires the isoassociative form, also with the correct Hermiticity property, $LN \rightarrow WLNW^{\dagger} = L' \hat{\times} N' = L' A N'$, $A = (WW^{\dagger})^{-1} = \hat{1}^{-1} = A^{\dagger}$; the Lie product and group then acquire the Lie-Santilli form; the Hilbert space becomes the isohilbert space, $|\hat{\psi}\rangle = W|\psi\rangle$, $\langle\hat{\phi}| = \langle\phi|W^{\dagger}$, $\langle\phi|\psi\rangle \rightarrow W\langle\phi|\psi\rangle W^{\dagger} = \langle\hat{\phi}|A|\hat{\psi}\rangle\hat{1} \in \hat{C}$; etc. (see 54] for details).

The above derivation of isotopic theories also assures the *preservation of conventional physical laws under a broader nonunitary structure*, an occurrence we cannot review here for brevity [46,54], with predictably intriguing and novel physical applications indicated below.

Note that the conventional and isotopic formulations coincide at the abstract realization-free level by conception and construction for all positive-definite isotopic element A. For this reasons Santilli insists that his isotopies *do not* characterize new theories, but merely *new realizations* of conventional abstract axioms [53,54].

Intriguingly, the reader should be aware that Santilli introduced the isotopies as a particular case of yet broader maps known as *genotopies* [30] in which the generalized units are no longer Hermitean, $\hat{1} \neq \hat{1}^{\dagger}$, with the emergence of the covering Lie-admissible algebras. In turn, the genotopies have resulted to be a particular case of still broader *multivalued hyperstructures with a left and right hyperunit*, in which the generalized units are characterized by an ordered *set* of nonhermitean elements, $\hat{1} = \{\hat{1}_1, \hat{1}_2, \ldots\}$, $\hat{1}_k \neq \hat{1}_k^{\dagger}$ [55].

Mathematically the isotopies are nontrivial. e.g., because they permit the turning of nonlinear systems into identical isolinear forms, or they permit the unification of all possible, simple, n-dimensional Lie algebras (over a field of characteristic zero) in Cartan's classification into one single, simple, Lie-Santilli isoalgebra of the same dimension. Its study has been initiated by Tsagas and Sourlas [61,62] via the use of one individual basis for all compact and noncompact simple algebras of the same dimension and the use of isounits with different signature. The isotopies also permit intriguing mathematical advances in functional analysis, differential calculus, geometry, topology, etc., which are now no longer restricted to the simplest possible unit essentially dating back to biblical times, but are formulated instead for arbitrary (well behaved) units with an evident broadening of their representational capabilities.

The physical nontriviality of the isotopies is also evident from their broader structure. In addition to the representation of all Hamiltonian interactions, the isotopies can represent nonlinear, nonlocal-integral and nonpotential-nonhamiltonian-nonunitary effects. This permits *novel* applications, i.e., applications beyond any possibility of quantitative treatment via conventional formulations.

Among a considerable number of applications in various fields available in the literature, it may be significant for the interested reader to mention: the isotopies $\hat{O}(3)$ of the rotational symmetry O(3) [36,37] originally applied to the lifting of Euler's theorem on *rigid* bodies with a fixed point into a form applicable to *deformable* bodies with a fixed point; the isotopies $\hat{SU}(2)$ of the SU(2)-spin and isospin symmetry [40] with the reconstruction of the exact isospin symmetry in nuclear physics; the isotopies $\hat{P}$(3.1- of the Poincare' symmetry P(3.1) [41] with the identification of the universal symmetry of gravitation or of locally varying speeds of electromagnetic waves within physical media; the isotopies of the spinorial covering of the Poincare' symmetry [44] with a quantitative representation of the synthesis of the neutron as occurring in stars at their formation, via the use of protons and electrons *only*; the isotopies $\hat{SU}(3)$ of the SU(3) symmetry, first sturdied in ref. [25], and today used as the foundation of the *isoquark theory* [43] i.e., a theory preserving all conventional quantum numbers, (from the evident isomorphism $\hat{SU}(3) \approx SU(3)$) yet admitting an exact confinement with an identically null probability of tunnel effects (from the incoherence of $\hat{\mathcal{H}}$ and $\mathcal{H}$ when interconnected by a suitable *nonunitary* transform); the isotopies of the Cooper pair in superconductivity [1], with the first explicit form of *attractive* interaction among the two *identical* electrons in remarkable agreement with experimental evidence; the first exact-numerical representation of nuclear magnetic moments under conventional angular momentum and spin [42], as permitted by the nonunitary structure of the theory; the isotopies of gauge theories [7,8,30]; a novel operator form of gravity verifying the same axioms of *relativistic* quantum mechanics [45]; an exact representation of the available experimental data on the anomalous behavior of the meanlives of unstable hadrons with energy [4,5]; a representation of the Bose-Einstein correlation as due to the nonlocality of the p-$\bar{p}$-fireball under the exact Poincarè-Santilli isosymmetry $\hat{P}(3.1)$ [38] in remarkable agreement with experimental evidence [6]; representation of the large difference in cosmological redshift between certain quasars and their associated galaxy when physically connected on grounds of photon spectrometry [2,27]; isotopies of the unitary scattering theory for the representation of nonunitary effects predicted in deep inelastic scattering, which was initiated in ref. [26] and continued in ref. [54]; and various other applications.

Acknowledgments

These Comments were submitted to *Letters in Mathematical Physics* to establish a record of the correct paternity of the algebra studied in the paper by Morosi and Pizzocchero [28], specifically, in the journal of its publication, as scientifically appropriate and customary, but they were rejected in the present or in any revised form, even though admitted in the rejection as being "certainly correct". The author would like to express his gratitude to the Editors and referees of *Algebras, Groups and Geometries* for correcting the misleading lack of paternity of the algebra studied in Ref. [28] apparently done in good faith by the authors, as well as for invaluable comments.

References

1. A. O. E. Animalu and R. M. Santilli, Nonlocal isotopic representation of the Cooper pair in superconductivity, Intern. J. Quantum Chem. 26, 175–187 (1995)
2. A. K. Aringazin, Lie-isotopic Finslerian lifting of the Lorentz group and Blochintsev/Redei-like behavior of the meson lifetime, *Hadronic J.* **12** (1989), 71–74.
3. A. K. Aringazin, A. Jannussis, D. F. Lopez, M. Nishioka and B. Veljanosky, *Santilli's Lie-isotopic Generalization of Galilei's and Einstein's Relativities*, Kostarakis Publisher, Athens, Greece (1991).
4. F. Cardone, R. Mignani and R. M. Santilli, On a possible non-Loretzian energy dependence of the K°_{S} lifetime, *J. Phys. G: Nucl. Part. Phys.* **18** (1992), L61–L65.
5. F. Cardone, R. Mignani and R. M. Santilli Lie-isotopic energy dependence of the K°_{S} lifetime, *J. Phys. G: Nucl. Part. Phys.* **18** (1992), L141–L152.
6. F. Cardone and R. Mignani, Nonlocal approach to the Bose-Einstein correlation, Univ. of Rome P/N 894, July 1992.
7. M. Gasperini, Elements for a Lie-isotopic gauge theory, *Hadronic J.* **6** (1983), 935–945.
8. M. Gasperini, Lie-isotopic lifting of gauge theories, *Hadronic J.* **6** (1983), 1462–1479.
9. A. Jannussis, Noncanonical quantum statistical mechanics, *Hadronic J. Suppl.* **1** (1985), 576–609.
10. A. Jannussis and R. Mignani, Algebraic structure of linear and nonlinear models of open quantum systems, *Physica A* **152** (1988), 469–476.
11. A. Jannussis, M. Miatovic and B. Veljanosky, Classical examples of Santilli's Lie-isotopic generalization of Galilei's relativity for closed systems with nonselfadjoint internal forces, *Physics Essays* **4** (1991), 202–211.
12. A. Jannussis and D. Skaltsas, Algebraic inconsistencies of a class of equations for the description of open systems and their resolutions via Lie-admissible formulations, *Ann. Fond.L. de Broglie* 18 (1993), 137–154
13. A. Jannussis, R. Mignani and R. M. Santilli, Problematic aspects of Weinberg nonlinear theory, *Ann. Fond. L. de Broglie* **18** (1993), 371–389

14. J. V. Kadeisvili, Elements of functional isoanalysis, *Algebras, Groups and Geometries* **9** (1992), 283–318.

15. J. V. Kadeisvili, Elements of the Fourier–Santilli isotransforms, *Algebras, Groups and Geometries* **9** (1992), 319–342.

16. J. V. Kadeisvili, Comments on the Lie–Santilli isotheory, *(Indian Math. J.) Jnanabha* **25** (1995), 7–55

17. J. V. Kadeisvili, Foundations of the Lie–Santilli isotheory, *Rendiconti Circolo Matematico Palermo, Suppl.* **42** (1996), 83–136

18. J. V. Kadeisvili, An introduction to the Lie–Santilli isotopic theory, *Math. Methods in Applied Sciences* **19** (1996), 1349–1395

19. J. V. Kadeisvili, Studies the Lie–Santilli isotopic theory, *Algebras, Groups and Geometries* **13** (1996), 171–281

20. J. V. Kadeisvili, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities*, second edition, Ukraine Academy of Sciences, Kiev, in press (first edition 1992)

21. J. V. Kadeisvili, N. Kamiya, and R. M. Santilli, A characterization of isofields and their isoduals, *Hadronic J.* **16** (1993), 168–185.

22. A. U. Klimyk and R. M. Santilli, Standard isorepresentations of isotopic/Q–operator deformations of Lie algebras, *Algebras, Groups and Geometries* **10** (1993), 323–333.

23. J. Lôhmus, E. Paal and L. Sorgsepp, *Nonassociative Algebras in Physics*, Hadronic Press, Palm Harbor, Florida, 1994.

24. D. F. Lopez, Problematic aspects of q–deformations and their isotopic resolutions, *Hadronic J.* **16** (1993), 429–457.

25. R. Mignani, Lie–isotopic lifting of SU_N symmetries, *Lett. Nuovo Cimento* **39** (1984), 413–416.

26. R. Mignani, Nonpotential scattering theory, Lett. Nuovo Cimento **39** (1984), 406–416

27. R. Mignani, Quasars' redshift in isominkowski space, *Physics Essays* **5** (1992), 531–539.

28. .C. Morosi and L. Pizzocchero, On the Euler equation: Bi–Hamiltonian structure and integrals in involution, *Lett. Math. Phys.* **37**, 117 (1996)

29. H. C. Myung and R. M. Santilli, Foundations of the hadronic generalization of atomic mechanics, II: Modular–isotopic Hilbert space formulation of the exterior strong problem, Hadronic J. **5**, 1277–1366 (1982)

30. M. Nishioka, An introduction to gauge fields by the Lie–isotopic lifting of the Hilbert space, *Lett. Nuovo Cimento* **40** (1984), 309–312.

31. R. M. Santilli, On a possible Lie–admissible covering of the Galilei relativity in Newtonian mechanics for nonconservative and Galilei noninvariant systems, *Hadronic J.* **1** (1978), 223–423; Addendum, *ibid.* **1** (1978), 1279–1342.

32. R. M. Santilli, Need for subjecting to an experimental verification the validity within a hadron of Einstein's special relativity and Pauli's exclusion principle, *Hadronic J.* **1** (1978), 574–902.

33. R. M. Santilli, Isotopic breaking of Gauge symmetries, *Phys. Rev. D* **20** (1979), 555–570.

34. R. M. Santilli, Lie-isotopic lifting of the special relativity for extended-deformable particles, *Lett. Nuovo Cimento* **37** (1983), 545–555.

35. R. M. Santilli, Lie-isotopic lifting or unitary symmetries and of Wigner's theorem for extended deformable particles, Lett. Nuovo Cimento 3 (1983), 509–521

36. R. M. Santilli, Lie-isotopic lifting of Lie symmetries, I: General considerations, *Hadronic J.* **8** (1985), 25–35.

37. R. M. Santilli, Lie-isotopic lifting of Lie symmetries, II: Lifting of rotations, *Hadronic J.* **8** (1985), 36–51

38. R. M. Santilli, Nonlocal formulation of the Bose-Einstein correlation within the context of hadronic mechanics, *Hadronic J.* **15** (1992), 1–50 and 79–134.

39. R. M. Santilli, Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isoduals and pseudoisoduals, and "hidden numbers" of dimension 3, 5, 6, 7, *Algebras, Groups and Geometries* **10** (1993), 273–322.

40. R. M. Santilli, Isotopies of SU(2)-spin symmetry, JINR Rapid Comm. **6** (1993), 24–32

41. R. M. Santilli, Nonlinear, nonlocal, noncanonical, axiom-preserving isotopies of the Poincaré symmetry, *J. Moscow Phys. Soc.* **3** (1993), 255–297.

42. R. M. Santilli, A quantitative isotopic representation of the deuteron magnetic moment, in *Proceedings of the International Symposium <Deuteron 1993>*, JINR, 1994

43. R. M. Santilli, Isotopic lifting of quark theories with exact confinement and convergentperturbative expansions, Comm. Theor. Phys. **4** (1995), 1–23

44. R. M. Santilli, Recent Theoretical and experimental evidence on the apparent synthesis of the neutron from protons and electrons, Chine J. Syst. Eng. & Electr. **6** (1995), 177–199

45. R. M. Santilli, Quantum-Iso-Gravity, Comm. Theor. Phys. **4** (1995), in press

46. R. M. Santilli, Nonlocal-integral isotopies of differential calculus, geometries and mechanics, Rendiconti Circolo Matematico di Palermo, Supplemento **42** (1996), 7–82 in press, 1996

47. R. M. Santilli, Isotopies of trigonometric and hyperbolic functions, *Ann. Univ. Constanta* 5 (1997), 13–26

48. R. M. Santilli, Problematic aspects of classical and quantum deformations, IBR preprint TH-S-97-09, submitted to *J. Math. Phys.*

49. R. M. Santilli, *Foundations of Theoretical Mechanics*, Vol. I: *The Inverse Problem in Newtonian Mechanics*, Springer-Verlag, Heidelberg, Germany (1978)

50. R. M. Santilli, *Foundations of Theoretical Mechanics*, Vol. II: *Birkhoffian Generalization of Hamiltonian Mechanics*, Springer-Verlag, Heidelberg, Germany (1983)

51. R. M. Santilli, *Isotopic Generalizations of Galilei's and Einstein's Relativities*, Vol. I: *Mathematical Foundations*, Hadronic Press, Palm Harbor, FL (1991)

52. R. M. Santilli, *Isotopic Generalizations of Galilei's and Einstein's Relativities*, Vol. II: *Classical Isotopies*, Hadronic Press, Palm Harbor, FL (1991)

53. R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. I: *Mathematical Foundations*, Ukraine Academy of Sciences, Kiev, Second Edition (1995)

54. R. M. Santilli, *Elements of Hadronic Mechanics*, Vol. II: *Theoretical Foundations*, Ukraine Academy of Sciences, Kiev, Second Edition (1995)

55. R. M. Santilli, *Isotopic, Genotopic and Hyperstructural Methods in Theoretical Biology*, Ukraine academy of sciences, Kiev(1996)

56. R. M. Santilli, *Isorelativities with applications to Quantum Gravity, Antigravity and Cosmology*, Balkan Geometry Society, Bucharest, Romania, in press

57. D. Schuch, On a logarithmic nonlinear Schroedinger equation compatible with hadronic mechanics and its connection with other approaches for the description of dissipative systems, in New Frontiers of Hadronic Mechanics, T. L. Gill, Editor, Hadronic Press (1996), pp. 113–131

58. D. Schuch, Nonunitary connection between explicitly time dependent and nonlinear approaches in the description of dissipative quantum systems, *Phys. Rev. A* 55 (1997), in press

59. G. T. Tsagas and D. S. Sourlas, Isomanifolds, Algebras, Groups and Geometries **12**, 1–65 (1995)

60. G. T. Tsagas and D. S. Sourlas, Isomappings between isomanifolds, Algebras, Groups and geometries **12** (1995), 67–88

61. G. T. Tsagas, Studies on the classification of the Lie-Santilli theory, Algebras, Groups and geometries **13** (1996), issue no. 2, in press

62. G. T. Tsagas, Isoaffine connections and Santilli isoriemannian metrics on an isomanifold, Algebras, Groups and geometries **13** (1996), issue no. 2, in press

63. D. S. Sourlas and G. T. Tsagas, *Mathematical Foundations of the Lie-Santilli Theory*, Ukraine Academy of Sciences, Kiev (1993).

ALGEBRAS GROUPS AND GEOMETRIES 15, 509–544 (1998)

FOUNDATIONS OF SANTILLI'S ISONUMBER THEORY, PART II: ISONUMBER THEORY OF THE SECOND KIND

Chun-xuan Jiang
P. O. Box 3924, Beijing 1200854, China

Received October 3, 1997; revised December 7, 1997

Abstract

In this paper we study, apparently for the first time, the foundations of Santilli's isonumber theory of the second kind, which is characterized by the axiom-preserving *isotopic* lifting of the unit via an element of the original field, with compatible lifting of multiplication, while the elements of the original field remain unchanged. In particular, we study the iosgroup, the isodivisibility, the prime number theorem of isoarithmetic progressions, the isocongruences, etc. We introduce a new branch of number theory called Santilli's isoadditive prime theory of the second kind. We prove about fifty theorems including the prime twins theorem, the Goldbach's theorem, k-tuples of primes, $(p_1+1)^{\hat{2}}+1$, $(p_1+1)^{\hat{4}}+1$, $(p_1+1)^{\hat{8}}+1$, $(p_1+1)^{\hat{16}}+1$, $(1+2)$, $(1+3)$, $(1+4)$, Rényi's theorem, etc., by using the arithmetic function $J_n(\omega)$. We present a generalization of Euler's proof for the existence of infinitely many primes. Finally, we disprove the Riemann's hypothesis.

AMS mathematics subject classification: primary 11P32, 11P99, 11A99, 11M26.

1 Introduction

In the seminal works[1,2] Santilli has introduced a generalization of real, comlex and quaternionic numbers $a = n, c, q$ based on the lifting of the unit 1 of conventional numbers into an invertible and well behaved quantity with arbitrary functional dependence on local variables

$$1 \to \hat{I}(t, x, \dot{x}, \cdots) = 1/\hat{T} \neq 1. \tag{1.1}$$

while jointly lifting the product $ab = a \times b$ of conventional numbers into the form

$$ab \to a * b = a\hat{T}b, \tag{1.2}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit

$$\hat{I} * a = \hat{T}^{-1}\hat{T}a = a * \hat{I} = a\hat{T}\hat{T}^{-1} = a \tag{1.3}$$

for all possible $a = n, c, q$.

Since the new multiplication $a * b$ is associative, Santilli has then proved that the new numbers verify all the axioms of a field.

Let $F(a, +, \times)$ be a conventional field with numbers $a = n,\ c,\ q$ equipped with the conventional sum $a + b \in F$, product $ab = a \times b \in F$, additive unit $0 \in F$ and their multiplicative unit $1 \in F$. The above liftings were then called *isotopic* in the Greek sense of being axiom-preserving. The prefix *iso* is then used whenever the axioms are preserved.

Definition 1.1. Santilli's isofields of the first kind $\hat{F} = \hat{F}(\hat{a}, +, *)$ are the rings with elements

$$\hat{a} = a\hat{I} \tag{1.4}$$

called *isonumbers*, where $a = n, c, q \in F$, $\hat{I} = 1/\hat{T}$ is a well behaved invertible and Hermitean quantity outside the original field $\hat{I} = 1/\hat{T} \notin F$ and $a\hat{I}$ is the multiplication in F equipped with the isosum

$$\hat{a} + \hat{b} = (a + b)\hat{I}, \tag{1.5}$$

with conventional additive unit $\hat{0} = \hat{0}\hat{I} = 0$, $\hat{a} + \hat{0} = \hat{0} + \hat{a} = \hat{a}$, $\hat{a} \in \hat{F}$ and the isoproduct

$$\hat{a} * \hat{b} = \hat{a}\hat{T}\hat{b} = a\hat{I}\hat{T}b\hat{I} = (ab)\hat{I}, \tag{1.6}$$

under which $\hat{I} = 1/\hat{T}$ is the correct left and right new unit ($\hat{I} * \hat{a} = \hat{a} * \hat{I} = \hat{a}$, $\forall \hat{a} \in \hat{F}$) called isounit.

Lemma 1.1. The isofields $\hat{F}(\hat{a},+,*)$ of Def.1.1. verify all the axioms of a field. The lifting $F \to \hat{F}$ is then an isotopy. All operations depending on the product must then be lifted in $\hat{F}$ for consistency.

The Santilli's commutative isogroup of the first kind

$$\hat{a}^{\hat{I}} = a\hat{I}, \quad \hat{a}^{-\hat{I}} = a^{-1}\hat{I}, \quad \hat{a}^{\hat{I}} * \hat{a}^{-\hat{I}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a}^{\hat{b}} = a^{b}\hat{I}, \quad \hat{a}^{-\hat{b}} = a^{-b}\hat{I}, \quad \hat{a}^{\hat{b}} * \hat{a}^{-\hat{b}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a}^{\widehat{c/b}} = a^{\frac{c}{b}}\hat{I}, \quad \hat{a}^{-\widehat{c/b}} = a^{-\frac{c}{b}}\hat{I}, \quad \hat{a}^{\widehat{c/b}} * \hat{a}^{-\widehat{c/b}} = \hat{a}^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\hat{a} * \hat{I} = \hat{a}\hat{T}\hat{I} = \hat{a}. \quad \hat{a} * \hat{b} = a\hat{I}\hat{T}b\hat{I} = ab\hat{I} = \hat{ab}.$$

$$\hat{a} * \hat{b}^{-\hat{2}} = a\hat{I}\hat{T}b^{-2}\hat{I} = a/b^{2}\hat{I}.$$

where $\hat{I}$ is called an isounit, $\hat{T}$ is called an isoinverse of $\hat{I}$; $\hat{a}^{-\hat{b}}$ is called an isoinverse of $\hat{a}^{\hat{b}}$; $(\hat{a}^{\hat{b}},*)$ is called the Santilli's commutative isogroup of the first kind.

Lemma 1.2. Sentilli's isofields of the second kind $\hat{F} = \hat{F}(a,+,*)$ (that is, $a \in F$ is not lifted to $\hat{a} = a\hat{I}$) also verify all the axioms of a field, if and only if the isounit is an element of the original field.

$$\hat{I} = 1/\hat{T} \in F \tag{1.7}$$

The isoproduct is defined by

$$a * b = a\hat{T}b \in \hat{F}. \tag{1.8}$$

Definition 1.2. Isodual isomultiplication is defined by

$$*^{d} = \times T^{d} \times = -*, \quad \hat{T}^{d} = -\hat{T} \tag{1.9}$$

We then have isodual isoproduct

$$a *^{d} b = a\hat{T}^{d}b = -a\hat{T}b. \tag{1.10}$$

The Santilli's commutative isogroup of the second kind

$$a^{\hat{I}} = a, \quad a^{-\hat{I}} = a^{-1}\hat{I}^{2}, \quad a^{\hat{I}} * a^{-\hat{I}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{2}} = a^{2}\hat{T}, \quad a^{-\hat{2}} = a^{-2}\hat{I}^{3}, \quad a^{\hat{2}} * a^{-\hat{2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{3}} = a^3\hat{T}^2, \quad a^{-\hat{3}} = a^{-3}\hat{I}^4, \quad a^{\hat{3}} * a^{-\hat{3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$\cdots\cdots$$

$$a^{\hat{n}} = a^n\hat{T}^{n-1}, \quad a^{-\hat{n}} = a^{-n}\hat{I}^{n+1}, \quad a^{\hat{n}} * a^{-\hat{n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/2}} = a^{\frac{1}{2}}(\hat{I})^{\frac{1}{2}}, \quad a^{-\widehat{1/2}} = a^{-\frac{1}{2}}(\hat{I})^{\frac{3}{2}}, \quad a^{\widehat{1/2}} * a^{-\widehat{1/2}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/3}} = a^{\frac{1}{3}}(\hat{I})^{\frac{2}{3}}, \quad a^{-\widehat{1/3}} = a^{-\frac{1}{3}}(\hat{I})^{\frac{4}{3}}, \quad a^{\widehat{1/3}} * a^{-\widehat{1/3}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{1/n}} = a^{\frac{1}{n}}(\hat{I})^{1-\frac{1}{n}}, \quad a^{-\widehat{1/n}} = a^{-\frac{1}{n}}(\hat{I})^{1+\frac{1}{n}}, \quad a^{\widehat{1/n}} * a^{-\widehat{1/n}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\widehat{c/b}} = a^{\frac{c}{b}}(\hat{I})^{1-\frac{c}{b}}, \quad a^{-\widehat{c/b}} = a^{-\frac{c}{b}}(\hat{I})^{1+\frac{c}{b}}, \quad a^{\widehat{c/b}} * a^{-\widehat{c/b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{b}} = a^b(\hat{I})^{1-b} = a^b\hat{T}^{b-1}, \quad a^{-\hat{b}} = a^{-b}(\hat{I})^{1+b}, \quad a^{\hat{b}} * a^{-\hat{b}} = a^{\hat{0}} = \hat{I} = \hat{T}^{-1} \neq 1.$$

$$a^{\hat{I}} * b^{\hat{I}} = a\hat{T}b, \quad a^{\hat{I}} * b^{-\hat{I}} = ab^{-1}\hat{I} = \hat{a/b}.$$

where $\hat{I}$ is called an isounit; $\hat{T}$ an isoinverse of $\hat{I}$; $a^{-\hat{b}}$ an isoinverse of $a^{\hat{b}}$; $(a^{\hat{b}}, *)$ the Santilli's commutative isogroup of the second kind. The following examples are devoted to an exposition of the simplest properties of isomultiplications.

Example 1:

$$a^{\widehat{3/2}} = a^{\hat{I}} * a^{\widehat{1/2}} = a^{\frac{3}{2}}(\hat{T})^{\frac{1}{2}}, a^{\widehat{3/2}} = a^{\widehat{1/2}} * a^{\widehat{1/2}} * a^{\widehat{1/2}} = a^{\frac{3}{2}}(\hat{T})^{\frac{1}{2}}, a^{\widehat{3/2}} = a^{\hat{2}} * a^{-\widehat{1/2}} = a^{\frac{3}{2}}(\hat{T})^{\frac{1}{2}}.$$

Example 2:

$$a^{-\widehat{3/2}} = a^{-\hat{I}} * a^{-\widehat{1/2}} = a^{-\widehat{1/2}} * a^{-\widehat{1/2}} * a^{-\widehat{1/2}} = a^{-\hat{2}} * a^{\widehat{1/2}} = a^{-\frac{3}{2}}(\hat{I})^{\frac{5}{2}}.$$

Example 3:

$$a^{\widehat{5/6}} = a^{\widehat{1/2}} * a^{\widehat{1/3}} = a^{\widehat{1/6}} * a^{\widehat{1/6}} * a^{\widehat{1/6}} * a^{\widehat{1/6}} * a^{\widehat{1/6}} = a^{\hat{I}} * a^{-\widehat{1/6}} = a^{\frac{5}{6}}(\hat{I})^{\frac{1}{6}}.$$

Example 4:

$$a^{-\widehat{5/6}} = a^{-\widehat{1/2}} * a^{-\widehat{1/3}} = a^{-\widehat{1/6}} * a^{-\widehat{1/6}} * a^{-\widehat{1/6}} * a^{-\widehat{1/6}} * a^{-\widehat{1/6}} = a^{-\hat{I}} * a^{\widehat{1/6}} = a^{-\frac{5}{6}}(\hat{I})^{\frac{11}{6}}.$$

Example 5:

$$a^{\hat{2}} = a^{\widehat{1/2}} * a^{\widehat{1/2}} * a^{\widehat{1/2}} * a^{\widehat{1/2}} = a^{\hat{3}} * a^{-\hat{I}} = a^2\hat{T}.$$

Example 6:

$$a^{\widehat{3/4}} = a^{\widehat{1/4}} * a^{\widehat{1/4}} * a^{\widehat{1/4}} = a^{\hat{I}} * a^{-\widehat{1/4}} = a^{\widehat{1/2}} * a^{\widehat{1/4}} * = a^{\frac{3}{4}}(\hat{I})^{\frac{1}{4}}.$$

Example 7:

$$a^{\widehat{13/6}} = a^{\widehat{2/3}} * a^{\widehat{3/2}} = a^{\hat{2}} * a^{\widehat{1/6}} = a^{\frac{13}{6}}(\hat{T})^{\frac{7}{6}}.$$

Example 8:

$$a^{-\widehat{1/6}} = a^{-\widehat{1/2}} * a^{\widehat{1/3}} = a^{\hat{I}} * a^{-\widehat{7/6}} = a^{-\frac{1}{6}}(\hat{I})^{\frac{7}{6}}.$$

Example 9:

$$a^{\widehat{1/2}} * b^{\widehat{1/3}} * c^{\widehat{1/5}} = a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{1}{5}} (\hat{T})^{\frac{1}{30}}.$$

Let $a = b = c$, we have $a^{\widehat{31/30}} = a^{\frac{31}{30}}(\hat{T})^{\frac{1}{30}}$.

Example 10:

$$a^{\widehat{1/2}} * b^{\widehat{1/3}} * c^{\widehat{1/5}} * d^{\widehat{1/6}} = a^{\frac{1}{2}} b^{\frac{1}{3}} c^{\frac{1}{5}} d^{\frac{1}{6}} (\hat{T})^{\frac{1}{5}}.$$

Let $a = b = c = d$, we have $a^{\widehat{6/5}} = a^{\frac{6}{5}}(\hat{T})^{\frac{1}{5}}$

Example 11:

$$a^{\hat{I}} = a^{\widehat{1/3}} * a^{\widehat{2/3}} = a\hat{I}\hat{T} = a.$$

Example 12:

$$a^{-\hat{I}} = a^{-\widehat{1/3}} * a^{-\widehat{2/3}} = a^{-1}\hat{I}^3\hat{T} = a^{-1}\hat{I}^2.$$

Example 13:

$$a^{\hat{2}} * b^{-\hat{2}} = (\frac{a}{b})^2 \hat{I}. \quad a^{\hat{2}} * b^{-\hat{3}} = a^2 b^{-3} \hat{I}^2.$$

Example 14:

$$a * y^{\hat{2}} + b * y + c = a(y\hat{T})^2 + b(y\hat{T}) + c.$$

Example 15:

$$a_3 * y^{\hat{3}} + a_2 * y^{\hat{2}} + a_1 * y + a_0 = a_3(y\hat{T})^3 + a_2(y\hat{T})^2 + a_1(y\hat{T}) + a_0.$$

Lemma 1.3. Santilli's isofields of the third kind $\hat{F} = \hat{F}(\bar{a}, +, *)$ (that is $a = p_1^{\alpha_1} \cdots p_n^{\alpha_n} \in F$ is lifted to $\bar{a}= p_1^{\widehat{\alpha_1}} * \cdots * p_n^{\widehat{\alpha_n}} \in \hat{F}$) verify all the axioms of a field if and only if the isounit is an element of the original field

$$\hat{I} = \frac{1}{\hat{T}} \in F.$$

The isoproduct is defined by

$$(\bar{a})^{\pm\hat{b}} = (\bar{a})^{\pm b}(\hat{T})^{\pm b-1}, a \neq 1.$$

$((\bar{a})^{\hat{b}}, *)$ is called the Santilli's commutative isogroup of the third kind.

Santilli's isonumber theory of the third kind is, without doubt, a very fascinating subject. Although it is the novel elementary mathematical system, the study of their properties has provided generations of mathematicians with problems of unending fascination. A fascinating example:

$$\bar{8}= 8\hat{T}^2, \quad \bar{2} + \bar{6}= 2 + 6\hat{T}, \quad \bar{3} + \bar{5}= 8, \quad \bar{4} + \bar{4}= 8\hat{T}.$$

Recently we make many of calculations and discover the Santilli's commutative isogroup of the first kind, the Santilli's commutative isogroup of the second kind and the Santilli's commutative isogroup of the third kind.

In previous paper [3] we study Santilli's isonumber theory of the first kind based on isofields $\hat{F} = \hat{F}(\hat{a}, +, *)$. In this paper we study Santilli's isonumber theory of the second kind based on isofields $\hat{F} = \hat{F}(a, +, *)$.

2. Foundations of Santilli's Isonumber Theory

By lifting $F(a, +, \times) \to \hat{F}(a, +, *)$ we study Santilli's isonumber theory of the second kind.

We can partition the positive integers into four classes:

1. The unit: 1,
2. The isounit: $\hat{I}$ or $\hat{T}$,
3. The prime numbers: $2, 3, 5, \ldots,$
4. The composite numbers: $4, 6, 8, \ldots.$

The Santilli's isonumber theory of the second kind is primarily concerned with isodivisibility properties of integers.

Definition 2.1. Definition of isodivisibility. We say that a nonzero integer a isodivides an integer b, if there exists an integer c such that $a * c = a\hat{T}c = b$, $c = b\hat{I}/a$. If a isodivides b, we write $a\hat{|}b = a|\hat{I}b$. Then we have $a|b$ or $a|\hat{I}$. If a does not isodivide b, we write $a \hat{\nmid} b = a \nmid b\hat{I}$. Then we have $a \nmid b$ and $a \nmid \hat{I}$.

The following theorem gives the key properties of isodivisibility.

Theorem 2.1.

(1) If a is a nonzero integer, then $a\hat{|}a = a|a\hat{I}$.
(2) If a is an integer, then $1\hat{|}a = 1|a\hat{I}$.
(3) If $a\hat{|}b = a|b\hat{I}$ and $b\hat{|}c = b|c\hat{I}$, then $a\hat{|}c = a|c\hat{I}$
(4) If $a\hat{|}b = a|b\hat{I}$ and c is a nonzero integer, then $ac\hat{|}bc = ac|bc\hat{I}$ and $a\hat{|}bc = a|bc\hat{I}$.
(5) If $a\hat{|}b = a|b\hat{I}$ and $a\hat{|}c = a|c\hat{I}$, then for all integer m and n we have $a\hat{|}(mb + nc) = a|\hat{I}(mb + nc)$.
(6) If $a\hat{|}b = a|b\hat{I}$ and $b\hat{|}a = b|a\hat{I}$, then $a = \pm b$ and $\hat{I} = 1$.
(7) If $a\hat{|}b = a|b\hat{I}$ and a and b are positive integers, then $a > b$, $a < \hat{I}$ and $a|\hat{I}$ or $a > \hat{I}, a < b$ and $a|b$.

Definition 2.2. If d divides two integer a and b, then d is called a common divisor of a and b. The number d is called the greatest common divisor (gcd) of a and b and is denoted by (a, b). If $(a, b) = 1$, then a and b are said to be relatively prime.

Theorem 2.2. The prime number theorem for isoarithmetic progressions

$$E_a(K) = \omega * K + a = \omega\hat{T}k + a, \tag{2.1}$$

where $k = 0, 1, 2, \ldots; (\omega\hat{T}, a) = 1$. We have

$$\pi_a(N) = \frac{1}{\phi(\omega\hat{T})}\frac{N}{\log N}(1 + O(1)), \tag{2.2}$$

where $\pi_a(N)$ denotes the number of prime in $E_a(K) \leq N$ and $\phi(\omega\hat{T})$ Euler's ϕ-function.

Santilli's isoadditive prime problems:

$$\hat{p}_2 = 2 * \hat{p}_1 + 1 = 2\hat{T}p_1 + 1, \quad p_3 = 4 * p_1 + 1 = 4\hat{T}p_1 + 1. \tag{2.3}$$

Let $\hat{T} = 1$, we have

$$p_2 = 2p_1 + 1, \quad p_3 = 4p + 1. \tag{2.4}$$

They cannot all be prime, for at least one of the three is divisible by 3.
There exist no 3-tuples of primes except $p_1 = 3$, $p_2 = 7$, $p_3 = 13$. Let $\hat{T} = 2$, and $\hat{I} = \frac{1}{2}$ we have

$$p_2 = 4p_1 + 1, \quad p_3 = 8p_1 + 1. \tag{2.5}$$

There exist no 3-tuples of primes.

Let $\hat{T} = 3$ and $\hat{I} = \frac{1}{3}$, we have

$$p_2 = 6p_1 + 1, p_3 = 12p_1 + 1. \tag{2.6}$$

There exist infinitely many 3-tuples of primes: 5, 31, 61; 13, 79, 157; 23, 139, 277; 61, 367, 733; ...

$$p_4 = (p_1 + p_2 + p_3 + 1)^{\hat{2}} + 1 = \hat{T}(p_1 + p_2 + p_3 + 1)^2 + 1. \tag{2.7}$$

Let $\hat{T} = \{\hat{T}_1, \ldots, \hat{T}_n\}$ and $\hat{I} = \{\hat{I}_1, \ldots, \hat{I}_n\}$. In (2.7) there are n additive prime equations. Every equation has an isounit.

Fermat-Santilli equations[4]:

$$x^{\hat{n}} + y^{\hat{n}} = 1. \tag{2.8}$$

From(2.8) we have

$$x^n + y^n = (\hat{I})^{n-1}, \quad (x, y) = 1. \tag{2.9}$$

For $n > 3$, (2.9) has no rational solutions.

Pell-Santilli equations:

$$x^{\hat{2}} - p * y^{\hat{2}} = \pm 1. \tag{2.10}$$

By Santilli's isonumber theory we can extend the additive prime equations and Diophantine equations.

Definition 2.3. Given integers a, b, m with $m > 0$. We say that a is isocongruent to b modulo m and we write

$$a \hat{\equiv} b \pmod{m}. \tag{2.11}$$

If m isodivides the difference $a-b$, the number m is called the modulus of isocongruence. The isocongruence (2.11) is equivalent to the isodivisibility relation

$$m \hat{|} (a-b) = m \mid \hat{I}(a-b). \tag{2.12}$$

If $m \hat{\not|}(a-b)$ we write

$$a \stackrel{\wedge}{\not\equiv} b \ (\mathrm{mod}\ m), \tag{2.13}$$

and we say that a and b are nonisocongruent (mod m).

Theorem 2.3. The isocongruence is an equivalence relation:

(1) $a \stackrel{\wedge}{\equiv} a \ (\mathrm{mod}\ m)$ (reflexivity)
(2) $a \stackrel{\wedge}{\equiv} b \ (\mathrm{mod}\ m)$ implies $b \stackrel{\wedge}{\equiv} a \ (\mathrm{mod}\ m)$ (symmetry)
(3) $a \stackrel{\wedge}{\equiv} b$ (mod) and$b \stackrel{\wedge}{\equiv} c$ (mod) implies $a \stackrel{\wedge}{\equiv} c(\mathrm{mod}\ m)$ (transitivity)

Definition 2.4. The quadratic isocongruence

$$x^{\hat{2}} \stackrel{\wedge}{\equiv} n \ (\mathrm{mod}\ p), \tag{2.14}$$

where p is an odd prime.Let $(\hat{I}, p) = 1$ so we can cancel $\hat{I}$. (2.14) can be written as

$$x^2 \equiv n\hat{I} \ (\mathrm{mod}\ p). \tag{2.15}$$

If congruence (2.15) has a solution and we say that n is a quadratic residue mod p and we write $(\frac{n\hat{I}}{p}) = 1$, where $(\frac{n\hat{I}}{p})$ is Legendre symbol. If (2.15) has no solution we say that n is a quadratic nonresidue mod p and we write $(\frac{n\hat{I}}{p}) = -1$.

Theorem 2.4.

$$\sum_{i=1}^{n} x_i \stackrel{\wedge}{\equiv} a \ (\mathrm{mod}\ p). \tag{2.16}$$

When $(\hat{I}, p) = 1$, we can cancel $\hat{I}$. (2.16) can be written as

$$\sum_{i=1}^{n} x_i \equiv a \ (\mathrm{mod}\ p). \tag{2.17}$$

where p is an odd prime.

(2.17) has exactly $J_n(p) + (-1)^n$ solutions, where $J_n(p) = \frac{(p-1)^n - (-1)^n}{p}$, if $p|a$ and (2.17) $J_n(p)$ solutions if $p \not| a$.

Theorem 2.5.

$$\hat{x}_1^2 + x_2 + \ldots + x_n \hat{\equiv} a \pmod{p}. \tag{2.18}$$

When $(\hat{I}, p) = 1$, we can cancel $\hat{I}$, (2.18) can be written as

$$x_1^2 + \hat{I}(x_2 + \ldots + x_n) \equiv \hat{I}a \pmod{p}. \tag{2.19}$$

(2.19) has exactly $J_n(p) - (-1)^n$ solutions if $(\frac{a\hat{I}}{p}) = 1$ and (2.19) $J_n(p) + (-1)^n$ solutions if $(\frac{a\hat{I}}{p}) = -1$ and $p|a$.

3. Santilli's Isoadditive Prime Theory

Definition 3.1. We define the arithmetic progressions[5,6]

$$E_{p_\alpha}(K) = \omega K + p_\alpha, \tag{3.1}$$

where $K = 0, 1, 2, \ldots$;

$$\omega = \prod_{2 \le p \le p_i} p; \ (\omega, p_\alpha) = 1;$$

$$p_i < p_\alpha = p_1, p_2, \ldots, p_{\phi(\omega)} = \omega + 1; \ \phi(\omega) = \sum_{\substack{(p_\alpha, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 1)$$

$\phi(\omega)$ is Euler's ϕ-function.

For every $Ep_\alpha(K)$, there exist infinitely many primes. We have

$$\pi_{p_\alpha}(N) = \frac{1}{\phi(\omega)} \frac{N}{\log N}(1 + O(1)), \tag{3.2}$$

where $\pi_{p_\alpha}(N)$ denotes the number of primes $p \le N$ in $E_{p_\alpha}(K)$. Since $\pi_{p_\alpha}(N)$ is independent of p_α, the primes seem to be equally distributed among the $\phi(\omega)$ reduced residue classes mod ω, and (3.2) is a precise statement of this fact.

We deal with the prime twins: $p_2 = p_1 + 2$. It can be written as the form of the arithmetic progressions

$$E_{p_\alpha + 2}(K) = E_{p_\alpha}(K) + 2. \tag{3.3}$$

We define the arithmetic function of the prime twins

$$J_2(\omega) = \sum_{\substack{(p_\alpha + 2, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 2). \tag{3.4}$$

Since $J_2(\omega) < \phi(\omega)$, it is a generalization of Euler's ϕ function $\phi(\omega)$. Since $(p_\alpha + 2, \omega) = 1$, (3.3) has the infinitude of the prime twins.

Let $p_i = 3$. From (3.1) we have

$$E_5(K) = 6K + 5, \quad E_7(K) = 6K + 7, \tag{3.5}$$

where $K = 0, 1, 2, \ldots$

From (3.4) we have $J_2(6) = 1$. From (3.5) we have one subequation of the prime twins

$$E_7(K) = E_5(K) + 2. \tag{3.6}$$

Since $(7, 6) = 1$, (3.6) has the infinitude of the prime twins.

Let $p_i = 5$. From (3.1) we have

$$E_{p_\alpha}(K) = 30K + p_\alpha, \tag{3.7}$$

where $K = 0, 1, \ldots;\ p_\alpha = 7, 11, 13, 17, 19, 23, 29, 31$.

From (3.4) we have $J_2(30) = 3$. From (3.7) we have three subequations of the prime twins

$$E_{13}(K) = E_{11}(K) + 2, E_{19}(K) = E_{17}(K) + 2, E_{31}(K) = E_{29}(K) + 2. \tag{3.8}$$

Since $(p_\alpha + 2, 30) = 1$, every subequation has the infinitude of the prime twins. The prime twins seem to be equally distributed among the $J_2(30)$ reduced residue classes mod 30. It is a generalization of Direchlet's theorem. For $J_2(\omega) \to \infty$ as $\omega \to \infty$, there exist infinitely many subequations of the prime twins, every subequation has the infinitude of the prime twins. By using this method and Santilli's isonumber theory we found the new branch of number theory: Santilli's isoadditive prime theory.

By lifting $F(a, +, \times) \to \hat{F}(a, +, *)$ from (3.1) we have isoarithmetic progressions

$$E_{p_\alpha}(K) = \omega * K + p_\alpha = \omega \hat{T} K + p_\alpha. \tag{3.9}$$

Let $\hat{T} = \omega^{m-1}$. From (3.9) we have

$$E_{p_\alpha}(K) = \omega^m K + p_\alpha, \tag{3.10}$$

where

$$p_i < p_\alpha = p_1, \ldots, p_{\phi(\omega^m)} = \omega^m + 1;\ \phi(\omega^m) = \sum_{(p_\alpha, \omega^m)=1} 1 = \omega^{m-1}\phi(\omega).$$

Theorem 3.1. If there exist infinitely many primes p_j (for $j = 1, \ldots, n-1$) such that the absolute values of polynomials $f_i(p_j)$ (for $i = 1, \ldots, k-1$) are all prime, then $f_i(p_j)$ must satisfy two necessary and sufficient conditions:

(I) Let $f_i(p_j)$ be $k-1$ distinct polynomials with integral coefficients irreducible over the integers.

(II) There exists an arithmeric function $J_n(\omega^m)$, that is to separate the number of k-tuples of subequations from (3.10). It is also the number of solutions of

$$\left(\prod_{i=1}^{k-1} f_i(p_{\alpha_j}), \omega^m\right) = 1, \tag{3.11}$$

where $1 \leq \alpha_j \leq \phi(\omega^m)$, $j = 1, \ldots, n-1$.

Since $J_n(\omega^m) \leq \phi^{n-1}(\omega^m)$, $J_n(\omega^m)$. can be expressed as the form

$$J_n(\omega^m) = \sum_{\alpha_{n-1}=1}^{\phi(\omega^m)} \cdots \sum_{\alpha_1=1}^{\phi(\omega^m)} \left[\frac{1}{(\prod_{i=1}^{k-1} f_i(p_{\alpha_j}), \omega^m)}\right]$$
$$= \omega^{(n-1)(m-1)} \prod_{3 \leq p \leq p_i} ((p-1)^{n-1} - H(p)), \tag{3.12}$$

where $H(p)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1} f_i(q_j) \equiv 0 \pmod{p}, \tag{3.13}$$

$q_j = 1, 2, \ldots, p-1; j = 1, \ldots, n-1$.

Since $(p-1)^{n-1} = \frac{(p-1)^n - (-1)^n}{p} + \frac{(p-1)^{n-1} - (-1)^{n-1}}{p}$, $J_n(\omega^m)$ can also be expressed as the form

$$J_n(\omega^m) = \omega^{(n-1)(m-1)} \prod_{3 \leq p \leq p_i} (\frac{(p-1)^n - (-1)^n}{p} - \chi(p)), \tag{3.14}$$

where $\chi(p) = 0, \pm 1, \ldots$

In the same way as in Ref.[3] we can derive the best asymptotic formula

$$\pi_k(N, n) = |\{p_j : p_j \leq N,\ f_i(p_j) = \text{prime}\}|$$
$$= \prod_{i=1}^{k-1} (\deg f_i)^{-1} \times \frac{J_n(\omega^m)(\omega^m)^{k-1}}{(n-1)!\phi^{n+k-2}(\omega^m)} \frac{N^{n-1}}{(\log N)^{n+k-2}} (1 + O(1)). \tag{3.15}$$

We have [3]

$$\frac{J_n(\omega^m)(\omega^m)^{k-1}}{\phi^{n+k-2}(\omega^m)} = \frac{J_n(\omega)\omega^{k-1}}{\phi^{n+k-2}(\omega)}. \tag{3.16}$$

Substituting (3.16) into (3.15) we have

$$\pi_k(N,n) = \prod_{i=1}^{k-1}(\deg f_i)^{-1} \times \frac{J_n(\omega)\omega^{k-1}}{(n-1)!\phi^{n+k-2}(\omega)} \frac{N^{n-1}}{(\log N)^{n+k-2}}(1+O(1)). \tag{3.17}$$

We prove that $\pi_k(N,n)$ is independent of m. (3.17) can be written as the form

$$\pi_k(N,n) = J_n(\omega)t_1, \tag{3.18}$$

where

$$t_1 = \prod_{i=1}^{k-1}(\deg f_i)^{-1} \times \frac{\omega^{k-1}N^{n-1}}{(n-1)!(\phi(\omega)\log N)^{n+k-2}}(1+O(1)). \tag{3.19}$$

t_1 denoting the number of k-tuples of primes in one k-tuple of subequations. t_1 can be applied to any k-tuple of subequations and is called the common factor in Santilli's additive prime theory. $t_1 = 0$ if $N < \omega$, $t_1 \neq 0$ if $N > \omega$ similar to (3.1). $t_1 \neq 0$ implies that there exist infinitely many prime solulions. If $J_n(\omega) = 0$ then $\pi_k(N,n) = 0$,there exist finitely many k-tuples of primes. If $J_n(\omega) \to \infty$ as $\omega \to \infty$,then there exist infinitely many k-tuples of prines. It is a generalization of Euler proof of the existence of infinitely many primes.

Let $n = 2$ and $k = 1$. From (3.19) we have

$$t_1 = \frac{N}{\phi(\omega)\log N}(1+O(1)). \tag{3.20}$$

It is the prime number theorem of the arithmetic progressions.

Since $k = 1$, we have $J_2(\omega) = \phi(\omega)$. Substituting (3.20) into (3.18) we have

$$\pi_1(N,2) = \frac{N}{\log N}(1+O(1)). \tag{3.21}$$

It is the prime number theorem.

(3.17) is a unified asymptotic formula in the Santilli's isoadditive prime theory. To prove it is transformed into studying the arithmetic functions $J_n(\omega)$. By using the $J_n(\omega)$ we prove the following Santilli's isoadditive prime theorems:

Theorem 3.1.1. $p_2 = a * p_1 + b = a\hat{T}p_1 + b$, where $(a\hat{T}, b) = 1$ and $2|a\hat{T}b$. We have

$$J_2(\omega) = \sum_{\substack{(a\hat{T}p_\alpha + b, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p-2) \prod_{p|a\hat{T}b} \frac{p-1}{p-2} \ne 0,$$

$$\pi_2(N,2) = 2 \prod_{3 \le p \le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|a\hat{T}b} \frac{N}{\log^2 N}(1 + O(1)).$$

Since $J_2(\omega) \ne= 0$, there exist infinitely many primes p_1 such that p_2 is also a prime. Let $a = 1$, it is the prime twins theorem.

Therem 3.1.2.

$$p_2 = a *^d p_1 + N = a\hat{T}\ ^d p_1 + N = -a\hat{T}p_1 + N,$$

Let $a = \hat{T} = 1$. We have the Goldbach's theorem: $p_2 = N - p_1$.
We have

$$J_2(\omega) = \sum_{\substack{(N - p_\alpha, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \ne 0,$$

$$\pi(N,2) = 2 \prod_{3 \le p \le p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N}{\log^2 N}(1 + O(1)).$$

Since $J_2(\omega) \ne 0$, every even number greater than 4 is the sum of two primes. It is the simplest theorem in Santilli's isoadditive prime theory.

Theorem 3.1.3. $p_2 = p_1^{\hat{2}} + p_1 + 1 = \hat{T}p_1^2 + p_1 + 1$, where $\hat{T}$ is an odd.

Since $J_2(\omega) \ne 0$, there exist infinitely many primes p_1 such that p_2 is slso a prime.

Let $\hat{T} = 1$. we have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^2 + p_\alpha + 1, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \ne 0,$$

where $\chi(3) = 0; \chi(p) = 1$ if $p \equiv 1 (\text{mod } 3)$, $\chi(p) = -1$ if $p \equiv -1\ (\text{mod } 3)$,

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.4. $p_2 = (p_1+1)^{\hat{2}} + 1 = \hat{T}(p_1+1)^2 + 1$.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\substack{((p_\alpha+1)^2+1,\omega)=1 \\ 1\le\alpha\le\phi(\omega)}} 1 = \prod_{3\le p\le p_i}\left(p-2-(-1)^{\frac{p-1}{2}}\right) \neq 0,$$

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{2\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.5. $p_2 = p_1^{\hat{3}} + 2 = \hat{T}^2 p_1^3 + 2$, where $\hat{T}$ is an odd.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^3+2,\omega)=1 \\ 1\le\alpha\le\phi(\omega)}} 1 = \prod_{3\le p\le p_i}(p-2-\chi(p)) \neq 0,$$

where $\chi(p) = 2$ if $2^{\frac{p-1}{3}} \equiv 1 (\mathrm{mod}\ p)$; $\chi(p) = -1$ if $2^{\frac{p-1}{3}} \not\equiv 1 (\mathrm{mod}\ p)$; $\chi(p) = 0$ otherwise.

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{3\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.6. $p_2 = (p_1+1)^{\hat{4}} + 1$.

If $J_2(\omega) \neq 0$, then there exist infinitely many primes p_1 such that p_2 is also *a* prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{((p_\alpha+1)^4+1,\omega)=1 \\ 1\le\alpha\le\phi(\omega)}} 1 = \prod_{3\le p\le p_i}(p-2-\chi(p)) \neq 0,$$

where $\chi(p) = 3$ if $p \equiv 1 (\mathrm{mod}\ 8)$; $\chi(p) = -1$ if $p \not\equiv 1 (\mathrm{mod}\ 8)$,

$$\pi_2(N,2) = \frac{J_2(\omega)\omega}{4\phi^2(\omega)}\frac{N}{\log^2 N}(1+O(1)).$$

Theorem 3.1.7. $p_2 = p_1^{\hat{5}} + 2$, where $\hat{T}$ is an odd.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^5 + 2, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 4$ if $2^{\frac{p-1}{5}} \equiv 1 (\bmod\ p)$; $\chi(p) = -1$ if $2^{\frac{p-1}{5}} \not\equiv 1 (\bmod\ p)$; $\chi(p) = 0$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{5\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.8. $p_2 = (p_1 + 4)^{\hat{6}} + 4$ where $\hat{T}$ is an odd.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{((p_\alpha + 4)^6 + 4, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 5$ if $(4)^{\frac{p-1}{6}} \equiv (-1)^{\frac{p-1}{6}} (\bmod\ p)$; $\chi(p) = -1$ if $(4)^{\frac{p-1}{6}} \not\equiv (-1)^{\frac{p-1}{6}} (\bmod\ p)$; $\chi(p) = (-1)^{\frac{p-1}{2}}$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{6\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.9. $p_2 = p_1^{\hat{6}} + p_1^{\hat{3}} + 1$.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^6 + p_\alpha^3 + 1, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(3) = 0$; $\chi(p) = 5$ if $p \equiv 1 (\bmod\ 18)$; $\chi(p) = -1$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{6\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.10. $p_2 = p_1^{\hat{7}} + 2$, where $\hat{T}$ is an odd.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^7 + 2, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 6$ if $2^{\frac{p-1}{7}} \equiv 1 (\text{mod } p)$; $\chi(p) = -1$ if $2^{\frac{p-1}{7}} \not\equiv 1 (\text{mod } p)$; $\chi(p) = 0$ otherwise.

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{7\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.11. $p_2 = (p_1 + 1)^{\hat{8}} + 1$.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{((p_\alpha + 1)^8 + 1, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0,$$

where $\chi(p) = 7$ if $p \equiv 1 (\text{mod } 16)$; $\chi(p) = -1$ if $p \not\equiv 1 (\text{mod } 16)$,

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{8\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.12. $p_2 = (p_1 + 1)^{\widehat{16}} + 1$.

Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.

Let $\hat{T} = 1$. We have

$$J_2(\omega) = \sum_{\substack{((P_\alpha + 1)^{16} + 1, \omega) = 1 \\ 1 \le \alpha \le \phi(\omega)}} 1 = \prod_{3 \le p \le p_i} (p - 2 - \chi(p)) \neq 0$$

where $\chi(p) = 15$ if $p \equiv 1 (\text{mod } 32)$; $\chi(p) = -1$ if $p \not\equiv 1 (\text{mod } 32)$,

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{16\phi^2(\omega)} \frac{N}{\log^2 N}(1 + 0(1)).$$

Theorem 3.1.3. $p_3 = p_1^{\hat{4}} + p_1^{\hat{3}} + p_1^{\hat{2}} + p_1 + 1$, where $\hat{T}$ is an odd.
Since $J_2(\omega) \neq 0$, there exist infinitely many primes p_1 such that p_2 is also a prime.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\substack{(p_\alpha^4 + p_\alpha^3 + p_\alpha^2 + p_\alpha + 1, \omega) = 1 \\ 1 \leq \alpha \leq \phi(\omega)}} 1 = \prod_{3 \leq p \leq p_i} (p - 2 - \chi(p)) \neq 0.$$

where

$$\chi(5) = 0; \chi(p) = 3 \text{ if } p \equiv 1 (\text{mod } 5); \chi(p) = -1 \text{ if } p \not\equiv 1 (\text{mod } 5),$$

$$\pi_2(N, 2) = \frac{J_2(\omega)\omega}{4\phi^2(\omega)} \frac{N}{\log^2 N}(1 + O(1)).$$

Theorem 3.1.14. $p_2 = 2 * p_1 + 1, \; p_3 = 4 * p_1 + 1.$

$$J_2(\omega) \neq 0 \text{ if } 3|\hat{T}; \; J_2(\omega) = 0 \text{ if } 3 \not| \hat{T}.$$

Let $\hat{T} = 3$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(6p_\alpha + 1, \omega) + (12p_\alpha + 1, \omega)} \right] = 2 \prod_{5 \leq p \leq p_i} (p - 3) \neq 0$$

Here [ ] denotes the greatest integer,

$$\pi_3(N, 2) = \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}(1 + O(1)).$$

Theorem 3.1.15. $p_2 = 3 * p_1 + 2, \; p_3 = 2 * p_1 + 3$, where $\hat{T}$ is a prime gerater than 3.
Since $J_2(\omega) \neq 0$, there are infinitely many 3-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(3p_\alpha + 2, \omega) + (2p_\alpha + 3, \omega)} \right] = 6 \prod_{7 \leq p \leq p_i} (p - 3) \neq 0.$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.16. $p_2 = 30 * p_1 + 1,\ p_3 = 60 * p_1 - 1$.

Since $J_2(\omega) \neq 0$, there are infinitely many 3-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(30p_\alpha + 1, \omega) + (60p_\alpha - 1, \omega)}\right] = 8 \prod_{7 \leq p \leq p_i} (p-3) \neq 0,$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{\phi^3(\omega)} \frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.17. $p_2 = p_1 + 4,\ p_3 = p_1^{\hat{2}} + 4$, where T is an odd and $3 \not| (\hat{T}+4)$.

Since $J_2(\omega) \neq 0$ there are infinitely many 3-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(p_\alpha + 4, \omega) + (p_\alpha^2 + 4, \omega)}\right] = 2 \prod_{7 \leq p \leq p_i} (p - 3 - (-1)^{\frac{p-1}{2}}) \neq 0.$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.18. $p_2 = p_1 + 2,\ p_3 = p_1^{\hat{2}} + 30$, where $\hat{T}$ is a prime greater than 5.

Since $J_2(\omega) \neq 0$, there are infinitely many 3-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(p_\alpha + 2, \omega) + (p_\alpha^2 + 30, \omega)}\right] = 3 \prod_{7 \leq p \leq p_i} \left(p - 3 - \left(-\frac{30}{p}\right)\right) \neq 0,$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{2\phi^3(\omega)} \frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.19. $p_2 = p_1^{\hat{2}} + 1,\ p_3 = p_1^{\hat{2}} + 3$.

$$J_2(\omega) \neq 0 \text{ if } 3|(\hat{T}-4);\ \ J_2(\omega) = 0 \text{ otherwise.}$$

Let $\hat{T}=4$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{2}{(4p_\alpha^2+1,\omega)+(4p_\alpha^2+3,\omega)}\right]$$

$$= 2\prod_{5\le p\le p_i}\left(p-3-(-1)^{\frac{p-1}{2}}-\left(\frac{-3}{p}\right)\right)\neq 0,$$

$$\pi_3(N,2) = \frac{J_2(\omega)\omega^2}{4\phi^3(\omega)}\frac{N}{\log^3 N}(1+O(1)).$$

Theorem 3.1.20. $p_2 = 2*p_1+1,\ p_3 = 6*p_1+1,\ p_4 = 8*p_1+1.$
Since $J_2(\omega)\neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T}=1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{(2p_\alpha+1,\omega)+(6p_\alpha+1,\omega)+(8p_\alpha+1,\omega)}\right]$$

$$= \prod_{5\le p\le p_i}(p-4)\neq 0.$$

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}(1+O(1)).$$

Theorem 3.1.21 $p_2 = 2*p_1+1,\ p_3 = 3*p_1+2,\ p_4 = 4*p_1+3$, where $\hat{T}$ is a prime greater than 3.

Since $J_2(\omega)\neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T}=1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{(2p_\alpha+1,\omega)+(3p_\alpha+2,\omega)+(4p_\alpha+3,\omega)}\right]$$

$$= \prod_{5\le p\le p_i}(p-4)\neq 0.$$

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)}\frac{N}{\log^4 N}(1+O(1)).$$

Theorem 3.1.22. $p_2 = 30 * p_1 + 1,\ p_3 = 60 * p_1 + 1,\ p_4 = 90 * p_1 + 1.$
Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(30ip_\alpha + 1, \omega)} \right] = 8 \prod_{7 \le p \le p_i} (p-4) \neq 0.$$

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.23. $p_2 = p_1^{\hat{2}} + 30,\ p_3 = p_1^{\hat{2}} + 60,\ p_4 = p_1^{\hat{2}} + 90,$ where $\hat{T}$, is a prime greater than 5.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(p_\alpha^2 + 30i, \omega)} \right]$$

$$= 8 \prod_{7 \le p \le p_i} \left(p - 4 - \left(\frac{-30}{p}\right) - \left(\frac{-15}{p}\right) - \left(\frac{-10}{p}\right) \right) \neq 0.$$

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{8\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.24. $p_2 = p_1^{\hat{3}} + 30,\ p_3 = p_1^{\hat{3}} + 60,\ p_4 = p_1^{\hat{3}} + 90,$ where $\hat{T}$ is a prime greater than 5.

Since $J_2(\omega) \neq 0$, there are infinitely many 4-tuples of primes.
Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{3}{\sum_{i=1}^{3}(p_\alpha^3 + 30i, \omega)} \right] = 8 \prod_{7 \le p \le p_i} (p - 4 - \chi_1(p) - \chi_2(p) - \chi_3(p)) \neq 0.$$

where $\chi_i(p) = 2$ if $B_i^{\frac{p-1}{3}} \equiv 1 (\bmod\ p)$; $\chi_i(p) = -1$ if $B_i^{\frac{p-1}{3}} \not\equiv 1 (\bmod\ p)$, $B_1 = 30,\ B_2 = 60,\ B_3 = 90$; $\chi(p) = 0$ otherwise.

$$\pi_4(N,2) = \frac{J_2(\omega)\omega^3}{27\phi^4(\omega)} \frac{N}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.25. $p_2 = 2 * p_1 + 1,\ p_3 = 6 * p_1 + 1,\ p_4 = 8 * p_1 + 1,\ p_5 = 12 * p_1 + 1.$

Since $J_2(\omega) \neq 0$, there are infinitely many 5-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{1}{(2p_\alpha + 1, \omega)\ (6p_\alpha + 1, \omega)\ (8p_\alpha + 1, \omega)\ (12p_\alpha + 1, \omega)}\right]$$

$$= \prod_{7 \leq p \leq p_i} (p - 5) \neq 0.$$

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.26. $p_2 = 30 * p_1 + 1,\ p_3 = 60 * p_1 + 1,\ p_4 = 90 * p_1 + 1,\ p_5 = 120 * p_1 + 1.$

Since $J_2(\omega) \neq 0$, there are infinitely many 5-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{4}{\sum_{i=1}^{4}(30ip_\alpha + 1, \omega)}\right] = 8 \prod_{7 \leq p \leq p_i} (p - 5) \neq 0.$$

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.27. $p_2 = p_1^{\hat{2}} + 30,\ p_3 = p_1^{\hat{2}} + 60,\ p_4 = p_1^{\hat{2}} + 90,\ p_5 = p_1^{\hat{2}} + 120,$ where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega) \neq 0$, then there are infinitely many 5-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{4}{\sum_{i=1}^{4}(p_\alpha^2 + 30i, \omega)}\right]$$

$$= 8 \prod_{7 \leq p \leq p_i} (p - 5 - 2(\frac{-30}{p}) - (\frac{-15}{p}) - (\frac{-10}{p})) \neq 0.$$

$$\pi_5(N, 2) = \frac{J_2(\omega)\omega^4}{16\phi^5(\omega)} \frac{N}{\log^5 N}(1 + O(1)).$$

Theorem 3.1.28. $p_2 = p_1^{\hat{3}} + 30,\ p_3 = p_1^{\hat{3}} + 60,\ p_4 = p_1^{\hat{3}} + 90,\ p_5 = p_1^{\hat{3}} + 120$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega) \neq 0$, then there are infinitely many 5-tuples of primes.If $J_2(\omega) = 0$, then there are no 5-tuples of primes.

Let $\hat{T} = 1$, we have $J_2(7) = 0$. then there are no 5-tuples of primes.

Theorem 3.1.29. $p_2 = 4 * p_1 + 1,\ p_3 = 6 * p_1 + 1,\ p_4 = 10 * p_1 + 1,\ p_5 = 12 * p_1 + 1,\ p_6 = 16 * p_1 + 1$.

Since $J_2(\omega) \neq 0$, there are infinitely many 6-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) =$$

$$\sum_{\alpha=1}^{\phi(\omega)} \left[\frac{1}{(4p_\alpha + 1, \omega)(6p_\alpha + 1, \omega)(10p_\alpha + 1, \omega)(12p_\alpha + 1, \omega)(16p_\alpha + 1, \omega)}\right]$$

$$= \prod_{7 \leq p \leq p_i} (p - 6) \neq 0.$$

$$\pi_6(N, 2) = \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Theorem 3.1.30. $p_2 = 30 * p_1 + 1,\ p_3 = 60 * p_1 + 1,\ p_4 = 90 * p_1 + 1,\ p_5 = 120 * p_1 + 1,\ p_6 = 150 * p_1 + 1$.

Since $J_2(\omega) \neq 0$, there are infinitely many 6-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{5}{\sum_{i=1}^{5}(30ip_\alpha + 1, \omega)}\right] = 8 \prod_{7 \leq p \leq p_i} (p - 6) \neq 0.$$

$$\pi_6(N, 2) = \frac{J_2(\omega)\omega^5}{\phi^6(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Theorem 3.1.31. $p_2 = p_1^{\hat{2}} + 30,\ p_3 = p_1^{\hat{2}} + 60,\ p_4 = p_1^{\hat{2}} + 90,\ p_5 = p_1^{\hat{2}} + 120.\ p_6 = p_1^{\hat{2}} + 150$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega) \neq 0$, then there are infinitely many 6-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{5}{\sum_{i=1}^{5}(p_\alpha^2 + 30i, \omega)}\right]$$

$$= 8 \prod_{7 \le p \le p_i} \left(p - 6 - 2\left(\frac{-30}{p}\right) - \left(\frac{-6}{p}\right) - \left(\frac{-10}{p}\right) - \left(\frac{-15}{p}\right)\right) \ne 0.$$

$$\pi_6(N,2) = \frac{J_2(\omega)\omega^5}{32\phi^6(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Theorem 3.1.32. $p_2 = p_1^{\hat{4}} + 30$, $p_3 = p_1^{\hat{4}} + 60$, $p_4 = p_1^{\hat{4}} + 90$, $p_5 = p_1^{\hat{4}} + 120$, $p_6 = p_1^{\hat{4}} + 150$, where $\hat{T}$ is a prime greater than 5.

If $J_2(\omega) \ne 0$, then there are infinitely many 6-tuples of primes.

Let $\hat{T} = 1$, we have

$$J_2(\omega) = \sum_{\alpha=1}^{\phi(\omega)} \left[\frac{5}{\sum_{i=1}^{5}(p_\alpha^4 + 30i, \omega)} \right] = 8 \prod_{7 \le p \le p_i} \left(p - 6 - \sum_{i-1}^{5} \chi_i(p) \right) \ne 0.$$

where $\chi_i(p) = 3$ if $B_i^{\frac{p-1}{4}} \equiv (-1)^{\frac{p-1}{4}} (\bmod p)$, $\chi(p) = -1$ if $B_i^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{4}} (\bmod p)$, $B_1 = 30$, $B_2 = 60$, $B_3 = 90$, $B_4 = 120$, $B_5 = 150$; $\chi_i(p) = (-\frac{B_i}{p})$ otherwise,

$$\pi_6(N,2) = \frac{J_2(\omega)\omega^5}{1024\phi^6(\omega)} \frac{N}{\log^6 N}(1 + O(1)).$$

Theorem 3.1.33. $p_3 = p_1 * p_2 + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.

We have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(\hat{T}p_{\alpha_1}p_{\alpha_2} + b, \omega)} \right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|\hat{T}b} \frac{p-1}{p-2} \ne 0,$$

$$\pi_2(N,3) = \prod_{3 \le p \le p_i} (1 - \frac{1}{(p-1)^2} \prod_{p|\hat{T}b} \frac{p-1}{p-2} \frac{N^2}{\log^3 N}(1 + O(1)).$$

This is three primes theorem called (1+2). It is the best asymptotic formula.

Theorem 3.1.34. $p_3 = N - p_1 * p_2$, $J_3(\omega) = 0$ if $(N, \hat{T}) > 1$; $J_3(\omega) \ne 0$ if $(N, \hat{T}) = 1$.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N - p_{\alpha_1}p_{\alpha_2}, \omega)} \right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|N} \frac{p-1}{p-2} \ne 0,$$

$$\pi_2(N,3) = |\{p_1, p_2 : p_1, p_2 \le N, p_3 = |N - p_1 p_2|\}|$$
$$= \prod_{3\le p\le p_i}\left(1 - \frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N^2}{\log^3 N}(1+O(1)).$$

This is three primes theorem called (1+2).

Theorem 3.1.35. $p_3 = (p_1+p_2)^{\hat{2}} + b$, where b is an odd and $3 \nmid (\hat{T}+b)$.
We have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(\hat{T}(p_{\alpha_1}+p_{\alpha_2})^2 + b, \omega)}\right] = \prod_{3\le p\le p_i}(p^2 - 3p + 3 - \chi(p)) \ne 0,$$

where $\chi(p) = (p-2)$ if $p|\hat{T}b$; $\chi(p) = (p-2)(\frac{-b\hat{T}}{p})$ otherwise,

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.36. $p_3 = (p_1+p_2)^{\hat{3}} + 3$, where $3 \nmid \hat{T}$.

Since $J_3(\omega) \ne 0$, there exist infinitely many primes p_1 and p_2 such that p_3 is also a prime.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{((p_{\alpha_1}+p_{\alpha_2})^3 + 3, \omega)}\right] = \prod_{3\le p\le p_i}(p^2 - 3p + 3 - \chi(p)) \ne 0,$$

where $\chi(p) = 2(p-2)$ if $3^{\frac{p-1}{3}} \equiv 1 (\bmod p)$; $\chi(p) = -(p-2)$ if $3^{\frac{p-1}{3}} \not\equiv 1 (\bmod p)$; $\chi(p) =$ 0 otherwise,

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{6\phi^3(\omega)}\frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.37. $p_3 = (p_1+p_2)^{\hat{4}} + 1$.

If $J_3(\omega) = 0$, then there are finitely many prime solutions. If $J_3(\omega) \ne 0$, then there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{((p_{\alpha_1}+p_{\alpha_2})^4 + 1, \omega)}\right] = \prod_{3\le p\le p_i}(p^2 - 3p + 3 - \chi(p)) \ne 0,$$

where $\chi(p) = 3(p-2)$ if $p \equiv 1 (\bmod 8)$; $\chi(p) = -(p-2)$ if $p \not\equiv 1 \pmod 8$.

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{8\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.38. $p_3 = p_1 * p_2^{\hat{2}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.
Since $J_3(\omega) \neq 0$ there exist infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^2 + b, \omega)}\right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{4\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.39. $p_3 = p_1 * p_2^{\hat{3}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^3 + b, \omega)}\right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{6\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.40. $p_3 = p_1 * p_2^{\hat{4}} + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.
Since $J_3(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_3(\omega) = \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^4 + b, \omega)}\right] = \phi(\omega) \prod_{3 \le p \le p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N,3) = \frac{J_3(\omega)\omega}{8\phi^3(\omega)} \frac{N^2}{\log^3 N}(1+O(1)).$$

Theorem 3.1.41. $p_4 = p_1 * p_2 * p_3 + b$, where $(\hat{T}, b) = 1$ and $2|\hat{T}b$.
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2} p_{\alpha_3} + b, \omega)}\right]$$

$$= \phi^2(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N, 4) = \frac{1}{3} \prod_{3 \leq p \leq p_i} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|b} \frac{p-1}{p-2} \frac{N^3}{\log^4 N}(1 + O(1)).$$

It is the four primes theorem called (1+3). It is the best asymptotic formula.

Theorem 3.1.42. $p_4 = N - p_1 * p_2 * p_3$.
$J_4(\omega) = 0$ if $(N, \hat{T}) > 1$; $J_4(\omega) \neq 0$ if $(N, \hat{T}) = 1$.
Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(N - p_{\alpha_1} p_{\alpha_2} p_{\alpha_3}, \omega)}\right]$$

$$= \phi^2(\omega) \prod_{3 \leq p \leq p_i} (p-2) \prod_{p|b} \frac{p-1}{p-2} \neq 0,$$

$$\pi_2(N, 4) = |\{p_1, p_2, p_3 : p_1, p_2, p_3 \leq N, p_4 = |N - p_1 p_2 p_3|\}|$$

$$= \frac{1}{3} \prod_{3 \leq p \leq p_i} (1 - \frac{1}{(p-1)^2}) \prod_{p|N} \frac{p-1}{p-2} \frac{N^3}{\log^4 N}(1 + O(1)).$$

It is the four primes theorem called (1+3). It is the best asymptotic formula.

Theorem 3.1.43. $p_4 = p_1 * p_2 + p_3 + b$, where $\hat{T}$ and b are both odds or both evens.
Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2} + p_{\alpha_3} + b, \omega)}\right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = p - 2$ if $p|b$; $\chi(p) = -1$ if $p \not| b$.

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{6\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.44. $p_4 = p_1 * p_2^2 + p_3 + b$, where $\hat{T}$ and b are both odds or both evens.

Since $J_4(\omega) \neq 0$, there are infinitely many prime solutions.

Let $\hat{T} = 1$, we have

$$J_4(\omega) = \sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{(p_{\alpha_1} p_{\alpha_2}^2 + p_{\alpha_3} + b, \omega)} \right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = p - 2$ if $p|b$; $\chi(p) = -1$ if $p \not| b$.

$$\pi_2(N,4) = \frac{J_4(\omega)\omega}{12\phi^4(\omega)} \frac{N^3}{\log^4 N}(1 + O(1)).$$

Theorem 3.1.45. $p_4 = (p_1 + p_2 + p_3 - 1)^2 + 1,\ p_5 = (p_1 + p_2 + p_3 + 1)^2 + 1.$

Let $\hat{T} = 1$, we have

$$J_4(\omega) =$$

$$\sum_{\alpha_3=1}^{\phi(\omega)} \sum_{\alpha_2=1}^{\phi(\omega)} \sum_{\alpha_1=1}^{\phi(\omega)} \left[\frac{1}{((p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}-1)^2+1,\omega)((p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+1)^2+1,\omega)} \right]$$

$$= \prod_{3 \le p \le p_i} \left(\frac{(p-1)^4 - 1}{p} - \chi(p) \right) \neq 0,$$

where $\chi(p) = (1 + 2(-1)^{\frac{p-1}{2}})(p^2 - 3p + 3)$,

$$\pi_3(N,4) = \frac{J_4(\omega)\omega^2}{4\phi^5(\omega)} \frac{N^3}{\log^5 N}(1 + O(1)).$$

Let $\hat{T}=4$, we have

$$J_4(\omega)=$$

$$\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(4(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}-1)^2+1,\omega)(4(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}+1)^2+1,\omega)}\right]$$

$$=\prod_{3\le p\le p_i}\left(\frac{(p-1)^4-1}{p}-\chi(p)\right)\neq 0,$$

where $\chi(5)=25,\ \chi(p)=(1+2(-1)^{\frac{p-1}{2}})(p^2-3p+3)$,

$$\pi_3(N,4)=\frac{J_4(\omega)\omega^2}{4\phi^5(\omega)}\frac{N^3}{\log^5 N}(1+O(1)).$$

Since $J_4(\omega)\neq 0$, there exist infinitely many prime solutions.

Theorem 3.1.46. $p_5=p_1*p_2*p_3*p_4+b$, where $(\hat{T},b)=1$ and $2|\hat{T}b$. Since $J_5(\omega)\neq 0$, there are infinitely many prime solutions.
Let $\hat{T}=1$, we have

$$J_5(\omega)=\sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}p_{\alpha_4}+b,\omega)}\right]$$

$$=\phi^3(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|b}\frac{p-1}{p-2}\neq 0,$$

$$\pi_2(N,5)=\frac{J_5(\omega)\omega}{24\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

It is the five primes theorem called (1+4). It is the best asymptotic formula.

Theorem 3.1.47. $p_5=N-p_1*p_2*p_3*p_4$.
$J_5(\omega)=0$ if $(\hat{T},N)>1$; $J_5(\omega)\neq 0$ if $(\hat{T},N)=1$.
Let $\hat{T}=1$, we have

$$J_5(\omega)=\sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(N-p_{\alpha_1}p_{\alpha_2}p_{\alpha_3}p_{\alpha_4},\omega)}\right]$$

$$=\phi^3(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|N}\frac{p-1}{p-2}\neq 0,$$

$$\pi_2(N,5) = |\{p_1,p_2,p_3,p_4 : p_1,p_2,p_3,p_4 \le N, p_5 = |N - p_1p_2p_3p_4|\}|$$
$$= \frac{1}{12}\prod_{3\le p\le p_i}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p|N}\frac{p-1}{p-2}\frac{N^4}{\log^5 N}(1+O(1)).$$

It is the five primes theorem called (1+4). It is the best asymptotic formula.

Theorem 3.1.48. $p_5 = p_1 + p_2 + p_3^{\hat{2}} * p_4^{\hat{3}}$, where $\hat{T}$ is an odd.
Since $J_5(\omega) \ne 0$ there are infinitely many prime solutions.
Let $\hat{T} = 1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}^2p_{\alpha_4}^3,\omega)}\right]$$
$$= \prod_{3\le p\le p_i}\left(\frac{(p-1)^5+1}{p}+p-2\right) \ne 0.$$
$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{144\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.49. $p_5 = p_1 + p_2 + p_3^{\hat{2}} * p_4^{\hat{3}} + b$, where b is an even and $\hat{T}$ is an odd.
Let $\hat{T} = 1$, we have

$$J_5(\omega) = \sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(p_{\alpha_1}+p_{\alpha_2}+p_{\alpha_3}^2p_{\alpha_4}^3+b,\omega)}\right]$$
$$= \prod_{3\le p\le p_i}\left(\frac{(p-1)^5+1}{p}-\chi(p)\right) \ne 0.$$

where $\chi(p) = 1$ if $p \nmid b$; $\chi(p) = -(p-2)$ if $p|b$.

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{144\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.50. $p_5 = p_1 + p_2 * (p_3 + p_4) + 2$.
Since $J_5(\omega) \ne 0$, there are infinitely many prime solutions.

Let $\hat{T}=1$, we have

$$J_5(\omega)=\sum_{\alpha_4=1}^{\phi(\omega)}\sum_{\alpha_3=1}^{\phi(\omega)}\sum_{\alpha_2=1}^{\phi(\omega)}\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(p_{\alpha_1}+p_{\alpha_2}(p_{\alpha_3}+p_{\alpha_4})+2,\omega)}\right]$$

$$=\prod_{3\le p\le p_i}\left(\frac{(p-1)^5+1}{p}-1\right)\neq 0.$$

$$\pi_2(N,5)=\frac{J_5(\omega)\omega}{24\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+O(1)).$$

Theorem 3.1.51. $p_n=p_1*p_2*\cdots*p_{n-1}+b$, where $(b,\hat{T})=1$ and $2|b\hat{T}$. We have

$$J_n(\omega)=\sum_{\alpha_{n-1}=1}^{\phi(\omega)}\cdots\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(\hat{T}^{n-2}p_{\alpha_1}\cdots p_{\alpha_{n-1}}+b,\omega)}\right]$$

$$=\phi^{n-2}(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|\hat{T}b}\frac{p-1}{p-2}\neq 0,$$

$$\pi_2(N,n)=\frac{2}{(n-1)!}\prod_{3\le p\le p_i}(1-\frac{1}{(p-1)^2})\prod_{p|\hat{T}b}\frac{p-1}{p-2}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

Let $n=2$, it is the prime twins theorem.

Theorem 3.1.52. $p_n=N-p_1*p_2*\cdots*p_{n-1}$.
$J_n(\omega)=0$ if $(N,\hat{T})>1$; $J_n(\omega)\neq 0$ if $(N,\hat{T})=1$.
Let $\hat{T}=1$, we have

$$J_n(\omega)=\sum_{\alpha_{n-1}=1}^{\phi(\omega)}\cdots\sum_{\alpha_1=1}^{\phi(\omega)}\left[\frac{1}{(N-p_{\alpha_1}\cdots p_{\alpha_{n-1}},\omega)}\right]$$

$$=\phi^{n-2}(\omega)\prod_{3\le p\le p_i}(p-2)\prod_{p|N}\frac{p-1}{p-2}\neq 0,$$

$$\pi_2(N,n)=|\{p_1,\cdots,p_{n-1}:p_1,\cdots,p_{n-1}\le N,p_n=|N-p_1\cdots p_{n-1}|\}|$$

$$=\frac{2}{(n-1)!}\prod_{3\le p\le p_i}(1-\frac{1}{(p-1)^2})\prod_{p|N}\frac{p-1}{p-2}\frac{N^{n-1}}{\log^n N}(1+O(1)).$$

It is the Rényi's theorem. Let $n = 2$, it is the Goldbach's theorem, see theorem 3.1.2.

Note. All sieve methods obtain only the upper estimates, but the lower estimates are more difficult[7]. Unfortunately, it turns out there is no method which will give such a formula for general sifting function. $J_n(\omega)$ is a precise sifting function. The Santilli's isoadditive prime theory will take the place of all sieve methods.

4. A Disproof of the Riemann's Hypothesis[8]

Let $s = \sigma + ti$, where σ and t are real, $i = \sqrt{-1}$. We have the Riemann's zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}, \tag{4.1}$$

where p ranges over all primes.

In 1859 Riemann [9] stated that nontrivial zeros of $\zeta(s)$ all lie on the line $\sigma = 1/2$ called the Riemann's hypothesis. In 1990 Hilbert listed the problem of proving or disproving the Riemann's hypothesis as one of the most important problems confronting twentieth century mathematicians. To this day it remains unsolved. The arithmetic (sifting) function $J_n(\omega)$ is able to take the place of the Riemann's hypothesis and the generalized Riemann's hypothesis.

Theorem 4.1. For $|\zeta(s)| = 0$, we have

$$N(0.5 + ti) < N(0.4 + ti) < N(0.3 + ti), \tag{4.2}$$

where $N(0.5 + ti)$ denotes the number of nontrivial zeros of $\zeta(0.5 + ti)$ with $t > 0$, $N(0.4+ti)$ the number of nontrivial zeros of $\zeta(0.4+ti)$ with $t > 0$, $N(0.3+ti)$. the number of nontrivial zeros of $\zeta(0.3 + ti)$ with $t > 0$. The nontrivial zeros of the Riemann's zeta function $\zeta(s)$ are independent of the real part $\sigma = 1/2$, but may well depend on the imaginary part t.

Proof. From (1) we have

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}) = Re^{\theta i}, \tag{4.3}$$

where

$$R = \prod_p R_p, \quad R_p = \sqrt{1 - \frac{2\cos(t \log p)}{p^\sigma} + \frac{1}{p^{2\sigma}}}, \tag{4.4}$$

$$\theta = \sum_p \theta_p, \quad \theta_p = \tan^{-1} \frac{\sin(t \log p)}{p^\sigma - \cos(t \log p)}. \tag{4.5}$$

$\zeta(s) = 0$ if and only if Re $\zeta(s) = 0$ and Im $\zeta(s) = 0$, that is $R = \infty$. From (4.4) we have that if $\cos(t\log p) \leq 0$ then $R_p > 1$ and if $\cos(t\log p) > 0$ then $R_p < 1$.

If

$$\cos(t \log p_1) > 0, \ \cdots, \ \cos(t \log p_2) > 0, \tag{4.6}$$

we have $p_2 < e^{\frac{\pi}{t}} p_1$ and $t \log p_2 < t \log p_1 + \pi$.

If

$$\cos(t\log p_1) < 0, \ \cdots, \ \cos(t\log p_2) < 0, \tag{4.7}$$

we have $p_2 < e^{\frac{\pi}{t}} p_1$ and $t\log p_2 < t\log p_1 + \pi$.

$\cos(t\log p)$ is independent of the real part σ, but may well depend on prime p and imaginary part t. We write $m_+(t)$ for the number of primes p satisfying $\cos(t\log p) > 0$, $m_-(t)$ for the number of primes p satisfying $\cos(t\log p) \leq 0$.

For $\cos(t\log p) > 0$ we have

$$1 > R_p(1 + ti) > R_p(0.5 + ti). \tag{4.8}$$

If $m_+(t_1)$ is much greater than $m_-(t_1)$ such that $R(0.5 + t_1 i)$=min. From (4.4), (4.5) and (4.8) we have for given t_1

$$\min R(\sigma_1 + t_1 i) > \min R(1 + t_1 i) > \min R(0.5 + t_1 i) > \min R(\sigma_2 + t_1 i) \to 0, \tag{4.9}$$

$$\theta(\sigma_1 + t_1 i) = \theta(1 + t_1 i) = \theta(0.5 + t_1 i) = \theta(\sigma_2 + t_1 i) = \text{const}, \tag{4.10}$$

where $\sigma_1 > 1$ and $0 < \sigma_2 < 0.5$.

Since $|\zeta(s)| = \frac{1}{R}$ from (4.9) we have

$$\max|\zeta(\sigma_1 + t_1 i)| < \max|\zeta(1 + t_1 i)| < \max|\zeta(0.5 + t_1 i)| < \max|\zeta(\sigma_2 + t_1 i)| \to \infty. \tag{4.11}$$

For $\cos(t\log p) < 0$ we have

$$1 < R_p(0.5 + ti) < R_p(0.4 + ti) < R_p(0.3 + ti). \tag{4.12}$$

If $m_-(t_1)$ is much greater than $m_+(t_1)$ such that $R(0.5 + t_1 i) = \max$.

From (4.4), (4.5) and (4.12) we have for given t_1

$$\max R(\sigma_1 + t_1 i) < \max R(0.5 + t_1 i) < \max R(0.4 + t_1 i)$$

$$< \max R(0.3 + t_1 i) < \max R(\sigma_2 + t_1 i) \to \infty, \tag{4.13}$$

$$\theta(\sigma_1+t_1 i) = \theta(0.5+t_1 i) = \theta(0.4+t_1 i) = \theta(0.3+t_1 i) = \theta(\sigma_2+t_1 i) = \text{const}, \tag{4.14}$$

where $\sigma_1 > 0.5$ and $0 < \sigma_2 < 0.3$.

Since $|\zeta(s)| = \frac{1}{R}$ from (4.13) we have

$$\min|\zeta(\sigma_1 + t_1 i)| > \min|\zeta(0.5 + t_1 i)| > \min|\zeta(0.4 + t_1 i)|$$

$$> \min|\zeta(0.3 + t_1 i)| > \min|\zeta(\sigma_2 + t_1 i)| \to 0. \tag{4.15}$$

From (4.15) we have that if $|\zeta(0.5 + t_1 i)| = 0$ then $|\zeta(0.4 + t_1 i)| = 0$ and if $|\zeta(0.4 + t_2 i)| = 0$ then $|\zeta(0.5 + t_2 i)| = 0$ or $|\zeta(0.5 + t_2 i)| \neq 0$. Therefore we have $N(0.4 + ti) > N(0.5 + ti)$. If $|\zeta(0.4 + t_1 i)| = 0$ then $|\zeta(0.3 + t_1 i)| = 0$ and if $|\zeta(0.3 + t_2 i)| = 0$ then $|\zeta(0.4 + t_2 i)| = 0$ or $|\zeta(0.4 + t_2 i)| \neq 0$. Therefore we have $N(0.3 + ti) > N(0.4 + ti)$. Q.E.D.

Corollary 4.1. If $|\zeta(\sigma_1 + t_1 i)| = \min$, where $\sigma_1 \geq 1$, then $|\zeta(0.5 + t_1 i)| = \min$. If $|\zeta(\sigma_1 + t_1 i)| \neq \min$ then $|\zeta(0.5 + t_1 i)| \neq \min$. For example, we study the nontrivial zeros of $\zeta(0.5 + t_1 i)$, where t_1=43.327073, 60.831779, 75.704691, 79.337375, 88.809111, 92.491899 and 95.870634 [10]. Since $|\zeta(1+t_1 i)| \neq \min$ [10], $|\zeta(0.5 + t_1 i)| \neq \min$, that is $|\zeta(0.5 + t_1 i)| \neq 0$. Both $\zeta(1 + ti)$ and $\zeta(0.5 + ti)$ have the same geometrical diagram [8]. In studying $\zeta(1 + ti)$ we follow that there are the finite nontrivial zeros of $\zeta(s)$ on the critical line $\sigma = 1/2$. Since $|\zeta(1 + 14.134725i)| \sim 0.3 < |\zeta(1 + ti)|$ with $t > 15$ [10], by using the first zero of $|\zeta(0.5 + 14.134725i)| = 0$ we study the nontrivial zeros of $\zeta(s)$. For example, if $|\zeta(\sigma_1 + 14.134725i)| \neq 0$, where $0.5 < \sigma_1 < 1$, then $|\zeta(\sigma_1 + ti)| \neq 0$ with $t > 0$ and if $|\zeta(\sigma_1 + 14.134725i)| = 0$, then $|\zeta(\sigma_1 + ti)|$ with $t > 0$ has the finite nontrivial zeros.

Using above methhod we may also disprove the generalized Riemann's hypothesis that nontrivial zeros of L-function all lie on the line $\sigma = 1/2$. Note. It was proved that $\zeta(1 + ti) \neq 0$ by employing the trigonometric inequality $3 + 4\cos\theta + \cos 2\theta \geq 0$. We disprove the Riemann's hypothesis and the generalized Riemann's hypothesis by using the trigonometric functions $\cos(t \log p) \leq 0$. We define [9]

$$C(\sigma + Ti) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}, \tag{4.16}$$

where $C(\sigma + Ti)$ is the number of the cycles of $\zeta(s)$ in $0 < \sigma, 0 < t < T$, or the number of $|\zeta(s)| = \max$ see (4.11) or the number of $|\zeta(s)| = \min$ see (4.15).

We have

$$C(0.3+Ti)=C(0.5+Ti)=C(10+Ti)=C(\sigma+Ti). \tag{4.17}$$

Brent[11] gave a computation which shows that the Riemann zeta function $\zeta(s)$ has exactly 75,000,000 cycles with the zeros and nonzeros in the region $0 < t < 32,585,736$ and $\sigma = 0.5$.

Achnowledgement. I have no words to express my deepest appreciation to Prof. R.M.Santilli for invaluable critical suggestions and encouragements in Beijing.

References

[1] R. M. Santilli, "Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isodual and pseudoduals, and 'hidden numbers' of dimension 3, 5, 6, 7." Algebras, Groups and Geometries, **10**, 273-322(1993).

[2] R. M. Santilli, *Elements of Hadronic Mechanics*, Vol.I: Mathematical Foundations, Ukraine Academy of Sciences, Kiev(1995).

[3] Chum-xuan Jiang, "Foundations of Santilli's isonumber theory, Part I: The Isonumber Theory of the First Kind". Algebras, Groups and Geometries, **15**, (1998).

[4] Chun-xuan Jiang, "On the Fermat-Santilli isotheorem". Ibid.

[5] Chun-xuan Jiang, "On the Yu-Goldbach prime theorem". Guangxi Sciences (in Chinese), **3**, 9-12 (1996).

[6] Chun-xuan Jiang, "On the k-tuples of primes in a table of primes", Preprint (1996).

[7] H. Halberstam and H. -E. Richert, *Sieve Methods*, Academic Press, London (1974).

[8] Chun-xuan Jiang, "The study of the Riemann's zeta function", unpublished, Oct. 1997.

[9] B. Riemann, "Über die Anzahl der Primzahlen unter einer gegebener Grösse". Monatsber. Akad. Berlin, 671-680(1859).

[10] C.B. Haslgrove, *Tables of the Riemann zeta function*, Roy. Soc. Math. Tables, Vol. 6, Cambridge Univ. Press, London and New York, 1960.

[11] Richard P. Brent, "On the zeros of the Riemann zeta function in the critical strip",Math. Comp.,**33**, 1361-1372(1979).

INFINITESIMAL MOTIONS ON SANTILLI-SOURLAS-TSAGAS ISOMANIFOLDS

Recep Aslander
İnönü Üniversitesi
Egitim Fakültesi
Matematik Egitimi Bölümü
44100 Malatya, Turkey
and
Sadik Keles
İnönü Üniversitesi
Fen-Edebiyat Fakültesi
Matematik Bölümü
44100 Malatya, Turkey

Received March 12, 1998
Revised November 15, 1998

Abstract

In this paper we study the axiom-preserving isotopies of fields, metric spaces, algebras and groups first constructed by R. M. Santilli [in ref.[2] (and then studied in more details in ref.[3,4,5]), as well as the isotopies of manifolds first constructed by Tsagas and Sourlas in ref.[6,7] via a conventional topology and isodifferential calculus, and then formulated via an integro-differential isotopology and isodifferential calculus by Santilli in the recent memoir[5]. In particular, in this paper we generalize the infinitesimal motions on Riemannian manifolds to isoriemannian manifolds. The significance of the generalization is pointed out.

1. Background notions on isotopies

The main idea of the isotopies studied by R. M. Santilli [2-5] is the lifting of the trivial n-dimensional unit I=diag(1, 1, ... , 1) of a conventional theory into a nowhere singular, symmetric, real-valued, positive-definite and n×n invertible matrix $\hat{I} = (\hat{I}_i{}^j) = \hat{T}^{-1} = (\hat{T}_i{}^j), i,j = 1,2, \dots, n,$ whose components have a smooth functional dependence on the local coordinates x, their derivatives $\dot{x}, \ddot{x}, \dots$, with respect to an independent variable t and any need additional local quantity,

$$I \rightarrow \hat{I} \doteq \hat{I}(t, x, \dot{x}, \ddot{x}, \dots), \qquad \dot{x} = \frac{dx}{dt} \text{ and so on.} \tag{1.1}$$

Since the new and old structures are indistinguishable at the abstract, realization-free level by construction, the lifting is a particular form of isotopy.

The fundamental isotopies are those of real field R. Let R= R$(x, +, .)$ be the field of real numbers x.

Definition 1.1[3]: "*Santilli's isofields*" $\hat{R} = R(\hat{x}, +, *)$ are rings with elements $\hat{x} = x\hat{I}$, called "*isonumbers*" where $x \in R$ and $\hat{I}$ is a positive-definite element generally outside R, equipped with two internal operations + and $*$, where + is definite by the conventional sum of R as follows:

$$+ : (\hat{x}, \hat{y}) \rightarrow \hat{x} + \hat{y} = (x + y)\hat{I} \tag{1.2}$$

and $*$ is a new multiplication definite by

$$* : (\hat{x}, \hat{y}) \rightarrow \hat{x} * \hat{y} = \hat{x}\, \mathrm{T}\, \hat{y}, \qquad \hat{\mathrm{T}}^{-1} = \hat{I} \tag{1.3}$$

called "*isomultiplication*", which is such that $\hat{I}$ is the left and right unit of $\hat{R}$,

$$\hat{I} * \hat{x} = \hat{x} * \hat{I} = \hat{x}, \qquad \forall \hat{x} \in \hat{R}, \tag{1.4}$$

called "*isounit*". Under these assumptions $\hat{R}$ is a field, i.e., it satisfies all properties of R in their isotopic form:

1. The set $\hat{R}$ is closed under addition, $\hat{x} + \hat{y} \in \hat{R}$, $\forall \hat{x}, \hat{y} \in \hat{R}$,
2. The addition is associative, $\hat{x} + (\hat{y} + \hat{z}) = (\hat{x} + \hat{y}) + \hat{z}$, $\forall \hat{x}, \hat{y}, \hat{z} \in \hat{R}$,
3. There is an element $\hat{0} = 0$, called "*additive unit*" such that

$\hat{x} + \hat{0} = \hat{0} + \hat{x} = \hat{x}, \forall \hat{x} \in \hat{R}$,

4. For each element $\hat{x} \in \hat{R}$, there is an element $-\hat{x} \in \hat{R}$, called the "*opposite of* $\hat{x}$" which is such that $\hat{x} + (-\hat{x}) = \hat{0}$,
5. The addition is commutative $\hat{x} + \hat{y} = \hat{y} + \hat{x}, \quad \forall \hat{x}, \hat{y} \in \hat{R}$,
6. The set $I\hat{R}$ is closed under isomultiplication, $\hat{x} * \hat{y} \in I\hat{R}, \forall \hat{x}, \hat{y} \in \hat{R}$,
7. The isomultiplication is generally non-isocommutative $\hat{x} * \hat{y} \neq \hat{y} * \hat{x}$, but isoassociative, $\hat{x} * (\hat{y} * \hat{z}) = (\hat{x} * \hat{y}) * \hat{z}, \quad \forall \hat{x}, \hat{y}, \hat{z} \in \hat{R}$,
8. The quantity $\hat{I}$ in factorization $\hat{x} = x\hat{I}$ is the "*multiplicative isounit*" of $\hat{R}$ as per Eq.s (1.4),
9. For each element $\hat{x} \in \hat{R}$, there is an element $\hat{x}^{-\hat{I}} \in \hat{R}$, called the "*isoinverse of* $\hat{x}$", which is such that $\hat{x} * \hat{x}^{-\hat{I}} = \hat{x}^{-\hat{I}} * \hat{x} = \hat{I}$,
10. All elements $\hat{x}, \hat{y}, \hat{z} \in \hat{R}$ verify the right and left "*distributive laws*"

$$\hat{x} * (\hat{y} + \hat{z}) = \hat{x} * \hat{y} + \hat{x} * \hat{z}, \quad (\hat{x} + \hat{y}) * \hat{z} = \hat{x} * \hat{z} + \hat{y} * \hat{z} \tag{1.5}$$

We therefore have the isofield $\hat{R}$ isoreal numbers $\hat{x}$. Since $\hat{R}$ preserves by construction all axioms of R, it is called an *isotope* of R and the lifting $R \to \hat{R}$ is called an *isotopy*. All conventional operations dependent on the multiplication on IR are generalized on $\hat{R}$, thus yielding isotopies of powers, quotients, square roots, etc. These isotopic operations are however such that $\hat{I}$ preserves all the original axiomatic properties of I, i.e.,

$$\hat{I}^{\hat{n}} = \underbrace{\hat{I} * \hat{I} * \cdots * \hat{I}}_{n-times} = \hat{I}, \quad \hat{I}^{\frac{\hat{1}}{2}} = \hat{I}, \quad \hat{I} \hat{/} \hat{I} = \hat{I} \quad \text{etc.} \tag{1.6}$$

The mathematical and physically most important implication of isofields is that they imply, for evident consistency, corresponding isotopies of all quantities definite over conventional fields. The second significant application of the isotopies is the lifting of the conventional vector and metric spaces, first presented in paper [2]. Let $E^n(x, \delta, R)$ be the n-dimensional Euclidean space, with local chart $x = (x^k)$, $k = 1, 2, \ldots, n$, and n-dimensional metric tensor $\delta = (\delta_{ij}) = diag(1, 1, \ldots, 1)$, and distance between two points $x, y \in E^n$

$$(x-y)^2 := (x^i - y^i)\delta_{ij}(x^j - y^j) \in R \tag{1.7}$$

over the real field R, where the convention on the sum of repeated indices is assumed hereon.

Definition 1.2[2]: "*Santilli's isoeuclidean spaces*" $\hat{E}^n(\hat{x},\hat{\delta},\hat{R})$ are n-dimensional metric spaces defined over an isoreal isofield $\hat{R} = \hat{R}(\hat{x},+,*)$ with an n-dimensional isounit $\hat{I}$, equipped with the "*isometric*"

$$\hat{\delta} = \left(\hat{\delta}_{ij}\right) = \hat{T}\left(t,x,\dot{x},\ddot{x},\ldots\right).\delta, \quad \hat{T}^{-1} = \hat{I}, \tag{1.8}$$

where δ is the conventional Euclidean metric, local chart in contravariant and covariant forms

$$\hat{x} = \left(\hat{x}^k\right) = \left(x^k \hat{I}\right), \quad \hat{x}_k = \hat{\delta}_{ki}\hat{x}^i = \hat{T}_k^{\ j}\delta_{ji}x^i\hat{I}, \quad x^k \in R, \tag{1.9}$$

and "*isoseparation*" among two points $\hat{x},\hat{y} \in \hat{E}^n$

$$(\hat{x}-\hat{y})^{\hat{2}} = \left[\left(\hat{x}^i - \hat{y}^i\right)\hat{\delta}_{ij}\left(\hat{x}^j - \hat{y}^j\right)\right]\hat{I} \in \hat{R} \tag{1.10}$$

The "*isoeuclidean geometry*" [4] is the geometry of the isoeuclidean spaces. The same apply for the definition of isominkowskian, isoriemannian and other isospaces and of related geometries.

2. Isomanifolds

The notion of an n-dimensional isomanifold was first studied by Tsagas and Sourlas [6,7]. Their study is referent to $M(\hat{R})$, rather than to $\hat{M}(\hat{R})$ because of the use of the conventional topology τ (i.e. a topology with the conventional n-dimensional unit I). The extension to $\hat{M}(\hat{R})$ of $M(\hat{R})$ with the isotopology $\hat{\Im}$ is introduced first time by R. M. Santilli [5]. For all additional aspects of isomanifolds and related topological properties we referred the interested reader to R. M. Santilli [5].

Let M be an n-dimensional manifold with local coordinates $x=(x^k)$ over the real field R. Let $\hat{M}$ denote the isotopic images of M with local coordinates $\hat{x}=(\hat{x}^k)$ over the isoreal isofield $\hat{R}$ with isounit $\hat{I}$. For simplicity we shall ignore the isounit in the definition $\hat{x}=x.\hat{I}$ because it cancels out in the isomultiplication with any quantity Q, $\hat{x}*Q=x.\hat{T}^{-1}\hat{T}Q\equiv xQ$. The isodifferential calculus on $\hat{M}$ is defined as an isotopic lifting of the conventional differential calculus on M under the condition of preserving the original axioms and properties of the ordinary differential calculus on M.

Definition 2.1[5]: The "*first order isodifferential*" of the local isocoordinates $\hat{x}^k$ on an isomanifold $\hat{M}$ of dimension n are given by:

$$\hat{d}\hat{x}^k = \hat{I}^k{}_i(t,x,\dot{x},\ddot{x},\dots)dx^i, \tag{2.1}$$

where the expression $\hat{d}\hat{x}^k$ are defined on $\hat{M}$ while the corresponding expression $\hat{I}^k{}_i dx^i$ are the projection on M. Let $\hat{f}(\hat{x})$ be a sufficiently smooth isofunction of isocoordinates $\hat{x}^k$ on $\hat{M}$. Then the "*isoderivative*" at the point $Q=(\hat{q}^k)\in\hat{M}$ is given by

$$\hat{f}'(\hat{q}^k)=\left.\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}\right|_{\hat{x}^k=\hat{q}^k}=\hat{T}_k{}^i\left.\frac{\partial f(x)}{\partial x^i}\right|_{\hat{x}^k=\hat{q}^k}=\lim_{\hat{d}\hat{x}^k\to\hat{0}^k}\frac{\hat{f}(\hat{q}^k+\hat{d}\hat{x}^k)-\hat{f}(\hat{q}^k)}{\hat{d}\hat{x}^k}$$

where $\frac{\hat{\partial}\hat{f}(\hat{x})}{\hat{\partial}\hat{x}^k}$ is computed on $\hat{M}$ and $\hat{T}_k{}^i\frac{\partial f(x)}{\partial x^i}$ is the projection on M. An isofunction $\hat{f}(\hat{x})$ is said to be $\hat{C}^\infty$ at $\hat{x}$ when it is ∞ - isodifferentiable at $\hat{x}$.

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various properties of the conventional differential calculus. We here mention for brevity only the following isotopic properties. The "*isodifferentials*" of an isofunction of isocoordinates $\hat{x}^k$ on $\hat{M}$ are defined according to the respective rules

$$\hat{d}\hat{f}(\hat{x}^k) = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k} * \hat{d}\hat{x}^k \equiv \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\hat{\mathrm{T}}^k{}_l\hat{d}\hat{x}^l$$

$$= \hat{\mathrm{T}}_k{}^i\frac{\partial f}{\partial x^i}\hat{\mathrm{T}}^k{}_l\hat{\mathrm{I}}^l{}_j dx^j = \hat{\mathrm{T}}_k{}^i\frac{\partial f}{\partial x^i}dx^k = \frac{\partial f}{\partial x^i}\hat{\mathrm{T}}^i{}_k dx^k ,$$

where the last expression originate from the fact that the contraction is in isomanifolds.

The moutrinality of the isodifferential calculus illustrated by the fact that isoderivatives commute on $\hat{M}$, $\hat{\partial}_i\hat{\partial}_j = \hat{\partial}_j\hat{\partial}_i$, but not in their projection on M, $\hat{\partial}_i\hat{\partial}_j = \mathrm{T}_i{}^r\partial_r\mathrm{T}_j{}^s\partial_s \neq \mathrm{T}_j{}^s\partial_s\mathrm{T}_i{}^r\partial_r = \hat{\partial}_j\hat{\partial}_i$.

3. Isovector Fields on Isomanifolds

Let $\hat{M}$ be an isodifferentiable Santilli- Sourlas -Tsagas isomanifold of dimension n over the isoreal numbers $\hat{R}$ with n-dimensional isounit $\hat{\mathrm{I}} = \left(\hat{\mathrm{I}}_i^j\right)$. Let $\hat{D}^o\left(\hat{M}\right)$ be the isoalgebra of all isodifferentiable isofunctions on $\hat{M}$. Every isoderivation on $\hat{D}^o\left(\hat{M}\right)$ is called "*isovector field*" on $\hat{M}$. If $\hat{X}$ is an isovector field on $\hat{M}$, then $\hat{X}$ has the following properties:

a. $\hat{X}:\hat{D}^o\left(\hat{M}\right) \to \hat{D}^o\left(\hat{M}\right), \quad \hat{X}:\hat{f} \to \hat{X}\left(\hat{f}\right)$

b. $\hat{X}\left(\hat{\alpha} * \hat{f} + \hat{\beta} * \hat{g}\right) = \hat{\alpha} * \hat{X}\left(\hat{f}\right) + \hat{\beta} * \hat{X}\left(\hat{g}\right)$ (3.1)

c. $\hat{X}\left(\hat{f} * \hat{g}\right) = \hat{f} * \hat{X}\left(\hat{g}\right) + \hat{g} * \hat{X}\left(\hat{f}\right)$

where $\hat{\alpha}, \hat{\beta} \in I\hat{R}$, $\hat{f}, \hat{g} \in D^o\left(\hat{M}\right)$ and $\hat{\alpha} * \hat{f}, \hat{f} * \hat{g} \in D^o\left(\hat{M}\right)$ definite by

$$\hat{\alpha} * \hat{f}:\hat{U} \overset{l}{\subset} \hat{M} \to I\hat{R}, \quad \hat{\alpha} * \hat{f}:\hat{P} \to \left(\hat{\alpha} * \hat{f}\right)\left(\hat{P}\right) = \hat{\alpha} * \hat{f}\left(\hat{P}\right)$$

$$\hat{f} * \hat{g}:\hat{U} \subset \hat{M} \to I\hat{R}, \quad \left(\hat{f} * \hat{g}\right):\hat{P} \to \left(\hat{f} * \hat{g}\right)\left(\hat{P}\right) = \hat{f}\left(\hat{P}\right) * \hat{g}\left(\hat{P}\right) \qquad (3.2)$$

The set of isovector fields on $\hat{M}$, denoted by $\hat{D}^1(\hat{M})$, is turned into a Lie-Santilli algebra [6,8] over $\hat{D}^o(\hat{M})$ by the following operations:

i. sum of two isovector fields: if $\hat{X},\hat{Y}\in\hat{D}^1(\hat{M})$, then $\hat{X}+\hat{Y}\in\hat{D}^1(\hat{M})$ definite by

$$\hat{X}+\hat{Y}:\hat{D}^o(\hat{M})\to\hat{D}^o(\hat{M}),\ (\hat{X}+\hat{Y}):\hat{f}\to(\hat{X}+\hat{Y})(\hat{f})=\hat{X}(\hat{f})+\hat{Y}(\hat{f}),$$

ii. isoscalar multiplication (the external operation): if $\hat{f}\in\hat{D}^o(\hat{M})$, and $\hat{X}\in\hat{D}^1(\hat{M})$, then $\hat{f}*\hat{X}$ definite by:

$$\hat{f}*\hat{X}:\hat{D}^o(\hat{M})\to\hat{D}^o(\hat{M}),\ \hat{f}*\hat{X}:\hat{g}\to(\hat{f}*\hat{X})(\hat{g})=\hat{f}*\hat{X}(\hat{g}), \quad (3.3)$$

iii. second internal operation (Lie-Santilli bracket): On $\hat{D}^1(\hat{M})$ we define a second internal operation, denoted by $[\,;]$, as follows:

$$[\,;]:\hat{D}^1(\hat{M})\times\hat{D}^1(\hat{M})\to\hat{D}^1(\hat{M}),\ [\,;]:(\hat{X},\hat{Y})\to[\hat{X};\hat{Y}]=\hat{X}\hat{T}\hat{Y}-\hat{Y}\hat{T}\hat{X}$$

where $\hat{T}$ is the inverse of the isounit $\hat{I}$ of the underlying isofield $\hat{R}$.

It can be easily proved that the Lie-Santilli bracket satisfies the following relations:

$$\begin{aligned} &a.\ \ [\hat{X};\hat{Y}]=-[\hat{Y};\hat{X}], \quad so \quad [\hat{X};\hat{X}]=0 \\ &b.\ \ [\hat{X};[\hat{Y};\hat{Z}]]+[\hat{Y};[\hat{Z};\hat{X}]]+[\hat{Z};[\hat{X};\hat{Y}]]=0 \end{aligned} \quad (3.4)$$

Hence $\hat{D}^1(\hat{M})$ becomes a Lie algebra over $\hat{D}^o(\hat{M})$.

The moutrinality of the Lie-Santilli isotheory is that it is nonlinear, nonl-ocal and non-Hamiltonian on M, yet it reconstructs linearity, locality and comonicity on $\hat{M}$. Also note that relations (3.4) are valid on $\hat{M}$ but yet necessarily are their projection on M .

4. Isomappings Between Isomanifolds

Let $\hat{M}$ and $\hat{N}$ be two differentiable isomanifolds. The mapping $\hat{\phi}$ of $\hat{M}$ onto $\hat{N}$, that is,

$$\hat{\phi}:\hat{M}\to\hat{N},\quad \hat{\phi}:P\to\hat{\phi}(P)$$

is called isomapping. This isomapping $\hat{\phi}$ is called differentiable at the point P, if for every neighborhood U of P, there exists a neighborhood V of the point $\hat{\phi}$(P) such that the isofunction

$$\hat{g} \circ \hat{\phi} \in \hat{D}^{\circ}(U), \forall \hat{g} \in \hat{D}^{\circ}(V).$$

If the isomapping $\hat{\phi}$ is differentiable for all points of $\hat{M}$, then $\hat{\phi}$ is called differentiable on the whole $\hat{M}$. If $\hat{\phi}$ is a homeomorphism then $\hat{\phi}$ is called isohomeomorphism.

Definition 4.1 [8]: Let $\hat{M}$ be a differentiable isomanifold. A differentiable isohomeomorphism of $\hat{M}$ onto $\hat{M}$ is called differentiable isotransformation or simply isotransformation.

We consider a differentiable isomapping $\hat{\alpha}$ of the open interval $I \subset \hat{R}$ into $\hat{M}$, that is

$$\hat{\alpha}: I \to \hat{M}, \hat{\alpha}: \hat{t} \in I \to \hat{\alpha}(\hat{t}) \in \hat{M}.$$

This isomapping is called isocurve on $\hat{M}$. In some cases we consider for the definition of the isocurve that the isointerval I is closed, that is

$$I = [\hat{a}, \hat{b}] \subseteq \hat{R},\ \hat{a}, \hat{b} \in \hat{R}.$$

This is true under the condition that the isomapping $\hat{\alpha}$ can be extended to an open isointerval $I \subset I_1$ of $\hat{R}$.

If $\hat{\alpha}$ is a differentiable isocurve on $\hat{M}$ and if $\hat{\alpha}'(\hat{t}) = \hat{X}_{\hat{\alpha}(\hat{t})}$ for all $\hat{t}$, then $\hat{\alpha}$ is called an integral isocurve of $\hat{X}$, where $\hat{X}$ is an isovector field on $\hat{M}$.

Theorem 4.1 [8]: Let $\hat{X}$ be an isovector field of the isomanifold $\hat{M}$. For every point $P \in \hat{M}$, there exists a unique integral isocurve $\hat{\alpha}(\hat{t})$ of $\hat{X}$ defined for $|t| \langle \varepsilon$, where $\varepsilon \rangle 0$ and such that $\hat{\alpha}(0) = P$.

Definition 4.2 [8]: Let $\hat{\phi}$ be a differentiable isomapping between two differentiable isomanifolds $\hat{M}$ and $\hat{N}$. Let (U, ϕ) be an isochart of $\hat{M}$ and $P \in U$. From $\hat{\phi}$ we obtain the point $\hat{\phi}(P) \in V \subset \hat{\phi}(U)$. Let $\hat{D}^{\circ}(U)$ and $\hat{D}^{\circ}(V)$ be the isoalgebras of the differentiable isofunctions on U and V respectively. If

$\hat{X}$ is an isovector field, then $\hat{X}$ can be considered as an isolinear operator on $\hat{D}^\circ(U)$, that means

$$\hat{X}:\hat{D}^\circ(U)\to\hat{D}^\circ(U),\ \hat{X}:\hat{f}\to\hat{X}(\hat{f}).$$

To the isovector $\hat{X}_P \in \mathrm{T}_P(\hat{M})$ we correspond the isovector $\hat{Y}_{\hat{\phi}(P)} \in \mathrm{T}_{\hat{\phi}(P)}(\hat{N})$ defined as an isooperator on $\hat{D}^\circ(V)$ as follows:

$$\hat{Y}:\hat{D}^\circ(V)\to\hat{D}^\circ(V),\ \hat{Y}:\hat{g}\to\hat{Y}(\hat{g})$$

where $\hat{Y}(\hat{g})$ is defined by the relation

$$\hat{Y}(\hat{g})_{\hat{\phi}(P)}=\hat{X}(\hat{g}\circ\hat{\phi})_P.$$

Now we have constructed an isomapping, denoted by $\hat{\phi}_{*P}$ of $\mathrm{T}_P(\hat{M})$ into $\mathrm{T}_{\hat{\phi}(P)}(\hat{N})$ defined as follows:

$$\hat{\phi}_{*P}:\mathrm{T}_P(\hat{M})\to\mathrm{T}_{\hat{\phi}(P)}(\hat{N}),\quad \hat{\phi}_{*P}:\hat{X}_P\to\hat{\phi}_{*P}(\hat{X})=\hat{Y}_{\hat{\phi}(P)}$$

This isomapping $\hat{\phi}_{*P}$ is called derivative of $\hat{\phi}$ at the point P. It is easy to show that the derivative isomapping $\hat{\phi}_{*P}$ is an isolinear mapping.

If $\hat{\phi}:\hat{M}\to\hat{N}$ is a differentiable isomapping and $\hat{X},\hat{Y}$ are $\hat{C}^\infty$ isovector fields on $\hat{M}$ and $\hat{N}$, respectively, we say that $\hat{X}$ and $\hat{Y}$ are $\hat{\phi}$-related when

$$\hat{\phi}_{*P}(\hat{X}_P)=\hat{Y}_{\hat{\phi}(P)}$$

for each $P\in\hat{M}$. If $\hat{g}:\hat{N}\to I\hat{R}$ is a $\hat{C}^\infty$ isofunction then

$$\hat{Y}_{\hat{\phi}(P)}(\hat{g})=\hat{\phi}_{*P}\hat{X}_P(\hat{g})=\hat{X}_P(\hat{g}\circ\hat{\phi})\ \text{so}\quad (\hat{Y}*\hat{g})\circ\hat{\phi}=\hat{X}\ (\hat{g}\circ\hat{\phi}).$$

Conversely, if this is true for all $\hat{C}^\infty$ isofunction $\hat{g}:\hat{N}\to I\hat{R}$, then $\hat{X}$ and $\hat{Y}$ are $\hat{\phi}$-related.

Proposition 4.3 [8]: Let $\hat{X}_i,\hat{Y}_i$, $i=1,2$ be two $\hat{\phi}$-related isovector fields of $\hat{M}$ and $\hat{N}$, respectively. The following relation

$$\hat{\phi}_*([\hat{X}_1\hat{;}\hat{X}_2])=[\hat{Y}_1\hat{;}\hat{Y}_2]$$

holds.

Proposition 4.4 [8]: Let $\hat{\phi}$ be isotransformation on $\hat{M}$. We set $f^* = \hat{f} \circ \hat{\phi}^{-1}$, where $\hat{f} \in \hat{D}^{\circ}\left(\hat{M}\right)$. Then,

$$\hat{\phi}_*\left(\hat{f}\hat{X}\right) = f^* \hat{\phi}_*\left(\hat{X}\right) \text{ and } f^*\left(\hat{X} * \hat{f}\right) = \hat{\hat{\phi}}_*\left(\hat{X}\right) f^*, \hat{X} \in D^1\left(\hat{M}\right).$$

5. One-Parameter Isotransformation Groups

Let $\hat{M}$ be a differentiable isomanifold, and $P \in \hat{M}$ and $\hat{X}$ be a $\hat{C}^{\infty}$ isovector field on $\hat{M}$. If V is an open set containing P and $\varepsilon > 0$, then there is a unique collection of isodiffeomorphims $\hat{\phi}_{\hat{t}} : V \to \hat{\phi}_{\hat{t}}(V) \subset \hat{M}$ for $|t| < \varepsilon$ with the following properties:

1. $\hat{\phi} : (-\varepsilon, \varepsilon) \times \hat{M} \to \hat{M}$, difined by $\hat{\phi}(\hat{t}, P) = \hat{\phi}_{\hat{t}}(P)$ is $\hat{C}^{\infty}$,
2. If $|\hat{s}|, |\hat{t}|, |\hat{s} + \hat{t}| < \varepsilon$, and $q, \hat{\phi}_{\hat{t}}(q) \in V$ then $\hat{\phi}_{\hat{s}+\hat{t}}(q) = \hat{\phi}_{\hat{s}} \circ \hat{\phi}_{\hat{t}}(q)$,
3. If $q \in V$, then $\hat{X}_q$ is the tangent isovector at $\hat{t} = \hat{0}$ of the isocurve $\hat{t} \to \hat{\phi}_{\hat{t}}(q)$.

Then the family $\left\{\hat{\phi}_{\hat{t}} \,/\, \hat{t} \in \hat{R}\right\}$ of isodiffeomorphims is called a *one-parameter Lie-Santilli isogroup* of isodiffeomorphims of the isomanifold generated by $\hat{X}$.

In what follows, we shall often call an isodiffeomorphism of $\hat{M}$ simply an *isotransformation* of $\hat{M}$.

If the family $\left\{\hat{\phi}_{\hat{t}} \,/\, \hat{t} \in \hat{R}\right\}$ is the one-parameter isogroup of isotransformation of $\hat{M}$, then $\hat{\phi}_{\hat{0}}$ is the identity map of $\hat{M}$ and $\hat{\phi}_{\hat{t}}^{-\hat{1}} = \hat{\phi}_{-\hat{t}}$. Now, for a one-parameter isogroup of isotransformations $\left\{\hat{\phi}_{\hat{t}}\right\}$ and $P \in \hat{M}$, we can define an isovector field on $\hat{M}$ by

$$\hat{X}_P \hat{f} = \left[\frac{\hat{d}\,\hat{f}\left(\hat{\phi}_{\hat{t}}(P)\right)}{\hat{d}\,\hat{t}}\right]_{\hat{t}=\hat{0}}.$$

This isovector field $\hat{X}$ is called the *infinitesimal isotransfomations* $\left\{\hat{\phi}_{\hat{t}}\right\}$. The isocurve defined by $\theta_p(\hat{t}) = \hat{\phi}_{\hat{t}}(p)$ is an integral isocurve of $\hat{X}$ such that $\theta_p(0) = p$. An isovector field $\hat{X}$, which is an infinitesimal isotransfomation of a one-parameter isogroup of isotransformations of $\hat{M}$, is said to be *complete*.

If $\hat{X}$ is complete , then there is only one one-parameter isogroup of isotransformations which has $\hat{X}$ as its infinitesimal isotransfomation.

6. Infinitesimal Motions on Isoriemannian Manifolds

Let M be a Riemann manifold over the reals R with local coordinates $x=(x^k)$, $k=1,2,\ \dots\ ,n$, and nowhere singular, symmetric, real-valued metric $g(x)=(g_{ij})=g^t$. Let $\hat{M}$ denote the isotopic images of M [5] such that $\hat{M}$ is defined over the isoreals $\hat{R}$ with common isounit $\hat{I}=(\hat{I}_i{}^j)$, local isocoordinates $\hat{x}=(\hat{x}^k)=(x^k\hat{I})$, and isometric $\hat{g}(x)=\hat{\mathrm{T}}(t,x,\dot{x},\dots)g(x)$. For isovector fields $\hat{X}$ and $\hat{Y}$ on $\hat{M}$, define a $\hat{C}^\infty$ isofunction on $\hat{M}$ by $P\rightarrow\hat{g}_P(\hat{X}_P,\hat{Y}_P)\in\hat{R}$. This isofunction is denoted by $\hat{g}(\hat{X},\hat{Y})$. Let $(\hat{x}_1,\hat{x}_2,\dots,\hat{x}_n)$be a local isocoordinate system on an open set U and let $\hat{g}_{ij},\hat{\xi}^i,\hat{\eta}^j$ be the components of $\hat{g},\hat{X},\hat{Y}$ respectively, with respect to the basis $\{\hat{\partial}_k\}$ of $\hat{D}^1(\hat{M})$. Then

$$\hat{g}(\hat{X},\hat{Y})=(\hat{\xi}^i\hat{g}_{ij}\hat{\eta}^j)\hat{\mathrm{I}}$$

on U.

Let $\hat{X}$ be a fixed isovector field on $\hat{M}$. If, for arbitrary isovector fields $\hat{Y},\hat{Z}$ on $\hat{M}$,

$$\hat{X}[\hat{g}(\hat{Y},\hat{Z})]=\hat{g}([\hat{X};\hat{Y}],\hat{Z})+\hat{g}(\hat{Y},[\hat{X};\hat{Z}]) \tag{6.1}$$

holds, then $\hat{X}$ is called an *infinitesimal motion* of the isoriemannian manifold $\hat{M}$ or *isokilling vector field.*

If we choose $\hat{Y}=\hat{\partial}_j,\hat{Z}=\hat{\partial}_k$ in (6.1), we obtain the system of isodifferential equations for $\hat{\xi}^i$:

$$\hat{g}_{ki}\hat{\partial}_j(\hat{\xi}^i)+\hat{g}_{ji}\hat{\partial}_k(\hat{\xi}^i)+\hat{\partial}_i(\hat{g}_{jk})\hat{\xi}^i=0,\ \ i,j,k=1,2,\dots,n \tag{6.2}$$

$\hat{X}$ is an infinitesimal motion if and only if the set of components $\hat{\xi}^i$ of $\hat{X}$ is a solution of the equations (6.2).

The reason why an isovector field $\hat{X}$ satisfying (6.2) is called an infinitesimal motion can be explained as follow.

We start with a well known definition in the conventional case . Let M and M^* be Riemannian manifolds with Riemannian metrics g and g^*, respectively. If a differentiable map Φ from M to M^* satisfies for any point P of M and any vector u of $T_p(M)$ the equality

$$\|\Phi_* u\| = \|u\|, \tag{6.3}$$

then Φ calls an *isometry* from M to M^*.

If a diffeomorphism Φ of a Riemannian manifold M onto itself is an isometry, then Φ calls a *motion* of M. The set of all motions of M forms a group called the group of motions of the Riemannian manifold M.

For a fixed point p of M for which (6.2) holds for all $u \in T_p(M)$, we have

$$g_{\Phi(p)}(\Phi_* u, \Phi_* v) = g_p(u, v) \text{ for all } u, v \in T_p(M).$$

The isotopic lifting of all remaining above aspect as well as the extension to isoriemannian manifolds will be left for brevity to the interested reader.

Theorem 6.1: Let $\hat{X}$ be a complete isovector field on an isoriemannian manifold $\hat{M}$. The isovector field $\hat{X}$ is an infinitesimal motion if and only if $\hat{\phi}_{\hat{t}}$ is a motion for each $\hat{t} \in \hat{R}$.

First we state the following lemma.

Lemma 6.2.[1]: For any given tangent isovector field $\hat{v}$ at a point p on an isomanifold $\hat{M}$, there is a $\hat{C}^\infty$ isovector field $\hat{X}$ on $\hat{M}$ such that $\hat{X}_p = \hat{v}$.

Proof of the theorem: By the lemma, the isomap $\hat{D}^1(\hat{M}) \to T_p(\hat{M})$, $\hat{X} \to \hat{X}_p$ is an onto map. Hence, setting $\hat{\phi}_{\hat{t}} = Exp(\hat{t} * \hat{X})$, the condition for $\hat{\phi}_{\hat{t}}$ to be a motion is that

$$\hat{g}_q(\hat{Y}_q, \hat{Z}_q) = \hat{g}_p((\hat{\phi}_{\hat{t}*})_q \hat{Y}_q, (\hat{\phi}_{\hat{t}*})_q \hat{Z}_q), \qquad q = \hat{\phi}_{\hat{t}}^{-1}(p) \tag{6.4}$$

holds for an arbitrary point p of $\hat{M}$ and arbitrary isovector fields $\hat{Y},\hat{Z}$. However, since $\left(\hat{\phi}_{\hat{t}*}\right)_q \hat{Y}_q=\left(\hat{\phi}_{\hat{t}*}\hat{Y}\right)_p$ and $\left(\hat{\phi}_{\hat{t}*}\right)_q \hat{Z}_q=\left(\hat{\phi}_{\hat{t}*}\hat{Z}\right)_p$, the equality above can be written as

$$\hat{g}(\hat{Y},\hat{Z})\left(\hat{\phi}_{\hat{t}}^{-\hat{1}}(p)\right)=\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y}_q,\hat{\phi}_{\hat{t}*}\hat{Z}\right)(p)$$

which means

$$\hat{\phi}^{*}_{-\hat{t}}\left(\hat{g}(\hat{Y},\hat{Z})\right)-\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}\right)=\hat{0} \tag{6.5}$$

where $\hat{\phi}^{*}_{-\hat{t}}:D^{\circ}(U)\to D^{\circ}(U)$ is a differentiable isofunction defined by $\hat{\phi}_{\hat{t}}^{\,*}(\hat{f})=\hat{f}\circ\hat{\phi}_{\hat{t}}$ for an isofunction $\hat{f}\in D^{\circ}(U)$.

Now, let us set

$$\hat{f}\left(\hat{t};\hat{Y},\hat{Z}\right)=-\hat{\phi}_{-\hat{t}}^{\,*}\left(\hat{g}(\hat{Y},\hat{Z})\right)+\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}\right) \tag{6.6}$$

and compute $\left.\frac{\hat{d}}{\hat{d}\hat{t}}\hat{f}\left(\hat{t};\hat{Y},\hat{Z}\right)\right|_{\hat{t}=\hat{0}}$. Differentiating the first term on the right side of (6.6) with respect to $\hat{t}$ and setting $\hat{t}=\hat{0}$, we get:

$$\begin{aligned}\left.\frac{\hat{d}}{\hat{d}\hat{t}}\left(-\hat{\phi}_{-\hat{t}}^{\,*}\left(\hat{g}(\hat{Y},\hat{Z})\right)\right)\right|_{\hat{t}=\hat{0}} &=-\lim_{\hat{t}\to\hat{0}}\frac{1}{\hat{t}}\left(\hat{\phi}_{-\hat{t}}^{\,*}\left(\hat{g}(\hat{Y},\hat{Z})\right)-\hat{g}(\hat{Y},\hat{Z})\right)\\ &=-\lim_{\hat{t}\to\hat{0}}\frac{1}{\hat{t}}\left(\left(\hat{g}\left(\hat{\phi}_{-\hat{t}*}\hat{Y},\hat{\phi}_{-\hat{t}*}\hat{Z}\right)\right)-\hat{g}(\hat{Y},\hat{Z})\right)\\ &=-\left.\frac{\hat{d}}{\hat{d}\hat{t}}\hat{g}\left(\hat{\phi}_{-\hat{t}*}\hat{Y},\hat{\phi}_{-\hat{t}*}\hat{Z}\right)\right|_{\hat{t}=\hat{0}}\\ &=-\left(-\hat{X}\left(\hat{g}(\hat{Y},\hat{Z})\right)\right)=\hat{X}\left(\hat{g}(\hat{Y},\hat{Z})\right)\end{aligned} \tag{6.7}$$

and for the second term

$$\begin{aligned}\left.\frac{\hat{d}}{\hat{d}\hat{t}}\left(\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}\right)\right)\right|_{\hat{t}=\hat{0}} &=\lim_{\hat{t}\to\hat{0}}\frac{1}{\hat{t}}\left(\left(\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}\right)\right)-\hat{g}(\hat{Y},\hat{Z})\right)\\ &=\lim_{\hat{t}\to\hat{0}}\frac{1}{\hat{t}}\left(\hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y}-\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}\right)+\hat{g}\left(\hat{Y},\hat{\phi}_{\hat{t}*}\hat{Z}-\hat{Z}\right)\right)\end{aligned}$$

$$= -\hat{g}\left(\lim_{\hat{t}\to\hat{0}} \frac{1}{\hat{t}} (\hat{Y} - \hat{\phi}_{\hat{t}*}\hat{Y}), \hat{\phi}_{\hat{t}*}\hat{Z} \right) - \hat{g}\left(\hat{Y}, \lim_{\hat{t}\to\hat{0}} \frac{1}{\hat{t}} (\hat{Z} - \hat{\phi}_{\hat{t}*}\hat{Z}) \right)$$

$$= -\hat{g}\left([\hat{X};\hat{Y}], \hat{Z} \right) - \hat{g}\left(\hat{Y}, [\hat{X};\hat{Z}] \right) \tag{6.8}$$

Hence, we have

$$\left. \frac{\hat{d}}{\hat{d}\hat{t}} \hat{f}(\hat{t};\hat{Y},\hat{Z}) \right|_{\hat{t}=\hat{0}} = \hat{X}\left(\hat{g}(\hat{Y},\hat{Z}) \right) - \hat{g}\left([\hat{X};\hat{Y}], \hat{Z} \right) - \hat{g}\left(\hat{Y}, [\hat{X};\hat{Z}] \right) \tag{6.9}$$

If $\hat{\phi}_{\hat{t}}$ is a motion, then, by (6.5) $\hat{f}(\hat{t}:\hat{Y},\hat{Z}) = \hat{0}$, so the left side of (6.9) is 0, and hence $\hat{X}$ is an infinitesimal motion.

Conversely, if $\hat{X}$ is an infinitesimal motion, then by (6.1) and (6.9), for arbitrary isovector fields $\hat{Y}$ and $\hat{Z}$ on $\hat{M}$ we have

$$\left. \frac{\hat{d}}{\hat{d}\hat{t}} \hat{f}(\hat{t};\hat{Y},\hat{Z}) \right|_{\hat{t}=\hat{0}} = 0. \tag{6.10}$$

After some computations we can see that

$$\hat{f}(\hat{s}+\hat{t};\hat{Y},\hat{Z}) = \hat{\phi}^{*}_{-\hat{s}}\left(\hat{f}(\hat{t};\hat{Y},\hat{Z}) \right) = \hat{f}\left(\hat{s}; \hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}\hat{Z} \right). \tag{6.11}$$

If we differentiate this equality with respect to $\hat{s}$ and set $\hat{s} = \hat{0}$, then, since $\left. \frac{\hat{d}}{\hat{d}\hat{s}} \hat{f}\left(\hat{s}; \hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}\hat{Z} \right) \right|_{\hat{s}=\hat{0}} = \hat{0}$, we have

$$\frac{\hat{d}}{\hat{d}\hat{t}} \hat{f}(\hat{t};\hat{Y},\hat{Z}) = -\hat{X}\, \hat{f}(\hat{t};\hat{Y},\hat{Z}). \tag{6.12}$$

On the other hand we have

$$\hat{X}\left(\hat{f}(\hat{t};\hat{Y},\hat{Z}) \right) = \hat{X}\left[-\hat{\phi}^{*}_{-\hat{t}}\left(\hat{g}(\hat{Y},\hat{Z}) + \hat{g}(\hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}\hat{Z}) \right] \right.$$

$$= -\hat{X}\left[\hat{\phi}^{*}_{-\hat{t}}\left(\hat{g}(\hat{Y},\hat{Z}) \right] + \hat{X}\left[\hat{g}(\hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}\hat{Z}) \right] \right. . \tag{6.13}$$

Since $\hat{\phi}_{\hat{t}*}\hat{X} = \hat{X}$, $\hat{X}\left[\hat{\phi}^{*}_{-\hat{t}}\left(\hat{g}(\hat{Y},\hat{Z}) \right] = \hat{\phi}^{*}_{-\hat{t}}\left(\hat{X}(\hat{g}(\hat{Y},\hat{Z}) \right) \right.$ and

$$\hat{X}\left(\hat{g}(\hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}\hat{Z}) \right) = \hat{g}\left(\hat{\phi}_{\hat{t}*}[\hat{X};\hat{Y}], \hat{\phi}_{\hat{t}*}\hat{Z} \right) + \hat{g}\left(\hat{\phi}_{\hat{t}*}\hat{Y}, \hat{\phi}_{\hat{t}*}[\hat{X};\hat{Z}] \right)$$

$$= \hat{f}(\hat{t};[\hat{X};\hat{Y}],\hat{Z}) + \hat{f}(\hat{t};\hat{Y},[\hat{X};\hat{Z}])$$
$$+\hat{\phi}^{*}_{-\hat{t}}(\hat{g}([\hat{X};\hat{Y}],\hat{Z})) + \hat{\phi}^{*}_{-\hat{t}}(\hat{g}(\hat{Y},[\hat{X};\hat{Z}]))$$

It follows from (6.12) and the fact that $\hat{X}$ is an infinitesimal motion that

$$\hat{X}(\hat{f}(\hat{t};\hat{Y},\hat{Z})) = \hat{f}(\hat{t};[\hat{X};\hat{Y}],\hat{Z}) + \hat{f}(\hat{t};\hat{Y},[\hat{X};\hat{Z}]).$$

Then for the isofunction $\hat{f}(\hat{t};\hat{Y},\hat{Z})$ satisfies, as an isofunction of $\hat{t}$, the isodifferential equation

$$\frac{\hat{d}}{\hat{d}\,\hat{t}}\hat{f}(\hat{t};\hat{Y},\hat{Z}) = -\hat{f}(\hat{t};[\hat{X};\hat{Y}],\hat{Z}) - \hat{f}(\hat{t};\hat{Y},[\hat{X};\hat{Z}]). \qquad (6.14)$$

From the definition of $\hat{f}(\hat{t};\hat{Y},\hat{Z})$ we can see the following properties:

$$\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \hat{f}(\hat{t};\hat{Z},\hat{Y})$$
$$\hat{f}(\hat{t};\hat{Y}+\hat{Y}',\hat{Z}) = \hat{f}(\hat{t};\hat{Y},\hat{Z}) + \hat{f}(\hat{t};\hat{Y}',\hat{Z})$$
$$\hat{f}(\hat{t};\hat{h}\hat{Y},\hat{Z}) = (\hat{\phi}^{*}_{-\hat{t}}\hat{h})\hat{f}(\hat{t};\hat{Y},\hat{Z}) \text{ for any isofunction } \hat{h} \in D^{\circ}(U).$$

Let $\hat{X}_i = \hat{\partial}/\hat{\partial}\hat{x}^i$ and $\hat{X} = \xi^i\hat{X}_i$ on U. Then $[\hat{X};\hat{X}_i] = h_i^{\,j}\hat{X}_j$ with $h_i^{\,j} = -\dfrac{\hat{\partial}\xi^j}{\hat{\partial}\hat{x}^i}$. We put $\hat{f}(\hat{t};\hat{X}_i,\hat{X}_k) = \hat{f}_{ik}(\hat{t};\hat{x})$ $(\hat{x} \in U)$. Then

$$\hat{f}(\hat{t};[\hat{X};\hat{X}_i],\hat{X}_k) + \hat{f}(\hat{t};\hat{X}_i,[\hat{X};\hat{X}_k]) = \hat{h}_i^{\,j}(\hat{\phi}_{-\hat{t}}(\hat{x}))\hat{f}_{jk}(\hat{t};\hat{x})$$
$$+\hat{h}_k^{\,j}(\hat{\phi}_{-\hat{t}}(\hat{x}))\hat{f}_{ij}(\hat{t};\hat{x})$$

and it follows from (6,14) that

$$\frac{\hat{d}\hat{f}_{ik}(\hat{t},\hat{x})}{\hat{d}\,\hat{t}} = -\hat{h}_i^{\,j}(\hat{\phi}_{-\hat{t}}(\hat{x}))\hat{f}_{jk}(\hat{t};\hat{x}) - \hat{h}_k^{\,j}(\hat{\phi}_{-\hat{t}}(\hat{x}))\hat{f}_{ij}(\hat{t};\hat{x}) \qquad (6.15)$$

for each $\hat{x} \in V$ and $|\hat{t}| < \varepsilon$, where V is the a neighborhood of p such that $\hat{\phi}_{-\hat{t}}(V) \subset U$. This shows that, for each $\hat{x} \in V$, the isofunctions $\hat{f}_{ik}(\hat{t};\hat{x})$ of $\hat{t}$

are the solution of (6.15) with the initial condition $\hat{f}_{ik}(\hat{0};\hat{x}) = \hat{0}$. Obviously, the isofunctions of $\hat{t}$ which are identically equal to zero form a solution of (6.15) with the same initial conditions. Hence by the uniqueness theorem of the solution of (6.15), we see that $\hat{f}_{ik}(\hat{t};\hat{x}) = \hat{0}$ for each $\hat{x} \in V$ and $|\hat{t}| < \varepsilon$. Let $\hat{Y} = \eta^i X_i$ and $\hat{Z} = \zeta^i X_i$ on U. Then

$$\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \eta^i(\hat{\phi}_{-\hat{t}}(\hat{x}))\zeta^k(\hat{\phi}_{-\hat{t}}(\hat{x}))\hat{f}_{ik}(\hat{t};\hat{x})$$

for $|\hat{t}| < \varepsilon$ on V and hence $\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \hat{0}$ on V for $|\hat{t}| < \varepsilon$ and for any $\hat{Y},\hat{Z} \in D^1(U)$. Then using (.) repeatedly, we see that $\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \hat{0}$ for any $\hat{t}$ In particular, $\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \hat{0}$ at p for any $\hat{t} \in I\hat{R}$ and any $\hat{Y},\hat{Z} \in D^1(U)$. Since p is an arbitrary point of $\hat{M}$, we get $\hat{f}(\hat{t};\hat{Y},\hat{Z}) = \hat{0}$ on $\hat{M}$. This proves that $\hat{\phi}_{\hat{t}}$ is an isometry for all $\hat{t} \in \hat{R}$.

The first significance of our study is that it permits an unrestricted functional dependence of isoriemann metric and all other quantities, by therefore broadening the applicability of the theory, e.g., from local-differential exterior gravitational problems in vacuum, to nonlocal-integral interior gravitational problems with an arbitrary nonlinearity in the velocities and other variables.

Another significance is that the formalism studied in this paper, that for a positive-definite isounit, is only the first of a chain of possible broader theories characterized by more general isounits, such as those resulting from the relaxation of its symmetric character or its assumption as a multi-valued, ordered set [5]. The latter aspects are contemplated for study in subsequent papers.

References

[1] Y.Matsushima, Trans. by E.T.Koboyashi, *Differentiable Manifolds*, Marsel Decker,Inc. New York 1972.

[2] R.M. Santilli, *Isotopic lifting of the special relativity for extented-deformable particles*, Lett.Nuovo Cimento 37, 545-555 (1983)

[3] R.M. Santilli, *Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isodual and pseudodual, and the "hidden numbers" of dimension 3, 5, 6, 7,* Algebras, Groups and Geometries 10 , 273-322 (1993)

[4] R.M. Santilli, *Elements of Hadronic Mechanics,* Vol. I: *Mathematical Foundations,* Ukraine Academy of Sciences, Kien, 1994

[5] R.M. Santilli, *Nonlocal-integral isotopies of differential calculus, Mechanics and Geometries,* Rendiconti del Circolo Matematico di Palermo Serie II Suppl. 42 (1996), pp. 7-82.

[6] G .Tsagas and D.S. Sourlas, *Mathematical foundations of the Lie-Santilli Theory,* Ukraine Aca. of Science (1993)

[7] G.Tsagas and D.S. Sourlas, *Isomanifolds and Their isotensor Fields,* Algebras, Groups and Geometries, 12 (1995)

[8] G.Tsagas and D.S. Sourlas,*Isomapping Between Isomanifolds* Algebras, Groups and Geometries, 12 (1995) pp 67-87

[9] J. V. Kadeisvili, *Foundations of the Lie-Santilli isotheory,* Rendiconti del Circolo Matematico di Palermo Serie II Suppl. 42 (1996), pp. 83-136.

[10] R.Aslaner, *İzomanifoldlar ve İzomanifoldların Eğrilikleri Üzerine*,Ph.D Thesis.İnönü Üniv., Fen-Bilimleri Enstitüsü, Malatya TURKEY (1996).

[11] R.Aslaner and S.Keleş, *Cartan's Structure Equations onSantilli-Tsagas-Sourlas Isomanifolds,* Algebras, Groups and Geometries, 14 (1997) in press

ALGEBRAS GROUPS AND GEOMETRIES 15, 563–568 (1998)

ON SANTILLI'S ONE–DIMENSIONAL ISODYNAMICAL SYSTEMS

M. R. Molaei
Faculty of Mathematics
Shahid Bahonar University of Kerman
P. O. Box 76135–133, Kerman, Iran
mrmolaei@arg3.uk.ac.ir

Received July 5, 1998;
Revised Nove,ber 3, 1998

Abstract

Certain dynamical aspects on one-dimensional spaces can be described via Santilli's one-dimensional isodynamical systems which are are characterized by axiom-preserving maps called *isotopies.* In this note we discuss discrete one-dimensional isodynamical systems, isohyperbolicity and isosensitivity. The isotopic form of Singer's theorem is also proved. A new isofamily and a number of one-dimensional isodynamical systems on $(-\hat{2}, +\hat{2})$ are introduced. Each of these systems is isosensitive and has three isodiscontinuities, one of them with isozero isoderivative.

1. Introduction

In 1978 Santilli introduced a new branch of mathematics, called isomathematics, based on maps called isotopies which preserving the axioms of a theory and have the ability to generalize a unitary, canonical or local theories to thrir nonunitary noncanonical or nonlocal versions. Santilli has shown that isotopies have another important property in symplectic geometry, they can change a (locally) nonhamiltonian system into an isohamiltonian system [5].

In this paper we consider some isotopic properties of discrete one dimensional dynamical systems. In another paper shall generalize constructions to higher dimensions.

We us begin by defining isodynamical systems. For this purpose assume that A is an interval of elements of the one-dimensional real Euclidean space $E(X, \delta, R(n, +, \times))$ where X, δ and $\mathbb{R}(n, +, \times)$ denote local coordinate, metric and the set of real numbers respectively. We also assume that $f : A \hookleftarrow$ is a one-dimensional dynamical system. By Santilli's isotopy of the unit $I = 1$ of one-dimensional Euclidean space into the positive isounit $\hat{I}$ [6] we have an isointerval $\hat{A}$ in Santilli's one-dimensional isoeuclidean space $\hat{E}(\hat{X}, \hat{\delta}, \hat{\mathbb{R}}(\hat{n}, \hat{+}, \hat{\times}))$ [3,7], with $\hat{X} = X \times \hat{I}$, $\hat{\delta} = (\hat{I})^{-1} \times \delta$, $\hat{+} = +$, $\hat{\times} = \times(\hat{I})^{-1}\times$, and $\hat{\mathbb{R}}(\hat{n}, \hat{+}, \hat{\times}) = \{\hat{n} = n \times \hat{I} : n \in \mathbb{R}(n, +, \times)\}$, the set of isoreal numbers which is a field under $\hat{+}$ and $\hat{\times}$ [4].

Definition 1.1: A one-dimensional isodynamical system on $\hat{A}$ is a mapping $\hat{f} : \hat{A} \hookleftarrow$ defined by $\hat{f}(\hat{x}) = f(x) \times \hat{I}$ where $\hat{x} = x \times \hat{I} \in \hat{A}$.

By definition 1.1 the discreteness rests in the dynamical system. So we have the following three results.

1. p is a fixed point (periodic point) of f if and only if $\hat{p} = p \times \hat{I}$ is a fixed point (periodic point) of $\hat{f}$.

$\hat{p}$ is called the isofixed point (isoperiodic point) of $\hat{f}$.

2. The positive isoorbit of the point $\hat{p} \in \hat{A}$, $\hat{O}(\hat{p}) := \{(\hat{f})^n(\hat{p}) : n = 0, 1, 2, \ldots\}$ is equal to the set $\{x \times \hat{I} : x \in O(p)\}$ where $O(p)$ is the orbit of p.

3. The iso-w-limit set of the point $\hat{p} \in \hat{A}$ is defined by $\hat{w}(\hat{p}) := \{\hat{x} \in \hat{A}$: There exist a sepuence $n_i \to \infty$such that $\lim_{n_i \to \infty} \hat{\delta}((\hat{f})^{n_i}(\hat{p}), \hat{x}) = 0\}$. It may be shown that $\hat{W}(\hat{p})$ is equal to the set $\{x \times \hat{I} : x \in w(p)\}$, where $w(p)$ is the w-limit set of p.

Let f be a differentiable map on an open subset of A containing $\{p\}$. Then by the definition of isoderivatives [5], the isoderivative of $\hat{f}$ at $\hat{p} = p \times \hat{I}$ is defined by the form $\hat{f}^{\hat{\prime}}(\hat{p}) := f'(p) \times \hat{I}$.

Definition 1.2: The isofixed point $\hat{p} = p \times \hat{I}$ of $\hat{f}$ is called an isohyperbolic isofixed point if $\hat{f}^{\hat{\prime}}(\hat{p}) \neq \hat{I}$.

It follows that if $\hat{I} \neq 1$, then an isohyperbolic isofixed point of $\hat{f}$ is not a hyperbolic fixed point for $\hat{f}$, and we have the following proposition.

Proposition 1.3: Nonidentity isotopies do not protect hyperbolicity.

2. Isotopic Singer's Theorem

In this section we restrict isodynamical systems on closed isointervals and without loss of generality we can assume that $\hat{A} = [-\hat{I}, \hat{I}]$. The Schwarzian derivative has an important role in the proof of Singer's theorem, hence we define this notion for isomaps.

Definition 2.1: Let $\hat{f} : [-\hat{I}, \hat{I}] \hookleftarrow$ be an isodynamical system of class $C^{\hat{3}}$. Then the isoschwarzian derivative of $\hat{f}$ at $\hat{p}$, denoted by $\hat{S}\hat{f}(\hat{x})$ is:

$$\hat{S}\hat{f}(\hat{x}) := \hat{f}^{\hat{\prime}\hat{\prime}\hat{\prime}}(\hat{x})\hat{/}\hat{f}^{\hat{\prime}}(\hat{x}) - (\hat{3}\hat{/}\hat{2})\hat{\times}(\hat{f}^{\hat{\prime}\hat{\prime}}(\hat{x})\hat{/}\hat{f}^{\hat{\prime}}(\hat{x}))^{\hat{2}}.$$

Proposition 2.2: Let $\hat{f}$ be an isodynamical system of class $C^{\hat{3}}$ and $\hat{S}\hat{f}(\hat{x}) < \hat{0}$ for all $\hat{x} \in [-\hat{I}, \hat{I}]$. Then $\hat{S}(\hat{f}^n)(\hat{x}) < \hat{0}$ for all $\hat{x} \in [-\hat{I}, \hat{I}]$ and all natural number n.

Proof: Let g be an isodynamical system of class $C^{\hat{3}}$. Then

$$\begin{aligned} \hat{S}(\hat{f}o\hat{g})(\hat{x}) &= S(fog)(x) \times \hat{I} \\ &= [(Sf)(g(x))g'(x)^2 + Sg(x)] \times \hat{I} \\ &= (\hat{S}\hat{f})(\hat{g}(\hat{x}))\hat{\times}[\hat{g}^{\hat{\prime}}(\hat{x})]^{\hat{2}} + \hat{S}\hat{g}(\hat{x}) \quad (1). \end{aligned}$$

So the proposition follows by induction.□

Let $\hat{p} \in [-\hat{I}, \hat{I}]$ be an isoperiodic isopoint of period N, *i.e.* N is the smallest positive integer such that $\hat{f}^N(\hat{p}) = \hat{p}$. Then $\hat{p}$ is called isostable if $|(\hat{f}^N)^{\hat{\prime}}(\hat{p})\hat{|} \leq \hat{I}$ where $|.\hat{|} = [\hat{\delta}(.,.)]^{\hat{1}\hat{/}\hat{2}}$.

Definition 2.3: The isostable manifold of $\hat{p}$, denoted by $\hat{W}^s(\hat{p})$, is the set of points $\hat{x}$ such that $(\hat{f})^m(\hat{x})\hat{\to}\hat{p}$ when $m \to \infty$.

Lemma 2.4: If $W^s(p)$ is the stable manifold of p for the mapping f, then $\hat{W}^s(\hat{p}) = \{x \times \hat{I} : x \in W^s(p)\}$.

Proof:

$$\begin{aligned} x \times \hat{I} = \hat{x} \in \hat{W}^s(\hat{p}) &\Leftrightarrow (\hat{f})^m(\hat{x})\hat{\to}\hat{p} \;\; when \;\; m \to \infty \\ &\Leftrightarrow f^m(x) \times \hat{I}\hat{\to}p \times \hat{I} \;\; \text{when} \;\; m \to \infty \\ &\Leftrightarrow f^m(x) \to p \;\; \text{when} \;\; m \to \infty \\ &\Leftrightarrow x \in W^s(p)\,. \;\; \square \end{aligned}$$

For the isotopic form of Singer's theorem we assume that $\hat{f}$ has the following properties.

I1. $\hat{f}(\hat{0}) = \hat{I}$;
I2. $\hat{f}^{\hat{'}}(\hat{x}) < \hat{0}$, for all $\hat{x} \in (\hat{0}, \hat{I})$ and $\hat{f}^{\hat{'}}(\hat{x}) > \hat{0}$, for all $\hat{x} \in (-\hat{I}, \hat{0})$;
I3. $\hat{S}\hat{f}(\hat{x}) < \hat{0}$ for all $\hat{x} \in [-\hat{I}, \hat{I}]$ and $\hat{S}\hat{f}(\hat{0}) = -\infty \times \hat{I}$ is also acceptable.

The statement of the isotopic form of Singer's theorem follows:

Theorem 2.5: Let $\hat{f}$ is of class C^3 and satisfies I1, I2 and I3, then the isostable manifold of every isostable isoperiodic isopoint contains at least one of the isopoints $-\hat{I}$, $\hat{0}$, $\hat{I}$.

Proof. By isotopy we conclude that
A. $\hat{f}$ satisfies I1 if and only if $f(0) = 1$;
B. $\hat{f}$ satisfies I2 if and only if $f'(x) < 0$; for all $x \in (0, 1)$;
C. $\hat{f}$ satisfies I3 if and only if $Sf(x) < 0$; for all $x \in [-1, 1]$. $Sf(0) = -\infty$ is also acceptable.

Now by Singer's theorem $W^s(p)$ contains at least one of the points -1, 0, 1. Hence by lemma 2.4, $W^s(\hat{p})$ contains at least one of the points $-\hat{I}$, $\hat{0}$, $\hat{I}$. □

3. Isosensitivity

The concept of isosensitivity is similar to that of sensitivity [1]. We have the following definition.

Definition 3.1: We say that the isodynamical system $\hat{f} : \hat{A} \hookleftarrow$ has isosensitivity if for all $\epsilon > 0$ there exist $C > 0$ and a subset $\hat{B}$ of $\hat{A}$ with positive Lebesgue measure satisfying the following condition:

$$\forall\ \hat{q} \in \hat{B}\ \ \exists\ n \geq 0\ \&\ \hat{s} \in \hat{A} \ni [|(\hat{f})^n(\hat{s}) - (\hat{f})^n(\hat{q})\hat{|} > C \times \hat{I}\ \&\ |\hat{s} - \hat{q}\hat{|} < \epsilon \times \hat{I}].$$

Lyapunov exponents have an essential role in the notion of sensitivity. We now define the isotopic form of the Lyapunov exponent.

Definition 3.2: Let $(\hat{f})^n$ be an isodifferentiable map on an open subset containing $\hat{x} \in \hat{A}$ $(n = 1, 2, 3, \ldots)$. If $\hat{\lambda}(\hat{x}) = \lim_{n \to \infty} [\hat{log}|((\hat{f})^n)^{\hat{'}}\hat{|}\hat{/}\hat{n}] \times \hat{I}$ exist, then it is called the isolyapunov exponent of $\hat{f}$ at $\hat{x}$.

Theorem 3.3: Suppose B is a subset of A with positive Lebesgue measure and the isolyapunov exponents of $\hat{f}$ on $\hat{B}$ are positive, then $\hat{f} : \hat{A} \hookleftarrow$ has isosensitivity.

Proof. Let $\epsilon > 0$ and $\hat{q} \in \hat{B}$ be given. Let $C = 1$ and choose $n \geq 0$ so long that

$$\hat{\lambda}(\hat{x}) \simeq \hat{log}[|((\hat{f})^n)^{\hat{'}}(\hat{x})\hat{|}/\hat{n}] \times \hat{I} \quad \text{or} \quad \hat{e}^{\hat{n} \hat{\times} \hat{\lambda}(\hat{x})} \simeq |((\hat{f})^n)^{\hat{'}}(\hat{x})\hat{|} \times \hat{I},$$

where $\hat{e}^{\hat{n} \hat{\times} \hat{\lambda}(\hat{x})} = e^{\hat{I}^{-1} \times (\hat{n} \hat{\times} \hat{\lambda}(\hat{x}))} \times \hat{I}$ [6].

We can now find $n_0 \geq 0$ such that $|((\hat{f})^{n_0})^{\hat{'}}(\hat{x})\hat{|} > \hat{1}\hat{/}di\hat{a}m\hat{A}$ for all $\hat{x} \in \hat{B}$, where $di\hat{a}m\hat{A}$ is the diameter of $\hat{A}$ in respect of $\hat{\delta}$.
Hence if $|\hat{s} - \hat{q}\hat{|} < \epsilon \times \hat{I}$, then $|\hat{f}^{n_0}(\hat{s}) - \hat{f}^{n_0}(\hat{q})\hat{|} > \hat{1}\hat{/}di\hat{a}m\hat{A} \hat{\times} di\hat{a}m\hat{A} = \hat{1}$. □

Now we would like to introduce a one-isoparameter isofamily of isomaps on $(-\hat{2},\hat{2}) \hookleftarrow$ such that each of them has three isodiscontinuities, one of them with isozero isoderivative. For this purpose consider the one-parameter family of maps on $(-2,2)$ of the form:

$$f_\mu(x) = \begin{cases} \psi(2+x)+\mu & if \quad -2 < x < -1 \\ -\psi(-x)-\mu & if \quad -1 < x < 0 \\ \psi(x)+\mu & if \quad 0 < x < 1 \\ -\psi(2-x)-\mu & if \quad 1 < x < 2 \end{cases}$$

where $1 < \mu < 2$ and $\psi : (0,1) \to (-2,0)$ is a C^2 map which satisfies the conditions A, B and C:
$A)$ $\lim_{x\to 1^-} \psi(x) = 0$ and $\lim_{x\to 0^+} \psi(x) = -2$;
$B)$ $\lim_{x\to 0^+} \psi'(x) = 0$ and $\frac{3}{2} < \lim_{x\to 1^-} \psi'(x) < 2$;
$C)$ $\psi''(x) > 0$ for all $x \in (0,1)$.
This family was introduced by this auther in [2]. Now we introduce the isotopic form of this family.

We define the isofamily $\{\hat{f}_{\hat{\mu}}\}$ of isomaps on $(-\hat{2},\hat{2})$ by the form:

$$\hat{f}_{\hat{\mu}}(\hat{x}) = f_\mu(x) \times \hat{I} \ .$$

Theorem 3.4: There exists $\hat{\mu}_0 > \hat{I}$ such that $\hat{f}^{\hat{\prime}}_{\hat{\mu}}$ has isosensitivity for all $\hat{I} < \hat{\mu} \leq \hat{\mu}_0$.

Proof. In [2] we have proved that there exists $\mu_0 > 1$ such that f'_μ has isosensitivity when $1 < \mu \leq \mu_0$, so the theorem follows from theorem 3.3. □
We finish this section by posing some open problems.
1. What is the supremum of the set $\{\hat{\mu}_0 : \hat{\mu}_0$ satisfies the properties of theorem 3.4$\}$?
2. What will happen to the isofamily if we change the condition $\lim_{x\to 1^-} \psi'(x) < 2$ to the condition $\lim_{x\to 1^-} \psi'(x) = \infty$?

Acknowledgment

I would like to thank professor G.F.Weiss for introducing me to this problem.

REFERENCES

1. J. Guckenheimer, Sensitive Dependence on Initial Conditions for One-Dimensional Maps, Comm. Math. Phys. 70 (1979).

2. B. Honary, M. R. Molaei, Positive Lyapunov Exponents for a Class of One-Parameter Families of Maps, Algebra, Groups and Geometries, Vol. 15 (1998).

3. R. M. Santilli, Nuovo Cimento Lett. 37, 545 (1983).

4. R. M. Santilli, Isonumbers and Genonumbers of Dimension 1, 2, 4, 8, Their Isoduals and Pseudoduals, and "Hidden Numbers" of Dimension 3, 5, 6, 7, Algebra Groups and Geometries 10, 273-322 (1993).

5. R. M. Santilli, Rendiconti Circolo Matematico Palermo, Suppl. 42, 7 (1996).

6. R. M. Santilli, Isotopic Grand Unification With the Inclusion of Gravity, Foundations of Physics Letters, Vol. 10, No. 4 (1997).

7. R. M. Santilli, Relativistic Hadronic Mechanics: Nonunitary, Axiom-Preserving Completion of Relativistic Quantum Mechanics, Foundations of physics. Vol. 27, No. 5 (1997).

8. D. Singer, Stable Orbits and Bifurcation of Maps of the Interval, SIAM J. Appl. Math. 35, 260 (1978).

ALGEBRAS, GROUPS AND GEOMETRIES

INDEX VOLUME 15, 1998