Need of subjecting to an experimental verification the validity within a hadron of Einstein's special relativity and Pauli's exclusion principle.

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Abstract

This paper is a call for theoretical and experimental studies on the problem whether the relativity and quantum mechanical laws which have proved so effective for the atomic as well as the nuclear constituents are truly verified also for the hadronic constituents. For the intent of stimulating these studies, this paper is devoted to the problem whether a violation of the laws considered in the arena considered is conceivable, plausible and quantitatively treatable on grounds of our current knowledge. This problem is studied according to a number of sequential steps.

First, we conduct a critical analysis of the quark models on the hadronic structure to the effect of indicating that, perhaps, their known problematic aspects are only the symptoms of a much more fundamental problem of consistency at the level of the basic laws. By noting that the available unitary models produce a Mendeleev-type classification of hadrons of unequivocal physical effetiveness and of virtually conclusive character, we search for a compatible but fundamentally different model of structure along much of the differentiation between the problem of classification and that of structure which resulted as necessary at the atomic level.

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We then enter into the study of a conceivable new model of hadronic structure which is capable, on one side, of achieving compatibility with the established models of unitary classification and, on the other side, of resolving the fundamental problematic aspect of available models of structure, the identification of the hadronic constituents with physical particles. According to our priorities, we assume as fundamental the problem of the nature of the forces of the hadronic constituents. The second problem in our priorities is that of the disciplines capable of treating the assumed type of strong hadronic forces. The last problem in our priorities is that of the construction of a structure model of hadrons and of its confrontation with physical reality.

A crucial experimental data of the hadronic phenomenology is that the charge volume of hadrons does not appreciably increase with mass (contrary to the correspondent occurrence at the nuclear level) and it is of the same order of magnitude of that of any other know, massive and charged particle. It then follows that, if the hadronic constituents are massive, charged and physical particles, that is, non-point-like, they are bounded according to a state of penetration of their charge volumes (or wave packets). This yields a realization of the strong hadronic forces as being nonlocal and nonderivable from a potential, that is, a type of force which is beyond our current knowledge at this time for any effective, quantitative treatment. We therefore approximate these forces with local forces nonderivable from a potential. This yields forces which, at the primitive Newtonian level, are the nonconservative forces of the systems of our everyday experience. The fundamental physical character of the assumed strong hadronic forces is therefore that of being nonconservative. These forces essentially constitute the simplest conceivable analytic generalization of the Lorentz force, in the sense that the Lorentz force is linearly dependent on the velocities and derivable from a potential (variationally selfadjoint forces), while our strong hadronic forces are dependent on the velocities in a generally nonlinear way and are non-derivable from a potential (variationally nonselfadjoint forces).

We then enter into the study of the quantization of nonconservative Newtonian forces in general and of strong nonselfadjoint hadronic forces in particular. For this purpose we briefly recall the dual methodologies for the classical treatment of the forces considered, as presented in details by the author in preceding papers and forthcoming monographs, those of the Inverse Problem and of the Lie-Admissible Problem. The paper essentially presents a study for the quantization of these methodologies which results in a proposed dual covering of Schrödinger's and Heisenberg's equations. A central result is that, under the condition that the quantum mechanical algorithms at hand (r, **p**, H, **M**, etc.) possess a direct physical significance, the brackets of the time evolution law must violate the Lie algebra identities as the fundamental condition for mathematical and physical consistency for the case of nonselfadjoint forces. Instead, the brackets considered can characterize a covering Lie-admissible algebra, in precisely the same way as it occurred at the classical level. Intriguingly, there is the emergence also of the Jordan algebras, which therefore acquire an apparent fundamental methodological role for the quantum mechanical treatment of nonconservative forces, perhaps equal to that of Lie algebras. Indeed, the brackets of the proposed covering of Heisenberg's equations result to be, jointly, Lie-admissible and Jordan-admissible. The epistemological lines for a possible covering of conventional quantum mechanics, here called hadronic mechanics, are presented. It is then pointed out in details, either via the generalized algebraic structure of the theory or via direct analysis of the dynamical behaviour, that the inflexible laws of quantum mechanics (here called atomic mechanics) for the treatment of selfadjoint forces are fundamentally inapplicable to the broader physical context constituted by strong nonselfadjoint hadronic forces. Instead, the familiar quantum mechanical laws appear to be replaced by covering laws capable of identically recovering the former at the limit of null forces non-derivable from a potential.

As a necessary complement to the above dynamical analysis of the problem, we then study the relativity laws which are applicable in nonconservative quantum mechanics (the hadronic mechanics in our terminology). This objective is achieved by quantizing the Lie-admissible covering of Galilei's relativity for nonconservative Newtonian mechanics proposed by the author in a recent paper. This results in the proposal of a quantum mechanical covering relativity for the hadronic constituents, under the assumed broader forces, which is Lie-admissible in algebraic character and, as such, capable of identically recovering the conventional relativity of atomic mechanics at the limit of null nonselfadjoint forces. It is pointed out that established relativities (Galilei's, Einstein's special and Einstein's general relativity for the interior problem) are inapplicable to the considered more general nature of the strong interactions. In particular, the proposed covering relativity results to be of non-Lie, non-conservative, non-inertial, non-linear, non-geodesic, non-symplectic and non-Riemannian character to technically characterize physical systems which are non-derivable from a variational principle. In conclusion, our studies on the dynamical profile of the problem of quantization of nonselfadjoint forces result to be in full agreement with the corresponding studies on the relativity profile.

We then enter into the study of an apparent dichotomy of physical laws for the hadronic phenomenology: the unequivocal validity of established laws for the behaviour of a hadron as a whole under ut most electromagnetic interactions and the conceivable applicability of covering laws for the hadronic constituents. This problem is studied via the use of nonintegrable, classical and quantum mechanical subsidiary constraints. In figurative terms, the established laws for the total physical quantities of a hadron are imposed as subsidiary constraints to the covering laws for the individual constituents. The emerging overdetermined systems of differential equations result to be consistent (that is, admitting a physically meaningful solution) under the proper selection of nonselfadjoint forces. It is this property which, in the final analysis, has allowed the presentation of the analysis of this paper. According to this approach, established laws are valid by construction for a hadron as a whole, and possible departures are admitted only for the hadronic constituents. In particular, the violation of these laws at the level of the constituents (only) emerges as a necessary condition for the existence of more general structure forces in order to attempt a real departure of the hadronic from the atomic structure. For instance, the imposition of Galilei's relativity at the structure level would imply conservative strong forces. We then argue that, under these conditions, the atomic and hadronic structures are dynamically equivalent.

As the last step in our priority, we finally consider the problem of the construction of a new structure model of hadrons based on this dichotomy of physical laws, and its confrontation with experimental data. The study essentially indicates that the identification of the constituents of unstable hadrons with suitably selected massive particles produced in their spontaneous decays, while prohibited by the conventional relativity and quantum mechanical laws of strict Lie algebraic character, becomes admissible under the proposed covering, relativity and quantum mechanical laws of joint Lie-admissible and Jordan-admissible algebraic character. The case of mesons is considered in detail and it is indicated that the model is capable of producing a quantitative representation of all the intrinsic characteristics of the particles, while offers some genuine hope for a quantitative interpretation of the decay modes and related fractions. Thus, the proposed new model of hadronic structure appears to resolve the fundamental problematic aspect of the quark models, the identification of the constituents with physical particles, while reaching full compatibility with the established unitary models of classification.

As we all know, our current theoretical knowledge can be interpreted as characterized by suitable implementations of the experimentally established knowledge for the electromagnetic interactions which preserve the underlying basic laws. Pending the verification by interested researchers, our analysis essentially indicates that such knowledge can be considered as applicable, provided that the forces (or couplings) are local and derivable from a potential, that is, the system is represented in its entirety via the simple Lagrangian structure $L_{tot} = L_{free} + L_{int}$. If the strong interactions are assumed as dynamically nonequivalent to the electromagnetic interactions and their forces are realized in a form analytically nonequivalent to the Lorentz force, they demand the abandonment of the virtual entirety of our current theoretical knowledge (such as: Galilei's and Einstein's relativities; Heisenberg's equations and Pauli's exclusion principle; scattering amplitude and Feynman diagrams; canonical field quantization and spin-statistics theorem; etc.). Instead, under the conditions indicated, the courageous construction of covering disciplines must be undertaken for the strong interactions in general and for the hadronic structure in particular, in exactly the same way as it occurred for the electromagnetic interactions in general and for the atomic structure in particular.

The paper concludes with remarks concerning the future orientation of experimental high energy physics which is needed to provide means for a physically effective selection among an ever increasing number of hadronic models. It is argued that, until the experimental efforts are essentially restricted to the identification of new particles, the problem of the hadronic structure will likely remain fundamentally unresolved, because the knowledge of new particles adds informations which are certainly useful for the classification of hadrons, but not necessarily for the structure. It is submitted that, jointly with the continuation of these valuable experiments, the fundamental problem of the validity or invalidity of established relativity and quantum mechanical laws for the hadronic constituents is confronted. In the final analysis, if the laws considered will eventually result to be valid in the arena considered, the quark models are likely to emerge as the only conceivable models at this time. On the contrary, if the laws considered will eventually emerge as being violated in the arena considered, the concept of quark as the constituent of hadrons is likely to be ruled out in a final form.

" In questions of science, the authority of a thousands is sometimes not worth the humble reasoning of one single individual."

GALILEO GALILEI

NEED OF SUBJECTING TO AN EXPERIMENTAL VERIFICATION THE VALIDITY WITHIN A HADRON OF EINSTEIN'S SPECIAL RELATIVITY AND PAULI'S EXCLUSION PRINCIPLE

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ERRATA - CORRIGE

R.M.Santilli, "On a possible Lie-admissible covering of the Galilei relativity in Newtonian mechanics for nonconservative and Galilei form-noninvariant systems", HADRONIC JOURNAL 1, 223-423 (1978) .... 902.

1. STATEMENT OF THE PROBLEM

One of the objectives of the HADRONIC JOURNAL is to attempt an active editorial policy in the sense that any member of the editorial organization of the journal may, at his own election, solicite contributions on open problems of general or specific interest in hadron physics.

Along these lines, the objective of this paper is to solicit contributions by interested researchers on the problem of the experimental verification (or test, if you prefer) of the validity within a hadron of established relativity and quantum mechanical formulations, with particular reference to Einstein's special relativity[1] and Pauli's exclusion principle[2].

In its simplest possible form, the argument is the following. The validity of the laws considered for the behaviour of a hadron as a whole under electromagnetic interactions appears to be established on solid experimental grounds. Neverthless, strictly speaking, this does not constitute evidence of the validity of the same laws for the hadronic constituents. Since a direct experimental evidence for the latter context is lacking at this moment, it follows that the current contributions on the problem of the hadronic structure are based on the (often tacit) assumption of the validity of the laws considered. This occurrence can be interpreted as an expression of clear plausibility of these laws and of the models based on them. Neverthless, without a direct experimental verification, this occurrence also cannot be interpreted as a final resolution of the problem considered.

After all, the historical nonapplicability of previously established knowledge to the problem of the atomic structure, or the more recent, but equally historical discovery of parity violation, should not be forgotten.

It is an easy prediction that an exhaustive study of the problem considered demands contributions from the scientific community at large. In particular, a number of complementary contributions appear recommendable. First of all, there is a truly realistic possibility that a proper reinterpretation of available experimental data (e.g., for deep inelastic scatterings) from the viewpoint of the validity of established laws could resolve the issue. At the same time, the issue whether the same experimental data can be quantitatively interpreted by models based on the inapplicability of established laws as currently known, should be confronted. Also, delicate mathematical issues in each of these opposite, yet complementary approaches should perhaps be first resolved at the pure mathematical level in order to reach the needed physical effectiveness. Finally, contributions by experts on the epistemological meaning of the terms "experimental verification" or "experimental test" should not be overlooked to avoid potential unnecessary controversies.

This completes the editorial part of this paper.

The rest of this paper is devoted to an initial identification of specific tests, via the study of the problem whether a violation of the laws considered within the arena considered is conceivable, plausible and quantitatively treatable on grounds of our current knowledge. In essence, we argue that an effective way to stimulate the necessary partecipation of independent researchers for a future resolution of the issue, is to initiate a study for the possible violation for the hadronic constituents of those laws which have proved to be so effective for the atomic (as well as nuclear) constituents. As such, the remaining content of this paper, besides being basically insufficient to resolve the issue considered, must be considered at this time an exercise of scientific curiosity.

2. A CRITICAL ANALYSIS OF THE QUARK MODEL

The period of the history of theoretical physics ranging from the awarding of the Davy medal in chemistry to D.I. MENDELÉYEV* in 1882 to N. H. D. BOHR's reception of the Nobel price in physics of 1922, as is known[3], was dominated by the problem of identifying methodological tools capable of providing an effective representation of the atomic structure.

It was not without controversy that the need of a radical departure from previously established classical formulations finally emerged. As a result of corageous contributions ranging over the first half of this century, a new mechanics, called quantum mechanics, was identified, conceived and constructed by founding fathers such as N. BOHR, M. BORN, L.V. DE BROGLIE, P.A.M. DIRAC, W. HEISENBERG, P. JORDAN, W. PAULI, M. PLANCK and E. SCHRÖDINGER.

The outcome was the establishing of one model of the hydrogen atom which proved to be fully consistent with physical reality. The advent of the special theory of relativity by A. EINSTEIN and the related implementation of quantum mechanics into a broder discipline, called relativistic quantum mechanics, resulted to be crucial for bringing the model of the atomic structure to its utmost maturity, as it is known nowadays.

The period of the history of theoretical physics ranging from 1922 to E. FERMI's award of the Nobel price in physics of 1938, as is also known[4], was dominated by the problem of the structure of the atomic nucleus. The fundamental discipline used for this objective was, and still is, quantum mechanics. As a result of a rather large number of contributions, an effective understanding of the nuclear structure was finally achieved.

The aspect of this latter period of the history of theoretical physics which is relevant for this paper is that quantum mechanics produced only one model on the structure of the hydrogen atom capable of interpreting the entirety of its phenomenological behaviour, while the same discipline, applied to the structure of the atomic nucleus, produced a variety of models, such as the shell model, the Fermi gas model, the optical model, etc., none of which seems to be individually capable of representing the entirety of the new physical context. Despite the subsequent advent of clustering models[5], this situation still persists as of today.

This situation seems to indicate that in the transition from the atomic to the nuclear level, nature suddenly becomes polyhedric by restricting our capability to effectively represent it to only few of its facets at a time. Our contention is that, on a comparative

* Other spellings in the English literature are: Mendeleev, or Mendeléeff or Mendelejeff

basis between atomic and nuclear physics, the latter is still fundamentally unresolved as of now. Indeed, the problem of the forces responsible for the atomic structure can be considered as resolved on ground on clear experimental evidence, while the problem of the nature of the nuclear forces is still open at this moment. We therefore argue that the chapter of the nuclear structure cannot be considered as closed until a final resolution of the problem of the nature of the nuclear forces and of the methodology for their treatment is achieved.

The period of the history of theoretical physics ranging from 1938 until now has been dominated, as it is also well known, by the problem of the structure of the strongly interacting particles, the hadrons. The fundamental discipline used in the virtual totality of the contributions in this topic is, again, quantum mechanics or its relativistic discrete or quantum field theoretical extension, but this time complemented with new tools, the unitary internal symmetry groups and, most notably, the SU(3) model by M. GELL-MANN and G. ZWEIG of 1964,[6] subsequently subjected to a series of implementations such as color, flavor, etc.[7]

The formidable task confronted by physicists in this latter period of the history of theoretical physics was in actuality two-fold. First, there was a clear need of producing an effective classification of an ever increasing number of hadrons, that is, to formulate the "Mendeléyev table"of the subnuclear particles . Secondly, there was the central problem of identifying the structure of the particles considered which, from mounting experimental indications, cannot be elementary.

As a result of brilliant contributions, it appears that the unitary models have indeed produced a "Mendeléyev classification" of hadrons of clear physical relevance*. Thus, the first of the indicated two objectives can nowadays be considered as accomplished to a major extent.

The same unitary models have been used to jointly attempt the construction of structure models of the hadrons, again within the context of quantum mechanics (or quantum field theory) as the fundamental discipline. The net outcome has been this time controversial. It would be of questionable scientific mentality to deny that some of the more recent contributions within the context of unitary structure models of hadrons constitute genuine progress in a rather formidable problem. But it would be unobjective to deny that the emerging models of structure have a lesser physical effectiveness on a comparative basis with nuclear and

* The Ω^- event reproduced in the front page of this journal has been selected to honor this achievement.

atomic models.

An aspect which is relevant for this paper is that, in the transition from the nuclear to the hadronic level (within the context of the same fundamental discipline) there has been a larger proliferation of models which appears to emphasize the departure from the unity of atomic physics. The interested reader is suggested to inspect, for instance, the review article of ref.[8] , identify the number of different models and ascertain the reasons for our inability to perform a clear selection of one model versus another. It seems as if, in the transition from the nuclear to the hadronic level, nature performs the transition from a polyhedric structure to a structure of undefined topology, with consequential loss of guidance for our efforts to effectively represent it.

This, however, is only part of the reasons of controversy indicated earlier. Other reasons can be identified as being a direct consequence of the discipline used in these studies. As is known, the fundamental representation of the SU(3) group, jointly with its use in the problem of the classification of hadrons, was assumed to represent a physical particle, called quark, and interpreted as the constituent of hadrons. The inflexible laws of quantum mechanics, however, directly imply certain special features of quarks, whenever they are assumed to be physical particles. For instance, in the models currently receiving the majority of consensus, the quarks possess a fractional charge. Since no such particle is produced in the spontaneous decays of unstable hadrons, a sizeable experimental effort has been implemented during the last decade and lately intensified aiming at the detection of quarks (or, more generally) fractionally charged particles) without achieving a final experimental resolution until now, despite contributions of unequivocal scientific value.[9] In turn, this situation has stimulated considerable theoretical efforts aiming at the so-called models of confinement.[10-19] Despite equally brilliant contributions of equally undeniable scientific value , no conclusive model of confinement which can be accepted by the physics community at large has been achieved until now.

But perhaps, more problematic aspects of the quark model are only surfacing at this time. An increasing experimental evidence indicates the existence of additional new (leptons and) hadrons, to the point that the possible existence of an infinite number of these particles cannot be ruled out at this moment. As it has been in the past, it is conceivable that the quark models on the hadronic structure must be subjected to a progressive number of sequential implementations. Technical arguments then suggest the need of introducing new quarks. At the possible limit of an infinite number of hadrons this could imply the assumption of an infinite number of different quarks. In any case, irrespective of whether this number is finite or infinite, any increase in the number of different quarks casts shadow on the conception that the quarks are the elemental constituents of the hadronic matter.

To summarize this brief historical perspective, the use of quantum mechanics as the fundamental discipline within the context of the currently identified layers of the microscopic reality

(I) the atomic layer,

(II) the nuclear layer, and

(III) the hadronic layer,

has produced

(I') one single model capable of effectively representing the entirety of the phenomenological behaviour for the layer for which it was conceived, the atomic structure,

(II') the lack of one single model for the subsequent layer of the nuclear structure, although the emerging models exhibit a clear physical effectiveness, and

(III') a proliferation of models for the subsequent layer of the hadronic structure of lesser physical effectiveness on a comparative basis with that of the nuclear and atomic models, to the point that no conclusive model of hadronic structure can be claimed at this moment.[20]

This situation calls for a critical analysis of the quark models aiming at the identification of conceivable alternatives which deserve a study, in the traditional spirit of unsolved physical problems.

One of the major difficulties of this task is the identification of an arena of unequivocal physical relevance of the quark models and of the underlying fundamental discipline. This problem is created by the clear achievements of clear physical relevance of the quark models. As a result, it is unlike that any final, future resolution of the problem of the hadronic phenomenology will be achieved without the use of unitary models. The problem therefore consists of the identification of a part of the hadronic phenomenology for which the unitary models unequivocally apply, and a part in which the search for fundamentally different models might have a scientific value. This is clearly a problem which cannot be resolved in an individual paper and will likely demand the partecipation of the scientific community

at large. The tentative answer which will be attempted in this paper is essentially the following. An arena of unequivocal applicability and physical relevance of the unitary models is that of hadronic classification only. In relation to the different problem of the hadronic structure, studies along the current unitary trends must continue. Jointly, fundamentally different models of structure should be attempted and subjected to a comparative confrontation with the hadronic reality.

Another major difficulty of the task under consideration is related to the identification of the reasons which prohibit the establishing of the quark models at this time as the conclusive models of the hadronic structure. This too, is not a problem which can be resolved in an individual paper and will also demand the contribution of the scientific community at large. The crucial nature of the problem considered is self-evident. If these difficulties can be identified in a way as detailed as possible, this would constitute a valuable basis for further progress on the problem of the hadronic structure.

This latter issue has clearly a complex structure owing to the inevitable enclosure of conceptual, technical and methodological aspects of the problem of the hadronic structure. The current problematic aspects of the unitary models of structure (lack of identification of quarks with physical particles, lack of a conclusive model of confinement, etc.) will not be considered in this paper. It is hoped that experts in the field will eventually study these issues under the profile considered. Our efforts will be centered instead on the methods used in the construction of unitary models of structure, in line with the epistemological considerations of Section 1. The issue which will be considered *is not* whether new models can be constructed within the context of the same fundamental discipline, quantum mechanics. Instead, the issue which will be considered is whether, in much the same way as it occurred at the atomic level, the problem of the hadronic structure might demand the construction of a new discipline, specifically conceived for the considered layer of the physical reality. It is this aspect which is here considered crucial for the objective of this paper: to stimulate the experimental verification of the validity (or invalidity?) of established laws for the hadronic structure. Indeed, if broader disciplines appears to be conceivable for the hadronic structure, the need of proving on clear experimental grounds the validity of established disciplines is consequential.

To avoid possible misrepresentations, it should be stressed already at the level of these introductory remarks that this conjectural possibility of broader disciplines will be studied only for the problem of the hadronic structure and not that of the behaviour of a hadron as a whole. This distinction between the methodological profile of the problem of structure and that of classification or "exterior behaviour" is suggested by the fact that, according to clear experimental evidence (e.g., in particle accelerators), established relativity and quantum mechanical laws unequivocally apply for a hadron as a a whole, such as, the conventionally quantized value of the spin, the relativistic behaviour under electromagnetic interactions, etc.

In conclusion, the exercise of scientific curiosity with which I hope to entertain the interested and open minded reader is whether the current problematic aspects of the current quark models of hadronic structure are only the symptoms of a much more fundamental problem of consistency at the level of the basic laws.

The theoretically inclined reader should keep in mind that this paper is primarily devoted to the researchers who will expectedly produce, in due time, the actual resolution of the issue: the experimentalists. As a result, I shall provide a genuine effort in presenting the main ideas in a language as simple as conceivably possible. A more technical study of the topic, which can indeed become quite involved on mathematical grounds (e.g., as the reader will see, via the use of non-Lie, but Lie-admissible algebras), is contemplated in subsequent papers.

For conciseness, I shall present the argument in sequential tables. A more detailed study is presented in the forthcoming monographs [21-22]. I have attemted to render this paper conceptually selfsufficient. Neverthless, the knowledge of the preceding paper [23] on the classical relativity profile appears to be recommendable.

TABLE 2.1: A FIRST POSSIBLE DICHOTOMY: THE ESTABLISHED UNITARY MODELS OF CLASSIFICATION VERSUS THE SEARCH FOR DIFFERENT MODELS OF STRUCTURE.

As recalled in the introduction of this section, the problem of the atomic phenomenology demanded two different but compatible models: The Mendeléyev model of classification, and the Bohr model of structure, later on generalized to what is currently known as the Thomas-Fermi model for higher atomic structures.

As a specific example, consider the palladium group, an octet. The identification of this group belongs to the Mendeléyev classification. Neverthless, the structure of each individual element of the group belongs to the (Bohr-) Thomas-Fermi model of structure.

ATOMIC PROBLEM

Classification : Mendeléyev model.

Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd

Structure : Bohr-Thomas-Fermi model.

The two approaches, the classification and the structure, are profoundly different on conceptual, technical and methodological grounds. Neverthless, they are compatible in the sense that the classification produces essential elements for the structure problem, while the structure models reproduce the classification.

When seen from this profile, the quark models exhibit a rather peculiar feature. They are customarily interpreted as providing a joint solution of both, the problem of classification and of structure.

As a specific example, consider the group of "stable" mesons, also an octet. The identification of this family was a result of the SU(3) model. The same model, under the assumption that the fundamental representation of SU(3) represents physical particles, the quarks, was then used to construct a structure model of each element of the group as a bound state of a quark and an antiquark.

HADRONIC PROBLEM

Classification: the SU(3) model.

π^0	π^+	π^-	K^+	K^-	K_S	K_L	η

Structure : the SU(3) model.

Our contention is that, perhaps, this occurrence could create a dichotomy of fundamental physical relevance. We argue that the physical significance of the SU(3) model for the considered hadronic classification is unequivocal. Thus, the SU(3) model can indeed be interpreted as providing the Mendeléyev classification of stable mesons. The situation for the corresponding problem of the structure of each individual element of the octet might instead result to be profoundly different thanthat currently conceived, in much the same way as it occurs at the atomic level.

In relation to the problem of structure two different alternatives are conceivable. First, there is no a priori reason why the situation which eventually resulted as necessary for the atomic phenomenology (a differentiation between the models of classification and of structure) should also occur within the context of the hadronicphenomenology. Thus, studies along the unitary models of joint classification and structure should continue. Secondly, there equally exists no a priory reason which prohibits, on grounds of our current knowledge, a differentiation of the model of classification and of structure in exactly the same way as it occurred for the atomic phenomenology. Thus, the search of new models of struoture which are fundamentally different than the unitary models (in a form to be yet indicated) appears to be recommendable.

We reach in this way our first

ASSUMPTION 2.1.1: The problem of the hadronic phenomenology demands two different but compatible approaches: the established unitary models of classification and new models of structure.

It should be stressed that, according to the above assumption, the available unitary models are assumed to be fully valid. Only their physical relevance is restricted to that

of hadronic classification only. As a matter of fact, most of the subsequent efforts along the line of study of this paper are devoted to the identification of the rudiments of a possible different model of structure under the uncompromisable condition that it should be able to recover the unitary classification in full. This is the meaning according to which the compatibility of possible new models of structure and the established unitary models is intended in Assumption 2.1.1. A problem which we shall leave open for the experts in the field is the selection, among the rather large variety of available models, of that model which is adequate and sufficient for an effective classification of hadrons (the originally conceived SU(3) model or some of its subsequent implementations via color, flavor, etc.).

It is advisable to stress one crucial implication of Assumption 2.1.1 already at this introductory stage. Strictly speaking, the assumption implies that the quark is only the fundamental representation of a Lie group, the SU(3) group, and does not represent a physical particle. This is clearly a crucial prerequisite for any search for possible, fundamentally different structure models. Additional implications will be pointed out during the course of our analysis.

TABLE 2.2: A SECOND POSSIBLE DICHOTOMY: THE ESTABLISHED CONCEPTS FOR THE UNITARY HADRON CLASSIFICATION VERSUS THE SEARCH OF NEW CONCEPTS FOR THE HADRONIC STRUCTURE.

Theoretical physics is essentially a quantitative representation via mathematical algorithms of primitive physical concepts or notions or insights. Before entering into the problem of the disciplines which are applicable to the hadronic structure, it appears then recommendable to devote some attention to the basic ideas. Again, a comparative analysis with the corresponding situation at the atomic level may serve as of valuable intuitional guidance.

As is well known, the Mendeléyev table is based on a number of fundamental concepts, such as the valence, the atomic weight, etc. The problem of the atomic structure demanded instead fundamentally different concepts as the central building blocks, such as the De Broglie's wavelenght, Einstein's frequency, etc., which however resulted to be able to achieve the compatibility between the classification and the structure models indicated in Table 2.1.

ATOMIC PROBLEM

Fundamental concepts for the classification: valence, atomic weight, etc.

Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd

Fundamental concepts for the structure: wavelenght, frequency, etc.

Again, the concepts and notions of the classification where recovered by the structural approach, but they played no constructive role. For instance, a structure model of, say, the helium atom which is based on the notion of the valence as the fundamental building block, was inconceivable at the time its structure was identified and remains to be so nowaday.

When seen from this profile, the quark model exhibit a second peculiar feature which is, in essence, a consequence of its customary dual interpretation as classification and structure model. In short, the same notions which are essential for the classification are used for the problem of structure. I am here referring in particular to the concept of particle multiplet (in any of its several variations) which is virtually dominating contemporary hadron physics.

HADRONIC PROBLEM

Fundamental concept for classification: multiplet.

π^o	π^+	π^-	K^+	K^-	K_s	K_L	η

Fundamental concept for structure: multiplet

The physical effectiveness of the concept of multiplet for the hadronic classification should be stressed. In the final analysis, the lack of use of this notion would likely prohibit any physically meaningful classification of hadrons. This is, for instance, the case for the notion of isotopic triplet for the pions.

In the transition to the problem of structure, again, two different alternatives are conceivable. First, it may be that the same fundamental concepts used in hadron classification are also essential for the problem of hadronic structure. Secondly, it may be that basically different concepts will result to be needed for the problem of the hadronic structure. We reach in this way our second

ASSUMPTION 2.2.1: The conventional concept of hadronic multiplet, while essential for the problem of classification, prohibits the proper formulation of the problem of the structure of each individual element of the multiplet and new, compatible concepts should be attempted.

In essence, we argue that a structure model of say, the π^o, which is centrally dependent on the notion of, say, the isotopic triplet of pions is conceptually equivalent to the attempt of constructing a structure model of the helium which is essentially dependent on the concept of valence as a building block. As we know, each element of an atomic group resulted to possess its own individual structure, even though a number of similarities emerged and allowed the classification into a group. On equivalent intuitional basis, we argue that, in the final analysis, the π^o and the $\pi^\pm$ may eventually result to have fundamentally different structures (in the sense, e.g., of demanding a different number of constituents), even though they exhibit clear elements of similarity to be consistently classified as an isotopic triplet.

One crucial implication of Assumption 2.2.1 should be here stressed. It essentially implies that our subsequent efforts will be devoted to the attempt of identifying a conceivable structure model of hadrons which, by central requirement, does not depend on the notion of multiplet and treats instead each hadron as an individual entity, but which, however, is capable of recovering the notion of multiplet of the corresponding unitary classification.

Of course, the above critical remarks on the conventional notion of multiplets is per sé sterile until new concepts for the structure problem are attempted and their plausibility assessed. The study of this problem will be initiated in Section 3.

Almost needless to say, the above critical remarks also apply to another notion which is also dominating current hadron physics, that of mass spectrum. A formula producing a mass spectrum which fits particle data by no means should be interpreted as providing the final solution of the structure problem, because subsequent studies may indicate that it is merely another way of formulating the classification problem.

TABLE 2.3: A THIRD POSSIBLE DICHOTOMY: STRONG HADRONIC FORCES DERIVABLE OR NON DERIVABLE FROM A POTENTIAL.

We are now sufficiently equipped to begin our analysis of a problem which appears to be the truly fundamental problem of the hadronic structure: the nature of the strong hadronic forces. Again, a comparative analysis with the corresponding situation at the atomic level might be of intuitional guidance.

In unpedagogical terms, we can say that one of the fundamental problems which was confronted for the identification of the atomic structure was that of the nature of the acting forces. The problem of the methods for their treatment was, to a considerable extent, sequential. As we all know, the founding fathers of atomic physics conjectured that the acting forces were of electromagnetic nature and then constructed a new discipline, quantum mechanics, for their treatment in the new layer of the physical reality then considered. As also familiar, this conjecture subsequently resulted to be consistent with physical reality, and it is now an established scientific truth.

ATOMIC STRUCTURE PROBLEM

Identification of the structure forces (electromagnetic)

↓

Construction of the applicable discipline (quantum mechanics)

↓

Construction of the structure model. (Bohr-Thomas -Fermi model)

When seen from this profile, the currently available unitary models of hadronic structure exhibit an additional peculiar feature. In essence, the attitude which has been implemented in the available studies has been that of first constructing unitary structure models and then studying the compatible realizations of the structure forces which, in this case, imply the additional presence of strong forces, besides the conventional

electromagnetic forces.

HADRONIC STRUCTURE PROBLEM

Assumption of the applicable discipline

↓

Construction of unitary structure models

↓

Identification of compatible realizations of the strong hadronic forces.

In short, a series of circumstances, essentially dictated by the lack of established experimental knowledge on the nature of the strong hadronic forces, has produced a third departure of the studies on the hadronic structure from the corresponding conceptual lines of studies on the atomic structure. In particular, the emerging realizations of the compatible strong hadronic forces are those derivable from a potential in the sense, for instance, of references[24]. It is significant here to indicate that the compatibility of this type of strong hadronic forces with the unitary models is of two-fold nature. First of all, the assumption that the forces considered are (local, of class C^{∞} and) derivable from a potential directly implies the full applicability of established quantization procedures. Thus, the forces considered are compatible with the used fundamental discipline to begin with. Secondly, the forces considered have exhibited a rather remarkable compatibility with the unitary models based on the indicated fundamental discipline. Indeed, upon a judicious selection of the potential, the agreement of the prediction of the theory with available experimental data is remarkable, although restricted to the case of the heavy hadron spectroscopy.

At this point a third possible dichotomy of fundamental physical relevance is conceivable. As we shall indicate during the course of our analysis, the problem of the laws which are applicable to the hadronic constituents is crucially dependent on the nature of the strong hadronic forces. If these forces are (local, of class C^{∞} and) derivable from a potential, established relativity and quantum mechanical laws are expected to apply in full, pending an explicit experimental verification. On the contrary, if the strong hadronic forces are structurally more general (in a form to be yet identified), then the problem of the applicable relativity and quantum mechanical laws appears to be open, pending theoretical verifications by interested researchers.

This situation suggests, as a possibly effective alternative, the reversing of the order of priority for the problem of the hadronic structure. We are here referring to the assumption of the problem of the nature of the strong hadronic forces as the fundamental problem. The problem of the relativity and quantum mechanical laws which are applicable to the assumed form of the forces is then second. Finally, the problem of the actual construction of structure models is third. This line of study, of course, is here advocated only as a complement to current trends in the hope that a possible, future, comparative confrontation with the physical reality may produce a resolution of the issue.

We reach in this way a truly crucial problem for the analysis of this paper: the identification of a conceivable more general form of the strong hadronic forces. Again, a comparative analysis with the corresponding atomic context may be valuable.

Let us consider on a comparative basis the lightest known atom, the positronium, and the lightest known hadron, the π^o. These two states exhibit a number of similarities, such as, they are both neutral, they have similar decays modes, and they can be both conceived as a bound state of one massive, charged particle and an antiparticle. The two states considered also exhibit fundamental physical differences, such as the charge radius of the π^o is substantially smaller than that of the positronium, the fractions of each decay mode is different, and, last but not least, the π^o demands the additional presence of short-range forces which we call "strong" We here ignore a number of additional technical implementations (e.g., the Bethe-Salpeter equation for the positronium and the use of the so-called "sea of gluons" for the π^o) because inessential for the intended level of these considerations.

As is well known, the force responsible for the positronium structure, the Coulomb force, is derivable from a potential. Contrary to previous occurrences, the currently used assumption on the nature of the strong hadronic forces is that it is <u>analytically equivalent</u> to that of the Coulomb force, that is, also derivable from a potential. This is epistemologically in line with the indicated similarities between the π^o and the positronium. The major differentiation is provided by the existence of unitary internal degrees of freedom which are absent for the positronium. This is epistemologically in line with the physical differences between the states considered.

A number of remarks are here in order to attempt an assessement of this situation. First of all, it should be recalled that the assumption that the strong hadronic forces are derivable from a potential does not appear to be in clear agreement with the <u>light</u> hadron spectroscopy. The agreement of the studies of refs.[24] with the experimental data refer only to the <u>heavy</u> hadron spectroscopy, while no study of comparable physical effectiveness is available as of now, despite a number of attempts, for the case of light hadrons. It is this point which renders potentially effective the studies of alternative forms of the strong hadronic forces.

Secondly, besides the problem of the explicit form of the potential, a more fundamental issue might be the question whether the unitary internal degrees of freedom are actually sufficient to represent the indicated physical differences between the π^o and the positronium. This problem is created by the fact that, again in an unpedagogical language, those degrees of freedom occur in a mathematical internal space combining the π^o with other mesons, while the physical differences considered occur in the physical space of the experimental detection (the Euclidean or Minkowski space). For instance, it may well be that a proper selection of the potential of the strong hadronic forces and the use of the internal unitary degrees of freedom may account for the different value of the fraction of the $\gamma\gamma$ decay mode of the π^o and of the positronium. But, on grounds of our current knowledge, we do not know whether the final interpretation of this difference will eventually demand a differentiation of the π^o and of the positronium structure in the physical space of the Euclidean (or Minkoswki) variables. In this latter case, the most likely candidate is a differentiation of the analytic structure of the acting forces.

In any case, if we intend to assess the possible physical relevance of Assumptions 2.1.1 and 2.2.1, the use of unitary internal degrees of freedom at the level of the structure problem is prohibited. The interpretation of the similarities as well as differences between the π^o and the positronium is then essentially reduced to the explicit form of the strong hadronic force in the space of the experimental detection. Under these circumstances, the assumption that the strong hadronic forces are also derivable from a poten tial, as it is the case for the Coulomb force, becomes somewhat unappealing on intuitional grounds. It then becomes recommendable the study of broader forces. In this respect two alternatives are conceivable. The first is that the strong hadronic forces responsible for the π^0 structure (besides the Coulomb force of its constituents assumed as charged) are nonlocal and not derivable from a potential. The second is that the forces considered are local but not derivable from a p otential. Clearly, the technical difficulties for the treatment of the former type of force are expected to be substantial (assuming that they are treatable in a physically effective way on grounds of our current knowledge). The second alternative therefore appears as more recommendable as an initial attempt of somewhat intermediate technical difficulties between local forces derivable from a potential and nonlocal forces not derivable from a potential. Besides, local forces not derivable from a potential as known to provide a good approximation of nonlocal forces.

We reach in this way our most crucial

ASSUMPTION 2.3.1: The strong hadronic forces which are responsible for the structure of the "stable" mesons are local, of class C^∞ and variationally nonselfadjoint, as an approximation of expected nonlocal forces.

The crucial problem of the currently identifyable degree of plausibility of this assumption will be studied in Section 3. In essence, in line with Assumptions 2.1.1 and 2.2.1, the above assumption is consider at this time on mere grounds of scientific curiosity. Also, the assumption is considered only as a complement of the con ventional assumption on the strong hadronic forces and not as a substitute. More specifically, our view is that studies on the current trends on the nature of the strong hadronic forces should continue while, jointly, studies on possible more general forces should be initiated.

Secondly, the reader should be aware that the assumption considered is introduced only for the stable mesons. Thus, the case of nucleons, hyperons, and other heavy hadrons are excluded. This attitude is motivated by the pragmatic intention of confronting first the problem of the lighter known hadrons and than that of heavier hadrons. After all, the former might ultimately result to have a considerably simpler structure than that of the latter (despite different plausible views). Also, the problem of the stability of the proton might well result to be of a complexity beyond our imagination at this time.

Finally, we assume that the reader is familiar, from refs.[21, 23], that the terms "variationally nonselfadjoint" are the technical characterization within the context of the Inverse Problem of forces not derivable from a potential. On a comparative basis, the established force of the positronium is "variationally selfadjoint".

TABLE 2.4: A FOURTH POSSIBLE DICHOTOMY: ESTABLISHED RELATIVITY LAWS VERSUS THE SEARCH FOR COVERING LAWS.

According to our line of study, once the nature of the strong hadronic forces has been assumed, the subsequent step is the identification of the applicable laws. This problem is (at least) two-fold in the sense that it demands the identification of the applicable relativity and quantum mechanical laws. It appears advisable to consider first the problem of the applicable relativity and than that of quantization. This is due to the possible deep impact on quantum mechanical formulations of the relativity laws (as we shall better illustrate in Section 3). In turn, the problem of the relativity laws is per sé multi-fold, in the sense that it can be of discrete non-relativistic or relativistic nature, inclusive or noninclusive of gravitational considerations, etc.

In this paper we shall only consider the most rudimentary possible relativity aspect, that of discrete nonrelativistic nature. A part from a few incidental comments, the problem of the relativistic, field theoretical and gravitational estensions will not be considered at this time.

Under these restrictions, our problem is that of identifying the relativity which is applicable to classical discrete systems under assumption 2.3.1. This is equivalent to the problem of the relativity which is applicable to local, class C^∞, regular, Newtonian systems with arbitrary forces, i.e., the nonconservative and Galilei form-noninvariant systems of our everyday experience.

The study of this problem was the subject of ref.[23]. The conclusion of this study was that the problem of the relativity which is applicable to the systems considered is still open as of today. This suggests our fouth

ASSUMPTION 2.4.1: Under a discrete, nonrelativistic, classical approximation, the Galilei relativity is applicable to a stable meson as a whole with ut most electromagnetic interactions, while a covering relativity must be specifically constructed for the motion of the constituents of the same hadron.

The hope of this assumption is to attempt another methodological differentiation between the problem of classification or "exterior behaviour" of the hadrons and that of the structure. Notice the dichotomy constituted by the assumption of established relativity ideas (under the restrictions considered) for the "exterior problem" of hadrons and the assumption of a covering relativity for the case of the "interior problem" (only).

In essence, we argue that charged stable mesons under ut most electromagnetic interactions must obey experimentally established relativity laws. This is the case, for instance, of the "pionic atom", i.e., the bound state under the Coulomb force (only and at distances >> 1F) of a π^+ and a π^-. At the discrete, nonrelativistic, classical limit this state is essentially that of a conservative Coulomb system, where the term "conservative" is intended to express the fact that the particles move in vacuum (and, thus, without dissipative forces) or, equivalently, that the Coulomb force is the only acting force. Such a system, besides being conservative, is manifestly invariant under the Galilei transformations. We are therefore within the context of an "arena of unequivocal applicability" of the Galilei relativity (in the language of ref.[23]). This, incidentally, implies (by using expansion techniques) the unequivocal applicability of Einstein special relativity under a relativistic extension for the same context, as in any case experimentally established. In turn, this implies the applicability of Einstein general relativity for the exterior problem.

In the transition to the problem of structure, the relativity profile results to be profoundly altered by Assumption 2.3.1. This is due to a number of aspects considered in ref.[23], such as the fact that the forces considered are now nonconservative and Galilei form-noninvariant. In turn, this creates the problem of the Newtonian relativity which is applicable to these forces. The contention of Assumption 2.4.1 is that a covering relativity specifically conceived for the forces considered must be built.

Among the numerous problems which are opened by Assumption 2.4.1, a fundamental one is whether such dichotomy of conventional relativity ideas for the "exterior problem" and a covering relativity for the "interior problem" can be technically realized in a consistent way. The line of study which will be attempted in this paper is via subsidiary constraints. In unpedagogical terms, we shall impose the validity of the conventional Galilei relativity for the indicated behaviour of a "stable" mesons as a whole as a "subsidiary constraint" to any conceivable generalized model of structure. The hope of this approach, in case results to be consistent, is to recover in full the available experimental evidence on the validity of established relativity laws for the "exterior problem" of the hadrons considered, while any departure from these laws should occur, by construction, only in the "interior problem".

Our epistemological argument is essentially the following. Consider, again, the π^0 as a bound state of a massive, charged particle and its antiparticle, and as isolated from the rest of the universe. Assumption 2.3.1 essentially implies that each constituent is in highly nonconservative conditions which, as we shall see better later on, result in the exchange (and thus, nonconservation) of its physical characteristic with the other constituent. But the system is isolated from the rest of the universe. Thus, as a whole, the system must be conservative. In different terms, the nonconservation of a physical quantity of one constituent must be entirely compensated by that of the other constituent in such a way to result in the conservation of the corresponding total quantity. Equivalently, this is a particular case of the conventional notion according to which the conservation of a total quantity does allow the nonconservation of its individual terms. The novelty is that it is realized via forces not derivable from a potential. It is this aspect, as we shall see, that demands the use of subsidiary constraints to ensure the conservation of total quantities, while such subsidiary constraints are unnecessary for the case of conservative forces.

Assuming that this dichotomy of relativity ideas for the exterior and the interior problem can be consistently resolved at the nonrelativistic level, the next fundamental problem is whether the approach will demand a consequential construction of a covering of the Einstein special relativity for the relativistic (classical, discrete) extension. This problem was only briefly touched in ref.[23]. In essence, it was recalled that local forces not derivable from a potential can be subject to a number of relativistic extensions. As a result, the issue here considered cannot be resolved without a specific study. It was however argued that the issue will fundamentally depend on the nature of the relativity which emerges to be applicable at the Newtonian level. For instance, if such relativity will result to be of Lie-admissible type, a consequential, necessary generalization of the special relativity appears to be conceivable (owing to the strict Lie algebraic character of the latter versus the non-Lie algebraic character of the former). In turn, this could imply the need for a consequential covering of the general theory of relativity (owing to an indicated possible incompatibility of the Riemannian geometry with Lie-admissible algebras) but, again, for the interior problem only. It is in respect to these latter issues that we shall restrict ourselves to only few incidental remarks in the hope that they can be of some assistance for the interested reader, and our analysis will be essentially restricted to only the discrete nonrelativistic profile.

With an open mind on these issues, an implication of Assumption 2.4.1 should be here indicated. The assumption essentially characterizes an attempt of differentiating the atomic and the hadronic structure via the applicable relativity laws. By again using the positronium and the π^o on a comparative basis, the Galilei relativity is fully applicable to the positronium structure, while for Assumption 2.4.1 the same relativity under the same classical, nonrelativistic and distrete approximation, is not applicable to the π^o structure. It is precisely in this differentiation that our hopes of attempting a representation of the differences of the structures considered rely.

Irrespective of Assumption 2.3.1, we argue that, perhaps, the assumption that the relativity for the π^o constituents is the same as that of the positronium constituents implies an equivalent dynamical characterization of these structures which, in turn, is insufficient to represent their physical differences.

At its extreme potential implications, Assumption 2.4.1 implies an attempt of differentiating the electromagnetic and the strong interactions in the physical space of their experimental detection via the applicable relativity laws: the established relativity laws for the former and possible covering laws for the latter.

Again, we argue that, if the strong interactions are assumed as obeying the same relativity of the electromagnetic interactions (e.g., Einstein special relativity) this necessary implies a form of equivalent dynamical behaviour. We then argue that, if the differentiation between these interactions is attempted with the only additional presence of unitary internal degrees of freedom for the strong, this is perhaps insufficient to represent the physical reality, because the rather profound differences between the interactions considered occur in the physical space of their experimental detection, while the unitary degrees of freedom could only result as acting in a mathematical internal space.

When the problem of the strong interactions in general, and that of the hadronic structure in particular, are inspected from this profile, the established relativity laws appear in a different light: perhaps, they constitute a major obstacle for a genuine differentiation of the electromagnetic and the strong interactions in general, and of the atomic and hadronic structures, in particular.

TABLE 2.5: A FIFTH POSSIBLE DICHOTOMY: ESTABLISHED QUANTUM MECHANICAL LAWS VERSUS THE SEARCH FOR COVERING LAWS.

The last aspect of our critical analysis of the quark models is that related to the problem of the applicable quantum mechanical laws. In essence, our analysis of the models considered is centered on a critical inspection of the used disciplines, rather than the models per sé. Our central contention is that the current problematic aspects of the quark models could in the final analysis only result to be the symptoms of a more fundamental consistency problem. Irrespective of whether this is the case or not, the most effective way to achieve, in due time, a final resolution either in favor of the quark models or against, is that of establishing the validity or invalidity of the used fundamental laws in an incontrovertible experimental form.

In essence, if the established relativity and quantum mechanical laws (which, as is well known, are at the foundation of the quark models in virtually all available variations) will result to be valid for the hadronic constituents, the quark models are likely the only conceivable structure models. Intriguingly, this possibility appears to be centrally dependent on the nature of the strong hadronic forces which, as such, acquires a fundamental physical role. The problem, in this case, will then be that of finding the "right quark model".

However, if the established relativity and quantum mechanical laws result to be inapplicable to the hadronic constituents and covering laws must be identified, the quark models on the hadronic structure are likely to be ruled out in a final form, and fundamentally different models are likely to be needed in this case. It should be stressed, in line with the remarks of Table 2.1. that, if this is the case, by no means this will necessarily rule out unitary models from the hadronic phenomenology. New insights in theoretical physics never "destroy" previous accomplishments of proved physical relevance. They only identify their arena of applicability. For the issue under consideration, the physical relevance of models for the problem of hadron classification appears to be established on rather unequivocal terms. The possible invalidity of the quark models of structure here referred to would leave this physical relevance unaffected. This is, in essence, the spirit of Assumption 2.1.1.

In conclusion, the experimental verification of established laws advocated in this paper could likely result to be a final test for the validity or invalidity of the quark models of structure only, while leaving the validity of the same models for the hadronic classification unaffected.

These remarks have here been made for the intend of stressing the crucial nature of the problem of the applicable quantum mechanical laws. The remarks of relativity nature of Table 2.4 are per sé inapplicable for an experimental verification because classical. Clearly, in order to attempt a possible, future, experimental verification, a consistent form of quantization is needed for the context of Assumption 2.3.1.

Along the line of study of this paper, the problem under consideration is whether conventional quantization techniques, laws, principles and insights are applicable or not to nonselfadjoint Newtonian systems (i.e., systems with local forces not derivable from a potential).

A simplistic solution of this issue based on mere relativity considerations would be in the negative. In essence, the transition from the Galilei relativity to the covering Einstein special relativity demanded the implementation of quantum mechanics into a covering relativistic discipline. If a nonconservative, classical, nonrelativistic covering of the Galilei relativity exists, this might inevitably imply the existence of a corresponding covering of quantum mechanics for which conventional laws, principles and insights are inapplicable as currently known in favor of covering notions.

It is in this respect where the algebraic profile might play a crucial role. We are all aware of the fundamental physical role of Lie algebras in quantum mechanics, e.g., for Heisenberg's time evolution law, for the construction of the conventional quantum mechanical notion of spin, etc. In the transition to relativistic quantum mechanics this algebraic character persists and therefore implies the preservation of a number of fundamental notions and only their generalization to a relativistic context. If a nonconservative, classical and nonrelativistic covering of the Galilei relativity exists and results to be of non-Lie type (e.g., Lie-admissible), its potential implications at a quantum mechanical level are predictably conspicuous, to the point of rendering conceivable the existence of a nonrelativistic covering of quantum mechanics for local forces not derivable from a potential, as we shall indicate in Section 4.

The problem under consideration, however, is such to also warrant the study of the quantum mechanical profile in a way independent from relativity considerations. A number of aspects in this case become relevent. Some of the most significant are the

quantization of nonconservative systems within the context of statistical-thermodynamical formulations and that of discrete nonrelativistic mechanics (along much of the lines of conventional quantum mechanics, but now referred to broader forces). Within the former context, major results have been achieved, most notably, by I. PRIGOGINE and his collaborators.[25] Within the latter context, the problem appears instead to be essentially open at this time, despite a number of attempts. It goes without saying that the analysis of Section 4 in relation to this issue is largely insufficient even for a partial study of the issue considered. It is primarily intended to stimulate the awareness of our community on the existence of the problem of quantization of forces not derivable from a potential.

Besides technical considerations, there is an epistemological aspect which should not be overlooked. Here a comparative analysis of the corresponding situation for the atomic structure might again provide valuable intuitional insights.

As it is known, the only disciplines available at the time of the inception of the Mendeléyev classification of atoms were classical (as well as nonrelativistic). The problem of the structure of the atoms was initially confronted with the use of the same disciplines which had proved so effective for the problem of classification. A number of consistency problems subsequently forced the acceptance of a radical departure from classical attitudes. The solution of the problem of the atomic structure simply demanded the courageous construction of a new discipline, quantum mechanics, specifically conceived for the interpretation of the atomic phenomenology (spectral lines, stability of the electron's orbits, etc.).

In conclusion, the problems of the atomic classification and structure, even though can be treated nowadays within the context of quantum mechanical formulations, were initially resolved with different but compatible disciplines. With reference to the group of the palladium considered earlier, we can therefore symbolically write

ATOMIC PROBLEM

Classification: classical formulations.

Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd

Structure: quantum mechanical formulations.

A comparative analysis of the above occurrence with the corresponding occurrence for the quark models of hadrons indicates the existence of an additional differentiation in attitude of the hadronic versus the atomic problem. In unpedagogical terms, a fundamental idea of the quark models is that of implementing with unitary internal degrees of freedom the discipline previously established for the electromagnetic phenomenology, quantum mechanics. The basic notions, laws and principles of this latter discipline, however, remain unaffected by such a unitary implementation. With reference to the previously considered family of "stable" mesons we can then symbolically write

HADRONIC PROBLEM

Classification: quantum mechanics.

π^0	π^+	π^-	K^+	K^-	K_S	K_L	η

Structure: quantum mechanics.

Our epistemological argument is that, irrespective of any preceding remark, a differentiation of the disciplines used in the hadronic classification and structure might in the final analysis result to be needed and, in any case, cannot be ruled out on grounds of our current knowledge. In essence, the issue here referred to is whether the available quantum mechanical formulations are capable of producing a final representation of the hadronic phenomenology. The major basis of judgment which is available at this time is that of the unitary models. But these models have been unable to achieve the objective considered in a final, incontrovertible form up to this moment, despite the partecipation of a rather significant part of the community of basic studies. Our attitude is by now familiar. Studies along the current unitary trends within the context of established disciplines must be continued because their possibility of achieving the objective considered cannot be ruled out. Jointly, different insights should be attempted.

Within the context of the latter alternative, a fundamental issue is the identification of a possible generalized discipline specifically conceived for the hadronic constituents which is capable of producing a representation of the hadronic structure in the desired incontrovertible form. Inevitably, this demands the joint resolution of the problem

of the hadronic constituents. We reach in this way our fifth

ASSUMPTION 2.5.1: Under a discrete nonrelativistic approximation, conventional quantum mechanics applies to a stable meson as a whole under ut most electromagnetic interactions, while a covering discipline specifically constructed for the constituents of the same hadron must be built.

The argument for the first part of this assumption is essentially the following. As indicated earlier (Table 2.4) a hadron, when considered as isolated from the rest of the universe, exhibits the conservation of all its physical characteristics. Conventional quantization procedures of these total quantities (e.g., the spin) must then apply, as experimentally established in any case. The situation for the hadronic constituents is potentially different. Again, if a total quantity is conserved, this does not necessarily imply that its individual components are conserved too or that the acting forces are (local and) conservative. Subject to an explicit verification to be attempted in Section 4, the possibility that the acting forces are (local but) not derivable from a potential cannot be ruled out. In this case, as we shall indicated, all the physical characteristics of an individual constituent are expected to be nonconserved, of course, in a way compatible with the conservation of the corresponding total quantities. It is this aspect which suggests the search for possible covering quantization rules for the "interior problem" only. In this way no deviation from established quantum mechanical laws is expected for the "exterior problem" of a hadron.

To attempt an initial identification of the problem to be confronted, consider a massive, charge "stable" particle with a finite charge volume which penetrates into an atomic structure. Owing to the large distances of the atomic constituents, the motion of such a particle while within the atomic structure can be well approximated as a motion in the physical vacuum (here intended as the absence of matter-to differentiate it from the quantum field theoretical vacuum) under electromagnetic interaction. The applicability of conventional quantum mechanical laws is in this case unequivocal. Suppose now that the same particle penetrates into a hadronic structure or, better, is created in the core of, say, a neutron star undergoing a phase transition to the hadronic constituents, and preserves its identity for a sufficiently long period of time (at the hadronic scale). In this case the approximation of the motion as occurring in the physical vacuum is questionable owing to finite charge volume of the particle and the high density of hadronic matter (whether a hadron or the core of a neutron star). As a matter of fact an opposite conjecture is now more plausible, that the motion occurs in a physical medium, the "hadronic medium". As we shall see in the next sections, Assumption 2.3.1 is considered in the hope of attempting a more realistic representation of this different context. In turn, it is this type of motion which suggests the possible nonconservative of all physical characteristics of the particle considered during the motion considered, including and, most importantly, the intrinsic characterists. The need of searching for possible covering quantum mechanical formulations is then consequential.

The major expectation for such a possible generalized approach is that of providing some hope for a resolution of the problem of the hadronic constituents. The line of study which will be attempted in Section 5 is essentially the following. An intriguing experimental data on the mesons considered is that they exhibit spontaneous decays. We therefore argue that the (massive and charged) constituents of unstable hadrons could, in the final analysis, be produced in these decays in much the same way as the positronium admits the decay mode e^+e^-. According to this view, the unstable hadrons would essentially be a bound state of suitably selected massive and charged particles produced in the spontaneous decays (e.g., those of the decay modes with the lowest fraction) by therefore removing in this way the need of confinement as well as of conjecturing yet unidentified constituents.

The crucial point (which the interested reader is here urged to verify) is that such a view of the hadronic structure appears to be inconsistent with conventional quantum mechanics for much of the same reasons which ruled out the old idea that the neutron is a bound state of a proton and an electron. Our contention is that, if a covering quantum mechanical discipline is constructed for the structure problem, such incompatibility could be removed. As a matter of fact, much of our subsequent efforts will be devoted to ascertain the possible existence of such a covering discipline which allows the consistent representation of an unstable hadron as a bound state of suitably selected massive and charged particles produced in its spontaneous decays.

In conclusion, besides technical aspects, the assumed mental attitude emerges as crucial. If a researcher insists in the preservation of established quantum mechanical laws for the hadronic structure problem, the ultimate implications are the current problematic aspects for the hadronic constituents. If another researcher, instead, admits the possibility that broader laws exhist, new, intriguing, perspectives appear to be open. When inspected from this profile, the established quantum mechanical laws appear in a different light: perhaps, they constitute a major obstacle for the identification of the hadronic constituents with physical particles.

3. THE EPISTEMOLOGICAL ARGUMENT FOR A POSSIBLE NEW MODEL OF HADRONIC STRUCTURE.

In order to attempt an initial assessement of the degree of plausibility of the assumptions of Section 2 a number of sequential steps will be implemented. The key idea is that, in order to stimulate the experimental verification of the validity of established laws within a hadron along the lines of Section 1, an effective way is that of ascertaining whether their violation is conceivable on grounds of our current knowledge. In turn, this appears to demand the attempt of the explicit construction of a structure model based on such conjectural violation and its confrontation with the experimental data.

In this section we shall introduce the epistemological argument for a possible structure model of "stable" mesons based on the assumptions on Section 2. The problem of the applicable relativity and quantum mechanical laws will be considered in Section 4. Finally, the study of the degree of plausibility of this approach to the hadronic structure will be conducted in Section 5.

The reader should be aware that the assumptions of Section 2 have been conceived for the intent of producing a radical departure from current trends, in the hope of stimulating the line of study of Section 1.

For instance, Assumption 2.2.1 prohibits the use for the structure problem of all the "chemical" quantum numbers, such as isospin, hypercharge, color, flavor, etc., which play an essential role for the problem of a Mendeléyev-type classification of hadrons. Thus, the only considered physical quantities are those definable in the carrier space of the experimental detection of each individual hadron (the Euclidean or Minkowski space). These physical quantities will be classified into two groups: the kinematical quantities (or characteristics) of the hadronic constituents, such as the kinetic energy, linear momentum and angular momentum, and the intrinsic quantities, such as mass, charge, spin, electric and magnetic moments, space and charge parity. All the remaining internal quantities , such as isospin, hypercharge, etc., will be ignored for the hadronic constituents and considered only for the classification of hadrons. This is intended to be in line with Assumption 2.1.1 according to which unitary models are fully applicable to the classification and new models are considered only for the structure problem under the condition that they are capable of reproducing the unitary classification.

Assumption 2.3.1 implies an additional considerable departure from current trends. Typically, a central starting point of the available quark models of structure is a Lagrangian of the form $L_{tot} = L_{free} + L_{int}$. The underlying tacit assumption is that such a Lagrangian is capable of representing the hadronic structure in its entirety. According to Assumption 2.3.1, such a Lagrangian is insufficient to represent the structure considered because the additive term L_{int} to the free term implies that the strong hadronic forces are (local and) derivable from a potential (see ref.[23] for details). But this is only a first conceptual departure from conventional trends. A deeper departure is given by the fact that Assumption 2.3.1 prohibits, in general, the use of any Lagrangian as a central starting point. The reason is simple. When a Lagrangian of the type $L_{tot} = L_{free} + L_{int}$ is considered, the explicit computation of the equations of motion is trivial and often unnecessary because the forces are derivable from L_{int}. When forces not derivable from a potential are used instead, the computation of a Lagrangian becomes, by far, a nontrivial problem, assuming that it exhists in the considered local variables. The study of the integrability conditions for the existence of a Lagrangian for the systems considered and the methods for its computation belong to a discipline called in refs[21] the Inverse Problem of classical mechanics. In conclusion, Assumption 2.3.1 implies a sort of return "ad originem": the equations of motion are assumed as the fundamental physical quantities, while their Lagrangian representation is assumed as a derived notion of primary methodological significance.

Finally, to keep in mind an intended continuity with Section 4, the reader should be aware that Assumptions 2.4.1 and 2.5.1 imply an additional radical departure from conventional attitudes. Typically, when considering equations of motion, the conventional attitude is that of ascertaining the compliance of the equations with the applicable, established, relativity (Galilei or Einstein). The case for Assumptions 2.4.1 and 2.5.1 is exactly the opposite of that, in the sense that now the efforts are devoted to the identification of Newtonian equations of motion (field equations) which violate the Galilei (Einstein) relativity in some of the mechanisms identified in ref.[23]. Clearly, this is essential for the objective of studying the plausibility for their violation.

TABLE 3.1: THE POSSIBLE PHYSICAL ORIGIN OF STRONG HADRONIC FORCES AS NON DERIVABLE FROM A POTENTIAL.

As indicated in Table 2.3, the problem of the nature of the strong hadronic forces appears to be the truly fundamental problem of the hadronic structure in the sense that different forms of these forces may imply different applicable methodologies and, thus, different models of structure. The study of the plausibility of Assumption 2.3.1 is therefore crucial for our program.

The problem under consideration is to see whether there exists any physical basis for the assumption that the strong hadronic forces are local but not derivable from a potential.

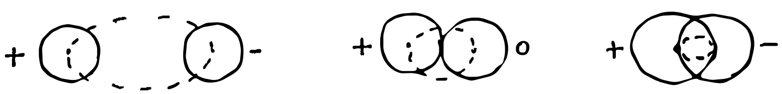
Our epistemological argument, in the simplest possible language, is the following. An intriguing experimental data of the hadronic phenomenology is that the charge volume of the hadrons does not sensibly increase with the mass and it is of the same order of magnitude of any other, charged, physically known particle ($\sim$ 1 F). If the hadronic constituents are assumed as massive, charged and physical particles, that is, also possessing a finite charge radius, the hadronic structure emerges as being substantially different than the atomic and the nuclear structure according to the following comparative lines.

(1) Atomic structure. From the established experimental data, the relative distances of the constituents are in this case much greater than the charge radius of the constituents.

(2) Nuclear structure. From the established experimental data we know that the nuclear charge volume is (approximately) proportional to the number of nucleons and such that the charge volumes of the constituents are "very close" to each other.

(3) Hadronic structure. From the established experimental data on the charge volume of the hadrons it follows that, if the constituents also possess a finite charge volume of the same order of magnitude, each constituent is bounded within the charge volumes of the others.[26]

In different terms, starting from the very large distances between the charge volumes of the constituents for the atomic structure, and going through the intermediate level of the nuclear structure in which the charge volumes of the constituents are very close to each other, we reach the hadronic level in which, to account for a non-point-like structure of their charged constituents, we have a "penetration" of the charge volume of each constituent with that of the others, according to the following schematic, comparative, view



Positronium   Deuteron π^{0}

This conception of the hadronic structure will inevitably imply nonlocal forces not derivable from a potential for any rigorous study. The technical problems for the treatment of these forces, however, are expected to be conspicuous (assuming that we possess the necessary knowledge at this time). In order to attempt any structure model which can be quantitevely treated at this time, an approximation of these forces is in order.

We reach in this way a crucial point of the analysis of this paper. We know from classical mechanics that local forces not derivable from a potential constitute a good approximation of nonlocal forces in general. We therefore argue that the possible physical origin of strong nonselfadjoint forces is that of constituting an approximation of more general forces which appear to be essential to account for a non-point-like structure of the charged constituents of hadrons. In essence, Assumption 2.3.1 emerges from these considerations as representing a sort of intermediate step between the conventional quark models with point-like constituents and strong selfadjoint forces, and expected nonlocal generalizations for non-point-like constituents. Equivalently, Assumption 2.3.1 can be considered as characterizing a first nontrivial generalization of available models but of approximate character, prior to the possible future study of more rigorous models.

A few comments are here in order. First of all, it should be indicated that Assumption 2.3.1 implies the possibility of interpreting the strong hadronic forces as constituted by the

superposition, in general, of two classes of forces, a first group (local and) derivable from a potential along much of the available models, and a second group (local and) non-derivable from a potential. In different terms, according to the assumption 2.3.1, the strong hadronic forces are only partially nonderivable from a potential, and ordinary forces are fully admissible. This is clearly dictated by nuclear considerations.

As we shall indicate in Section 4, this latter feature has potentially crucial methodological implications because one can start from available models of bound states with forces derivable from a potential and of Lie algebraic character and then add suitably selected local forces non-derivable from a potential by therefore rendering applicable the Lie-admissible formulations along the lines of ref.[23]. In particular, one can also attempt as a first study the addition of these latter forces with infinitesimal coefficients by therefore rendering potentially applicable a Lie-admissible (rather than Lie) deformation theory, in this case, of the first order.[23]

Another crucial implication of the picture of the hadronic structure proposed in this table is that it implies a differentiation between the nuclear and the strong hadronic forces as a direct consequence of the difference of charge volume distribution per each structure. More intriguingly, if the strong hadronic forces will eventually result as being nonderivable from a potential, this will likely call for a reinspection of the problem of the nuclear forces. In this case the contribution from the more general forces considered can ut most be of infinitesimal nature, by therefore bringing into focus in a natural way the potential significance in nuclear physics of a first order Lie-admissible deformation theory, as conjectured in ref.[23] It is of course understood that, whether nuclear of hadronic, the component of the forces which is derivable from a potential can be of either central or noncentral type. In other words, we are here referring only to a possible implementation of the available nuclear forces with contributions from terms not derivable from a potential, and not to a modification of these established forces.

A further crucial implication of the hadronic structure model here considered is that the strong hadronic forces are not expected to be unique. Instead, a hierarchy of forces of increasing complexity and methodological needs is expected to occur, as the reader can see by inspecting, for instance, the transition from two to three constituents, each bounded within the charge radius of the other. When reinspected from this profile, the indicated possible differentiation between the nuclear and the hadronic forces appears as a first indication of possible deeper differentiations within the hadron, depending on the complexity of the structure considered.

The indicated view of the strong hadronic forces implies a differentiation with the electromagnetic forces on a number of counts. First of all, the electromagnetic and the strong forces emerges as being analytically nonequivalent in the sense that the former are fully derivable from a potential, while the latter are not. Secondly, while the explicit form of the electromagnetic forces (e.g., the Lorentz force) is unique, this is not the case for the strong forces considered, because several functionally nonequivalent forms non-derivable from a potential become admissible, depending on the structure considered. Nevertheless, according to this view, the electromagnetic and the strong forces are both defined in terms of the local variables of the experimental detection and their derivatives (or the field components and their derivatives), without any recursion to unitary internal degrees of freedom. As by now familiar, this point appears to be crucial to attempt a dynamical differentiation of the electromagnetic and the strong interactions in the physical space of their experimental detection.

TABLE 3.2: THE CONCEPT OF HADRONIC CONSTITUENT, CALLED ELETON, AS A MUTATION OF A CONVENTIONALLY QUANTIZED PARTICLE.

To make further progress, it is here essential to identify in more details the notion of hadronic constituent which emerges from the analysis of the preceding table. A comparative analysis of simple atomic and hadronic structures is of some intuitional value.

Consider, again, the positronium. It is known that electromagnetic interactions do allow the exhange of kinematical quantities (kinetic energy, linear momentum and angular momentum) between the constituents while prohibit the exchange of intrinsic quantities (spin, charge, etc.). In different terms, the notion of positronium constituent we are here referring to is that of a particle (an electron or a positron) whose kinematical characteristics are generally nonconserved, but in such a way that the corresponding total quantities are conserved. The point which is relevant for the analysis of this paper is that the positronium constituents preserve their individual intrinsic characteristic during the entire life of the system. Most importantly, the constituents are spin $1/2$ Fermions and remain spin $1/2$

Fermions.

Consider now the π^o as a bound state of a massive, charged, non-point-like particle and its antiparticle under Assumption 2.3.1 for the strong hadronic forces (local and not entirely derivable from a potential). As it is well known in Newtonian mechanics, these forces are nonconservative. As a result, they also imply the nonconservation of the kinematical quantities of each individual constituent (the problem of the conservation of the corresponding total quantities will be considered in Table 3.4). An issue of fundamental physical relevance is, however, whether the intrinsic quantities of each individual constituent are conserved or not. Our contention is that, if these quantities are conserved, as for the positronium constituents, we imply a dynamical equivalence between the electromagnetic and the strong interactions which, perhaps, is not sufficient to represent their physical differences (e.g., the differences between the positronium and the π^o, both considered as a bound state of a charged particle and an antiparticle).

We reach in this way our sixth

> ASSUMPTION 3.2.1: The constituents of the "stable" mesons are massive, charged, spinning and non-point-like particles of charge radius of the order of 1F for which all physical characteristics of both kinematical and intrinsic type are generally nonconserved, but such that the corresponding total quantities are conserved and quantized according to conventional rules.

One of the primary implications for which the above assumption is in the final analysis conceived is that if a hadronic constituent is, say, a spin $1/2$ Fermion at a given instant of time, it does not necessarily preserve the value of the spin and, thus, its statistical character during its evolution in time, even though the hadron as a whole has a fixed, conventionally quantized value of the spin.

The epistemological argument which suggests the study of the above assumption is essentially the following. At the level of the atomic structure the preservation of the "perennial value" of the spin of the constituents is fully admissible on intuitional grounds owing to the large distances involved. In the transition to the hadronic case the situation is different. Consider, for instance, a particle produced in the core of a neutron star under the assumption that it is massive, charged and non-point-like. But then the preservation of the value of the spin becomes unappealing on intuitional grounds, owing to the penetration of its charge volume with those of other hadronic constituents during its evolution in time, i.e., the motion of a state of finite dimension in a medium of high density. Mutual interferences in this case could well result in a "mutation" of the value of the spin.

In conclusion, Assumption 3.2.1 essentially implies that the same massive, charged, spinning and non-point-like particle can exists in a hierarchy of dynamical conditions. First, there is the conventionally quantized state under electromagnetic interactions (only) as experimental detected with currently available techniques. In this case its kinematical characteristics are generally nonconserved, but the intrinsic characteristics are strictly conserved. In particular, and most importantly, the particle preserves the value of its spin and, thus, its statistical character. Secondly, the same particle can exhists in a "mutated" form when penetrates into a hadronic medium (only) in which case, in addition to the nonconservation of the kinematical quantities, there is the nonconservation of the intrinsic quantities. In particular, and most importantly, the value of the spin and, thus, the statistical character, is not necessarily preserved during the motion within a hadronic medium (only).

It might be of some value to indicate that this concept of "mutation" of a conventionally quantized and experimentally detected particles (under electromagnetic interactions) is suggested, besides the indicated epistemological arguments, by one of the most significant nonassociative algebras of potential applicability, the Lie-admissible mutation algebras[23,27] $A(\lambda,\mu)$, as we shall indicated in more details in Section 4. In other words, the concept of mutation we are here attempting is inspired by algebraic consideration suggested by the forces here considered (not derivable from a potential) and the corresponding algebras which have been indicated in ref.[23] as being in this case directly applicable (the strictly non-Lie, but Lie-admissible algebras).

Somewhat reluctantly owing to the recent proliferation of new words in physics, it appears advisable to here introduce a new name for the proposed concept of hadronic constituent for both, concise subsequent reference, as well as for its necessary differentiation from the known concepts of hadronic constituents, such as quark and parton.

After some selection and for reason which will be indicated later on during the course of our analysis, I hereby submit the name of

ELETON (or ANTIELETON)

from the greak meaning (with some linguistic licence)

έλη ύλη ≈ corpuscle of matter

(I would like here express my gratitude to C. N. Ktorides for assistance in this greek meaning of the term).

Thus, the term eleton (or antieleton) stands for a massive, charged, spinning, non-point-like particle within a hadronic medium in such high dynamical conditions, to have all its physical characteristics, of both kinematical and intrinsic nature, generally nonconserved. For instance, the difference between an electron (as commonly understood) and an eleton is that the former refers to a conventionally quantized particle under electromagnetic (long-range) interactions only which exhibits a "perennian value" of the spin and of its statistical character, while the latter refers to a particle under electromagnetic and strong hadronic forces (the latter being short-range and not derivable from a potential) whose value of the spin and statistical character is not necessarily preserved. It should be stressed that, when the short-range forces not derivable from a potential are null, the conventionally quantized notion of particle under electromagnetic interactions only is recovered in full. Thus, the concept of eleton is a covering of that of conventionally quantized particles. It should also be stressed that, according to our assumption, the same particle can exists in both, the conventionally quantized state and the "eletonic state", depending on the region of space in which exists (an atomic cloud or, say, the core of a neutron star).

As the attentif reader has by now identified, the notion of eleton has numerous implications from a relativity profile. We reach in this way the essence of this paper. As by now familiar, an objective of this paper is to stimulate the awareness of the physics community on the need to verify with experiments the validity of Einstein special relativity within a hadron, as a fundamental prerequisite for the future achievement of a final resolution of the problem of the hadronic structure. This objective is attempted by studying the plausibility of a possible violation of the relativity considered in the arena considered. The reader should therefore be aware that the mental attitude implemented in this paper is different than that of current papers in hadron physics. Here, all our efforts are centered in studying the maximum possible violation of established relativity ideas which can be identified on grounds of our technical knowledge at this time.

On grounds of pure scientific curiosity, we can therefore say that the concept of eleton is attempted as a realization of Assumption 2.4.1 (on the nonapplicability of the relativity considered in the arena considered). More technically, we can say that for a (local) field to be a genuine representative of an eletonic particle it must not transform covariantly under the Poincaré group. This aspect will be studied in more details in Sections 4 and 5. At this point a conceptual argument should be sufficient to identify our objective.

In essence, if a(local) field is subjected to the condition of transforming covariantly under the Poincaré group according to conventional approaches, the net effect is the preservation of the value of the spin of the state considered and, thus, its statistical character in the conventionally known way. In turn, this implies the lack of representation of an eletonic particle. On the contrary, to achieve a technical characterization in field theory of a particle with varying spin and statistical character, the lack of covariance of the representative field under the Poincaré group appears to be a necessary prerequisite.

This situation is created not only by the strict Lie algebraic character of the Einstein special relativity, but more specifically, by the SU(2)-spin (Lie) subalgebra of the Poincaré (Lie) algebra. A study of the representations of the SU(2) algebras then indicates the inability of this algebra to characterize a varying spin.

Of course, it remains to be seen whether this notion of eleton can be technically realized in due time. The conceivable line of study which is advocated in this paper is an embedding of the SU(2) Lie-spin algebra into a covering SU(2)-Lie-admissible-spin algebra, along the lines of ref.[23]. For the reader in search of a rudimentary Newtonian correspondent of this context, we can indicate the transition from the SO(3)-Lie treatment of the conservative top, versus the SO(3)-Lie-admissible treatment of the spinning top with additional drag torques not derivable from a potential and responsible for the decaying in time of the angular momentum, as conform to experimental evidence.

The attentif reader has by now also identified technical inconsistencies of the notion of eleton with conventional quantum mechanical notions, laws and principles. The point is that this is precisely the desired occurrence. It is sufficient at this point to say that the concept of a particle with varying spin does not appears to be treatable with conventional quantum mechanical rules. Of course, it remains to be seen whether a generalization of quantum mechanics capable of properly characterizing the concept of eleton can be consistently achieved. On grounds of scientific curiosity, this is perhaps one of the most intriguing problems we have identified until now.

TABLE 3.3: THE PRIMITIVE NEWTONIAN FORM OF THE EQUATIONS OF MOTION FOR AN ELETON.

In order to attempt a technical characterization of the concept of eleton it appears advisable to first identify the equations of motion at the primitive Newtonian level. This may provide invaluable intuitional guidance, as well as an initial basis for the identification of the technical problems to be confronted for a consistent quantization. The reader should be aware, from the remarks in the introduction to this section, that the conventional starting point of virtually all contemporary physical contributions, a Lagrangian, is inapplicable and potentially misleading in our case. It is inapplicable because the assumed forces are not derivable from a potential and, thus, the problem of the construction of a Lagrangian (when and if it exhists) is by far nontrivial. It is potentially misleading because, as indicated in ref.[23], a possible, admissible, Lagrangian has an arbitrary functional structure (the conventional structure $L_{tot} = L_{fre} + L_{int}$ is here prohibited) which, in turn, exhibits rather complex degrees of functional arbitrariness (the isotopic transformations of refs[21,23]). The former occurrence creates problems of the direct physical interpretation of the acting forces (which is absent in the conventional conservative case case-the forces in this case being uniquely characterized by L_{int}). The latter occurrence creates additional, unnecessary technical problems for a consistent quantization (also present in the conservative case), as will be indicated in Section 4.

Let us begin from established grounds. The positronium structure, at a primitive Newtonian limit, is a two-body conservative Coulomb system in vacuum (that is, the particles move in vacuum under only the Coulomb force-thus yielding a conservative system). In their simplest possible form, the equations of motion can be written

$$\left[m_e \ddot{r}_{k} - f_{k}(r) \right]_{SA}^{C^\infty, R} = 0, \quad k = 1, 2, \quad m_e = m_{electron}, \tag{3.3.1a}$$

$$f_{k} = f_{k}^{Coulomb} = -\frac{\partial V(r)}{\partial r^k}, \quad r = |r_1 - r_2|, \tag{3.3.1b}$$

where C^∞ indicates that the acting forces are infinitely differentiable, R denotes that the functional determinant of the system (given by a power of the mass) is nonnull (and thus, the masses are nonnull) and SA denotes variational selfadjointness. This latter property is the technical characterization in the language of the Inverse Problem of the fact that the Coulomb force is derivable from a potential.[28]

If the more general Lorentz force is considered, we shall write

$$\left[m_e \ddot{r}_{\mu k} - f_{\mu k}(t, r, \dot{r})\right]_{SA}^{C^\infty, R} = 0, \quad k = 1, 2, \tag{3.3.2a}$$

$$f_{\mu k} = f_{\mu k}^{Lorentz} = -\frac{\partial U(t, r, \dot{r})}{\partial r^k} + \frac{d}{dt}\frac{\partial U(t, r, \dot{r})}{\partial \dot{r}^k}, \tag{3.3.2b}$$

where the reader should keep in mind the crucial preservation of the selfadjointness property (the Lorentz force is variationally selfadjoint in Euclidean, Minkowski and Riemannian spaces).[28] Also, the reader should recall that, upon quantization, systems (3.3.1) and (3.3.2) describe conventionally quantized systems.

We now implement these systems in accordance with Assumption 2.3.1. This simply demand the addition of local class C^∞ forces which, as a fundamental requirement, this time are not derivable from a potential and we shall write

$$\left\{\left[m \ddot{r}_{\mu k} - f_{\mu k}(t, r, \dot{r})\right]_{SA}^{C^\infty, R} - F_{\mu k}(t, r, \dot{r})\right\}_{NSA}^{C^\infty, R} = 0, \tag{3.3.3}$$

where NSA stands for nonselfadjointness and constitutes the technical characterization in the language of the Inverse Problem of the property that the F-forces are not derivable from a potential.[28]

System (3.3.3) constitutes the primitive Newtonian characterization of the motion of eletonic particles. The reader should be aware that numerous technical implementations are needed to actually achieve a technical characterization of an eleton. System (3.3.3) is only the most rudimentary possible first step.

It might be of some assistance for the interested reader to recall that the Inverse Problem provides a dual characterization of systems (3.3.3) into

NONESSENTIALLY NONSELFADJOINT SYSTEMS.[23] These are, in essence, systems (3.3.3) which, even though prohibit a direct analytic representation of the type possible for systems (3.3.1) and (3.3.2), i.e.,

$$\frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{r}^k} - \frac{\partial L_{tot}}{\partial r^k} \equiv \left[m \ddot{r}_{\mu k} - f_{\mu k}\right]_{SA}^{C^\infty, R}, \tag{3.3.4a}$$

$$L_{tot} = L_{free} + L_{int}, \quad L_{int} = -V \text{ or } = -U, \tag{3.3.4b}$$

they are such to admit an indirect analytic representation via a class C^∞, regular matrix of factor terms, i.e.

$$\frac{d}{dt}\frac{\partial L^{gen}}{\partial \dot{r}^{ka}} - \frac{\partial L^{gen}}{\partial r^{ka}} = \left\{h_{ka}^{\ \ jb}(t, r, \dot{r})\left[\left(m \ddot{r}_{jb} - f_{jb}\right)_{SA}^{C^\infty, R} - F_{jb}\right]_{NSA}^{C^\infty, R}\right\}_{SA}^{C^\infty, R},$$

$$k, j = 1, 2, \ldots, \quad a, b = x, y, z, \tag{3.3.5}$$

in which case the emerging Lagrangian can be computed via the techniques of the Inverse Problem and result to have a generalized structure of the type

$$L^{gen}(t, r, \dot{r})$$

$$= \sum_{k} \sum_{a=x,y,z} L^{(ka)}_{int, I}(t, r, \dot{r})\, L^{(ka)}_{free}(\dot{r}^{ka}) + L_{int, II}(t, r, \dot{r})$$

$$= \frac{1}{2}\left[\dot{r}^{ia} G_{ia;jb}(t, r, \dot{r})\dot{r}^{jb} + 2\dot{r}^{ia} F_{ia;jb}(t, r) r^{jb} + r^{ia} E_{ia;jb}(t, r) r^{jb}\right]. \tag{3.3.6}$$

ESSENTIALLY NONSELFADJOINT SYSTEMS.[23] These are systems (3.3.3) which are such to violate the integrability conditions for an indirect analytic representation of type (3.3.5). In this case a Lagrangian for the analytic representation of the systems considered in the variable considered does not exist.

Before proceeding with the consideration of subsequent aspects, a few comments appear to be recommendable. In particular, a comparative analysis of the Lorentz force and the "eletonic forces" (i.e., the F-forces) may be of some value.

Intriguingly, the Lorentz force and the eletonic forces have several features in common. They are both local and of class C^∞ and, most importantly, they have the same functional dependence, i.e., they both depend explicitly on time, coordinates and velocities. Their first major difference is that they are analytically nonequivalent, that is, the Lorentz force is derivable from a potential (selfadjoint) while the eletonic forces are not (nonselfadjoint). By recalling that the former (latter) are representative of the electromagnetic interactions (strong hadronic forces), this is in essence our idea of achieving a dynamical differentiation between the interactions considered in the (Euclidean, in this case) space of their experimental detection.

Notice, in conformity with Assumption 2.2.1, the complete absence of internal unitary degrees of freedom. This absence is crucial to attempt a characterization of the structure of each hadron of a given unitary multiplet of classification as an individual, unique, entity per sé, in much the same way as it occurred at the atomic level. Therefore, the reader should expect that all our efforts will be devoted in attempting the complete absence of all unitary numbers (isospin, hypercharge, color, flavor, ecc.) at each and every level of our analysis for the structural part.

The methods of the Inverse Problem produce significant informations on the indicated analytic nonequivalence of the forces considered. In essence, a necessary condition for a force $f(t, \underset{\sim}{r}, \dot{\underset{\sim}{r}})$ to be variationally selfadjoint is that it is linear in the velocity.[28] This is exactly the case for the Lorentz force. The eletonic forces are therefore expected to be generally nonlinear in the velocities.[28]

This is a realization of our idea of attempting the simplest possible generalization of the analytic structure of the electromagnetic forces for the strong interactions, as an approximation of an expected, substantially more complex physical reality. In essence, we argue that the velocity dependence of the acting forces is already present at the electromagnetic level. It only occurs in its simplest possible form, the linear form. The logical, simplest possible generalization is then that of a nonlinear dependence in the velocities for the strong hadronic forces. But this implies the lack of derivability of the forces considered from a potential (we here refer to the forces themselves and not to an indirect Lagrangian representation of the complete systems). In turn, at a primitive Newtonian level, this implies that the forces considered are nonconservative. Still in turn, this has fundamental methodological implications (to be indicated in the next section) because it implies the transition from the typical methodological setting of conservative mechanics to broader disciplines for the treatment of nonconservative mechanics.

Another crucial differentiation of the Lorentz force and the eletonic forces is that the functional dependence of the former is unique, while this is not the case for the latter, and a virtually infinite variety of different functional structures are in principle conceivable. On technical grounds, this is expressed by the classification of the eletonic forces into nonessentially and essentially nonselfadjoint, with each group having its own subclassification depending on a number of technical aspects here inessential.[21,23]

This is a primitive realization of our idea that, in the transition from, say, the π^0 (135MeV) to the K* (1420 MeV), a possible increase in complexity of the acting force should not be excluded on grounds of the increase in the rest energy (by a factor of the order of 10) without any appreciable increase of the charge volume.

It then follows that one of the most crucial aspects of the line of study of this paper is the proper selection of the explicit form of the eletonic forces $\underset{\sim}{F}_k$. This problem is practically rendered more complex by the fact that these F-forces are in actuality composed of two groups, one fully derivable from a potential and one not, as indicated in Table 3.2. Almost needless to say, to even partially confront this problem, the prior identification of the methods for their treatment is necessary. Thus, the selection of an explicit form of the eletonic forces is expected to be of (at least) dual and sequential nature. A first selection is expected to be of methodological character (on pragmatic grounds, this implies the restriction to forces which are treatable on grounds of available knowledge). The second selection is expected to be of phenomenological nature, that is, originating from the confrontation of the predictions of the theory with the experimental data.

TABLE 3.4. THE PROBLEM OF THE COMPATIBILITY OF A HADRON AS A WHOLE WITH ESTABLISHED RELATIVITY LAWS.

System (3.3.3) does not represents the(Newtonian limit of) the motion of eletons on numerous counts. A most important reason is that it violates Assumption 2.4.1 on the (uncompromisable) condition that an eletonic bound state must obey, as a whole, established relativity laws, and possible violations should occur only at the level of each individual hadronic constituent.

This crucial point can be easily identified. System (3.3.3) is an ordinary Newtonian nonconservative system. Thus, the total physical characteristics are not conserved. This is in violation of the conservation laws of the Galilei relativity. Also, it is inconsistent within the context of our model of hadronic structure because when a hadron is interpreted as a bound state of particles, irrespective of the nature of the acting forces its total physical characteristics must be conserved whenever it is isolated from the rest of the universe.

We reach in this way another most crucial point of our analysis. Experimental evidence of incontrovertible nature indicates that hadrons under ut most electromagnetic interactions obey established relativity laws (Galilei or Einstein special relativity). No structure model of the hadron can achieve even a minimum degree of plausibility unless this property is recovered in its entirety.

As indicated in Section 2.4, the idea which we shall attempt to achieve this consistency, is via subsidiary constraints. To present this possibility, it is advisable to reinspect the Coulomb system (3.3.1). It is manifestly invarian t under the Galilei group. This implies (via Noether's theorem) the existence of ten conservation laws (for the total energy, linear momentum, angular momentum and uniform motion of the center of mass).

By including these conservation laws, system (3.3.1) can be explicitly written

$$\begin{cases} \left[m \ddot{r}_k - f_k^{Coulomb}(r) \right]_{SA}^{Positronium} = 0, \quad k=1,2, & (3.4.1a) \\ \frac{d}{dt} E_t = \frac{d}{dt}\left(T + V_{Coulomb}\right) = 0, & (3.4.1b) \\ \frac{d}{dt} P_t = \frac{d}{dt}\sum_k p_k = \frac{d}{dt}\sum_k m \dot{r}_k = 0, & (3.4.1c) \\ \frac{d}{dt} M_t = \frac{d}{dt}\sum_k r_k \times p_k = 0, & (3.4.1d) \\ \frac{d}{dt} G_t = \frac{d}{dt}\left(\sum_k m r_k - P_t t\right) = 0, & (3.4.1e) \end{cases}$$

For reason which will appear selfevident in a moment, we shall say that the above system is a trivially constrained system, in the sense that Eqs. (3.4.1b) through (3.4.1e), which ensure the compliance with the physical laws of the Galilei relativity, are not subsidiary constraints. Instead, they are simply first integrals of the equations of motion.

This is sufficient to provide the needed intuitional element. To attempt consistency with the Galilei relativity for an eletonic bound state, we simply impose the Galilei conservation

laws for total quantity as nontrivial subsidiary constraints.

The primitive Newtonian form of an eletonic bound state for, say the π^o, interpreted as a bound state of an eleton and an antieleton, can be then written

$$\begin{cases} \left\{ \left[m \ddot{r}_k - f_k^{elm}(r) \right]_{\underline{SA}}^{Positronium} - F_k^{strong}(t, r, \dot{r}) \right\}_{\underline{NSA}}^{\pi^o} = 0, \quad k=1,2, & (3.4.2a) \\ \frac{d}{dt} E_t = \frac{d}{dt}\left(T + V_{tot}\right) = 0, & (3.4.2b) \\ \frac{d}{dt} P_t = \frac{d}{dt}\sum_k p_k = \frac{d}{dt}\sum_k m \dot{r}_k = 0, & (3.4.2c) \\ \frac{d}{dt} M_t = \frac{d}{dt}\sum_k r_k \times p_k = 0, & (3.4.2d) \\ \frac{d}{dt} G_t = \frac{d}{dt}\left(\sum_k m \dot{r}_k - P_t t\right) = 0. & (3.4.2e) \end{cases}$$

It results to be a nontrivially constrained system, in the sense that, unlike the case of Eqs. (3.4.1), now Eqs. (3.4.2b) through (3.4.2e) constitute bona fide subsidiary constraints to Eqs. (3.4.2a). The generalization to more than two eletons is trivial and will be tacitly implemented in the following.

In model (3.4.2), upon a number of technical implementations, relies our hopes of stimulating the awareness of the physics community on the need of verifying the established relativity laws within a hadron.

In essence, model (3.4.2) realizes, this time at a primitive, classical and Newtonian level, the dichotomy of relativity profiles according to Assumption 2.4.1. The state as a whole is fully consistent with the Galilei relativity because it obeys the physical laws of this relativity by construction. Neverthless, as we shall elaborate better later on, the Galilei relativity is violated at the level of each individual constituent, also by construction. It is sufficient to say, at this point, that the force F_1 can be, for instance, of the type

$$\underset{\sim}{F}_1 = g\, f(t)\, \underset{\sim}{r}_1\, \dot{\underset{\sim}{r}}_2^{\,2}, \qquad (3.4.3)$$

which implies the form-noninvariant under the complete Galilei group.

It might be of some value to reelaborate our epistemological argument in terms of model (3.4.2). In essence, for the Newtonian limit of the positronium structure, experimental evidence dictate that the equations of motion must be conform to the Galilei relativity. The situation for the π^o is different. Here, our incontrovertibly established knowledge is minimal, as far as the structure is concerned. The net result is that at least two possibilities are conceivable : the Galilei relativity applies and does not apply for the interior case (with an understanding, stressed earlier, that the relativity applies in both cases for the state as a whole). Clearly, to conduct a study of the problem of the structure of the hadrons of sufficient depth, both possibilities must be studied and subjected to a comparative confrontation with physical reality. The former possibility (the case of validity of established laws for the structure problem) has been studied in details along the available unitary models, but has resulted in a number of problematic aspects which are still unresolved as of today. Thus, the second possibility (invalidity of the established relativity laws for the structure problem only) should deserve a study.

In the most simplest possible form, our argument in favor of such a latter study is the following. The imposition of the Galilei form-invariance on the equations of motion for the structure directly implies a substantial restriction of the admissible forces. In turn, this directly impiles the analytic equivalence of the electromagnetic and the strong hadronic forces (in the case at hand, both must be conservative). In turn, we argue that this is insufficient to represent the physical differences between the interacti ons considered.

Explicitly, our contention is that, perhaps, if we implement the Coulomb system with conservative, Galilei form-invariant forces, we do not achieve the departure from the positronium structure which is needed to properly represent a strongly interacting particle, the π^o.

On the contrary, if the Galilei relativity is assumed as being violated for the structure problem (only), the situation is different. Now we have no restriction on the explicit form of the strong hadronic forces. Thus, we have full freedom to attempt a profound departure from the positronium structure for the structure of the π^o.

Permit me also to elaborate on the concept of "mutation" of a conventionally quantized particle or, in this instance, of a bound state. An inspection of model (3.4.2) indicates that this structure of the π^o is conceived as a "mutation of that of the positronium" in the sense that it is realized via the addition of Galilei form-noninvariant forces. The important point is that, at the limit of null F-forces, the positrinium structure i s recovered is full. And indeed, under this limit Eqs. (3.4.2b) through (3.4.2e) recover their significance of being first integrals of the equations of motion and the system is no longer with subsidiary constraints.

We cannot close this table without considering the problem whether a constrained system of type (3.4.2) is actually consistent, i.e., admits solutions (in the sense of the theory of overdetermined systems of ordinary differential equations). Equivalently, this problem consists of the study whether there exist nontrivial forces nonderivable from a potential for which system (3.4.2) is consistent for $N \geq 2$.

This is a typi cal case to illustrate the relevance of mathematics for physical problems. Indeed, if system (3.4.2) can be mathematically established as being inconsistent for forces nonderivable from a potential, the physical issue considered in this paper is closed, in the sense that there would be no ground of suspecting a possible violation of established relativity laws for the hadronic constituent at the level here considered (Galilean).

Regrettably, while the theory of determined systems of ordinary differential equations (i.e., systems in which the number of equations is equal to the number of unknown) is an established mathematical discipline by today's standard, this is not the case for over-determined systems (i.e., system in which the number of equations exceeds that of the unknowns which is precisely the case for systems (3.4.2)). To the best of my knowledge, this is a topic with a number of aspects which are still open on grounds of pure mathematics. This is also an area, as indicated in Section 1, in which contributions by mathematicians in the field are a necessary prerequisite to attempt physical conclusions (a number of additional areas in which this interplay between mathematicians and physicists is essential to attempt physical progress will be indicated later on).

With an understanding that the issue must be resolved on rigorous mathematical grounds, a few rudimentary remarks are essential for the content of this paper. In essence, the answer to the problem considered appears to be in the affirmative, in the sense that there exist nontrivial forces nonderivable from a potential under which system (3.4.2) is consistent. As a matter of fact, it appears that there exists three classes of consistent models of increasing complexity of the nonconservative forces (and, thus, of methodological needs), according to the following outline.

(A) Models in which simple nonconservative forces are selected with empirical means, e.g., their direct compatibility with the subsidiary constraints (for details, see below).

(B) Models in which the nonconservative forces are selected in compliance with the Levi-Civita sufficiency method. T. LEVI-CIVITA[29] identified (in 1906) a simple but quite effective method for the study of the consistency of overdetermined systems. It is essentially given by a sufficient condition for which the subsidiary constraints are particularized first integrals of the equations of motion. The consistency of the system is then selfevident. The physical effectiveness of the method is that it allow the use of forces more general that those identifyable with empirical means and, most importantly, the method can be applied without any knowledge of the solution. By recalling that the equations considered are generally nonlinear in both the coordinates and the velocities, the importance of this latter property for practical applications is also selfevident. It should be indicated for clarity that this Levi-civita method is only sufficient and not necessary and sufficient. This is an indication that yet broader forces are conceivable for the consistency of system (3.4.2). The only recent account on this method of which I am aware is that by S. SHANMUGADHASAN[30]. Oddily, there has been a rather intensive study of constrained systems in the recent physical literature (mostly in relation to the so-called Dirac theory of systems with subsidiary constraints), but the problem of consistency of the systems considered is not treated in most of these studies, to the best of my knowledge. For more specific contributions, the interested reader may consult the easily identifyable literature of the Calculus of Variations, with particular reference to the so-called Problem of Bolza in general, and the multiplier rule in particular.

(C) Models in which the nonconservative forces can be selected in compliancewith the available theorems of the existence theory of overdetermined systems. The reader should be aware that the conventional existence theory of ordinary differential equations (for the case of determined systems) is not applicable to systems (3.4.2), even though they satisfy the continuity and regularity conditions of such existence theory, because the systems considered are overdetermined. In this case, a number of valuable studies by mathematicians have been conducted to extend the existence theory to the broader context here considered. For the interested reader, refs.[31] gives the specialized literature of which I am aware at this time (I would like to thank D. C. Spencer for indicating to me these papers). The

aspect which is relevant for the analysis of this paper is that, apparently, and pending verifications by interested mathematicians in the field, these existence theorem for the theory of overdetermined systems (the so-called theorems of formal integrability), allow a further broadening of the admitted forces, on a comparative basis with approaches A and B.

In conclusion, it appears that our dichotomy of relativity laws (validity of the Galilei relativity for the exterior behaviour and its invality for the interior problem) is technically realizable with consistent differential equations. It is this property which has allowed the presentation of the argument of this paper.

Permit me to illustrate the case with one of the simplest possible examples along the empirical methods of approach A. First, to avoid complexities beyond our capabilities at this time, we assume that system (3.4.2a) is nonessentially nonselfadjoint, i.e., it does indeed admit a Lagrangian according to Eqs. (3.3.5). Secondly, we content ourselves with a nontrivially generalized structure of the Lagrangian of the simplest possible type

$$L^{gen} = L_{int,I}\, L_{free} + L_{int}^{Coulomb}, \qquad (3.4.4)$$

where the multiplicative term L_{int} is different than one. The reader sould be aware that for $L_{int,I} = 1$, the system is the conventional Colomb system. Thus, the multiplicative interaction term to the term representing the free motion is essential to achieve a "mutation" of the Coulomb system in the sense indicated earlier. This is sufficient to indicate the departure from conventional models which are all based, in general, on the addition of interaction terms to that representing the free motion. Also, the reader should be aware that this approach implies a further broadening of the functional dependence of the F-forces because for Lagrangian (3.4.4) they are also linearly dependent on the accelerations. Since this dependence is (necessarily) linear, we are still within the context of Newtonian forces (we have, in essence, the so-called acceleration couplings of the theory of coupled oscillators).

We now select the F-forces with empirical means, in such a way to allow the verification of the Galilei conservation laws for the total quantities, while yielding a generalized structure of a Lagrangian of type (3.4.4). By summing up the equations of motion (3.4.2a), the conservation of the total linear momentum is satisfied if and only if $\underset{\sim}{F}_1 = -\underset{\sim}{F}_2$. In turn, this yields a motion in a plane, as for the conventional Coulomb system. To treat the system in the center of mass coordinates, we assume that $p^t = 0$. The uniform motion of the center of mass, Eqs. (3.4.2e) is now automatically verified, and system (3.4.2) can be rewritten

$$M \ddot{R} = 0, \quad \mu \ddot{r} - f - F = 0, \quad M = 2m, \mu = \tfrac{1}{2} m,$$
$$\frac{d}{dt} \mathcal{E}_t = 0, \quad \frac{d}{dt} M_t = 0, \quad r = r_1 - r_2, \qquad (3.4.5)$$
$$f = f_1 = -f_2, \quad F = F_1 = -F_2.$$

In view of the property

$$\dot{M}_t = r \times \dot{p} = r \times f + r \times F = r \times F, \qquad (3.4.6)$$

the conservation law of the total angular momentum holds if and only if

$$r \times F = 0. \qquad (3.4.7)$$

Similarly, in view of the property

$$\dot{\mathcal{E}}_t = \dot{r} \cdot (\mu \ddot{r} - f) = \dot{r} \cdot F, \qquad (3.4.8)$$

the conservation of the total energy holds if and only if

$$\dot{r} \cdot F = 0. \qquad (3.4.9)$$

Thus, system (3.4.5) can be rewritten

$$\begin{cases} \mu \ddot{r} - f - F = 0, \quad M \ddot{R} = 0, \\ r \times F = 0, \quad \dot{r} \cdot F = 0. \end{cases} \qquad (3.4.10)$$

Our problem is the identification of a force F of the above system which yields consistency as well as a generalized Lagrangian. Our simplest possible solution is given by

$$F = g \ddot{r} = \frac{g}{\mu - g} f, \quad r = (r_x, r_y), \qquad (3.4.11)$$

where we have ignored the third component under the assumption that the motion is in the (x, y) plane, and where the second form of the forces is computed by using the

equations of motion. Our contention is that the emerging system

$$\begin{cases} M \ddot{R} = 0, & (3.4.12a) \\ (\mu - g) \ddot{r} - f = 0, \quad f = \dfrac{e^2}{r^2}, & (3.4.12b) \\ \dot{r} \cdot r = 0, & (3.4.12c) \end{cases}$$

has a physically meaningful solution. To see it, consider the following generalized form of the Runge-Lenz vector

$$\mathfrak{M} = M_t \times p, \quad \dot{\mathfrak{M}} = M_t \times (f + F). \qquad (3.4.13)$$

Then

$$A \cdot r = A r \cos\theta = -M_t^2 + \frac{\mu^2 e^2}{\mu - g} r, \qquad (3.4.14a)$$
$$A = \mathfrak{M} + \frac{\mu^2 e^2}{\mu - g} \hat{r}, \quad \hat{r} = \frac{r}{r}, \quad r = |r|. \qquad (3.4.14b)$$

A solution of Eqs. (3.4.12b) is given by the conventional form of a conic

$$\frac{1}{r} = \frac{\mu^2 e^2}{(\mu - g) M_t^2} (1 - \varepsilon \cos\theta), \qquad (3.4.15)$$

but with different coefficient than that of the conventional Coulomb system and a different value of the excentricity

$$\varepsilon = \frac{\mu - g}{e^2 \mu^2} A = \sqrt{1 + \frac{2(\mu - g)^2}{\mu^3 e} |M_t| \mathcal{E}_t}. \qquad (3.4.16)$$

The important point is that constraint (3.4.12c) now restricts the only admissible orbit to the circle

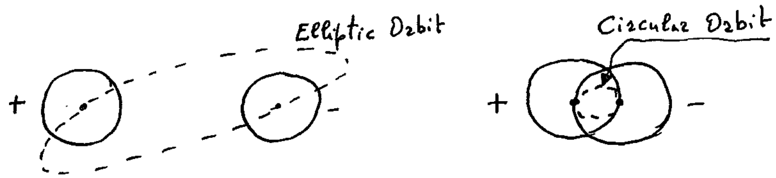
$$\varepsilon = 0, \quad r = \frac{(\mu - g)^2}{\mu^2 e^2} M_t^2, \quad \mathcal{E}_t = -\frac{\mu^3 e}{2(\mu - g) |M_t|}. \qquad (3.4.17)$$

The other important point is that a Lagrangian for the relative motion is indeed generalized

$$L^{gen} = \frac{\mu - g}{\mu} T - V^{Coulomb}. \qquad (3.4.18)$$

This concludes the example.

The rather intriguing conclusion of the above simplest possible "mutation" of the positronium structure is therefore that, while for the positronium both elliptic and circular orbit are allowed, the only admissible orbit for the π^o is the circle, of course at this primitive Newtonian level. Neverthless, as we shall elaborate in more details in Section 5, there could be some value in this occurrence. The argument is quite simple. In the positronium the charge volumes of the constituents are at very large distances from each other. Thus, elliptical orbits are fully admissible.
In the transition to the π^o the situation is, again, different. Under the assumption that the constituents are in a state of penetration of the charge volumes



Positronium $\qquad\qquad$ π^o

the elliptical orbits are expected to be highly unstable. We then remain only with the circle as the orbit with a possibility of yielding a stable structure of the π^o. Of course, this is a first,rudimentary, classical, look at the π^o as a bound state of an eleton and an antieleton which, as such, leaves open numerous other possibilities (e.g., an infinitesimal value of the excentricity).

Another significant indication which emerges from the above model is that, in all expectations, this structure of the π^o appears as rather unique, in the sense that in the transition to the structure of the charges pions, $\pi^{\pm}$, a substantially different structure might eventually emerge as necessary, despite the fact that all these particles constitute an isotopic triplet. This is a first elaboration of our argument that, perhaps, in much the occurrence at the atomic level, each hadron possesses its own unique structure which is distinct from that of others to the point of even demanding a different number of constituents (e.g., starting from two eletons for the π^o, we might need three eletons for the $\pi^{\pm}$). The reader can thus begin to see the departures from current trends which are implies by the model under consideration (for the quark model all pions have the same number of quarks - plus the sea of gluons).

In closing this table, it might be of some value to reexamine the argument on the physical origin of more general forces for the hadronic constituents (Table 3.1). Again, Newtonian mechanics can prove to be of invaluable intuitional possibilities. Let us recall that one of the conceptual foundations of the theory of nonconservative Newtonian systems, as elaborated in refs.[21,22,23], is given by the notion of physical medium (liquid, gas, or plasma) in which the motion occurs. This is clearly a necessary prerequisite for the dissipation of energy by the particle considered. In particular, the medium is the receptacle of the energy lost by the particle. In turn, the contact forces produced by the motion in this medium emerge as being nonderivable from a potential, that is, of a class more general than the action-at-a-distance forces (e.g., the Coulomb force).

The problem we are here referring to is a conceptual identification of the hadronic equivalent of this Newtonian nonconservative context as the conjectured physical origin of nonselfadjoint structure forces. Here a variety of situations may occur, depending on the case at hand. The first occurrence can be that of a (charged) particle penetrating into a hadron (e.g., as for deep inelastic scatterings). In this case the hadron itself can be conceived as the hadronic medium owing to its high density of matter.
We then argue that it is this medium which is responsible of local nonselfadjoint forces acting on the particle considered and only while within the hadron considered, again, as an approximation of expected more complex forces.

In the transition to the problem of structure the situation is different and depends on the considered number of charged, non-point-like constituents (eletons and antieletons). We are here interested to the case of model (3.4.2), i.e., when there are two constituents. In this case we argue that each constituent constitutes the hadronic medium of the other because each non-point-like constituent moves within the charge volume of the other. In turn, this situation opens rather delicate problems concerning the proper systems of coordinates, which are absent for conventional systems.

Let us recall in this respect that in conservative mechanics it is costumary to move from one inertial reference system to another without any risk to alter the dynamics (i.e., the forces). In the transition to nonconservative mechanics the situation is, again, different.[21,23] This is due to the fact that the forces now are not form-invariant under familiar transformations (e.g., the Galilei transformations). The net effect is that the transition from one reference frame to another generally imply a change in the structure of the forces not

not derivable from a potential. For instance, the following two forces in different reference frames

$$\underset{\sim}{F}_1 = g f(t)\, \underset{\sim}{r}_1\, \dot{\underset{\sim}{r}}_2^{\,2} , \quad \underset{\sim}{F}'_1 = g f(t)\, \underset{\sim}{r}'_1\, \dot{\underset{\sim}{r}}'^{\,2}_2 + 2g f(t)\, \underset{\sim}{r}'_1\, \dot{\underset{\sim}{r}}_2 \cdot \underset{\sim}{V}_0 + g f(t)\, \underset{\sim}{r}'_1 \underset{\sim}{V}_0^2 \tag{3.4.19}$$

are related by a Galilei boost, even though they appear as structurally nonequivalent.

The net effect is that care must be exercised in the selection of the explicit form of nonselfadjoint forces because such form depends on the selected system of coordinates. In this respect too Newtonian mechanics can be of assistance. The most natural way to identify nonconservative (e. g., drag) forces is by using a reference system at rest with respect to the medium in which the motion occurs. This suggests that for the study of the motion of an eleton-antieleton bound state a valuable reference frame is that of the relative coordinates because each particle is assumed as the medium in which the motion of the other occurs. In turn, this aspect may have considerable implications for relativity problems, as we shall indicate in Table 4.4.

TABLE 3.5: THE PROBLEM OF THE RELATIVISTIC, FIELD THEORETICAL AND GRAVITATIONAL EXTENSIONS OF THE EQUATIONS OF MOTION OF AN ELETON.

As stressed earlier, the analysis of this paper is essentially restricted to the first and simplest conceivable profile, that of discrete nonrelativistic particles. It is an easy prediction that the problem under consideration cannot be exhaustively studied without relativistic extensions, owing to the expected high value of the kinetic energy of the hadronic constituents. We shall attempt to consider these aspects in some subsequent paper. Neverthless, few introductory remarks can be of some value for this paper, because technical difficulties which might be overlooked at the discrete nonrelativistic level, are better focused when studying relativistic extensions.

On classical methodological grounds, the first recommendable step is to extend the methodology of the Inverse Problem to the broader arena under consideration (Minkowski

space, local field theory and Riemannian manifolds). Indeed, this approach, as it was the case for the Newtonian context, provides valuable tools for the study of the structure of the forces (or couplings) considered as well as methods for the computation of an action functional (when it exists in the considered system of local variables). These extensions of the Inverse Problem heve been studied in refs.[21, 22] -as well as in the quoted papers

We shall not enter here in the problem of a relativistic discrete extension of model (3.4.2) because of the alternatives for such an extension of forces non derivable from a potential, which demand a specific study. For later needs we simply restrict ourselves to the remark that, for a conceivable relativistic extension of model (3.4.2) the four-momentum of an individual eleton may obey a generalization of the energy-momentum law of the type

$$(p_\mu - eA_\mu)(p^\mu - eA^\mu) = f(m^2, x^\alpha, p^\alpha), \qquad \alpha = 0, 1, 2, 3, \tag{3.5.1}$$

while the total four-momentum obeys, as a subsidiary constraint, the conventional law

$$P_{tot\,\mu}\, P_{tot}^{\ \mu} = M^2, \tag{3.5.2}$$

where: (a) the symbols $p_k^{\ \mu}$ represent the canonical four-momenta of the maximal associated selfadjoint subsystem (i. e., the system without forces not derivable from a potential), (b) A^α is the conventional electromagnetic potential and (c) all strong hadronic forces are assumed as not derivable from a potential (if this is not the case, Eq. (3.5.1) exhibits a simple modification).

The field theoretical extension of model (3.4.2) is less subject to technical alternatives, as typical for most of the field theoretical extensions of Newtonian mechanics. If we consider the case of the motion of a particle moving in a hadronic medium, the simplest possible equations can be written in the second order form

$$\left\{ \Big[(\Box + m^2)\,\varphi - f(\varphi, \varphi_{;\alpha}) \Big]_{SA}^{C^\infty, R} - F(x^\alpha, \varphi, \varphi_{;\alpha}) \right\}_{NSA}^{C^\infty, R} = 0, \qquad \varphi_{;\alpha} = \partial\varphi/\partial x^\alpha, \tag{3.5.3}$$

where SA and NSA represent again selfadjointness and nonselfadjointness, but now referred to a different methodology. In this respect, the interested reader may inspect the explicit form of the conditions of selfadjointness as identified in refs.[21-III] (see[28]).

Notice the crucial appearance of an explicit dependence of the nonselfadjoint forces in the Minkowski coordinates, which is customarily absent in conventional models. It is a direct generalization of the Newtonian occurrence whereby nonconservative forces may depend explicitly on time.

This state of affairs suggest the following interpretation of Eqs. (3.5.3).

(1) It constitutes an approximation of the field theoretical description of the motion of massive, charged and non-point-like particle moving within a hadron.

(2) The admitted interactions can be classified into two groups, depending on their character under the conditions of selfadjointness. The first group is represented by conventional couplings (e.g., electromagnetic) which are identified by the Inverse Problem as being selfadjoint. The second group is represented by nonselfadjoint couplings and is here interpreted as characterizing strong interactions.

(3) The Minkowski coordinates x^α are assumed as characterizing a reference frame whose space component is at rest with respect to the hadronic medium considered (the hadron itself), to comply with the conjectured physical origin of the generalized couplings of Table 3.4.

For instance, the following extension

$$\left\{ \left[(\Box + m^2)\varphi + \lambda\varphi^3 \right]_{SA} + \mu\, \varphi_{;\alpha}\, \varphi^{;\alpha} \right\}_{NSA} = 0, \qquad (3.5.4)$$

of the familiar equation for nonlinear models

$$\left[(\Box + m^2)\varphi + \lambda\varphi^3 \right]_{SA} = 0, \qquad (3.5.5)$$

is an example of Eq. (3.5.3). Notice, however, the difference in interpretation. Customarily, Eq. (3.5.5) is interpreted as characterizing a "selfcoupled" field. In our model the corresponding nonselfadjoint extension (3.5.4) characterizes a strong interaction.

As for the Newtonian case, Eq. (3.5.3) can be classified as being of either nonessentially nonselfadjoint type (i.e., admitting an indirect Lagrangian representation) or of essentially nonselfadjoint type (in which case a Lagrangian, in the considered local variables, does not exist). Intriguingly, the Lagrangian densities sometimes emerging for the former case

are of the type

$$\mathcal{L} = \varphi_{;\mu}\, G(\varphi)\, \varphi^{;\mu} + P(\varphi). \qquad (3.5.6)$$

(also called chiral Lagrangians), while the most general Lagrangian density is of the type

$$\mathcal{L} = \varphi_{;\mu}\, G(x^\alpha, \varphi, \varphi_{;\alpha})\, \varphi^{;\mu} + \varphi_{;\mu}\, F^\mu(x^\alpha, \varphi) + E(x^\alpha, \varphi). \qquad (3.5.7)$$

The point is that Lagrangians with the simpler structure (3.5.6) yield nontrivial nonselfadjoint models according to the structure

$$\frac{d}{dx^\mu}\frac{\partial \mathcal{L}}{\partial \varphi_{;\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi} = \left\{ h(\varphi, \varphi_{;\alpha}) \left[\left((\Box + m^2)\varphi - f \right)_{SA} - F \right]_{NSA} \right\}_{SA}, \quad h \neq 0. \qquad (3.5.8)$$

If a field theoretical extension of the complete, constrained model (3.4.2) is desired, it results to be of the form

$$\begin{cases} \left\{ \left[(\Box + m_k^2)\varphi_k - f_k(\varphi, \varphi_{;\alpha}) \right]_{SA}^{C^\infty, R} - F_k(x^\alpha, \varphi, \varphi_{;\alpha}) \right\}_{NSA}^{C^\infty, R} = 0, & (3.5.9a) \\ \partial_\mu \Theta^{\mu\nu}_{tot} = 0, \quad k = 1, 2, \ldots, N, & (3.5.9b) \\ \partial_\mu M^{\mu\nu\rho}_{tot} = 0, \quad \nu, \rho = 0, 1, 2, 3, & (3.5.9c) \end{cases}$$

where Eqs. (3.5.9b) and (3.5.9c) represent the conservation laws of the energy-momentum and of the angular momentum of the total quantities and are here interpeted as providing the compliance of the state as a whole with the physical laws of the Poincaré symmetry in Einstein special relativity.

Again, model (3.5.9) is of the class with subsidiary constraints. The problem of its consistency, however, now becomes rather substantial and we shall not consider it. Also, the problem of the computation of explicit solutions appears to be beyond our knowledge at this time. It is here appropriate to note that the solutions of even a "simple" system of type (3.5.4) imply a rather conspicuous technical implementation of the currently available methods for the computation of solutions of the selfadjoint model (3.5.5) (whether in Euclidean four-space or in a reduced Minkowski space). This is clearly due to the additional presence of the nonlinear c oupling in the partial derivatives.

The extension of the proposed realization of strong interactions to the case of first order field equations is particularly intriguing because it involves a generalization of Dirac's equations for a charged particle in an (external) electromagnetic field and under strong nonselfadjoint interactions

$$\left\{ \left[\begin{pmatrix} -\bar{e}_{;\mu}\gamma^{\mu} + m\bar{e} \\ \gamma^{\mu} e_{;\mu} + m e \end{pmatrix} - \begin{pmatrix} f_{\bar{e}} \\ f_{e} \end{pmatrix}^{elm} \right]_{\underline{SA}} - \begin{pmatrix} F_{\bar{e}} \\ F_{e} \end{pmatrix}^{strong} \right\}_{\underline{NSA}} = 0, \tag{3.5.10a}$$

$$f_{\bar{e}} = -\frac{\partial \mathcal{L}^{elm}_{int}}{\partial e}, \quad f_{e} = -\frac{\partial \mathcal{L}^{elm}_{int}}{\partial \bar{e}}, \quad \mathcal{L}^{elm}_{int} = -i e \bar{e}(x) A_{\mu}(x) \gamma^{\mu} e(x), \tag{3.5.10b}$$

$F_{\bar{e}}, F_e$, nonderivable from a density.

In turn, this opens a number of intriguing problems which we hope to consider in detail in some subsequent paper (e.g., whether the matrices γ^{μ} of Eqs. (3.5.10a) are Dirac's matrices or not). The reader should be, however, aware that, even though Dirac's equations in their conventional form verify the condition of selfadjointness (as it is the case for all models with the trivial Lagrangian density $\mathcal{L}_{tot} = \mathcal{L}_{free} + \mathcal{L}_{int}$), the explicit form of the conditions of selfadjointness in the transition from second-order to first order partial differential equations is substantially altered.[28]

A most intriguing point is that, in order to achieve a nonselfadjoint generalization of Dirac's equation, the extra additive couplings generally depend on the partial derivatives of the fields. This is a field theoretical extension of the primitive Newtonian notion that nonconservative forces are generally dependent on the velocity. As we shall see later on (Table 4.20), this feature produces a rather profound "mutation" of a conventional Dirac's field, e.g., because the Dirac's term $\gamma^{\mu} e_{;\mu}$ now loses its dominance for the characterization of the intrinsic quantities in favor of a corresponding dominance by the strong couplings. In turn, this has a number of implications for relativity aspects. In conclusion, Eqs. (3.5.10) appears to be a genuine representation of a Dirac particle in an "eletonin state", according to the terminology of Table 3.2 .

But, perhaps, one of the most intriguing problems which is identified by our line of study is that of a possible gravitational extension of model (3.4.2) whereby, as by now familiar, the conventional Einstein's equations for the exterior problem are recovered in full (to comply with the available experimental evidence), and a departure from Einstein's gravitational approach is attempted only for the interior problem. As indicated in ref.[23], a conceivable approach is that of imposing the conventional Einstein's equations for the exterior problem as subsidiary constraints to a more general model for the interior problem specifically constructed for forces non-derivable from a potential.

After all, model (3.4.2) could in the final analysis emerge as a primitive Newtonian limit of such possible generalized approach to gravitation, in which the subsidiary constraints ensuring the compliance with the Galilei relativity laws are a nongeometrical limit of Einstein's exterior equations.

4. THE EPISTEMOLOGICAL ARGUMENT FOR A POSSIBLE LIE-ADMISSIBLE COVERING OF QUANTUM MECHANICS FOR FORCES NONDERIVABLE FROM A POTENTIAL.

Clearly, no assessement of any effectiveness of the model for the hadronic structure proposed in Section 3 can be attempted without identifying, at least in a rudimentary way, the classical and quantum mechanical methods for the treatment of local forces not derivable from a potential, and then confronting the predictionsof the theory with physical reality. Inevitably, this problem demands the study of the relativity and quantum mechanical laws which are applicable to the systems considered.

My studies on the classical profile of this problem are presented in monographs[21,22]. For a summary, see ref.[23] The first tables of this section will summarize, for the reader's convenience, only the very essential aspects.

The quantum mechanical profile of the problem appears to be open as of now, to the best of my knowledge. It should be here recalled, as stressed in Table 2.5, that the open aspect of the issue is that of the problem of each individual particle under the broader forces considered, and not the statistical - thermodynamical treatment of nonconservative systems which has seen major advances.[25]

At a closer look, the quantum mechanical methods which are needed to achieve an assessement of the possible physical relevance of the proposed model of hadron structure appear to be differentiated into the following classes of increasing complexity.

(I) The possible existence and physical consistency of a quantum mechanical formulation of nonconservative Newtonian systems without subsidiary constraints. We are here referring to the case, say, of one particle under action-at-a-distance and contact forces not derivable from a potential. It is customarily believed that contact forces of frictional nature have no place in the microscopic world. Our contention is that this belief should be reexamined prior to a final judgment. Our argument is by now familiar. There is no doubt that these forces have no place in the physical arena for which quantum mechanics was conceived, the atomic structure. In the transition to the hadronic structure the situation is different, provided that particles are assumed to be physical, that is, non-point-like. In this case a quantum mechanical extension of frictional, contact forces could well emerge in the final analysis as existing and being physically relevant, of course, as an approximation of expected more complex forces. In any case, the motion of a massive, charged and estended particle in a hadronic medium is purely nonconservative in conception,

and it is sufficiently intriguing to deserve a study for its possible quantum mechanical formulation.

(II) The possible existence and physical consistency of a quantum mechanical formulation of nonconservative Newtonian systems with subsidiary constraints. As familiar, Newtonian systems are often constrained in practice. If the quantum mechanical problem of aspect (I) admits a solution, the next logical step is the study of the possible extension of the emerging quantum mechanical context to the case with subsidiary constraints. By inspecting the Newtonian form of our model of hadronic structure, Eqs. (3.4.2), the study of this more general context appears to be essential.

(III) The possible existence and physical consistency of relativistic, quantum mechanical extensions. As indicated in Section 3, the speed of our hadronic constituents, the eletons and the antieletons, is expected to be of the order of that of light (for details, see Section 5). Therefore, if the quantum mechanical aspects (I) and (II) can be successfully achieved, this will provide the necessary tools for only a crude assessement of the problem considered. The study of possible relativistic extensions will then be essential.

In this section I shall present an initial, tentative study of Problem (I) and only few incidental remarks related to the more general Problem (II). No relativistic quantum mechanical (or quantum field theoretical) extension will be considered at this time. The study is, in essence, an attempt of quantizing the methodologies of the Inverse Problem[21] and of the Lie-admissible problem[22] for nonconservative Newtonian systems.

Before entering into the presentation of this study, a few introductory remarks could be of some value. In essence, besides predictable technical difficulties, the major difficulty in the problem of quantization of nonconservative forces appears to be of conceptual nature. In ref.[23] we have stressed the rather profound conceptual departures from conventional conservative mechanics which is needed for the study of nonconservative systems. In the transition to a possible quantum mechanical extension, a fully equivalent situation occurs. As we all know, quantum mechanics was conceived for the quantization of conservative forces, the Coulomb force of an atomic structure, even though subsequent studies indicated that the emerging methods are indeed applicable to broader forces derivable from a potential. Neverthless, owing to extended use, the primary emphasis still rests on the originally conceived conceptual foundations, such as the stability of the electon's orbits in an atomic structure. The major conceptual difficulty we referred earlier is given by the fact that, a departure from these conventional notions is needed, as a necessary prerequisite to achieve a true nonconservative quantum mechanics. For instance, one of the problems we shall be confronted with is that of achieving a genuine quantum mechanical description of nonstationary orbits, even for the case when the acting forces do not depend explicitly on time (but are non derivable from a potential). Indeed, the stationarety of the orbit of a particle moving, say, within the core of a neutron star, would now be conceptually inconsistent, to begin with.

Here, again, an inspection of the corresponding occurrence in Newtonian mechanics can prove to be of invaluable assistance. Consider the conservative case of a particle moving in vacuum under a central force, say, the gravitational force. Suppose that the orbit is "closed" (elliptical or circular). As familiar, the emphasis in this case in the interpretation of the stability of the orbit (e.g., the stability of the orbit of a satellite in the -Newtonian- gravitational field of Earth). This is a typical conservative setting. Perform now the transition to a corresponding nonconservative system, e.g., the motion of the same particle under the same central force, but now moving in a physical, dissipative medium. Experimental evidence in this case indicates that the orbit decays in time. No representation of the system can be attempted in a way conform to physical reality unless the mental attitude is now shifted to the proper representation of the non stationarety of the orbit. As we shall see, in the transition to a corresponding quantum mechanical context, the situation is fully equivalent on conceptual grounds.

In conclusion, in the study of the possible existence of a quantum mechanical formulation of nonconservative mechanics the necessary mental attitude is that of searching for all conceivable departures from conventional quantum mechanical notions, laws and principles which are induced by nonconservative forces. On the contrary, if the mental attitude of preserving "as much as possible" conventional quantum mechanical notions, laws and principles is implemented, the net outcome, as we shall indicate, is a number of physical inconsistencies. In essence, this situation can be predicted from the profound physical differences between conservative and nonconservative settings.

As of now, this may appear as a reasonable mental framework to the curious and open minded researcher. In practice, however, the attitude may result of somewhat difficult realization because of the nature of the methodological alternatives which are inherent in the study of the problem considered, such as, preservation or violation

of the customary notion of wave packet?, preservation or violation of established relativity laws ? preservation or violation of Pauli's exclusion principle?, etc.

We reach in this way the essence of this paper. We shall claim no final resolution of these issues. Neverthless, the situation is such to apparently suggest the experimental verification of the validity of established relativity and quantum mechanical laws within a hadron.

The technically oriented reader should be made aware that we shall provide our best efforts in avoiding all technical aspects which we believe to be unnecessary for an initial, mainly intuitional study of the problem considered. For instance, notions such as rigged Hilbert spaces, C* algebras, von Neumann algebras, etc. are extraneous to the objectives of this section and the technical language which will be used is that, say, of the lectures in quantum mechanics by W. PAULI in Zürich of 1956-1957.

This is suggested by a number of aspects. First of all, we believe that during the intuitional phase of study of an open physical problem the primary emphasis should be put in the identification of the emerging physical laws, while the study of the rigorous mathematical language for the proper representation of these laws should be sequential. Secondly, we believe that the use of excessively complex mathematical algorithms during the intuitional phase of an open physical problem might result in an obfuscation of the physical issue, to the detriment of the solution of the problem itself. Thirdly, we believe that our best efforts in using the most modern possible mathematical tools might eventually result to be premature, in the sense that the rigorous treatment of the problem considered may eventually demand the use of mathematical tools which would not be conventionally used by today's standard.

In conclusion, this section must be considered an epistemological rather than a technical study of the problem. In essence, we had the alternatives of (A) conducting a detailed technical study of each major aspect in a series of papers, or (B) conducting an epistemological study of the perspective for a generalized quantum mechanics. We selected alternative (B) because alternative (A) would have delayed the program for years, while restricting the possible partecipation of interested researchers due to the lack of an overview (as the reader will see, each aspect of the quantum mechanical treatment of nonconservative forces is so different from conventional conservative lines, to the point that may appear absurd on an isolated basis -- the hopes then emerge from the unit of diversified aspects). In the final analysis, we belive that the construction of a possible generalization of quantum mechanics for nonconservative forces is not a job for one isolated researcher.

TABLE 4.1: THE OPEN NATURE OF THE PROBLEM OF QUANTIZATION OF NON-CONSERVATIVE NEWTONIAN SYSTEMS.

One arena of "unequivocal applicability" of conventional quantization procedures[32] which is identified by the methodology of the Inverse Problem is that of essentially selfadjoint unconstrained Newtonian systems (i. e, unconstraned systems which, as derived from the second law - - ma - F = 0 - - are variationally selfadjoint[28])

$$\left[m_k \ddot{r}_k - f_k (t, r, \dot{r}) \right]^{C^\infty, R}_{SA} = 0, \quad k = 1, 2, \ldots, N. \qquad (4.1.1)$$

They include the virtual totality of systems consistently quantized until now. These systems, in essence, verify the integrability conditions for the existence of an ordered direct Lagrangian representation

$$\left[\frac{d}{dt} \frac{\partial L_{tot}}{\partial \dot{r}^k} - \frac{\partial L_{tot}}{\partial r^k} \right]^{C^\infty, R}_{SA} \equiv \left[m_k \ddot{r}_k - f_k \right]^{C^\infty, R}_{SA}, \qquad (4.1.2)$$

$$L_{tot} = L_{free} + L_{int}.$$

The corresponding Hamiltonian, as constructed via the familiar Legendre transform,

$$H_{tot} = H_{free}(p) + H_{int}(t, r, p) \qquad (4.1.3)$$

exhibits a crucial physical property: it represents the total energy of the system considered. The use of conventional quantization procedures, e. g., via the Hamilton-Jacobi equation, then yields a quantum mechanical context in which the mathematical operator "H" has a direct physical significance. Most importantly, the canonical momentum p represents the physical linear momentum $m_k \dot{r}_k$ whenever conservative forces are considered. Thus, the emerging quantum mechanical formulation results to be consistent on both mathematical and physical grounds. As we shall elaborate during the analysis of this section, the physical consistency originates from the fact that the expectation values of the mathematical algorithms "H" and "p" possess a direct, clear and unequivocal physical significance (total energy and linear momentum, respectively). It then follows that the quantum mechanical algorithm $M = r \times p$ also possess a clear physical meaning, the angular momentum.

One arena identified by the methodology of the Inverse Problem for which the problem of quantization is open on both mathematical and physical grounds is that of

nonessentially or essentially nonselfadjoint, unconstrained Newtonian systems (Table 3.3)

$$\left\{ \left[m_k \ddot{r}_k - f_k (t, r, \dot{r}) \right]^{C^\infty, R}_{SA} - F_k (t, r, \dot{r}) \right\}^{C^\infty, R}_{NSA} = 0, \qquad (4.1.4)$$

which is precisely representative of the nonconservative systems under consideration.

The open nature of the problem can be easily indicated via the methodology of the Inverse Problem. The essentially nonselfadjoint systems are such that they violate the integrability conditions for the existence of an indirect Lagrangian representation of type (3.3.5). This means that a Hamiltonian for the representation of the systems considered in the coordinate considered does not exist. It is hoped that the lack of existence of a Hamiltonian establishes the open nature of the problem considered (e. g., because of lack of applicability of, say, quantization via the Hamilton-Jacobi equation). The reader should be aware that the methods of the Inverse Problem are indeed capable of producing a Hamiltonian representation for the system considered, but a necessary condition is that of transforming the system in a new set of coordinates[21] $r' = r'(t, r, \dot{r})$ with an essential, generally nonlinear dependence on the velocities. The problem of the quantization of the systems considered in this mathematical representation will be ignored in this section and left to the interested reader for the reasons indicated in ref.[23] (such mathematical new systems of coordinates are generally noninertial and nonrealizable in experiments owing to the nonlinear dependence of the new coordinates in the old velocities). It is hoped that these remarks establish the need of confronting the problem of quantization in the coordinate system of direct physical relevance. Once this problem is successfully solved, then the problem of quantization in mathematical systems of new coordinates can acquire its proper role.

The case of the simpler nonessentially nonselfadjoint systems is equivalent. These systems are such that they do verify the integrability conditions for the existence of an analytic representation in the coordinates of direct physical significance. However, since the forces are not derivable from a potential, such representation must necessarily be indirect, i. e., of the type (3.3.5), i.e.,

$$\left[\frac{d}{dt}\frac{\partial L^{gen}}{\partial \dot{r}^{ka}} - \frac{\partial L^{gen}}{\partial r^{ka}}\right]_{SA}^{C^\infty,R} = \left\{ h_{ka}^{\ jb}(t,r,\dot{r}) \left[\left(m_j \ddot{r}_{jb} - f_{jb} \right)_{SA}^{C^\infty,R} - F_{jb} \right]_{NSA}^{C^\infty,R} \right\}_{SA}^{C^\infty,R},$$

$$k = 1,2,\dots,N, \quad a,b = x,y,z, \tag{4.1.5}$$

with a regular matrix of factor functions $(h(t, r, \dot{r}))$. The net effect is that a necessary condition for the existence of such a Lagrangian representation is that the Lagrangian has a generalized structure, say, of the type (3.3.6), i.e.

$$L^{gen}(t, r, \dot{r}) \tag{4.1.6}$$

$$= \sum_k \sum_{a=x,y,z} L_{int,I}^{(ka)}(t,r,\dot{r})\, L_{free}^{(ka)}(\dot{r}^{ka}) + L_{int,II}(t,r,\dot{r}),$$

where the multiplicative terms to the free terms are essential. In turn, a necessary condition for such analytic representation is that the canonical momentum p, the canonical Hamiltonian H and the canonical angular momentum $M = r \times p$ do not represent physical quantities, that is, they are mathematical algorithms deprived of the conventional physical meaning of the corresponding quantities of conservative classical mechanics. An example is here in order to illustrate this occurrence. The familiar linearly damped oscillator belongs to the class of nonessentially nonselfadjoint systems, can be written in the language of characterization (4.1. 4)

$$\left[\left(\ddot{r} + r \right)_{SA}^{C^\infty,R} + \gamma \dot{r} \right]_{NSA}^{C^\infty,R} = 0, \quad m = 1, \ k = 1, \tag{4.1.7}$$

and admit a Hamiltonian representation of the type

$$H = \ln\left[\cos(\omega r p)\right] - \ln r + \frac{1}{2}\gamma r p, \quad p = \frac{\partial L^{gen}}{\partial \dot{r}}, \quad \omega^2 = 1 - \frac{\gamma}{4}. \tag{4.1.8}$$

The nontriviality of the quantization problem is then selfevident.

It is hoped that these remarks establish the open nature of the problem of quantization also for the simpler nonessentially nonselfadjoint systems because, even assuming that a quantum mechanical formulation of the algorithm H can be achieved with a mathematical consistency, the problem of physical consistency persists (e.g., which is the physical meaning of the expectation values of a quantity H which, at the Newtonian limit under the correspondence principle, does not represent a physical quantity by central assumption?).

TABLE 4.2: THE DUAL METHODOLOGIES FOR THE CLASSICAL TREATMENT OF NONCONSERVATIVE SYSTEMS: THE INVERSE PROBLEM AND THE LIE-ADMISSIBLE PROBLEM.

As indicated by the remarks of Table 4.1, one of the most insidious and potentially misleading aspects of the problem of quantization of nonconservative systems is the direct physical significance of the algorithms at hand, p, H, M, etc. In turn, this aspect has fundamental methodological implications which can essentially be classified into the following research attitudes.

(A) One may attempt to preserve the central methodological tool of established disciplines, the Lie algebra, for the treatment of nonconservative systems too. At the prequantized level, this implies the characterization of the time evolution law via the familiar Lie (Poisson) brackets

$$\dot{A}(r,p) = [A, H]_{(r,p)} = \frac{\partial A}{\partial r^{ka}}\frac{\partial H}{\partial p_{ka}} - \frac{\partial A}{\partial p_{ka}}\frac{\partial H}{\partial r^{ka}}. \tag{4.2.1}$$

The methodology of the Inverse Problem does indeed render this attitude practically realizable (for a review, see ref.[23], while for a detailed treatment see monographs[21]). However, the direct consequence of this approach to the systems considered is the loss of direct physical significance of the algorithms at hand. More specifically, a necessary condition for such algebraic characterization of essentially nonselfadjoint (nonessentially nonselfadjoint) systems is that both sets of canonically conjugate quantities r and p (the canonical momenta) do not possess a direct physical significance. At the level of quantization, this creates the problem of physical consistency of the emerging formulations.

(B) Owing to the novelty of the physical context, one may also attempt the identification of a non-Lie algebra for the characterization of the systems considered. And indeed, an inspection of the problem indicates that a necessary condition for the algebraic characterization of the time evolution law of nonconservative systems under the condition that the algorithms at hand (r, p, H, M, etc.)

have a direct physical significance, is that the basic algebra is (nonassociative and) NON-LIE, i.e., the time evolution law is expressed in terms of brackets, say, $(A,H)_{(r,p)}$

$$\dot{A} = (A, H)_{(r,p)} \qquad (4.2.2)$$

which violate the Lie algebra identities as the fundamental dynamical prerequisite. This second attitude is rendered practically realizable by the methodology of the Lie-admissible problem (for a review see ref.[23] while for a detailed treatment see monographs[22]). The main results are that such a non-Lie algebraic characterization of nonconservative systems does indeed exhist, results to be directly applicable to the systems considered (in exactly the same measure as Lie algebras are directly applicable to systems with forces derivable from a potential), possesses an analitic origin via a suitable generalized form of Hamilton's equations (in exactly the same measure as Lie algebras originate from the conventional Hamilton's equations), results to be of the so-called Lie-admissible type and, last but not least, emerges as being a covering of the conventional, Lie, algebraic approaches (in the sense of being directly applicable to a physically broader context, while capable of recovering the conventional, Lie formulations identically at the limit of null forces not derivable from a potential). At the level of quantization, this second approach exhibits open mathematical problems, but its physical consistency is expected to be ensured by the direct physical significance of the algorithms at hand.

(C) The mental attitude which is advocated in this paper is that of using both approaches (A) and (B), whenever applicable. In essence, the two approaches, even though profoundly different in inspiration and technical realization, are ultimately complementary because they study the same system. Owing to the complexity of the problems to be confronted and, in due time, eventually solved, each approach can in the final analysis exhibit a methodological backing or verification for the other.

As we shall see, the implementation of approach (C) indicates that the known Schrödinger and Heisen berg approaches individually exhibit a dual (rather than single) generalization for nonconservative systems.

TABLE 4.3: THE ANALYTIC ORIGIN OF THE LIE-ADMISSIBLE ALGEBRAS.

As indicated in ref.[23], it appears that the methodologies of the Inverse Problem and of the Lie-admissible problem for the treatment of nonconservative systems have different physical relevance, even though mathematically complementary to each other. For instance, the Inverse Problem provides valuable tools for the study of the structure of the forces considered or for the identification of the mechanisms of violation of the Galilei symmetry by the considered physical context, but it does not exhibit (to the best of my understanding at this time) a genuine constructive physical role, e.g., for the identification of the broader relativity laws which are expected to apply to nonconservative systems. The situation for the Lie-admissible problem is somewhat the opposite. It exhibits no methodological role for the study of the structure of the forces considered, while it appears to provide intriguing constructive capabilities for a classical, nonrelativistic, covering of the Galilei relativity in nonconservative mechanics. It may then be of some assistance for the interested reader to briefly review the analytic origin of Lie-admissible algebras in Newtonian mechanics.

The starting point is nonselfadjoint system (4.1.4). To avoid indirect analytic representation of type (4.1.5) which are responsible for the lack of direct physical meaning of the algorithms, the applicable analytic equations must be nonselfadjoint too. The conventional Lagrange's equations without external terms are unable to satisfy this fundamental requirement because they are always variationally selfadjoint for all Lagrangians of at least class C^4 and regular.[21,23] Therefore, they must be modified into a more general form capable of producing the desired direct Lagrangian representation. Lagrange's equations with external terms must be excluded too because, after a Legendre tramsform, the brackets of the time evolution law do not satisfy the conditions for a consistent algebra[23]. The covering of Lagrange's equations which was suggested in ref.[23] (for the case of conservative maximal selfadjoint subsystems) is given by

$$\left[\frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{\xi}^{ka}} - \frac{\partial E_{tot}}{\partial \xi^{k}}\right]^{C^\infty, R}_{NSA} = 0, \quad k = 1, 2, \ldots, \qquad (4.3.1a)$$

$$E_{tot} = \dot{\xi}^k \cdot \frac{\partial L_{tot}}{\partial \dot{\xi}^k} - L_{tot}, \qquad (4.3.1b)$$

$$L_{tot} = T(\dot{\xi}) - V(\xi), \qquad (4.3.1c)$$

where $E(r,\dot{r})$ is the energy integral of the conservative subsystem and $Z^k_\mu(t,r,\dot{r})$ is a set of functions (uniquely) characterized by the ordered direct analytic representations

$$\left[\frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{z}^k} - \frac{\partial E_{tot}}{\partial Z^k}\right]^{C^\infty,R}_{NSA} \equiv \left[\left(m_k \ddot{z}_k - f_k\right)^{C^\infty,R}_{SA} - F_k\right]^{C^\infty,R}_{NSA} . \qquad (4.3.2)$$

The compatibility with the conventional Lagrange's equations is then established by the fact that Eqs. (4.3.1a) recover these equations identically at the limit of null forces not derivable from a potential,

$$\lim_{F_{ka}\to 0}\left[\frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{z}^{ka}} - \frac{\partial E_{tot}}{\partial Z^{ka}}\right]$$
$$= \lim_{Z^{ka}\to z^{ka}}\left[\frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{z}^{ka}} + \frac{\partial z^{ib}}{\partial Z^{ka}}\frac{\partial L_{tot}}{\partial z^{ib}} - \left(\frac{\partial}{\partial Z^{ka}}\frac{\partial L_{tot}}{\partial \dot{z}^{ib}}\right)\dot{z}^{ib}\right]$$
$$= \frac{d}{dt}\frac{\partial L_{tot}}{\partial \dot{z}^{ka}} - \frac{\partial L_{tot}}{\partial z^{ka}} . \qquad (4.3.3)$$

For this reason, Eqs. (4.3.1a) were called in ref.[23] the Lagrange-admissible equations.

The conventional Legendre transform applied to Eqs. (4.3.1) yields the following covering of Hamilton's equations

$$\dot{b}^\mu - \frac{\partial H_{tot}}{\partial Z_\mu} = \dot{b}^\mu - \frac{\partial b^\nu}{\partial Z_\mu}\frac{\partial H_{tot}}{\partial b^\nu} = 0, \qquad (4.3.4a)$$
$$\mu = 1, 2, \dots, 6N,$$
$$\{b^\mu\} = \{z^{ka}, p_{ka}\}, \quad \{Z_\mu\} = \{p_{ka}, Z^{ka}\}, \qquad (4.3.4b)$$
$$p_{ka} = \frac{\partial L_{tot}}{\partial \dot{z}^{ka}}, \quad H_{tot} = \dot{z}^{ka} p_{ka} - L_{tot} = H_{free} + H_{int}, \qquad (4.3.4c)$$

which can also be written in a variety of equivalent covariant and contravariant forms, e.g.,

$$\dot{b}^\mu - S^{\mu\nu}\frac{\partial H_{tot}}{\partial b^\nu} = \dot{b}^\mu - \frac{\partial b^\mu}{\partial R_\nu}\frac{\partial H_{tot}}{\partial b^\nu} = 0, \quad \det(S^{\mu\nu}) \neq 0, \qquad (4.3.5a)$$
$$\left[S_{\mu\nu}\dot{b}^\nu - \frac{\partial H_{tot}}{\partial b^\mu}\right]^{C^\infty,R}_{NSA} = \left[\frac{\partial R_\mu}{\partial b^\nu}\dot{b}^\nu - \frac{\partial H_{tot}}{\partial b^\mu}\right]^{C^\infty,R}_{NSA} = 0, \qquad (4.3.5b)$$
$$(S_{\mu\nu}) = (S^{\mu\nu})^{-1}, \qquad (4.3.5c)$$

where the new functions considered are (uniquely) characterized by the functions $Z_\mu(t,b)$, and where the regularity of the matrix $(S^{\mu\nu})$ is always realizable.[23]

Eqs. (4.3.5) recover the conventional Hamilton's equations identically at the limit of null nonconservative forces

$$\lim_{F_{ka}\to 0}\left[S_{\mu\nu}\dot{b}^\nu - \frac{\partial H_{tot}}{\partial b^\mu}\right]^{C^\infty,R}_{NSA} = \left[\omega_{\mu\nu}\dot{b}^\nu - \frac{\partial H_{tot}}{\partial b^\mu}\right]^{C^\infty,R}_{SA}, \qquad (4.3.6a)$$
$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{3N\times 3N} & -1_{3N\times 3N} \\ 1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}, \qquad (4.3.6b)$$

and, for this reason, they were called in ref.[23] the Hamilton-admissible equations.

There is no need of going through the intermediate step of a Lagrangian representation (as it is also the case for the Inverse Problem[21]), because Eqs. (4.3.5) can be directly constructed from the equations of motion in a first-order vector field form via the solution of the equations

$$\frac{\partial R_\mu(t,b)}{\partial b^\rho}\left[f^\rho(b) - F^\rho(t,b)\right] = f_\mu(b), \qquad (4.3.7a)$$
$$f^\rho = \omega^{\rho\nu}\frac{\partial H_{tot}}{\partial b^\nu}, \quad f_\mu = \frac{\partial H_{tot}}{\partial b^\mu}, \quad \{F^\rho\} = \{0, F_{k}\}, \quad (\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1}. \qquad (4.3.7b)$$

The reasons why this covering analytic context was selected over other conceivable alternatives are basically the following.

The fundamental algebraic property of the Hamilton-admissible equations is that the brackets of the time evolution law

$$\dot{A}(b) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}\frac{\partial H_{tot}}{\partial b^\nu} = \frac{\partial A}{\partial b^\mu}\frac{\partial H_{tot}}{\partial Z_\mu} = (A, H_{tot}), \qquad (4.3.8)$$

characterize a (nonassociative, non-Lie) Lie-admissible algebra, that is, the attached brackets

$$[A, B]^* = (A, B) - (B, A) \qquad (4.3.9)$$

are Lie, i.e., verify the Lie algebra identities. This means that the Lie algebras are not lost in the approach. Instead, they are preserved in full according to

(a) An embedded form, technically realizable via a Lie-admissible generalization of the notion of enveloping algebra, which essentially expresses the property that a (nonassociative) Lie-admissible algebra U with brackets (A, B) possesses a well defined content of Lie algebra in the attached form U^- characterized by brackets $[A,B]^* = (A,B) - (B,A)$. See Tables 4.14 through 4.19 for technical details.

(b) A limiting form, in the sense that, at the limit of null forces not derivable from a potential, the Lie-admissible brackets (A, B) recover the conventional Poisson brackets identically

$$\lim_{F_\mu \to 0} (A,B) = \lim_{F_\mu \to 0} \frac{\partial A}{\partial b^\mu} S^{\mu\nu} \frac{\partial B}{\partial b^\nu} = \frac{\partial A}{\partial b^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial b^\nu} = [A,B]. \quad (4.3.10)$$

In turn, this is an indication that the nonconservative forces are embedded into the algebraic tensor $S^{\mu\nu}$ of the approach. And indeed, the deviation of the Lie-admissible over the Lie algebras is a direct representative of these broader forces, for instance, according to the rule

$$(S^{\mu\nu} - \omega^{\mu\nu}) \frac{\partial H_{tot}}{\partial b^\nu} = F^\mu. \quad (4.3.11)$$

The above properties were the fundamental requirements of selection of the considered covering analytic context. In particular, the Lie-admissible algebras are the only algebras capable of satisfying the joint requirements (a) and (b).

The reader should be aware that requirement (b) is crucial for the consistency of any possible generalization of quantum mechanics for nonconservative systems because, at the limit of null forces not derivable from a potential, conventional quantum mechanics must be recovered in full.

The reader can now see that the objectives which motivated the study of the Lie-admissible approach, the direct physical significance of the algorithms at hand (Table 4.2), is indeed achieved in full. The mathematical algorithm "p" now does represent the physical linear momentum $m\dot{r}$, the algorith "H" represents the total energy (see Table 4.7 for more details), and the algorithm "M" (= $r \times p$) represents the angular momentum.

Equally importantly, this Lie-admissible approach to nonconservative systems emerges as being directly universal, in the sense of being applicable without redefinition of the variables r and p (as necessary for Lie formulations) and being always applicable to the system considered because, under the assumed conditions, Eqs. (4.3.7) always admit a solution.

In ref.[23] I then presented a summary of my tentative studies for the possible existence of a Lie-admissible generalization of conventional formulations for inspection, assessement, and possible technical finalization by interested researchers. The objective was to indicate the existence of genuine hopes of achieving, in due time, a body of Lie-admissible formulations consisting of the following three essential aspects.

(I) Covering Lie-admissible analytic formulations, which, besides the equations recalled in this section, include a generalization of the canonical transformation theory, canonical perturbation theory, etc.

(II) Covering Lie-admissible algebraic formulations, essentially consisting of a possible covering of Lie's theory.

(III) Covering Lie-admissible geometrical formulations, essentially consisting of a possible covering of the symplectic geometry capable of producing an effective geometrical characterization of Lie-admi sssible algebras.

TABLE 4.4: THE NONCONSERVATIVE, NON-LIE, NONINERTIAL, NONGEODESIC, NONLINEAR, NONSYMPLECTIC AND NONRIEMANNIAN CHARACTER OF THE RECENTLY PROPOSED LIE-ADMISSIBLE COVERING OF THE GALILEI RELATIVITY.

A classical nonrelativistic covering of the Galilei relativity was proposed in ref.[23] for nonconservative systems by using the Lie-admissible formulations. It should be stressed that the study was tentative because in need of inspection and assessement by interested researchers. Neverthless, an outline of this covering relativity might here have some value, e. g., as a starting ground for subsequent possible implementations, modifications and eventual finalization of the issue.

The proposed Lie-admissible covering of the Galilei relativity, called Galilei-admissible relativity, is essentially characterized by the connected, Lie-admissible group of transformations[23]

$$\hat{G}(3.1): \ \hat{b}^\mu = \left(e^{\theta^i S_{(i)}^{\alpha\beta}(t,b)\frac{\partial X_i}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\right) b^\mu, \qquad (4.4.1)$$

$$\{b^\mu\} = \{r^{ka}, p_{ka}\}, \ \mu = 1,2,\ldots,6N; \ k = 1,2,\ldots,N; \ a = x,y,z, \ i = 1,2,\ldots,10,$$

where the X's are the generator of the conventional Galilei algebra for the conservative, maximal, selfadjoint subsystem of system (4.1.4), i. e.

$$X_1 = H_t \ , \qquad t = total \ , \qquad (4.4.2a)$$

$$\{X_2, X_3, X_4\} = \{P_{tx}, P_{ty}, P_{tz}\}, \qquad (4.4.2b)$$

$$\{X_5, X_6, X_7\} = \{M_{tx}, M_{ty}, M_{tz}\}, \qquad (4.4.2c)$$

$$\{X_8, X_9, X_{10}\} = \{G_{tx}, G_{ty}, G_{tz}\}, \qquad (4.4.2d)$$

the θ's are the conventional, corresponding, parameters of the Galilei group, i. e.,

$$\{\theta^i\} = \{t_0; \underset{\sim}{r}_0; \alpha, \beta, \gamma; \underset{\sim}{v}_0\}, \qquad (4.4.3)$$

and the $S_i^{\mu\nu}$ constitute the Lie-admissible tensors as constructed via analytic representations of type (4.3.5) (which are generally different for different generators)

The Lie-admissible character of this context is established by the fact that the brackets of the infinitesimal, first-order transformations are (nonassociative) Lie-admissible, i. e.,

$$\hat{b}^\mu \approx b^\mu + \theta^i\left(S_{(i)}^{\alpha\beta}\frac{\partial X_i}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}\right)b^\mu = b^\mu + \theta^i (b^\mu, X_i)_{(i)} \, . \quad (4.4.4)$$

Some of the key properties of this Lie-admissible relativity are the capability of recovering the conventional Galilei relativity identically at the limit of null forces not derivable from a potential, i. e.,

$$\lim_{\underset{\sim}{F} \to 0} \hat{G}(3.1) = G(3.1) \ : \ b'^\mu = \left(e^{\theta^i \omega^{\alpha\beta}\frac{\partial X_i}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\right) b^\mu, \qquad (4.4.5)$$

and the capability of apparently yielding generalized transformations which leave form-invariant the equations of motion considered, i. e.,

$$\hat{G}(3.1): \ \dot{b}^\mu - \Xi^\mu(t,b) = 0 \longrightarrow \dot{\hat{b}}^\mu - \Xi(\hat{t},\hat{b}) = 0, \quad (4.4.6a)$$

$$\left(\Xi^\mu\right) = \begin{pmatrix} p_{ka} \\ f_{ka} + F_{ka} \end{pmatrix}. \qquad (4.4.6b)$$

The former constitutes what was called an "uncompromisable requirement of compatibility" with the conventional Galilei relativity. The latter is a crucial requirement for any possible relativity, that is, a form-invariant description of physical reality (the reader should recall that one of the central properties of the systems considered is that of being form-noninvariant under the conventional Galilei transformations).

It may be of some value for the context of this paper to identify the departures from conventional relativity ideas which are inherent in this Lie-admissible approach, hoping that this may eventually result to be of some assistance for a resolution of the issue.

(1) The nonconservative character. The argument is that the systems considered are nonconservative and, thus, any applicable relativity must be able to characterize this physical profile in its entirety. In particular, under nonconservative forces, all the the

ten physical quantities of the conventional conservative case are generally nonconserved,

$$\dot{X}_i(t,b) = \frac{\partial X_i}{\partial b^\mu}\dot{b}^\mu + \frac{\partial X_i}{\partial t} \neq 0, \qquad (4.4.7)$$

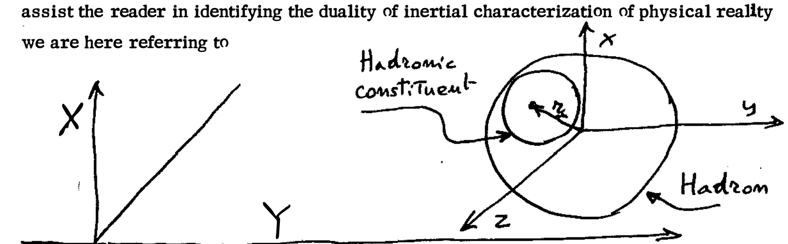
as a necessary condition to comply with physical reality. For instance, in the case of a two-body nonconservative system (i.e., the conventional Kepler system in a physical, dissipative medium) the total energy, linear momentum and angular momentum are nonconserved while the center of mass motion decays in time, because the system tends to rest in a finite period of time.

(2) The non-Lie character. As recalled earlier, we argue that this is a necessary condition for any applicable relativity, under the requirement that the algorithms at hand (p, H, M etc.) possess a direct physical meaning. This implies that the algebraic characterization of nonconservation laws (4.4.7) is done via brackets (A, H_t)

$$\dot{X}_i = (A, H_t) + \frac{\partial X_i}{\partial t} \neq 0, \qquad (4.4.8)$$

which violate the Lie algebra identities. The algebraic character of the proposed relativity is that of Lie-admissibility.

(3) The noninertial character. This character appears to be crucially dependent on the selected coordinates. The assumption which is here implemented is that these coordinates are those of a frame at rest with respect with the medium in which the motion occurs. In turn, any such medium (e.g., the Earth atmosphere or, more specifically for this paper, a hadron) is in generally noninertial conditions. The following picture might assist the reader in identifying the duality of inertial characterization of physical reality we are here referring to



Conventional, inertial, Galilean characterization for the exterior behaviour of a hadron.

Assumed, noninertial, Galilei-admissible characterization of the interior behaviour of a hadronic constituent.

where the noninertial reference frame is at rest with respect to the hadron.

Thus, the refence frame assumed for the Galilei-admissible relativity is generally noninertial by conception. This noninertial character is complemented by the fact that the new coordinate frame induced by a Galilei-admissible transformation (4.4.1) is generally noninertial, irrespective of whether the original system is inertial or not, because of a generally nonlinear dependence of the new coordinates $\hat{r}$ in the old momenta p.

The reader should be aware that, since the systems considered are not form-invariant under the Galilei transformations or any, more generally, arbitrary (but of class C^∞ and invertible) transformation of the coordinates, if one represents both, the hadron as a whole and its constituents in an inertial frame, this implies a different form of the structure forces. In other words, in the transition from the characterization of the structure equations in the reference frame at rest with respect to the hadron in noninertial conditions (say, a nucleon of the nucleus of the palladium) to the characterization in an outside inertial frame, the structure equations are subject to a nontrivial change of their form.

The reader should also be aware of the argument of ref.[23] to the effect that, after all, an inertial characterization of physical reality is a conceptual abstraction in the sense that no experiment has been conducted to date in an actual inertial frame and none will be conducted until a sophisticated interplanetary (or interstellar, or even intergalattic--under the assumption that the universe is in a nonuniform expansion) technology is available. The net effect is that the problem of a generalized relativity which is noninertial by central conception must sooner or later be confronted.

(4) The nongeodesic character. Conservative systems have a geodesic character in the Euclidean space of their detection in the sense that, at the limit of null action-at-a-distance force, the motion is geodesic. On more technical grounds, the property can be expressed by saying that the action of the Galilei group in its topological manifold is geodesic. Both these features are generally lost for our Lie-admissible relativity. First of all, nonconservative systems have a nongeodesic character in the Euclidean space of their detection in the sense that, at the limit of null action-at-a-distance forces (but nonnull dissipative medium), the trajectory is not necessarily geodesic. On more technical grounds, the action of the Lie-admissible group on its topological manifold is not necessarily geodesic. At a more detailed study, this nongeodesic character appears as necessary for a genuine representa-

tion of nonconservative systems under our by now familiar assumptions.

(5) The nonlinear character. This charater emerges in a number of contexts. For instance, the equations of motion are generally nonlinear in the velocities. A most intriguing nonlinear character of the Galilei-admissible relativity is however, that the representations of the underlying Lie-admissible envelop are nonlinear (because of their nonassociative nature), while linear representations are fully admissible for the universal enveloping algebra of the Galilei context (because it is associative). Apparently, this latter aspect is again related to the nonconservative nature of the systems considered or, equivalently, is another technical aspect to characterize the presence of forces not derivable from a potential.[33]

(6) The nonsymplectic character. This character emerges from the fact that the symplectic geometry, as currently known, does not result to be compatible with the Lie-admissible algebras (other then Lie) because the Lie-admissible tensor $S^{\mu\nu}$ is neither totally symmetric nor totally antisymmetric. In particular, the construction of an exterior two-form via the tensor $S_{\mu\nu}$, i.e.,

$$S_{(2)} = S_{\mu\nu}\, db^{\mu} \wedge db^{\nu} \equiv \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right) db^{\mu} \wedge db^{\nu} \qquad (4.4.9)$$

yields the "Lie-content" of the tensor itself. Thus, the symplectic geometry does not appear to be able to characterize the symmetric part of the Lie-admissible tensor. This creates the intriguing problem of the possible existence of a covering geometry, which, after all, is expected from the preceding characters of the Lie-admissible approach.

(7) The non-Riemannian character. If the symplectic geometry does not result to be applicable, one might then suspect the applicability of the Riemannian geometry as currently known (the crucial role of the former geometry for Galilei relativity and of the latter for Einstein general relativity should be here recalled). Intriguingly, this does not appear to be the case on a number of counts. On geometrical grounds, the Riemannian geometry does not appear to be applicable for the reason complementary to that of the symplectic geometry, that is, the inability for a proper characterization, this time, of the antisymmetric part of the Lie-admissible tensor. On dynamical grounds the occurrence can be seen from the fact that, when a Hamiltonian exists without redefinition of the Euclidean coordinates and use of the conventional Hamilton's equations, it can be written in the form

$$H^{gen} = p_{ia}\, G^{ia\,jb}(t, \underline{r}, \underline{p})\, p_{jb} + p_{ia}\, g^{ia}{}_{jb}(t, r)\, r^{jb} + g(t, \underline{r}), \qquad (4.4.10)$$

that is, the tensor G exhibits an essential dependence in the derivatives of the coordinates, which is outside the context of Riemannian geometry as currently known. The reader should be aware that the context for these geometrical profiles we are here referring to is related to our concept of representation with a direct physical significance of the algorithms at hand. If coordinate transformations or other concepts of representations are assumed (e.g., via locally Hamiltonian vector fields[23]), the applicability of established geometries becomes admissible.

But, besides these aspects, one of the most representative departures from conventional formulations which is inherent in the systems considered is that, in general, they do not admit a representation via variational principles in the coordinates of direct physical relevance, i.e., in the language of the Inverse Problem, the systems considered are in general essentially nonselfadjoint.

Illustrative examples of our Galilei-admissible relativity are presented in paper[23] and in monograph[22-II]. It may be of some assistance for the interested reader to present one of the simplest possible examples and verify the properties considered.

Consider a two-body conservative system. Suppose that the (central) force is, for simplicity but without loss of generality, of the oscillator type. In relative coordinates the equation of motion are given by the first order (vector field) form

$$\left(\dot{b}^{\mu}\right) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -r \end{pmatrix}, \quad \mu = 1,\ k = 1, \quad r = r_1 - r_2. \qquad (4.4.11)$$

The above system possesses an exact symmetry under translations in time which constitute the one-parameter subgroup $T_1(t)$ of the Galilei group. In canonical realization it is given by

$$T_1(t): \quad b'^{\mu} = e^{t\,\omega^{\alpha\beta}\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\, b^\mu, \tag{4.4.12a}$$

$$H_t = \frac{1}{2}(p^2 + r^2), \tag{4.4.12b}$$

and explicitly reads

$$r' = r + \frac{t}{1!}[r, H_t] + \frac{t^2}{2!}[[r, H_t], H_t] + \cdots \tag{4.4.13a}$$
$$= r + \frac{t}{1!}p - \frac{t^2}{2!}r - \frac{t^3}{3!}p + \cdots$$

$$p' = p + \frac{t}{1!}[p, H_t] + \frac{t^2}{2!}[[p, H_t], H_t] + \cdots \tag{4.4.13b}$$
$$= p - \frac{t}{1!}r - \frac{t^2}{2!}p + \cdots$$

The series then converge into the finite transformations

$$\begin{pmatrix} r' \\ p' \end{pmatrix} = \begin{pmatrix} r\cos t + p\sin t \\ -r\sin t + p\cos t \end{pmatrix}, \tag{4.4.14}$$

which leave form-invariant the vector field

$$\begin{pmatrix} \dot r' \\ \dot p' \end{pmatrix} = \begin{pmatrix} -r\sin t + p\cos t \\ -r\cos t - p\sin t \end{pmatrix} = \begin{pmatrix} p' \\ -r' \end{pmatrix}, \tag{4.4.15}$$

while constituting a group with trivial composition law

$$b' = f(b;t), \quad b'' = f(b';t') = f(b; t+t'), \tag{4.4.16a}$$

$$\begin{pmatrix} r'' \\ p'' \end{pmatrix} = \begin{pmatrix} r'\cos t' + p'\sin t' \\ -r'\sin t' + p'\cos t' \end{pmatrix} = \begin{pmatrix} r\cos(t+t') + p\sin(t+t') \\ -r\sin(t+t') + p\cos(t+t') \end{pmatrix}. \tag{4.4.16b}$$

The physical implication of the above form-invariance is that of characterizing a <u>physical law</u>, the conservation of the energy. This can be seen either from the form-invariance of H_t

$$H'_t(b') = e^{t\,\omega^{\alpha\beta}\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\, H_t(b) = H(b'), \tag{4.4.17}$$

or from the Lie characterization

$$\delta H_t = t\left(\omega^{\alpha\beta}\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}\right) H_t = t[H_t, H_t] = 0. \tag{4.4.18}$$

The above relativity characterization of system (4.4.11) has

(1')- a conservative character, the system being conservative by assumption;

(2') a Lie character in the dual meaning that structure (4.4.12a) forms a (connected) Lie group of transformations and the physical law (4.4.18) is expressed via the product of a Lie algebra;

(3') inertial character, in the sense that all inertial systems can be equivalently used <u>without altering the form of the equations of motion;</u>

(4') a geodesic character in the sense that, at the limit of null force $K \to 0$, the trajectory is a geodesic (a straight line) or, more rigorously, the action of the group $T_1(t)$ in its topological manifold is geodesic

$$e^{t\,\omega^{\alpha\beta}\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}} = \gamma(t) = \text{geodesic}; \tag{4.4.19}$$

(5') a linear character in the sense that linear representations of the group $T_1(t)$ are fully admitted because the underlying algebraic envelop is associative;

(6') a symplectic character because the fundamental symplectic form

$$\frac{1}{2}\omega_2 = \frac{1}{2}\omega_{\alpha\beta}\, db^\alpha \wedge db^\beta = dr \wedge dp \tag{4.4.20}$$

directly inters into the very structure of $T_1(t)$;

(7') a Riemannian character in the sense that, say, the system is represented by a Hamiltonian with a trivially Riemannian structure

$$H_t = \frac{1}{2}\, p_i G^{ii} p_i + G_0(r_1, r_2), \tag{4.4.21a}$$

$$G^{ii} = \frac{1}{\mu_i}\delta^{ij}, \tag{4.4.21b}$$

or in the sense of more formal treatments.[34]

We now assume that the two-body system (4.4.11) enters into a physical medium (say, gas or plasma) which results in the nonconservation of the energy. The system then tends to rest in a finite period of time. This broader physical context can be represented with the addition to the equations of motion of a nonconservative force-F.

Eqs. (4.4.11) are then replaced by the broader equations

$$(\dot{b}^\mu) = \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p \\ -r-F \end{pmatrix}, \tag{4.4.22}$$

where F is tacitly understood to comply with physical evidence (instability of the orbits and achievement of the rest status in a finite period of time).

The Galilei relativity is no longer applicable for a physically meaningful characterization of nonconservative system (4.4.22). First of all, the system now becomes form-noninvariant under the Galilei transformations (4.4.14), trivially, because

$$\begin{pmatrix} \dot{r}' \\ \dot{p}' \end{pmatrix} = \begin{pmatrix} -r\sin t + p\cos t \\ -r\cos t + p\sin t \end{pmatrix} = \begin{pmatrix} p' \\ -r' \end{pmatrix} \neq \begin{pmatrix} p' \\ -r'-F \end{pmatrix}. \tag{4.4.23}$$

Irrespective from that, the Galilei relativity presents a number of deficiencies to provide a consistent characterization of the new, broader physical situation, the NON-conservation of the energy, as analysed in details in references[22,23]. After all, this is not surprising. The Galilei relativity was specifically conceived for conservative settings. The broader physical context we are considering is profoundly different than that.

This situation creates the problem of searching for a covering relativity, that is, a mathematical formulation which produces a consistent characterization of the nonconservation of the energy via form-invariance under broader transformations (to qualify as relativity) and which, most importantly (particularly for the problem of the hadronic structure -- see the crucial implications of Section 5) is capable of recovering the Galilei relativity identically at the limit of null nonconservative force F.

For the case considered our Galilei-admissible covering relativity satisfies these properties. Consider first the simplest possible case of the nonconservative force F, a costant force (functional dependences will be considered later).

The simplest possible representation of Eqs. (4.4.22) via our Lie-admissible covering of Hamilton's equations is given by

$$\dot{b}^\mu = S^{\mu\nu}\frac{\partial H_t}{\partial b^\nu}, \quad H_t = \frac{1}{2}(p^2 + r^2), \tag{4.4.24a}$$

$$\{b^\mu\} = \{r, p\}, \quad (S^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ s & 0 \end{pmatrix}, \quad s = -\left(1 + \frac{F}{r}\right). \tag{4.4.24b}$$

The knowledge of the Lie-admissible tensor $S^{\mu\nu}$ then immediately yields our Galilei-admissible covering of the Galilei's group of translations in time

$$\hat{T}_1(t): \quad \hat{b}^\mu = e^{t S^{\alpha\beta}(b)\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}\, b^\mu, \tag{4.4.25}$$

consisting of the separate series

$$\begin{aligned} \hat{r} &= r + \frac{t}{1!}(r, H_t) + \frac{t^2}{2!}((r, H_t), H_t) + \cdots \\ &= r + \frac{t}{1!}(p) + \frac{t^2}{2!}(-r-F) + \cdots \end{aligned} \tag{4.4.26a}$$

$$\begin{aligned} \hat{p} &= p + \frac{t}{1!}(p, H_t) + \frac{t^2}{2!}((p, H_t), H_t) + \cdots \\ &= p + \frac{t}{1!}(-r-F) + \frac{t^2}{2!}(-p) + \cdots \end{aligned} \tag{4.4.26b}$$

$$(A, B) = \frac{\partial A}{\partial b^\alpha} S^{\alpha\beta} \frac{\partial B}{\partial b^\beta} \tag{4.4.26c}$$

which now converge into the finite transformations

$$\begin{pmatrix} \hat{r} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} -F + (r+F)\cos t + p\sin t \\ -(r+F)\sin t + p\cos t \end{pmatrix}. \tag{4.4.27}$$

First, we must verify the fundamental requirement for any formulation to qualify as "relativity," the form-invariant description of nature. This requirement is indeed satisfied by generalized transformations (4.4.27) because, trivially,

$$\begin{pmatrix} \dot{\hat{r}} \\ \dot{\hat{p}} \end{pmatrix} = \begin{pmatrix} -(r+F)\sin t + p\cos t \\ -p\sin t - (r+F)\cos t \end{pmatrix} = \begin{pmatrix} \hat{p} \\ -\hat{r} - F \end{pmatrix}. \tag{4.4.28}$$

Secondly, we must verify the covering nature of the group $\hat{T}_1(t)$ over $T_1(t)$, to qualify as a covering relativity group. This implies (at least) two aspects. First, the capability of recovering the group $T_1(t)$ identically at the limit $F \to 0$. This is trivially satisfied by our transformations (4.4.27)

$$\lim_{F\to 0} \begin{pmatrix} \hat{r} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} r\cos t + p\sin t \\ -r\sin t + p\cos t \end{pmatrix} = \begin{pmatrix} r' \\ p' \end{pmatrix}, \tag{4.4.29}$$

Secondly, we must inspect the methodological context to see whether a group structure exists. Transformations (4.4.27) are trivially connected because for $t \approx 0$

$$\begin{pmatrix} \hat{r} \\ \hat{p} \end{pmatrix} \cong \begin{pmatrix} -F + (r-F) \\ p \end{pmatrix} = \begin{pmatrix} r \\ p \end{pmatrix}, \tag{4.4.30}$$

also, they trivially satisfy the composition law

$$\hat{b} = f(\hat{b};t), \quad \hat{\hat{b}} = f(\hat{b};t) = f(b;t+t'), \tag{4.4.31a}$$

$$\begin{pmatrix} \hat{\hat{r}} \\ \hat{\hat{p}} \end{pmatrix} = \begin{pmatrix} -F + (\hat{r}+F)\cos t' + \hat{p}\sin t' \\ -(\hat{r}+F)\sin t' + \hat{p}\cos t' \end{pmatrix} \tag{4.4.31b}$$

$$= \begin{pmatrix} -F + (r+F)\cos(t+t') + p\sin(t+t') \\ -(r+F)\sin(t+t') + p\cos(t+t') \end{pmatrix}.$$

Thus, our structure $\hat{T}_1(t)$ constitutes a connected Lie group of transformations in exactly the same way as that of $T_1(t)$. The structure of these two group, however, is different. Indeed, $\hat{T}_1(t)$ is defined in terms of the same generator (H_t), the same parameter (t) and the same manifold (b^μ) as those for $T_1(t)$. As a matter of fact, it is precisely this property which qualifies $\hat{T}_1(t)$ as a covering of $T_1(t)$, not in the topological sense, but in our physical sense that

$$\lim_{F\to 0} \hat{T}_1(t) = T_1(t). \tag{4.4.32}$$

Owing to these differences, we have called $\hat{T}_1(t)$ a Lie-admissible group,[23] in the sense that it admit a different Lie group under a limit procedure, while its behaviour in the neighborhood of the identity produce an algebra which is NON-LIE, but Lie-admissible

$$\delta A = t\left(S^{\alpha\beta}(b)\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}\right)A = t(A,H_t), \tag{4.4.33}$$

or equivalently, Lie-admissible algebras directly enter into the structure of $\hat{T}_1(t)$ according to the embedding

$$\underbrace{e^{t\,\omega^{\alpha\beta}\frac{\partial H_t}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}}_{\text{LIE}} \Longrightarrow \underbrace{e^{t\,S^{\alpha\beta}(b)\frac{\partial H_b}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}}}_{\text{LIE-ADMISSIBLE}}. \tag{4.4.34}$$

The non-Lie character of the covering relativity transformations has a direct physical significance which is crucial for the consistency with physical evidence: the NON-conservation of the energy. Indeed, it is precisely the loss of the Lie algebras which allows the direct algebraic representation of the nonconservation law

$$\delta H_t = t\left(S^{\alpha\beta}\frac{\partial H_b}{\partial b^\beta}\frac{\partial}{\partial b^\alpha}\right)H_t = t\,(H_t,H_t) \neq 0, \tag{4.4.35}$$

as a covering of conservation law (4.4.18). Equivalently, the property can be seen from the form-noninvariance of H_b under $\hat{T}_1(t)$.

In conclusion, while the Galilei group $T_1(t)$ is unable to provide both, a form-invariant description of the system and a characterization of the nonconservation of the energy, our Galilei-admissible group $\hat{T}_1(t)$ is not only capable of satisfying both these fundamental requirements, but it is jointly capable of recovering the group $T_1(t)$ identically at the limit of null nonconservative force.

We consider now the case of a functional dependence of the F-force. The extension of the results to the case when F explicitly depends on time, $F = F(t)$, is trivial and will be left to the interested reader (we assume, however, that the reader is familiar with the fact that in the recomputation of Eqs. (4.4.15) the time derivative is performed with respect to parameter t of $T_1(t)$, while the quantity t' of the equation of motion becomes an initial value and, as such, it is constant). The results are exactly the same as those for a constant F.

The situation for the case when F has a broader functional dependence, say, $F = F(t, \dot{r})$ and $F = F(t, r, \dot{r})$, is different. Here the reader should be warned against the expectation of simple solutions. The reason is the following. For conservative systems, the Galilei transformations are manifest symmetries, that is, symmetries of simple, direct identification. For nonconservative systems the situation is profoundly different. Indeed, in this case the relativity transformations which leave form-invariant the equations of motion are always nonmanifest symmetries, that is, transformations of rather complex structure which under no circumstance can be identified via a visual inspection of the system or simple empirical techniques. The reason is trivial: the equations under consideration are nonconservative, generally nonlinear and generally dependent explicitly on time. The construction of transformations which leave form-invariant these systems is not an easy task. This is precisely the effectiveness of our covering relativity because it provides specific rules for the explicit construction of these nonmanifest symmetries. Indeed, for system (4.4.22) the information requested was:

(a) the conventional Galilei group;

(b) the Hamiltonian for the conservative system;

(c) the nonconservative force.

The Lie-admissible formulations then produced the desired result: the nonmanifest symmetry group (4.4.27).

The interested reader is therefore encouraged to work our cases with more complex functional dependences of the nonconservative forces, with the expectation that for any functional dependence other than F(t), the emerging transformations are not given by Eqs. (4.4.27) and that their explicit form is quite complex indeed.

We must now identify the implications of our covering relativity. The case (4.4.22) is sufficient. In short, the form-invariance characterization of system (4.4.22) and of the underlying nonconservation law of the energy via a covering of the corresponding conservative setting implies the abandonment of the conservative, Lie, inertial, geodesic, linear, symplectic and Riemannian character of the original (but now inapplicable) relativity according to

(1") nonconservative character, as the fundamental condition for physical consistency and as expressed by nonconservation law (4.4.35);

(2") non-Lie character, as characterized by the algebraic structure of law (4.4.35) or the new behaviour in the neighborhood of the origin (4.4.33) (as recalled in Table 4.2, a necessary condition to characterize such nonconservation law when all the algorithms at hand have a direct physical significance is that the brackets of the time evolution law violate the Lie algebra identities);

(3") noninertial character, e.g., now the transition from one inertial system to another violates, in general, the form-invariance of the equations of motion;

(4") nongeodesic character, in the sense that when the conservative force is null, the motion is not necessary a geodesic (it is not, in general, a straight line due to dissipative effects), or, more technically, because the action of the Galilei-admissible group in its topological manifold is not necessary geodesic

$$e^{t\, S^{\alpha\beta}(b) \frac{\partial H_t}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} = f(\gamma(t)) \neq \text{geodesic}; \quad (4.4.36)$$

(5") nonlinear character, in the sense that the representation of the Galilei-admissible group $\hat{T}_1(t)$ are nonlinear because of the nonassociative nature of the underlying algebraic envelop (see Tables 4.14 through 4.19);

(6") nonsymplectic character, in the sense that symplectic structures of type (4.4.9) do not characterize the Lie-admissible tensor $S^{\alpha\beta}$ and only its antisymmetric part, while the broader geometry which appears as applicable is the so-called symplectic-admissible geometry with nowhere degenerate exterior admissible two-form [23]

$$\hat{S}_2 = S_{\mu\nu}\, db^\mu \circ db^\nu$$
$$= \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right) db^\mu \times db^\nu + \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right) db^\mu \wedge db^\nu, \quad (4.4.37)$$

for which

$$\hat{S}_2 - \hat{S}_2^T = \Omega_2 = \frac{1}{2}\left(S_{\mu\nu} - S_{\nu\mu}\right) db^\mu \wedge db^\nu = \text{Symplectic} \quad (4.4.38)$$
$$d\hat{S}_2 \neq 0, \quad d\left(\hat{S}_2 - \hat{S}_2^T\right) = 0;$$

(7") non-Riemannian character, in the sense that now the system admits a Hamiltonian with the non-Riemannian structure (4.4.10), or that structure (4.4.37), besides being non-symplectic, is also non-Riemannian, or in the sense to be indicated in Table 4.20 for detailed treatment at some later paper.

In addition to the indicated departures from the familiar settings of the conservative Galilei's relativity, our Galilei-admissible relativity indicates still another departure: it suggests the existence of a priviledged class of reference frames, those at rest with respect to the medium in which the motion occurs. In relation to the hadronic structure, this implies that for the case, say, of the motion of one particle within hadronic matter, the forces are computed in the reference frames at rest with such medium. We shall not attempt a rigorous justification of this additional departure from conventional relativity ideas (for an analysis see ref.[23]). We shall simply assume it and assess its plausibility from its consequences.

The proposed Galilei-admissible relativity is at the very foundation of our model of hadronic structure to be presented in Section 5. We shall therefore complement the brief review of this table with additional comments throughout our analysis. Of course, one of the central objectives of this paper is to achieve quantization of this relativity, in order to reach a form which is directly applicable to hadron structure. This problem will be studied in Table 4.18.

TABLE 4.5: A FIRST DICHOTOMY FOR QUANTIZATION: NONESSENTIALLY NONSELFADJOINT VERSUS ESSENTIALLY NONSELFADJOINT SYSTEMS.

We are now equipped to begin the study of an intriguing but delicate research project: the quantization of nonconservative Newtonian systems in general, and of strong nonselfadjoint forces in particular.

The problem under consideration will be essentially studied from two profiles: the possible existence of a generalization of Schrödinger's and Heisemberg's approach for the quantum mechanical treatment of local, class C^∞ forces no derivable from a potential. Tables 4.5 through 4.12 are devoted to the first profile, while Tables 4.14 through 4.19 are devoted to the second profile. Tables 4.20 and 4.21 outline the implications of the emerging formulations.

Let us begin with the first profile of the program. A tentative formulation of the problem is the following. It consists of the study of the possible existence and consistency, on both mathematical and physical grounds, of a covering of Schrödinger's equation for the quantization of local, class C^∞ nonconservative (nonselfadjoint) Newtonian systems

$$\left[\left(m_k \ddot{\mathbf{r}}_k - \mathbf{f}_k\right)_{SA}^{C^\infty, R} - \mathbf{F}_k\right]_{NSA}^{C^\infty, R} = 0, \quad k = 1, 2, \ldots, N, \quad (4.5.1)$$

which satisfies Bohr's correspondence principle. The following three properties are expected to be crucial:

(I) The desired generalization of Schrödinger's equation should be nontrivially different than the conventional equation (as currently known) for a nontrivially different physical context (forces derivable from a potential versus forces not derivable from a potential).

(II) The possible generalizazion of Schrödinger's equation should be able to recover the conventional equation identically at the limit of null forces not derivable from a potential.

(III) The expected generalization of Schrödinger's equation should recover the

Hamilton-Jacobi equation for the nonconservative systems considered at the limit of large value of the action functional as compared to Planck's constant (correspondence principle).

The above properties I, II and III are here intended as a specification of the "covering" nature of the expected generalization versus the conventionally known form of the equation considered, as well as well as a more specific identification of the objectives to be attempted.

A number of restrictions on the equations to be quantized, Eqs. (4.5.1) is now in order. The first important restrictions can be presented as follows.

(1) Restrictions on the acting forces. As by now familiar, the symbol SA (NSA) in Eqs. (4.5.1) represents the technical characterization in the language of the Inverse Problem for the forces f_k (F_k) of being derivable from a potential, i.e., selfadjoint (nonderivable from a potential--nonselfadjoint). These forces must be more specifically identified. Their explicit functional dependence which will be considered can be written

$$f_k = f_k(t, r, \dot{r}) = -\frac{\partial U}{\partial r^k} + \frac{d}{dt}\frac{\partial U}{\partial \dot{r}^k}, \qquad (4.5.2a)$$

$$F_k = F_k(t, r, \dot{r}, \ddot{r}, \gamma) \neq -\frac{\partial U'}{\partial r^k} + \frac{d}{dt}\frac{\partial U'}{\partial \dot{r}^k}, \qquad (4.5.2b)$$

where a typical representative of the f_k forces is the Lorentz force

$$f_k^{Lorentz} = e\left[E(t, r) + \frac{1}{c}\dot{r} \times B(t, r)\right] = f_k(t, r, \dot{r}), \qquad (4.5.3a)$$

$$U^{Lorentz}(t, r, \dot{r}) = e\varphi(t, r) - \frac{e}{c} A(t, r) \times \dot{r}, \qquad (4.5.3b)$$

while a typical representative of F_k is that of, say, occurring in the theory of damped, coupled oscillators. Most importantly, the dependence of the F-forces in the accelerations must be ut most linear (to qualify as a Newtonian force), with a known example being given by the so-called acceleration couplings occurring in the theory of oscillators. Also, the F-forces will be assumed to be explicitly dependent on a number of parameters, here denoted with the symbol γ, in such a way that

$$\lim_{\gamma \to 0} F_k = 0. \qquad (4.5.4)$$

This is the case, for instance, of the acceleration coupling parameters, or the damping parameters of Newtonian drag forces. At this Newtonian level, no intrinsic characterization (e.g., the spin) will be introduced, to be in line with the (early) formulation of quantum mechanics. The reader should be aware, from preceding remarks, that selfadjoint forces

can be nonlinear in the coordinates r^k, but they are necessarily linear in the velocities $\dot{r}^k$. The nonselfadjoint (Newtonian) forces, instead, are generally nonlinear in both r^k and $\dot{r}^k$ and necessarily linear only in $\ddot{r}^k$.[28]

(2) Restrictions on the coordinate system The coordinates r^k of system (4.5.1) are assumed as representing the Cartesian coordinates of a three-dimensional Euclidean system which is at rest with respect to the medium in which the motion occurs (the hadronic medium, under our assumption). In other words, Eqs. (4.5.1) are assumed as the Newtonian limit of an expected discrete, quantum mechanical approximation of the motion of charged particles in hadronic matter. By following a customary approach in Newtonian mechanics, the explicit form of the nonconservative forces is then identified within the reference frame at rest with respect to the dissipative medium. The reader should be aware, from preceding remarks, that this system of coordinates is in generally noninertial conditions and it is generally different than that used for quantum mechanical experiments. The reader should also be aware that now the transition to another system of coordinates implies a nontrivial change in the structure of the nonconservative forces. In turn, this assumption exhibits nontrivial epistemological and technical implications for the measurement theory. At this stage, the assumption is considered for the specific intent of probing the plausibility of a dichotomy of relativity laws, as outlined in Table 4.4.

(3) Restrictions of variational character. One of the fundamental conditions we have imposed on the expected generalization of Schrödinger's equation is that it recovers the Hamilton-Jacobi equation under the correspondence principle. In turn, this implies that one of the crucial restrictions on the systems considered is that they must satisfy the integrability conditions of the Inverse Problem for the existence of a Hamiltonian representation.[21,23] As recalled earlier, two classes of systems can be identified in this respect: those which admit an indirect analytic representation of type (4.1.5) without redefinition of the variables r^k (nonessentially nonselfadjoint systems) and those which admit indirect analytic representation with a necessary redefinition of the variables r^k (essentially nonselfadjoint systems). Our analysis for the possible existence of a covering of Schrödinger's equation is restricted to the nonessentially nonselfadjoint systems, while the more general case of essentially nonselfadjoint systems will be left to the interested researcher (a part from few, incidental remarks related to the second profile, a possible covering of Heisen berg's approach).

TABLE 4. 6: A SECOND DICHOTOMY FOR QUANTIZATION: CONVENTIONAL LAGRANGIAN STRUCTURES VERSUS ISOTOPICALLY MAPPED STRUCTURES.

The first restriction on variational grounds (nonessentially nonselfadjoint systems) is still insufficient for our analysis, and a number of additional restrictions will be needed. The next restriction is related to the admissible Lagrangian structure. As recalled in Table 3.3 , the structure of the emerging Lagrangians is generalized, i. e. , of the type (3. 3 . 6). In particular, the dependence on the velocities is generally higher than the conventional quadratic form of the kinetic term of the trivial conservative systems. In turn, this implies additional technical implications at a possible quantum mechanical level which are unnecessary for this initial treatment of the problem (e. g. , a generalization of the Schrödinger's equation of order higher than the second). We shall therefore assume that all admitted systems are such to yield a generalized Lagrangian structure which is quadratic in the velocities, i. e. ,

$$\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{ka}} - \frac{\partial L}{\partial r^{ka}}\right]_{SA}^{C^\infty,R} = \left\{ h_{ka}^{\ \ jb}(t,r,\dot{r})\left[\left(m_j \ddot{r}_{jb} - f_{jb}\right)_{SA}^{C^\infty,R} - F_{jb}\right]_{NSA}^{C^\infty,R}\right\}_{SA}^{C^\infty,R}, \quad (4.6.1a)$$

$$L^{gen}(t,r,\dot{r};\mathcal{F}) = \frac{1}{2}\dot{r}^{ia}G^{-1}_{ia\,jb}(t,r;\mathcal{F})\dot{r}^{jb} - G_0(t,r;\mathcal{F}). \quad (4.6.1b)$$

$$(G^{-1}) = (G^{-1})^T, \quad \det(G^{-1}) \neq 0.$$

In turn, this implies the restriction to nonconservative forces which are ut most quadratic in the velocities. Even though restrictive, this subclass of the class of nonessentially nonselfadjoint forces is genuinely nonconservative and sufficient for our objectives. Upon use of the conventional Legendre transform, the admitted Hamiltonians are therefore of the type

$$H(t,r,p;\mathcal{F}) = \frac{1}{2}p_{ia}G^{ia\,jb}(t,r;\mathcal{F})p_{jb} + G_0(t,r;\mathcal{F}), \quad (4.6.2a)$$

$$p_{ia} = \frac{\partial L}{\partial \dot{r}^{ia}} = \dot{r}^{jb}G^{-1}_{jb\,ia}, \quad (G)^{-1} = (G^{-1}), \quad (G) = (G)^T. \quad (4.6.2b)$$

Next, in order to comply with the uncompromisable requirement that the expected generalized Schrödinger's equation recovers the conventional equation identically at the limit of null nonconservative forces, it is advantageous to introduce the corresponding restriction at the Newtonian level, i. e. ,

$$\lim_{F_{\mu k}\to 0} L = \lim_{\mathcal{F}\to 0} L = T - U, \quad (4.6.3a)$$

$$\lim_{F_{\mu k}\to 0} H = \lim_{\mathcal{F}\to 0} H = T + U. \quad (4.6.3b)$$

In other words, the only admitted Lagrangians or Hamiltonians for nonconservative systems are those capable of recovering the maximal selfadjoint subsystem under the limit of null forces non derivable from a potential. In turn, technical arguments imply the elimination of the so called isotopic degrees of freedom of a Lagrangian or Hamiltonian structure.[21,22,23]

As we shall see later on, this aspect appears to have nontrivial implication for the problem of quantization of both conservative and nonconservative systems and, as such, it deserve an elaboration. Let us recall that the methodology of the Inverse Problem allows the construction of a Lagrangian representation of all equivalent selfadjoint formsof the equations of motion. In essence, when a Lagrangian $L(t,r,\dot{r})$ is known, a family of equivalent Lagrangians $L^*(t,r,\dot{r})$ (called isotopically mapped Lagrangians) may exist according to the rule

$$\left[\frac{d}{dt}\frac{\partial L^*}{\partial \dot{r}^{ka}} - \frac{\partial L^*}{\partial r^{ka}}\right]_{SA}^{C^\infty,R} \equiv \left\{ h_{ka}^{\ \ jb}\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{r}^{jb}} - \frac{\partial L}{\partial r^{jb}}\right]_{SA}^{C^\infty,R}\right\}_{SA}^{C^\infty,R}, \quad (4.6.4)$$

where the selfadjointness of the Lagrange's equations in L is always verified[21,23], while that of the equivelent system with factor terms $h_{ka}^{\ \ jb}$ is imposed. This implies a system of (quasilinear) partial differential equations in the unknown factor terms $h_{ka}^{\ \ jb}$ for a fixed L. When this system is consistent, it customarily admit a solution with functional degree of arbitraryness (as typical for partial, rather than ordinary, differential equations). This in turn implies a family of structurally different Lagrangians L* because dependent on the explicit form of the factors $h_{ka}^{\ \ jb}$. An important aspect is that rule (4.6.4) applies irrespective of whether the represented system is conservative or not. The Legendre transform then yields a corresponding family of Hamiltonians. In turn, this creates a considerable ambiguity at the quantum mechanical level.

Simple, but explicit examples of these degrees of freedom at both the conservative and nonconservative level are here in order. First, for the case of a free particle we have

$$(\ddot{r})_{SA} = 0 , \quad (m=1, \dot{r} \neq 0), \qquad (4.6.5a)$$

$$L_{free} = \frac{1}{2}\dot{r}^2 , \quad H_{free} = \frac{1}{2} p^2 , \qquad (4.6.5b)$$

$$L^*_{free} = \dot{r} \ln \dot{r} \quad H^*_{free} = e^{p'-1} . \qquad (4.6.5c)$$

The symplest nonconservative extension is that of a particle with a linear velocity drag force for which we have

$$[(\ddot{r})_{SA} + \gamma \dot{r}]_{NSA} = 0, \qquad (4.6.6a)$$

$$L = e^{\gamma t}\frac{1}{2}\dot{r}^2 , \qquad H = e^{-\gamma t}\frac{1}{2} p^2 , \qquad (4.6.6b)$$

$$L^* = \dot{r} \ln \dot{r} - \gamma r , \quad H^* = e^{p'-1} + \gamma r . \qquad (4.6.6c)$$

The reader should be aware that we have here indicated only one isotopically mapped Lagrangian or Hamiltonian. For additional forms, see ref.[21-II]

The quantum mechanical ambiguities are now selfevident. Indeed, the quantization of, say, H_{free} is established, but that of H^*_{free} is unknown at this time, to the best of my knowledge. As we shall see, a fully equivalent situation occurs at the nonconservative level for structures (4.6.6).

As an example with a conservative force, we have the one-dimensional harmonic oscillator for which[21,23]

$$(\ddot{r} + r)_{SA} = 0 , \quad (m=1, k=1, \dot{r} \neq 0) , \qquad (4.6.7a)$$

$$H = \frac{1}{2}\left(p^2 + r^2\right) , \qquad (4.6.7b)$$

$$H^* = \ln \left| r \sec \frac{1}{2} r p' \right| . \qquad (4.6.7c)$$

Its simplest nonconservative extension is given by the linearly damped oscillator for which[21,23]

$$[(\ddot{r} + r)_{SA} + \gamma \dot{r}]_{NSA} = 0 , \qquad (4.6.8a)$$

$$H = e^{-\gamma t}\frac{1}{2} p^2 + e^{\gamma t}\frac{1}{2} r^2 , \qquad (4.6.8b)$$

$$H^* = \ln \left| \cos(\omega p' r) \right| + \ln r + \frac{1}{2}\gamma r p', \quad \omega = 1 - \gamma^2/4 . \qquad (4.6.8c)$$

A comparison of systems (4.6.7) and (4.6.8) then confirms the full analogy between conservative and nonconservative systems, as far as the isotopic degrees of freedom are concerned. The quantum mechanical ambiguity is again selfevident. The quantization of the harmonic oscillator with Hamiltonian (4.6.7b) is established, but that with Hamiltonian (4.6.7c) is an open problem. We then argue that the quantization of the two Hamiltonians (4.6.8b) and (4.6.8c) for the nonconservative extension of the same system consistutes two different problems. In essence, according to restriction (4.6.3), only structures (4.6.6b) and (4.6.8b) are admitted in our analysis, while the corresponding analytically equivalent structures (4.6.6c) and (4.6.8c) should be considered as a separate problem of quantization.

To summarize, the study of the analytic representations of nonconservative systems brings into focus a class of degrees of freedom of a Lagrangian or Hamiltonian structure which is customarily ignored for conservative mechanics. These degrees of freedom are new in the sense that they are not of the currently established type. More specifically, the transition from L to an isotopic image L* is not recoverable through the trivial Newtonian "gauge" transform

$$L \rightarrow L' = L + \frac{d}{dt} G(t, \underline{r}) , \qquad (4.6.9)$$

and, similarly, the transition from H to H* is not a canonical transformation. Instead, the degrees of freedom considered originate directly from the integrability conditions for the existence of an analytic representation. It is this novelty which creates the problem of quantization. The restriction of this table is introduced to separate the problem of quantization of structures satisfying limit (3.6.3) from that of quantization of analytically equivalent but functionally nonequivalent structures.

TABLE 4.7: A THIRD DICHOTOMY FOR QUANTIZATION: CANONICAL VERSUS PHYSICAL QUANTITIES.

The restrictions introduced in Tables 4.5 and 4.6 to identify the class of nonconservative system admitted for quantization via an expected covering of Schrödinger's equations, are inspired to the intent of achieving mathematical consistency in the quantization process, but they are highly insufficient to achieve consistency on physical grounds.

The methodology of the Inverse Problem proves in this respect too of valuable assistance. In essence, it identifies a dichotomy between canonical versus physical quantities which is absent in the conventional treatment of conservative systems. The idea here is that a necessary prerequisite for the physical interpretation of quantum mechanical algorithms is a precise identification of the physical significance of the corresponding algorithms at the purely Newtonian level. In the final analysis, this is dictated by our condition that any generalized form of Schrödinger's equations should satisfy the correspondence principle. At a closer look, this implies that, not only the classical limit of the Schrödinger's equation is the Hamilton-Jacobi equation, but also fundamental physical quantities, such as the energy, linear momentum and angular momentum should preserve their physical meaning at the classical limit under consideration.

In the conventional, classical and quantum mechanical treatment of conservative systems the problem under consideration does not exist. In this case, the function $H_{tot} = H_{free} + H_{int}$ has the symbiotic character of (a) representing the system via Hamilton's (Schrödinger's) equation, (b) constituting the canonical (quantum mechanical) generator of translations in time and (c) characterizing the classical (quantum mechanical) energy of the system. Similarly, the mathematical algorithm $\underset{\sim}{p}$, for the conservative systems considered, (a) represents the physical, classical (quantum mechanical) linear momentum, and (b) constitutes the classical (quantum mechanical) generator of translations in space. Finally, also for the systems considered, the mathematical algorithm $\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{p}$ (a) represents the physical, classical (quantum mechanical) angular momentum, and (b) constitutes the classical (quantum mechanical) generator of rotations. In conclusion, in the conventional treatment of conservative systems via the structure $H_{tot} = H_{free} + H_{int}$ the canonical and physical quantities coincide, by acquiring the known symbiotic meaning of being the generators of physically meaningful transformations, while jointly representing physical quantities.

In the transition to the classical, canonical treatment of nonconservative systems this crucial property is necessarily lost in the sense that a necessary condition for the Hamiltonian representation of nonconservative systems is that the canonical functions H, $\underset{\sim}{p}$ and $\underset{\sim}{M} = \underset{\sim}{r} \times \underset{\sim}{p}$ do not represent the physical energy, linear momentum and angular momentum, respectively.

For instance, a direct consequence of the generalized nature of the Lagrangian implies that the canonical momentum

$$p_{ka} = \frac{\partial L}{\partial \dot{r}^{ka}}, \quad L = \frac{1}{2}\dot{r}^{ia} G^{-1}_{ia;jb} \dot{r}^{jb} - G_0, \quad (4.7.1)$$

cannot coincide with the physical momentum ($m \dot{\underset{\sim}{r}}$) as a necessary condition for consistency. It then follows that the canonical angular momentum $\underset{\sim}{r} \times \underset{\sim}{p}$ does not coincide with the physical angular momentum ($\underset{\sim}{r} \times m\dot{\underset{\sim}{r}}$), also as a necessary condition for consistency. A simple inspection of the generalized Hamiltonian structure , say, Eq. (4.6.8b) then indicates that this canonical Hamiltonian cannot represent the physical energy (T+U), also as a necessary condition of consistency for nonconservative systems. Indeed, at the limit when the Hamiltonian represents the sum of the kinetic energy and the potential energy, the forces are all derivable from a potential and no nonselfadjoint force is admitted.

As indicated earlier, this is clearly a fundamental point for the physical consistency of any quantization of nonconservative systems. It should be stressed that what is here at stake is the physical content of the theory and not its mathematical profile. Specifically, unless this problem is confronted and resolved, any comparison of the prediction of the theory (say, the expectation values of the quantum mechanical algorithm "p") with the physical reality (experimental data on the physical linear momentum) could in the final analysis result to be vacuous (because already at a classical level the mathematical algorithm "p" does not represent the physical linear momentum, as a necessary condition for consistency).

The resolution of this issue which is here proposed is that suggested by the methodology of the Inverse Problem: for the Hamiltonian characterization of nonconservative systems the canonical and physical quantities are necessarily different in their functional structure, have different methodological implications and (under certain conditions) their functional relationship can be uniquely identified.

To avoid possibilities of notational confusion, it is here advantageous to differentiate the symbols used for canonical versus physical quantities. For the former case we shall use the symbols H^{can} , $\underset{\sim}{p}^{can}$ and $\underset{\sim}{M}^{can}$, while for the latter case we shall use the symbols

H^{phys}, $\mathbf{p}^{phys}$ and $\mathbf{M}^{phys}$. More specifically,

(a) H^{can} represents the generalized Hamiltonian for nonconservative (nonessentially nonselfadjoint) systems, as constructed via the Inverse Problem.

(b) $\mathbf{p}^{can}$ represents the canonical (generalized) momentum according to the conventional rule (4.7.1).

(c) $\mathbf{M}^{can}$ represents the vector product $\mathbf{r} \times \mathbf{p}^{can}$ of the Euclidean coordinate vector $\mathbf{r}$ and the canonical vector $\mathbf{p}^{can}$.

(d) H^{phys} represents the physical energy, that is, the sum of the kinetic energy and the potential energy of all forces derivable from a potential, $H^{phys} = T + U$.

(e) $\mathbf{p}^{phys}$ represents the physical linear momentum, that is, $m\dot{\mathbf{r}}$.

(f) $\mathbf{M}^{phys}$ represents the physical angular momentum, that is, $\mathbf{r} \times \mathbf{p}^{phys} = \mathbf{r} \times m\dot{\mathbf{r}}$.

The difference in the methodological implications of the canonical versus physical quantities should be identified. In essence, the canonical transformation theory is basically insensitive to the explicit functional structure of the generators. Thus, a Hamiltonian is the generator of translations in time irrespective of whether the represented system is conservative ($H = H^{phys}$) or nonconservative ($H \neq H^{phys}$). The net effect is that the canonical quantities of nonconservative systems H^{can}, $\mathbf{p}^{can}$ and $\mathbf{M}^{can}$ are the generators of the corresponding physical transformations, translation in time, translations in space and rotations, in precisely the same measure as that of the corresponding quantities for conservative systems.

Notice the terms "physical transformations" which are absent in conventional, classical and quantum mechanical treatments of conservative systems. These terms are suggested by the fact that the generators of canonical transformations are arbitrary functions of phase space coordinates (satisfying the needed continuity conditions). Thus, the physical quantities of nonconservative systems H^{phys}, $\mathbf{p}^{phys}$ and $\mathbf{M}^{phys}$, when expressed in the phase space coordinates (see below) are the generator of canonical transformations which are fully defined on mathematical grounds. The point is that, as another necessary condition of consistency, these transformation exhibit no direct connection or resemblance with physical transformations (translations in time, translations in space and rotation). The interested reader is urged to work out specific examples, e.g., to compute the canonical transformation induced by the energy $\frac{1}{2}(m\dot{r}^2 + k\,r^2)$ for the damped oscillator (4.6.8).

To summarize, in the Hamiltonian treatment of nonconservative systems, the generators of physical transformations are nonphysical (that is, do not possess a direct physical meaning), while the physical quantities are generators of nonphysical transformations (that is, canonical

transformations other than those of conventional direct, space-time, physical significance).

One can now begin to see the possible crucial meaning of restriction (4.6.3) for the expected covering of the Schrödinger equation. Indeed, this restriction on the admissible form of the generalized Hamiltonian structure can now be rewritten

$$\lim_{\mathbf{F}_k \to 0} \left(H^{can}, \mathbf{p}^{can}, \mathbf{M}^{can} \right) = \left(H^{phys}, \mathbf{p}^{phys}, \mathbf{M}^{phys} \right) \qquad (4.7.2)$$

and implies that at the limit of null nonconservative forces the canonical and physical quantities coincide by therefore reacquiring the familiar symbiotic character of representing both, physical quantities, as well as the generator of physical transformations.

We now remain with the problem of the functional relationship between the canonical and the physical quantities. This is easily solved by the methods of the Inverse Problem yielding, under the assumed continuity and regularity conditions, the unique solution in phase space coordinates

$$L^{gen} = \frac{1}{2} \dot{r}^{ia} G^{-1}_{ia\,jb} \dot{r}^{jb} - G_0 \;, \quad (G^{-1}) = (G^{-1})^T, \; \det(G^{-1}) \neq 0, \qquad (4.7.3a)$$

$$p^{can}_{ka} = \frac{\partial L^{gen}}{\partial \dot{r}^{ka}} = \dot{r}^{jb} G^{-1}_{jb\,ka} \;, \qquad (4.7.3b)$$

$$H^{can} = \frac{1}{2} p^{can}_{ia} G^{ia\,jb} p^{can}_{jb} + G_0 \;, \quad (G)^{-1} = (G^{-1}), \qquad (4.7.3c)$$

$$\mathbf{M}^{can}_k = \mathbf{r}_k \times \mathbf{p}^{can}_k \;, \qquad (4.7.3d)$$

$$p^{phys}_{ka} = m_k \dot{r}_{ka} = m_k G_{ka}{}^{jb} p^{can}_{jb} \; (\text{no } k\text{-sum}), \qquad (4.7.3e)$$

$$H^{phys} = \frac{1}{2m_k} \mathbf{p}^{phys}_k \cdot \mathbf{p}^{phys}_k + V = \frac{1}{2} p^{can}_{ia} D^{ia\,jb} p^{can}_{jb} + V, \qquad (4.7.3f)$$

$$\mathbf{M}^{phys}_k = \mathbf{r}_k \times \mathbf{p}^{phys}_k(t, \mathbf{r}, \mathbf{p}^{can}; \gamma), \qquad (4.7.3g)$$

$$D^{ia\,jb} = m_k^2 G_{ka}{}^{ia} G^{ka\,jb} \;, \qquad (4.7.3f)$$

$$\lim_{\gamma \to 0} G^{ia\,jb}(t, \mathbf{r}; \gamma) = \frac{1}{m_i} \delta_{ij} \delta_{ab} \;, \quad \lim_{\gamma \to 0} G_0(t, \mathbf{r}; \gamma) = V(\mathbf{r}), \qquad (4.7.3e)$$

with corresponding reformulation in the physical (but noncanonical) coordinates $\underset{\sim}{r}$ and $\underset{\sim}{p}^{phys}$

$$p^{can}_{ka} = p^{can}_{ka}(t, \underset{\sim}{r}, \underset{\sim}{p}^{phys}; \gamma) , \qquad (4.7.4a)$$

$$H^{can}(t, \underset{\sim}{r}, \underset{\sim}{p}^{can}; \gamma) = H'^{can}(t, \underset{\sim}{r}, p^{phys}; \gamma), \qquad (4.7.4b)$$

$$\underset{\sim}{M}^{can}_{k} = \underset{\sim}{r}_k \times \underset{\sim}{p}^{can}_{k}(t, \underset{\sim}{r}, \underset{\sim}{p}^{phys}; \gamma), \qquad (4.7.4c)$$

$$\underset{\sim}{p}^{phys}_{k} = m_k \dot{\underset{\sim}{r}}_k , \quad H^{phys} = \frac{1}{2m_k} \underset{\sim}{p}^{phys}_{k} \cdot \underset{\sim}{p}^{phys}_{k} + V, \qquad (4.7.4d)$$

$$\underset{\sim}{M}^{phys}_{k} = \underset{\sim}{r}_k \times \underset{\sim}{p}^{phys}_{k} . \qquad (4.7.4e)$$

A crucial alternative of the assumed coordinates then emerges. The canonical and physical quantities can be both expressed in terms of either the canonical coordinates $\underset{\sim}{r}$ and $\underset{\sim}{p}^{can}$ or in terms of the physical coordinates $\underset{\sim}{r}$ and $\underset{\sim}{p}^{phys}$, with a corresponding change in the functional structure of the quantities in the transition from one system of coordinates to the other. In turn, this alternative creates the crucial problem to be confronted in the next table: where conventional quantization rules should be applied to, the canonical or the physical quantities?

Notice that, under the assumption that the forces derivable from a potential are conservative, the physical quantities are the canonical quantities of the maximal selfadjoint associated subsystem.

As a final incidental remark, the reader might be amused to know that the notion of physical quantity for a nonconservative system is not immune of controversy. In other words, the concept of energy for a nonconservative system could, in principle, be defined in different ways. The definition here assumed is as naive as possible (actually derived from undergraduate textbooks[35], owing to the virtual complete silence in current advanced literature on nonconservative mechanics): the total physical energy is the sum of the kinetic energy and the potential energy of all forces derivable from a potential, irrespective of whether the system is conservative or not. In the former case such energy is conserved. In the latter case it is not, as conform to physical reality in nonconservative Newtonian mechanics. The study of other definitions is left to the interested reader.

TABLE 4.8: QUANTIZATION FOR SCHRÖDINGER-TYPE FORMULATIONS.

A fundamental problem for the quantization of nonconservative Newtonian systems with a Schrödinger-type approach is whether the conventional (naive) quantization rules

$$\text{momentum} \rightarrow \frac{\hbar}{i} \underset{\sim}{\nabla} , \qquad \text{Hamiltonian} \rightarrow i\hbar \frac{\partial}{\partial t} , \qquad (4.8.1)$$

should be applied to the canonical or physical quantities of the Hamiltonian representation of nonessentially nonselfadjoint systems.

A study of this problem indicates that, in order for the emerging generalized Schrödinger's equation to recover the Hamilton-Jacobi equation under the classical limit (i.e., for the compliance with the correspondence principle), quantization rules (4.8.1) should be applied to the canonical quantities, and we write

$$\underset{\sim}{p}^{can} \rightarrow \frac{\hbar}{i} \underset{\sim}{\nabla} , \qquad H^{can} \rightarrow i\hbar \frac{\partial}{\partial t} , \qquad (4.8.2)$$

while the application of the same rules to the physical quantities, i.e.,

$$\underset{\sim}{p}^{phys} \rightarrow \frac{\hbar}{i} \underset{\sim}{\nabla} , \qquad H^{phys} \rightarrow i\hbar \frac{\partial}{\partial t} , \qquad (4.8.3)$$

leads to inconsistencies.

In unpedagogical terms, we can say that the established methods of quantization of conservative systems for the Schrödinger's representation[32] emerge as being applicable, mathematically consistent and in compliance with the correspondence principle also for the Hamiltonian representations of the nonselfadjoint systems satisfying the restrictions of Table 4.5, 4.6 and 4.7. It should be stressed that this solves only half of the problem, the mathematical consistency of the quantization. The second and equally crucial part of the problem, the physical consistency, must be separately studied.

In this table we would like to present an epistemological argument, besides that of the compliance with the correspondence principle, which support quantization via rules (4.8.2).

During the preceding parts of our analysis we have stresses the similarities as well as differences between nonconservative forces and the Lorentz force. A brief anamnesis of the quantization of this force may be of valuable intuitional guidance.

The Newtonian motion of charged particles under an electromagnetic field can be described via the familiar minimal coupling rule

$$\underset{\sim}{p} \longrightarrow \underset{\sim}{p} - e\underset{\sim}{A}, \qquad (c=1) \qquad (4.8.4)$$

and the Hamiltonian

$$H = \frac{1}{2m}\left(\underset{\sim}{p} - e\underset{\sim}{A}\right)^2 + e\varphi. \qquad (4.8.5)$$

But the Lorentz force exhibits an explicit dependence on the velocities (although only linear). This is sufficient to imply the property (which is crucial for the analysis of this paper) according to which the canonical momentum of charged particles under the Lorentz force does not coincide with the physical linear momentum. This is precisely an established case of dichotomy of canonical versus physical quantities, but only restricted to the case of the linear (and angular) momentum. At the level of the Hamiltonian the dichotomy does not occurs because the forces considered are derivable from a potential and, thus, Hamiltonian (4.8.5) does indeed represent the total physical energy.

The important point is that the methods for the quantization of the Lorentz force have proved to be conform with physical reality under quantization rules (4.8.2) and not rules (4.8.3). In other words, to achieve a consistent quantization of the velocity-dependent Lorentz force, the quantization rules must be applied to the canonical momentum $\underset{\sim}{p}^{can}$, even though it does not coincide with the physical momentum $\underset{\sim}{p}^{phys}$.

Our epistemological argument can then be presented as follows. In the transition from the Lorentz force to the nonconservative Newtonian forces nonderivable from a potential the functional dependence remains the same and so does the fact that the canonical momentum is not directly representative of the physical linear momentum. We simply have a more generalized form of the same occurrence, in the sense that the functional relationship between the canonical and physical linear (and angular) momentum simply becomes more complex, while the dichotomy necessarily extends also to the Hamiltonian, because the forces considered are nonderivable from a potential.

We then argue that mathematical consistency, as well as compliance with the correspondence principle, can be achieved for a Schrödinger-type formulation of nonselfadjoint Newtonian systems by essentially using the canonical quantization rules established for the Lorentz force.

TABLE 4.9: THE PROPOSED DUAL COVERING OF SCHRÖDINGER'S EQUATION FOR NONCONSERVATIVE FORCES.

The Hamilton-Jacobi description of the nonconservative, nonessentially nonselfadjoint Newtonian systems satisfying the assumed restrictions can be written

$$\frac{\partial A}{\partial t} + H^{can}(t, \underset{\sim}{r}, \underset{\sim}{p}^{can}; \gamma) = 0, \qquad (4.9.1a)$$

$$\underset{\sim}{p}^{can} = \frac{\partial A}{\partial \underset{\sim}{r}^k}. \qquad (4.9.1b)$$

The use of the established quantization rules (e.g., via the second Beltrami's procedure) along the lines of Table 4.7, then yields the following generalized Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = H^{can}\psi = -\left[\frac{\hbar^2}{2G^{1/2}}\nabla_{ia}\left(G^{\frac{1}{2}}G^{ia;jb}\nabla_{jb}\right) - G_0\right]\psi, \qquad (4.9.2)$$

$$G = \det(G),$$

which satisfies the correspondence principle, that is, recovers Hamilton-Jacobi equation (4.9.1a) at the limit $\hbar/A \longrightarrow 0$, as the reader can verify by inspection.

Our contention is that Eq. (4.9.2) is mathematically consistent, but physically inconsistent for the quantum mechanical description of nonconservative systems in the sense that the expectation values of the operator H^{can} do not describe energy levels, unless all forces nonderivable from a potential are null.

To achieve a quantum mechanical description of the systems considered which is consistent on both mathematical and physical grounds, the only alternative of which I am aware is by constructing a dual generalization of the Schrödinger's equation, as a direct generalization of the dual classical context identified by the Inverse problem. This essentially implies a distinction between the quantity H^{can} which characterizes the time evolution of the system and the quantity H^{phys} which represents the total physical energy. Quantization rule (4.8.2) then implies that both these quantities should be expressed in terms of the canonical variables $\underset{\sim}{r}^k$ and $\underset{\sim}{p}_k^{can}$. In turn, this implies that, in addition to Eq. (4.9.2), a second quantum mechanical equation can be constructed via the quantum mechanical $\underset{\sim}{p}^{can}$ applied to $H^{phys}(t, \underset{\sim}{r}, \underset{\sim}{p}^{can}; \gamma)$ of Eq. (4.7.3f).

Under the assumption that the operators H^{can} and H^{phys} are Hermitian and commute we reach in this way the following <u>dual generalization of Schrödinger's equation for non-conservative systems</u>

$$i\hbar \frac{\partial}{\partial t}\psi = H^{can}\psi = -\left[\frac{\hbar^2}{2G^{\frac{1}{2}}}\nabla_{ia}\left(G^{\frac{1}{2}} G^{iajb}\nabla_{jb}\right) - G_0\right]\psi, \quad (4.9.3a)$$

$$H^{phys}\psi = -\left[\frac{\hbar^2}{2D^{\frac{1}{2}}}\nabla_{ia}\left(D^{\frac{1}{2}} D^{iajb}\nabla_{jb}\right) - G_0\right]\psi = E^{phys}\psi, \quad (4.9.3b)$$

subject to the following interpretation: H^{can} characterizes the time evolution of the system (e.g, via a unitary transformation) while H^{phys} characterizes the total physical energy (e.g., via expectation values).

Eqs. (4.9.3) are complemented with a dual generalization of quantum mechanical momenta

$$p^{can}_{ka} = \frac{\hbar}{i}\nabla_{ka}, \quad (4.9.4a)$$

$$p^{phys}_{ka} = \frac{\hbar}{2i} m_k \{G_{ka}{}^{ib}, \nabla_{jb}\}, \quad (4.9.4b)$$

where $\underset{\sim}{p}^{can}$ characterizes translations in space and $\underset{\sim}{p}^{phys}$ characterizes the physical linear momentum, as well as a dual generalization of quantum mechanical angular momenta

$$\underset{\sim}{M}^{can}_{k} = \underset{\sim}{r}_k \times \underset{\sim}{p}^{can}_{k}, \quad (4.9.5a)$$

$$\underset{\sim}{M}^{phys}_{k} = \underset{\sim}{r}_k \times \underset{\sim}{p}^{phys}_{k}, \quad (4.9.5b)$$

where $\underset{\sim}{M}^{can}$ characterizes rotations while $\underset{\sim}{M}^{phys}$ characterizes the physical angular momenta.

It is here essential to assess the plausibility of Eqs. (4.9.3) with explicit examples. But first, the conceptual framework for which they are intended should be recalled. As by now familiar, we argue that quantum mechanics unequivocally applies for the arena for which it was conceived, the atomic structure, while a generalized mechanics could emerge for the new layer of the hadronic structure. In the transition from the context of the conventional Schrödinger's equation (as currently known) and that of the proposed generalization, the mental attitude should therefore shift from massive, charged and extended particles moving in vacuum, to that of the same particles moving in hadronic matter, in which case the extended nature of the particles considered results in nonlocal forces here <u>approximated</u> with local forces nonderivable from a potential (with an understanding that these new forces do not appear at the atomic context owing to the large distances between the constituents of the atomic structure).

To emphasizes this distinction, we shall then call <u>atomic mechanics</u>, the quantum mechanics as currently known, and <u>hadronic mechanics</u> its expected covering. For instance, the first step we have identified until now, the (naive) quantization rules, can be differentiated into the <u>quantization rules of physical quantities for the atomic mechanics</u>

$$\underset{\sim}{p}^{phys} \rightarrow \frac{\hbar}{i}\underset{\sim}{\nabla}, \quad (4.9.6a)$$

$$H^{phys} \rightarrow i\hbar\frac{\partial}{\partial t}, \quad (4.9.6b)$$

$$\underset{\sim}{M}^{phys} \rightarrow \underset{\sim}{r} \times \frac{\hbar}{i}\underset{\sim}{\nabla}, \quad (4.9.6c)$$

and the <u>covering quantization rules of physical quantities for the hadronic mechanics</u>

$$\underset{\sim}{p}^{phys} \rightarrow \underset{\sim}{\mathcal{P}}\left(t, \underset{\sim}{r}, \frac{\hbar}{i}\underset{\sim}{\nabla}, \ldots\right), \quad (4.9.7a)$$

$$H^{phys} \rightarrow \mathcal{E}\left(t, \underset{\sim}{r}, i\hbar\frac{\partial}{\partial t}, \ldots\right), \quad (4.9.7b)$$

$$\underset{\sim}{M}^{phys} \rightarrow \underset{\sim}{\mathcal{M}}\left(t, \underset{\sim}{r}, \underset{\sim}{r} \times \frac{\hbar}{i}\underset{\sim}{\nabla}, \ldots\right), \quad (4.9.7c)$$

where the covering nature is indicated by the fact that the latter rules are intended for a more complex physical context, the hadronic context, while capable of recovering the conventional rules identically at the limit of null nonconservative forces (or hadronic medium),

$$\lim_{\underset{\sim}{F}\to 0} \{ \underset{\sim}{\mathcal{P}}, \mathcal{E}, \mathcal{M} \} = \{ \frac{\hbar}{i} \underset{\sim}{\nabla}, i\hbar\frac{\partial}{\partial t}, \underset{\sim}{r} \times \frac{\hbar}{i} \underset{\sim}{\nabla} \}. \quad (4.9.8)$$

In particular, the same particle can perform the transition from motion in vacuum to motion within a hadronic medium (or viceversa), in which case the quantization rules are expected to perform a corresponding transition from the atomic to the hadronic rules. Most importantly, while the conceptual emphasis in the atomic mechanics is in the conservation of total physical quantities, that of the hadronic mechanics is in the nonconservation of the physical characteristic of a particle while moving in a hadronic medium.

The simplest conceivable example to assess the plausibility of Eqs. (4.9.3) is therefore that of the motion of an extended but neutral particle which penetrates into the hadronic matter and, in doing so, experiences a "hadronic drag force". In other words, we can ignore, in a first step, electromagnetic forces because assumed as the same in both the atomic and the hadronic mechanics, and consider instead the new forces expected to occur in the new layer of physical reality. The simplest conceivable, local, drag force is that proportional to the velocity. Thus, the Newtonian limit of the motion considered is

$$\left[(m\ddot{r})_{SA} + \gamma\dot{r} \right]_{NSA} = 0, \quad (4.9.9)$$

and admits the Hamiltonian representation satisfying the restrictions of the preceding tables

$$L = e^{\gamma t/m} \frac{1}{2} m\dot{r}^2, \quad (4.9.10a)$$

$$p^{cam} = e^{\gamma t/m} m\dot{r} = e^{\gamma t/m} p^{phys}, \quad (4.9.10b)$$

$$H^{cam} = e^{-\gamma t/m} \frac{1}{2m} (p^{cam})^2 = e^{\gamma t/m} H^{phys}. \quad (4.9.10c)$$

Eqs. (4.9.3) in this case yield the quantum mechanical formulation

$$i\hbar\frac{\partial}{\partial t}\psi = H^{cam}\psi = -e^{-\gamma t/m}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi, \quad (4.9.11a)$$

$$H^{phys}\psi = -e^{-2\gamma t/m}\frac{\hbar}{2m}\frac{\partial^2}{\partial r^2}\psi = E\psi. \quad (4.9.11b)$$

here interpreted as a covering of the corresponding atomic context (free motion in this case) because applying for a force which does not exist within the atomic context, while being able to recover the corresponding atomic context, i. e.,

$$i\hbar\frac{\partial}{\partial t}\psi = H_{free}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\psi, \quad (4.9.12)$$

at the limit of null value of the hadronic force ($\gamma \to 0$, that is, when the particle considered exits the hadronic medium by reacquiring motion in vacuum). The considered hadronic covering is of dual (rather than single) nature because of the intent of recovering the Hamilton-Jacobi equation at the classical limit via the use of the canonical Hamiltonian, and its necessary differentiation with the physical energy, the kinetic energy for the case at hand. The point is that this dual nature disappears at the limit of null hadronic forces, because under this limit the canonical and physical Hamiltonians coincide.

The physical assessement of Eqs. (4.9.11) should therefore be based on the problem whether their are capable of producing a physically consistent characterization of the nonconservation of the physical characteristics of the particle while moving within a hadronic medium and not their compliance with established laws at the atomic level. It is in this respect where the attitude of searching for departures from established atomic settings indicated earlier should be implemented.

The space dependence of the atomic and hadronic equations is the same. It is therefore conceivable to assume that the integral $\int_V \psi^*\psi\, dV$ is constant in time. This allows the preservation (in this case) of the probabilistic interpretation of the atomic mechanics and the computation of expectation values. For the physical linear momentum we have

$$\langle p^{phys} \rangle = \int \psi^* e^{-\gamma t/m} \frac{\hbar}{i} \nabla \psi\, dV, \nabla = \frac{\partial}{\partial r}. \quad (4.9.13)$$

Thus

$$\frac{d}{dt}\langle p^{phys} \rangle = -\frac{\gamma}{m}\langle p^{phys} \rangle, \langle p^{phys} \rangle = e^{-\frac{\gamma}{m}t}\langle p_o^{phys} \rangle, \quad (4.9.14)$$

where $\langle p_o^{phys} \rangle$ can be assumed as the expectation value of the linear momentum, say, at the time of penetration in the hadronic matter. For the physical energy we have

$$< H^{phys} > = \int \psi^* \left(- e^{-\frac{2\gamma}{m}t} \frac{\hbar^2}{2m} \Delta \right) \psi \, dV, \quad \Delta = \frac{\partial^2}{\partial r^2} . \quad (4.9.15)$$

Then

$$\frac{d}{dt} < H^{phys} > = - \frac{2\gamma}{m} < H^{phys} >, \qquad (4.9.16)$$

and therefore

$$< H^{phys} > = e^{-\frac{2\gamma}{m}t} < H_0^{phys} >, \qquad (4.9.17)$$

where $< H_0^{phys} >$ can be assumed as the expectation value of the energy also at the time of penetration into a hadronic matter.

As a result, Eqs. (4.9.11) not only characterize nonconserved expectation values of the linear momentum and the energy, as desired, but their behaviour in time is exactly equal to the corresponding Newtonian behaviour. We thus argue that the quantization of the nonconservative Newtonian system (4.9.9) via Eqs. (4.9.11) is plausible.

The departure from the established laws of the atomic mechanics is however substantial. The hadronic "wave function" ψ, after solving Eq. (4.9.11a), is of the type

$$\psi \approx e^{\frac{i}{\hbar}\left(pr - E^{con} \frac{m}{\gamma} \exp \frac{\gamma}{m} t\right)}, \quad (4.9.18)$$

where p is the eigenvalue of the canonical momentum. Such a "wave" does not appear to be compatible with established relativity laws, as we shall elaborate in Table 4.20.

This point is clearly such, to deserve a verification of structure (4.9.18). We shall then ignore the preceding quantization of the nonconservative system (4.9.9) and search for an independent quantization with a function of the type

$$\psi(t,r) = N \int f(E) \, e^{\frac{i}{\hbar}[\alpha(p,r) - \beta(E,t)]} dE, \quad (4.9.19)$$

as for conventional quantization approaches, where p is the canonical momentum and E the energy. At the limit (hadronic mechanics) $\rightarrow$ (atomic mechanics) (i.e., $\gamma \rightarrow 0$), the phase

satisfies the relations

$$\frac{\partial \beta}{\partial t} = \frac{1}{2m}\left(\frac{\partial \alpha}{\partial r}\right)^2, \qquad (4.9.20a)$$

$$\alpha(p,r) - \beta(E,t) = pr - Et, \qquad (4.9.20b)$$

$$\frac{\partial \beta}{\partial t} = E = \frac{p^2}{2m}. \qquad (4.9.20c)$$

For an effective quantum mechanical description of the nonconservation of the energy and, thus, of the damping, we assume the equation for the hadronic phase

$$\frac{\partial \beta}{\partial t} = e^{-\gamma t/m} \frac{1}{2m}\left(\frac{\partial \alpha}{\partial r}\right)^2; \qquad (4.9.21)$$

which now satisfies the equations

$$\frac{\partial \beta}{\partial t} = e^{\frac{\gamma}{m}t} E = e^{-\frac{\gamma}{m}t} \frac{1}{2m}\left(\frac{\partial \alpha}{\partial r}\right)^2 = e^{-\frac{\gamma}{m}t} \frac{1}{2m} p^2, \qquad (4.9.22a)$$

$$\alpha(p,r) - \beta(E,t) = pr - E\frac{m}{\gamma} e^{\frac{\gamma}{m}t}, \qquad (4.9.22b)$$

and, therefore, results to satisfy the correspondence principle under which it recovers the Newtonian context of Eq. (4.9.9). This yields again Eqs. (4.9.11). In particular,

$$pr - E\frac{m}{\gamma} e^{\frac{\gamma}{m}t} = (pr - Et) - E\frac{m}{\gamma} - E\frac{\gamma}{m}\frac{1}{2!}t^2 + \ldots \qquad (4.9.23)$$

Thus, to first order terms, the hadronic and atomic phase coincide, up to a scale (dimensionless) term Em/γ.

In conclusion, we have a case in which two opposite mental attitudes can be implemented. One can assume that established relativity laws are valid within a hadron. This necessarily implies quantization via a relativistically invariant phase. The consequence is the violation of the correspondence principle, in the sense that (assuming such a quantization exists) its Newtonian limit is incompatible with that of Eq. (4.9.9). Another researcher may instead acknowledge the limitations of our current knowledge on the hadronic reality, and search for new conceivable insights. The assumption of Newtonian mechanics as an intuitional basis and the compliance with the correspondence principle then implies a phase in violation of relativity laws which, strictly speaking, are established on clear experimental grounds only at the atomic level, or for the behaviour of a charged particle as a whole under (ut most) electromagnetic interactions.

The extension of hadronic system (4.9.11) to the broader class of systems

$$i\hbar\frac{\partial}{\partial t}\psi = H^{can}\psi = f(t)\left[-\frac{\hbar^2}{2m}\Delta + V(r)\right]\psi, \qquad (4.9.24)$$

is trivial and will be left to the interested readers. More technically involved is the class of systems for which the canonical Hamiltonian yields a generalized Schrödinger's equation of the type

$$i\hbar\frac{\partial}{\partial t}\psi = H^{can}\psi = \left[-f(t)\frac{\hbar^2}{2m}\Delta + g(t)V(r)\right]\psi, \qquad (4.9.25)$$
$$f(t) \neq g(t).$$

Still more technically involved is the class of systems for which the mutation terms are dependent on the coordinates. In this case the full symmetrized equations (4. 9.3) must be used. For brevity we shall not enter into a treatment of these broader systems.

It should be stressed that the systems considered as are such to yield the simplest possible generalization of Schrödinger's equation. and that numerous further generalizations are conceivable. Some of the most relevant cases are

(a) the time-evolution canonical operator H^{can} is non-Hermitian;[36], *

(b) the generalized Schrödinger's equation is of order higher than the second (this case occurs when the mutation parameter depent on the canonical momenta, e.g., for spin-orbit couplings which multiply the kinetic term -- see Table 5.2);

(c) the generalized Schrödinger's equation becomes nonlinear.

* Almost needless to recall, the first quantum mechanical chara cterization of nonconservative effects (e.g., the absoption of waves) has been done via the use of non-Hermitian Hamiltonians, although realized via the simple assumption of a complex potential and, thus, without generalized Hamiltonian structures of our type.

TABLE 4.10: THE PROPOSED HADRONIC MUTATION OF DE BROGLIE'S, EINSTEIN'S AND HEISENBERG'S PRINCIPLES.

One of the direct implications of the dual covering (4.9.3) of Schrödinger's equation is that the de Broglie's wavelength principle

$$p = \hbar k = \hbar\frac{2\pi}{\lambda}, \qquad (4.10.1)$$

Einstein's frequency principle

$$E = \hbar\omega = \hbar 2\pi\nu, \qquad (4.10.2)$$

and Heisenberg's uncertainty principle

$$\Delta r\Delta p \geq \tfrac{1}{2}\hbar, \quad \Delta t\Delta E \geq \tfrac{1}{2}\hbar, \qquad (4.10.3)$$

are inapplicable, as currently understood, to the intended hadronic mechanics.

The argument is by now familiar. Since λ is a physically measured quantity, principle (4.10.1) is crucially dependent on the fact that the mathematical symbol "p" represents the physical linear momentum under quantized form $\frac{\hbar}{i}\nabla$, while these features are lost for nonselfadjoint strong hadronic forces. Similarly, since ν is a physical quantity, principle (4.10.2) is crucially dependent on the property that the mathematical symbol "H" represents the total physical energy under the quantized form $i\hbar\frac{\partial}{\partial t}$, while these features are also lost for the hadronic context under consideration. Finally, principle (4.10.3) is crucially dependent not only on the indicated physical meaning and quantization of the algorithms at hand, but also on the property that the physical quantities (r, p) and (t, E) are canonically conjugate (e.g., in the sense of a variational principle with end points contributions), while all these features are also lost for the considered forces (for nonconservative systems the physical linear momentum cannot be canonically conjugate to the coordinate as a necessary condition for consistency, and a similar relationship occurs between the physical energy and time [21]).

This situation creates the problem of searching for a possible covering of the laws considered for the hadronic structure under the assumption of forces nonderivable from a potential.

The answer we here proposed is that based on the hadronic quantization rules of physical quantities, Eqs. (3.9.7), under the restrictions on the admitted class of nonselfadjoint forces identified earlier, plus the condition that the emerging quantum mechanical operators are Hermitian and the mutation terms do not depend on coordinates.

Indeed, under rule (4.9.7a), we have the hadronic wavelength principle

$$p^{phys} = f(t, \hbar k), \qquad (4.10.4a)$$

$$\lim_{F \to 0} f = \hbar k, \qquad (4.10.4b)$$

under rule (4.9.7b) we have the hadronic frequency principle

$$E^{phys} = g(t, \hbar\omega), \qquad (4.10.5)$$

$$\lim_{F \to 0} g = \hbar\omega, \qquad (4.10.5b)$$

and, similarly, under both these rules we have a hadronic uncertainty principle of the form

$$\Delta r \, \Delta p^{phys} \geq h_1(t, \hbar), \quad \Delta t \, \Delta E^{phys} \geq h_2(t, \hbar), \qquad (4.10.6a)$$

$$\lim_{F \to 0} h_1, h_2 = \frac{1}{2}\hbar, \qquad (4.10.6b)$$

where h_1 and h_2 are suitable functions (uniquely) characterized by the functional structure of rules (4.9.7a) and (4.9.7b) and where the limits are intended to recall that, by construction, the generalized principles recover the conventional principles identically at the limit of null nonconservative forces.

It should be stressed that principles (4.10.4), (4.10.5) and (4.10.6) are introduced on purely empirical grounds, along the epistemological lines of this paper. A more technical study is contemplated in a separate paper. At this time, we content ourselves with an illustration of these principles as simple as possible, hoping that might be of some value to assess their plausibility.

Consider again the hadronic quantization of the one-dimensional motion of a particle with linear damping, Eqs. (4.9.11). The canonical quantization rules (4.8.2) yield the following realization of the "wave function" (4.9.19)

$$\psi(t, E) = N \int f(E) \, e^{i\left(k r - \omega \frac{m}{\gamma} e^{-\frac{\gamma}{m} t}\right)} dE \qquad (4.10.7)$$

for which

$$p^{phys} = e^{-\frac{\gamma}{m} t} \hbar k, \qquad (4.10.8a)$$

$$E^{phys} = e^{-\frac{2\gamma}{m} t} \hbar\omega, \qquad (4.10.8b)$$

$$\Delta r \, \Delta p^{phys} \geq \frac{1}{2}\hbar e^{-\frac{\gamma}{m} t}, \quad \Delta t \, \Delta E^{phys} \geq \frac{1}{2}\hbar e^{-\frac{2\gamma}{m} t}, \qquad (4.10.8c)$$

This illustrates the generalized principles under consideration.

The argument for their plausibility is also familiar by now. Since the physical linear momentum is (exponentially) decaying in time, the wavelength is also expected to decay with a similar occurrence for the frequency. Principles (4.10.8a) and (4.10.8b) simply indicate this occurrence. But then trivial calculations yield principle (4.10.8c). Notice that the quantities k and ω of these generalized principles are constant and verifying the relation

$$\omega = \frac{1}{2m} \hbar k. \qquad (4.10.9)$$

Thus, they can be considered as representing the correspondent conserved values prior to penetration of the particle into the hadronic matter. Despite that, Eq. (4.10.7) does not represents a wave packet as currently understood.

The epistemological implications of generalized principle (4.10.8c) are intriguing. In unpedagogical terms, it implies a weakening of the strictly undeterministic nature of the atomic mechanics because, at the limit $t \to \infty$, principle (4.10.8c) recovers the deterministic nature of classical mechanics. The reader should be however aware that the hadronic system under consideration, system (4.9.11), is noting but an extremely rudimentary approximation of the motion of a particle in hadronic matter because of numerous approximation.

Principles (4.10.4), (4.10.5) and (4.10.6) will be called the hadronic mutation of the de Broglie's, Einstein's and Heisenberg's principles, respectively. This is a first step for a possible realization of the notion of hadronic constituent (the eleton) as a mutation of a conventionally quantized particle, according to Section 3. The term "mutation" is also used in the expectation of algebraic treatments to be considered later on in this section.

It is of some relevance to indicate that principle (4.10.8c) implies a damping of the spreading of the "wave packet". This damping was also obtained by independent treatments, e.g., the nonlinear generalization of the Schrödinger's equation of ref.[37] for the quantization of the same system (4.9.9). Thus, it is of some value to indicate that a similar damping is obtained in our linear quantization of the same system.

Consider the expression

$$\psi(t,r) = \int A(k)\, e^{i\left(kr - \frac{h k^2}{2\gamma} e^{-\frac{\gamma}{m} t}\right)}\, dk. \quad (4.10.10)$$

By assuming a Gaussian distribution

$$A(k) = (2\pi a)^{-\frac{1}{4}}\, e^{-\frac{k^2}{4a}}, \quad a = \frac{1}{4\langle r^2\rangle}, \quad (4.10.11)$$

we have

$$|\psi|^2 = \left(\frac{2a}{\pi}\right)^{\frac{1}{2}} \left[1 + \left(\frac{2ah}{\gamma}\, e^{-\frac{\gamma}{m}t}\right)^2\right]^{-\frac{1}{2}} e^{-\alpha(t) r^2}, \quad (4.10.12a)$$

$$\alpha(t) = \frac{2a}{1 + \left(\frac{2ah}{\gamma}\right)^2 e^{-\frac{2\gamma}{m}t}}. \quad (4.10.12b)$$

Thus

$$\langle(\Delta r)^2\rangle = \frac{1}{4\langle(\Delta k)^2\rangle} + \frac{\hbar^2 \langle(\Delta k)^2\rangle}{\gamma^2}\, e^{-\frac{2\gamma}{m}t} \quad (4.10.13)$$

$$= \frac{1}{4\langle(\Delta p)^2\rangle}\, e^{-\frac{2\gamma}{m}t} + \frac{\hbar^2\langle(\Delta p)^2\rangle}{\gamma^2}$$

namely, the dispersion of the "wave packet" decreases in time. This feature has intriguing potential implications for our model of hadronic structure as bound states of eletons.

It should be stressed that the proposed generalized principles (4.10.4), (4.10.5) and (4.10.6) are restricted to the class of systems of type (4.9.24). For more general nonconservative systems the proposed principles should be considered as purely indicative, that is, as a mere indication of the existence of nontrivial departures from the conventional principle of atomic mechanics. The explicit computation of these departures will not be considered at this time.

TABLE 4.11: THE PROPOSED HADRONIC MUTATION OF THE CONVENTIONAL QUANTUM MECHANICAL NOTIONS OF ANGULAR MOMENTUM AND SPIN.

Another direct implication of the dual covering (4.9.3) of Schrödinger's equation for nonselfadjoint strong hadronic forces is that the familiar angular momentum rules of atomic mechanics

$$\underline{M}^2\,\psi = \hbar^2\,\ell(\ell+1)\,\psi\,, \qquad \ell = 0, 1, 2, \ldots, \tag{4.11.1a}$$

$$M_z\,\psi = \hbar\, m\,\psi\,, \qquad m = \ell, \ell-1, \ldots, -\ell, \tag{4.11.1b}$$

$$\underline{M} = \underline{r} \times \frac{\hbar}{i}\,\underline{\nabla}\,, \tag{4.11.1c}$$

are inapplicable to the hadronic context considered.

The argument is also familiar by now. A prerequisite for the conventional physical interpretation of rules (4.11.1) is that the quantum mechanical algorithm $\underline{M} = \underline{r} \times \underline{p}^{can}$ represents the physical angular momentum, which is not the case, as a necessary condition of consistency, for the nonselfadjoint forces considered.

Under the restrictions considered earlier on the class of admissible forces, plus the condition that the quantum mechanical operators $\underline{M}^{can}$ and $\underline{M}^{phys}$ are Hermitian and commute, Eqs. (4.9.7) imply the following hadronic angular momentum rules

$$\underline{M}^{phys\,2}\,\psi = \alpha(t;\hbar,\ell)\,\psi\,, \tag{4.11.2a}$$

$$M_z^{phys}\,\psi = \beta(t;\hbar,m)\,\psi\,, \tag{4.11.2b}$$

$$\lim_{\underline{F}\to 0} \alpha = \hbar^2\,\ell(\ell+1)\,, \qquad \lim_{\underline{F}\to 0} \beta = \hbar m\,, \tag{4.11.2c}$$

where α and β are suitable functions (uniquely) characterized by the functional relationship between $\underline{p}^{can}$ and $\underline{p}^{phys}$. Notice that the crucial limiting properties (4.11.2c) are satisfied by construction for the admitted forces and that we have excluded a dependence on the coordinates from the condition of commutativity of the operators $\underline{M}^{can}$ and $\underline{M}^{phys}$.

The purely empirical way of derivation of rules (4.11.2) should be here stressed. By delaying some more technical treatment at a later time, let us consider an example to ascertain the plausibility of the approach. In essence, we are interested in probing the idea according to which the conventional quantum mechanical notion of angular momentum

of the atomic mechanics is subjected to a mutation in the transition to the motion of extended particles within hadronic matter. The computation of the explicit form of the mutation for forces more general than those here admitted is, in any case, of such a complexity to go beyond the rudimentary level of this presentation.

We shall consider one of the simplest possible cases of the dual covering (4.9.3) of the Schrödinger's equation which, precisely in view of its simplicity, is perhaps potentially more misleading than others.

Consider the following mutation of the conventional Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\,\psi(t,\underline{r}) = g^{-1}(t)\left[-\frac{\hbar^2}{2m}\Delta + V(r)\right]\psi(t,r)\,, \tag{4.11.3}$$

where, as by now familiar, the mutation occurs because of a nontrivial term (the function $g^{-1}(t)$) which multiplies the kinetic term.

Eq. (4.11.3) is separable with the trivial solution

$$\psi(t,\underline{r}) = e^{-\frac{i}{\hbar}E\int_0^t g(t)\,dt}\,\psi(r,\theta,\varphi)\,, \tag{4.11.4a}$$

$$\left\{-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{(\underline{M}^{can})^2}{r^2}\right] + V\right\}\psi = E\,\psi\,, \tag{4.11.4b}$$

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r}\,Y(\theta,\varphi)\,, \tag{4.11.4c}$$

$$\left\{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \left[V(r) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right]\right\}R(r) = E\,R(r)\,, \tag{4.11.4d}$$

$$(\underline{M}^{can})^2\,Y_{\ell m}(\theta,\varphi) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell m} = \hbar^2\ell(\ell+1)\,Y_{\ell m}\,, \tag{4.11.4e}$$

$$M_z^{can}\,Y_{\ell m} = -i\hbar\frac{\partial}{\partial\varphi}Y_{\ell m} = \hbar m\,Y_{\ell m}\,. \tag{4.11.4f}$$

Since the space part of the equation is fully conventional (i.e., that for systems with forces derivable from the rotationally invariant potential V), one might be tempted to conclude that the atomic notion of angular momentum applies. Our contention is that this conclusion is mathematically consistent but physically vacuous in the case considered.

One way to illustrate this contention is by reversing the quantization, i.e., via the correspondence principle. This yields the Hamilton-Jacobi equation as the classical limit of Eq. (4.11.3)

$$\frac{\partial A}{\partial t} + H^{can} = 0, \quad \mathbf{p}^{can} = \frac{\partial A}{\partial \mathbf{r}}, \tag{4.11.5a}$$

$$H^{can} = g^{-1}(t)\left(\frac{1}{2m}\mathbf{p}^{can\,2} + V\right). \tag{4.11.5b}$$

The Lagrangian, via the inverse Legendre transform, is then given by

$$L^{gen} = g(t)\,\frac{1}{2}\,m\,\dot{\mathbf{r}}^{2} - g^{-1}(t)\,V(r). \tag{4.11.6}$$

Thus, Newton's equations of motion which corresponds to the mutation (4.11.3) of Schrödinger's equation are given by

$$\left[\left(m\,\ddot{\mathbf{r}} + \frac{1}{g^{2}(t)}\frac{\partial V}{\partial \mathbf{r}}\right)_{SA} + m\,\frac{\dot{g}(t)}{g(t)}\,\dot{\mathbf{r}}\right]_{NSA} = 0, \tag{4.11.7}$$

and they result to be genuinely nonconservative. In particular, the nonconservative character of the system is such that, not only the total energy, but also the physical angular momentum is nonconserved. And indeed, we have

$$\frac{d}{dt}\mathbf{M}^{phys} = \frac{d}{dt}\left(\mathbf{r}\times m\,\dot{\mathbf{r}}\right) = -\gamma(t)\,\mathbf{M}^{phys}, \tag{4.11.8a}$$

$$\mathbf{M}^{phys} = \mathbf{M}_{0}\,e^{-\int_0^t \gamma(t)\,dt}, \quad \gamma(t) = m\,\frac{\dot{g}(t)}{g(t)}. \tag{4.11.8b}$$

We then argue that Eqs. (4.11.4) are mathematically consistent in the sense that they represent a consistent quantization of the canonical, conserved, angular momentum

$$\mathbf{M}^{can} = \mathbf{r}\times\mathbf{p}^{can} = g(t)\left(\mathbf{r}\times m\,\dot{\mathbf{r}}\right), \tag{4.11.9a}$$

$$\frac{d}{dt}\mathbf{M}^{can} \equiv 0. \tag{4.11.9b}$$

Neverthless, the same equations are physically inconsistent in the sense that they do not represents the physical angular momentum which is given instead by

$$Q.M.: \quad \mathbf{M}^{phys} = g^{-1}(t)\,\mathbf{M}^{can}, \quad [\mathbf{M}^{phys}, M^{can}] = 0, \tag{4.11.10a}$$

$$\left(\mathbf{M}^{phys}\right)^{2}\psi = g^{-2}(t)\,\hbar^{2}\,\ell(\ell+1)\,\psi, \tag{4.11.10b}$$

$$M_{3}^{phys}\,\psi = g^{-1}(t)\,\hbar\,m\,\psi. \tag{4.11.10c}$$

This yields an illustration of the hadronic rules (4.11.2).

A few comments are here in order. The potentially misleading nature of the example considered exists already at the purely Newtonian level and simply carries over to the hadronic quantization. It is constituted by the fact that the classical Hamiltonian (4.11.5b) is fully invariant under rotation and, equivalently, equations of motion (4.11.7) are fully covariant under rotations. The use of conventional rules of conservative (classical or quantum) mechanics would then imply that the physical angular momentum is conserved and, thus, conventionally quantized.

The methodology of the Inverse Problem has disproved this familiar association of symmetries to conservation laws, as summarized in ref.[23] and treated in details in monographs[21,22]. A brief summary may here be of some assistance for the interested reader. In essence, when a system exhibits a conventional symmetry (translations in time, translations in space and rotations), by no means this implies that the conventionally associated physical quantity (energy, linear momentum and angular momentum, respectively) is necessarily conserved. A distinction between the case of the equations of motion and their analytic representation is here useful.

(A) Symmetries of the equations of motion. The fact that the form-invariance of the equations of motion under translations in time does not necessarily imply the conservation of the physical energy is clearly established by the linearly damped particle (4.9.9) which is manifestly form-invariant under translations in time, neverthless, the physical energy is nonconserved, as experimentally established. The same equations are manifestly invariant under translations in space. Neverthless, again, the linear momentum is nonconserved, as a necessary condition of compliance with physical reality. For the case of rotations, consider the spinning top under gravity, for simplicity (but without loss of generality), with one degree of rotational freedom. The assumption that the physical angular momentum is in this case conserved would literally imply the acceptance of the perpetual motion in our environment. To comply with physical evidence, the equation of motion must contain a drag torque responsible for the decaying in time of the angular momentum, as experimentally established. The typical functional dependence of the drag torque is that on the angular velocity. This yields the equation for the nonconservative spinning top

$$\left[(I \ddot{\theta})_{SA} + T(\dot{\theta}) \right]_{NSA} = 0, \qquad (4.11.11)$$

which, as such, is fully form-invariant under rotation. Neverthless, the physical angular momentum is necessarily nonconserved, as experimentally established. This latter case is precisely that of Eqs. (4.11.7): the equations of motion are form-invariant under rotations, but this doe not imply the conservation of the physical angular momentum.

(B) Symmetries of the Lagrangian (or Hamiltonian) representations. In the transition from the equations of motion to their analytic representation, the relationship between symmetries and conservation laws becomes considerably weaker. This is due to the fact, also established by the Inverse Problem, according to which the class of manifest symmetries of the equations of motion is generally larger than that of each individual Lagrangian for their analytic representation. This in practice means that if the equations of motion possess a manifest symmetry, such a symmetry is not necessarily carried over at the level of the analytic representations, trivially, because of the functional dependence of the integrating factors in representations (4.1 . 5). It is in this sense that the relationship between manifest symmetry and the conservation laws become even weaker at the Lagrangian level. For instance, as indicated earlier, the equation for the damped particle (4.9.9) is manifestly invariant under translations in time and space. Neverthless, at the level of the Lagrangian or Hamiltonian representation (4.9.10) the invariance under translations in time is lost, while only that under translations in space is preserved. Despite that, this symmetry does not imply that the physical linear momentum is conserved, for the same reason indicated earlier. In conclusion, as it is the case for the equations of motion, conventional symmetries

of a Hamiltonian (translations in time, translations in space and rotations), by no means necessarily imply conventional, physical, conservation laws (energy, linear momentum and angular momentum, respectively). It is this feature which carries over in its entirety at the quantum mechanical level.

(C) Symmetries via Noether's theorem . The above remarks are fully compatible with Noether's theorem. As a matter of fact, Noether's theorem is one of the most effective tools to identify the crucial dichotomy of nonconservative mechanics: canonical versus physical quantities. In essence, Noether's theorem establishes that, when a Lagrangian possesses a symmetry under an n-dimensional (connected) Lie group, there exist n quantities which are conserved along the actual path (first integrals of the equations of motion). The key point for the problem under consideration is that Noether's theorem, by no means, enters into the physical nature of these first integrals which, as such, rests on the sole nature of the acting forces. For instance, Noether's theorem yields, for the form-invariance of the damped particle under under translations in space, the following first integral

$$\frac{d}{dt} I \equiv 0, \qquad I = e^{\frac{\gamma}{m} t} m \dot{\underline{r}}, \qquad (4.11.12)$$

but this first integral is not representative of a physical law. It is only a mathematical occurrence. In conclusion, when a Lagrangian or Hamiltonian exhibits a conventional symmetry (translations in time, translations in space and rotations) Noether's theorem ensures the existence of corresponding first integrals. Their physical nature solely depends on the physical character of the acting forces. For Hamiltonian (4.11.5b), the invariance under rotations does indeed yield a first integral which, in this case, is the canonical angular momentum, as already empirically established by Eqs. (4.11.9). The point, again, is that the system (4.11.7) is nonconservative and such that, in particular, the physical angular momentum is nonconserved. One of the central contentions of this paper, which leads to the notion of mutation of the atomic angular momentum, is that this dichotomy between canonical and physical angular momentum must be carried over to the quantum mechanical formulation.

In conclusion, nonselfadjoint forces produce a breaking of the methodologically most significant part of Galilei's relativity or Einstein's special relativity: the group of rotations. The mechanisms according to which this fundamental symmetry is broken by the forces considered has been the subject of an extensive study in paper[23] and in monographs[21,22]. This has resulted in the identification of five classes of breakings. Those which are crucial for the analysis of this paper are the following three.

(I) Semicanonical breaking. This is the case when the equations of motion (or a generalized Hamiltonian) are form-invariant under the symmetry considered, but the physical conservation laws are lost. This is precisely the case of the physical spinning top (4.11.11). The forces in this case are restricted to be nonessentially nonselfadjoint, by therefore allowing the construction of the canonical realization of the original symmetry. For the case of the rotations this implies that the operator $M^{can} = r \times p^{can}$ is indeed the generator of rotations. This is the simplest possible breaking of space-time Lie symmetry induced by nonconservative forces and will be extensively used in the following.

(II) Canonical breaking. This case occurs when there is both, the loss of form-invariance of the equations of motion (or a generalized Hamiltonian) and the loss of the physical conservation laws. The forces, however, are still nonessentially nonselfadjoint. In this case the breaking occurs at the central methodological level: the canonical formalism (from the non-invariance of the Hamiltonian) and this is the reason for the selected term "canonical breaking". In comparison, the "semicanonical breaking" still preserves the canonical formalism and loses only the direct physical meaning of representing physical conservation laws. A trivial example is when Eq. (4.11.11) is implemented in the form

$$\left[(I\ddot{\theta})_{SA} + T(\theta, \dot{\theta}) \right]_{NSA} = 0. \qquad (4.11.13)$$

(III) Essentially nonselfadjoint breaking. This is the most powerful breaking of the space-time symmetries, inclusive and most importantly of the group of rotations, which is produced by the nonselfadjoint forces. It occurs when, not only the equations of motion are form-noninvariant under the original symmetry and there is the loss of physical conservation laws, but the forces are such to violate the integrability conditions for the existence of a Hamiltonian in the coordinate frame of the experimental detection of the system. As a result, for such coordinate frame of direct physical relevance, the entire canonical formalism in general, inclusive of the Lie algebras, cannot be even defined, let alone the formalism of the individually considered space-time symmetry.

TABLE 4.12: THE PROPOSED HADRONIC MUTATION OF THE CONVENTIONAL QUANTUM MECHANICAL TWO-BODY AND RESTRICTED THREE BODY-SYSTEMS.

One of the fundamental problems for an assessement of the plausibility of the proposed model of hadron structure as bound states of eletons and antieletons is the quantization of the Newtonian equations (3.4.2) for the two- and three-body cases.

As by now familiar, these equations constitute a system of variationally nonselfadjoint, ordinary, differential equations subject to a system of subsidiary, nonintegrable, constraints. As a result, the technical difficulties for the quantization of such systems are two-fold. First, we have the problem of quantizing unconstrained but nonselfadjoint equations of motion. Second, we have the additional problem of a quantum mechanical description of the constrained generalization of the system.

Almost needless to say, each of these problems goes beyond the capability of an isolated researcher. Thus, the content of this table must be considered as tentative. We will have achieved our objectives if, besides identifying the problems considered, our remarks will result of some value as a first step.

To proceed in subsequent stages, the first aspect of the problems considered is that of the quantization of the nonconservative, Newtonian, two-body system, which, in by now familiar notations, can be written

$$\left[\begin{pmatrix} m_1 \ddot{r}_1 + \frac{\partial V}{\partial r^1} \\ m_2 \ddot{r}_2 + \frac{\partial V}{\partial r^2} \end{pmatrix}_{SA}^{C^\infty, R} - \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \right]_{NSA}^{C^\infty, R} = 0. \qquad (4.12.1)$$

The second aspect is that of the constrained extension of the emerging quantum mechanical two-body system. The third aspect is that of unconstrained and constrained, quantum mechanical, three-body extensions.

Before entering into the problems considered, a brief review of the conceptual lines may be valuable for the identification of the quantum mechanical objectives to be attempted.

The central hypothesis of this paper is

that hadronic matter (whether a single hadron or the core of a neutron star) is a physical, penetrable medium, here called hadronic medium. In turn, this implies that in the transition from motion in the physical vacuum to penetration and motion in such hadronic medium, extended, charged particles exhibit the action of nonconservative nonlocal forces, here approximated with local, nonconservative, nonselfadjoint forces.[38] The preceding content of this section is intended to attempt a quantum mechanical description of such nonconservative motion of one particle within hadronic matter. It should be stresses, at the risk of being pedantic, that the objective, by no means, is that of attempting a rigorous description of such a motion. Such a rigorous treatment appears to necessarily demand the use of nonlocal forces and, as such, it is beyond our current knowledge both classically and quantum mechanically. Our objective is, instead, the identification of the departures from conventional concepts, notions and laws, which appear to be requested by such type of motion, as well as of hadronic structure, already at the level of local nonselfadjoint forces, with an understanding that even greater conceptual, technical and methodological departures are expected to occur when we will be in a position to effectively treat the full, nonlocal, nonconservative setting.

The first step of this table, i.e., quantization of system (4.12.1), can be conceptually conceived as constituting the transition from the motion of one particle within hadronic matter to the motion of a bound state of two particles within the same medium. As such, this first step is fundamentally insufficient to characterize the structure of a hadron as a bound state of an eleton and an antieleton. Neverthless, this type of motion might be of some value per sé, e.g., for the motion of a hadron within hadronic matter under the assumption that such a hadron is a bound state of one charged, extended particle and an antiparticle and with an understanding that the life of such a system within such a medium is expected to be very short at the time scale of the same hadronic world.

These remarks are intended to identify the objectives of this first step. One of the central objectives of the Newtonian description of a two-body Kepler system in a physical dissipative medium (e.g., a gas) is the technical characterization of the violation of the ten Galilei conservation laws (conservation of energy, linear momentum and angular momentum, as well as uniform motion of the center of mass), of course, with an understanding that, say, the energy lost by the two-body system is acquired by the dissipative medium. The objective of the first step of this table is that of attempting a quantum mechanical description of precisely this Newtonian context. Specifically, the center of mass motion of

such a quantum mechanical two-body system is not intended to be uniform, its total, quantum mechanical, physical, energy is not expected to be conserved, etc.

Now that the objectives have been conceptually identified, we can outline our treatment. The first alternative is whether to attempt quantization via a covering of Schrödinger's or Heisenberg's approach. We here select the former. The joint conditions of preservation of the Cartesian coordinates r^k and of quantization in compliance with the correspondence principle (i.e., admitting a limiting Hamilton-Jacobi Newtonian description) implies a first substantial restriction on the structure of the admissible nonconservative forces. As outlined earlier, they must be such to satisfy the integrability conditions for the existence of a Hamiltonian representation without redefinition of the r-coordinates (nonessentially nonselfadjoint forces). The condition that the emerging generalized Schrödinger equation is of second order as well as capable of recovering the conventional equation identically at the limit of null nonselfadjoint forces then implies the restrictions of Tables 4.7 and 4.8.

In short, to avoid considerable technical difficulties which are inessential for this first treatment, we restrict the nonselfadjoint forces of system (4.12.1) to be such to admit a generalized Hamiltonian (computable via the techniques of the Inverse Problem) of the type

$$H^{\mathrm{can}} = g_1(t,r;\gamma)\frac{1}{2m_1}p_1^{\mathrm{can}\,2} + g_2(t,r;\gamma)\frac{1}{2m_2}p_2^{\mathrm{can}\,2} \tag{4.12.2a}$$

$$+ g_{12}(t,r;\gamma)\, p_1^{\mathrm{can}}\cdot p_2^{\mathrm{can}} + \rho(t,r;\gamma)V,$$

$$\lim_{\gamma\to 0} g_1, g_2, \rho = 1; \quad \lim_{\gamma\to 0} g_{12} = 0, \tag{4.12.2b}$$

which, under the redefinition

$$P^{\mathrm{can}} = p_1^{\mathrm{can}} + p_2^{\mathrm{can}}, \quad p^{\mathrm{can}} = \frac{m_2 p_1^{\mathrm{can}} - m_1 p_2^{\mathrm{can}}}{m_1 + m_2}, \quad M = m_1 + m_2 \tag{4.12.3}$$

can be written

$$H^{\mathrm{can}} = \frac{1}{2M^2}\left(g_1 m_1 + g_2 m_2 + g_{12} m_1 m_2\right) P^{\mathrm{can}\,2} \tag{4.1.4}$$

$$+ \frac{1}{2}\left(\frac{g_1}{m_1} + \frac{g_2}{m_2} - 2g_{12}\right) p^{\mathrm{can}\,2} + (g_1 - g_2)\frac{1}{M} P^{\mathrm{can}}\cdot p^{\mathrm{can}} + \rho V$$

To further semplify the treatment, we assume an ever simpler, but genuinely nonselfadjoint form of the forces for which

$$g_1 = g_2, \quad g + g_{12}\mu = \rho(t;\gamma), \quad \lim_{\gamma\to 0}\rho = 1, \tag{4.12.5a}$$

$$\alpha = 1 - 3\frac{g_{12}}{\rho}\mu = \alpha(r;\gamma), \quad \lim_{\gamma\to 0}\alpha = 1. \tag{4.12.5b}$$

In conclusion, the nonconservative two-body Newtonian system we are here considering is such to be characterizable by the canonical quantities

$$H^{can} = \rho(t;\gamma)\left[\frac{1}{2M} \underset{\sim}{P}^{can\,2} + \frac{1}{2\mu}\,\alpha(\underset{\sim}{r};\gamma)\,\underset{\sim}{p}^{can\,2} + V(r)\right], \quad (4.12.6a)$$

$$\underset{\sim}{P}^{can} = \frac{1}{\rho} M\dot{\underset{\sim}{R}}, \quad \underset{\sim}{p}^{can} = \frac{1}{\rho\alpha}\mu\dot{\underset{\sim}{r}}, \quad \underset{\sim}{r} = \underset{\sim}{r}_1 - \underset{\sim}{r}_2, \quad (4.12.6b)$$

$$\underset{\sim}{M}^{can}_{tot} = \underset{\sim}{R} \times \underset{\sim}{P}^{can} + \underset{\sim}{r} \times \underset{\sim}{p}^{can}, \quad r = |\underset{\sim}{r}|, \quad (4.12.6c)$$

with corresponding physical quantities

$$H^{phys} = \frac{1}{2M}\underset{\sim}{P}^{phys\,2} + \frac{1}{2\mu}\underset{\sim}{p}^{phys\,2} + V, \quad (4.12.7a)$$

$$\underset{\sim}{P}^{phys} = M\dot{\underset{\sim}{R}} = \rho\,\underset{\sim}{P}^{can}, \quad \underset{\sim}{p}^{phys} = \rho\alpha\,\underset{\sim}{p}^{can} = \mu\dot{\underset{\sim}{r}}, \quad (4.12.7b)$$

$$\underset{\sim}{M}^{phys}_{tot} = \underset{\sim}{R} \times \underset{\sim}{P}^{phys} + \underset{\sim}{r} \times \underset{\sim}{p}^{phys}. \quad (4.12.7c)$$

Our hadronic quantization rules then yield the dual covering of . Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\Psi = H^{can}\Psi = \rho\left[-\frac{\hbar^2}{2M}\Delta_{\underset{\sim}{R}} - \frac{\hbar^2}{2\mu}\alpha\Delta_{\underset{\sim}{r}} + V\right]\Psi, \quad (4.12.8a)$$

$$H^{phys}\Psi = \left[-\frac{\hbar^2}{2M}\rho^2\Delta_{\underset{\sim}{R}} - \frac{\hbar^2}{2\mu}\rho^2\alpha^2\Delta_{\underset{\sim}{r}} + V\right]\Psi, \quad (4.12.8b)$$

under the further assumption that the operators H^{can} and H^{phys} commute. Finally, me make the assumption that the strong hadronic forces are such that Eq. (4.12.8a) satisfies the conditions of the separability theorems[39]. This yields the solution

$$\Psi(t,\underset{\sim}{R},\underset{\sim}{r}) = \psi(t,\underset{\sim}{R})\,e(\underset{\sim}{r}) = f(t)\,U(\underset{\sim}{R})\,e(\underset{\sim}{r}), \quad (4.12.9a)$$

$$i\hbar\,\rho^{-1}(t;\gamma)\frac{\partial}{\partial t}f(t) = E_t f(t), \quad (4.12.9b)$$

$$-\frac{\hbar^2}{2M}\Delta_{\underset{\sim}{R}}U(\underset{\sim}{R}) = E_1 U(\underset{\sim}{R}), \quad (4.12.9c)$$

$$\left[-\frac{\hbar^2}{2\mu}\alpha(\underset{\sim}{r};\gamma)\Delta_{\underset{\sim}{r}} + V\right]e(\underset{\sim}{r}) = E_2\,e(\underset{\sim}{r}), \quad (4.12.9d)$$

$$\psi(t,\underset{\sim}{R}) = \int A(E)\,e^{\frac{i}{\hbar}\left(\underset{\sim}{P}\cdot\underset{\sim}{R} - E_t\int_0^t \rho\,dt\right)}dE. \quad (4.12.9e)$$

Our contention is that this quantum mechanical system does indeed achieve the desired objectives, that is, <u>a characterization of the simplest possible violation of the ten Galilei conservation laws</u> according to the spirit of Section 3, that is, as a necessary prerequisite for the existence of forces non-derivable from a potential. Indeed, the violation of the conservation of the total physical energy is trivially established by the time dependence of the ρ term for which (in terms of expectation values)

$$\dot{H}^{phys} \neq 0. \quad (4.12.10)$$

Equivalent reasons then imply the nonconservation of the total linear momentum and angular momentum

$$\dot{\underset{\sim}{P}}^{phys}_{tot} \neq 0, \quad \dot{\underset{\sim}{M}}^{phys}_{tot} \neq 0. \quad (4.12.11)$$

Finally, the lack of uniform motion of the center of mass can be seen, for instance, from the fact that function (4.12.9e) <u>does not</u> represent a free wave.

We shall call the emerging system a <u>hadronic mutation of the two-body, conservative, quantum mechanical system.</u>

The second step contemplated in this table, a constrained, quantum mechanical extension of model (4.12.8), is considerably more involved on technical grounds even for the simplest possible cases. We shall therefore restrict to only few aspects.

The reader should recall that this second step is related to the problem whether a quantum mechanical formulation of the concept of bound state of an eleton and an anti-eleton can be consistently achieved. We are here referring to a state which is conservative as a whole, when isolated. Nevertheless, the forces are not entirely derivable from a potential. The hopes for the existence of such a hadronic state are given by the consistency at the classical level, as indicated in Table 3.4. The objective of such a possible state is that of attempting a structure model of the π^0, to be considered in Section 5. A further restriction on the type of admitted nonselfadjoint forces is expected to occur for the two-body case while, as it was the case at the classical level, more general nonselfadjoint forces are expected to be admitted for the case of a higher number of constituents.

In its most rudimentary possible form, the <u>discrete, nonrelativistic, quantum mechanical, bound state of an eleton and an antieleton</u> can be written according to a constrained implementation of the mutated Schrödinger's equation (4.12.8a) of the (indicative) type

$$i\hbar\frac{\partial}{\partial t}\Psi = H^{can}\Psi = \rho\left[-\frac{\hbar^2}{2M}\Delta_{\underset{\sim}{R}} - \frac{\hbar^2}{2\mu}\alpha\Delta_{\underset{\sim}{z}} + V\right]\Psi, \quad (4.12.12a)$$

$$H^{phys}\Psi = \left[-\frac{\hbar^2}{2M}\rho^2\Delta_{\underset{\sim}{R}} - \frac{\hbar^2}{2\mu}\rho^2\alpha^2\Delta_{\underset{\sim}{z}} + V\right]\Psi = E^{phys}\Psi, \quad (4.12.12b)$$

$$\underset{\sim}{M}^{phys^2}\Psi = \hbar^2 \ell(\ell+1)\Psi, \quad M_z^{phys}\Psi = \hbar m\Psi, \quad (4.12.12c)$$

$$\ell = 0, 1, 2, \ldots; \qquad m = \ell, \ell-1, \ldots, -\ell,$$

$$\dot{H}^{phys} = \dot{\underset{\sim}{M}}^{phys} = \dot{\underset{\sim}{P}}^{phys} = 0, \quad (4.12.12d)$$

where the last equations are intended, e.g., via expectation values.

A few comments are here in order. First, it should be stressed that in the transition from the unconstrained model (4.12.8) to its constrained extension (4.12.12) the equation on

the physical Hamiltonian, Eq. (4.12.12b), is now intended to be a <u>subsidiary constraint</u> to the "time volution" equation (4.12.12a). This implies, in particular, that the eigenvalue E^{phys} is imposed as a condition on the operators. Similarly, the eigenvalues of the physical angular momentum equations are conventional, but they are conditions on the corresponding operators. The argument is familiar. Since the bound state is isolated, its total, physical, energy and angular momentum are conserved and, thus, they must be <u>conventionally</u> quantized, as a fundamental prerequisite for a hadron as a whole to comply with established experimental evidence. The objective of this exercise of scientific curiosity is then to see whether more general structure forces are admissible by such conventionally quantized total quantities. In the final analysis, this is the scope of this paper.

Intriguingly, there are indications that this dichotomy of conventional quantum mechanics for total quantities versus a "mutation" for the behaviour of the individual constituents cannot be excluded, pending verifications by interested researchers. The conservation of H^{phys} implies that the term ρ must be independent of time. This implies that function (4.12.9e) reacquires its conventional form of a free wave, i.e.,

$$\psi(t, \underset{\sim}{R}) = \int A(E)\, e^{\frac{i}{\hbar}(\underset{\sim}{P}\cdot\underset{\sim}{R} - E_t\rho t)}\, dE, \quad (4.12.13)$$

a part from a scale (dimensionless) term. In turn, this means the regaining also of the uniform motion of the center of mass of the system, as well as the conservation of the total linear momentum. Finally, the eigenvalues of the physical angular momentum reacquire their conventional form, for instance, when $\alpha = \rho^{-1}$. The use of a reference frame at rest with the center of mass of the system is here assumed. In conclusion, it appears that values of the terms ρ and α other then unit, and yielding a consistent system (4.12.12) exists. This is sufficient for the objectives of this paper because, as the reader may recall from Section 3, such values imply, at the Newtonian limit, the presence of nonconservative forces.

Perhaps more intriguing is a detailed study whether a possible functional dependence of the "mutation factors", i.e., the ρ and α terms, is compatible with a consistent system (4.12.12). This study will not be conducted at this time. Notice that it implies a departure from the conventional structure of the eigenfunction Ψ which emerges for system (4.12.12). As a matter of fact, it is precisely such departure which opens the possibility of achieving consistency.

In conclusion, our classical structure model for the eletons, Eqs. (3.4.2), appears to admit a consistent quantum mechanical (hadronic in our terminology) formulation. Our structure model of hadrons of Section 5 is based on such formulation.

The extension of the analysis to the restricted three -body system is self-evident and will be considered later on in Table 5.2 to avoid unnecessary repetitions.

TABLE 4.13: MISCELLANEOUS OPEN PROBLEMS.

Almost needless to say, the number of problems which we have left open in regards to a generalized Schrödinger's formulation (when it exists) of nonconservative systems is rather large indeed. In any case, these problems will not escape the attentif reader. Some of the open problems merit a specific mention.

(I) The statistical-thermodynamical profile. Permit me to stress that the central objective of this paper is to attempt a generalization of the atomic structure for the case of hadrons beginning, most importantly, at the simplest possible level of the two-body system. For this objective it is vital to achieve a quantum mechanical formulation of the behaviour of one particle under nonselfadjoint forces. This approach might appear to be fundamentally non-statistical in spiration. In our opinion, this is not the case. The crucial methodological function of statistical and thermodynamical formulation for conservative quantum mechanics is well known. In our view, this function becomes even more essential in the transition to nonconservative quantum mechanics (hadronic mechanics in our terminology). It is then predictable that our approach to the quantization of nonselfadjoint forces cannot be considered as established until its statistical-thermodynamical profile is studied in details and proved to be consistent. The most intriguing line of study along this profile is to attempt the identification of a possible relationship between our approach to nonconservative forces and the thermodynamical approach by I. PRIGONINE and his collaborators[25]. This aspect will not be considered at this time. It is however significant to point out that the studies of ref.[25] and those of this paper are devoted to essentially the same class of physical systems, only from different profiles. The minor difference is that ref.[25] treats dissipative forces (for which the energy monotonically decreases in time), while we treat nonconservative forces (for which the energy can arbitrarily vary in time, because the variation of the physical characteristics of our hadronic constituents cannot be restricted to a decrease only). The point is that both these forces are nonselfadjoint. Also, the studies of ref.[25] appear to be extendable to all nonselfadjoint forces in general without any fundamental modification of the underlying methodology.

But there is another profile which, in our opinion, may eventually render mandatory the use of statistical-thermodynamical considerations in hadron physics. As we all know, the quark models consist of a few-body characterization of the hadronic structure, in the sense that mesons are represented as two-quark systems while baryons are three-quark systems (plus the sea of gluons of quantum field theoretical character). This few body characterization of hadron structure has inevitably implied a lack of emphasis on the statistical-thermodynamical profile. However, the quark models have not reached until now the needed degree of unequivocal physical validity. The joint study of fundamentally different structure models then becomes recommandable. Besides, the quarks models are vitally dependent on selfadjoint structure forces and related conventional relativity and quantum mechanical laws.

When the problem of hadron structure is seen under the assumption of broader forces, the situation becomes fundamentally different, and the potential, direct, physical relevance of statistical-thermodynamical methods emerges in its full light. This is due to a number of reasons. First of all, the assumption of nonselfadjoint forces directly implies the focusing of irreversible processes. This is trivially due to, say, the breaking of the symmetry under translations and inversions in time realized at the level of the motion of each individual constituent (as by now familiar, the case for an isolated hadron as a whole is different) or by the motion of one particle within hadronic matter, Eqs. (4.9.24). The net effect is that irreversible processes are expected to play a crucial role, jointly with their methodology, in the structure of hadrons under nonselfadjoint strong forces. But there is still another reason which focuses the attention on statistical-thermodynamical profiles. As we shall see in Section 5, the hadronic structure will emerge, under strong nonselfadjoint forces, as considerably equivalent to the atomic and the nuclear structures as far as the number of constituents is concerned. Explicitly, we shall identify the need, under the forces considered, of an increase of the number of constituents with mass. In conclusion, two-body systems are expected to apply for the lightest known hadrons, while the structure of heavy hadrons will definitively emerge as genuine many-body systems governed in both their structure and their spontaneous decays by irreversible processes. The potential direct physical relevance of statistical-thermodynamical methods is then self-evident.

(II) The measurement profile. The attentif reader has by now noticed the complete absence of the term "observable" in our study. This absence will be kept for the entire paper. This is due to the fact that the problem of the measurement theory must be reinspected

from its very foundation for the case of nonselfadjoint forces before the term "observable" can acquire a physical meaning. The problems created by nonselfadjoint forces for this profile are numerous. For instance, what is the meaning, as far as observability in concerned, of a Hermitian operator H^{can} which, by central property, does not represent the physical energy? But, a part from technical problems, there is an epistemological problem which we believe to have a fundamental character. When considering the measurement theory, as a result of extended use, we are naturally lead to a conceptual representation of the conventional setting of atomic mechanics whereby we have a measuring apparatus and a physical event, say, the Compton scattering. The physical context we are here considering is profoundly different. The fundamental objective of the nonconservative quantum mechanics we are interested in is the representation of the motion of massive, extended particles within hadronic matter, say, the core of a neutron star. What is then the meaning of the conventional measurement setting of atomic mechanics for this layer of physical reality? does it implies that we have a measuring apparatus within the core of a neutron star ? and, if the measuring apparatus is put outside of such neutron star, how do we "measure" the motion of a particle within its core ?

We consider these questions outside of the objectives of this paper and, as such, the problem will be left entirely open. The mental attitude which is however recommended to the interested researcher is to expect profound departures from conventional approaches, as it has been the case for numerous other profiles, whenever nonselfadjoint forces are considered (breaking of space-time symmetries, non-inertial character of a possible covering relativity, departures from Heisemberg's uncertainty principle, etc.).

To avoid possible misunderstandings, it should be stressed that this problem of the measurement theory does not exist for our structure model of hadrons. When considering, say, the problem of the structure of pions, what is measured and, most importantly, conventionally measured is the set of physical characteristics of these particles (mass, spin, charge, etc.). Those are total quantities and, as such, represented by the subsidiary operational constraints in our structure model (4.12.12). We shall therefore be interested in recovering the conventional observability of these total quantities. The problem of observability at the level of each individual constituent will be ignored.

(III) The Marmo-Saletan problem. As indicated in Table 4.6. the Inverse Problem produces, in general, a family of structurally different, but analytically equivalent Lagrangians or Hamiltonians. In our quantization approach we have selected only one element of this family, that which satisfies limits (4.7.2). The problem of quantization of the so-called isotopically mapped Lagrangians and Hamiltonians constitutes a different problem.

It should be here recalled that these new degrees of freedom of Lagrangian and Hamiltonian representations were identified, apparently for the first time, by E.J. SALETAN[40] and his collaborators via conventional techniques. R.M.SANTILLI[21,23] then identified, via the methods of the Inverse Problem, the integrability conditions for the existence of these equivalent Lagrangians and the methods for their construction.

In a recent paper, G. MARMO and E.J.SALETAN[41] have initiated the study of the problem of quantization of isotopically mapped (in our language, q-equivalent in the language of these authors) Hamiltonians for conservative systems, e.g., for the case of the two-dimensional harmonic oscillator

$$L^{gen} = \frac{1}{2}\left(\dot{r}_1^2 - \dot{r}_2^2\right) - \frac{1}{2}\left(r_1^2 - r_2^2\right), \qquad (4.13.1a)$$

$$H^{can} = \frac{1}{2}\left(p_1^2 - p_2^2\right) + \frac{1}{2}\left(r_1^2 - r_2^2\right), \qquad (4.13.1b)$$

$$p_k = p_k^{can} = \partial L^{gen} / \partial \dot{r}^k, \quad k = 1, 2. \qquad (4.13.1c)$$

The results are that, even though at the classical elevel we have full equivalence between the conventional Hamiltonian of a conservative system and one of its isotopic images, the corresponding quantum mechanical formulations appear to be inequivalent.

The problem of quantization of isotopically mapped Hamiltonians for conservative systems appears to be quite valuable for that of quantization of nonconservative Newtonian systems, and therefore, further studies are encouraged. Indeed, structures of type (4.13.1b) exhibit most of the lines of study advocated in this paper. For instance, Hamiltonian (4.13.1b) does not represent the physical energy and it is only the time evolution operator, in exactly the same way as it coccurs for nonconservative systems. A similar situation occurs for the canonical momenta. In principle, our differentiation between canonical and physical quantities may be of assistance in the possible removal of these quantum mechanical ambiguities.

TABLE 4.14: SIMPLE ALGEBRAIC NOTIONS FOR LIE-ADMISSIBLE QUANTIZATION

We are now equipped to study the algebraic profile of the problem of quantization of forces not derivable from a potential. This is clearly a necessary complement to the Schrödinger-type analysis of the preceding tables of this section. In particular, this algebraic profile is crucial for the proposed structure model of hadrons on numerous counts, i.e., whether the notion of spin of the eletons can be algebraically formulated in a consistent way as a covering of the conventional Lie characterization of the familiar quantum mechanical spin. In turn, this profile inevitably demands the study of the problem whether a non-Lie covering of Heisenberg's approach for forces non derivable from a potential exists. The argument is selfevident. As it was the case at the Newtonian level, in order for any algebraic approach to physical systems, whether Lie or non-Lie, to have a clear physical role, it must exhibit a dynamical origin, that is, it must originate via the time evolution law.

Predictably, a study of this nature involves numerous technical problems, some of which are of pure mathematical nature. In particular, the study necessarily demands the use of the theory of Abstract Algebras, with particular reference to the part of this discipline devoted to the study of algebraic coverings of the Lie algebras. In the following we shall restrict ourselves to only truly rudimentary remarks and to the identification of only those notions which appear to be essential. More technical treatments are contemplated in subsequent papers.

In its simplest possible formulation, the statement of the problem is the following. It consists of the study whether there exists a generalization of Heisenberg's law for nonself-adjoint forces

$$\dot{X}_i = (X_i, H), \quad i = 1, 2, 3, \ldots, \qquad (4.14.1)$$

such that:

(I) all operators have a direct physical significance (i.e., H is the total physical energy, and the X's are physical quantities, such as the linear momentum, angular momentum, or the energy itself);

(II) the product (X, H) violates the Lie algebra identities and characterizes instead a broader nonassociative algebra other than Lie;

(III) law (4.14.1) is a covering of the conventional Heisenberg's law

$$\dot{X}_i = [X_i, H], \qquad (4.14.2)$$

in the sense of applying to a broader physical context (quantization of nonselfadjoint forces) while being able to recover law (4.14.2) identically at the limit of null forces non derivable from a potential; also,

the generalized Heisenberg's approach recovers, at the classical limit, an algebraically equivalent generalized Hamiltonian approach.

In this table we shall ignore the physical profile and consider only the algebraic aspect of the problem. It essentially consists of the problem whether there exists an algebraic covering of Lie algebras, that is, a nonassociative algebra with elements, say, A, B, C, ... and product, say, (A, B) which violates the Lie algebra identities but which is such to recover the Lie product [A, B] identically under a limiting procedure. This algebraic problem is known to admit a solution given by the Lie-admissible algebras.[27,23] As a matter of fact, under certain technical restrictions, the Lie-admissible algebras constitute the only solution of the problem considered.[42]

In the reader's convenience we here review the main ideas. According to A. A. ALBERT,[43] a Lie-admissible algebra U over a field F (of characteristic here assumed zero) is a vector space of elements A, B, C, ... equipped with the (abstract) product (A, B) such that the attached algebra U^-, which is the same vector space as U but equipped with the product

$$[A, B]_U = (A, B) - (B, A) \qquad (4.14.3)$$

is a Lie algebra. Clearly, if (A, B) is the ordinary associative product AB (say, of matrices), then product (4.14.3) characterizes a Lie algebra as conventionally used in quantum mechanics. Thus, associative algebras are Lie-admissible. As a matter of fact, we can say that the notion of Lie-admissibility is at the very foundation of conventional quantum mechanics, and only expressed in its simplest possible form, the associative form. We are however interested in nonassociative Lie-admissible algebras in accordance with condition (II) above. Suppose than that the product (A, B) is Lie, i.e., [A, B] = (A, B). Then [A, B]-[B, A] = 2[A, B]. Thus, Lie algebras are Lie-admissible. However, still in line with condition (II) above, the product (A, B) must characterize a nonassociative algebra other than Lie. This implies

that the desired product (A, B) is neither totally antisymmetric nor totally symmetric, but still such to satisfy the rule of Lie-admissibility, Eq. (4.14.3). As we shall see in a moment, this is precisely the case for the product of Lie-admissible algebras in general.

To summarize this first step, the theory of Abstract algebras indicates that the fundamental quantum mechanical rule for the construction of a Lie algebra via the associative product AB admits an algebraically consistent covering via nonassociative products (A, B), according to the schematic view

Algebraic structure of the conventional Heisenberg's law for selfadjoint forces AB - BA = LIE, AB = associative product
Algebraic structure of the intended generalization of Heisenberg's law for nonselfadjoint forces (A, B) - (B, A) = LIE, (A, B) = nonassociative product
Compatibility condition between the generalized and conventional law LIMIT (A, B) = AB - BA = LIE NSA forces → 0

The second step consists of an inspection of the Lie-admissible identities (or axioms, or laws) to verify the covering nature of the Lie-admissible over the Lie algebras. According to R. M. SANTILLI,[27] the Lie-admissible algebras can be classified into the following three classes of decreasing complexity and methodological needs.

I. General Lie-admissible algebras. These are all (nonassociative) algebras U over a field F satisfying the so-called general condition of Lie-admissibility

$$[A,B,C]_U + [B,C,A]_U + [C,A,B]_U - [C,B,A]_U - [B,A,C]_U - [A,C,B]_U = 0, \qquad (4.14.4)$$

where the quantity

$$[A,B,C]_U = (A,(B,C)) - ((A,B),C) \qquad (4.14.5)$$

is called the associator and expresses the amount by which the elements miss obeying the associative law.

II. Flexible Lie-admissible algebras. These are all algebras U over a field F satisfying the laws

$$[A,B,A]_U = 0, \qquad (4.14.6a)$$

$$[A,B,C]_U + [B,C,A]_U + [C,A,B]_U = 0, \qquad (4.14.6b)$$

where Eq. (4.14.6a) is called the flexible law and Eq. (4.14.6b) is called the flexible condition of Lie-admissibility.

III. Lie algebras. These are all algebras U over a field F verifying the familiar laws

$$[A,B] + [B,A] = 0, \qquad (4.14.7a)$$

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0. \qquad (4.14.7b)$$

Our objective is that of indicating that the general and flexible Lie-admissible laws are a covering of the Lie algebra laws. For this purpose we first note that the flexible law (4.14.6a) is a covering of the anticommutative law (4.14.7a). Indeed, all anticommutative algebras are flexible, trivially, because

$$[A,[B,A]] = -[A,[A,B]] = [[A,B],A]. \qquad (4.14.8)$$

Neverthless, flexible algebras are not necessarily anticommutative and numerous examples to this effect exist in the literature of Abstract Algebras (e.g., the so-called quasiassociative algebras). Secondly, the flexible condition of Lie-admissibility (4.14.6b) is a covering of the Jacobi law (4.14.7b). This can be seen by the fact that Lie algebras satisfy law (4.14.6b), i.e.,

$$[A,B,C]_{Lie} + [B,C,A]_{Lie} + [C,A,B]_{Lie} = 2\{[A,[B,C]] + [B,[C,A]] + [C,[A,B]]\} = 0. \qquad (4.14.9)$$

Neverthless, flexible Lie-admissible algebras are not necessarily Lie, and explicit examples of the product illustrating this feature exist, as we shall recall below. In conclusion, the laws of flexible Lie-admissibility, Eqs. (4.14.6), do constitute an algebraic covering of the Lie algebra laws.

In turn, the general condition of Lie-admissibility constitutes an algebraic covering of the conditions of flexible Lie-admissibility. Indeed, flexible Lie-admissible algebras are also general, that is, satisfy law (4.14.4). Neverthless, the algebras characterized by law (4.14.4) are not necessarily flexible. As a result, law (4.14.4) can also be interpreted as a further generalization of the Lie algebra laws. A study of the problem indicates that law (4.14.4) characterizes the most general possible nonassociative algebras capable of satisfying the fundamental rule (4.14.3). Notice that, under the limit of anticommutativity of the product, law (4.14.4) reduces to four time the Jacoby law, i.e.,

$$[A,B,C]_{Lie} + [B,C,A]_{Lie} + [C,A,B]_{Lie} - [C,B,A]_{Lie} - [B,A,C]_{Lie} - [A,C,B]_{Lie} = 4\{[A,[B,C]] + [B,[C,A]] + [C,[A,B]]\} \underset{Lie}{=} 0. \qquad (4.14.10)$$

To summarize this second step, the theory of Abstract Algebras indicates that the Lie algebras, as conventionally used in quantum mechanics, are in actuality the simplest possible nonassociative Lie-admissible algebras and that two additional classes of Lie-admissible algebras exist with the enclosure properties

$$\left\{\frac{\text{Lie}}{\text{algebras}}\right\} \subset \left\{\frac{\text{Flexible Lie-admissible}}{\text{algebras}}\right\} \subset \left\{\frac{\text{General Lie-admissible}}{\text{algebras}}\right\}$$

The fundamental algebraic character is that of Lie-admissibility, that is, the capability of characterizing a Lie algebra via rule (4.14.3). This indicates that the Lie algebras are not lost in the transition to their flexible and general Lie-admissible covering. Instead, they are preserved in full in a double form: the attached form $U^- \approx L^*$ and the limiting form under anticommutativity of the product $U \to L$.

Our third step is the identification of the explicit form of the product of the flexible and general Lie-admissible algebras. This is clearly a crucial aspect for the study of the possible application of the algebraic context under consideration for the quantum mechanical treatment of nonselfadjoint forces. The reader should be aware that the realization of the product of an (associative or nonassociative, and Lie or non-Lie) algebra is not unique and that several products satisfying the same laws can, in general, exist. Thus, we here present a form of the product per each class of algebra which we assume of primary rilevance mainly on physical grounds to be yet indicated (at an abstract algebraic level any form of the product which satisfies the laws considered is perfectly admissible).

I. Fundamental realization of the product of general Lie-admissible algebras. It is given by

$$(A,B) = ARB - BSA, \quad R \neq \pm S \text{ and nonsingular}, \tag{4.14.11}$$

where AR, RB, etc. is associative and, thus, $(AR)B = A(RB) = ARB$. Also, R and S are invertible, fixed, elements of U, that is, in the above product the elements A and B can span all the elements of U, including R and S, but the elements R and S, as well as the structure of the product, are kept fixed. It is a tedious but straightforward exercise to prove that the product (4.14.11) satisfies the general condition of Lie-admissibility, Eq. (4.14.4). This can be seen by computing first the associator

$$\begin{aligned}[A,B,C]_u &= (A,(B,C)) - ((A,B),C) = (A, BRC - CSB) \\ &- (ARB - BSA, C) = A\cancel{RBRC} - BRCSA - ARCSB \\ &+ CS\cancel{BSA} - A\cancel{RBRC} + CSARB + BSARC - \cancel{CSBSA},\end{aligned} \tag{4.14.12}$$

and then all remaining associators needed for law (4.14.4). Equally important, is the proof that product (4.14.11) violates the flexibility law (4.14.6a) and, thus, both Lie algebra laws (4.14.17).

Finally, product (4.14.11) satisfies the distributive and scalar rules

$$(A, B+C) = (A,B) + (A,C), \tag{4.14.13a}$$

$$(A+B, C) = (A,C) + (B,C), \tag{4.14.13b}$$

$$(A, \alpha B) = (\alpha A, B) = \alpha(A,B), \quad \alpha \in F, \tag{4.14.13c}$$

to properly qualify as the product of an algebra. In conclusion, product (4.14.11) characterizes a general non-flexible Lie-admissible algebra. To the best of my knowledge, this product is here introduced for the first time.

II. Fundamental realization of the product of flexible Lie-admissible algebras. It is given by the product of the (λ,μ)-mutation algebras

$$(A,B) = \lambda AB - \mu BA, \quad AB = \text{Assoc.}, \quad \lambda \neq \pm\mu, \quad \lambda,\mu \neq 0, \tag{4.14.14}$$

where λ and μ are elements of the field or functions of independent parameters (e.g., functions of time). It is an easy exercise to see that product (4.14.14) satisfies the flexibility law

$$((A,B),A) = (A,(B,A)) = (\lambda^2+\mu^2)ABA - \lambda\mu(AAB + BAA), \tag{4.14.15}$$

and the flexible condition of Lie-admissibility

$$\begin{aligned}&[A,B,C]_u + [B,C,A]_u + [C,A,B]_u \\ &= (\lambda+\mu)^2 \{[A,[B,C]] + [B,[C,A]] + [C,[A,B]]\} = 0.\end{aligned} \tag{4.14.16}$$

It is an instructive exercise to see that product (4.14.14) satisfies also the general condition of Lie-admissibility as well as, of course, the distributive and scalar rules (4.14.13) but violates the Lie algebra laws. Thus, product (4.14.14) characterizes a flexible, non-Lie, Lie-admissible algebra. To the best of my knowledge, product (4.14.14) was introduced for the first time by R. M. SANTILLI and G. SOLIANI in the unpublished note[45] and by R. M. SANTILLI in paper[27] (1967).

III. Fundamental realization of the product of Lie algebras. It is given by the familiar form

$$[A,B] = AB - BA, \quad AB = \text{Assoc.} \qquad (4.14.17)$$

As stressed numerous times, no generalization of quantum mechanics for forces not derivable from a potential can be consistent unless it is capable of recovering conventional quantum mechanics identically at the limit of null nonselfadjoint forces. At the level of the expected generalization of Heisenberg's law, this implies that law (4.14.1) must recover law (4.14.2) identically under the limit considered. In turn, an algebraic prerequisite to reach this physical notion, is that the product of the assumed generalized algebraic structure must be able to reduce to the Lie product identically under a limiting procedure, and such procedure results to be representative of the limit of null nonselfadjoint forces.

By again ignoring in this table the physical profile, the assumed fundamental realizations of the products of the general and flexible Lie-admissible algebras do indeed satisfy this uncompromisable requirement because

$$\lim_{R,S=1} (A,B) = \lim_{R,S=1} ARB - BSA = AB - BA = [A,B], \qquad (4.14.18a)$$

$$\lim_{\lambda,\mu=1} (A,B) = \lim_{\lambda,\mu=1} \lambda AB - \mu BA = AB - BA = [A,B]. \qquad (4.14.18b)$$

As a matter of fact, products (4.14.11) and (4.14.14) have been selected over other conceivable products precisely in view of their capability of satisfying limits (4.14.18). For instance, the following product introduced by A. A. ALBERT [43]

$$(A,B) = \lambda AB + (1-\lambda) BA, \quad \lambda \in F, \qquad (4.14.19)$$

does characterize a fully acceptable flexible Lie-admissible algebra, as the reader can verify. Neverthless, product (4.14.19) fails to recover the Lie product under a (finite) limit of the value of the λ parameter and, as such, must be rejected for our objectives.

To summarize, this third algebraic step indicates the existence of consistent realizations of the product of the general and flexible Lie-admissible algebra which, most importantly

from a physical profile, are capable of recovering the fundamental realization of the Lie product identically under a limit procedure. In turn, as we shall see in the subsequent tables, this algebraic property sets the way for the physical profile, that is, the study whether the limits considered are representatives of the limit of null value of the nonselfadjoint forces. If this is the case, we then expect a property of crucial physical meaning: the representation of the broader forces under consideration via a departure from the conventional Lie product represented by the operators R and S for the general Lie-admissible algebras and the parameters λ and μ for the flexible Lie-admissible algebras.

We are however not yet ready to initiate the study of this physical profile because in need of additional algebraic notions which appear to be essential. In simplistic terms, the explicit form of the Lie algebra product conventionally used in quantum mechanics is unique (i.e., form (4.14.17)) in the sense that such product is fully sufficient for the considered layer of systems (with selfadjoint forces). In the transition to more general systems the situation is different in the sense that such a study brings into focus aspects of the theory of Lie algebras which are generally ignored in conventional treatments because inessential. We are here referring in particular to the degrees of freedom of the realization of the Lie algebra product. This problem is clearly created by the fact that for the general Lie-admissible algebras, (A, B) - (B, A) is indeed Lie, but in no way it coincides with the conventionally used product AB-BA. As a matter of fact, it is possible to prove that a necessary condition for an algebra to be a general Lie-admissible algebra is that $(A,B) - (B,A) \not\equiv AB - BA$.

Our fourth step is therefore the study of the degrees of freedom of the realization of the product satisfying given algebraic laws. Within the context of the theory of Abstract Algebras, this problem is resolved by the so-called notion of algebraic isotopy.[23] Let U be any algebra (i.e., associative, Lie, Lie-admissible etc.) with elements A, B, C, ... and (abstract) product $A \cdot B$ over a field F (of characteristic zero). An isotopic extension or simply an isotope of U is any new algebra U* which can be constructed in terms of the same elements of U and of a modification of its product induced by elements of the field and/or the algebra U itself which is such to preserve the algebraic laws of U.

Simple examples are here in order to illustrate this notion. Suppose first that U is an associative algebra with the ordinary (associative) product AB. Then a simple isotope

U^* of U is given by

$$AB \longrightarrow A * B = \lambda AB \quad , \lambda \in F. \qquad (4.14.20)$$

Indeed, A*B is still associative, while the original product AB can be recovered via the isotopic mapping induced by λ^{-1}. Thus, the notion of algebraic isotopy stands to indicate an invertible axiom-preserving mapping of the product.

The isotopy is not restricted to elements of the field nor to any particular transformation of the product. For instance, any of the following new products constructed with the elements of U with fixed, invertible elements C, ... $\in$ U

$$A\cdot B \to A \overset{*}{\cdot} B = A \cdot C \cdot B \,, \qquad (4.14.21a)$$

$$A \cdot B \to A \overset{*}{\cdot} B = A \cdot C \cdot B - B \cdot (1-C) \cdot A, \qquad (4.14.21b)$$

etc.,

characterizes an isotope of U provided that the algebraic laws of U and U* coincide. Thus, the class of all possible algebraic isotopies of an algebra exhausts the class of all possible forms of the product which are admitted by the defining algebraic laws.

We are now equipped to give an algebraic differentiation of the Lie algebras emerging from the Lie-admissible rule (4.14.3) and the Lie algebras of conventional use in mechanics. The classical aspect of this problem has been studied in details in ref.[23] Let us here only recall that the transition from the conventional Poisson to the generalized Poisson brackets

$$[A,B]_{ce} = \frac{\partial A}{\partial b^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial b^\nu} \longrightarrow [A,B]^*_{ce} = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu}, \qquad (4.14.22a)$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, (\Omega^{\mu\nu}) = (\Omega_{\mu\nu})^{-1} = \left(\frac{\partial R_\mu}{\partial b^\nu} - \frac{\partial R_\nu}{\partial b^\mu}\right)^{-1},$$

$$\mu,\nu = 1,2,\dots,6N \,, \{b^\mu\} = \{\underset{\sim}{r}, \underset{\sim}{p}\}, \Omega^{\mu\nu} \in C^\infty, \det(\Omega) \neq 0. \qquad (4.14.22b)$$

(which is the algebraic aspect of the transition from Hamilton's to Birkhoff's equations) is precisely an example of Lie algebra isotopy. Both brackets $[A,B]_{ce}$ and $[A,B]^*_{ce}$

satisfy the Lie algebra identities. Thus, both of them are perfectly acceptable realizations of the Lie algebra product in classical mechanics. The invertible algebraic isotopy is in this case done via functions of the phase space coordinates because, under the regularity and continuity conditions here assumed, the generalized Lie tensor $\Omega^{\mu\nu}$ can be factorized in the form

$$\Omega^{\mu\nu}(b) = g^\mu_{\ \rho}(b)\, \omega^{\rho\nu}. \qquad (4.14.23)$$

Thus, the invertible Lie algebra preserving mapping $[A,B]_{ce} \rightarrow [A,B]^*_{ce}$ can be equivalently written

$$\omega^{\mu\nu} \longrightarrow g^\mu_{\ \rho}(b)\, \omega^{\rho\nu} = \Omega^{\mu\nu}(b) . \qquad (4.14.24a)$$

$$\Omega^{\mu\nu} + \Omega^{\nu\mu} = 0 \,, \qquad \Omega^{\mu\rho}\frac{\partial \Omega^{\nu\tau}}{\partial b^\rho} + \Omega^{\nu\rho}\frac{\partial \Omega^{\tau\mu}}{\partial b^\rho} + \Omega^{\tau\rho}\frac{\partial \Omega^{\mu\nu}}{\partial b^\rho} = 0. \qquad (4.14.24b)$$

A fully equivalent situation exists at the level of the classical realizations of the general Lie-admissible algebras. Indeed the transition from the Lie-admissible brackets $(A,B)_{ce}$ of Table 4.3 to the following more general form of Lie-admissible brackets

$$(A,B)_{ce} = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t,b) \frac{\partial B}{\partial b^\nu} \rightarrow (A,B)^*_{ce} = \frac{\partial A}{\partial b^\mu} S^{*\mu\nu}(t,b) \frac{\partial B}{\partial b^\nu}, \qquad (4.14.25a)$$

$$S^{\mu\nu} = \frac{\partial b^\mu}{\partial R_\nu} \,, \{R_\nu\} = \{\underset{\sim}{R}(t,\underset{\sim}{r},\underset{\sim}{p}), \underset{\sim}{r}\}, \; S^{*\mu\nu} = \frac{\partial b^\mu}{\partial R^*_\nu}, \qquad (4.14.25b)$$

$$\{R^*_\nu\} = \{\underset{\sim}{R}(t,\underset{\sim}{r},\underset{\sim}{p}), \underset{\sim}{R}'(t,\underset{\sim}{r},\underset{\sim}{p})\},$$

is precisely an example of Lie-admissible isotopy. Both brackets $(A,B)_{ce}$ and $(A,B)^*_{ce}$ satisfy the general Lie-admissible condition, Eq. (4.14.3) and, thus, they are both perfectly admissible realizations of the general Lie-admissible product in classical mechanics. The isotopy is again induced by functions of the $\underset{\sim}{r}$ and $\underset{\sim}{p}$ coordinates as for the Lie case, Eq. (4.14.22). Intriguingly, the isotopy emerges as representing a degree of freedom of the time evolution law for both selfadjoint systems (for the Lie case) and nonselfadjoint systems (Lie-admissible case).[23]

The aspect of this classical context which is here significant is that the Lie algebra which is attached to the brackets $(A,B)_{cl}$ via rule (4.14.3) is that expressed by the generalized Poisson brackets and not the conventional Poisson brackets, i.e.,

$$(A,B)_{cl} - (B,A)_{cl} = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu}, \qquad (4.14.26a)$$

$$(\Omega^{\mu\nu}) = (S^{\mu\nu} - S^{\nu\mu}) = \left(\frac{\partial R_\mu}{\partial b^\nu} - \frac{\partial R_\nu}{\partial b^\mu}\right)^{-1}. \qquad (4.14.26b)$$

We can therefore conclude by saying that the classical realization of the Lie algebra attached to a general Lie-admissible algebra constitutes an isotopic image of the conventional realization via Poisson brackets. For a detailed study of this algebraic profile at the classical level we here refer the interested reader to refs.[21,23]

In the transition to the abstract treatment of Lie-admissible algebras for possible quantum mechanical use, the situation is fully equivalent. In few words, the conventional realization of the Lie algebra product, i.e., $[A,B] = AB - BA$ is, by no means, unique (in precisely the same measure as the classical product $[A,B]_{cl}$ is not). Among the numerous conceivable forms of isotopy of the product $[A,B]$ we are interested the following forms

$$[A,B] \rightarrow [A,B]^* = A\tau B - B\tau A = \tau(AB - BA), \qquad (4.14.27a)$$

$$[A,B] \rightarrow [A,B]^* = ATB - BTA. \qquad (4.14.27b)$$

Consider now the fundamental realization of the product of the flexible Lie-admissible algebras, Eq. (4.14.14). Then rule (4.14.3) reads

$$\begin{aligned}[A,B]_u = (A,B) - (B,A) &= (\lambda+\mu)(AB - BA) = (\lambda+\mu)[A,B] \\ &= [A,B]^*. \end{aligned} \qquad (4.14.28)$$

Thus, the Lie algebras which are attached to the fundamental realization of flexible Lie-admissible algebras are those in the simple isotopic form with product $[A,B]^*$, trivially, with $\tau = \lambda + \mu$. This can be also seen by noting that product (4.14.14) is neither totally antisymmetric nor totally symmetric and, thus, it is decomposable in a combination of commutators and anticommutators, i.e.,

$$(A,B) = \lambda AB - \mu BA = \frac{1}{2}\tau[A,B] + \frac{1}{2}\nu\{A,B\}, \qquad (4.14.29a)$$

$$\tau + \nu = 2\lambda, \quad \tau - \nu = 2\mu. \qquad (4.14.29b)$$

The Lie-admissible rule (4.14.3) then simply eliminates the symmetric part of the product.

On fully equivalent grounds, consider the fundamental realization of the product of the general Lie-admissible algebra, Eq. (4.14.11). Then rule (4.14.3) yields

$$\begin{aligned}[A,B]_u = (A,B) - (B,A) &= ARB - BSA \\ &\quad - BRA + ASB \\ &= A(R+S)B - B(R+S)A. \end{aligned} \qquad (4.14.30)$$

Thus, the Lie algebras which are attached to the fundamental realization of the general Lie-admissible algebras are expressed in terms of isotopy (4.14.27b), trivially, for $T = R + S$. And indeed, rule (4.14.3) can now be, in general, written

$$[A,B]_u = [A,B]^*. \qquad (4.14.31)$$

This yields the desired interpretation of the Lie algebras emerging from rule (4.14.3) as needed for the analysis of the subsequent tables. The notion of isotopy, as we shall see, will also result to be valuable for other aspects, i.e., the transformation theory of Hamilton's and Hamilton-admissible equations as well as of Heisenberg's law (4.14.2) and the intended generalization (4.14.1). The study of the isotopic degrees of freedom of the Lie-admissible products (4.14.11) and (4.14.14) is here left as an exercise for the interested reader.

To summarize, this forth algebraic step indicates that the Lie algebras emerge from the Lie-admissible algebras in a dual form:

(A) the attached form U^- which is an isotopic form L^* of a conventional Lie form L;

(B) the limit form U → L under Eqs. (4.14.18) which is a conventional Lie form. As we shall see, this dual algebraic profile appears to be crucial for the study of the quantum mechanical treatment of nonselfadjoint forces.

To begin this study, however, we still need one last algebraic notion. In essence, at the classical level, the Lie-admissible characterization of nonselfadjoint forces is done by preserving all physical quantities of conservative mechanics and by changing instead the algebraic structure of the analytic equations, i.e., by performing the transition from Hamilton's equations to our Hamilton-admissible equations[23]

$$\dot{b}^\mu - \omega^{\mu\nu}\frac{\partial H_{tot}}{\partial b^\nu} = 0 \longrightarrow \dot{b}^\mu - S^{\mu\nu}(t,b)\frac{\partial H_{tot}}{\partial b^\nu}, \quad (4.14.32)$$

where the former emerge as characterizing the maximal associated selfadjoint subsystem. This is equivalent to saying that the Lie-admissible characterization of forces not derivable from a potential is not a Lie isotopy, as a fundamental dynamical prerequisite for the preservation of the direct physical significance of all the algorithms at hand, as stressed earlier during the course of our analysis.

This situation calls for an algebraic characterization of the transition from a Lie algebra to a Lie-admissible algebra or, more generally, from a given algebra U to a generally different algebra $\hat{U}$, where U and $\hat{U}$ are characterized by different laws while they coincide as vector spaces (i.e., have the same elements). Within the context of the theory of Abstract Algebras, this problem is resolved by the notion of so-called algebraic genotopy.[23] Let U be any algebra (i.e., associative, Lie, Lie-admissible, etc.) with elements A, B, C, ... and (abstract) product A·B over a field F (of characteristic zero). A genotopic extension or simply a genotope of U is any new algebra $\hat{U}$ which can be constructed in terms of the elements of U and a modification of its product induced by elements of the fields and/or the algebra U itself, which is such to violate the algebraic laws of U.

The specialization of the notion of algebraic genotopy which is needed for our analysis is that when the axioms of the new algebra $\hat{U}$ are given. In this case the mapping U → $\hat{U}$ can be interpreted as invertible, axiom-inducing mapping of the product (while, on a comparative basis, the isotopic mapping U → U* is an invertible axiom-preserving mapping

of the product). Still more particularly, the type of genotopic mapping in which we are interested is that from a Lie algebra to a Lie-admissible algebra of general or flexible type. In this case we shall say that we have a Lie-admissible genotopic mapping of a Lie algebra, that is, an invertible mapping of the Lie algebra product which induces a Lie-admissible algebra within the context of the same vector space.

At a classical level, it is easy to see that, from an algebraic profile, mapping (4.14.32) from Hamilton's equations to their covering Hamilton-admissible form without changing the physical quantities ($\underset{\sim}{r}$, $\underset{\sim}{p}$, H and $\underset{\sim}{M}$) is precisely a case of Lie-admissible genotopic mapping of the time evolution law, i.e.,

$$[A,B]_{ce} = \frac{\partial A}{\partial b^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial b^\nu} \longrightarrow (A,B)_{ce} = \frac{\partial A}{\partial b^\mu}S^{\mu\nu}(t,b)\frac{\partial B}{\partial b^\nu}. \quad (4.14.33)$$

As a result, at this classical level, the algebraic notion of genotopy has a crucial physical meaning: it represents the nonselfadjoint forces. Indeed, since all physical quantities are preserved by assumption in the transition from a selfadjoint system to a nonselfadjoint form with forces not derivable from a potential, these additive forces are precisely represented by the Lie-admissible genotopic mapping $\omega^{\mu\nu} \rightarrow S^{\mu\nu}$. One of the fundamental technical problems of this paper is to see whether this crucial aspect of classical mechanics admits a consistent quantum mechanical extension which preserves the algebraic character.

At this point, by ignoring the physical profile, we can say that the transition from the fundamental (abstract) realization of Lie algebras to that of flexible and general Lie-admissible algebras are precisely cases of Lie-admissible genotopic mappings according to the rules

$$[A,B] = A(1)B - B(1)A \rightarrow (A,B) = A\lambda B - B\mu A, \quad (4.14.34a)$$

$$[A,B] = A(1)B - B(1)A \rightarrow (A,B) = ARB - BSA. \quad (4.14.34b)$$

The inverse mappings are then given by

$$(A,B) = A\lambda B - B\mu A \rightarrow [A,B] = A\lambda\lambda^{-1}B - B\mu\mu^{-1}A, \quad (4.14.35a)$$

$$(A,B) = ARB - BSA \rightarrow [A,B] = ARR^{-1}B - BSS^{-1}A. \quad (4.14.35b)$$

The preservation of the underlying vector space in the genotopic (as well as isotopic) mapping has a crucial role on both mathematical and physical grounds which should be here stressed. On mathematical grounds it implies that the elements of the algebras U and $\hat{U}$ (or U and U*) are the same and only the product is changed. On physical grounds, it implies that the physical quantities are unchanged in the transition from a conservative to a nonconservative system, as a crucial prerequisite to comply with physical reality (in the sense that, e.g., the physical angular momentum is $\underset{\sim}{r} \times m\dot{\underset{\sim}{r}}$ and this definition occurs irrespective of whether the acting forces are derivable from a potential or not). In conclusion, the notion of genotopic mapping results to have a crucial physical meaning for the classical, algebraic, characterization of the additive forces non derivable from a potential. The problem to be studied in the next tables is that whether the notion admits a consistent, algebraically equivalent, quantum mechanical formulation.

To summarize, the fifth and last algebraic step of this table indicates that the Lie-admissible algebras can be constructed as a vector space preserving, invertible, "mutation" of conventional realizations of Lie algebras.

We would like to conclude this table by recalling a few additional aspects which have been investigated in the existing literature, but without entering into a technical treatment.

At a deeper study, the Lie-admissible algebras of general and flexible type emerge as being induced by a genotopic mapping of the universal enveloping associative algebra of a Lie algebra. We assume that the reader is familiar with the fact that the fundamental physical and mathematical role of the algebraic structure of currently available physical models is played by the universal enveloping algebra, rather than the Lie algebra, e.g., because the former contains the latter as a vector space, while powers of the base elements (e.g., X^2) can <u>only</u> be computed within the context of the former (X^2 is definable via the

associative product and it is identically null if defined in terms of the Lie product). At the quantum mechanical level we can thus say that, without the use of the universal enveloping associative algebra of a Lie algebra (say, SU(2)-angular momentum-spin) we would be unable to compute some of the most significant physical quantities and, in general, <u>all</u> polinomials in the powers of the elements of the basis. On mathematical grounds, the universal enveloping associative algebras are equally crucial for Lie's theory. For instance, they constitute an essential link between the Lie algebras and the corresponding Lie groups, because in the exponential law

$$e^{\theta X} = 1 + \frac{\theta^1}{1!}X + \frac{\theta^2}{2!}X^2 + \cdots \quad (4.14.36)$$

all elements from the second order term on are outside of the Lie algebra and only definable in its envelop. It then follows that such (associative) envelop has fundamental implications for the representation theory.[46]

In the transition to the Lie-admissible formulation, a deeper study indicates that it is such associative envelop which is mapped into a nonassociative but still Lie-admissible envelop.[23] Thus, in addition to Eqs. (4.14.34), the Lie-admissible algebras can be more properly characterized as the mapping

$$AB \longrightarrow (A,B) = ARB - BSA. \quad (4.14.37)$$

In turn, this second approach has fundamental implications of mathematical and physical nature. On mathematical grounds it implies the rather unpredictable (but crucial) occurrence of the Lie-admissible formulations whereby the Lie algebra structure in the neighborhood of the identity is lost by central condition. Nevertheless, a connected Lie group (which we have called <u>Lie-admissible group</u> owing to its special structure) persists! On physical grounds it is precisely this dual covering of Lie's formulations (in both, the local behaviour in the neighborhood of the identity and the global group structure) which has allowed the conjecture of a Lie-admissible covering of the Galilei relativity of ref.[23] And indeed, our Galilei-admissible group, Eqs. (4.4.1), is a genotope of Galilei's relativity which is such that (1) it has a Lie-admissible structure in the neighborhood of the identity, (2) it has a global, connected Lie group structure although of rather special type (because defined in terms of the base manifold, the generators and the parameters of a different group), and (3) it has a Lie-admissible, nonassociative, envelop which, in the final analysis, is precisely the origin of properties (1) and (2).

It should be here recalled that an essential element of these recent studies on Lie-admissible algebras is a generalization of the Poincaré-Birkhoff-Witt theorem to flexible Lie-admissible algebras formulated by C. N. KTORIDES[47] (with a further generalization to the general Lie-admissible algebras[23]).

One aspect of these studies which is relevant for this paper is that the algebra for the quantum mechanical treatment of selfadjoint systems (that is, the associative envelop) is infinite-dimensional to begin with and remains infinite-dimensional in the transition to the intended Lie-admissible covering (that is, the nonassociative but Lie-admissible envelop).

To summarize, the Lie-admissible algebras can be introduced via the notion of algebraic genotopy in a dual way;

(A) as a genotopic mapping of a Lie algebra, Eqs. (4.14.34), and

(B) as a genotopic mapping of the universal enveloping associative algebra of a Lie algebra, Eq. (4.14.37).

Approach (A) is preferable for a first quantum mechanical study of the problem because simpler and of more pragmatic character. Approach (B) is preferable for a more technical formulation of the problem and, thus, for a verification of initial results obtained via approach (A).

The most comprehensive studies on flexible Lie-admissible algebras of which I am aware are those by H. C. MYUNG[48] which the interested reader is here urged to consult. As we shall see, these mathematical studies emerge as possessing a direct, potential, physical significance in both hadron and nuclear physics. Regrettably, no paper by mathematicians on the general (nonflexible) Lie-admissible algebras has appeared as of now and the only available studies on these broader algebras available at this moment are those by R.M.SANTILLI[22, 23]. We have no words to stress the need for the line of study of this paper of contributions by mathematicians for the study of the general Lie-admissible algebras at the pure mathematical level.

TABLE 4.15: <u>THE DYNAMICAL ORIGIN OF THE LIE-ADMISSIBLE ALGEBRAS FOR THE QUANTUM MECHANICAL TREATMENT OF NONCONSERVATIVE FORCES VIA A COVERING OF HEISENBERG'S EQUATIONS.</u>

Despite the inevitable technical complexity in a rigorous treatment, the physical motivation of the Lie-admissible algebras is as pragmatic as possible: to provide a direct characterization of forces not derivable from a potential via a generalization of the time evolution law without altering the physical quantities. The validity of the approach for the classical profile has been studied in details in refs.[22,23]. In this paper we shall indicate that the approach is also applicable for the quantum mechanical profile of the problem. For this objective we shall consider the following three approaches:

(A) quantization of the Hamilton-admissible equations;

(B) quantization via the transformation theory; and, as a refinement,

(C) quantization via the enveloping algebra.

Approaches (A) and (B) are considered in this table. Approach (C) is presented in Table 4.16.

Let us first review the conventional Heisenberg's laws for the case of selfadjoint forces and for the case of one space component, for simplicity. It is advantageous to use the following unified notation of the conventional Hamilton's equations

$$\dot{b}^\mu = \omega^{\mu\nu} \frac{\partial H^{phys}}{\partial b^\nu}, \quad \{b^\mu\} = \{r, p^{phys}\}, \quad (\omega^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mu = 1, 2. \tag{4.15.1}$$

The <u>fundamental commutation rules of the atomic mechanics</u> can then be written in the unified form

$$[b^\mu, b^\nu] = b^\mu b^\nu - b^\nu b^\mu = i\hbar\, \omega^{\mu\nu}, \tag{4.15.2}$$

that is, more explicitly,

$$\begin{pmatrix} [r, r] & [r, p] \\ [p, r] & [p, p] \end{pmatrix} = i\hbar \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.15.3}$$

Under the transformation of H^{phys} into a Hermitian operator via the replacement of the phase space variables b^μ with the corresponding operators and suitable symmetrization, if needed, Heisenberg's laws coincide with Eqs. (4.15.1).

This is tantamount to saying that within the context of Heisenberg's approach the quantum mechanical equations formally coincide with the corresponding Newtonian equations at the classical limit, a part from terms originating from possible symmetrizations of the Hamiltonian. In turn, this is one way to directly express the compliance of Heisenberg's equations with the correspondence principle for the case of selfadjoint forces.

The use of expectation values yields the same result. By using the familiar equation

$$i\hbar\,\dot{\psi}(t) = H^{phys}(t)\,\psi(t), \qquad (4.15.4)$$

we can write

$$\begin{aligned}\frac{d}{dt}\langle b^\mu\rangle &= \int dv\, \dot{\psi}^* b^\mu \psi + \int dv\, \psi^* b^\mu \dot{\psi} \\ &= \int dv\, \psi^* \frac{1}{i\hbar}[b^\mu, H^{phys}]\,\psi \\ &= \int dv\, \psi^* \omega^{\mu\nu}\frac{\partial H^{phys}}{\partial b^\nu}\psi = \langle \omega^{\mu\nu}\frac{\partial H^{phys}}{\partial b^\nu}\rangle,\end{aligned} \qquad (4.15.5)$$

by therefore recovering Eqs. (4.15.1) again. For instance, for the case of a free particle we have

$$\frac{d}{dt}\langle r\rangle = \langle \frac{\partial H^{phys}}{\partial p}\rangle = \langle p\rangle, \quad (m=1) \qquad (4.15.6a)$$

$$\frac{d}{dt}\langle p\rangle = \langle -\frac{\partial H^{phys}}{\partial r}\rangle = 0, \qquad (4.15.6b)$$

and corresponding equations occur for the case of forces derivable from a potential.

As indicated earlier, the objective of the Lie-admissible approach is as pragmatic as possible. It essentially consists of the identification of the mutation of the conventional Heisenberg's approach which is induced by the addition of forces not derivable from a potential to a conventionally quantized, selfadjoint, Newtonian system.

By its very nature, such an approach is therefore different in inspiration that the Schrödinger's approach to nonselfadjoint systems considered early in this section. The attitude now is that of first quantizing the maximal associated selfadjoint subsystem according to conventional rules, and then identifying the departure from Heisenberg's law which is induced by the additional nonselfadjoint forces.

This yields the following fundamental difference. In the generalized Schrödinger's approach the quantization rules are applied to the canonical variables r and p^{can}, Eqs. (4.8.2). In the generalized Heisenberg's approach we are looking for, the quantization rules are applied to the noncanonical, but physical variables p^{phys} and H^{phys} (under the assumption that the H^{phys} represents a conservative system)

$$p^{phys} = m\dot{r} \rightarrow \text{operator}, \quad H^{phys} \rightarrow \text{operator}. \qquad (4.15.7)$$

The need of searching for a possible consistent quantization of nonconservative systems via rules (4.15.7) is the following. First of all, as recalled in Table 4.1, a Hamiltonian for the representation of the class of systems under consideration does not exist in general (essentially nonselfadjoint systems), unless velocity-dependent transformations of the coordinates are considered. This renders inapplicable the quantization via Schrödinger-type equations and canonical rules (4.8.2). In turn, this renders inapplicable the quantum mechanical characterization of the systems considered via the conventional Heisenberg's law (with generalized Hamiltonians) in the coordinates r of direct physical relevance. Still in turn, this renders inapplicable the Lie algebras as a methodological tool for the quantum mechanical characterization of the systems considered under the condition considered, in full analogy with the corresponding classical occurrence. Even when a Hamiltonian exists and can be computed (nonessentially nonselfadjoint systems), Heisenberg's approach has a number of problematic aspects which will be indicated below in this table.

These occurrences suggest that, in order to achieve a quantum mechanical description of nonconservative systems which is applicable to the entire class of the systems considered (local, of class C^∞ and nonselfadjoint), rather than to specific subsystems, the approach must be independent from conventional canonical formulations by conception, and must be of non-Lie algebraic character as a fundamental dynamical condition, although capable of recovering conventional formulations at the limit of null nonselfadjoint forces.

Clearly, a generalization of Heisenberg's laws which is Lie-admissible in algebraic character is attractive because ensuring the covering nature of the approach over conventional formulations, while at the same time offering the minimal possible mutilation of established techniques.

The simplest and most rudimentary possible Lie-admissible quantization of forces not derivable from a potential is that via a direct quantization of our canonical-admissible equations and use of the conventional quantization rule for physical quantities, i.e.,

$$\dot{b}^{\mu} = S^{\mu\nu}(t,b)\frac{\partial H^{phys}}{\partial b^{\nu}}, \qquad (4.15.8a)$$

$$[b^{\mu}, b^{\nu}] = i\hbar\,\omega^{\mu\nu}. \qquad (4.15.8b)$$

In essence we argue that the quantization of Eqs. (4.15.1) is done (a) by quantizing the r and p variable via rules (4.15.2), (b) by transforming the Hamiltonian into a Hermitian operator which formally coincides with the classical operator, a part from possible symmetrization, and (c) the Lie tensor $\omega^{\mu\nu}$ (i.e., the fundamental symplectic form ω_2) remains unchanged in the quantization process because of real valued numerical elements. In the above Lie-admissible quantization, steps (a) and (b) remain unchanged and only the crucial algebraic step (c) is changed in the sense that the Lie tensor $\omega^{\mu\nu}$ is now replaced by the Lie-admissible tensor $S^{\mu\nu}$ (i.e., the fundamental symplectic-admissible form S_2, in the language of ref.[23]) which, being explicitly dependent on the b-quantities, in general, is now quantized by using the same rule as that for the Hamiltonian, including a possible symmetrization, when needed.

This is, in essence, a straightforward implementation of the notion of Lie-admissible genotopic mapping (Table 4.14), but now at the quantum mechanical level. According to this approach one actually quantizes the maximal selfadjoint associated subsystem. Then the mapping considered is implemented, resulting in a Lie-admissible algebraic structure. The departure from the Lie structure is then a direct measure of the nonselfadjoint forces, according to Eqs. (4.3.11), i.e.,

$$(S^{\mu\nu} - \omega^{\mu\nu})\frac{\partial H^{phys}}{\partial b^{\nu}} = F^{\mu}, \quad \{F^{\mu}\} = \{0, F_{NSA}\}. \qquad (4.15.9)$$

The approach exhibits a direct compliance with the correspondence principle in the sense that the quantum mechanical equations formally coincide with the classical equations, in

precisely the same measure as that of Eqs. (4.15.1).

The approach is also implementable at the level of expectation values, i.e.,

$$\frac{d}{dt}\langle b^{\mu}\rangle = \int dV\,\psi^{*}\,\omega^{\mu\nu}\frac{\partial H^{phys}}{\partial b^{\nu}}\psi \Longrightarrow \int dV\,\psi^{*}S^{\mu\nu}(t,b)\frac{\partial H^{phys}}{\partial b^{\nu}}\psi, \qquad (4.15.10)$$

with an understanding that the symbol ψ represents an eigenstate of H^{phys} only and, as such, it is not an eigenstate in the conventional sense (it does not contain representatives from the nonselfadjoint forces). Despite that, the Lie-admissible genotopic mapping (4.15.10) does indeed achieve the intended result: a quantum mechanical characterization of the nonconservation laws. For instance, for the case of a particle with velocity damping (Table 4.9) we have the correct expressions of the behaviour in time of the physical energy and momentum

$$\ddot{r} + \gamma\dot{r} = 0, \quad m = 1, \qquad (4.15.11a)$$

$$\{b^{\mu}\} = \{r, p\}, \quad H^{phys} = \tfrac{1}{2}p^2, \quad p = \dot{r}, \quad (S^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}, \qquad (4.15.11b)$$

$$\frac{d}{dt}\langle p\rangle = \int dV\,\psi^{*} - \gamma\frac{\partial H}{\partial p}\psi = -\gamma\langle p\rangle, \qquad (4.15.11c)$$

$$\frac{d}{dt}\langle H^{phys}\rangle = \int dV\,\psi^{*}\frac{\partial H^{phys}}{\partial p}\left(-\gamma\frac{\partial H^{phys}}{\partial p}\right)\psi = -2\gamma\langle H^{phys}\rangle. \qquad (4.15.11d)$$

The interested reader is urged to work out more complex cases.

To summarize, this simplest possible Lie-admissible quantization exhibits the following features.

(1) It is based on the conventional Hamiltonian representation and quantization of the maximal associated selfadjoint subsystem and then the mapping from the Lie tensor $\omega^{\mu\nu}$ to the Lie-admissible tensor $S^{\mu\nu}(t,b)$ which is representative of the forces not derivable from a potential according to our Hamilton-admissible equations.

(2) It is entirely independent from the possible existence of the Hamiltonian for the complete system (inclusive of the nonselfadjoint forces) via the techniques of the Inverse Problem and, as such, it is directly applicable to the class of systems considered, including and most importantly, the essentially nonselfadjoint systems.

(3) It satisfies the correspondence principle.

(4) It is fundamentally non-Lie in algebraic character and, instead, of Lie-admissible type.

(5) It characterizes a covering of Heisenberg's approach in the sense of directly applying to a broader physical context, while being able to recover the conventional Heisenberg's approach at the limit $S^{\mu\nu} \rightarrow \omega^{\mu\nu}$ of null nonselfadjoint forces.

Despite these features, the above Lie-admissible approach exhibits precise limitations. The most important one is that the conventional canonical commutation rules (4.15.8b), while consistent for the selfadjoint systems represented by $H^{phys}(b)$ and $\omega^{\mu\nu}$, are not expected to be necessarily consistent for the more general systems represented by $H^{phys}(b)$ and $S^{\mu\nu}(t, b)$. A simple argument is sufficient to indicate this expectation. Consider a conservative system. Its quantization demands the conventional canonical rules for $\underset{\sim}{r}$ and $p^{phys} = m\dot{\underset{\sim}{r}}$. Add now to this system the Lorentz force. Then the quantization rules hold for $\underset{\sim}{p}^{can} = \underset{\sim}{p}^{phys} + e\underset{\sim}{A}$. This means that the addition of a velocity-dependent (in a linear way) force has changed the quantum mechanical form of p^{phys}. The further addition of non-linear, velocity dependent, nonconservative, forces is therefore expected to result in a further change of the quantum mechanical form of p^{phys}, which is contrary to rules (4.15.8b). We shall consider this problem below in this table. At this moment we content ourselves with the fact that the preservation of conventional rules (4.15.8b) also for the considered broader systems is, in the final analysis, a direct consequence of the rudimentary nature of the approach, a simple algebraic mutation of a conventionally quantized selfadjoint system.

Our next objective is that of achieving a matrix form of Lie-admissible equations (4.15.8a). This is clearly a crucial step to attempt further progress and it is dictated by the fact that Heisenberg's laws, besides the conventional Hamilton's form (4.15.1), can also be written in the methodologically more significant form

$$\dot{A} = \frac{1}{i\hbar}[A, H^{phys}] = \frac{1}{i\hbar}(A H^{phys} - H^{phys} A). \qquad (4.15.12)$$

In turn, if a Lie-admissible covering of Heisenberg's approach for nonselfadjoint forces exists, it is expected to exist for both forms (4.15.1) and (4.15.12).

Let us first review the derivation of Eqs. (4.15.12) via the transformation theory. At the classical level of selfadjoint systems the Hamiltonian H^{phys} is the generator of a canonical transformation which, for an arbitrary (polynomial) function of the phase space variables can be written

$$A'(b') = e^{t\,\omega^{\mu\nu}\frac{\partial H^{phys}}{\partial b^\nu}\frac{\partial}{\partial b^\mu}} A(b), \qquad (4.15.13)$$
$$b = b(0), \quad b' = b'(t).$$

In the transition to the corresponding quantum mechanical context, the canonical transformation is mapped into a unitary transformation generated now by the quantum mechanical H^{phys}, i.e., the familiar form

$$A(t) = e^{-\frac{t}{i\hbar}H^{phys}} A(0)\, e^{\frac{t}{i\hbar}H^{phys}}. \qquad (4.15.14)$$

The total time derivative then yields law (4.15.12), i.e.,

$$\dot{A} = \frac{1}{i\hbar}[A, H^{phys}]. \qquad (4.15.15)$$

The construction of a possible Lie-admissible covering of these fundamental steps demands the prior knowledge of the transformation theory of the classical Lie-admissible equations (4.15.8a). An initial study of this problem is presented in ref.[23]. The covering of Eq. (4.15.13) is in this case given by (Table 4.4)

$$\hat{A}(\hat{b}) = e^{t\,S^{\mu\nu}(t,b)\frac{\partial H^{phys}}{\partial b^\nu}\frac{\partial}{\partial b^\mu}} A(b), \qquad (4.15.16)$$
$$b = b(0), \quad \hat{b} = \hat{b}(t),$$

namely, it is a particular form of a connected Lie group constructed in terms of the base manifold, the generator and the parameter of a different group and such to yields a Lie-admissible algebra in the neighborhood of the identity, i.e.,

$$\delta^{(t)} A = t\, S^{\mu\nu} \frac{\partial H^{phys}}{\partial b^\nu} \frac{\partial}{\partial b^\mu} A = t\,(A, H^{phys})_{\text{classical}} \quad (4.15.17)$$

This is, in essence, the notion of Lie-admissible group of ref.[23]. The important point is that the transformations preserve the Lie-admissible character of the equations (i.e., the form of our Hamilton-admissible equations). In turn, this ensures the preservation of the time evolution of the system. Transformations (4.15.16) with an arbitrary generator X(t, b), rather than H^{phys} only, have been called canonical-admissible transformations[23] to express their meaning as a generalization of the conventional canonical transformations. In particular, the reader should be aware that our Galilei-admissible group (4.4.1) is precisely a group of canonical-admissible transformations. The underlying idea is quite simple. It is known that one of the fundamental properties of the Galilei transformations is that of being canonical transformations, i.e., they preserve the Lie time evolution law for conservative and Galilei form-invariant systems. It then follows that, in order to attempt a covering relativity for nonselfadjoint and Galilei form-noninvariant systems, the emerging covering transformations are conceived as canonical-admissible transformations, that is, as transformations which preserve, this time, the Lie-admissible time evolution law of the broader systems considered.

Before entering into a possible quantum mechanical extension of transformations (4.15.16) the reader should be aware of the rather profound departure from canonical transformations which is here involved. In the language of ref.[23] these latter transformations are Lie identity isotopic transformations in the sense that they not only preserve the Lie algebra (Lie isotopy) but actually preserve the value of the Lie product (Lie identity isotopy) according to the familiar rule

$$b' = b'(b) \; : \; \omega^{\mu\nu} \longrightarrow \frac{\partial b'^\mu}{\partial b^\rho} \omega^{\rho\sigma} \frac{\partial b'^\nu}{\partial b^\sigma} \equiv \omega^{\mu\nu}. \quad (4.15.18)$$

It should be here recalled that the preservation of the value of the product is crucial for the physical consistency of the theory. In essence, the Lie formulations reach the climax of their physical significance in the characterization of exact space-time symmetries and conservation laws, i.e.,

$$\dot{X}_i(b) = [X_i, H^{phys}]_{\text{classical}} = 0, \quad (4.15.19)$$

$$i = 1, 2, 3, \ldots$$

In order for any transformation theory to be physically significant, the value zero of the above product must be preserved. On relativity grounds, if a physical quantity is conserved within the context of a given reference frame it must remain so for any other admissible frame.

In the transition to the Lie-admissible formulations the situation is profoundly altered. Their central objective is that of characterizing broken space-time symmetries (e.g., broken Galilei symmetry) and the consequential nonconservation laws due to forces non-derivable from a potential. Thus, Eq (4.15.19) are now mapped into the Lie-admissible form

$$\dot{X}_i(b) = (X_i, H^{phys})_{\text{classical}} \neq 0, \quad (4.15.20)$$

where, as by now familiar, the physical quantities are unchanged and their nonconserved character is expressed by the Lie-admissible structure of the product.

The rather unpredictable result is that a transformation theory based on the preservation of the value of the Lie-admissible brackets (4.15.20) would be inconsistent. This can be easily seen for the case of transformations in time. In this case Eqs. (4.15.20) express the time rate of variation of the energy which, as such, is function of time. It then follows that such rate of variation at one given value of time is generally different than the corresponding rate at a different value of time. A transformation theory based on the preservation of such time rate of variation (i.e., the value of brackets (4.15.20)) would therefore be inconsistent with physical evidence. The case for other transformations is equivalent.

The canonical-admissible transformations are intended to characterize precisely this broader physical context, that is, the representation of the dynamical evolution of the system in a way consistent with the rate of change of the physical quantities. A study of this problem indicates that, for nonnull nonselfadjoint forces, the physically significant transformations are nonidentity Lie-admissible isotopic, that is, they preserve the Lie-admissible character of the time evolution law (Lie-admissible isotopy) but they do not preserve the value of the brackets (nonidentity Lie-admissible isotopy) as a fundamental requirement for physical consistency.

The above features of the transformation theory of our Hamilton-admissible equations are sufficient to provide the needed intuitional elements for a possible quantization. Our main contention is that the quantization of transformations (4.15.16) must be attempted via nonunitary transformations. In essence, we argue that if one implements the attitude of preserving conventional quantum mechanical notions as much as possible also for the broader context considered, the net result might be physical inconsistency. For the case at hand, the use of unitary transformations would indeed result in a violation of the correspondence principle in the sense of yielding transformations which preserve the time rate of change of physical quantities under nonconservative forces, contrary to physical evidence.

Once we have identified (on purely intuitional grounds and without any technical elaboration) the type of transformations which is expected to generalize Eqs. (4.15.14), the next problem is that of its explicit form. Our criteria of selection of such explicit form are the following.

(a) The generators of the nonunitary transformations should be expressed in terms of the product of generally nonHermitian operators, time the Hamiltonian H^{phys}, in analogy to the classical case where the exponent of Eqs. (4.15.16) can be written

$$t\, S^{\mu\nu}(t,b)\frac{\partial H^{phys}}{\partial b^\nu}\frac{\partial}{\partial b^\mu} = t\, g^\mu_\rho(t,b)\,\omega^{\rho\nu}\frac{\partial H^{phys}}{\partial b^\nu}\frac{\partial}{\partial b^\mu}, \quad (4.15.21)$$

(b) The nonunitary transformations should yield a Lie-admissible algebra for the time evolution of the system in analogy to the algebraic character of the classical case, and

(c) The nonunitary transformations should produce a nonindentity Lie-admissible isotopy of the generalized time evolution law, also in analogy with the classical case.

An explicit form of the desired quantum mechanical, nonunitary transformations in time of physical quantities under nonselfadjoint forces which satisfies the above requirements is given by

$$A(t) = e^{-\frac{t}{i\hbar}H^{phys}S}\, A(0)\, e^{\frac{t}{i\hbar}RH^{phys}}, \quad (4.15.22a)$$

$$R \neq S,\quad S \neq S^\dagger,\quad R \neq R^\dagger,\quad [S, H^{phys}] \neq 0,\quad [R, H^{phys}] \neq 0, \quad (4.15.22b)$$

where R and S are generally nonHermitian operators representing the forces non derivable from a potential, whose explicit form should be determined from the equations of motion as indicated below.

Transformations (4.15.22) do indeed satisfy requirements (a), (b) and (c). First of all they constitute a covering of the conventional transformations (4.15.14) in the sense of representing the additional presence of nonselfadjoint forces, while being able to recover conventional unitary transformations at the limit of null value of these forces, i.e.,

$$\lim_{F^{NSA}\to 0} A(t) = \lim_{R,S\to 1} A(t) = e^{-\frac{t}{i\hbar}H^{phys}}\, A(0)\, e^{\frac{t}{i\hbar}H^{phys}}. \quad (4.15.23)$$

Secondly, they produce a Lie-admissible covering of Heisenberg's law (4.15.12) because, trivially, the product emerging from the total time derivative

$$\dot{A} = \frac{1}{i\hbar}(A, H^{phys}) = \frac{1}{i\hbar}\left(ARH^{phys} - H^{phys}SA\right), \quad (4.15.24)$$

is Lie-admissible (Table 4.14). Thirdly, they induce a nonidentity Lie-admissible isotopy of the time evolution law according to our uncompromisable condition for the lack of constancy of the rate of change of physical quantities. Explicitly, the Lie-admissible product (4.15.24) under transformations (4.15.22) in mapped into the form

$$\mathfrak{S}(A,B)\mathcal{R} = (A', B')^* = A'R^*B' - B'S^*A', \quad (4.15.25a)$$

$$\mathfrak{S} = e^{-\frac{t}{i\hbar}H^{phys}S}, \quad \mathcal{R} = e^{\frac{t}{i\hbar}RH^{phys}}, \quad (4.15.25b)$$

$$P' = \mathfrak{S}P\mathcal{R},\quad P = A, B, R, S;\quad Q^* = \mathcal{R}^{-1}\mathfrak{S}^{-1}Q'\mathcal{R}^{-1}\mathfrak{S}^{-1},\quad Q = R, S, \quad (4.15.25c)$$

which is fully Lie-admissible. Neverthless the value of the brackets is not preserved because of the new operators R^* and S^*. This is precisely in line with the classical occurrence according to which the Lie-admissible tensor $S^{\mu\nu}$ is transformed into a new tensor under the canonical-admissible transformations, i.e.,

$$\hat{b}^\mu = \hat{b}^\mu(b):\ S^{\mu\nu}(t,b) \to \frac{\partial \hat{b}^\mu}{\partial b^\rho} S^{\rho\sigma}[t, b(\hat{b})]\frac{\partial \hat{b}^\nu}{\partial b^\sigma} = S^{*\mu\nu}(t,\hat{b})$$

$$= \frac{\partial \hat{b}^\mu}{\partial R^*_\nu} \neq S^{\mu\nu}(t,\hat{b}). \quad (4.15.26)$$

We shall call transformations (4.15.22) unitary-admissible transformations in the sense that they admit the conventional unitary transformations under the limit (4.15.23). The generator of these transformations is still the Hamiltonian H^{phys} in the sense that the operator R and S are fixed for a given system. This is precisely in line with the corresponding classical case whereby the generator of the canonical-admissible transformation (4.15.16) is still the total physical energy, as for the conventional canonical case, but now referred to a broader transformation law. More generally, this is in line with one of the central objectives of the Lie-admissible transformations, the use of physical quantities as the generators of physically significant transformations (which is prohibited for the Lie approach to the same systems via the Inverse Problem, as stressed earlier during our analysis). On relativity grounds, the generators of our Galilei-admissible transformations (4.4.1) are precisely the conventional Galilei generators and, as such, quantities with a direct physical significance. The covering transformation law for the preservation of the time evolution of the systems and their form invariance is then constructed via a Lie-admissible mutation of the structure of the transformation to which such physical generators are applied. These notions are clearly crucial to attempt a covering of the Galilei relativity.

As indicated earlier, for a given system, the explicit form of the R and S operators must be computed via the equations of motion. Alternatively, the assumption of an explicit form of these generators in according with the general properties (4.15.22b) implies given nonselfadjoint forces. A formal solution of the former case is the following. Eq. (4.15.24) applies for arbitrary (polynomial, for simplicity) quantities A. Thus it applies in particular for the variables b^μ. This yields the desired alternative form of Eqs. (4.15.8a), i.e.,

$$\dot{b}^\mu_{\mu=1,2} = \frac{1}{i\hbar}(b^\mu, H^{phys}) = \frac{1}{i\hbar}(b^\mu R H^{phys} - H^{phys} S b^\mu), \quad (4.15.27)$$

as a covering of the corresponding Heisenberg's form

$$\dot{b}^\mu_{\mu=1,2} = \frac{1}{i\hbar}[b^\mu, H^{phys}] = \frac{1}{i\hbar}(b^\mu H^{phys} - H^{phys} b^\mu). \quad (4.15.28)$$

Suppose now that the equations of motion are known and the explicit form of the R and S operators is needed. Then we can write in a selfexplanatory notation

$$\dot{r} = \frac{1}{i\hbar}(r, H^{phys}) = p, \quad (\mu = 1), \quad (4.15.29a)$$

$$\dot{p} = \frac{1}{i\hbar}(p, H^{phys}) = f_{SA} + F_{NSA}. \quad (4.15.29b)$$

The formal solution is then given by the system

$$R = i\hbar r^{-1} p H^{-1} + r^{-1} H S r H^{-1}, \quad (4.15.30a)$$

$$S - p r^{-1} H S r = i\hbar p r^{-1} p - i\hbar H^{-1}(f+F) p^{-1}. \quad (4.15.30b)$$

We are now equipped to consider the problem of the expected generalization of the canonical quantization rules (4.15.8b) for the broader systems considered. With an understanding that such a problem is of considerable complexity, the following considerations might have some value for the interested reader.

One of the fundamental dynamical aspects of the classical treatment of Lie-admissible formulations is the loss of the familiar, fundamental Poisson brackets

$$[b^\mu, b^\nu]_{cl} = \frac{\partial b^\mu}{\partial b^\rho}\omega^{\rho\sigma}\frac{\partial b^\nu}{\partial b^\sigma} = \omega^{\mu\nu}, \quad (4.15.31)$$

in favor of their Lie-admissible covering

$$(b^\mu, b^\nu)_{cl} = \frac{\partial b^\mu}{\partial b^\rho} S^{\mu\nu}(t,b)\frac{\partial b^\nu}{\partial b^\sigma} = S^{\mu\nu}(t,b). \quad (4.15.32)$$

Under the assumption that the $(b^\mu) = (r, p^{phys})$ variables have the by now familiar interpretation and nonselfadjoint forces occur, the inapplicability of brackets (4.15.31) is a fundamental dynamical prerequisite.[23] At this classical level, brackets (4.15.31) can in principle be formally introduced, but they have no dynamical role because the Lie brackets have lost their meaning as characterizing the time evolution law. As a matter of fact, their use could be potentially misleading, if not properly interpreted.

An elaboration of this latter case is in order. Consider a Kepler system in a dissipative medium represented with Lie-admissible formulations and let M^{phys} and H^{phys} be the physical angular momentum and energy, respectively. In principle one can certainly compute the Poisson brackets between these quantities

$$\dot{\underset{\sim}{M}}^{phys} = [\underset{\sim}{M}^{phys}, H^{phys}]_{ce} = 0 . \qquad (4.15.33)$$

This computation, however, is mathematically consistent but physically vacuous for the system considered. Indeed, if physically interpreted, Eqs. (4.15.33) imply the conservation of the angular momentum which is contrary to experimental evidence (the system considered tends to rest in a finite period of time). In actuality, Eqs. (4.15.33) express a known property of the maximal associated selfadjoint system (the conservative Kepler system) and not the true physical property of the complete system inclusive of nonconservative forces. In conclusion, the use of the Lie product for nonconservative systems represented with Lie-admissible formulations has no physical meaning, even though formally definable. This is the case in general of the Lie product of arbitrary quantities and thus, in particular, of the (r, p^{phys}) quantities. It then follows that the correct expression under the conditions considered are Eqs. (4.15.32) and not Eqs. (4.15.31).

In the transition to a quantum mechanical formulation a fully parallel situation is expected to occur. Specifically, we expect that for the Lie-admissible context of covering laws (4.15.27) the Lie product of the operators b^μ and b^ν can indeed be formally defined, as in Eqs. (4.15.8b) but, first of all, we do not necessarily expect that such product has the value $i\hbar\omega^{\mu\nu}$ and, secondly, such a product is not expected to have physical meaning because the formulation is non-Lie as a fundamental dynamical prerequisite.

It seems therefore natural to search for a possible Lie-admissible generalization of the fundamental canonical commutation rules, as the basic condition for the quantum mechanical characterization of the quantities b^μ. The objective is therefore that of identifying a quantum mechanical covering of the classical Lie-admissible equations (4.15.32).

Our conjectural argument is mainly by analogy with the classical case. We note that the Lie-admissible structure of the product of the time evolution law, Eqs. (4.15.8a), and of the fundamental dynamical brackets, Eqs. (4.15.32), coincide. We therefore argue that at the quantum mechanical level the Lie-admissible structure of the product of the time evolution law, Eqs. (4.15.27), and that of the fundamental brackets should also coincide.

We reach in this epistemological way the proposed Lie-admissible covering of Heisenberg's equations

$$\dot{b}^\mu = \frac{1}{i\hbar}(b^\mu, H^{phys}) = \frac{1}{i\hbar}(b^\mu R H^{phys} - H^{phys} S b^\mu), \qquad (4.15.34a)$$

$$(b^\mu, b^\nu) = b^\mu R b^\nu - b^\nu S b^\mu = i\hbar\, S^{\mu\nu}(t, b), \qquad (4.15.34b)$$

$$\mu = 1, 2,$$

where $S^{\mu\nu}(t, b)$ is the operational form of the Lie-admissible tensor of Eqs. (4.15.32). Besides a full analogy with the classical case, an intriguing aspect which support the Lie-admissible quantization rules, Eqs. (4.15.34b), is that they allow the preservation of the classical limit in full analogy with the conventional Lie case (the so-called Dirac's limit), under certain restrictions on the R and S operators.

In our unified notation, the conventional case can be written (for polynomial functions)

$$\begin{aligned}\lim_{\hbar\to 0}\frac{1}{i\hbar}[A, B] &= \lim_{\hbar\to 0}\frac{1}{i\hbar}[A(1)B - B(1)A] \\ &= \lim_{\hbar\to 0}\frac{1}{i\hbar}\left\{\left(\bar{A} + \frac{\partial \bar{A}}{\partial b^\mu} b^\mu\right)(1)\left(\bar{B} + \frac{\partial \bar{B}}{\partial b^\nu} b^\nu\right) - \left(\bar{B} + \frac{\partial \bar{B}}{\partial b^\nu} b^\nu\right)(1)\left(\bar{A} + \frac{\partial \bar{A}}{\partial b^\mu} b^\mu\right)\right\} \\ &= \lim_{\hbar\to 0}\frac{1}{i\hbar}\frac{\partial \bar{A}}{\partial b^\mu}\frac{\partial \bar{B}}{\partial b^\nu}\left[b^\mu(+1)b^\nu - b^\nu(+1)b^\mu\right] = \lim_{\hbar\to 0}\frac{1}{i\hbar}\frac{\partial \bar{A}}{\partial b^\mu}\frac{\partial \bar{B}}{\partial b^\nu}\, i\hbar\,\omega^{\mu\nu} \\ &= [\bar{A}, \bar{B}]_{classical} \qquad (4.15.35)\end{aligned}$$

where, as well known, the quantities $\bar{A}$, $\bar{B}$ and their partial derivatives are classical functions and the b-variables are operators. In full analogy we have for our Lie-admissible covering case

$$\begin{aligned}\lim_{\hbar\to 0}\frac{1}{i\hbar}(A, B) &= \lim_{\hbar\to 0}\frac{1}{i\hbar}(ARB - BSA) \\ &= \lim_{\hbar\to 0}\frac{1}{i\hbar}\left\{\left(\bar{A} + \frac{\partial \bar{A}}{\partial b^\mu} b^\mu\right) R \left(\bar{B} + \frac{\partial \bar{B}}{\partial b^\nu} b^\nu\right) - \left(\bar{B} + \frac{\partial \bar{B}}{\partial b^\nu} b^\nu\right) S \left(\bar{A} + \frac{\partial \bar{A}}{\partial b^\mu} b^\mu\right)\right\} \\ &\qquad (4.15.36)\end{aligned}$$

$$= \lim_{\hbar \to 0} \frac{1}{i\hbar} \Big\{ \bar{A}\bar{B}(R-S) + \bar{A}\frac{\partial \bar{B}}{\partial b^\mu}(R b^\mu - b^\mu S) + \bar{B}\frac{\partial \bar{A}}{\partial b^\nu}(b^\nu R - S b^\nu) + \frac{\partial \bar{A}}{\partial b^\mu}\frac{\partial \bar{B}}{\partial b^\nu}(b^\mu R b^\nu - b^\nu S b^\mu) \Big\} \tag{4.15.36}$$

$$= \frac{\partial \bar{A}}{\partial b^\mu}\frac{\partial \bar{B}}{\partial b^\nu} S^{\mu\nu}(t,b) + \lim_{\hbar \to 0} \Big\{ \bar{A}\bar{B}(R-S) + \bar{A}\frac{\partial \bar{B}}{\partial b^\mu}(R b^\mu - b^\mu S) + \bar{B}\frac{\partial \bar{A}}{\partial b^\nu}(b^\nu R - S b^\nu) \Big\}.$$

Therefore, under the conditions that the R and S operators depend on Planck 's constant in such a way that

$$\lim_{\hbar \to 0} (R-S) = 0, \tag{4.15.37}$$

the classical Lie-admissible brackets are recovered identically, i.e.,

$$\lim_{\hbar \to 0} \frac{1}{i\hbar}(A,B) = (A,B)_{classical} \tag{4.15.38}$$

The covering nature of Eqs. (4.15.34) over the conventional Heisenberg's equations

$$\dot{b}^\mu = \frac{1}{i\hbar}[b^\mu, H^{phys}] = \frac{1}{i\hbar}\left(b^\mu H^{phys} - H^{phys} b^\mu\right), \tag{4.15.39a}$$

$$[b^\mu, b^\nu] = b^\mu b^\nu - b^\nu b^\mu = i\hbar\, \omega^{\mu\nu}, \tag{4.15.39b}$$

is expressed by the following properties. First of all, Eqs. (4.15.34) satisfy our "uncompromisable condition of compatibility" with Eqs. (4.15.39), that is, the capability of recovering the latters identically at the limit of null nonselfadjoint forces, i.e.,

$$\lim_{F^{NSA}_\mu \to 0} (A,B) = \lim_{R,S \to 1} (A,B) = [A,B]. \tag{4.15.40}$$

Secondly, Eqs. (4.15.34) apply to a class of physical systems (nonselfadjoint) which is broader than that of Eqs. (4.15.39) (selfadjoint). Thirdly, Eqs. (4.15.34) are nontrivially different than Eqs. (4.15.39) in the sense of demanding broader methodological tools for their proper treatment. In this latter respect, we have emphasized in this paper only the algebraic profile, in the sense that Eqs. (4.15.39) are stricly Lie in algebraic structure while Eqs. (4.15.34) demand the use of the covering Lie-admissible algebras. Nevertheless, the reader should be aware that, at a deeper study, Eqs. (4.15.34) also demand the use of broader operational approaches than those of Eqs. (4.15.39), the latter being essentially those of Hermitian operators in Hilbert spaces. Indeed, there is no a priori reason why the operators b^μ, solution of Eqs. (4.15.34b), should be Hermitian and more general operator structures (capable of recovering conventional Hermitian structures at the limit of null nonselfadjoint forces) become admissible. It then follows that the basic carrier space is not expected to be necessarily of Hilbert type and, again, more general spaces become conceivable. These latter aspects are considerably involved on technical grounds and they will be left as open problems in this paper.

In essence, one of the objectives of this paper is to stimulate the experimental verification of the validity of established quantum mechanical laws within a hadron. For this purpose, as we shall indicate in Table 4.21, it is sufficient to know that generalizations of the Heisenberg's equations which:

(1) apply to nonselfadjoint systems,

(2) are a covering of the conventional equations (in the sense indicated earlier) and

(3) are strictly non-Lie in algebraic character,

are conceivable on grounds of our current knowledge. Indeed, these occurrences are sufficient to cast shadow on the currently assumed "universality" of the established quantum mechanical laws, that is, their unlimited applicability to the entire microscopic structure of the physical universe, beyond the arena for which they were conceived and experimentally proved to be conform to physical reality: the atomic structure.

To emphasize this profile, we shall continue to preserve the terms of atomic mechanics to denote the conventional quantum mechanics, but restricted to the arena in which it has been exprerimentally proved to be conform to physical reality in its entirety. We shall then also preserve the terms of hadronic mechanics to first stress the lack of currently available unequivocal experimental verification of the former mechanics for the hadronic constituents, as well as the possibility of the emergence, in the final analysis, of a covering mechanics for the new layer of physical reality. To complete our epistemological analysis, we shall then use the terms nuclear mechanics to denote the possible existence of an intermediate layer of formulations between those of atomic and hadronic type. The reader should be aware of the preceding analysis in this respect. Conventional quantum mechanics as currently known, when applied to the nuclear structure, produces results in excellent agreement with experimental data. Neverthless, the problem of the nuclear forces is still open as of now. If the strong hadronic forces will result to be of nonselfadjoint type, this will inevitably demand a reinspection of the laws used in nuclear physics. The net effect is that the departures of a conceivable, broader, nuclear mechanics from the atomic mechanics can ut most be of very small value. This is precisely the nature of the intended intermediate character of the nuclear mechanics with respect to the conceivable finite and nontrivial departures from established laws we are here considering for the hadronic structure. Also, this is in line with our epistemological argument related to the possible physical origin of forces not derivable from a potential (Table 3.1). Indeed, if these more general forces are due to the extended nature of the constituents when bounded at distances of the same order of magnitude of their charge volumes, the contributions which can be expected at the nuclear level can ut most be very small, while large contributions can be expected only at the hadronic level.

In line with these remarks, it is of some significance to indicate possible variations or particularizations of Eqs. (4.15.34). First of all, when reinterpreted from this profile, the conventional Heisenberg's equations emerge as the algebraically fundamental equations of atomic mechanics, while Eqs. (4.15.34) emerges as the intended equations of corresponding fundamental algebraic character for the hadronic mechanics, and we shall write

$$\dot{b}^{\mu} = \frac{1}{i\hbar}[b^{\mu}, H^{phys}]_{AM} \quad , \quad \dot{b}^{\mu} = \frac{1}{i\hbar}(b^{\mu}, H^{phys})_{HM} . \tag{4.15.41}$$

To keep the proper algebraic perspective, the reader should however be aware that the product of an algebra, as stressed in Table 4.15, is by no means unique. This implies that what we are primarily referring to when considering hadronic equations is the Lie-admissible character and not the explicit structure of the product as given in Eqs. (4.15.34). To be specific in this truly crucial point, the structure of the product characterized by the time evolution law (4.15.34), besides having been conceived for the case with one space dimension, is not intended to be unique and other explicit forms are conceivable. What appears to be of fundamental relevance on a number of counts (such as, compliance with the correspondence principle, preservation of the Lie-admissibility rule, relativity and other aspects which will be indicated in the subsequent tables) is that such a product is Lie-admissible. The study of these alternative forms of the product will be left to the interested reader.

What is more significant for the objectives of this paper is the study of conceivable particularized forms of Eqs. (4.15.34). These are essentially the following.

(A) Particularized forms of Lie type. When studying Lie-admissible formulations, both classically and quantum mechanically, the reader should always keep in mind that Lie algebras are Lie-admissible (Table 4.14). Thus, Eqs. (4.15.34) can indeed be Lie as a particular case other than the limit case (4.15.40). Indeed, for R = S Eqs. (4.15.34) become Lie in algebraic character in the sense that the product

$$[A, B]^* = ARB - BRA \tag{4.15.42}$$

satisfies the Lie algebra identities, although it is predictably of a nature more general than that conventionally used in the quantum mechanical treatment of simple selfadjoint systems. A second Lie subcase is given when R = S = f(t) in which case the product

$$[A, B]^* = f(t)\,[A, B] \tag{4.15.43}$$

is trivially Lie. The third and last subcase is when R = S = α = constant, in which case the product

$$[A, B]^* = \alpha\,[A, B] \tag{4.15.44}$$

is also trivially Lie. On algebraic grounds all these cases belong to the isotopic degrees of freedom of the universal enveloping associative algebra. If we denote with $\mathcal{A}$ such an algebra with associative product AB, the conventionally used Lie product in contemporary quantum mechanics can be more properly written in the language of Lie-admissibility

$$[A, B]_{\mathcal{A}} = AB - BA \quad , \qquad (4.15.45)$$

The mappings

$$\mathcal{A} : AB \Rightarrow \mathcal{A}^* : A*B = ARB,\ f(t)AB,\ \alpha AB, \quad R, f, \alpha = \text{fixed and invertible}, \qquad (4.15.46)$$

characterize isotopes $\mathcal{A}^*$ of $\mathcal{A}$ because they preserve the associativity of the product. Thus, all Lie subcases of our Lie-admissible equations (4.15.34) can be written

$$[A, B]_{\mathcal{A}^*} = A*B - B*A = [A, B]^* \qquad (4.15.47)$$

In conclusion, our Lie-admissible equations indicates the possible existence of dynamically meaningful degrees of freedom of the Lie algebra product in quantum mechanics, of potential interest for the transition from the atomic to the nuclear or hadronic level. The interested reader is urged to inspect the corresponding classical context of ref.[23], where the conventional Hamilton's equations are replaced by the broader Birkhoff's equations which emerge as being an analytic, Lie and symplectic covering of the former (in the sense of being more general than the former, neverthless capable of preserving (a) analytic mechanics in its entirety, including the characterization of the time evolution law, derivability from variational principles, transformation theory, etc., (b) the Lie algebra, in the sense that the generalized brackets characterized by Birkhoff's equations are perfectly acceptable classical realizations of the Lie product and (c) symplectic geometry, in the sense that in the transition from Hamilton's to Birkhoff's equations we simple have the transition from globally Hamiltonian to locally Hamiltonian vectorfields without altering the symplectic nature of the applicable geometry).

If we recall the relationship between our Hamilton-admissible equations and Birkhoff's equations (for simplicity we here consider the autonomous case only)[23]

$$S_{\mu\nu}(b)\,\dot{b}^{\nu} - \frac{\partial H^{phys}}{\partial b^{\nu}} = 0, \quad S_{\mu\nu} = \frac{\partial R_{\mu}}{\partial b^{\nu}}, \quad \{b^{\mu}\} = \{r, p^{phys}\}, \qquad (4.15.48a)$$

$$\Omega_{\mu\nu}(b)\,\dot{b}^{\nu} - \frac{\partial H^{gen}}{\partial b^{\nu}} = 0, \quad \Omega_{\mu\nu} = S_{\mu\nu} - S_{\nu\mu}, \qquad (4.15.48b)$$

we can conclude by saying that the following Lie particularization of Eqs. (4.15.34)

$$\dot{b}^{\mu} = \frac{1}{i\hbar}[b^{\mu}, H^{gen}]^* = \frac{1}{i\hbar}(b^{\mu} T H^{gen} - H^{gen} T b^{\mu}), \qquad (4.15.49)$$

emerges as a <u>conceivable quantum mechanical form of Birkhoff's equations.</u>

(B) <u>Particularized forms of finite, flexible, mutational nature.</u> Let us recall that one of the most significant, nontrivial Lie-admissible algebras (i.e., those of non-Lie type) is given by the so-called $\mathcal{A}(\lambda, \mu)$-mutation algebras of an associative algebra with product (4.14.14), where λ and μ are functions of time or elements of the field (assumed of characteristic zero throughout this paper). Thus, a nontrivial particularization of Eqs. (4.15.34a) is given by

$$\dot{b}^{\mu} = \frac{1}{i\hbar}(b^{\mu}, H^{phys}) = \frac{1}{i\hbar}(\lambda b^{\mu} H^{phys} - \mu H^{phys} b^{\mu}), \qquad (4.15.50)$$

$$\lambda \neq \pm\mu; \quad \lambda, \mu \neq 0.$$

The above equations can be significant on a number of counts, as we shall indicate better later on. At this point let us stress that the R and S quantities, in a genuine Lie-admissible context, are expected to be <u>operators</u> satisfying Eqs. (4.15.22b). Thus, Eqs. (4.15.50) can be interpreted as an <u>approximation</u> of Eqs. (4.15.34a). In principle, Eqs. (4.15.50) could be derived via the following procedure. Classical brackets are often computed along the actual path (i.e., the solution of the analytic equations inducing the brackets). For the Lie-admissible brackets this implies the reduction of the Lie-admissible tensor $S^{\mu\nu}(t, b)$ to only a dependence on time, i.e.,

$$(A,B)_{classical}\Big|_{Actual\ Path} = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t) \frac{\partial B}{\partial b^\nu} . \qquad (4.15.51)$$

A Lie-admissible quantization of this context could conceivably give Eqs. (4.15.50). The point is that the nonselfadjoint equations we are primarily interested in are generally nonlinear in the velocity (to yield a nontrivial generalization of the Lorentz force). Thus, their solutions are of quite difficult computation indeed. Lacking the knowledge of the solutions, empirical approximation techniques might be of some value, e.g., for qualitative results, particularly for the line of study of this paper (study of conceivable departures from established laws characterized by nonselfadjoint forces). In conclusion, Eqs. (4.15.50) might be of some value as an approximation of the expected more general equations (4.15.34a).

The reader should be aware that Eqs. (4.15.50) are genuine Lie-admissible in algebraic character and, thus, they are outside the class of equations characterized by the isotopies of the associative algebra. This can be seen also from the fact that the fundamental rule of Lie-admissibility for Eqs. (4.15.50) can be written, instead of Eqs. (4.15.47), in the form

$$[A,B]_{\mathcal{U}} = (A,B) - (B,A) = [A,B]^*, \qquad (4.15.52)$$

where $\mathcal{U}$ is the nonassociative algebra $\mathcal{A}(\lambda,\mu)$. In actuality, the equations under consideration belongs to the covering Lie-admissible genotopic degrees of freedom of the associative algebra to characterize a Lie algebra (which are inclusive of the simpler isotopic degrees of freedom considered earlier). In the final analysis, this is the fundamental algebraic aspect of the Lie-admissible approach to nonselfadjoint system: the characterization of a Lie algebra via a product which is not necessarily associative.

(C) Particularizations of infinitesimal, mutational character. Eqs. (4.15.34) and (4.15.50) produce a finite nontrivial departure from the conventional methodological context of atomic mechanics and, as such, they are expected to be inconsistent with the nuclear phenomenology. (they are strictly hadronic in intended use). It might therefore be of some value to identify an algebraic characterization of the aforementioned notion of very small departure from atomic mechanics as the only conceivable character of a possible nuclear mechanics. This objective is rendered algebraically possible by the infinitesimal mutation algebras

$\mathcal{A}(1+\varepsilon, 1-\varepsilon)$. The particularization of Eqs. (4.15.34a) we are here referring to is therefore of the type

$$\dot{b}^\mu = \frac{1}{i}(b^\mu, H^{phys}) = \frac{1}{i\hbar}\left[b^\mu(1+\varepsilon)H^{phys} - (1-\varepsilon)H^{phys}b^\mu\right],$$
$$= \frac{1}{i\hbar}[b^\mu, H^{phys}] + \frac{\varepsilon}{i\hbar}\{b^\mu, H^{phys}\}, \quad \varepsilon \approx 0. \qquad (4.15.53)$$

Intriguingly, the above equations can be interpreted as the quantum mechanical version of the so-called Duffin's equations[49]

$$\dot{r} = \frac{\partial H^{phys}}{\partial p^{phys}}, \quad \dot{p}^{phys} = -(1+\varepsilon)\frac{\partial H^{phys}}{\partial r}, \qquad (4.15.54)$$
$$\varepsilon \approx 0$$

which were conceived and studied to treat electrical circuits with infinitesimal internal losses. The important point is that the algebra characterized by Duffin's equations via the brackets

$$(A,B)_{classical} = \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} - (1+\varepsilon)\frac{\partial A}{\partial p}\frac{\partial B}{\partial r}, \qquad (4.15.55)$$

is genuinely non-Lie, but Lie-admissible, in algebraic character[27], although the departure from the Lie-structure is infinitesimal by conception. This is precisely the intended algebraic character for a conceivable nuclear mechanics, as preserved by Eqs. (4.15.53) in its entirety. The reader should be aware, from the analysis of Section 3, that all generalized equations considered in this table are referred to the motion of one particle within a hadronic or nuclear context and not to the characterization of a nucleus or a hadron as a whole. The reason is by now familiar. The motion of a particle under the conditions considered is expected to be genuinely nonconservative in the sense of being characterized by generalized forces of either finite (hadronic layer) of infinitesimal (nuclear layer) nature which are nonconservative at the Newtonian limit. This does not prohibit that a hadron or a nucleus as a whole is in a strictly conservative conditions as far the total quantities are concerned and when the state is considered as isolated from the rest of the universe. It is merely our way of expressing the expectation that the hadronic and nuclear constituents are in a dynamical state more general than that of the atomic constituents. In turn, the only conceivable way to attempt a genuine characterization of these broader dynamical conditions is by attempting the construction of broader laws.

To summarize (and somewhat elaborate) the content of this and the preceding table, the following three complementary approaches to the quantization of nonselfadjoint forces are conceivable.

I - Quantization via the theory of abstract algebras. This is the approach which has been studied in this paper for systems with local, class C^{∞} and arbitrary Newtonian forces. It is essentially characterized by the mutation of the Lie algebraic character of Heisenberg's laws into a covering Lie-admissible character, where the departure of the generalized product from the conventional Lie product is a direct representative of the forces not derivable from a potential, according to the schematic view[50]

$$\begin{tikzcd}[column sep=huge, row sep=huge]
& \text{Selfadjoint systems} & \text{Nonselfadjoint systems} \\
\text{Classical level} & \boxed{\dot b^{\mu} = [b^{\mu}, H^{phys}]_{ce}} \arrow[r, "\text{Lie-admissible genotopy}"] \arrow[d, "\text{Lie isotopy}"] & \boxed{\dot b^{\mu} = (b^{\mu}, H^{phys})_{ce}} \arrow[d, "\text{Lie-admissible isotopy}"] \\
\text{Quantum mechanical level} & \boxed{\dot b^{\mu} = \frac{1}{i\hbar}[b^{\mu}, H^{phys}]} \arrow[r, "\text{Lie-admissible genotopy}"] & \boxed{\dot b^{\mu} = \frac{1}{i\hbar}(b^{\mu}, H^{phys})}
\end{tikzcd}$$

II - Quantization via the differential geometry. The currently available studies of quantization via the symplectic geometry[32] are expected to be extendable to the quantization of Birkhoff's equations (owing to the preservation of the symplectic character of Hamilton's equations indicated earlier.). It is hoped that this study is indeed performed by the experts of the field. The reason is that the Lie content of Eqs. (4.15.34a) is of the quantum mechanical Birkhoff's type (4.15.49) and not of the conventional Hamilton-Heisenberg's type (4.15.39a), the latter being identically recovered only at the limit of

null nonselfadjoint forces. Therefore, the study of the quantization of Birkhoff's equations is expected to be of valuable methodological nature for Lie-admissible approaches.

It should be stressed that such symplectic quantization (for either Hamilton's or Birkhoff's equations) is expected to be fundamentally inapplicable to the Lie-admissible equations owing to their nonsymplectic character, as studied in refs.[23] and briefly indicated in Table 4.4.

It is hoped that the the construction of the covering geometry which appears to be needed for the Hamilton-admissible equations (tentatively called symplectic-admissible geometry in ref.[23]) is actually performed by the experts of the field, possibly, under the classical and quantum mechanical profile. This would provide invaluable methodological insights for the study of forces more general than those derivable from a potential. In any case, the study of the geometrical profile appears to be a necessary complement to the algebraic approach to quantization, perhaps, in a more pressing form than that for the conventional quantization of selfadjoint forces via the trivial Lie product AB-BA.

III - Quantization via the functional analysis in abstract spaces. It is an easy prediction that the quantization of nonselfadjoint forces cannot be considered as completed until the operational character of the symbols at hand and the space in which they act (inclusive of the field itself-wheter that of complex numbers, or quaternions or octonions)are properly identified. This is the area in which this paper is most deficient. It is hoped that this aspect too is sufficiently studied in due time.

In relation to the physical arena of applicability of the considered formulations the following classification might have an epistemological value.

LIE ALGEBRAS-ATOMIC MECHANICS with dynamical equations

$$\dot b^{\mu} = \frac{1}{i\hbar}\left(b^{\mu}, H^{phys}\right)^{LA}_{AM} \overset{def}{\equiv} \frac{1}{i\hbar}\left(b^{\mu}H^{phys} - H^{phys}b^{\mu}\right), \tag{4.15.56}$$

FLEXIBLE LIE-ADMISSIBLE ALGEBRAS-NUCLEAR MECHANICS with dynamical equations

$$\dot b^{\mu} = \frac{1}{i\hbar}\left(b^{\mu}, H^{phys}\right)^{LA}_{NM} \overset{def}{=} \frac{1}{i\hbar}\left[b^{\mu}(1+\varepsilon)H^{phys} - H^{phys}(1-\varepsilon)b^{\mu}\right]; \tag{4.15.57}$$

GENERAL LIE-ADMISSIBLE ALGEBRAS-HADRONIC MECHANICS, with dynamical equations

$$\dot b^{\mu} = \frac{1}{i\hbar}\left(b^{\mu}, H^{phys}\right)^{LA}_{HM} \overset{def}{\equiv} \frac{1}{i\hbar}\left(b^{\mu}RH^{phys} - H^{phys}Sb^{\mu}\right), \tag{4.15.58}$$

where LA stands for the common algebraic character, Lie-Admissibility. Intriguingly, according to this view, each of the three algebraically meaningful classes of (nonassociative) Lie-admissible algebras (Table 4.14) has a corresponding layer of the microscopic world. It should be stressed that the approximate character of Eqs. (4.15.56) (being nonrelativistic) becames stronger in the transition to the covering equations (4.15.57) and (4.15.58) in the sense that, besides their nonrelativistic nature, these equations are based on the approximation of expected nonlocal forces not derivable from a potential via local forces of the same analytic character. A tentative identification of the arena of potential applicability of these latter equations is presented in Section 5.

We cannot close this table without touching on the representation of nonconservative systems with the conventional Heisenberg's equations but, of course, via generalized Hamiltonian structures. This possibility is offered by the very existence of generalized Schrödinger's equations (Table 4.8). Consider nonessentially nonselfadjoint systems satisfying the restrictions of Tables 4.5, 4.6 and 4.7. Suppose that the conventional canonical quantization of the generalized Hamiltonian representations produced by the Inverse Problem results into a Hermitian operator H^{gen} with Schrödinger-type equation (4.9.2). Then the conventional methods for the transition to Heisenberg's representation are applicable on formal grounds, yielding Heisenberg's equations

$$\dot{a}^{\mu} = \frac{1}{i\hbar}\left[a^{\mu}, H^{gen}\right] = \frac{1}{i\hbar}\left(a^{\mu}H^{gen} - H^{gen}a^{\mu}\right), \tag{4.15.59a}$$

$$\left[a^{\mu}, a^{\nu}\right] = a^{\mu}a^{\nu} - a^{\nu}a^{\mu} = i\hbar\,\omega^{\mu\nu}, \tag{4.15.59b}$$

$$\{a^{\mu}\} = \{\underset{\sim}{r}, \underset{\sim}{p}^{can}\}, \quad \underset{\sim}{p}^{can} = \frac{\partial L^{gen}}{\partial \dot{\underset{\sim}{r}}}. \tag{4.15.59c}$$

On a comparative basis between Eqs. (4.15.34) and (4.15.59), the latter possess the conventional structure of conservative quantum mechanics and therefore preserve the underlying methodology. Neverthless the algorithms at hand lose their direct physical significance as a necessary condition of physical consistency. The opposite situation occurs for the former equations in the sense that all the algorithms at hand have a direct physical significance by construction, but the applicable methodology must be necessarily generalized to account for the nonselfadjoint forces. It is hoped that a judicious interplay between both these equations, wheneber jointly applicable, might be of assistance in avoiding a most insidious trap of nonconservative mechanics: the physical meaning of the mathematical algorithms at hand.

But perhaps the most significant differentiation between the equations considered is that Eqs. (4.15.59) are vitally dependent on the capability of computing a Hamiltonian via the methods of the Inverse Problem and, thus, on the related integrability conditions for the existence of such Hamiltonian.[21] On the contrary, Eqs. (4.15.34) have been conceived to be independent from such a Hamiltonian representation of the complete system (inclusive of nonselfadjoint forces) and result to be characterized by the total energy (H^{phys}) and the nonselfadjoint forces responsible for its nonconservation (the R and S operators). Thus, the latter equations are potentially applicable also to the class of systems which violate the integrability conditions of the Inverse Problem.

Irrespective of these differentiations or possible complementarity, the reader should be aware of a number of problematic, yet unresolved aspects of Eqs. (4.15.59) whenever used for forces not derivable from a potential. These aspects have been studied in details in refs.[51]. Here is a brief summary. The Schrödinger's and Heisenberg's approach, by no means, exhaust the available possibility of quantum mechanical formulations. For instance, another significant approach is offered by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underset{\sim}{r}}} - \frac{\partial L}{\partial r} = 0, \tag{4.15.60}$$

of course here interpreted in an operational form equivalent to that of Heisenberg's equations. The reader should recall in this respect that the Lagrangian approach becomes fundamental in quantum field theory as currently known. Thus, the following remarks have a particular meaning for the quantization of nonselfadjoint field equations, besides that for Newton's equations.

One of the fundamental conditions to reach a consistent quantization via Eqs. (4.15.59) is the equivalence with Eqs. (4.15.60). The central problematic aspect we are here referring to is that such equivalence is generally lost at the quantum mechanical level, whenever nonselfadjoint forces are considered. It is here crucial to indicate that when the forces are selfadjoint, the quantum mechanical Lagrangian and Hamiltonian formulations are fully equivalent. Thus, the consistency problem we are here referring to is strictly dependent on the nonselfadjoint character of the forces and it is strictly a quantum mechanical occurrence.

More specifically, at the classical level, it is easy to prove that Lagrange's and Hamilton's equations are equivalent for all Lagrangians of class C^{∞} and regular and for corresponding Hamiltonians constructed via the conventional Legendre transform, irrespective of the functional structure of these functions. This implies that generalized structures of type (4. 6 . 1b) are fully admissible, without affecting the equivalence of the two approaches considered. This crucial aspect can, of course, be better studied within the context of the integrability conditions for the existence of a Lagrangian or a Hamiltonian and we refer the interested reader to refs.[21] for a detailed analysis.

In the transition to the quantum mechanical level, the situation results to be different. In turn, this is another indication of the fact stressed troughout our analysis, that conventional quantum mechanical knowledge can be considered solidly established at both the quantum mechanical and quantum field theoretical level ONLY for systems representable in their entirety with the trivial structure $L_{tot} = L_{free} + L_{int}$, that is, when the forces are derivable from L_{int}, while the case of more general systems demands a fundamentally broader approach.

First, it is essentially to indicate that the consistency problem under consideration is absent for the most representative force derivable from a potential, the Lorentz force. The conventional Lagrangian structure for the quantum mechanical treatment of this case has a (symmetrized) structure of the type (with L_{int} necessarily linear in the velocities)

$$L = \frac{1}{2}\alpha \dot{r}^2 + \frac{1}{2}\beta\{A(r), \dot{r}\} + \gamma\, G_0(r)\,; \alpha, \beta, \gamma \in F. \qquad (4.15.61)$$

The corresponding (symmetrized) Hermitian Hamiltonian is induced by the Legendre transform

$$H^{phys} = \frac{1}{2}\{p^{can}, \dot{r}\} - L. \qquad (4.15.62)$$

The quantum mechanical Lagrange's and Hamilton's equations

$$\left\{ \dot{p}^{can} = \frac{1}{i\hbar}[p^{can}, H^{phys}]\right\} \approx \left\{ \frac{d}{dt} p^{can} = \frac{\partial L}{\partial r}\right\}, \qquad (4.15.63)$$

can then be trivially proved to be equivalent. Conventional quantization techniques emerges again as being consistent for the case of the Lorentz forces. At a corresponding field theoretical level, conventional quantization techniques can correspondigly be proved to be consistent for gauge theories (whose Lagrangian densities L_{int} are also linear in the field derivatives).

As by no familiar, a sufficient condition for a force to be nonselfadjoint is that it is non linear in the velocity. We consider then the simplest possible analytic generalization of the Lorentz force, that is, a force which is quadratic in the velocity. This is precisely the simplest possible eletonic force of Section 3. By ignoring forces derivable from a potential because inessential for our context, the simplest possible generalized Lagrangian is therefore of the type

$$L^{gen} = \frac{1}{2}\dot{r}\, G(r)\, \dot{r}. \qquad (4.15.64)$$

The (Hermitian) canonical momentum is then given by

$$p^{can} = \frac{1}{2}\{\dot{r}, G(r)\}. \qquad (4.15.65)$$

The use of Lagrange's equations then yields the equations

$$\frac{d}{dt} p^{can} - \frac{\partial L^{gen}}{\partial r} = \dot{p}^{can} - \frac{1}{2}\dot{r}\frac{\partial G}{\partial r}\dot{r} = 0. \qquad (4.15.66)$$

Consider now the Hamiltonian operator (different Hermitian forms yield equivalent results)

$$H^{gen} = \frac{1}{2}\{p^{can}, \dot{r}\} - L = \frac{1}{4}\left(G\dot{r}^2 + \dot{r}^2 G\right)$$
$$= \frac{1}{4}\left(G^{\frac{1}{2}} p^{can} G^{-1} p^{can} G^{-\frac{1}{2}} + h.c.\right). \qquad (4.15.67)$$

Heisenberg's equations then yield

$$\dot{p}^{can} = \frac{1}{i\hbar}[p^{can}, H^{gen}] \qquad (4.15.68)$$
$$= \dot{p}^{can} - \frac{1}{2}\dot{r}\frac{\partial G}{\partial r}\dot{r} - \frac{\hbar^2}{4}\left[\frac{\partial G}{\partial r}\frac{\partial^2 G}{\partial r^2}G^{-3} - \left(\frac{\partial G}{\partial r}\right)^3 G^{-4}\right] = 0,$$

and, as such, they are inequivalent to Lagrange's equations (4.15. 66), unless a number of restrictions on the G-function are introduced. This is here interpreted as a rather nontrivial indication that conventional quantization rules which have proved to be consistent for the atomic structure and underlying forces, fail to apply to expected more general forces for the hadronic(as well as nuclear) structure and that the study of fundamentally broader quantization techniques should be undertaken for the intent of identifying the departures (rather than the preservation as much as possible) from conventional laws which are expected for the broader physical context considered.

The technical reasons for the breakdown of the equivalence of conventionally quantized Lagrange's and Hamilton's equations for nontrivial generalizations of the Lorentz force are studied in details in refs.[51] and they will not be reported here for brevity (they are related also to the breakdown of the quantum mechanical version of the chain rule of differentiation).

Most significantly for the analysis of this paper, the above occurrence indicates that, despite a number of papers on this subject, extreme scientific caution should be exercised in quantum field theory before claiming a consistent quantization of the so-called chiral Lagrangians

$$L^{gen} = \frac{1}{2} \varphi_i{}^{;\mu} G^{ij}(\varphi) \varphi_{j;\mu} + G^i_\mu(\varphi) \varphi_i{}^{;\mu} + G(\varphi) \qquad (4.15.69)$$

$$= 1, 2, \ldots, N \ ; \quad \varphi^{;\mu} = \frac{\partial \varphi}{\partial x^\mu}, \quad G^{ij} = G^{ji}, \quad \det(G^{ij}) \neq 0,$$

via customary quantization techniques which are established for the trivial structure $L_{tot} = L_{free} + L_{int}$. Indeed, Lagrangians (4.15.69) are precisely representative of genuine nonselfadjoint field equations (3.5.9a), that is, field equations with nonlinear derivative couplings. The reader is here encouraged to properly interpreted the implications of this occurrence. Strictly speaking, <u>the inconsistency here considered for the conventional quantization of Lagrangians of type (4.15.69) implies that the entire methodological horizon of conventional quantum field theory (propagators, Feynman diagrams, scattering amplitudes, dispersion relations, etc.) is in question.</u> A detailed critical inspection of this quantum field theoretical profile will not be conducted here to avoid a prohibitive length of the paper.

In conclusion, interested and receptive researchers are discouraged from the use of the conventional Heisenberg's equations (4.15.59) for nonselfadjoint forces until the problematic aspects here considered are resolved in their entirety. Almost needless to say, a similar scientific caution is suggested for our Lie-admissible equations (4.15.34) until studied to the necessary extent.

<u>TABLE</u> 4.16: <u>KTORIDES LIE-ADMISSIBLE QUANTIZATION.</u>

As indicated earlier (and as studied in details in ref.[22]) the notion of Lie-admissibility

$$AB - BA = \text{LIE}, \qquad AB = \text{associative} \qquad (4.16.1)$$

is at the very foundation of Lie's theory, only expressed in its simplest possible form, the associative form. In turn, the same notion is clearly at the foundation of quantum mechanics as currently known (the atomic mechanics in our terminology).

The study of the more general notion of Lie-admissibility

$$(A,B) - (B,A) = \text{LIE}, \qquad (A,B) = \text{nonassociative} \qquad (4.16.2)$$

brings into focus a number of aspects of the abstract approach to Lie's theory and, thus, of quantum mechanics, which are customarily ignored in conventional treatments.

The most significant of these aspects is the crucial physical and mathematical role of the truly fundamental algebra, the <u>universal enveloping associative algebra</u> $\mathcal{A}$ of a Lie algebra $\mathcal{L}$ [46]. This role is properly identified in the mathematical literature, but conventional treatments in the physical literature on both Lie's theory <u>and</u> quantum mechanics are often conducted without reference to the envelop $\mathcal{A}$. It then follows that quantities of fundamental physical significance, such as the square of the total angular momentum J^2, remain algebraically undefined because they are clearly outside the Lie algebra itself, the SU(2) algebra for the case considered, while the algebra which actually allows the computation of the product for physical applications, the associative envelop, ($J^2 = 0$ for SU(2)), has not been properly treated.

In the transition to the Lie-admissible quantization of nonselfadjoint forces this occurrence acquires its proper light. Indeed, now the product (A, B) is not even Lie, to begin with. Thus, its interpretation as the product of a nonassociative but Lie-admissible envelop of a Lie algebra becomes crucial for an in depth study.

In short, <u>in the transition from the Lie quantization of selfadjoint systems to the Lie-admissible quantization of the more general nonselfadjoint systems it is the universal enveloping associative algebra of the former which is actually mapped into a nonassociative Lie-admissible genotope.</u>

In turn, this purely algebraic aspect has crucial physical implications related to the problem whether fundamental physical quantities such as $\underset{\sim}{J}^2$, under the presence of nonselfadjoint forces, should be computed in terms of the associative Lie-admissible envelop, i.e.,

$$\underset{\sim}{J}^2 = J_x J_x + J_y J_y + J_z \quad , \quad J_x J_x = \text{Associative}, \qquad (4.16.3)$$

or in terms of the nonassociative Lie-admissible envelop, i.e.,

$$\underset{\sim}{J}^2 = (J_x, J_x) + (J_y, J_y) + (J_z, J_z), \quad (J_x, J_x) = \text{Nonassociative}. \qquad (4.16.4)$$

Still in turn, this has fundamental physical implications with respect to a crucial part of our analysis, to be studied in Table 4.19, whether in the transition from motion in vacuum under electromagnetic forces to motion within a hadron under electromagnetic and strong nonselfadjoint forces, the conventional atomic notion of perennial value of the spin persists or it is subject to a finite mutation.

A number of fundamental contributions for the proper treatment of these issues have been made by C. N. KTORIDES in a first paper of 1975[47] and in a recent paper of 1978[52]. In essence, this author has first indicated the existence of a consistent nonassociative Lie-admissible generalization of the Poincaré-Birkhoff-Witt theorem (which is at the foundation of the theory of $\mathcal{A}$), i.e., the Poincaré-Birkhoff-Witt-Ktorides theorem of ref.[23] in which the approach is extended to general Lie-admissible envelops. As we shall indicate during the course of our analysis, this first contribution by Ktorides has fundamental physical implications because it established the rigorous methodological context for the proper computation of powers of the of the elements of the basis. A second contribution by Ktorides has been the reinspection of Dirac's quantization for the conventional case of selfadjoint systems to identify the role of the associative envelop $\mathcal{A}$ for the proper quantum mechanical treatment of powers of operators.[52] The emerging broader quantum mechanical context recovers the results of Dirac's quantization in their entirety. We simple have the shift of the methodological emphasis from a Lie algebra $\mathcal{L}$ (with consequential exclusion of all polynomials forms in its basis, including the Casimir invariants) to its universal enveloping associative algebras (with consequential inclusion of the Lie algebra basis plus all polynomial functions of the elements of the basis in general, and of the Casimir invariants

in particular). A third contribution by C. N. Ktorides has been the embedding of such (quantum mechanical) associative envelop into a flexible Lie-admissible extension[52] for the case of nonlinear field theories, as one way to circumvent the known restrictions of conventional axiomatic field theory toward the construction of a genuine interacting field theory with (in principle) unrestricted couplings, via the minimal possible mutilation of established notions (the nontriviality of Ktorides approach is that the emerging quantum fields "cut through" Borchers classes). Most importantly, this approach avoids the problematic aspects indicated in the last part of Table 4.15 because it is based on the conventional canonical quantization of the free fields (the difficulties indicated emerges when attempting the quantization of the true interacting field under nonselfadjoint couplings). In other words, the starting ground is the established free ground and quantum field theoretical effects are introduced via a Lie-admissible genotope of the underlying associative algebra of (time or normal orderings of) polynomials in the fields. This is precisely in line with the idea independently introduced by R. M. SANTILLI[23] of representing nontrivial forces or couplings via Lie-admissible formulations, but now specifically worked out in details for quantum field theory.

We shall call this approach to quantum field theory Ktorides Lie-admissible quantization. Regrettably, we cannot review this approach at the necessary technical level to avoid a prohibitive length of this paper. We shall therefore restrict ourselves to only those aspects which are essential for the completion of the analysis of this paper.

In essence, the emerging methodological context, at the abstract (classical and quantum mechanical) level, can be classified into the following three layers.

I- UNIVERSAL ENVELOPING ASSOCIATIVE ALGEBRA OF A LIE ALGEBRA.[46] This is the conventional Lie's context. The basic algebra is the associative envelop $\mathcal{A}$. The costumarily used Lie algebra is constructed via the attached algebra $\mathcal{A}^-$ of $\mathcal{A}$. The crucial Poincaré-Birkhoff-Witt theorem provides a basis of $\mathcal{A}$ given by the cosets of I and the standard monomials. Under certain technical steps, such a basis is expressible in terms of the quantities

$$\mathcal{A}: 1, \; X_i, \; \underset{i \le j}{X_i X_j}, \; \underset{i \le j \le k}{X_i X_j X_k}, \; \dots, \quad i, j, k = 1, 2, \dots, n, \qquad (4.16.5)$$

and, as such, it is infinite-dimensional already at the Lie level. The basis of A^- (here tacitly assumed to be finite-dimensional), i.e., $\{ X_i \mid i = \text{ordered set of } (1, n) \}$

is of course contained in full in the basis of $\mathcal{A}$. The dynamical law, i.e., the time evolution law, is recovered via the Lie-admissible-associative rule with respect to the generator X = H. The underlying dynamical equations are Hamilton's equations for the classical case and Heisenberg's equations for the quantum mechanical case. Dirac's quantization is extended to $\mathcal{A}$. This includes the conventional Dirac's quantization of $\mathcal{L}$ and extends it to arbitrary elements of $\mathcal{A}$

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \cdots X_{i_m}^{k_m}, \quad k_1, k_2, \ldots, k_m = 0, 1, 2, \ldots, \quad i_1, i_2, \ldots, i_m = 1, 2, \ldots, n \tag{4.16.6}$$

which are outside of $\mathcal{L}$ and, thus, cannot be, strictly speaking, quantized within the context of $\mathcal{L}$ only. This is the first part of Ktorides analysis of ref.[52]

R. M. SANTILLI[23] has independently pointed out that the Poincaré-Birkhoff-Witt theorem can be extended to include the isotopic degrees of freedom of the associative product of $\mathcal{A}$. The emerging basis is given by the cosets of 1 and the so-called standard isotopically mapped monomials, that is, the monomials of $\mathcal{A}$ under the product of the isotope $\mathcal{A}^*$. Upon certain technical steps, such a basis is given by

$$\mathcal{A}^*: 1, \; X_i, \; X_i * X_j, \; X_i * X_j * X_k, \ldots \quad i \leq j, \quad i \leq j \leq k \tag{4.16.7}$$

and, as such, remains infinite-dimensional. It is simply expressed in terms of the basis of the original Lie algebra $\mathcal{L}$ but now referred to isotopically mapped associative products of type (4.15.46). Elements of the type (4.16.6) have now no meaning within the context of $\mathcal{A}^*$ and they are replaced by

$$X_{i_1}^{k_1} * X_{i_2}^{k_2} * \cdots * X_{i_m}^{k_m}, \tag{4.16.8}$$

where powers are computed in $\mathcal{A}^*$, i.e.,

$$X_{i_1}^{k_1} = X_{i_1} * X_{i_1} * \cdots * X_{i_1}, \quad k_1\text{-times} \tag{4.16.9}$$

while no product ambiguity arises because the * product is associative as a condition for the isotopic nature of the construction.

The mathematical nontriviality of the approach is expressed by the fact that $\mathcal{A}^{*-}$ is generally <u>nonisomorphic</u> to $\mathcal{A}^-$. This is, in essence, a characterization of the property according to which nonisomorphic Lie algebras (e.g., SO(3) and SO(2.1)) can be realized in terms of the <u>same generators but different Lie products</u>, that is, isotopically mapped Lie products, as illustrated in details in ref.[23]

The physical significance of this isotopic extension of the Poincaré-Birkhoff-Witt theorem appears to be of diversified nature. First of all, the extension is essential for Lie-admissible formulations, as indicated below. Secondly, the extension appears significant for the classical and quantum mechanical treatment of the very conservative systems. One area is the nonuniqueness of an exact Lie symmetry for the characterization of physical conservation laws. For instance the isotopically mapped Hamiltonians

$$H^{phys} = \frac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}\left(r_x^2 + r_y^2 + r_z^2\right), \tag{4.16.10a}$$

$$H^* = \frac{1}{2}\left(p_x^2 - p_y^2 + p_z^2\right) + \frac{1}{2}\left(r_x^2 - r_y^2 + r_z^2\right), \tag{4.16.10b}$$

trivially represent the same system. Neverthless the symmetry which induces, via Norther's theorem, the conservation of the angular momentum is SO(3) for H^{phys} and SO(2.1) for H*. In this case the construction of the conventional Lie algebra of SO(2.1) and its envelop $\mathcal{A}$(SO(2.1)) would yield inconsistent physical results in the sense of producing generators other than those actually conserved, the angular momentum components. Thus, to achieve a mathematical characterization of the physical conservation law of the angular momentum for H*, the generators of SO(2.1) must be the components of the conventional angular momentum. To achieve with these generators a true SO(2.1) algebra, rather than SO(3), the only possibility is that of performing a change of the Lie product, i.e., our Lie isotopic mapping. This inevitably demands the construction of an associative envelop $\mathcal{A}^*$(SO(3)), that is, an algebra induced by the generators of SO(3) (the angular momentum components), but referred to an isotopically mapped associative product (rather than the conventional associative product M_iM_j). Under a suitable selection of such isotopic image of the product, worked out in details in ref.[23], the attached algebra $\mathcal{A}^{*-}$ is isomorphic to SO(2.1) and not to SO(3).

The dynamical law, i.e., the time evolution law, is recovered also via the Lie-admissible associative rule, but now referred to a generalized Hamiltonian H^{gen} as generator. The underlying analytic equations are therefore Birkhoff's equations, and not Hamilton's equations, owing to the isotopically mapped nature of the product.

Conceivably, an isotopic image of Ktorides reinspection of Dirac's quantization could produce the quantization of Birkhoff's equations, our Eqs. (4.15.49).

II. FLEXIBLE LIE-ADMISSIBLE GENOTOPIC MAPPING OF THE ENVELOPING ASSOCIATIVE ALGEBRA OF A LIE-ALGEBRA. This is the context of the generalization of the Poincaré-Birkhoff-Witt theorem of refs[47,23]. The basis is given by the union of the F -linearly independent standard genotopically mapped monomials and standard isotopically mapped monomials. Upon a number of technical steps, they can be expressed in terms of the union of independent combinations of all possible associations of the type

$$\mathcal{U}_{\lambda,\mu}: 1,\ X_i,\ X_i\cdot X_j,\ (X_i\cdot X_j)\cdot X_k,\ X_i\cdot(X_j\cdot X_k),\ \dots \qquad (4.16.11)$$

$$X_i * X_j,\ i\le j, \quad X_i * X_j * X_k,\ i\le j\le k, \quad X_i * X_k = \alpha X_i X_j,\ \alpha\in F$$

where

$$X_i\cdot X_j = \lambda X_i\cdot X_j - \mu X_j X_i,\quad \lambda\neq\pm\mu, \qquad (4.16.12)$$

is the flexible Lie-admissible mapping of the product of $\mathcal{A}$. The use of the conventional associative product for the characterization of powers has no meaning for this context (it would imply the exact $\mathcal{L}$ symmetry or the lack of nonselfadjoint forces). An arbitrary element of $\mathcal{U}$ is therefore expressible in the form

$$X_{i_1}^{k_1}\cdot X_{i_2}^{k_2}\cdot\ldots\cdot X_{i_m}^{k_m},\quad X_i^k = X_i\cdot X_i\cdots X_i,\ k\text{-times}. \qquad (4.16.13)$$

A most relevant aspect of $\mathcal{U}$ is that $\mathcal{U}^-\simeq\mathcal{A}^-$, that is, the Lie algebra attached to $\mathcal{U}$ is isomorphic to the original Lie algebra attached to $\mathcal{A}$. This is due to the fact that product emerging from the Lie-admissible nonassociative rule

$$[X_i, X_j]_{\mathcal{U}} = (\lambda+\mu)\,[X_i, X_j]_A \qquad (4.16.14)$$

is the conventional Lie product multiplied by a constant. In turn this constant can be trivially eliminated via the isotopy

$$\mathcal{A}: X_i X_j \longrightarrow \mathcal{A}^*: X_i * X_j = (\lambda+\mu) X_i X_j \qquad (4.16.15)$$

Then $\mathcal{U}^-$ formally coincides with $\mathcal{A}^{*-}$.

The dynamical content of the approach is expressed by the parameters of the mutation,

λ and μ, which are directly representative of the nonselfadjoint forces, along the lines of the preceding table. The dynamical equations are Lie-admissible nonassociative at both the classical and quantum mechanical level, although the approach must be interpreted as producing, in general, an approximation of the equations of motion.

Ktorides Lie-admissible quantization in field theory is based on the mapping[52]

$$\mathcal{A}\longrightarrow\mathcal{U}_{\lambda,\mu} = \mathcal{A}(\lambda,\mu), \qquad (4.16.16)$$

where polynomials in the fields are computed via rule (4.16.13). The interested reader is urged to work out in details the reduction of Ktorides quantization to the case of a Lie-admissible flexible quantum mechanics. As we shall indicate later on, this appears as particularly promising in nuclear physics for $\lambda = 1+\varepsilon$, $\mu = 1-\varepsilon$.

It should be here recalled that the algebras $\mathcal{A}(\lambda,\mu)$ were introduced by R. M. SANTILLI[27] to work out the construction of the Gell-Mann-Okubo mass formula without recursion to the associative envelop $\mathcal{A}$ (SU(3)) which, strictly speaking, can only characterize an exact, equal-mass, SU(3) symmetry. Its embedding into $\mathcal{A}(\lambda,\mu)$, while reproducing the Gell-Mann-Okubo mass formula, assured that the underlying SU(3) context is actually broken.[42]

For further comments along these lines, ses Table 4.19.

III-GENERAL LIE-ADMISSIBLE GENOTOPIC MAPPING OF THE ENVELOPING ASSOCIATIVE ALGEBRA OF A LIE ALGEBRA. This constitutes the most general possible extension of Ktorides approach to enveloping algebras which preserve the fundamental rule of Lie-admissibility, as summarised in ref.[23] and worked out in details by R. M. SANTILLI[22-II]. The emerging generalization of the Poincaré-Birkhoff-Witt theorem produces a basis as the cosets of I and the union of F*-linearly independent standard genotopically mapped and standard isotopically mapped monomials. Upon a number of technical implementations, such a basis can be expressed as the union of the independent elements of the type

$$\mathcal{U}: 1,\ X_i,\ X_i\cdot X_j,\ (X_i\cdot X_j)\cdot X_k,\ X_i\,(X_j\cdot X_k),$$

$$X_i * X_j,\ i\le j, \quad X_i * X_j * X_k,\ i\le j\le k, \qquad (4.16.17)$$

where

$$X_i \cdot X_j = X_i R X_j - X_j S X_i, \quad R, S = \text{fixed}, \quad (4.16.18)$$

is our Lie-admissible product of general type here interpreted in its abstract form, and

$$X_i * X_j = X_i (R+S) X_j - X_j (R+S) X_i \quad (4.16.19)$$

is the associated isotope.

The dynamical content of the approach is expressed by the R and S elements as representative of forces not derivable from a potential or as forces which break the Lie symmetry of the original algebra $\mathcal{L}$. The dynamical equations result to be our Hamilton-admissible equations at the classical level and, expectedly, our Lie-admissible covering of Heisenberg's equations at the quantum mechanical level.

The emerging nonassociative nenvelop $\mathcal{U}$ is a Lie-admissible genotope of the original algebra $\mathcal{A}$. This implies that it preserves the generators of $\mathcal{L}$ as the fundamental building blocks and implements them with a mutation of the algebraic structure to represent the broader physical context they are referred to. Unlike the case of the flexible mutation, the Lie algebra attached to $\mathcal{U}$, i.e., $\mathcal{U}^-$, is no longer necessarily isomorphis to the original algebra $\mathcal{L}$. This is clearly due to the nontrivial structure of the isotope (4.16.19). In turn, this implies that the breaking of the original Lie algebra is much deeper, to the point that such an algebra can only be recovered at the limit $\mathcal{U} \rightarrow \mathcal{A}$, i.e., at the limit of null symmetry breaking forces.

As we shall indicate later on, these generalizations of the Poincaré-Birkhoff-Witt theorem have fundamental physical implications, e.g., for the problem of the relativity which is applicable to the quantum mechanical treatment of nonselfadjoint forces (Table 4.18) or the notion of spin under strong nonselfadjoint hadronic forces (Table 4.19). It is hoped that Ktorides Lie-admissible quantization (of flexible type) will result to be extendable to the general Lie-admissible algebras. It is also hoped that the rather involved technical profile is treated by mathematicians for the resolution of the mathematical issues at the mathematical level. In any case, as stressed in ref.[23], the approach here considered via enveloping algebras represents the true technical characterization of the notion of Lie-admissibility on both mathematical and physical grounds.

TABLE 4.17: THE EMERGENCE OF THE JORDAN ALGEBRAS FOR THE QUANTUM MECHANICAL TREATMENT OF NONCONSERVATIVE FORCES.

The Jordan algebras, has is well known, had their birth in the studies by P. JORDAN[53] of 1932 and 1933 concerning certain statistical aspects of the measurement theory. The celebrated subsequent paper by P. JORDAN, J. von NEUMANN and E. WIGNER of 1934[54] studied these algebras in great details under the name of "r-number algebras", which later on became known under the name of "Jordan algebras". The title of the latter paper, "On an algebraic generalization of the quantum mechanical formalism", clearly indicates the desire by these authors of attempting the reformulation of the foundations of quantum mechanics via the nonassociative and commutative product

$$J: \quad \{A, B\} = \frac{1}{2}(AB + BA), \quad AB = \text{Associative}. \quad (4.17.1)$$

It is here appropriate to quote the first paragraph of the latter paper

> "One of us has shown that the statistical properties of the measurements of a quantum mechanical system assume their simplest form when expressed in terms of a certain hypercomplex algebra which is commutative but not associative. This algebra differs from the noncommutative but associative matrix algebra usually considered in that one is concerned with the commutative expression $\frac{1}{2}$(AxB + BxA) instead of the associative product AxB of two matrices. It was conjectured that the laws of this commutative algebra would form a suitable starting point for a generalization of the present quantum mechanical theory. The need of such a generalization arises from the (probably) fundamental difficulties resulting when one attempts to apply quantum mechanics to questions in relativistic and nuclear phenomena." [55]
>
> P. JORDAN, J. von NEUMANN and E. WIGNER (1934)

Since that time, the Jordan algebras have been subjected to intensive studies from both a mathematical and a physical profile. Within the former context the Jordan algebras have actually generated an entirely new line of study of the theory of Abstract Algebras and have nowadays reached a degree of sophystication comparable to that of the Lie algebras.[56] Within the latter context the originally intended objective (a generalization of quantum mechanics) has remained fundamentally unachieved as of now. Nevertheless, the Jordan algebras have seen their appearance in contemporary theoretical physics, mainly in relation to the problem of the hadronic structure, as a result of the contributions by a number of authors such as, A. PAIS (1961),[57] H. H. GOLDSTINE and L. P. HORWITZ (1962) and (1964),[58] A. GAMBA (1965),[59] L. P. HORWITZ and L. C. BIEDENHARN (1965),[60] M. GÜNAYDIN and F. GÜRSEY (1973),[61] and others. For a recent account, including a list of more recent contributions, see for instance M. GÜNAYDIN (1977).[62]

Despite these quite valuable studies, it does not appear that the Jordan algebras have reached a level of physical applications which is comparable to that of Lie algebras, up to this moment. An epistemological study of this occurrence is here of some significance. In essence, as stressed throughout our analysis, in order for any algebra to become solidly established in physics, it must exhibit a direct dynamical origin via the brackets of the time evolution law, as it is the case for the Lie approach to selfadjoint systems at both the classical and quantum mechanical level. If this fundamental dynamical origin is not established, physical applications are indeed still possible. Nevertheless, they cannot achieve a fundamental relevance, in the sense of replacing the Lie algebras and their universal enveloping associative algebras.

At the classical level, a study of the problem by R. M. SANTILLI[22-II] indicates a number of rather substantial difficulties in the attempt of achieving a time evolution law whose brackets satisfy the Jordan algebra identities

$$\{A,B\} = \{B,A\}, \qquad (4.17.2a)$$

$$\{\{\{A,A\},B\},A\} = \{\{A,A\},\{B,A\}\}. \qquad (4.17.2b)$$

The reasons are of both physical and algebraic nature. Under the assumption that the brackets are totally symmetric (commutative), there exist physical difficulties in constructing a time evolution law of the conventional structure

$$\dot{A} = \{A, H^{gen}\}_{classical}, \qquad (4.17.3)$$

that is, via the "anticommutator" with a function H^{gen} which substitutes the Hamiltonian H^{phys} of the Lie time evolution law. Assuming that this difficulty is resolved, there exist rather considerable mathematical difficulties in identifying an explicit form of the product

$\{A, H^{gen}\}$. They are essentially due to the fact that brackets of the type

$$\{A,B\} = \frac{\partial A}{\partial r}\frac{\partial B}{\partial p} + \frac{\partial A}{\partial p}\frac{\partial A}{\partial r}, \qquad (4.17.4)$$

even though commutative and, thus, satisfying law (4.17.2a), do not characterize a Jordan algebra because they violate the Jordan law (4.17.2b) as the reader is encouraged to verify.

At the quantum mechanical level (still for selfadjoint systems), the studies by C. N. KTORIDES[52] can be used to identify the reasons why the original objective by P. Jordan, J. von Neumann and E. Wigner (the replacement of the associative product of quantum mechanics with the commutative nonassociative product (4.17.1)) has not been consistently achieved. Indeed, by following Ktorides reinspection of Dirac's quantization (Table 4.16), we have seen that the truly fundamental algebraic structure in quantum mechanics is the universal enveloping associative algebra $\mathcal{A}$. The dynamical content of the theory, that is, the time evolution law, is then recovered via the trivial (associative) Lie-admissible rule $\mathcal{A} \to \mathcal{A}^-$. Now, Ktorides flexible mutations $\mathcal{A}(\lambda, \mu)$ of an associative enveloping algebra $\mathcal{A}$ do indeed contain the Jordan algebras as a particular case, trivially, for $\lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$. The crucial point is that, under these values, the nonassociative envelop $\mathcal{A}(\lambda, \mu)$ becomes singular in the sense of losing its enveloping character. This is trivially due to the very commutative nature of the Jordan product for which

$$[A, B]_J = \{A,B\} - \{B,A\} \equiv 0. \qquad (4.17.5)$$

Thus, the Jordan algebras, under the Lie-admissible rule, yield a zero Lie algebra, that is, an algebra which is not physically meaningful. The net effect is the loss of the dynamical content of the theory as characterized via the associative Lie-admissible rule. We can therefore conclude by saying that <u>a fundamental generalization of quantum mechanics via the replacement of the associative product with the Jordan product in such a way to preserve the conventional, Lie, dynamical content, is impossible.</u>

All this occurs for the conventional treatment (via the trivial Lie product AB-BA) of conventional systems (selfadjoint).

In the transition to the quantum mechanical treatment of nonselfadjoint (nonconservative) systems, the situation is profoundly different. It is a pleasure for me to report that for these broader systems there is the apparent emergence of the Jordan algebras as methodological tools of fundamental significance, perhaps, equal to that of the Lie algebras.

To present this occurrence, the first point which should be stressed is that such an emergence is not direct, that is, via a generalized time evolution law for nonconservative systems of type (4.17.3) which is Jordan in algebraic character. As a matter of fact, it is possible to prove that, under the condition that the algorithms at hand possess a direct physical significance (linear momentum, energy, angular momentum, etc.) the algebra characterized by the time evolution law cannot be Jordan. The proof is essentially the same as that of the corresponding exclusion, under the indicated conditions, of the Lie algebras. In essence it emerges that, to consistently represent the equations of motion, such a product must be neither totally symmetry (commutative) nor totally antisymmetric (anti-commutative). The argument therefore implies a simultaneous, exclusion of both the Lie algebras and the Jordan algebras for the direct algebraic characterization of nonconservative systems under the condition of direct physical significance of the algorithms at hand.

The emergence of the Jordan algebras we are here referring to is instead of indirect nature. But this does not diminish its fundamental character. To understand this occurrence, let me here recall that the Lie algebras are not lost in the Lie-admissible treatment of nonselfadjoint systems. Instead, they preserve their fundamental methodological value via the rule of Lie-admissibility, in the sense that any admissible generalized product (A, B) of the necessarily broader time evolution law for the systems considered must be able to characterize a Lie algebra via the rule (A, B) - (B, A) as a crucial prerequisite of physical consistency. It then follows that the Lie algebras have a fundamental methodological function for the study of the covering Lie-admissible algebras, as elaborated in details in refs[22,23]

The emergence of the Jordan algebras in the Lie-admissible approach to nonseldajoint systems is fully parallel to that of the Lie algebras. We simply have that the Lie-admissible product (A, B) of our generalization (4.15.34) of Heisenberg's equations exibits a precise Jordan algebra content in the sense that the attached algebra (A, B) + (B, A) does indeed characterize a fully admissible realization of the Jordan algebras. The net effect is that the Jordan algebras emerge in this way jointly with the Lie algebras and via complementary criteria of construction. It then follows that the Jordan and Lie algebras have equivalent methodological functions in our Lie-admissible approach to nonselfadjoint systems. The original difficulties which have prohibited the emergence of the Jordan algebras at a fundamental

level of quantum mechanics, as indicated earlier, are now circumvented by our main algebraic idea for the characterization of the systems considered: the algebraic character of the time evolution law must be non-Lie as the fundamental dynamical prerequisite under broader forces. Therefore, the emergence of a physically consistent presence of the Jordan algebras in our approach is a direct consequence of the complete loss of dynamical functions of the Lie algebras and their restriction to a methodological function only. Indeed, the reader can see that, whenever the Lie product reacquires its dynamical meaning as characterizing the product of the time evolution law, the Jordan algebra content is lost.

In the following we shall present the rudiments of the algebraic characterization of the Jordan algebras in our treatment of nonselfadjoint systems.

In his paper of 1948, quite significantly, A. A. ALBERT[43] introduced jointly the notion of Lie-admissibility and of Jordan-admissibility, although without an extensive treatment. The notion of Lie-admissibility according to Albert has been recalled in Table 4.15. That of Jordan-admissibility according to the same author is the follow.

A Jordan-admissible algebra U over a field F (again assumed here of characteristics zero) is a vector space of elements A, B, C, ... equipped with the (abstract) product AB such that the attached algebra U^+, which is the same vector space as U (that is, the elements of U and U^+ are the same) but equipped with the product

$$\{A, B\}_U = \frac{1}{2}(AB + BA), \qquad (4.17.6)$$

is a (commutative) Jordan algebra. Clearly, associative algebras are Jordan-admissible. As a matter of fact, associative algebras constitute the fundamental algebras for the construction of a class of Jordan algebras called special. By recalling the analysis of Table 4.14, we can more generally say that associative algebras are jointly Lie-admissible and Jordan-admissible. The important point for the context of this table is that the algebra U can also be nonassociative, but such that U^+ is Jordan. A trivial example is given by product (4.17.1). Thus, Jordan algebras are Jordan-admissible. However, they constitute a trivial class of nonassociative Jordan-admissible algebras because their product is commutative. The physically most significative Jordan-admissible algebras are therefore those which are neither commutative nor anticommutative. The subclass of these (non-associative) algebras in which we are interested is, in particular, that which is jointly Lie-admissible and Jordan-admissible.

If we recall that the universal enveloping associative algebras are precisely of this type (jointly Lie- and Jordan-admissible), we expect that this algebraic character is preserved by the general and flexible Lie-admissible genotopes of the associative envelop, because such a property is expected to be a general property of all enveloping algebras of a Lie algebra.

A study of the problem indicates that this is indeed the case. According to R. M. SANTILLI[27] the Jordan-admissible algebras can be classified into the following three layers, parallel to the corresponding layers of the Lie-admissible algebras.

I. General Jordan-admissible algebras. These are all algebras U over F verifying the law

$$(A^2B)A + A(BA^2) + (BA^2)A + A(A^2B) = A^2(BA) + (AB)A^2 + A^2(AB) + (BA)A^2, \tag{4.17.7}$$

here called general Jordan-admissible law.

II. Flexible Jordan-admissible algebras. These are all algebras U over F verifying the laws

$$(AB)A = A(BA), \tag{4.17.8a}$$

$$(A^2B)A + A(A^2B) = A^2(BA) + A^2(AB), \tag{4.17.8b}$$

where Eq. (4.17.8a) is the flexibility law (4.14.6a) and Eq. (4.17.8b) is called the flexible Lie-admissible law and it is a particularization of the general law under the flexible property.

III. Jordan algebras (of commutative type). These are all algebras U over F verifying laws (4.17.2), i.e.,

$$AB = BA, \tag{4.17.9a}$$

$$(A^2B)A = A^2(BA). \tag{4.17.9b}$$

The reader is urged to verify that, in full analogy to the corresponding Lie case, the flexible and general laws of Jordan-admissibility constitute an algebraic covering of the Jordan laws. Thus, most significant on physical grounds is the property that the flexible law

is a covering of both the commutative and the anticommutative laws.

Our next objective is that of the realization of the product. The reader should again recall that such a product, for given algebraic laws, is not uniquely characterized and several different products can satisfy the same laws. We shall therefore assume below the indicated products as fundamental mainly on physical (rather than algebraic) grounds.

I'. Fundamental realization of the product of general Jordan-admissible algebras. It is given by

$$(A,B) = ARB - BSA, \quad \begin{array}{l} R, S = \text{fixed}, \\ AR, \text{etc} = \text{Associative}, \\ R \neq \pm S \text{ and non-singular}, \end{array} \tag{4.17.10}$$

that is, it coincides with the corresponding form of the general Lie-admissible algebras which therefore emerge as being jointly Lie-admissible and Jordan-admissible as desired. The proof that product (4.17.10) does indeed satisfy law (4.17.7) is a tedious but instructive exercise for the interested reader. A more straighforward proof is given by computing the Jordan-admissible rule

$$\{A,B\}_U = \frac{1}{2}\{A(R-S)B + B(R-S)A\} \tag{4.17.11}$$

and then showing that such commutative product obeys the Jordan law (4.17.2b). Notice that we could have considered the product

$$(A,B) = ARB + BS'A \tag{4.17.12}$$

instead of product (4.17.10). But, since $R \neq \pm S$, products (4.17.10) and (4.17.12) are algebraically equivalent.

II'. Fundamental realization of the product of the flexible Jordan-admissible algebras. It is given by

$$(A,B) = \lambda AB - \mu BA, \quad \begin{array}{l} \lambda \neq \pm\mu \\ \text{and non-null} \\ AB = \text{Assoc.} \end{array} \tag{4.17.13}$$

that is, it coincides with the product of the flexible Lie-admissible algebras which therefore also emerge as being (in the considered realization of the product) jointly Lie-admissible and Jordan-admissible. Again we could have written

$$(A,B) = \lambda AB + \mu' BA, \tag{4.17.14}$$

But, since $\lambda \neq \pm\mu$ and their value is arbitrary (at this algebraic level), products (4.17.13) and (4.17.14) are algebraically equivalent. It should be here recalled that the realization of the product of the Jordan-admissible algebras of class II which has been most extensively used in the mathematical literature is that of the λ-mutation algebras, Eq. (4.14.19), i.e.,

$$(A,B) = \lambda AB + (1-\lambda)BA, \quad AB = \text{Assoc.} \qquad (4.17.15)$$

However, we have excluded this product on physical grounds, owing to its inability to recover the Lie product under a physically meaningful limit.

III'. Fundamental realization of the Jordan algebras. It is given by the familiar form

$$\{A,B\} = \frac{1}{2}(AB+BA), \quad AB = \text{Assoc.} \qquad (4.17.16)$$

The reader should be aware that this product is admissible only for the special commutative Jordan algebras (and not for the exceptional commutative Jordan algebras which are related to the exceptional simple Lie algebras). Also, the reader should by aware that the terms "Jordan algebras" of this table refer to the "commutative Jordan algebras". The so-called noncommutative Jordan algebras have been extensively studied in the literature and they often result to be jointly Lie-admissible and Jordan-admissible (e.g., the quasiassociative algebras).[22-II]

In conclusion, the proposed hadronic mechanics (as well as nuclear mechanics) with generalized time evolution law

$$\dot{A} = \frac{1}{i\hbar}(A,H) = \frac{1}{i\hbar}(ARH - HSA) \qquad (4.17.17)$$

for finite strong hadronic (infinitesimal strong nuclear) nonselfadjoint forces results to be of joint Lie-admissible and Jordan-admissible character. This implies the joint emergence of the Lie algebras and of the Jordan algebras via the corresponding rules

$$[A,B]_u = (A,B) - (B,A) = A(R+S)B - B(R+S)A, \qquad (4.17.18a)$$

$$\{A,B\}_u = \frac{1}{2}[(A,B) + (B,A)] = \frac{1}{2}[A(R-S)B + B(R-S)A], \qquad (4.17.18b)$$

which therefore acquire a complementary, fundamental, methodological function.

A presentation of the current state of the art on the methodological function of the Jordan algebras for the study of the Lie-admissible algebras (via use of idempotents, Pierce decomposition, radical, etc.) is given by R. M. SANTILLI in the monograph[22-II].

Expectedly, the Jordan algebras do not emerge in the form of the product originally conceived by P. Jordan, Eq. (4.17.1). Instead, they emerge in the generalized form (4.17.18b) which is apparently proposed here for the first time. The reader should be aware that this demands a reinspection and a suitable generalization of a rather significant portion of the theory of Jordan algebras, to properly account for the generalized structure of the product. It is regrettable that the virtual totality of the rather vast amount of literature on both Lie algebras and Jordan algebras is restricted to the trivial realizations of the product, AB-BA and $\frac{1}{2}(AB+BA)$, respectively, and the study of these algebras for more general realizations of the product is truly limited.

A most intriguing aspect is that the emergence of the Jordan algebras for the treatment of nonselfadjoint forces appears to be a purely quantum mechanical effect. Indeed, the classical Lie-admissible limit of product (4.17.10), i.e., the Lie-admissible brackets

$$(A,B) = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t,b)\frac{\partial B}{\partial b^\nu}, \quad S^{\mu\nu} = \frac{\partial b^\mu}{\partial R_\nu} \qquad (4.17.19)$$

are not Jordan-admissible. Of course, this is only the result of a first look at the problem. The finalization of this intriguing point demands a detailed study of the classical limit of product (4.17,10) beyond the rudimentary step (4.15.36). Neverthless, owing to the difficulties for the existence of a classical realization of the Jordan product via brackets of type (4.17.4), as indicated earlier, the Jordan algebras are not expected to appear in a direct form under this classical limit. This does not exclude the possibility of an algebraically associated form. We here only restrict ourselves to the indication for the interested researcher of the existence of the so-called bonded algebras.[22-II] The subclass of these algebras which is here potentially significant is that produced by a mutation of the product of a Lie-admissible algebra of the type

$$U:(A,B) \rightarrow U^*(T,U^+): \; A\cdot B = [A,B]_u + \{A,B\}_u T, \qquad (4.17.20)$$

where T is a linear mapping (the bonding mapping) of U^+ into U. The preservation of the Jordan algebras also in nonconservative Newtonian mechanics via indirect mechanisms of this type cannot be, in principle, excluded and studies to this effect are solicited.

TABLE 4.18: THE PROPOSED LIE-ADMISSIBLE COVERING OF THE GALILEI RELATIVITY FOR THE QUANTUM MECHANICAL TREATMENT OF NONCONSERVATIVE FORCES.

As by now familiar, one of the central objectives of this paper is to stimulate the experimental verification of the validity (or invalidity) of established relativity laws for the hadronic constituents. It is therefore significant, both per sé as well as for this objective, to outline the covering of the Galilei relativity which is expected to apply to our Lie-admissible covering of Heisenberg's equations.

Let us recall that the relativity which is applicable to classical, conservative, Hamilton's equations

$$\dot{b}^\mu = [b^\mu, H^{phys}]_{class} = \frac{\partial b^\mu}{\partial b^\rho}\, \omega^{\rho\sigma}\, \frac{\partial H^{phys}}{\partial b^\sigma}, \quad (4.18.1a)$$

$$\{b^\mu\} = \{\underset{\sim}{r}, \underset{\sim}{p}\}, \quad (\omega^{\rho\sigma}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mu = 1, 2, \ldots, 2m, \quad (4.18.1b)$$

is the conventional Galilei relativity. It is given by the connected Lie group of canonical transformations

$$A'(b') = e^{\theta_i\, \omega^{\alpha\beta} \frac{\partial X_i}{\partial b^\beta} \frac{\partial}{\partial b^\alpha}} A(b), \quad (4.18.2a)$$

$$\{X_i\} = \{H^{phys}; \underset{\sim}{P}^{phys}; \underset{\sim}{M}^{phys}; \underset{\sim}{G}^{phys}\}, \quad (4.18.2b)$$

$$\{\theta_i\} = \{t_0; \underset{\sim}{r}_0; \alpha, \beta, \gamma; \underset{\sim}{v}_0\}. \quad (4.18.2c)$$

The fundamental algebraic structure, according to the view of Table 4.16, is the universal enveloping associative algebra of the Galilei Lie algebra. Its (infinite-dimensional) basis, from Eqs. (4.16.5) is given by elements of the type (in the abstract treatment)

$$\mathfrak{A}(\underset{\sim}{G}(3.1)): \; 1, \; X_i, \; X_i X_j, \; X_i X_j X_k, \; \ldots \quad (4.18.3)$$
$$i \leq j \qquad i \leq j \leq k$$
$$i, j, k = 1, 2, 3, \ldots, 10$$

Then the algebra $\mathfrak{A}$ includes the basis of $\underset{\sim}{G}(3.1)$, reproduces $\underset{\sim}{G}(3.1)$ via the associate Lie-admissible rule $\mathfrak{A}^- \simeq \underset{\sim}{G}(3.1)$, and characterizes all the physically significant quantities, such as the Casimir invariants.

The relativity which is applicable to the Heisenberg's equations for conservative selfadjoint forces

$$\dot{b}^\mu = \frac{1}{i\hbar}[b^\mu, H^{phys}], \quad (4.18.4a)$$

$$[b^\mu, b^\nu] = i\hbar\, \omega^{\mu\nu}, \quad (4.18.4b)$$

is given by a quantum mechanical version of the Galilei relativity. It is essentially characterized by the connected Lie group of unitary transformations

$$A'(b') = e^{-\frac{\theta_i}{i\hbar} X_i} A(b)\, e^{\frac{\theta_i}{i\hbar} X_i}, \quad (4.18.5)$$

where now, of course, the X's are Hermitian operators, with composition law

$$A''(b'') = e^{-\frac{\theta'_i}{i\hbar} X_i}\, e^{-\frac{\theta_j}{i\hbar} X_j} A(b)\, e^{\frac{\theta_i}{i\hbar} X_i}\, e^{\frac{\theta'_j}{i\hbar} X_j}$$
$$= e^{-\frac{\theta_i + \theta'_i}{i\hbar} X_i - \frac{\theta_i \theta_j}{\hbar^2}[X_i, X_j] - \cdots} A(b)\, e^{\frac{\theta_i + \theta'_i}{i\hbar} X_i - \frac{\theta_i \theta_j}{\hbar^2}[X_i, X_j]} \quad (4.18.6)$$

originating from the use of the Baker-Campbell-Hausdorff formula

$$e^{tA}\, e^{tB} = e^{t(A+B) + \frac{t^2}{2}[A, B] + O^{(3)}}, \quad t \approx 0. \quad (4.18.7)$$

The fundamental algebraic structure is again given by the universal enveloping associative algebra which, according to the view of Table 4.16, makes possible the computation of physically significant quantities other than the basis of the (quantum mechanical) $\underset{\sim}{G}(3.1)$ algebra. Its basis is again given by Eqs. (4.18.3), under the indicated reinterpretation of the generators.

In conclusion, in the transition from a classical-conservative to the corresponding quantum mechanical setting, the Galilei relativity persists in its entirety, a part from technical implementations due to the new nature of the basis. The important point is that the classical and quantum mechanical versions of the Galilei relativity can be jointly treated with a formally unified abstract formulation. This is an expression of the preservation of the relativity laws (conservation of the energy, linear momentum, etc.) in the transition from Eqs. (4.18.1) to Eqs. (4.18.4), and only their reinterpretation within a quantum mechanical setting.

The problem of the relativity laws which are applicable to classical, nonconservative Newtonian systems has been studied in details by R. M. SANTILLI[23]. The Galilei relativity emerges as being violated on a number of counts, and a covering relativity is proposed for the Hamilton-admissible implementation of Eqs. (4.18.1)

$$\dot{b}^{\mu} = (b^{\mu}, H^{phys})_{class.} \tag{4.18.8a}$$

$$(b^{\mu}, b^{\nu}) = S^{\mu\nu}(t, b) = \partial b^{\mu} / \partial R_{\nu} \tag{4.18.8b}$$

to account for the additional forces not derivable from a potential. As recalled in Table 4.4, the relativity which is characterized by the Lie-admissible formulations is essentially given by the Lie-admissible group

$$\hat{A}(\hat{b}) = e^{\theta_i S_i^{\alpha\beta} \frac{\partial X_i}{\partial b^{\beta}} \frac{\partial}{\partial b^{\alpha}}} A(b), \tag{4.18.9a}$$

$$\{X_i\} = \{H^{phys}; \underline{P}^{phys}; \underline{M}^{phys}; \underline{G}^{phys}\}, \tag{4.18.9b}$$

$$\{\theta_i\} = \{t_0; \underline{r}_0; \alpha, \beta, \gamma; \underline{v}_0\}, \tag{4.18.9c}$$

where the Lie-admissible tensors $S_i^{\mu\nu}$ are generally different for different generators and must be individually computed. In Table 4.4 we also recalled that such relativity is non-Lie, noninertial, nongeodesic, nonlinear, nonsymplectic and non-Riemannian, to stress the departure from the corresponding Galilei relativity which is Lie, inertial, geodesic, linear (in the admissible representation theory) and symplectic (the symplectic tensor directly enters into the very structure of Eqs. (4.18.1)) although of V trivial Riemannian nature under the absence of gravitational considerations. It was then indicated that these departures from conventional relativity settings are specifically intended as representative of the departures from conservative mechanics which is implied by nonconservative forces. Let us recall that the group (4.18.9a) results to be in actuality a group of canonical-admissible transformations

(nonidentity Lie-admissible isotopic transformations which preserve the time evolution law of the Lie-admissible equations, as indicated in Table 4.15 -- for details see ref.[23]). Also, the fundamental algebraic structure results to be a general Lie-admissible genotopic image of the algebra $\mathcal{A}(\underline{G}(3.1))$.

This latter feature is truly crucial on both mathematical and physical grounds. On mathematical grounds it is such an algebraic structure which allows the loss of the Lie algebras in the neighborhood of the origin in favor of the Lie-admissible algebras, while preserving at the same time a global, connected group structure, although of predictable generalized structure. In turn, this has crucial implications for the representation theory, such as the general lack of linear representations.[33]

On physical grounds it is precisely the genotopic nature of the Lie admissible enveloping algebra $\mathcal{U}(\underline{G}(3.1))$ which permits the complyance with our "uncompromisable condition of compatibility": to recover the conventional Galilei relativity identically under the limit of null nonconservative forces. By reinspecting this feature with the algebraic analysis of Table 4.16, the reader can indeed see that this compatibility condition is rendered possible by the fact that $\mathcal{U}(\underline{G}(3.1))$ is precisely constructed in terms of the basis of $\underline{G}(3.1)$ and only embedded into a nonassociative mutation of the product of $\mathcal{A}(\underline{G}(3.1))$ which is capable of representing the Galilei relativity breaking forces. The compatibility then becomes algebraically trivial and it is given by the limit $\mathcal{U}(\underline{G}(3.1)) \to \mathcal{A}(\underline{G}(3.1))$. This is precisely the spirit of the Lie-admissible genotopic mapping.

We are now equipped to consider the problem of the relativity which is applicable to our Lie-admissible hadronic equations

$$\dot{b}^{\mu} = \frac{1}{i\hbar}(b^{\mu}, H^{phys}), \tag{4.18.10a}$$

$$(b^{\mu}, b^{\nu}) = i\hbar S^{\mu\nu}. \tag{4.18.10b}$$

Since the above equations are an algebra preserving quantization of the Hamilton-admissible equations (4.18.8), it is natural to search, as the most likely candidate, for a Lie-admissible quantization of the structure (4.18.9).

Let us begin with the case of the covering of the one-dimensional subgroup of translations in time. It is easy to see that the unitary-admissible transformations (4.15.22)

$$\hat{A}(\hat{b}) = e^{-\frac{t}{i\hbar} H^{phys} S} A(b)\, e^{\frac{t}{i\hbar} R H^{phys}}, \qquad (4.18.11a)$$

$$R \neq \pm S,\ R \neq R^{\dagger},\ S \neq S^{\dagger},\ [S, H] \neq 0,\ [R, H] \neq 0,\ R, S \text{ nonsingular}, \qquad (4.18.11b)$$

already provides the desired covering. As a matter of fact, by looking in retrospective, the reader can see that the Lie-admissible covering of Heisenberg's equations was reached by first generalizing the relativity profile (restricted in this case only to translations in time) and then studying the emerging dynamical context. Trivially, Eqs. (4.18.10) are not only compatible with transformations (4.18.11), but actually demand the latter for their very derivation via the transformation theory.

The reader should keep in mind the considerations of Table 4.15 to the effect that conventional unitary transformations must be excluded in our Lie-admissible formulations because they imply the constancy of the rate of variation of physical quantities, contrary to the classical experimental evidence and the desired quantum mechanical context. Once unitary transformations are excluded, our unitary-admissible transformations become conceivable candidates for the covering Lie-admissible formulations. Again the reader should be aware that the structure of such transformations is not expected to be unique because the product of an algebra is nonunique. What is therefore important is to first identify one explicit structure of the transformations yielding the desired algebraic structure. The study of all other structures which yield the same algebra is then a problem of algebraic isotopy and, as such, of secondary rilevance.

One of the intriguing properties of the unitary-admissible transformations (4.18.11) is that they do form a connected Lie group, although not in the conventionally known meaning of these terms. Indeed, transformations (4.18.11) are manifestly connected and they satisfy the composition law

$$\begin{aligned}\hat{\hat{A}}(\hat{\hat{b}}) &= e^{-\frac{t'}{i\hbar} H^{phys} S}\, e^{-\frac{t}{i\hbar} H^{phys} S} A(b)\, e^{\frac{t}{i\hbar} R H^{phys}}\, e^{\frac{t'}{i\hbar} R H^{phys}} \\ &= e^{-\frac{(t+t')}{i\hbar} H^{phys} S} A(b)\, e^{\frac{(t+t')}{i\hbar} R H^{phys}}. \qquad (4.18.12)\end{aligned}$$

As a result, they are a connected Lie-admissible group in the language of ref.[23]. In particular, they trivially admit the original Galilei group at the limit R, S → 1.

The most physically significant function of these unitary-admissible translations in time is that of producing an algebraic-group theoretical characterization of the quantum mechanical law of nonconservation of the energy. As such, they produce, also in the language of ref.[23]

a covering-breaking of the exact symmetry under conventional unitary translations in time

$$H(b') = e^{-\frac{t}{i\hbar} H} H(b)\, e^{\frac{t}{i\hbar} H}, \quad H = H^{phys}. \qquad (4.18.13)$$

First of all, transformations (4.18.11) ensure a breaking of this exact symmetry. This is clearly an elemental prerequisite for any nonconservative quantum mechanics. Notice that the breaking occurs at the level of the fundamental methodological tools: the Lie algebras and their associative envelop, which become inapplicable "ab initio" in the Lie-admissible formulations. Secondly, in the covering formulation such a breaking does not remain algebraically and group theoretically undefined, as typically of most of contemporary mechanisms of symmetry breaking. Instead, the Lie algebras, the enveloping algebra, the dynamical equations, the unitary transformations, the global group structure, etc., are all replaced by corresponding covering formulations for the intent of providing methodological tools for broken space-time symmetries under nonself-adjoint forces. For a detailed elaboration of these aspects, the reader is referred to for conciseness to refs.[22,23]

Our next problem is that of identifying the Lie-admissible quantization of the remaining subgroups of the Galilei-admissible group. An implementation of the analysis then yields the structure

$$\hat{A}(\hat{b}) = e^{-\frac{\theta_i}{i\hbar} X_i S_i} A(b)\, e^{+\frac{\theta_i}{i\hbar} R_i X_i}, \qquad (4.18.14)$$

$$i = 1, 2, \ldots, 10,$$

where the R_i and S_i quantities are the quantum mechanical operators which correspond to the $S_i^{\mu\nu}$ classical tensors and, as such, they are generally different for different generators. The reader should be aware from the analysis of ref.[23], that this dependence of the Lie-admissible product on the generators, rather then being a drowback, results to be a necessary condition of consistency for the approach. The reason is essentially that nonconservative forces always produce the nonconservation of the energy, but not necessarily that of other quantities. An implementation of the full Galilei-admissible group whould then imply the nonconservation of all the Galilei generators, contrary to assumption. The net effect is that our Galilei-admissible relativity is conceived to allow only a partial implementation of the Galilei group, in which case the R and S operators for the unchanged part of the structure are the unit. This is ultimately allowed by the fact that Lie algebras are Lie-admissible and it is an indication of the possibilities of the approach.

The composition law for the complete structure (4.18.14) reads

$$\hat{A}(\hat{b}) = e^{-\frac{\theta'_i}{i\hbar} X_i S_i} e^{-\frac{\theta_j}{i\hbar} X_j S_j} A(b)\, e^{+\frac{\theta_i}{i\hbar} R_i X_i} e^{+\frac{\theta'_j}{i\hbar} R_j X_j} \qquad (4.18.15)$$

$$= e^{-\frac{(\theta_i+\theta'_i)}{i\hbar} X_i S_i - \frac{\theta_i \theta'_i}{2\hbar^2}[X_i S_i, X_j S_j] + O^{(3)}} A(b)\, e^{+\frac{(\theta_i+\theta'_i)}{i\hbar} R_i X_i - \frac{\theta_i \theta'_i}{2\hbar^2}[R_i X_i, R_j X_j] + O^{(3)}},$$

by again using Eq. (4.18.7). Its inspection indicates that the enveloping algebra was infinite-dimensional at the associative level (exact Galilei symmetry) and remains infinite-dimensional at the nonassociative level (broken Galilei symmetry). This is additional indication that we have a mapping of the associative envelop (rather than the Lie algebra) into a nonassociative form. The abstract treatment of the composition law (4.18.15) as well as of the corresponding form for the classical case, and the problem of the reduction of arbitrary elements to standard monomials indicates that the genotopically mapped monomials alone are not sufficient to constitute a basis of $\mathcal{U}(G(3.1))$ and the need of isotopic mappings of the associative standard monomials emerges. This is also indicated by the need of reducing to the base elements the new elements originating from the contributions from the Baker-Campbell-Hasdorff formula, yielding the (abstract) basis (4.16.11).

Structure (4.18.15) is sufficient for our subsequent needs and, therefore, we shall not enter into a more detailed study at this time. Almost needless to say, the problems which we leave open are rather numerous indeed. The most important one is the identification of the operational character of the Galilei generators in the transition from structure (4.18.5) to its covering (4.18.14), as well as the carrier space in which they act.

As a final note, relativity (4.18.14) is of joint Lie-admissible and Jordan-admissible algebraic character.

Structure (4.18.14) is here assumed as the covering relativity for the strong nonselfadjoint forces or, equivalently, for the characterization of our hadronic constituents, the eletons and antieletons, via covering relativity laws. As such, such covering relativity will play a fundamental role in our structure model of hadrons of Section 5.

TABLE 4.19: THE PROPOSED LIE-ADMISSIBLE HADRONIC MUTATION OF THE CONVENTIONAL ATOMIC NOTION OF SPIN.

It is an easy prediction that no genuinely new quantum mechanical formulations for the hadronic structure can be achieved without reaching a consistent generalization of the conventional atomic concept of spin. The same need also emerges on phenomenological grounds to attempt a truly new structure model for the hadrons, as will be indicated in Section 5.

This situation is better identified via relativity considerations. The truly fundamental part of the Galilei (and Einstein special) relativity is that identifying the intrinsic characteristics of the states and, thus, most importantly the spin (via the SU(2) group). The remaining components of the relativity considered essentially characterize the kinematical characteristics of the same state. This distinction is known. It is simply better focused by the study of nonselfadjoint strong hadronic forces. Indeed, as recalled in Section 3, the Lorentz force is already capable of producing the nonconservation of the kinematical characteristics of each individual constituent of a bound state and such nonconservation trivially persists for any force which is analytically more general than the Lorentz force. The fundamental physical difference therefore occurs at the level of the intrinsic characteristics. The Lorentz force (or more generally, the electromagnetic interactions) preserve the intrinsic characteristics of the constituents of a bound state, with numerous fundamental implications (Pauli's exclusion principle, lack of charge-exchange processes, etc.). If a nonconservative quantum mechanics is only capable of producing the nonconservation of the kinematical characteristics of a particle, while preserve the conventional atomic notions of intrinsic quantities (spin, magnetic moments, etc.), it is not a fundamentally new theory. As a matter of fact, the emerging dynamical context is ultimately equivalent to that of the electromagnetic interactions. The net result is that, in this case, we fail to achieve a dynamical departure from the electromagnetic interactions for the intended characterization of the strong interactions.

A crucial contention of this paper is therefore that a generalization of the atomic notion of spin and other intrinsic quantities is a necessary prerequisite to attempt a genuine differentiation between the electromagnetic and the strong interactions as it occurs in the physical reality.

As it will be selfevident in a moment, all the preceding studies of this paper and most of the studies of refs.[21,22,23] have been conducted by the author for the primary objective of attempting a generalization of the atomic concept of spin.

By recalling the crucial methodological role of conventional relativities for the technical characterization of the conventional notion of spin, a necessary prerequisite to attempt the objective considered was the identification of a possible covering of the Galilei relativity. Thus, the primary intended use of the Lie-admissible covering of the Galilei relativity worked out in ref.[23] and reelaborated in this paper is that of producing a technical characterization of a generalized notion of spin.

The objective of this table is therefore that of conducting a preliminary study of the angular momentum-spin component of our Galilei-admissible relativity

$$A' = e^{-\frac{\theta_i}{i\hbar} J_i S_i} \, A \, e^{+\frac{\theta_i}{i\hbar} R_i J_i}, \qquad (4.19.1)$$

where the J's are the generators of the SU(2) Lie algebra. and the θ 's are Euler's angles.

A study of the full structure (4.19.1) is somewhat premature at this moment, owing to the prior need of an extended knowledge of Lie-admissible algebras and groups, which is simply lacking at this time (recall from Table 4.15 that no paper by mathematicians has appeared up to this moment on the general Lie-admissible algebras).

We are therefore forced to study, as the first step, the problem in its simplest conceivable, but genuinely Lie-admissible form. In any case, this is fully sufficient for the objectives of this paper. If a nontrivial generalization of the atomic notion of spin emerges from such simplest possible approach, a deeper generalization is then expected for lesser trivial Lie-admissible formulations. Most importantly, if the simplest possible treatment indicates nontrivial departures from established quantum mechanical laws, a <u>greater</u> departure is expected for more realistic models. The mental attitude implemented troughout this paper therefore reaches its climax in this table: the search for a <u>departure</u> from the atomic notion of spin in the transition from electromagnetic to the strong interactions now becomes vital for the intended differentiation between these interactions.

To proceed, it is useful to reach a more detailed <u>statement of the problem.</u> The objective is that of identifying the breaking of the SU(2)-angular momentum-spin Lie symmetry due to strong nonselfadjoint forces. The applicability of the Lie-admissible formulations for the treatment of such broken SU(2) context is then expected to be consequential.

The following three aspects of the problem deserve an elaboration.

(a) <u>Nature of the SU(2)-angular momentum-spin breaking.</u> Let us recall that the notion of broken Lie symmetry dates back to the times of the very inception of the Lie symmetries in physics. Nowadays, the breaking of Lie symmetries can be proved to be essential in numerous physical situations. For instance, an exact SU(3) symmetry would yield equal-mass multiplets in disagreement with physical evidence. Thus, the physically relevant context is that of the <u>broken</u> SU(3) Lie symmetry (due to strong interactions). A deeper inspection indicates that the virtual totality of the (rather numerous) broken Lie symmetries in the existing literature are restricted to symmetries other than those of connected space-time nature (the celebrated parity violation in weak interactions is of discrete nature). This is clearly due to the intent of avoiding the violation of established relativities, as tacitly implemented in the available physical lietrature. In this and in the preceding paper[23] we are essentially extending, apparently for the first time, the notion of broken Lie symmetries to those of fundamental relevance, the connected space-time symmetries, along the by now familiar lines (to probe the relativity laws for the hadronic constituents and not for the behaviour of a hadron as a whole under ut most electromagnetic interactions). In this table we are interested to the breaking of the central part of these connected space-time symmetries: the SU(2)-angular momentum-spin.

(b) <u>Nature of the forces responsible for the SU(2)-angular momentum-spin breaking</u> The virtual totality of the available classical, Lie, symmetry breakings is achieved by adding a symmetry breaking term to the Hamiltonian

$$H_{tot}^{\text{Exact Symm.}} \longrightarrow H_{tot}^{\text{Broken Symm.}} = H_{tot}^{\text{Exact Symm.}} + H'_{\substack{\text{Symm.}\\ \text{breaking}\\ \text{term.}}}, \qquad (4.19.2)$$

with the understanding that the underlying dynamical equations remain Hamilton's (or Heisenberg's) equations. Such an approach is fundamentally insufficient for the breaking of the SU(2)-angular momentum- spin. The reason is due to the fact that, according to mechanism (4.19.2), the breaking is due to (local) forces fully derivable from a potential. The point is that these forces are compatible with established relativity laws. In order to reach a genuine breaking of the SU(2)-angular momentum-spin symmetry, the forces responsible for the breaking must be incompatible with established relativity laws. The line of study of this and of the preceding paper[23] is that of using the simplest class of these forces, the local forces not derivable from a potential, as an intermediate step prior to more general forces, such as nonlocal and nonderivable from a potential.

(c) <u>The methodology for the treatment of the broken SU(2) angular momentum-spin symmetry.</u> The conventional approach (4.19.2) to symmetry breaking is fundamentally inapplicable to the SU(2)-angular momentum-spin breaking also on methodological grounds. The best way to illustrate this occurrence is by conducting a critical examination of the currently used methods to treat the broken SU(3) symmetry. This is a typical case of mechanism (4.19.2). The symmetry is semiphenomenologically broken at the level of the Hamiltonian. However, since the dynamical equations (Heisenberg's equations) remain unchanged, the Lie algebras preserve their fundamental methodological role in their entirety. The net result is that physically meaningful results, such as the Gell-Mann-Okubo mass formula, are computed by using the enveloping associative algebra $\mathcal{A}$(SU(3)) via scalar expresions of the type

$$I = X_i X^i, \quad i = 1, 2, 3, \ldots, 8, \tag{4.19.3}$$

where, most importantly, the product $X\ X$ is precisely that of $\mathcal{A}$ (SU(3)). Our contention is that we are here facing a fundamental methodological inconsistency. As elaborated in details in ref.[22-I,23] and as recalled in Table 4.16, the use of the enveloping algebras $\mathcal{A}$ (SU(3)) is the most technically effective way to ensure an <u>exact</u> SU(3) symmetry, because the algebraic structure which characterizes in a symbiotic way the infinitesimal algebraic and finite group character of the exact SU(3) symmetry is precisely its enveloping associative algebra. The net result is the following. On one side the physical validity of the Gell-Mann-Okubo mass formula is unequivocal. On the other side, the methods conventionally used to construct this formula are those of the <u>exact</u> SU(3) symmetry, only empirically modified via the introduction of parameters which, in the final analysis, remain algebraically unjustified. What is therefore at stake is the depth of the physical insight.

We therefore argue that a profound revision of the methods for the treatment of the broken Lie symmetries is in order, irrespective of whether of space-time nature of not. The uncompromisable condition for any applicable methodology is that of truly ensuring that the Lie symmetry is indeed broken at the algebraic level. This is clearly dictated to avoid the occurrence of incompatible structures, such as a genuine breaking at the Hamiltonian level and the use of the exact $\mathcal{A}$(SU(3)) algebra for physical computations. In turn, the only way known to me to ensure that a Lie symmetry is indeed algebraically broken is to <u>avoid</u> the use of the associative enveloping algebra of the exact, Lie, symmetry algebra. Indeed, whenever such algebra $\mathcal{A}$ is used in physical computations, the associative Lie-admissible rule $\mathcal{A} \to \mathcal{A}^-$ ensures that the original, exact, Lie symmetry is preserved in its entirety and no breaking has been actually implemented, despite empirical procedures.

As elaborated in details in ref.[23], the Lie-admissible formulations have been conceived precisely to yield a methodological treatment of <u>broken</u> Lie symmetries in general and those of space-time nature in particular. A brief outline may be here useful. A first central objective to reach a nontrivial departure from the exact Lie context is to ensure that the Lie algebras are inapplicable "ab initio" as a methodological tool for the broken context. The inapplicability of $\mathcal{A}^-$ then necessarily imply that of $\mathcal{A}$. This first part, however, is per sé sterile, particularly on physical grounds. If the associative envelop $\mathcal{A}$ is inapplicable, we clearly need a substitute to compute physically significant quantities. It is at this point where the true algebraic notion of (nontrivial) Lie-admissible algebras, that of <u>non</u>associative enveloping algebras according to the Poincaré-Birkhoff-Witt-Ktorides theorem, becomes fundamental. Indeed, the replacement of the associative envelop $\mathcal{A}$ with the Lie-admissible nonassociative genotope $\mathcal{U}$ (1) ensures that the original exact symmetry is indeed broken (e.g., because in general $\mathcal{U}^- \not\approx \mathcal{A}^-$); (2) offers the broader algebraic context for practical calculations with the simple replacement of the associative product $X\ X$ in $\mathcal{A}$ with the nonassociative product $X \cdot X$ in $\mathcal{U}$; (3) the exact context is recoverable identically at the limit $\mathcal{U} \to \mathcal{A}$, i.e., at the limit of null symmetry breaking forces, (4) the approach enjoys a compatible dynamical backing via dynamical equations of the same Lie-admissible character (to avoid incompatibilities between different methodological treatments) and, last but not least, (5) the approach allows the study of the implications of the symmetry breaking forces for the physical quantities, such as the relation between the parameters of the Gell-Mann-Okubo mass formula and the SU(3) breaking forces (when these forces are null, we have equal mass multiplets).

Such a broader methodological approach to SU(3) breaking by strong interactions was worked out by this author at the Center for Theoretical Studies of Coral Gable in 1967-1968 and presented at the Indiana Conference of June 1978 (see the proceedings[42]). Subsequently, the approach was reinspected in great details by C. N. KTORIDES.[47] The Lie-admissible algebra used was the $\mathcal{U} = A(\lambda, \mu)$ mutation of the associative envelop $\mathcal{A}$ (SU(3)). The approach reproduces the Gell-Mann-Okubo mass formula identically. However, the parameters of this formula now acquire a precise dependence on the λ and μ parameter, that is, the algebraic quantities which are representative of the symmetry breaking forces.

It should be here recalled that the dynamical origin of this broader methodological approach rests on the crucial generalization of Hamilton's (or Heisenberg's) equations which must be non-Lie in algebraic character. Explicitly, we abandon the notion of breaking at the level of the Hamiltonian and consider instead that at the level of the equations of motion

$$\left[\omega_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H^{ES}}{\partial b^{\mu}}\right]^{ES}_{SA} = 0 \longrightarrow \left\{\left[\omega_{\mu\nu}\dot{b}^{\nu} - \frac{\partial H^{ES}}{\partial b^{\mu}}\right]^{ES}_{SA} - F_{\mu}\right\}^{BS}_{\substack{NSA\\SA}} = 0,$$

$$ES = \text{Exact Symm.}, \quad BS = \text{Broken Symm.} \qquad (4.19.4)$$

The problem whether the symmetry breaking forces are derivable from a potential or not, then becomes secondary. The net effect is that the methods are indeed applicable for selfadjoint forces (when the preservation of the space-time connected symmetries is desired) but now they naturally allow the inclusion of the most general possible local forces, the nonselfadjoint forces (when the breaking of the space-time connected symmetries is desired).

To reach a consistent algebraic structure, the broken equations of motion are written in our Hamilton-admissible form

$$\left[S_{\mu\nu}(t,b)\,\dot{b}^{\nu} - \frac{\partial H^{ES}}{\partial b^{\mu}}\right]^{BS}_{\substack{NSA\\SA}} = 0, \qquad (4.19.5a)$$

$$S_{\mu\nu} = \frac{\partial R_{\mu}}{\partial b^{\nu}} : \quad \frac{\partial R_{\mu}}{\partial b^{\nu}}\left(\omega^{\nu\rho}\frac{\partial H^{ES}}{\partial b^{\rho}} - F^{\nu}\right) = \frac{\partial H^{ES}}{\partial b^{\mu}}. \qquad (4.19.5b)$$

The departure from conventional approaches to symmetry breaking now become visible. Typically, according to mechanism (4.19.2) the Lie symmetry is broken at the level of the Hamiltonian, while the analytic equations remain strictly Lie in algebraic character.

In our approach we have exactly the opposite occurrence. The Hamiltonian remains fully invariant under the original symmetry group, while the analytic equations are generalized into a non-Lie form to ensure the breaking at the algebraic level. In this way the Lie-admissible tensor $S^{\mu\nu}$ becomes a direct representative of the symmetry breaking forces. In turn, these symmetry breaking forces characterize the structure of the nonassociative Lie-admissible enveloping algebra [22-II, 23]

$$\mathcal{U} = \hat{\mathcal{T}} / \hat{\mathcal{R}} \qquad (4.19.6a)$$

$$\hat{\mathcal{T}} : F \oplus \underline{\mathcal{L}} \oplus \underline{\mathcal{L}} \odot \underline{\mathcal{L}} \oplus \underline{\mathcal{L}} \odot \underline{\mathcal{L}} \odot \underline{\mathcal{L}} \oplus \cdots \qquad (4.19.6b)$$

$$\hat{\mathcal{R}} : \left\{ [X_i, X_j]_{\mathcal{U}} - (X_i \circ X_j - X_j \circ X_i) \right\}. \qquad (4.19.6c)$$

This has the, by now, dual familiar effect. First, it produces a Lie-admissible characterization in the neighborhood of the identity

$$\delta A = \theta_i A \circ X_i = \theta_i \frac{\partial A}{\partial b^{\mu}} S_i^{\mu\nu} \frac{\partial X_i}{\partial b^{\nu}}, \qquad (4.19.7)$$

and second, under integrability conditions, produces a connected group of finite transformations

$$A' = e^{\theta_i S_i^{\mu\nu} \frac{\partial X_i}{\partial b^{\nu}} \frac{\partial}{\partial b^{\mu}}} A, \qquad (4.19.8)$$

which we have called of Lie-admissible type. It is in this way that the Lie-admissible formulations offer a possibility of treating broken Lie symmetry by attempting the compatible generalization of each major methodological aspect for the conventional treatment of exact symmetries.

To summarize, the apparent novelty of our analysis rests on

(a) the extension of the conventional notion of broken Lie symmetries to the space-time connected Lie symmetries in general, and the SU(2)-angular-momentum-spin case in particular,

(b) a generalization of the analytic structure of the forces currently used in symmetry breakings (selfadjoint) into the broadest admissible local forces (nonselfadjoint) to truly achieve the breakings considered, and

(c) the embedding of the conventional Lie formulations for the exact symmetries into covering Lie-admissible formulations.

Once the conceptual and methodological context has been identified, the presentation of the desired simplest possible breaking of the SU(2)-angular momentum-spin symmetry is straightforward.

On physical grounds we restrict the symmetry breaking forces to produce only a semicanonical breaking of the symmetry considered (Table 4.11), that is, when the equations of motion are form-invariant under rotations, but the physical angular momentum is nonconserved. This is the case, at a primitive Newtonian level, of the physical spinning top under gravity, Eqs. (4.11.11), i.e.,

$$\begin{array}{c}\text{SO(2)}\\ \text{SEMICANONICAL}\\ \text{BREAKING}\end{array} : \left[(I\ddot{\theta})^{ES}_{SA} - T(\dot{\theta}) \right]^{BS}_{NSA} = 0. \quad (4.19.9)$$

The fundamental point is that the physical angular momentum is <u>non</u>conserved. This is clearly a crucial prerequisite to attempt a technical characterization of the spin of an eleton (Table 3.2). The implications of the selected simplest possible class of symmetry breaking forces is that they allow the full formulation of the <u>canonical</u> angular momentum (which, as by now familiar, for nonconservative systems must necessarily be different that the physical angular momentum $\underset{\sim}{r} \times m \dot{\underset{\sim}{r}}$). In turn, upon quantization, this allows for the use of the entire machinery of the SU(2) group as known, including and most importantly the Pauli matrices. However, for nonselfadjoint forces such machinery has only a physical meaning for the maximal associated selfadjoint subsystem. For the complete system it must be embedded in the Lie-admissible context.

It is here significant to stress that the broader classes of forces producing a canonical and an essentially nonselfadjoint breaking imply such a breaking of the symmetry under rotations to be beyond our knowledge for an effective treatment at this time.

Once the symmetry breaking forces have been restricted on physical grounds, we remain with a further restriction of algebraic nature. We here assume, for simplicity, that the forces for a semicanonical breaking of SU(2) yield a (λ(t), μ (t))-mutation of the universal enveloping associative algebra $\mathcal{A}$(SU(2))

$$\mathcal{A}(SU(2)) : J_i J_j \longrightarrow \mathcal{U}(SU(2)) : J_i \circ J_j = \lambda(t) J_i J_j - \mu(t) J_j J_i, \quad (4.19.10)$$

that is, a mutation via functions λ(t) and μ(t) of time (which can be, as a particular case, constants).

This produces the desired result. Physical quantities must now be computed by using the product of the Lie-admissible enveloping algebra and <u>not</u> the original associative envelop (which would imply the preservation of the exact SU(2) symmetry in its entirety). Thus, the value of the angular momentum is mapped from the familiar expression to a function of time according to

$$\sum_{i=1}^{3} J_i J_i = \hbar^2 j(j+1) \longrightarrow \sum_{i=1}^{3} J_i \circ J_i = [\lambda(t) - \mu(t)] \hbar^2 j(j+1). \quad (4.19.11)$$

This is precisely the intended simplest possible case of hadronic mutation of the atomic concept of spin, that is, the dynamical implications at the level of the spin which are expected to result in the transition of the motion of physical, extended particles in vacuum under ut most electromagnetic forces to motion within hadronic matter. It should be here stressed that we are here referring to the simplest possible nonselfadjoint forces. In principle, the broadest admissible forces are expected to have such implications to, perhaps, render questionable the preservation of the very term "spin".

Eq. (4.19.11) is referred to the motion of extended particles within hadronic matter and not to the notion of spin of an eleton, that is, of an extended hadronic <u>constituent</u> according to Section 3. The reason is that, to achieve a Lie-admissible characterization of such a notion, there is the dual need of reaching a departure from the atomic spin for the individual constituents of a hadron, while recovering the conventionally quantized total angular momentum for a hadron as a whole.

When the simplest possible hadronic mutation (4.19.10) is used, this essentially implies that the mutation parameters must be not only constant, but actually capable of producing a total, conventionally quantized, angular momentum.

As we shall see in Section 5, the case which is most significant from the structure viewpoint is that of a hadron as a bound state of three eletons which, at the limit of null strong hadronic forces, are conventional spin $1/2$ particles. The use of the semicanonical breaking then allows the preservation of the Pauli's spin matrices σ_i, $i = 1, 2, 3$.

but now embedded in a Lie-admissible context. Our problem is to identify the admissible value of the total, conventionally quantized spin.

As we shall see in Section 5, this constitutes a truly crucial test of our entire exercise of scientific curiosity. According to conventional quantum mechanics in its arena of unequivocal applicability (the atomic mechanics in our language), a bound state of three spin $1/2$ particles can only yield a state with half-odd-integer total angular momentum. Our contention is that this is true, provided that the admissible forces are derivable from a potential (the typical atomic and most of the nuclear setting). If forces nonderivable from a potential are admitted, we have no clear experimental evidence (strictly speaking, none at all, to the best of my knowledge) of the validity of conventional quantum mechanics for these broader forces. The net result, and this must be stressed, is that, under these broader forces, the restriction to only half-odd-integer values of the total angular momentum for a bound state of particles which, under ut most electromagnetic forces, exhibit a spin $1/2$, is a MERE CONJECTURE, because experimentally unsubstantiated. Trivially, the fact that these particles have spin $1/2$ under ut most electromagnetic interactions, by no means, constitutes evidence that such value of the spin is preserved under the additional presence of short-range forces not derivable from a potential.

Once the lack of established experimental knowledge has been stressed, we are free to formulate any conceivable conjecture, with the firm understanding that the final solution must be provided by experiments and not by theoretical considerations alone.

Again, the problem consists of the study of the value of the total, conventionally quantized angular momentum which is admissible in a bound state of three particles under nonselfadjoint forces, when these particles have spin $1/2$ at the limit of only electromagnetic forces.

An exhaustive study of this problem (which is vital for our hadronic structure model) demands the study of all plausible alternatives and their comparative confrontation with experiments. These alternatives are essentially two.

CASE 1: The particles preserve the value $1/2$ of their spin also under nonselfadjoint forces. The total angular momentum of a bound state of three constituents of this type can then only be half-odd-integer. The study of this case is left to the interested researcher. It is essentially based on the mental attitude of preserving conventional quantum mechanical notions as much as possible, also for the broader physical context considered. The interested reader should be aware that at a close inspection, the theoretically consistent proof of the occurrence considered under the forces considered is by far nontrivial.

CASE 2: The value of the spin of the constituents changes in the transition from the case of ut most electromagnetic interactions to that of additional strong nonselfadjoint interactions (hadronic mutation of spin). Then the total, conventionally quantized angular momentum of a bound state of three particles of this type can be arbitrary, that is either half-odd-integer or integer. This is the line of study of this paper.

At the risk of being pedantic, it should be stressed that the final resolution of the issue must be that via experiments, and not that via theoretical considerations alone which are conjectural in both cases.

The simplest possible technical characterization of the second occurrence via the (λ, μ)-mutation algebras is quite simple. Assume (for simplicity, but without loss of generality) that the angular momentum is null both canonically and physically. Then the total, conventionally quantized, value of the spin of the bound state of three eletons is given by

$$\underset{\sim}{J}^2 \Big|_{\mathcal{U} = A(\lambda,\mu)} = (\lambda - \mu)\, j\,(j+1) = 0, \tfrac{1}{2}, \ldots \qquad (\hbar = 1) \qquad (4.19.12a)$$

$$\lambda - \mu = \frac{2}{m}\, j\,(j+1), \quad j\,(j+1) = \underset{\sim}{J}^2\Big|_A = (\underset{\sim}{\sigma}^1 + \underset{\sim}{\sigma}^2 + \underset{\sim}{\sigma}^3)^2 = \begin{cases} \frac{15}{2}, \\ \frac{3}{4}. \end{cases} \qquad (4.19.12b)$$

In conclusion, it appears that the use of more general strong hadronic forces and of their representation via Lie-admissible formulations allows the construction of a bound state with an arbitrary, conventionally quantized value of the total angular momentum in terms of constituents (the eletons and antieletons) which, at the limit of only electromagnetic interactions, have an arbitrary value of the spin. For instance, two spin 1/2 particles (under electromagnetic interactions) can produce, according to this view, a spin 1/2 bound state under nonselfadjoint forces, or three spin 1/2 particles, under the same conditions, can produce a bound state of either spin zero or 1/2, as the lowest admissible levels.

These dynamical effects of the strong nonseldadjoint forces will be at the basis of our model of hadronic structure, as outlined in Section 5. As we shall see, they emerge as apparently being a necessary prerequisite for the identification of the hadronic constituents with physically established, already known particles.

TABLE 4.20: THE INDICATIONS FOR A CONCEIVABLE INAPPLICABILITY OF ESTABLISHED RELATIVITY LAWS FOR THE HADRONIC CONSTITUENTS.

The results of our studies for the relativity profile can be expressed with the following

PROPOSITION 4.20.1: Nonselfadjoint strong hadronic forces are generally incompatible with established relativities (Galilei's, Einstein's special and Einstein's general relativity for the interior problem) in the space of their realization (Euclidean, Minkowskian and Riemannian, respectively).

We shall consider the indicated relativities on a sequential basis.

I. The case of Galilei's relativity. The incompatibility of the forces considered with Galilei's relativity was the subject of an extensive study by R. M. SANTILLI[23] via the use of the methodology of the Inverse Problem and the identification of the mechanisms of relativity breaking produced by these forces. This study (which will be tacitly assumed forthe remaining analysis) identified five classes of Galilei's relativity breakings in Newtonian mechanics: the isotopic, selfadjoint, semicanonical, canonical and essentially nonselfadjoint breakings.

In the transition to a quantum mechanical formulation of nonconservative forces, the breakings of Galilei's relativity persist (with the possible exception of the isotopic breaking, owing to the open nature of the problem of quantization of isotopically mapped Hamiltonians - Table 4.13). This can be seen in a number of ways. At the level of the basic laws, the conservation laws are violated by a quantization of nonconservative forces under the condition of satisfying the correspondence principle. At the level of the transformation theory, the occurrence can be seen from the fact that the "wave packets" emerging from the quantization of these forces, e.g., (Tables 4.9 and 4.12)

$$\psi = \int dE\, A(E)\, \exp\left\{ \frac{i}{\hbar} \left(\underset{\sim}{p} \cdot \underset{\sim}{r} - E \int_0^t f(t)\, dt \right) \right\} \qquad (4.20.1)$$

lose the customary invariance of the phase as it occurs, say, in atomic mechanics.

II. The case of Einstein's special relativity. The epistemological argument is that the breaking of Galilei's relativity will inevitably imply, on compatibility grounds, that of Einstein's special relativity under a relativistic extension of the forces considered.

The discrete relativistic case will be considered in details in a separate paper.[63] We here consider the case of field theories. The familiar chiral Lagrangians

$$\mathcal{L}^{gen} = \frac{1}{2} \varphi^{i}_{;\mu} G^{ij}(\varphi) \varphi_{j}^{;\mu} + \varphi^{i}_{;\mu} G^{\mu}_{i}(\varphi) + G_0(\varphi) \qquad (4.20.2)$$

$$i, j = 1, 2, \ldots, n$$

are admissible under Proposition 4.20.1 because they represent genuine couplings not derivable from a potential. Indeed, upon elimination of the regular matrix of integrating factors, the underlying field equations are of type (3.5.9a), i.e.,

$$\left\{ \left[(\Box + m_k^2)\varphi_k - f_k(\varphi, \varphi_{;\mu}) \right]_{SA} - F_k(\varphi, \varphi_{;\mu}) \right\}_{NSA} = 0. \qquad (4.20.3)$$

The couplingsa are genuine nonselfadjoint couplings. In particular, they are necessarily quadratic in the field derivatives. However, they constitute a subclass of the class of nonselfadjoint couplings because they verify, by assumption, the integrability conditions of the Inverse Problem for the existence of an indirect Lagrangian representation (non-essentially nonselfadjoint coupli ngs).[28] Despite this restrictive character, Eqs. (4.20. 3) are sufficient for our needs.

It is customarily believed that Lagrangians of type (4.20. 2) are invariant under Lorentz transformations because of a structure which formally verifies this requirement. We shall here indicate that this is not necessarily the case. A fundamental condition for the invariance of any Lagrangian under the Lorentz transformations is that the fields transform covariantly under these transformations. When this crucial requirement is met, then the selection of a compatible Lagrangian structure is significant.

Our problem is therefore that of seeing whether the fields φ_k, solutions of Eqs. (4.20. 3) under the condition of admitting Lagrangian (4.20. 2), verify the fundamental Lorentz covariance rule [64]

$$\varphi'(x') = S(\Lambda)\varphi(x), \quad x' = \Lambda x, \quad S \in L^{\uparrow}_{+} \qquad (4.20.4)$$

for some representation of $L^{\uparrow}_{+}$ or not.

An effective way of studying this problem is by linearizing Eqs. (4.20. 3). It is sufficient for us to consider the case n = 1. Suppose that the selfadjoint couplings of Eqs.

(4.20. 3) are the conventional electromagnetic couplings. Then the linearization of the maximal associated selfadjoint subsystem is given by the conventional Dirac's equation under an external electromagnetic field. The linearization of the full equation (4.20. 3), including the nonselfadjoint couplings, then demands the implementation of such Dirac's equations with additional terms not derivable from a Lagrangian density (as it can be proved via the techniques of the Inverse Problem).

We reach in this way, as a linearization of Eq. (4.20.3) for n = 1 and $f = f^{elm}$, the following nonselfadjoint generalization of Dirac's equation

$$\left\{ \left[\begin{pmatrix} -\bar{e}_{;\mu}\gamma^{\mu} + m\bar{e} \\ \gamma^{\mu} e_{;\mu} + m e \end{pmatrix}_{SA} - \begin{pmatrix} f_{\bar{e}} \\ f_e \end{pmatrix}^{Elm} \right]_{SA} - \begin{pmatrix} F_{\bar{e}} \\ F_e \end{pmatrix} \right\}_{NSA} = 0 \qquad (4.20.5)$$

This is precisely our proposed generalization of Dirac's equation under nonselfadjoint strong interactions, Eqs. (3. 5.10).

Our problem is now reduced to see whether the fields e and $\bar{e}$, under extension (4.20. 5) of the familiar Dirac's equations, necessarily preserve Lorentz covariance or a violation of this covariance is admissible.

Let us recall that the nonselfadjoint couplings of the original, second-order equation are quadratic in the field derivatives. This implies that the nonselfadjoint couplings of its linearized form must necessarily depent on the field derivatives, although in a linear form, i.e., they are of the type

$$F_e = \Gamma^{\mu}(e, \bar{e}, A) e_{;\mu}, \quad F_{\bar{e}} = \bar{e}_{;\mu} \bar{\Gamma}^{\mu}(e, \bar{e}, A), \qquad (4.20.6)$$

$$\Gamma^{\mu} \neq \bar{\Gamma}^{\mu},$$

where the γ's and Γ's satisfy the familiar rule

$$\left\{ \left[(\gamma^{\mu} e_{;\mu} - f_e - F_e) + m \right] \left[(\gamma^{\mu} e_{;\mu} - f_e - F_e) - m \right] \right\} e$$

$$= \left\{ \left[(\Box + m^2)\varphi - f \right]_{SA} - F \right\}_{NSA} e = 0, \qquad (4.20.7)$$

but now extended, of course, to include broader couplings.

The net effect is that the familiar terms $\gamma^\mu e_{;\mu}$ of Dirac's equation now lose their dominance for the characterization of the spin of the particle in favor of a corresponding dominance of the new terms originating from the nonselfadjoint couplings (our strong couplings). Indeed, the extended equations can be written

$$\begin{pmatrix} -\bar{e}_{;\mu}(\gamma^\mu+\bar{\Gamma}'^\mu) - f_{\bar{e}} + m\bar{e} \\ (\gamma^\mu-\Gamma'^\mu)e_{;\mu} - f_e + m e \end{pmatrix}_{NSA} = 0. \quad (4.20.8)$$

This implies the mapping of the spin tensor

$$\sigma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\mu\right) \quad (4.20.9)$$

into a generalized spin tensor of the type (for $\bar{\Gamma}'^\mu = -\Gamma'^\mu$)

$$\Gamma'^{\mu\nu} \propto (\gamma^\mu-\Gamma'^\mu)(\gamma^\mu-\Gamma'^\nu) - (\gamma^\nu-\Gamma'^\nu)(\gamma^\mu-\Gamma'^\mu). \quad (4.20.10)$$

The net result is that the "spin" of the generalized state becomes generally dependent on the parameters of the nonselfadjoint couplings as well as the fields themselves. This is sufficient to indicate that the Lorentz covariance is, in general, lost. For instance, in the simplest possible realization of the nonseladjoint couplings in terms of the conventional γ-matrices

$$\Gamma'^\mu = f(x)\gamma^\mu, \quad \bar{\Gamma}'^\mu = -\Gamma'^\mu, \quad (4.20.11)$$

the spin tensor is mapped into a form of the type

$$\Gamma'^{\mu\nu} = f^2(x)\,\sigma^{\mu\nu}, \quad (4.20.12)$$

which expresses a dependence of the "spin" on the Minkowski coordinates. Even when the g-quantity reduces to a constant we have a mutation of the value of the spin along the lines of Table 4.19. Such a context is simply outside of Einstein's special relativity in field theories.

The above situation is subjected to the following physical interpretation. In all currently available models of implementations of Dirac's equations with strong interactions, the latter are always realized in a form derivable from a potential density. The methodology of the In verse Problem identifies all these model with the class of selfadjoint extensions of Dirac's equations, i.e.,

$$\left\{\left[\begin{pmatrix} -\bar{e}_{;\mu}\gamma^\mu + m\bar{e} \\ \gamma^\mu e_{;\mu} + m e\end{pmatrix}_{SA} - \begin{pmatrix} f_{\bar{e}} \\ f_e\end{pmatrix}^{Elm}\right]_{SA} - \begin{pmatrix} f'_{\bar{e}} \\ f'_e\end{pmatrix}^{strong}\right\}_{SA} = 0. \quad (4.20.13)$$

Still the same methods imply that the strong couplings must be independent from field derivatives (this is necessarily the case for first order partial differential equations and related selfadjoint additive couplings)[28]. The net result is that the solutions of Eqs. (4.20.13) under the assumption of a selfadjoint realization of the strong couplings, do preserve the Lorentz covariance of the fields in its entirety. The reason is quite simple. Since the strong couplings are independent (again, at this first order level) from the field derivatives, Dirac's $\gamma^\mu e_{;\mu}$ term preserves its dominance for the characterization of the spin (as well as other intrinsic quantities). The preservation of the Lorentz covariance under the assumed realization of the strong interactions is consequential.

A central contention of this paper is that selfadjoint models (4.20.13) for strong interactions constitute an approximation for point-like particles, and that when particles are considered as physical, that is, as extended, the strong interactions demand couplings which are analytically more general than the electromagnetic interactions.

In different terms, until only the long-range, "action-at-a-distance," electromagnetic interactions are considered, the point-like characterization of particles produced by Dirac's equations is fully admissible and conform with physical evidence, as experimentally established in any way. When the short-range, strong interactions are considered, the situation is different. Here the point-like characterization becames questionable, particularly when the distances involved are smaller than the charge diameter of particles ($\approx$ 1F). To account for the non-point-like character of physical particles while preserving a local theory there appears to be only one way: realize the strong interactions via nonselfadjoint couplings. The methods of the Inverse Problem then indicates that such couplings depend on the field derivatives.[28] In turn, this implies the loss of the familiar $\gamma^{\mu} e_{\mu}$ terms for the complete characterization of intrinsic characteristics, in favor of a characterization by the strong couplings. Still in turn, this implies a dynamical dominance of the strong over the electromagnetic interactions at the level of their impact in the intrinsic characteristics of the particle, which is precisely our criterion of differentiation between the interactions considered.

In particular, the idea of a perennial value of spin, as in atomic physics, becomes simply unappealing on intuitional grounds, in favor of an expected dependence of all intrinsic characteristics on the geometry of penetration of the charge volume of the particle with hadronic matter. Eqs. (4.20.12) with a dependence of the "spin" tensor on the local coordinates are intended precisely to express this broader notion.

Almost needless to say, a mutation of the spin implies a mutation also of other intrinsic quantities. For instance, for model (4.20.12) we can write

$$\left\{\left[\left(\gamma^{\mu}\frac{\partial}{\partial y^{\mu}} + ie\gamma^{\mu}A_{\mu}\right) - m\right]\left[\left(\gamma^{\mu}\frac{\partial}{\partial y^{\mu}} + ie\gamma^{\mu}A_{\mu}\right) + m\right]\right\} e \qquad (4.20.14a)$$

$$= \left[\left(\frac{\partial}{\partial y^{\mu}} + ieA_{\mu}\right)\left(\frac{\partial}{\partial y_{\mu}} + ieA^{\mu}\right) + \frac{1}{2} e\sigma^{\mu\nu}\hat{F}_{\mu\nu} + m^2\right] e = 0,$$

$$y^{\mu} = \frac{x^{\mu}}{1+g}, \quad \hat{F}_{\mu\nu} = \frac{\partial A_{\mu}(x)}{\partial y^{\nu}} - \frac{\partial A_{\nu}(x)}{\partial y^{\mu}}. \qquad (4.20.14b)$$

The magnetic and electric dipole moments are then characterized by the generalized term

$$\frac{1}{2} e\sigma^{\mu\nu}\hat{F}_{\mu\nu} = \hat{\underset{\sim}{\mu}} \cdot \underset{\sim}{H} + \hat{\underset{\sim}{m}} \cdot \underset{\sim}{E}, \qquad (4.20.15)$$

and they result in the predictable mutated form

$$\hat{\mu}^{k} = -(1+g)\frac{e\hbar}{2mc} s^{k}, \quad \hat{m}^{k} = i(1+g)\frac{e\hbar}{2mc}\alpha^{k}, \qquad (4.20.16a)$$

$$(s) = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \quad (\alpha) = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}. \qquad (4.20.16b)$$

For a detailed study of these other aspects, we refer the reader to ref.[22-II]

In conclusion, Lagrangians of the generalized type (4.20.2) were expected to yield only a semicanonical breaking of Einstein's special relativity (covariance of the equations of motion under the Poincaré transformations, but loss of conservation of the physical energy-momentum and angular momentum tensor, the couplings being nonconservative). This class of breakings cannot be excluded. Neverthless, there is the emergence of the more significative canonical breaking of Einstein's special relativity for the structure considered (lack of both, covariance and conservation laws), under suitably selected explicit forms of the nonselfadjoint couplings.

The reader should be aware that the models considered are still restrictive because they satisfy the integrability conditions of the Inverse Problem by assumption. This leaves an additional, fundamentally broader class of Eqs. (4.20.3): that which violate such integrability conditions. For these models (essentially nonselfadjoint strong interactions) a Lagrangian representation does not exist. This implies the loss of the entire canonical formalism and related Lie algebra. The incompatibility of these broader models with Einstein's special relativity is then consequential and its study is left to the interested reader.

III. The case of Einstein's general relativity for the interior problem. It should be stressed that the broader forces under consideration are fully compatible with the general theory for the exterior problem of gravitation (the field and the behaviour of test particles at a distance from a massive object), in the sense that they are null outside hadrons. (or for distances between particles > 1F). Our considerations are solely restricted to the interior problem of gravitation along the lines of ref.[23]. It is easy to see that the incompatibility of the Galilei's and Einstein's special relativities with the forces considered carries over to their gravitational covering, already at the level of simple consistency arguments.

This problem is studied in refs.[22-I,II] and will not be studied in details here. In essence, there is the emergence of the following three layers of models according to

$$\left[m \frac{d^2 x^\mu}{ds^2} - e F^{\mu\nu}(x) \frac{x_\nu}{ds} \right]_{\underline{\underline{SA}}} = 0, \qquad (4.20.17a)$$

$$\left\{ \left[m \frac{d^2 x^\mu}{ds^2} - e F^{\mu\nu}(x) \frac{x_\nu}{ds} \right]_{\underline{\underline{SA}}} + m \Gamma^{\mu}_{\rho\sigma}(x) \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} \right\}_{\underline{\underline{NSA}}} = 0, \qquad (4.20.17b)$$

$$\left\{ \left[m \frac{d^2 x^\mu}{ds^2} - e F^{\mu\nu}(x) \frac{dx_\nu}{ds} \right]_{\underline{\underline{SA}}} + m \Gamma^{\mu}_{\rho\sigma}(x) \frac{dx^\rho}{ds} \frac{dx^\sigma}{ds} - F^\mu\left(s, x, \frac{dx}{ds}\right) \right\}_{\underline{\underline{NSA}}} = 0. \qquad (4.20.17c)$$

Eq. (4.20.17a) is the conventional realization of the electromagnetic forces in special relativity which is characterized as selfadjoint by the Inverse Problem. Eq. (4.20.17b) is the covering characterization in general relativity.* It is nonselfadjoint, but of non-essential type because admitting the indirect representation via an action principle.[34]

Eq. (4.20.17c) represents the additive presence of non-selfadjoint forces. It emerges as being incompatible with the general relativity for the interior problem on a number of conceptual and technical grounds. For instance, the equation is generally <u>non</u> derivable from a variational principle, contrary to a familiar property of general relativity. There is the lack of meaningful characterization of curvature as geodesic deviation, because the motion is nongeodesic even for null selfadjoint forces. There is the lack of conservation laws, etc.

These occurrences, however, do not prohibit the implementation of the intriguing possibility indicated in Table 3.5 : the studies of the possible construction of a generalized theory of gravitation, inclusive of more general hadronic forces, which exhibits the conventional Einstein's equations for the exterior problem as subsidiary constraints and recovers under a Newtonian approximation our model (3.4.2).

* Notice that essentially nonselfadjoint forces quadratic in the velocities are fully admitted by conventional Riemannian approaches to the interior problem, as evident in Eqs. (4.20.17b), but the admitted form is highly restrictive for the line of study of this paper

It is here tempting to conclude that our study, if proved physically consistent in due time, demands the reconstruction of a theory for strong interactions virtually "ab initio", while leaves the established knowledge for the electromagnetic interactions entirely unaffected.

To achieve, in due time, such a goal, the fundamental step is that of the applicable relativities. The problem of quantization is second.

The objective of this paper on the relativity profile was to elaborate one point of ref.[23] to the effect that our current knowledge on the relativity laws of the physical universe, rather than having reached a terminal stage, appears to be potentially open to new, intriguing horizons.

Equivalently, one objective of this paper was to elaborate, on the relativity profile, the belief expressed in ref.[23] according to which Theoretical Physics is a Science which will never admit terminal disciplines.[65]

TABLE 4.21: THE INDICATIONS FOR A CONCEIVABLE INAPPLICABILITY OF ESTABLISHED QUANTUM MECHANICAL LAWS FOR THE HADRONIC CONSTITUENTS.

The results of our studies for the quantum mechanical profile can be expressed with the following

PROPOSITION 2.21.1: Nonselfadjoint strong hadronic forces are generally incompatible with established quantum mechanical laws.

The studies have been conducted along the following complementary profiles.

I. The relativity profile. The impact of relativity ideas in our representation of physical reality is well known. The need of a generalization of Galilei's relativity for the proper characterization of nonconservative forces inevitably implies the need for a corresponding generalization of quantum mechanics as currently known for the proper treatment of the forces considered. In particular, our studies indicate the possible emergence of a covering of Galilei's relativity which is non-Lie in algebraic character, as the fundamental condition for the preservation of the direct physical significance of the algorithms at hand. Owing to the truly fundamental role of Lie algebras in quantum mechanics as currently known, this is sufficient to indicate the incompatibility of the forces considered with the discipline considered.

II. The dynamical profile. The problem of the quantum mechanical laws for the hadronic constituents under forces more general than those derivable from a potential is too crucial to be reduced to only relativity considerations. As such, it demanded the independent, but complementary study of the dynamical profile, that is, the behaviour of particles under the broader forces considered. Pending a verification by interested researchers, it appears that the dynamical analysis of this paper is in full agreement with the relativity profile. The forces considered, even in their simplest possible form (the nonessentially nonselfadjoint form satisfying the restrictions of Tables 4.5, 4.6 and 4.7) imply a departure from a number of established quantum mechanical principles, such as De Broglie's wavelength principle, Einstein's frequency principle, Heisenberg's undeterminacy principle, etc.

To complete our study, in this table we shall present an epistemological analysis of the most representative principle of conventional quantum mechanics, Pauli's exclusion principle. 66

To avoid possible misrepresentations of the scope of our speculative analysis, let us considered separately the following three layers of the microscopic reality.

A. Atomic structure. The validity of Pauli's principle within this physical context is simply unequivocal, as established by a rather large amount of incontrovertible experimental evidence (the principle is crucial for several central features of the Mendeléyev table, such as the existence of the long periods containing the iron, platinum and palladium groups, and even those of the 14 rare earths). Thus, under no circumstance our analysis for nonselfadjoint forces should be interpreted as applicable to the atomic structure.

B. Nuclear structure. In the transition from the atomic to the nuclear structure considerable scientific caution must be exercised to avoid possible prejudices. The reason is that Pauli's principle was conceived for a purely electromagnetic setting, the atomic structure, while the deeper layer of the nuclear structure implies the nontrivial, additional, presence of strong nuclear forces. What can be safely stated is that the use of Pauli's exclusion principle in nuclear physics produces an excellent agreement with the experimental data. This, however, does not constitute evidence that Pauli's principle is exactly valid in the nuclear structure, that is, it is valid in exactly the same measure as that of the atomic structure, or very small deviations are consistent with physical reality. As we shall elaborate better in Table 5.5, the experimental resolution of the issue is strongly advocated because of crucial physical and methodological significance for both, nuclear and hadronic physics.

The first aspect of our study can therefore be formulated as follows.

STATEMENT OF THE NUCLEAR PROFILE OF THE PROBLEM: Is Pauli's exclusion principle exactly valid for the nuclear constituents, or very small deviations can be theoretically and experimentally established ?

Equivalently, is our current knowledge on the validity of Pauli's principle in nuclear physics quantitatively comparable to the current knowledge of the PCT symmetry in particle physics, or is it at a stage prior to the discovery of parity violation? As we did in similar instances, the studies on the exact validity of Pauli's principle in nuclear physics will be left to the interested researchers. In this paper we are interested to the possibility of very small deviations.

Our first contention is that the problem considered will remain basically unsolved on theoretical grounds until the problem of the nature of the nuclear forces is resolved in a way conceptually and methodologically equivalent to the available knowledge on the nature of the atomic forces. To state it explicit, we believe that the problem of the validity of Pauli's principle in nuclear physics is fundamentally dependent on the nature of the nuclear forces. The reason is by now familiar. If the final resolution of the problem of the nuclear forces will unequivocally establish that these forces are entirely derivable from a potential, the exact applicability of quantum mechanics (as currently known) in nuclear physics is consequential. This will inevitably imply the exact applicability of Pauli's principle in nuclear physics. However, if the final resolution of the problem of the nuclear forces will establish the presence of terms not derivable from a potential, the situation becomes fundamentally different because in this case the applicability of the entire quantum mechanics, let alone Pauli's principle, is in question.

Our second contention is that presented and elaborated in this paper: on grounds of our current knowledge, the possible presence of infinitesimal contributions in the nuclear forces originating from terms not derivable from a potential, and additive to the established (central and noncentral) nuclear forces derivable from a potential, cannot be excluded and specific theoretical and experimental studies to this effect should be conducted. The epistemological argument is, by now, familiar. Nucleons <u>are not</u> point-like particles, but instead extended objects possessing a well defined charge volume, as experimentally established. Available experimental data on the nuclear volume indicates that the charge volumes of nucleons are very close to each other. Within the context of such geometry of charge volumes, the possibility of a small penetration of the charge volumes of the nuclear constituents during their evolution in time must be expected. This yields infinitesimal contributions of forces which are

sufficiently well approximated with local forces not derivable from a potential (nonlinear in the velocities and variationally nonselfadjoint). This is the simplest conceivable generalization of the Lorentz force (linear in the velocities and variationally selfadjoint). In this case, quantum mechanics in general is not expected to be exactly valid for the nuclear structure. Very small deviations from Pauli's exclusion principle are then expected.

Our third contention is that that the proposed "nuclear mechanics" could in the final

analysis result to be of some assistance for the theoretical treatment of the problem. The idea is to construct a quantum mechanics which, by central requirement, produces only infinitesimal deviations from the established mechanics, which is then called "atomic mechanics".

The analytic realization of such a nuclear mechanics is via multiplicative terms to the free terms of the conventional Hamiltonian which are infinitesimally close to the unit, e.g.,

$$H^{phys}_{AM} = T + V = -\frac{\hbar^2}{2m}\Delta + V \longrightarrow H^{gen}_{NM} = e^{\delta V'} T + V = -\frac{\hbar^2}{2m} e^{\delta V'} \Delta + V, \quad \delta \approx 0, \tag{4.21.1}$$

i.e., via infinitesimal mutation terms in our terminology. The algebraic realization of such nuclear mechanics is an infinitesimal mutation of Heisenberg's law of joint Lie-admissible and Jordan-admissible algebraic character

$$\dot{A} = \frac{1}{i\hbar}(A, H^{phys}) = \frac{1}{i\hbar}[A, H^{phys}] + \varepsilon \frac{1}{i\hbar}\{A, H^{phys}\}, \varepsilon \approx 0. \tag{4.21.2}$$

The expected implications from the viewpoint of Pauli's principle are the following. Lie-admissible algebras are fundamentally incompatible with totally antisymmetric solutions because they are neither totally antisymmetric not totally symmetry. However, for an infinitesimal mutation of type (4.21.2) only infinitesimal deviations from totally antisymmetric solutions are admitted. Thus, the wavefunction of, say, two "identical" nucleons under only infinitesimal nonselfadjoint forces is expected to be of the type

$$\Psi = \frac{1}{\sqrt{2}}\left[\psi_{m_1}(1)\psi_{m_2}(2) - (1+\tau)\psi_{m_1}(2)\psi_{m_2}(1)\right], \quad \tau \approx 0. \tag{4.21.3}$$

<u>This is the type of very small deviations from Pauli's exclusion principle which is hereby submitted for experiments.</u>* The physical motivation for such a departure from totally antisymmetric wavefunctions is the following. At the atomic level, electrons are indistinguishable particles because their wave packets do not overlap (appreciably) owing to the very large distances here involved. In the transition to the nuclear level the situation is expected to be, again, different. In few words, identical nucleons are expected to lose their indistinguishability while in a state of penetration of their charge volumes with other nucleons. The reason is simply due to the fact that the dynamical effects

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* See page 882 for more specific suggestions.

originating from such penetration, as elaborated troughout out our analysis, are dependent on the degree of penetration. A proton in such a state of penetration then simply becomes infinitesimally different than another proton whose charge volume is not overlapping with that of others. Pauli's principle, under these circumstances, is not violated. It is simply rendered inapplicable because it demands the strict identity of the particles for its very application.

C. Hadronic structure. In the transition from the nuclear to the hadronic structure the situation becomes profoundly different, both conceptually and quantitatively. Here, extreme scientific caution must be exercised, in the traditional spirit of fundamentally unsolved physical problems, before claiming any scientific truth. The main argument of this paper is by now familiar. Experimental data indicates that the charge volumes of hadrons does not increase appreciably with mass (contrary to the corresponding occurrence at the nuclear level), and it is of the same order of magnitude of any other known massive particle. If the hadronic constituents are physical particles, that is non-point-like, we must have a state of penetration of the charge volume of each constituent with those of the others during the entire life of the systems (contrary to only an infinitesimal occurrence at the nuclear level).

This situation creates a framework, from the viewpoint of Pauli's principle, which is basically different than the corresponding nuclear framework, by therefore calling for a different statement of the problem.

> STATEMENT OF THE HADRONIC PROFILE OF THE PROBLEM: Given a system of massive, charged and extended Fermions under electromagnetic interactions at large mutual distances which strictly obeys Pauli's exclusion principle, does the same system exactly obeys this principle when bounded under additional strong interactions at distances smaller than their charge diameter, or finite departures can be theoretically and experimentally established?

As we all know, the assumption of the exact validity of Pauli's principle within the hadronic structure is studied by the color implementations of quark models.[7-8] The study of the validity of the same models under the condition that the hadronic constituents are not point-like will be left to the interested researchers (these models are vitally dependent on Pauli's exclusion principle). To complement these approaches in this paper we are solely interested in the study of the possible violation. Such a study can be conducted in a way fully parallel to the corresponding nuclear case and essentially via the transition from an infinitesimal to a finite contribution from local nonselfadjoint forces (again, as an approximation of expected nonlocal forces) with consequential transition from an infinitesimal to a finite departure from Pauli's principle.

As a matter of fact, such a parallelism between the nuclear and hadronic case is so significant to render fundamental the resolution of the issue at the nuclear level first. The argument is simple. The technological complexity of the experimental resolution of the hadronic problem is such, to conceivably be beyond our present experimental capabilities. In the absence of the possibility of an immediate experimental resolution at the level of the hadronic structure, that at the nuclear level acquires a fundamental role. Indeed, the current technological level of nuclear experiments is rather advanced and, in any case, the experimental resolution of the problem of the validity of Pauli's principle at the nuclear level may well result "easier" than that at the hadronic level. The following two possibilities are then conceivable.

(1) Very small deviations from Pauli's principle in the nuclear structure are experimentally established. This would undoutedly constitute indications for a possible finite departure at the hadronic level, with un understanding that such possible resolution at the nuclear level cannot be considered as the final resolution also for the hadronic problem.

(2) The exact validity of Pauli's principle for the nuclear structure is experimentally established (we are here referring to the experimental identification of a quantitative validity, say, similar to that of PTC in particle physics -- thus, the term "exact" should not be taken "ad litteram"). This would undoutedly constitute indications for the possible lack of nonselfadjoint terms in the nuclear forces by therefore resulting to be invaluable also for the problem of the nuclear forces. However, such a resolution would not constitute evidence that Pauli's principle is exactly valid also at the hadronic level, howing to the profound physical differences between the nuclear and hadronic structures stressed throughout our analysis.

Our first contention is that the problem of Pauli's principle in hadronic settings will remain fundamentally unsolved on theoretical grounds until the problem of the strong hadronic forces is resolved in a way equivalent to that of the atomic and nuclear structures. Again, we intend here to stress an even greater dependence of the problem of Pauli's principle and the nature of the acting forces, than that at the nuclear level.

Our second contention is that elaborated throughout this paper: at the level of the hadronic structure, strong nonselfadjoint forces are conceivable on grounds of our current knowledge in such a form to dominate the dynamical behaviour of the constituents over all other forces derivable from a potential.

Our third contention is that our Lie-admissible approach to nonselfadjoint forces, if properly studied and developed, could be valuable for the theoretical study of the problem. We are here referring to our "hadronic mechanics" conceived to produce a finite departure from the "atomic mechanics", with the "nuclear mechanics" playing the intermediate role. The analytic realization of such a hadronic mechanics is via the "multiplication" of finite interaction terms to the free term of the conventional Hamiltonian

$$\begin{aligned} H^{phys}_{AM} &= -\frac{\hbar^2}{2m}\Delta + V \longrightarrow \\ \longrightarrow H^{gen}_{NM} &= -\frac{\hbar^2}{2m} e^{\varepsilon V'} \Delta + V \longrightarrow \\ \longrightarrow H^{gen}_{HM} &= -\frac{\hbar^2}{2G^{\frac{1}{2}}m}\nabla\left[G^{\frac{1}{2}} G \frac{\nabla}{m}\right] + V, \end{aligned} \qquad (4.21.4)$$

(although this approach is applicable only for the subclass of nonessentially nonselfadjoint forces). The G-terms are the hadronic mutation terms in our terminology. The algebraic realization of such hadronic mechanics is based on an algebraic mutation of Heisenberg's equations of finite character which is also jointly Lie-admissible and Jordan-admissible

$$\begin{aligned} \dot{A} &= \frac{1}{i\hbar}[A, H]_{AM} = \frac{1}{i\hbar}(AH + HA) \longrightarrow \\ \longrightarrow \dot{A} &= \frac{1}{i\hbar}(A, H)_{NM} = \frac{1}{i\hbar}\left(A(1+\varepsilon)H - H(1-\varepsilon)A\right) \longrightarrow \\ \longrightarrow \dot{A} &= \frac{1}{i\hbar}(A, H)_{HM} = \frac{1}{i\hbar}\left(ARH - HSA\right), \end{aligned} \qquad (4.21.5)$$

(this approach is expected to be independent of the integrability conditions of the Inverse Problem and, thus, applicable, at least in principle, to all nonseradjoint forces). The implications for Pauli's principle are substantial. To state it explicitly, it appears that,

under the conditions considered, all the technical ingredients for the construction of the atomic context of Pauli's principle are lost. Here is a nontechnical summary.

(1) The forces considered generally violate the integrability conditions for the existence of a Hamiltonian (at the classical level). This implies that the Schrödinger-type representations cannot generally be constructed via the Hamilton-Jacobi approach, and in the form currently known. In turn, this implies that the very concept of statevector for a system of hadronic constituents is in question, let alone its totally antisymmetric character.

(2) Under the restriction that the forces considered satisfy the integrability conditions for an indirect Hamiltonian representation without redefinition of the space-coordinates, the Hamiltonian of the emerging generalized Schrödinger equations is generally non-Hermitian. Assuming that the statevector can be proved to be totally antisymmetric, it loses its physical meaning in relation to the observability.

(3) Under the further restriction that the emerging canonical Hamiltonian is Hermitian, it generally violates the conditions of the separability theorems[39]. Thus, a separation of the hadronic statevector

$$\Psi^{HM}_{m_1 m_2 \cdots m_k}(1, 2, \ldots, k) \propto \psi_{m_1}(1)\,\psi_{m_2}(2)\cdots\psi_{m_k}(k) \qquad (4.21.6)$$

does not generally exist. In the final analysis, this is the minimal expected mutilation of the atomic settings because, after all, we are treating the "strongest" interactions in nature known until now. Assuming that such nonseparable statevector can be proved to be antisymmetric for a given hadronic mutation term, this still does not imply Pauli's principle on numerous counts, such as, Ψ^{HM} is the eigenstate of an operator H^{gen} which does not represent the physical energy, the orbits of the constituents are nonstationary, etc.

(4) Under the further restriction that the canonical Hamiltonian (exists, is Hermitian and) separable (i.e., satisfies the theorems of refs.[39]), it does not, in general, commute with the physical Hamiltonian (the energy). This implies that the eigenfunctions of the physical energy can be written in terms of the eigenfunctions of the canonical Hamiltonian say, for the case of two particles, via expressions of the type

$$\Psi^{HM}_{phys} = \alpha_{m_1 m_2}(1,2)\,\psi^{can}_{m_1}(1)\,\psi^{can}_{m_2}(2) + \beta_{m_1 m_2}(1,2)\,\psi^{can}_{m_2}(2)\,\psi^{can}_{m_2}(1). \qquad (4.21.7)$$

It should be stressed that such a scripture has only a formal value. It merely indicates the covering nature of the hadronic mechanics over the atomic mechanics. Indeed, at the limit of null nonselfadjoint forces, the canonical and physical Hamiltonians coincide by recovering the typical setting of Pauli's principle

$$\lim_{F_{NSA}\to 0} \psi^{HM} = \lim_{H^{can}\to H^{phys}} \psi^{HM} = \lim_{\alpha\to\frac{1}{\sqrt{2}},\ \beta\to\pm\frac{1}{\sqrt{2}}} \psi^{HM} = \tag{4.21.8}$$
$$= \frac{1}{\sqrt{2}}\left(\psi_{m_1}(1)\,\psi_{m_2}(1) \pm \psi_{m_1}(2)\,\psi_{m_2}(1)\right).$$

(5) Under the still further assumption that the canonical Hamiltonian (exists, is Hermitian, is separable and) commutes with the physical Hamiltonian, the eigenfunctions are not expected to be either totally antisymmetric or totally symmetric. One of the reasons is due to the expected high value of the spin-orbit couplings directly via the hadronic mutation terms, that is, multiplicative to the kinetic term (rather than additive, as in all currently available models), e.g.,

$$H = -\frac{\hbar^2}{2G^{\frac{1}{2}}}\,\underset{\sim}{\nabla}\left[G^{\frac{1}{2}}G\,\underset{\sim}{\nabla}\right] + V, \tag{4.21.9}$$

$$G = \alpha\,\underset{\sim}{\sigma}_1\cdot\underset{\sim}{M}_2 + \beta\,\underset{\sim}{\sigma}_2\cdot M_1 + \cdots . \quad \alpha\neq\beta.$$

Indeed, depending on the geometry of penetration of the charge volumes, the value of the angular momentum of a particle (as an example) may depend on the value of the spin of another given particle, as well as other elements.

The list could continue. But, rather than doing this, it might be of some value to point out the reason for the conceivable nonapplicability of Pauli's exclusion principle for the hadronic constituents which we consider the fundamental one. It is based on relativity considerations.

Quantum mechanics is, in essence, an articulated body of deeply interrelated and inflexible laws which see their ultimate characterization in the applicable relativity, Galilei's (or Einstein's special) relativity. In this setting, the characterization of a Fermion demands the knowledge of four quantities: one intrinsic (the spin) and three kinematical (say, the three components of the physical linear momentum)

Characterization of a Fermion under electromagnetic interactions only	$\{p_x, p_y, p_z$ (kinematical)	$s\}$. (intrinsic)	(4.21.10)

The conceivable nonapplicability of established relativity laws for the hadronic constituents (Table 4.20) directly implies a consequential, conceivable nonapplicability of Pauli's exclusion principle. To see this occurrence, one must not interpreted the breaking of these space-time symmetries in a marginal way (say, only the breaking of translations in time). Instead, the breaking must be brought to the utmost of its possibilities: the breaking of the group of rotations. This directly implies that the atomic characterization (4.21.10) is unsufficient to characterize the same "Fermion" under the additional presence of strong nonselfadjoint forces because now the quantum number "s" (which has a complete meaning under forces derivable from a potential) is insufficient to characterize the intrinsic quantity considered (the state is no longer an eigenstate of SU(2) invariant operators).

For the characterization of a hadronic state under the conditions considered, the identification of a covering relativity then becomes vital. The proposed Lie-admissible covering of the Galilei's relativity essentially provides the following characterization. The kinematical quantities of the state remain the same as those for conventional atomic settings (the linear momentum is generally nonconserved -- for one particle in a state of many -- under both selfadjoint and nonselfadjoint forces). The mutation occurs at the level of the intrinsic characteristics (which are strictly conserved for atomic settings). Thus, in addition to the value "s" under only electromagnetic interactions, at least one additional quantity is needed to characterize the dynamical effects of the strong interactions in such intrinsic quantity: its departure from the value under electromagnetic interactions only. We reach in this way, as the simplest possible case, five quantities for the characterization of our hadronic constituent

Characterization of a Fermion under electromagnetic and strong nonselfadjoint forces.	$\{p_x, p_y, p_z$ (kinematical)	$s, m_s\}$, (intrinsic)	(4.21.11)

where "m_s" is the mutation term, that is, it measures the departure from the value s under electromagnetic interactions only. At a deeper analysis it emerges that this must be the case also for all other intrinsic quantities, such as the charge, rest mass, magnetic moments, etc., because the mutation of one intrinsic quantity is not necessarily proportional to that of another. Thus, the characterization of one of our hadronic constituents demands several more quantities. This aspect will be studied in some future paper.[63]

The simplest possible characterization (4.20.11) of our hadronic constituent under the proposed Lie-admissible covering of the Galilei relativity is sufficient for the objectives of this paper. Indeed it produces the inapplicability of Pauli's exclusion principle. The reason is so simple to appear trivial: Pauli's exclusion principle is not expected to be applicable for the hadronic constituents under nonselfadjoint strong forces because, if the constituents are Fermions under electromagnetic interactions only, THEY ARE NO LONGER NECESSARILY FERMIONS under the forces considered. As it was the case at the nuclear level, Pauli's principle is not violated (no principle of this magnitude can be easily "violated"). Instead, it is rendered fundamentally inapplicable by the acting relativity because such relativity alters the statistical character of the particles as experimentally detected with currently available experimental techniques (i.e., under electromagnetic interactions only).

In conclusion, one of the most intriguing problems which is opened by our study is that of the covering statistics which are expected to be applicable to particles under the forces considered. This problem will be left open for the interested researcher at this time. However, permit me to indicate that initial studies on Lie-admissible covering statistics have been conducted by R.M.SANTILLI and G. SOLIANI in an unpublished note of 1967 [45]. The emerging statistics were called "formal unification of Bose-Einstein and Fermi-Dirac statistics" because the used basic tool, a Lie-admissible covering of the Lie's triple systems

$$(N_k^{gen}, a_k) = (\alpha\, a_k a_k^{+} + \beta\, a_k^{+} a_k,\ a_k) = \gamma\, a_k, \qquad (4.21.12)$$
$$(a, b) = \lambda ab + \mu ba, \quad \alpha, \beta, \gamma, \lambda, \mu \in F,$$

was capable of producing either Fermions or Bosons at the limit to Lie triple systems (now reinterpretable as the limit to null nonselfadjoint forces). A rather crucial result was achieved on purely algebraic grounds by this note. It can now be subjected to a reinterpretation in light of the subsequent studies. Irrespective of whether the hadronic constituents are Fermions or Bosons under electromagnetic forces only, strong nonselfadjoint forces render them statistically equivalent. As we shall see in the next section, these concepts appear to be crucial to attempt a structure model of the hadrons based on the identification of the constituents with particles which are already fully known at the limit of electromagnetic interactions only. It is therefore hoped that the line of study of note [45] can now be completed, at least up to an initial stage of insights.

As a personal note, permit me to indicate that, after learning Pauli's exclusion principle during my graduate studies in physics (at the University of Torino, Italy, 1963-1966), I was surprised to see that the principle was used also under strong interactions. Note[45] was intended in the hope of indicating that this universal applicability of Pauli's principle in the microscopic world is not fully convincing, as it is the case for all physical laws, because they inevitably have precise boundaries of applicability (e.g., Galilei's relativity laws are not universal, Schrödinger's equation is not universal, etc.). If the boundaries of applicability of one physical law have not been discovered by one researcher, likely they will be discovered by another in due time. Along this spirit, the objective of note[45] was that of indicating that generalized statistics beyond those currently known are after all conceivable (this includes conventional statistics and parastatistics, all being a subcase of our Lie-admissible ansatz(4.20.12)). Thus, Pauli's exclusion principle, as it is the case for so many physical laws, is not expected to be necessarily universal. A number of years have now passed since the time of note[47] (as well as of my first note on Lie-admissible formulations[27]). Since my doubts have increased, rather than decreased in time, this has lead to my efforts to identify the boundaries of applicability of Pauli's principle. The answer which is hereby submitted is by now selfevident:

> Pauli's exclusion principle is necessarily applicable, provided that the acting forces are variationally selfadjoint under the correspondence principle.[67]

Permit me to conclude this section with the following remark. Assuming that a Lie-admissible covering of the methodological context of Pauli's principle (relativity-mechanics-statistics) can, in due time, be consistently achieved and results to be conform to experimental evidence, by no means such covering context should be considered as a terminal description. Indeed, its primary limitations (approximation of expected nonlocal settings) have been identified at the very time of the inception of these possible formulations. This point must be stressed because too often in the history of physics the establishing of new physical laws has later on proved to be a major obstacle for further basic advancements in broader physical arenas. To state it explicitly, another objective of this paper was to elaborate, this time from a quantum mechanical profile, the belief expressed in ref.[23] according to which Theoretical Physics is a Science which will never admit terminal disciplines.

5. THE EPISTEMOLOGICAL ARGUMENT FOR A POSSIBLE IDENTIFICATION OF THE HADRONIC CONSTITUENTS WITH PHYSICAL PARTICLES.

As recalled in Section 2, a number of fundamental problematic aspects forced the acceptance, not without initial controversy, of a radical departure from established classical disciplines in order to achieve a physically consistent model of the atomic structure.

A central contention of this paper is that, perhaps, the fundamental problematic aspects of the quark models will eventually demand a radical departure from established quantum mechanical disciplines in order to achieve the final solution of the problem of the hadronic structure.

The reason for such quite delicate contention is related to the truly fundamental issue: the identification of the notion of constituent which is applicable for the case of the hadronic structure. In turn, this notion can be reduced to the question: what is a particle ?

Our studies essentially indicate that the notion of a particle in general, and that of a constituent of a bound state in particular, may exhibit a profound differentiation depending on whether the particle (or constituent) is under electromagnetic interactions only (as in the atomic structure) or under electromagnetic and strong interactions (as in the hadronic structure). Clearly, this issue is at the foundation of the central problem of hadron physics: the identification of the hadronic constituents with physical particles. Indeed, unless the notion of constituent is properly defined, the problem of the structure of hadrons may be inconsistent in its very formulation.

Permit me to present my view. By "physical particle" I mean an "experimentally established particle other than free". As we all know, the quantitative formulation of the physical particles which is currently adopted is crucially dependent on the established relativity and quantum mechanical laws. From a relativity profile we use the applicable relativity (Galilei's or Einstein's special) to characterize a particle via a suitably selected representation of the relativity Lie group. From a quantum mechanical profile we use a number of fundamental notions, such as spin, wave packet, etc. to achieve the needed quantitative characterization.

This contemporary notion of particle has proved to possess an unequivocal physical consistency for the case of all particles under electromagnetic interactions in general and for the case of the constituents of the atomic structure in particular. In conclusion, the

identification of the notion of constituent for the case of the hydrogen atom can be considered as accomplished in a physically incontrovertible form. We then argue that the problem of the structure of the hydrogen atom is consistently formulated because we possess a quantitative characterization of the notion of constituent under electromagnetic interactions. The net result is that the statement

" the electron and the proton are the constituents of the hydrogen atom "

is physically consistent because we possess a physically consistent notion of particle under electromagnetic interactions.

In the transition to the problem of the structure of the hadrons the physical context is profoundly altered. The reason is quite simple. In the atomic structure we have particles under long-range, action-at-a-distance interactions. In the hadronic structure we have "particles" bounded at extremely small distances under "strong" interactions, as well as conventional electromagnetic interactions. Our contention is that this new physical context demands a reinspection of the notion of particle to consistently define the problem of structure.

In conclusion, what appears to be the fundamental prerequisite for the very formulation of the problem of hadronic structure, let alone its treatment, is the identification of a physically consistent notion of particle under strong interactions as well as, of course, under joint electromagnetic interactions. In turn, as it has been the case for the constituents of the atomic structure, the quantitative formulation of this notion crucially depends on the relativity and quantum mechanical disciplines which are applicable to strong interactions. This is the reason why the problem of hadronic structure has been considered the very last in our priorities, and all our efforts have been primarily devoted to the problem of the disciplines which are applicable to the strong interactions.

The implications of our studies in regards to this issue are essentially given by the following alternatives.

ALTERNATIVE I: The strong hadronic forces are local and derivable from a potential. In this case conventional relativity and quantum mechanical disciplines unequivocally apply. The direct consequence is that the notion of constituent of hadrons and that of constituent of atoms are dynamically equivalent, in the sense that they are characterized by exactly the same disciplines. As a result, the hypothesis

" the quarks are the constituents of hadrons "

is vitally dependent on the assumption that the relativity and quantum mechanical disciplines which have been constructed for the atomic structure identically apply (that is, apply without any basic modification) also for the hadronic structure, because the quantitative characterization of the concept of quark vitally depends on these disciplines (a fractional charge is irrelevant for this context as compared, say, to the value 1/2 of the spin).

As recalled in Section 2, despite a rather substantial effort, this approach to hadron structure has not reached the needed level of unequivocal physical consistency. This situation then demands, in our view, the study of other alternatives.

ALTERNATIVE II: The strong hadronic forces are local and non-derivable from a potential. As stressed throughout our analysis, conventional relativity and quantum mechanical disciplines do not apply to this broader approach to strong interactions. In this case there simply is the need for the courageous abandonment of established disciplines and for the search of covering disciplines specifically conceived for the broader physical arena considered. In turn, the knowledge of these disciplines is clearly needed before the terms "constituents of hadrons" can acquire a physical meaning under the conditions considered.

As by now familiar, in the preceding paper[23] and in this paper (see monographs[21-22] for a detailed presentation) I have proposed covering, Lie-admissible and Jordan-admissible, relativity and quantum mechanical formulations for the treatment of the considered type of strong interactions. Even though this proposal is the result of a number of years of isolated and silent effort (I have been working at this project, although not on a full time basis, since the time of my graduate studies of 1963-1966), under no circumstance the proposed covering formulations should be considered as the final formulations which are applicable to the considered physical context, until proved so by independent interested researchers. In essence, I will have achieved my objective if I succeed in formulating the problem in a way sufficiently clear and if the proposed formulations will emerge as being a good starting ground.

With a clear understanding on these points, this section is devoted to an epistemological study of the structure of the hadrons which emerges from the proposed covering disciplines. In view of the preceding remarks, the fundamental prerequisite for any treatment of the problem is the quantitative characterization of the concept of constituent which emerges from the considered more general nature of the strong interactions. The answer is by now predictable:

" the eletons and antieletons are the constituents of hadrons "

where the term "constituents" now implies the characterization via the proposed covering

formulations.

The argument is also by now familiar. When the strong interactions are realized in a way analytically equivalent to the electromagnetic interactions (selfadjoint) they have exactly the same dynamical implications of the latter interactions at the level of the intrinsic characteristics. Thus, an electron of an atomic cloud has the perennial value 1/2 of the spin which is preserved for the entire life of the particle as a member of such atomic system or, equivalently, the statistical character of the particle is also unchanged during the entire life of the system. In the transition to the hadronic structure under strong selfadjoint interactions the situation is exactly the same. A constituent is assumed as possessing perennial values of the intrinsic characteristics which are simply unaltered by the strong interactions. This is exactly the case for the quark hypothesis.

When the strong interactions are considered as dynamically and analytically nonequivalent to the electromagnetic interactions and, thus, realized in terms of nonselfadjoint forces (or couplings), the situation is profoundly different. These forces produce a breaking of the central part of the atomic relativities, the SU(2)-spin part. The net effect is that strong nonselfadjoint interactions produce a change (mutation in our terminology), in general, of all the intrinsic characteristics of the particles under electromagnetic interactions only. In turn, this produces a notion of constituent for the hadronic structure which is fundamentally different than that for the atomic structure. To be quite specific in this crucial point, the idea that a hadronic constituent has perennial values of the spin, magnetic moment, charge (irrespective of whether integral or fractional), etc. becomes vacuous under the conditions considered.

The notion of eleton (and antieleton) has been conceived in the intent of achieving (in due time) a quantitative characterization of precisely this broader notion of "constituent", that of a particle under strong nonselfadjoint forces (as well as conventional electromagnetic forces). By keeping into account that a rigorous treatment of the problem demands the prior in depth knowledge of the Lie-admissible algebras (which is simply lacking at this time), we content ourselves with the characterization of an eleton according to our Assumption 3.2.1.

The reader should recall the crucial restriction that the mutation of the intrinsic characteristics of a hadronic constituent must always be compatible with the strictly conserved (again, under electromagnetic interactions only) nature of the total intrinsic characteristics of a hadron.

In turn, the central hope of the notion of eleton is that of achieving the resolution of the fundamental problem of hadron physics: the identification of the hadronic constituents with physical particles. Indeed, by its very conception, an eleton can be produced as free. The reason is simple. Whether selfadjoint or nonselfadjoint, strong interactions are short -range interactions. When dynamical conditions allow an eleton to be outside the range of the strong interactions it must reacquire its conventional state under electromagnetic interactions, a part secondary effects (as we shall see, mostly realized via the emission of neutrinos).

The technical realization of this occurrence, the reduction of an eleton to a conventional particle under electromagnetic interactions only, is fundamentally dependent on the capability of the covering relativity and quantum mechanical disciplines to recover identically the conventional disciplines. This is the reason why we have considered this limit property an uncompromisable condition for the construction of the covering disciplines. In turn, this is the reason which uniquely selects (jointly with few technical aspects here inessential) the Lie-admissible algebras as the only admissible algebraic structure of the covering disciplines (both for relativity and quantum mechanical profiles).

We are now equipped to perform a crucial step for the analysis of this paper: the identification of the hadronic constituents with physical particles, as suggested by Lie-admissible formulations.

To begin, permit me to recall that beauty and simplicity have often resulted to be an invaluable guide in our efforts to represent the physical world. During numerous occurrences in the history of physics, complicated initial hypothesis were subsequently forced to leave the way to the simplect possible hypothesis, of course, after the methods for their treatment had been identified.

In relation to the fundamental problem of hadron physics, the identification of the hadronic constituents with physical particles, our attitude is as simple as conceivably possible. Consider the mesons

$$\pi^o, \pi^{\pm}, K^{\pm}, K^o_{S,L}, \eta .$$

A dominant physical characteristic of these particles is that they exhibit spontaneous decays. We therefore simply assume that the mesonic constituents are produced in these spontaneous decays in precisely the same way as the positronium exhibits the decay into e^+e^-. We reach in this way our most crucial

ASSUMPTION 5.1: The constituents of mesons are massive and charged particles produced in their spontaneous decays.

There is no doubt that this assumption is fundamentally simpler than the assumption of yet unidentified constituents (the quarks) under the further assumption of a yet unachieved confinement, and complemented by the still further assumption of complex decay processes mediated by sometimes yet unidentified additional particles.

Despite its simplicity, the quantitative treatment of Assumption 5.1 is by far non-trivial. As we shall see, the assumption considered is simply incompatible with the established relativity and quantum mechanical disciplines and renders simply mandatory the construction of covering disciplines for any assessement of its plausibility via a quantitative treatment.

The objective of this section is therefore that of studying whether Assumption 5.1, while prohibited by conventional, relativity and quantum mechanical disciplines of strict Lie algebraic character, becomes admissible, is quantitatively treatable and results to be in agreement with the experimental data under the assumption of the proposed covering, relativity and quantum mechanical formulations of joint Lie-admissible and Jordan-admissible algebraic character. It is understood that the term "constituents" of Assumption 5.1 is that characterized by those covering disciplines (i.e., the particles are in an eletonic state).

On more specific grounds, the statement of the problem can be formulated as follows. It constists of the study whether it is possible to construct a new structure model of mesons under the proposed Lie-admissible and Jordan-admissible formulations which

(A) satisfies our uncompromisable requirement of physical consistency, the identification of the mesonic constituents with physical particles;

(B) provides a quantitative representation of the following mesonic phenomenology:

(B-1) all the intrinsic characteristics of the particles considered (rest energy, mean life, spin, charge, space-parity, charge-parity, electric and magnetic dipole moments, electric and magnetic quadrupole moments, etc.);

(B-2) spontaneous decays and related fractions;

with an understanding that the remaining part of the mesonic phenomenology

(B-3) inelastic and elastic scattering processes involving mesons

is simply beyond our knowledge at this time on numerous grounds (need of the final solution of the problem of structure of all other hadrons; need of identifying a covering of the scattering theory which is truly applicable under nonselfadjoint forces;*etc.); and

(C) reaches full compatibility with the established unitary models of Mendeléyev-type classification of hadrons.

* The event $pp \rightarrow pp\pi^{\pm}\pi^{\pm}\pi^{\pm}\pi^{\pm}\pi^{\pm}\pi^{\pm}$ has been reproduced in the front page of this journal to remaind all of us that the problem of scattering processes is still open as of now, perhaps, primarily on methodological grounds.

Notice that this statement of the problem is entirely in line with the epistemological analysis of Section 2. In particular, established unitary models of classification not only are assumed as valid, but they are expected to provide invaluable elements for the structure (in much of the interplay between classification and structure which occurred for the atoms). We simply identify their arena of unequivocal physical relevance.

It should be stressed that the problem of the structure of nucleons and other heavier hadrons must be considered extraneous to the objective of this section as well as of this paper.[68] The reason is that the problem of a true, quantitative interpretation of the stability of the proton may well result to be of a complexity beyond our most vivid imagination.

In turn, this situation creates the problem of identifying the boundary of possible physical relevance of Alternative II and related methodologies. The tentative answer which I here submit is:

> The arena of possible physical relevance of the assumption that the strong hadronic forces are local and variationally nonselfadjoint is that of the mesonic phenomenology only.

I have taken the liberty in the preceding paper[23] and in this paper to stress my belief that Theoretical Physics is a Science which will never admit terminal disciplines. Numerous times in the history of theoretical physics the establishing of a methodological context for one given physical arena subsequently emerged as a major obstacle for fundamental progresses in a yet broader physical arena. I have no words to stress the fact that, in the event that the Lie-admissible formulations emerge as possessing physical relevance for the mesonic phenomenology, under no circumstance they should be considered as necessarily applicable to the entire hadronic phenomenology. On still more specific grounds, the problem of the stability of the proton is such that the need for still more general methods cannot be excluded at this time. This leads to the last

ALTERNATIVE III: The strong hadronic forces are non - local and non-derivable from a potential. In essence, my studies indicates (see, for instance, Table 4.20) that the conceptual and methodological context of Alternative I can be considered as an approximation of that of Alternative II at the limit of point-like constituents. In turn, Alternative II was conceived as an approximation of Alternative III. It is substantially premature to attempt the identification of which alternative is promising for the problem f the structure of the proton. The only point

I intended to stress is that the Lie-admissible formulations, as presented in this paper and in the other references,[21,22,23] apply only for local forces. The existence of still more general formulations for non-local forces cannot therefore be excluded.

To conclude this introductory part, permit me to indicate that in this section I shall implement the preceding parts of the analysis often in a tacit form, whenever repetitions are unnecessary. Owing to the rather radical departures called by the approach with respect to conventional trends in hadronic physics, we expect that the interested reader becomes familiar with the preceding analysis, particularly with its conceptual profile. For instance, a most crucial prerequisite for a consistent treatment of Assumption 5.1 is the total lack of unitary quantum numbers (isospin, hypercharge, color, etc.) at the structure level. The idea is that the notion, say, of isotopic triplet of pions, while vital for the classification of pions, prohibits the proper formulation and treatment of the problem of structure of each individual pion. This notion may be debatable if considered only within the context of its presentation, Table 2.2. The point is that the notion becomes mandatory when the Lie-admissible formulations are assumed. Indeed, these algebras produce such a breaking of unitary Lie algebras to the point of rendering the notion of unitary multiplet meaningless.

But the conceptual (as well as technical) profile which appears to be truly crucial is that of the proposed covering relativity of joint Lie-admissible and Jordan-admissible algebraic character. Indeed, this relativity appears to express in a symbiotic way the virtual entirety of the notions considered in this paper. As a matter of fact, the relevance of this relativity for the structure of mesons worked out in the following tables is so prominent, that the physical consistency of the model can, in the final analysis, be reduced to that of the applied relativity.

In conclusion, as anticipated in ref.[23], what I am attempting in this section is a differentiation of the atomic and the hadronic structure via the applicable relativity laws.

TABLE 5.1: THE PROPOSED STRUCTURE MODEL OF THE π^o PARTICLE WITH PHYSICAL CONSTITUENTS

The central intuitional grounds for our structure model of the π^o particle rest on the similarities as well as differences between the particle considered and the positronium.

The physical characteristics of the π^o are:

(1) mass: 134.96 MeV;

(2) mean life: 0.828 x 10^{-16} sec;

(3) charge radius: $\approx 10^{-13}$ cm;

(4) charge: zero;

(5) spin: zero;

(6) space-parity: negative;

(7) charge-parity: positive;

(8) electric and magnetic dipole moments: null;

(9) electric and magnetic quadrupole moments: null;

(10) decays modes and related fractions:

$\gamma\gamma$	98.85 % ;
γe^+e^-	1.15 % ;
$\gamma\gamma\gamma$	$< 5 \times 10^{-6}$;
$e^+e^-e^+e^-$	3.3.2 x 10^{-5} ;
$\gamma\gamma\gamma\gamma$	$< 6 \times 10^{-5}$;
e^+e^-	$< 2 \times 10^{-6}$.

An inspection of these data indicates a number of similarities with the positronium. Those which are considered here important are the similarities in the decay modes. Indeed, the positronium also admits the decays

$$\gamma\gamma\,(^1S)\ ;\ \gamma\gamma\gamma\,(^3S)\ ;\ e^+e^-;$$

the latter one being that with the lowest fraction (tunnel effect for the positronium constituents).

On grounds of these similarities, we argue that the positronium and the π^o have the same constituents, only embedded in different dynamical conditions.

This leads to the study of the following

HYPOTHESIS 5.1.1: The constituents of the π^o are one electron and one positron bounded in an eletonic state.

By keeping in mind the indicated similarities with the positronium, we note that the π^o exhibits spontaneous decays. The simplest possible hypothesis is therefore that its constituents are produced in these spontaneous decays (in exactly the same way as it occurs for the positronium). Hypothesis 5.1.1 then follows from the fact that the only massive and charged particles produced in these decays are the electrons and positrons. This excludes any other possibility of directly identifying the π^o constituents with physically known particles. The hypothesis is then completed by the fact that the decay of the π^o into its constituents carries the lowest fraction (in precisely the same way as it occurs for the positronium).

It should be indicated that Hypothesis 5.1.1 would have appeared simply paradoxical only a few years ago. Not so lately. Indeed, the hypothesis that leptons are also strongly interacting particles has began to appear more frequently in the literature[69]. It should also be indicated that Hypothesis 5.1.1 has been apparently formulated for the first time (in 1974) by R.M. SANTILLI[70], not surprisingly, within the context of a study of the structure of the π^o from a gravitational viewpoint. We shall comment on this aspect later on.

The second phase of the needed intuitional elements is that related to the differences between the π^o and the positronium. Those which are considered here important (besides the fact that the positronium forces are only electromagnetic while the π^o forces and electromagnetic and strong) are

(a) the charge radius of the π^o is much smaller than that of the positronium (for a factor of the order of 10^{-5});

(b) the positronium can exists in a state which is either singlet (J = 0) or triplet (J = 1) while the π^o exhibits only the value J = 0.

(c) the fractions of the same decays of the positronium and of the π^o are different.

Our contention is that, a covering of the conventional atomic mechanics is needed to interpeted these differences under the condition that the constituents of both, the π^o and the positronium are the electrons and positrons.

This point will appear in much more imperative terms for other mesons. We therefore restrict ourselves to only few remarks. On epistemological grounds, a covering of quantum mechanics appears to be needed by the lack of presence of the dual state of singlet and of triplet under the "same" constituents. In essence, we argue that the laws applying to the positronium must be subjected to a mutation in order that only the singlet state is stable when the same constituents are bounded at very small distances. On quantitative grounds, the conventional Schrödinger's equation (in relative coordinates -- μ being the reduced mass)

$$\left\{ - \frac{\hbar}{2\mu} \Delta + V(r) \right\} \psi = E \psi , \qquad (5.1.1)$$

sometimes yields only complex values of the energy when the total energy of the state is bigger than the rest energy of the constituents.[71] In other words, the typical setting in which Eq. (5.1.1) produces physically consistent results is that in which the total energy of a bound state is smaller than than the rest energy of the constituents, as it occurs in the deuteron. In our case we have exactly the opposite situation. The rest energy of the constituents of the π^o in our case is 1.2 MeV (the sum of the rest energy of one electron and one positron) while the total energy is 134 MeV (the π^o mass).

In conclusion, we expect that Hypothesis 5.1.1 is incompatible with conventional quantum mechanics (the atomic mechanics in our terminology). This is precisely the desired occurrence for the context of this paper.

Rather than considering this situation reason for withdrewal, we consider it must significant on methodological grounds. Indeed, if the atomic mechanics does not yield consistent results, by no means, this constitutes evidence that a covering mechanics more specifically conceived for the structure considered cannot emerge as valid. This leads in a natural way to the study of the problem whether our proposed hadronic relativity and mechanics, if applied to Hypothesis 5.1.1, leads to a consistent, quantitative interpretation of the π^o phenomenology listed at the beginning of this table (the entire π^o phenomenology, less the part related to scattering processes which will not be considered at this time).

Our fundamental assumption is that the ordinary electrons and positrons, when bounded at very small distances of the order of magnitude of their wave packets, exhibit local forces non-derivable from a potential, as representative of the dynamical effects resulting from the state of penetration of these wave packets which is entirely absent in the positronium.

In turn, this assumption has a number of consequences. First of all, under a proper selection of the nonselfadjoint forces, the assumption remouves the expected inconsistencies

of the conventional Schrödinger's equation under the indicated conditions (total energy >> rest energy of constituents), as we shall see in this table and in Table 5.2. The net effect is the emergence of the possibility for a quantitative treatment of Hypothesis 5.1.1 which would be prohibited, in general, under conventional quantum mechanics.

The second implication is that the broader nature of the forces implies the applicability of the proposed Lie-admissible covering of Galilei's relativity and related hadronic mechanics, as presented in the preceding section of this paper.

Finally, the assumed broader nature of the forces implies that the electron and the positron, when bounded at very small distances according to Hypothesis 5.1.1, are not identical to the same particles when bounded in a positronium state at large distances. Instead, they are in a mutated form, our eletonic state. Indeed, we can now indicate that the term "eleton" has been conceived as a mutation of the term "electron".

Before entering into the explicit construction of the model, we must elaborate one additional aspect which we consider crucial for the general lines of study of this paper, as well as for the explicit selection of our strong nonselfadjoint forces.

A point which we have stressed during the course of our analysis is the possibility that a profound differentiation between the models of classification and those of structure may eventually emerge as necessary for the hadrons, in much the same way as it occurred at the atomic level. This expectation implies a radical departure from the basic notions for classification in order to attempt a different but compatible model of structure. Once this approach is studied in more details, it suggests the complete lack of unitary quantum numbers (isospin, hypercharge, color, etc.) in the intended structure model. Also, the approach implies that the familiar notion of mass spectrum (that is, a mathematical algorithm producing the mass of different hadrons) may eventually result to be another formulation of the problem of classification and not that of structure. Clearly, these ideas must be better focused.

Another differentiation between the π^o and the positronium which we consider fundamental is that the the former does not exhibit exhited states (that is, quantum states with total energy close to that of the ground state by a multiple of $\hbar$, which therefore excludes mesonic resonances), contrary to the established exhited states of the latter. We therefore interpret this occurrence "ad litteram". We here assume that one electron and one positron, when bounded in an eletonic state, produce ONLY ONE STABLE STATE, the π^o. In other words, the structure model of the π^o we are interested in should exhibit the COMPLETE ABSENCE OF MASS SPECTRUM , and produce only one value of the mass: 134.96 MeV. It is this requirement which, when realized in practice for the π^o as well as for the other mesons (Tables 5.2 and 5.3) renders mandatory the complete absence of unitary quantum numbers for the structure problem, and identifies familiar spectrum producing formulae as belonging to the problem of classification.

We are now equipped to construct the model. The statement of the problem can be formulated as follows. It consists of identifying a spectrum suppressing hadronic mutation of the positronium model . By recalling that the notion of energy spectrum is so rooted in atomic mechanics (in the final analysis it was conceived for this purpose), the suppression of such spectrum is then considered as a manifestation of a covering mechanics.

We shall study this problem by using first our hadronic mutation of the atomic two-body system, Eqs. (4.12.12). As by now familiar, this model implies first a generalization of Shrödinger's equations (5.1.1) via mutation terms which multiply the free term $-\frac{\hbar^2}{2\mu}\Delta$ and then by a second generalization via nonintegrable subsidiary constraints. As a matter of fact, the subsidiary constraints were conceived precisely in the hope of achieving the indicated suppression of the energy spectrum of atomic mechanics.

The practical realization of a consistent model (4.12.12) with nonselfadjoint forces which is readily computable in a closed form (we are not interested in numerical approximations in this first analysis) is expected to be rather complex. We therefore content ourselves with the simplest possible form of the eletonic forces which leads to the desired result (one value of the energy) via readily solvable equations.

Let us denote with the symbols $\varepsilon^\pm$ the electrons $e^\pm$ in our eletonic state. The bound state of these particles under electromagnetic interactions only, the positronium, will be denoted with the symbol $|e^+, e^-|$, while our structure model of the π^o under Hypothesis 5.1.1 will be denoted with the symbol $|\varepsilon^+, \varepsilon^-|$. A number of restrictions on the eletonic forces are now in order. First of all, our uncompromisable condition of compatibility of the models $|\varepsilon^+, \varepsilon^-|$ and $|e^+, e^-|$ demands that the the former be a covering of the latter in the sense that

$$\lim_{F_{NSA}\to 0} |\varepsilon^+, \varepsilon^-| \equiv |e^+, e^-| . \qquad (5.1.2)$$

This implies in particular that

$$\lim_{F_{NSA}\to 0} \varepsilon^\pm \equiv e^\pm , \qquad (5.1.3)$$

that is, the eletons of the π^o coincide with the electrons at the limit of null eletonic forces. This implies, as expected, that the eletonic forces are short-range, that is, they must decay at least exponenentially for distances greater than the charge radius of the π^o. Condition (5.1.3) also implies that we are here considering the simplest possible mutation of the electrons in the sense that its intrinsic characteristics (spin, charge and magnetic moment) are unchanged, and only the kinematical quantities are subjected to a mutation. Indeed the value 1/2 of the spin for the ε^- is admitted to recover, in a "singlet" state, the zero spin of the π^o . From the viewpoint of our Lie-admissible relativity this implies that the operators R and S related to the $SU(2)$ part are 1 (i.e., the Lie-admissible relativity is Lie in its spin part) and the true Lie-admissible embedding occurs only for the other generators. The reader should keep in mind that the π^o emerges as the simplest possible eletonic bound state and that nontrivial mutation of the intrinsic characteristics are expected to be needed for more complext structure (see next table).

In conclusion, these initial considerations indicate that the eletonic forces should be short-range as well as such that they do not alter the intrinsic characteristics of the electrons.

Our second step in the selection of these forces is that of dividing them into two groups one derivable from a potential (along much of the nuclear approach) and one non-derivable from a potential.

The primitive Newtonian form of our structure model of the π^o , Eqs. (3.4.2), can be then written

$$\begin{cases} m_e \ddot{r}_{\mu k} - f^{Coulomb}_{\mu k \underline{\underline{SA}}} - f^{strong}_{\mu k \underline{\underline{SA}}} - F^{strong}_{\mu k \underline{\underline{NSA}}} = 0, \quad k = 1, 2, & (5.1.4a) \\ \frac{d}{dt}\mathcal{E}_t = \frac{d}{dt}(T + V_t) = 0 , & (5.1.4b) \\ \frac{d}{dt} P_{\mu t} = \frac{d}{dt}\sum_{k=1}^{2} m_e \dot{r}_{\mu k} = 0 , & (5.1.4c) \\ \frac{d}{dt} M_{\mu t} = \frac{d}{dt}\sum_{k=1}^{2} r_{\mu k} \times m_e \dot{r}_{\mu k} = 0 , & (5.1.4d) \\ \frac{d}{dt} G_{\mu t} = \frac{d}{dt}\left(\sum_{k=1}^{2} m_e r_{\mu k} - P_{\mu t}\, t\right) = 0 , & (5.1.4e) \end{cases}$$

where, as by now familiar, the subsidiary constraints guarantee that the π^o structure as a whole satisfies the laws of Galilei's relativity, while the same relativity is violated by construction for the behaviour of the individual constituents to allow broader forces (if Galilei's relativity is imposed also for the structure, the only admissible forces would be conservative and, as such, fundamentally unable to produce a quantitative characterization of Hypothesis 5.1.1).

Our next step is to restrict the strong forces in such a way that we have a consistent system (5.1.4) already at the classical level. This is clearly crucial, not only to comply with the correspondence principle, but also to produce a model consistent with the general lines of this paper according to which departures from established laws can be expected only at the structure level and not for the behaviour of the state as a whole (the inconsistency of Eqs. (5.1.4) would imply the violation of Galilei's relativity for the state as a whole).

A study of this problem suggests the use of central forces for the strong forces. In relation to the strong nonselfadjoint forces we content ourselves with the simplest possible realization which we have proved in Table 3.4 as yielding a consistent system, i.e., the acceleration dependent (in relative coordinates) force

$$F^{strong}_{NSA} = \gamma \ddot{r} , \quad r = |r_1 - r_2| , \quad \gamma > 0. \tag{5.1.5}$$

For reason to be justified a posteriori, we select as the radial part of the strong selfadjoint force that derivable from Hulthen's potential

$$F^{strong}_{SA} = -\frac{\partial V^{Hulthen}}{\partial r} , \quad V^{Hulthen} = -V_o \frac{e^{-br}}{1-e^{-br}} . \tag{5.1.6}$$

The selected forces satisfy all the restrictions of Tables 4.5, 4.6 and 4.7, yielding a Hermitian canonical Hamiltonian operator H^{can} for the time evolution (only). The emerging Schrödinger-type equation for the relative motion is then given by (Table 4.9)

$$H^{can}\psi = \left[\rho \frac{\hbar^2}{m_e} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d}{dr} \right) + \frac{e^2}{r} + V_o \frac{e^{-br}}{1-e^{-br}} \right] \psi = E^{can}\psi , \tag{5.1.7a}$$

$$\rho = \frac{m_e}{m_e + \gamma} < 1 , \tag{5.1.7b}$$

and constitutes the simplest possible mutation of the conventional equation via a constant mutation term (the reader shoul keep in mind that for more complex models demanding a mutation also of the intrinsic characteristics of the eletons, this mutation term will indeed acquire a nontrivial functional dependence -- see Table 5.2 and 5.3).

The forces selected imply motion in a plane and conservation of the angular momentum (Table 3.4). Thus the angular and spin part of the state eigenfunction can be separated from the radial part and treated according to conventional quantum mechanical rules (again, this is not the case for more complex models). Under the values $\ell = 0$ for the angular momentum, $s = 1/2$ for the spin of the eletons and the singlet state (see below why the triplet state is unstable), this recovers the value zero of the π^o

$$\Psi(r; \ell, \theta; s) = \psi(r)\, \eta(\ell, \theta; s) , \tag{5.1.8a}$$

$$J^2_{tot}\, \eta = 0 . \tag{5.1.8b}$$

The value zero of the total charge is trivially recovered because, under the assumption of the lack of mutation of the intrinsic characteristics, the charge of the eletons is $\pm$ e = const. (again, this is not necessarily the case for more complex models where the nonconservation of the charge of the individual eletons is needed, of course in a way compatible with the conservation of the total charge -- see Table 5.3).

In conclusion, the assumed forces imply a trivial representation of the zero value of the spin and charge of the π^o which will therefore be ignored from here on.

We now consider the crucial problem of the physical energy levels. One of the central points of the hadronic mechanics is that generalized Hamiltonians H^{can} do not represent the physical energy, as a necessary condition of consistency (if H^{can} = total energy, all forces nonderivable from a potential are identically null). This implies that Eq. (5.1.7) represents only the dynamical evolution of the system. To compute the energy we must compute the value of H^{phys} in the canonical coordinates, according to Eqs. 4.7.3. Recall that

$$\dot{r} = \frac{\partial H^{can}}{\partial p^{can}} = \frac{2\rho}{m_e} p^{can} . \tag{5.1.9}$$

Thus, the relationship between the physical and canonical linear momentum is given by

$$p^{phys} = \frac{m_e}{2} \dot{r} = \mu \dot{r} = \rho\, p^{can} . \tag{5.1.10}$$

This yields the (unique) form of the physical Hamiltonian in canonical coordinates

$$H^{phys} = \frac{1}{2\mu}(p^{phys})^2 + V^{Coul.} + V^{Hulthén} = \frac{\rho^2}{m_e}(p^{can})^2 + V^{Coul.} + V^{Hulthén} \qquad (5.1.11)$$

Thus, within the context of our hadronic mechanics, the computation of the energy must be done with the equation

$$H^{phys}\psi = \left\{ -\rho^2 \frac{\hbar^2}{m_e}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{e^2}{r} + V_0 \frac{e^{-br}}{1-e^{-br}} \right\}\psi = E^{phys}\psi, \qquad (5.1.12)$$

rather than Eq. (5.1.7a). This is trivially made possible for the case at hand because H^{can} and H^{phys} are both Hermitian and commute. This computation will be done later on.

Next, we remain with the problem of a quantitative representation of the mean life of the π^0 (no structure model of any hadron can achieve a minimum degree of plausibility unless the mean life of the particle considered is quantitatively represented). There exist numerous methods for the computation of mean lifes of bound states in the literature. However, these methods are generally based on transition matrix elements. A study of this problem indicates that these methods do not appear as necessarily applicable under nonselfadjoint forces (what is a scattering amplitude under these forces? and, to begin with, can it be consistently defined?). Pending a reinspection and possible generalization of these methods, we are forced to use simpler methods. In the following we shall use the known formula

$$\tau^{-1} = \frac{\pi}{4}\lambda\!\!\!^{-2} |\psi(0)|^2 \frac{\alpha^2 E^{Kin}}{\hbar} \qquad (5.1.13)$$

which produces acceptable results in a number of cases.

Our structure model of the π^0 as a bound state of an eleton and an entieleton can therefore be written

$$\begin{cases} \left\{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right) + \frac{m_e}{\rho^2\hbar^2}\left[E^{phys} + \frac{e^2}{r} + V_0\frac{e^{-br}}{1-e^{-br}}\right]\right\}\psi(r) = 0, & (5.1.14a) \\ E^{tot}_{\pi^0} = 2E^{Kin}_{\varepsilon} - E^{phys} = 135 \text{ MeV}, & (5.1.14b) \\ \tau^{-1}_{\pi^0} = 4\pi\lambda\!\!\!^{-2} |\psi(0)|^2 \frac{\alpha E^{Kin}_{\varepsilon}}{\hbar} = 10^{16} \sec^{-1}, & (5.1.14c) \\ b^{-1}_{\pi^0} = 10^{-13} \text{ cm}, & (5.1.14d) \end{cases}$$

where Eqs. (5.1.14a), (5.1.14b) and (5.1.14c) should be interpreted as subsidiary constraints (with an understanding that they may be trivial in this simplest possible case, but not necessarily so for more general models). The value of 10^{-13} cm for the charge radius of the π^0 is consistent with available experimental data (the value 0.6×10^{-13} cm is often quoted, but the factor 0.6 is inessential for our analysis). Also, the rest energy of the constituents (1.2 MeV) is ignored in the expression of the total energy (5.1.14b). The interested reader can trivially modify the following calculations with the inclusion of these terms.

Model (5.1.14) on the structure of the π^0 was proposed and studied in details by R.M. SANTILLI[70]. A reinspection of this study in light of the subsequent analysis is in order.

First of all, Hultén's potential decays exponentially with the increase of the distance beyond the charge radius. Thus, it constitutes an acceptable potential for the compatibility conditions (5.1.2) and (5.2.3), of course, at the discrete nonrelativistic level of this analysis. Secondly, Hulthén's potential behaves precisely like the Coulomb potential at small distances

$$V^{Hulthén} \approx -\frac{V_0}{b}\frac{1}{r}, \quad r \approx 0. \qquad (5.1.15)$$

This implies that the Coulomb potential can be effectively ignored in Eq. (5.1.14a) via a redefinition of the factor V_0 which will be tacitly implemented.

Under these conditions, by putting

$$\psi(r) = e^{-|A|^{1/2} br} \frac{1}{r} S(br), \quad x = 1 - e^{-br}, \tag{5.1.16a}$$

$$A = \frac{m_e}{\rho^2 \hbar^2 b^2} E^{phys} < 0, \quad B = \frac{m_e V_o}{\rho^2 \hbar^2 b^2}. \tag{5.1.16b}$$

Eq. (5.1.14a) becomes

$$\left[x(1-x)\frac{d^2}{dx^2} - (2|A|^{1/2}+1)\frac{d}{dx} + B \right] S(x) = 0, \tag{5.1.17}$$

with boundary conditions

$$S(0) = 0, \quad \lim_{br \to \infty} e^{-|A|^{1/2} br} S(br) = 0 \tag{5.1.18}$$

and S(l) finite for A = 0. The solutions of Eq. (5.1.17) are known. They are given by the familiar Jacobi polynomials

$$G_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \binom{n+k+2|A|^{1/2}-1}{k} x^k. \tag{5.1.19}$$

The physical (binding) energy then acquires the typical spectrum of atomic mechanics

$$BE_n = -E_n^{phys} = \rho^2 \frac{\hbar^2 b^2}{4 m_e} \left(\frac{m_e V_o}{\rho^2 \hbar^2 b^2 n} - n \right)^2. \tag{5.1.20}$$

Since we intend to avoid an energy spectrum, this occurrence demands an inspection. Solution (5.1.19) can be more generally written in terms of the hypergeometric function

$$\psi(r) = {}_2F_1\left(2\alpha+1+n, 1-n, 2\alpha+1, e^{-br}\right) e^{-\alpha br} (1-e^{-br})/r,$$

$$\alpha^2 = -\frac{m_e E^{phys}}{\rho^2 \hbar^2 b^2} = \frac{1}{2}|A|. \tag{5.1.21}$$

The boundary conditions then demand

$$\alpha = \frac{\beta^2 - n^2}{2n}, \quad \beta^2 = \frac{m_e V_o}{\rho^2 \hbar^2 b^2}. \tag{5.1.22}$$

Since $\alpha > 0$, the system considered necessarily demands that

$$\beta^2 = \frac{m_e V_o}{\rho^2 \hbar^2 b^2} > n^2. \tag{5.1.23}$$

This is the crucial property of Hulthén's potential which justifies a posteriori its choice. In short, while the Coulomb potential admits an infinite number of energy states, Hulthén's potential admits instead a finite number of these states. In particular, when $\beta = 1$, Hulthen's potential admits only ONE ENERGY LEVEL. This property is well known. See, for instance, references[72].

We reach in this way a most crucial test of our ideas. It is constituted by the fact whether the values 135 MeV, 10^{-16} sec and 10^{-13} cm for the mass, mean life and charge radius of the π^o, respectively, leads to only one admissible energy level, or more than one is admitted. At the risk of being pedantic, permit me to stress that this is a truly crucial point. Any structure model of the π^o as a bound state of two particles yielding a ground state and admissible exited states close to it (irrespective of whether finite or infinite in number), is inconsistent with experimental evidence because the π^o, according to our knowledge, does not admit exited states of this type (the possibility of exited energy levels of the π^o which are infinitesimally close to 134.9645 ± 0.0074 MeV and, as such, have escaped the experimental detection until now, cannot be ruled out, but it will not be considered).

To conduct this crucial test, we must reach the numerical solution of the model. First, we note that, for the $\varepsilon^{\pm}$ particles to be bounded in a sphere of radius b^{-1}, their mutated de Broglie's wavelength (Table 4.10) must be of the order of b^{-1}. We can therefore put

$$\lambda_\varepsilon = (k_1 b)^{-1}, \tag{5.1.24}$$

where K_1 is an unknown (real, positive) number. Notice that, since the state is stationary, the mutation of the conventional de Broglie's wavelength can at most be via a constant.

For the kinetic energy of the $\varepsilon^{\pm}$ particles we can then put

$$E_\varepsilon^{kin} = K_1 \hbar b c = \rho^2 \frac{\hbar^2 b^2}{2 m_e} = \frac{1}{2 m_e} (p^{phys})^2, \tag{5.1.25}$$

where the last identifications are crucially dependent on the use of the physical linear momentum (the use of the canonical momentum would be physically inconsistent in nonconservative mechanics for the computation of the kinetic energy).

The desired value to avoid an energy spectrum is

$$\frac{m_e V_0}{p^2 \hbar^2 b^2} = K_2 = 1 + \varepsilon , \quad 0 \leq \varepsilon < 1 , \qquad (5.1.26)$$

for which

$$V_0 = K_2 \frac{\hbar^2 p^2 b^2}{m_e} = 2 K_1 K_2 \hbar b c = 2 K_2 E^{kin}_{\varepsilon} \approx 2 E^{kin}_{\varepsilon} , \qquad (5.1.27)$$

which is fully conceivable (V_0 being of the order of magnitude of the total kinetic energy), but again crucially dependent on the use of the physical linear momentum.

Since we have two unknown constants (K_1 and K_2) we need two equations. The first equation is provided by the value of the π^0 mass

$$E^{tot}_{\pi^0} = 2E^{kin}_{\varepsilon} - BE = 2K_1 [1 - (K_2 - 1)^2] \hbar b c = 2K_1 (1 - \varepsilon^2) \hbar b c = 135 \text{ MeV}. \qquad (5.1.28)$$

A second relation originates from the mean life (5.1.14c). The normalized eigenfunction for the ground state is given by

$$\psi(r) = \left[\frac{\Gamma(2|A|^{1/2} + 3)}{\Gamma(3)\,\Gamma(2|A|^{1/2})} \right]^{1/2} e^{-|A|^{1/2} b r} \, \frac{1 - e^{-br}}{r} , \qquad (5.1.29)$$

and becomes at the origin

$$|A|_{n=1} = \frac{m_e}{p^2 \hbar^2 b^2} BE_1 = \frac{1}{4} (K_2 - 1)^2 , \qquad (5.1.30a)$$

$$\psi(0) = \left[\frac{\frac{1}{2}(K_2 - 1) \; \Gamma[\frac{1}{2}(K_2 - 1) + 2]}{3! \; \Gamma[\frac{1}{2}(K_2 - 1)]} \right]^{\frac{1}{2}} b$$

$$= \left[\frac{\frac{1}{4}(K_2 - 1)^2 \, \Gamma[\frac{1}{2}(K_2 - 1) + 1]}{6 \, \Gamma[\frac{1}{2}(K_2 - 1)]} \right]^{\frac{1}{2}} b = \frac{(K_2 - 1)^{3/2}}{(48)^{1/2}} b = \frac{\varepsilon^{3/2}}{(48)^{1/2}} b . \qquad (5.1.30b)$$

Eq. (5.1.14c) then becomes

$$\tau^{-1} = \frac{4\pi}{K_1^2 b^2} \, \frac{(K_2 - 1)^3}{48} \, \frac{K_1 \hbar b c}{(137)^2 \hbar} = \frac{4\pi}{48\,(137)^2} \, \frac{(K_2 - 1)^3}{K_1} \, b c = 10^{16} \text{ cm}. \qquad (5.1.31)$$

Under the value $b^{-1} = 10^{-13}$ cm, the two unknown quantities of the model are characterized by the system

$$\begin{cases} K_1 [1 - (K_2 - 1)^2] = K_1 (1 - \varepsilon^2) = \dfrac{1}{2\hbar c} (E^{tot}_{\pi^0} b^{-1}) & (5.1.32a) \\ \qquad = 0.25 \times 10^{11} (E^{tot}_{\pi^0} b^{-1}_{\pi^0}) = 0.33 , & \\ \dfrac{(K_2 - 1)^3}{K_1} = \dfrac{\varepsilon^3}{K_1} = \dfrac{48\,(137)^2}{4\pi c} (\tau^{-1}_{\pi^0} b^{-1}_{\pi^0}) & (5.1.32b) \\ \qquad = 2.38 \times 10^{-6} (\tau^{-1}_{\pi^0} b^{-1}_{\pi^0}) = 2.38 \times 10^{-3} , & \end{cases}$$

which is consistent and admits the solutions

$$\varepsilon = 4.27 \times 10^{-2} , \qquad (5.1.33a)$$

$$K_1 = 0.34 . \qquad (5.1.33b)$$

Since K_2 is close to 1 (ε is smaller than 1), we conclude that the proposed structure model of the π^0 does indeed achieve the desired result, only one energy level of 134.96 MeV (states with $\ell > 0$ are expected to produce the positronium structure due to centrifugal effects -- if not, they can be eliminated via a total angular momentum constraint). Equivalently, we can say that our hadronic mutation of the positronium structure achieves the intended suppression of the energy spectrum. The π^0 emerges as a unique structure, fundamentally independent from that of other mesons.

Another aspect which may have some significance is that identified in Table 3.4 according to which, at the classical level, the only admitted orbit is the circle. We believe that this occurrence provides additional indication on the uniqueness of the structure considered in the sense that an eletonic bound state of an electron and a positron, not only provides a state with one admissible value of the total energy, but also with one admissible orbit. In turn, this may provide information on the dynamical effects which are at the base of the structure (penetration of wave packets and eletonic forces) as well as on the relative phase of the spin of the constituents (see below) which is needed to provide stability.

A brief inspection of the reason why generalized Schrodinger's equation (5.1.12) is consistent despite the high value of the total energy and the small value of the rest energy of the constituents is in order.

In essence, possible technical difficulties may arise in Eq. (5.1.12) when, at the atomic limit $\rho = 1$,

$$E^{tot} \approx 2E^{kin} = \frac{(p^{can})^2}{m_e} = \frac{(p^{phys})^2}{m_e} >> 2m_e c^2. \quad (5.1.34)$$

Under hadronic conditions the situation is different because

$$E^{tot} \simeq 2E^{kin} = \rho^2 \frac{(p^{can})^2}{m_e}, \quad \rho^2 = 1.73 \times 10^{-3} \quad (5.1.35)$$

In other words, the physical kinetic energy can be interpreted as the canonical expression but now referred to a much higher mass

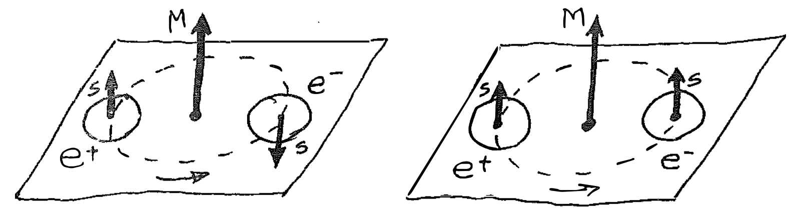
$$E^{tot} \simeq 2E^{kin} = \frac{(p^{can})^2}{m'}, \quad m' = \frac{m_e}{\rho^2} = 294 \text{ MeV}. \quad (5.1.36)$$

Thus, the effect of the mutation term is that of bringing our hadronic form of the equation into a mathematically equivalent conventional form in which

$$E^{tot}_{\pi^0} = 134 \text{ MeV} < 2m' = 588 \text{ MeV}. \quad (5.1.37)$$

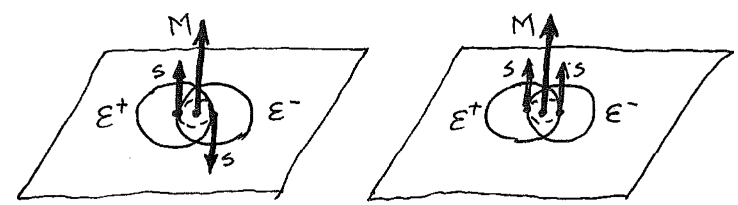
We must now attempt an understanding of the reason why only the state J = 0 is admitted. A comparative analysis of the positronium structure (with related atomic mechanics) and our π^0 structure (with related covering hadronic mechanics) is of valuable intuitional guidance in this respect.

As is well known, the positronium admits both, the singlet and triplet states according to the pictures



$L = 0, \; J = 0 \qquad\qquad L = 0, \; J = 1$

The corresponding states for the same constituents but bounded at a distance of the order of their charge radius is instead given by



$M = 0, \; J = 0 \qquad\qquad M = 0, \; J = 1$

that is, the dynamical evolution occurs with a condition of continuous penetration of the charge volume (or hadronic wave packet) of one constituent into that of the other.

This is sufficient to provide an intuitive understanding of the exclusion of the value J = 1 for the π^0. In essence, at the level of the positronium, the mutual orientation of the spin is insensitive to the dynamical evolution in the sense that the charge volumes of the constituents are at large distances from each other. In the transition to the π^0 the situation is different. Here, the state J = 0 is admitted because the relative spinning of the constituents is in phase. The state J = 1, however, is highly unstable because the spinning of the constituents would oppose each other under the conditions considered. This produces a qualitative representation of the fact that the decay $\gamma\gamma$ carries the highest fraction (98.85 %), while the decay $\gamma\gamma\gamma$ carries a very small fraction ($< 5 \times 10^{-6}$). A quantitative interpretation of the fractions of these decays can be attempted by considering the wave function of π^0 as a mixture of a dominant singlet state and of a very small contribution ($\sim 10^{-6}$) of triplet state which is subjected to the subsidiary constraint (5.1.8b). This aspect will be left to the interested reader (the methods are essentially the same as those for the model of $\pi^{\pm}$ of the next table).

As an incidental remark, notice that model (5.1.14) can be interpreted as producing the value of 68 MeV as an upper bound for the mass of the constituents. Then the only known massive and charged particles capable of satisfying this upper bound are the electrons and positrons. This yields again Hypothesis 5.1.1 and constitutes the way according to which it was derived in ref. 70

The remaining decays of the π^0, that is

$$\gamma e^+ e^- ; \quad e^+ e^- e^+ e^- ; \quad \gamma\gamma\gamma\gamma ; \quad e^+ e^- ; \qquad (5.1.38)$$

can be qualitatively interpreted as follows. The last decay is fundamental from a structure viewpoint because it constitutes the tunnel effect of the constituents. Indeed, it carries the lowest fraction of all π^0 decays ($< 2 \times 10^{-6}$). The other decays demand the use of pair creation, but not according to conventional quantum electrodynamics. The reason is that the pairs e^+e^- are expected to be created while within hadronic matter and, thus, the creation is in actuality for $\epsilon^+\epsilon^-$ which then become e^+e^- after exiting the hadronic medium, or annihilate into $\gamma\gamma$ while still within such a medium.

The quantitative treatment of this new situation, while particularly promising and intriguing on methodological grounds, is quite complex indeed in practical realization. It literally demands the reinspection of quantum electrodynamics and the study whether it must be generalized into a covering discipline under the presence of nonselfadjoint forces. The reader should be aware that, in principle, the entire methodological context of quantum-electrodynamics is here at stake. The questionable nature of conventional canonical quantization under nonselfadjoint forces has been indicated at the end of Table 4.15. A preliminary study

of this situation (which is here omitted) indicates that, irrespective of whether conventional quantization procedures are still valid or not, the physical effectiveness of the Feynman diagrams is in doubt under the considered broader forces. At the limit case of essentially nonselfadjoint forces the dynamical effects of these forces are such that, perhaps, the very notion of diagram (and related computation) is inconsistent as currently known. This should not be surprising. Feynman diagrams were conceived for a purely electromagnetic setting and later on proved to be unequivocally valid also for the broader class of local couplings derivable from a potential (again, for system representable in their entirety with the simple Lagrangian structure $L_{tot} = L_{free} + L_{int}$, where L_{int} is ut most linear in the derivative, as in the unified gauge theories of weak and electromagnetic interactions). The physical context we are here referring to is fundamentally broader than that and, as such, there is no a priory reason why the Feynman diagrams should still hold. Indeed, structures of the type $L_{tot} = L_{free} + L_{int}$ now becomes basically insufficient to represent physical reality (the often used trick of adding and substracting a free term to recover conventional settings, i.e., $L^{gen} = L_{free} + L'_{int}$, $L'_{int} = L^{gen} - L_{free}$, can be questioned on a number of grounds, assuming that a generalized Lagrangian exists).

In conclusion, the proposed discrete, nonrelativistic, structure model of the π^0 may indeed provide a quantitative interpretation of the spontaneous decays and their fractions

$$\gamma\gamma \; ; \; \gamma\gamma\gamma \; ; \; e^+ e^- ; \qquad (5.1.39)$$

which we here call primary decays, in the sense that they can be interpreted without the intervention of intermediary processes. Indeed, these decays can be accounted for as directly originating from the nature of the constituents (electrons and positron). The quantitative interpretation of the remaining decays

$$\gamma e^+ e^- ; \quad e^+ e^- e^+ e^- ; \quad \gamma\gamma\gamma\gamma ; \qquad (5.1.40)$$

appears to be, instead, a quite complex problem, although particularly promising to stimulate the study for generalizations of currently available methods (which is the central objective of this paper). In particular, decays (5.1.40) are here called secondary hadronic processes, in the sense that they demand intermediate processes within hadronic matter. Their study is expected to be beyond the discrete nonrelativistic approximation of the model proposed and it will not be considered at this time.

We now remain with the interpretation of properties (6) through (9) of π^o, as listed at the beginning of this table. At a deper analysis, the quantitative interpretation of the negative space parity of the π^o can also result considerably involved, because for such a treatment the use of a hadronic quantum field theoretical level appears to be needed. Within the context of our discrete nonrelativistic approximation we are then forced to a simplistic solution. Since the total angular momentum is null, its parity is positive. However, the constituents are Fermions of spin 1/2 at the limit of null strong forces. It is known that these particles can have imaginary parity. See, for instance, ref [73]. The simplest possible quantitative representation of the negative space parity of the π^o is therefore provided by the condition that the eletons have imaginary parity $\pm i$. Then $P_{tot} = P^2 = (\pm i)^2 = -1$. More elaborate representation of the space parity are left to the interested reader.

The interpretation of the positive charge parity is trivial. Equally trivial is the interpretation of null electric and magnetic dipole moment (the state being stationary) and of null electric and magnetic quadrupole moments (the angular momentum being smaller than one).

The preceding analysis was based on the Schrödinger-type approach of our hadronic mechanics. For completeness, the model should also be inspected within the context of our algebraic approach, that is, the Lie-admissible covering of Heisenberg's equations. This problem will not be considered at this time to avoid a prohibitive length of this paper. Nevertheless, a few remarks might be of some value for the interested reader.

The statement of the problem can be formulated as follows. It consists of attempting a Lie-admissible mutation of the positronium structure which yields only one level of energy, 134.96 MeV. On physical grounds, let us recall that our structure model of π^o has been conceived as a mutation of that of the positronium, in the sense that the constituents are the same by central requirement and simply subjected to additional strong (selfadjoint and nonselfadjoint) forces. Thus, the central constructive idea of our model is that the proposed structure of π^o is a covering of that of the positronium in the sense of Eqs. (5.1.2) and (5.1.3).

The algebraic treatment of this problem is quite intriguing indeed. Let us recall that the hydrogen atom (by ignoring spin) can be fully and consistently treated within the context of Lie algebras. Indeed, it is known that the SO(4) Lie algebra with generators (the components of the angular momentum and of the Runge-Lenz vector)

$$\{X_i\} = \{\underset{\sim}{L} = \underset{\sim}{r} \times \underset{\sim}{p} \; ; \; \underset{\sim}{A} = \frac{1}{2}(\underset{\sim}{p} \times \underset{\sim}{L} - \underset{\sim}{L} \times \underset{\sim}{p}) - \frac{Z\underset{\sim}{r}}{r}\}, \quad i = 1, 2, \ldots, 6 \tag{5.1.41}$$

and Casimir invariants

$$I_1 = \frac{1}{2}(\underset{\sim}{L}^2 + \underset{\sim}{A}^2), \tag{5.1.42a}$$

$$I_2 = \underset{\sim}{L} \cdot \underset{\sim}{A}, \tag{5.1.42b}$$

is a spectrum generating algebra in the sense that it produces the Bohr spectrum identically. [74]

The problem we are here referring to is to see whether there exists a Lie-admissible spectrum suppressing mutation of this SO(4) Lie algebra, as the fundamental dynamical prerequisite to represent the lack of energy spectrum for the π^o discussed earlier.

Again, in line with the analysis in this respect of Section 4, the fundamental algebra is the associative envelop $\mathcal{A}$(SO(4)), and not the Lie algebra SO(4), because the quantities which produces the spectrum (the Casimir invariants) are definable only at the level of the associative envelop. The mechanism of suppression of the Bohr spectrum is therefore expected to occur at the level of the transition from the associative form $\mathcal{A}$(SO(4)) to the Lie-admissible nonassociative mutation $\mathcal{U}$(SO(4)), where, as by now familiar, the basis $\{X_i\}$ remains unchanged and the strong forces are represented via the mutation of the product of $\mathcal{A}$. Thus, the new scalar quantities are computed in $\mathcal{U}$ (to ensure the algebraic breaking of the original Lie symmetry). It is in this computation where the spectrum is expected to be suppressed.

This problem is here left to the interested reader. If it admits a consistent solution, it would constitute the most direct way of representing our structure model of π^o in the sense that it would provide a direct algebraic formulation of the fact that in the transition from the positronium structure to the proposed π^o structure the additional strong forces are responsible for the suppression of the mass spectrum.

We can therefore conclude by saying that, under the assumption of the hadronic covering of the conventional atomic mechanics, the proposed structure model of the π^o particle offers genuine hopes for

(A) providing a direct identification of the π^o constituents with physical particles, those produced in the spontaneous decay with the lowest fraction, by therefore avoiding the conjecture of unidentified constituents and the need of mediation by further particles for the interpretation of the decays;

(B) providing a quantitative interpretation of all the intrinsic characteristics of the π^o, that is,

(1) mass;
(2) mean life;
(3) charge radius;
(4) charge;
(5) spin;
(6) space-parity;
(7) charge-parity;
(8) electric and magnetic dipole moments;
(9) electric and magnetic quadrupole moments;

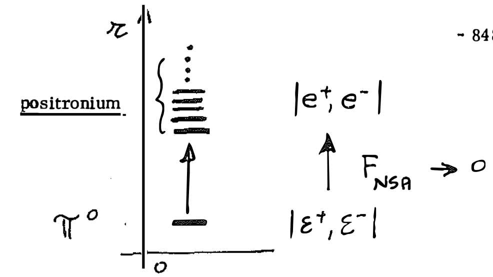
and, of the primary decays

$$\gamma\gamma \, ; \; \gamma\gamma\gamma \, ; \; e^+e^- \, ; \tag{5.1.43}$$

with an understanding that the quantitative interpretation of the secondary decays, Eq. (5.1.40), might well demand the prior construction of a covering of quantum electrodynamics;

(C) characterizing the π^o structure as a mutation of the positronium structure under strong nonselfadjoint forces.

According to this model, one electron and one positron emerge as being capable of producing two bound states, one under electromagnetic interactions only, at large distances and with a spectrum of energy levels, and one under electromagnetic and strong interactions at very small distances without a spectrum of energy levels, according to the view



Perhaps, one of the most significant features of the model is that it characterizes a very small binding energy between the π^o constituents (because, for $K_2 \approx 1$, from Eq. (5.1.28), BE $\approx$ 0). This is here interpreted as a prerequisite for physical consistency because the π^o exhibits spontaneous decays and a small mean life (in the sense of being much smaller than that of the positronium). In the final analysis, it is the joint quantitative interpretation of the π^o mass, mean life and charge radius which has produced, via model (5.1.14), the indicated very small value of the binding energy. Equivalently, we argue that under the condition of a much higher binding energy, we would expect a much smaller mean life than that experimentally detected because, after all, the constituents are one particle and its antiparticle which, as such, annihilate.

It is of some significance to indicate that the small value of the binding energy for the hadronic constituents is not new. Indeed, it occurs also for a number of quark models such as the MIT bag model.[13] There is however a fundamental difference of mental attitude which must be stressed.

Since in the MIT bag model (on in similar models) the constituents are assumed to be quarks, that is, yet unidentified particles which have escaped a rather intensive search until now, the small value of the binding energy forces into the idea of confinement (also unresolved until now in a form acceptable by the scientific community al large).

In our model, instead, the low value of the binding energy among the constituents is interpreted as a clear indication that these constituents must be produced free in the spontaneous decays. In turn, this directly implies the identification of the constituents with the massive and charged particles produced in the decays with lowest fraction, Hypothesis 5.1.1. This is, in essence, our physical framework. However, to reach a consistent quantitative representation of this "simplest" possible interpretation of the spontaneous decays, we had to abandon the relativity and quantum mechanical laws of atomic mechanics and enter into a laborious construction of possible covering laws specifically constructed for the hadronic sublayer of physical reality.

TABLE 5.2: THE PROPOSED STRUCTURE MODEL FOR THE $\mu^{\pm}$ AND $\pi^{\pm}$ PARTICLES WITH PHYSICAL CONSTITUENTS.

The central intuitional grounds for our structure model of the $\mu^{\pm}$ and $\pi^{\pm}$ particles rests on the similitarities as well as differences with the proposed structure model for π^{o}.

The physical characteristics of the $\mu^{\pm}$ particles are:

(1) mass 105.65 MeV;

(2) mean life: 2.19×10^{-6} sec;

(3) charge: $\pm$ e;

(4) spin: 1/2;

(5) spontaneous decay modes and their fractions

$e \nu \bar{\nu}$	100,
$e \gamma \gamma$	$< 4 \times 10^{-6}$,
$e \gamma$	$< 2.2 \times 10^{-8}$,
$3e$	$< 6 \times 10^{-9}$,

while the physical characteristics of the $\pi^{\pm}$ particles are:

(1) mass: 139.56 MeV;

(2) mean life: 2.60×10^{-8};

(3) charge: $\pm$ e;

(4) spin: zero;

(5) space-parity: negative;

(6) spontaneous decay modes and their fractions

$\mu \nu$	100,
$e \nu$	1.26×10^{-4},
$\mu \nu \gamma$	1.24×10^{-4},
$\pi^{o} e \nu$	1.02×10^{-8},
$e \nu \gamma$	3.0×10^{-8},
$3 e \nu$	$< 3.4 \times 10^{-8}$.

On similarity grounds with the π^{o}, we argue that the $\mu^{\pm}$ and $\pi^{\pm}$ particles exhibit spontaneous decays. Therefore, there are composite. The constituents are then expected to be produced free under these decays. An inspection of the decay modes indicates that electrons and positrons are the only emitted massive and charged particles. An inspection of the decay modes with lowest fractions indicates that both particles $\mu^{\pm}$ and $\pi^{\pm}$ emit three electrons, that is, either two e^{+} and one e^{-} or two e^{-} and one e^{+}, depending on the total charge. We reach in this way

HYPOTHESIS 5.2.1: The constituents of the μ^{-} and π^{-} (μ^{+} and π^{+}) particles are two electrons and one positron (one electron and two positrons) bounded in different eletonic states.

It should be stressed from the outset that the above hypothesis is fundamentally incompatible with the atomic mechanics. Indeed, for the case of the $\pi^{\pm}$ it implies that a bound state of three particles which have spin 1/2 under electromagnetic interactions only yields a state with zero total spin. In order to quantitatively treat the hypothesis for an assessement of its plausibility, the use of a covering of the atomic mechanics then becomes mandatory. In short, at the level of the simplest possible eletonic state, that of one constituent and its antiparticle, the need of a covering mechanics emerged only after the inspection of the technical context. In the transition to the more complex eletonic state with three constituents, the emergence of the need for a covering mechanics is direct and immediate. This is here assumed to be an indication that in the transition from the eletonic bound state $|\varepsilon^{+}, \varepsilon^{-}|$ to its three-body generalization $|\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}|$ we have an increase in the complexity of the acting forces, with consequential larger departures from the atomic setting. This is the primitive form of the hierarchy of strong nonselfadjoint forces which is inherent in the proposed Lie-admissible relativity.

On a comparative basis between the $\mu^{\pm}$ and $\pi^{\pm}$, this notion of dynamical hierarchy provides elements of valuable intuitional guidance for the pratical construction of a structure model according to Hypothesis 5.2.1. The dominant physical difference between these particles is that the former are leptons while the latter are hadrons. Additional physical differences are provided by the particle data. Indeed, the former have a smaller smass and a higher mean life than those of the latter. We then argue that these differences provide the basic intuitional elements for the differentiation between the two models of structure in terms of the same constituents.

In essence we argue that a comparative analysis of the two models

$$\mu^{\pm} = \left|\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}\right|_{J=\frac{1}{2}} , \quad \pi^{\pm} = \left|\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}\right|_{J=0} , \qquad (5.2.1)$$

indicates the need of a considerably smaller mutation for the constituents of the $\mu^{\pm}$ than that which is needed for the constituents of the $\pi^{\pm}$. Indeed, in the former case (at least for a first inspection-- see below for further comments) the intrinsic characteristics of the electrons need not to be mutated, while the mutation of these intrinsic characteristics is crucial for the latter case. This is precisely in line with a central contention of this paper, that electromagnetic and strong interactions are differentiated primarily at the level of their dynamical effects on the intrinsic characteristics (strictly conserved and conventionally quantized for the former interations while generally mutated for the latter). As a matter of fact, we argue that the structure $\mu^{\pm} = \left|\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}\right|_{J=\frac{1}{2}}$ is a lepton precisely because the constituents do not need a mutation of their intrinsic characteristics, while the structure $\pi^{\pm} = \left|\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}\right|_{J=0}$ is a hadron precisely because of the need of a mutation of the intrinsic characteristics and, most importantly, of the spin (Tables 4.11 and 4.19). Equally important is the fact that a lesser mutation calls for lesser dynamical effects, that is, weaker short-range forces. In turn, this is expected to be fully in line with the smaller mass and higher mean life of the $\mu^{\pm}$ on a comparative basis with the $\pi^{\pm}$.

To reach more intuitional elements, a schematic, epistemological view of the bound state of three eletons is in order. We here select that of a restricted three body in which one eleton is at rest, that providing the total charge and magnetic moment, and the remaining two eletons evolve in a state of continuous penetration of their charge volumes (or hadronic wave packets) into that of the eleton at rest. The idea of hierarchy of dynamical conditions then suggests the following schematic view of the structure of the particles considered

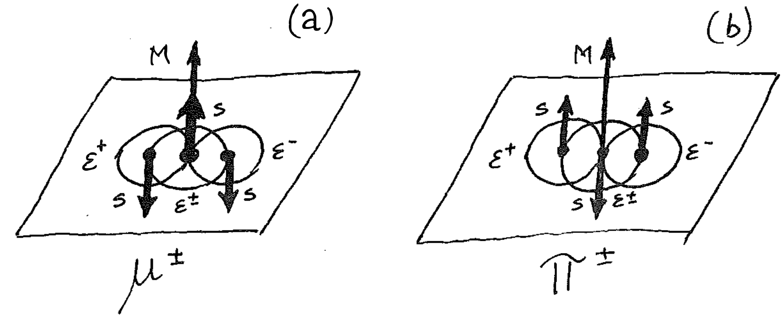


FIGURE 1

with an understanding that it is intended on pure grounds of intuitional guidance.

The reason for the association of the $\mu^{\pm}$ with the left-hand-side model and not the right-hand-side one is that it presents much weaker spin-orbit effects than those of the latter, due to the fact that the orbital angular momentum is parallel to the spin of the central eleton. In the other case, instead, the antiparallel nature of these quantities jointly with the condition that each pair of eletons must be in phase as per the π^{0}, implies spin-orbit effects which are expected to be substantially larger than those of the atomic mechanics. At the limit, these spin-orbit couplings and phase effects are expected to be so high to allow a zero value of the total angular momentum via a mutation of the spin and orbital angular momentum of the constituents.

Although substantially unrealistic, these spin-orbit and phase effects can be intuitively seen at the primitive Newtonian level by considering the coupling and evolution of gears with the same configuration as those of the figure, i. e.,

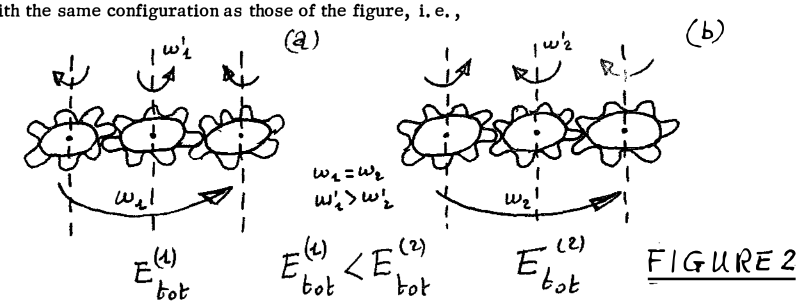
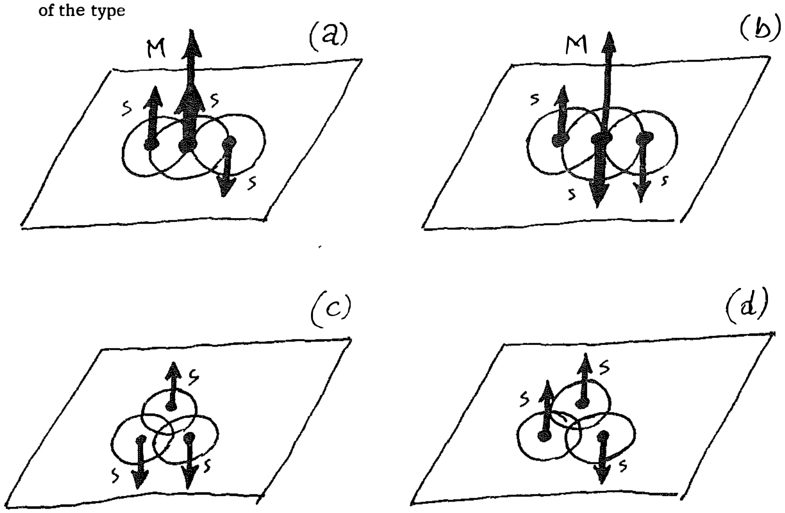


FIGURE 2

In essence, we argue that, even though wave packets are penetrable, the gear-type models of Figure 2 provide the conceptual configuration of the states with the largest stability possible. It is on these epistemological grounds that we exclude configurations of the type



Indeed, they are all unstable under the condition that the spinning of coupled eletons is in phase and that at least one eleton has a nonzero spin. Configurations 3c-d are excluded on grounds of the fact that the same conditions would imply that all three eletons have zero spin (as it can be better seen in terms of the gears). Such structures are expected to be highly unstable on a comparative basis with those of Figure 1.

In conclusion, restrictions on the relative phase of the spin of the constituents in a state of mutual penetration of their wave packets, and related stability considerations, jointly with the dynamical effects of the expected high values of the spin-orbit couplings suggest the study of the structures of Figure 1 as the only expected stable configurations.

Intriguingly, the model under consideration prohibits the existence of a neutral muon, in agreement with experimental evidence.

We now have sufficient elements for the explicit construction of the models in the intent of assessing the plausibility of Hypothesis 5.2.1. We shall, in essence, attempt a spectrum suppressing hadronic mutation of the atomic model for helium. For this objective we first recall that in the transition from the positronium to the helium there is a larger proliferation of energy states. In particular, helium atoms in a singlet state are called parahelium, while those in a triplet state are called orthohelium.

The statement of the problem can then be formulated as follows. The structure of the $\mu^{\pm}$ and $\pi^{\pm}$ particles will be attempted as a spectrum suppressing mutation of that of the orthohelium The hadronic mutation of the parahelium is expected to be highly unstable (see in this respect the configurations of Fig. 1).

To quantitatively construct these models, the reader should be warned against excessive expectations. The conventional Schrödinger's equation for atomic mechanics cannot be directly solved, even by numerical methods, for all atoms containing more than one electron. As a result, the energy levels and wave functions are both computed in these cases with approximation techniques, such as the so-called self-consistent methods. In the transition to the hadronic mutation of these equations (that is, via terms which multiply the free term) these difficulties are expected to be magnified. A detailed technical treatment of this problem would render the length of this paper prohibitive. We shall therefore restrict ourselves to the simplest possible treatment. In any case, this is sufficient for our objective of assessing the plausibility of Hypothesis 5.2.1.

We shall consider first the case of our structure model of the $\mu^{\pm}$ particles. An area of potential, unnecessary controversy is that related to the charge radius of these particles. As we all know, electrons and muons are treated in contemporary theoretical physics on much of the same line, via quantum electrodynamics. In particular, the experimentally observed electromagnetic interactions of both electrons and muons have resulted to be in a complete (fashinating) agreement with the predictions of this discipline down to the recently reached distances of 7×10^{-17} cm. The problem which then emerges is to identify a charge radius of the $\mu^{\pm}$ which is compatible with these results. Since we have no experimental data available at this time on this crucial quantity (to the best of my knowledge), we must assume a value of such radius. Let me indicate from the outset that the proposed model of eletonic structure is apparently consistent down to charge radii as low as 10^{-50} cm, as we shall indicate below. We can therefore select a charge radius of the muons virtually at will. Without any experimentally established need of going down to excessively small distances (relative to our current knowledge), we shall assume the value of

$$b_{\mu}^{-1} = 10^{-19}\ cm \tag{5.2.2}$$

for the charge radius of the muons, with an understanding that such value can be decreased (as well as increased) according to the desire of the individual researcher.

An incidental comment is here in order. The equivalence of the electrons and muons under electromagnetic interactions, as recalled earlier, is unequivocal. But there is a fundamental physical difference between these particles: the former are stable, while the latter are not and exhibit spontaneous decays. This implies that the muons have a composite structure. The existence of a finite charge radius (which characterizes the size of the eleton orbits in our model) then appears to be unequivocal. What can be debated is the value of such charge radius. Also, the point-like characterization of these particles in quantum electrodynamics is not in contradiction with our model because at the quantum mechanical (or hadronic mechanical) level these particles possess a wave packet which is not point-like. This is sufficient to recover our argument on the physical origin of nonselfadjoint forces when these wave packets are in a state of penetration.

Our structure model $\mu^{\pm} = |\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}|$ is realized in a way as simple as possible. In particular, we assume that: (a) the eletons do not exibit a mutation of their intrinsic characteristics; (b) the eleton providing the value of the total charge, $\varepsilon^{\pm}$, is at rest by therefore characterizing a restricted three body structure; and (c) the canonical and physical angular momenta are null. In this structure the central eleton can be ignored in the differential equations for the dynamical evolution of the peripheral eletons and only its interaction with these latter eletons can be taken into account.

In order to construct a hadronic mutation of the orthohelium the first problem is that of the selection of the nonselfadjoint forces at the classical level. We here assume for simplicity that these forces satisfy all the restrictions of Tables 4.5, 4.6 and 4.7. This yields, after hadronic quantization, a time evolution Hermitian Hamiltonian H^{can} and therefore, allows the construction of a state vector. Since our setting is discrete nonrelativistic, we assume the

familiar convention of separability of the spin part from the space part of the eigenfunctions of H^{can}, i.e.,

$$i\hbar \frac{\partial}{\partial t}\Psi = H^{can}\Psi = E^{can}\Psi, \tag{5.2.3a}$$

$$\Psi = \Psi(r_1, r_2; s_1, s_2, \ldots) = \psi(r_1, r_2)\,\eta(s_1, s_2, \ldots). \tag{5.2.3b}$$

For an initial study (see below for a refinement related to the problem of the difference between the anomalies of the magnetic moments of the electrons and muons), the spin part of the structure can then be treated with conventional quantum mechanical rules, yielding the desired spin 1/2 (the magnetic moment is then that of the ordinary electrons and so is the related anomaly).

We are then left with the space part of the equations. For our objectives (as well as to avoid technical difficulties beyond our knowledge at this time) we assume the simplest possible mutation term, a constant as in model $\pi^{o} = |\varepsilon^{+}, \varepsilon^{-}|$ (the reader should however keep in mind that, even though constant, this term implies a nontrivial eletonic force).

Our structure model of the $\mu^{\pm}$ is essentially given by an "intersection" of two of our models of the π^{o} with one common constituent. It therefore appears reasonable to assume for the strong selfadjoint eletonic force that derivable from the Hulthén potential. This implies that the time evolution operator H^{can} and the energy operators H^{phys} are both Hermitian and commute. The model can then be explicitly written

$$\left\{\begin{array}{ll}
\Big[-\rho^{2}\frac{\hbar^{2}}{m_e}\Delta_1 - \rho^{2}\frac{\hbar^{2}}{\mu_e}\Delta_2 + V^{Coul}(r_1) - V^{Coul}(r_2) + V^{Hulthen}(r_1) + V^{Hulthen}(r_2) & \\
\quad + V^{Coul}(r_1 - r_2) - V^{Hulten}(r_1 - r_2)\Big]\psi(r_1, r_2) = E^{phys}\psi(r_1, r_2), & (5.2.4a) \\
E_{\mu}^{tot} = 2E_{\varepsilon}^{kin} - E^{phys} = 106\ MeV, & (5.2.4b) \\
\tau_{\mu}^{-1} = 4\pi\lambda^{2}|\psi(0,0)|^{2}\frac{\alpha E_{\varepsilon}^{kin}}{\hbar} = 10^{6}\ sec^{-1}, & (5.2.4c) \\
b_{\mu}^{-1} = 10^{-19}\ cm, & (5.2.4d)
\end{array}\right.$$

where in the total energy we have ignored the rest energy of the eletons (.51 MeV each) because considerably smaller than the kinetic energy, as it was the case for the π^o.

As it is the case for the Helium (ρ = 1 and $V^{Hulthen}$ = 0) we must resort to approximation. We assume that the repulsive electric and eletonic interactions are negligible with respect to their attractive counterparts. Also, we assume that the Coulomb part of the potential can be neglected in favor of a redefinition of the constant factor of the Hulthen potential, as in the π^o case. It is understood that these approximations should be compensated by a perturbative treatment, as in the conventional self-consistent approach.

Under these conditions Eq. (5.2.4a) is separable into two π^o-like equations

$$\left[\frac{1}{r_1^2}\frac{d}{dr_1}\left(r_1^2\frac{d}{dr_1}\right)+\frac{me}{\rho^2\hbar^2}\left(E^{phys}+V_0\frac{e^{-br_1}}{1-e^{-br_1}}\right)\right]u(r_1)=0, \qquad (5.2.5a)$$

$$\left[\frac{1}{r_2^2}\frac{d}{dr_2}\left(r_2^2\frac{d}{dr_2}\right)+\frac{me}{\rho^2\hbar^2}\left(E^{phys}+V_0\frac{e^{-br_1}}{1-e^{-br_1}}\right)\right]u(r_2)=0. \qquad (5.2.5b)$$

The ground state eigenfunction is then approximated by the product of two ground state eigenfunctions for the π^o, Eq. (5.1.29), i.e.,

$$u(r_1,r_2)_{n=1}\approx\left[b^{1/2}\frac{\Gamma(2|A|^{1/2}+3)}{\Gamma(3)\,\Gamma(2|A|^{1/2})}\right]_{n=1} e^{-2|A|^{1/3}b(r_1+r_2)}\frac{(1-e^{-br_1})(1-e^{-br_2})}{r_1 r_2}. \qquad (5.2.6)$$

The physical quantities of the $\mu^{\pm}$ are then represented via the solution of the equations in the K_1 and K_2 parameters, as for the π^o case

$$E^{tot}_{\mu}=2K_1\left[1-(K_2-1)^2\right]\hbar bc=106\ MeV, \qquad (5.2.7a)$$

$$\tau^{-1}_{\mu}=\frac{4\pi bc}{(48)^2(137)^2}\,\frac{(K_2-1)^6}{K_1}=10^6\ sec^{-1}, \qquad (5.2.7b)$$

that is

$$K_1\left[1-(K_2-1)^2\right]=\frac{1}{2\hbar c}\left(E^{tot}_{\mu}b^{-1}_{\mu}\right)=0.25\times10^{-11}\left(E^{tot}_{\mu}b^{-1}_{\mu}\right) \qquad (5.2.8a)$$
$$=2.62\times10^{-7},$$

$$\frac{(K_2-1)^6}{K_1}=\frac{(48)^2(137)^2}{4\times\pi\times3\times10^{10}}\left(\tau^{-1}_{\mu}b^{-1}_{\mu}\right)=1.14\times10^{-4}\left(\tau^{-1}_{\mu}b^{-1}_{\mu}\right) \qquad (5.2.8b)$$
$$=1.14\times10^{-17}.$$

The solution is then given by the values

$$K_2=1+\varepsilon,\quad \varepsilon=\sqrt[6]{2.98}\times10^{-4}, \qquad (5.2.9a)$$

$$K_1=2.62\times10^{-7}. \qquad (5.2.9b)$$

Since K_1 results to be close to one, as for the π^o, we conclude that the proposed structure model of the muons as a hadronic mutation of that of the helium without a mutation of the intrinsic characteristics of the electrons (to preserve the electromagnetic character of the particle) does indeed achieve the desired suppression of the atomic spectrum. The bound state of three eletons under the indicated condition for the intrinsic characteristics then emerges as a unique state. In particular, the only allowed orbit is again the circle.

In relation to the admissible values of the charge radius of $\mu^{\pm}$, notice that

$$(K_2-1)^6=\varepsilon^6=0.25\times10^{-11}\times1.14\times10^{-4}\left(E^{tot}_{\mu}\tau^{-1}_{\mu}b^{-2}_{\mu}\right)$$
$$=2.98\times10^{-15}\left(E^{tot}_{\mu}\tau^{-1}_{\mu}b^{-2}_{\mu}\right) \qquad (5.2.10)$$
$$=3.13\times10^{-8}\,b^{-2}_{\mu}.$$

Thus, $\varepsilon\to0$ as $b^{-1}\to0$. It is this property which allows charge radii for the $\mu^{\pm}$ particles considerably smaller than value (5.2.2) without affecting the consistency of the model as well as its spectrum suppressing nature.

Notice that the model provides again a very low value of the binding energy

$$BE = \frac{1}{2} K_1 (K_2 - 1) \hbar b c \approx 0 \text{ for } K_2 \approx 1, \qquad (5.2.11)$$

and the following value of the kinetic energy of the peripheral eletons

$$E_{\varepsilon}^{kin} = K_1 \hbar b c \cong 52 \text{ MeV}, \qquad (5.2.12)$$

which can be again interpreted as an upper bound for the mass of the constituents under value (5.2.11). These two combined features again reproduce Hypothesis 5.2.1. Indeed, since the constituents are ligthly bound, a structure model of $\mu^{\pm}$ must demand that these particles are produced free in the spontaneous decays (unless one prefers to resort to experimentally unproved, theoretically unsettled and conceptually questionable mechanisms of confinement). This is the physical implication of value (5.2.11). Upper bound (5.2.12) then severely restricts the admissible constituents. Since the only known physical particles satisfying upper bound (5.2.12) are the ordinary electrons and positrons, and since (independently from that) the electrons and positrons are the only massive, charged particles produced in the spontaneous decays, Hypothesis 5.2.1 emerges as the only admissible on grounds of our current knowledge (unless one intends to resort to the assumption of yet unidentified constituents and therefore lose the direct physical plausibility of the structure).

As is well known in the self-consistent approximation, the ground state wave function (5.2.6) is only a trial function and the numerical results (5.2.9) must be subjected to a number of corrections due to different physical effects. This study is here left to the interested readers.

As by now familiar, we have assumed that the intrinsic characteristics of the eletons coincide with those of the electrons. As a result, the model produces the identity of the intrinsic characteristics of charge, spin and magnetic moments of $\mu^{\pm}$ with those of $\varepsilon^{\pm}$. We would like to subject this assumption to an epistemological study to see whether it is actually realized in an exact way.

A quite intriguing differentiation between the electrons and the muons is that their anomalies in the magnetic moments, according to recent experimental data[75], are given by

$$e^{\pm}: \quad g = 2(1.00119 \pm 0.00005) \qquad (5.2.13a)$$

$$\mu^{\pm}: \quad g = 2(1.001168 \pm 0.0000, \qquad (5.2.13b)$$

and, as such, they do not coincide. As is well known, quantum electrodynamics is indeed capable of accounting for this difference. Neverthless, this disciple treats the muons as point - like particles. According to our view, this restriction on the physical nature of the muons is incompatible with its extended structure as manifestly indicated by the spontaneous decays. It is therefore tempting to predict that, in due time, the familiar derivation of the anomalous magnetic moment of the muons via quantum electrodynamics will be replaced by a covering approach which takes in due account the structure of these particles.

From the viewpoint of our structure model of the muons, it might be of some significance to remark that the difference in the anomalous magnetic moments of muons and electrons

$$g_{\mu} - g_e = 0.00003 \qquad (5.2.14)$$

is an indication of the possibility that the intrinsic characteristics of the eletons are subjected to a small mutation with respect to the corresponding values of the electrons which is precisely characterized by value (5.2.14). This does not affect the value of the spin of the state as a whole via the use of the mutational approach of Table 4.19 (see below).

On relativity grounds, this implies that the R and S spin operators for the Lie-admissible relativity (4.18.14) are close to unit (flexible Lie-admissible mutation of the same order of magnitude as that submitted for experimental verification in Table 4.21 in nuclear physics in relation to the problem of Pauli's exclusion principle).

This situation might appear to be of marginal significance. In actuality, it seems to be of primary relevance to attempt a quantitative interpretation of the spontaneous decays of muons and related fractions. Indeed, under the condition that all the three eletons are not subjected to a mutation of their intrinsic characteristics, the dominant decay would be expected to be

$$\mu^{\pm} \longrightarrow e^{\pm} \gamma \gamma, \qquad (5.2.15)$$

which is not the case, since muons decays virtually 100 %

$$\mu^{\pm} \rightarrow e^{\pm} \nu \bar{\nu} . \qquad (5.2.16)$$

. We therefore argue that decay (5.2.15) is prohibited when the eletons differ from electrons in a small, but finite amount. Indeed, such decay demands the use of the costumary decay for singlets

$$|e^{+}, e^{-}| \rightarrow \gamma \gamma , \qquad (5.2.17)$$

which is allowed, provided that $e^{\pm} \equiv \mathcal{E}^{\pm}$. As a matter of fact, we argue that the ratio

$$\frac{\mu \rightarrow e \gamma \gamma}{\mu \rightarrow e \nu \bar{\nu}} \qquad (5.2.18)$$

is a measure of the fraction of time during which eletons are electrons.

As indicated in Table 5.1, a quantitative treatment of these problems demands a prior substantial effort in the identification of the applicable methodology. The working out of the specific calculations for the problem considered is only secondary. As a result, lacking at this time a scattering theory which is truly applicable for forces nonderivable from a potential, we are regrettably forced to leave open the indicated problem.

We can therefore conclude by saying that Hypothesis 5.2.1, under the assumption of covering relativity and quantum mechanical laws, appears to be capable of producing a quantitative interpretation of all the intrinsic characteristics of the muons and offers some genuine hope of reaching, in due time, a quantitative interpretation of the spontaneous decays of the particles considered and their fractions. As such, we believe that Hypothesis 5.2.1 deserves further study.

There is one aspect of the proposed structure model of muons which deserves an elaboration. In essence, it provides the first indication for the possible nonapplicability for the constituents of composite particles of established relativity and quantum mechanical laws. This is a direct consequence of the possibility that the intrinsic characteristics of the electrons are subjected to a small, but finite mutation when bounded at very small distances. Indeed, such a mutation is fundamentally incompatible with both, established relativity laws and established quantum mechanical laws.

We consider now our structure model for $\pi^{\pm}$. It is essentially based a mutation of that of the muons induced by the subsidiary constraint that the total angular momentum is null. Under the assumption that the acting forces are such to admit a state eigenfunction, as in Eq. (5.1.8), such a constraint reads

$$\left(J^{phys}_{tot} \right)^2 \psi = 0 . \qquad (5.2.19)$$

A number of configurations then becomes admissible. For instance, we can assume $\ell = 0$, and no mutation of the peripheral eletons in a triplet state. The value $J = 0$ is then recovered via a mutation of the spin of the central eletons $\mathcal{E}^{\pm}$. Another model is $\pi^{\pm} = |\pi^{0}, \mathcal{E}^{\pm}|$ in which case $J = 0$ is recovered via a different mutation. In the final analysis we expect these models to be equivalent (Table 5.3). The mutation of the spin for the former case

$$e^{\pm} \left| \begin{array}{l} q = \pm e \\ M = 0.5 \text{ MeV} \\ S = 1/2 \end{array} \right. \longrightarrow \mathcal{E}^{\pm} \left| \begin{array}{l} q = \pm e \\ M = 0.5 \text{ MeV} \\ S = 1 \end{array} \right. \qquad (5.2.20)$$

can be treated via our Lie-admissible covering relativity. It essentially implies a mutation of the associative product in which square of physical quantities are computed. In its simplest possible form such a mutation is of flexible Lie-admissible type

$$A : X_i X_j \rightarrow U = A(\lambda, \mu) = \lambda X_i X_j - \mu X_j X_i \qquad (5.2.21)$$

where the λ and μ quantities are representative of the forces nonderivable from a potential. By using Eqs. (4.19.12), the value zero of the total angular momentum is reached when

$$\lambda - \mu = \frac{8}{3} \qquad (5.2.22)$$

It should be however stressed that the underlying assumption (the lack of mutation of the spin of the two peripheral eletons) is highly restrictive within the context of our hadronic mechanics. A more expected occurrence is that all three eletons have a mutated value of the spin which is such to produce a null total angular momentum. Neverthless, the model we are considering based on the sole mutation (5.2.20) is sufficient for an initial study of the plausibility of Hypothesis 5.2.1, and other possibilities will be left open for interested researchers.

We are here interested in an epistemological study of the strong forces which are necessary to achieve mutation (5.2.20). As by now familiar, such a mutation demands the breaking of the SU(2)-spin Lie symmetry via nonselfadjoint forces. It should be again stressed that selfadjoint forces (such as those of atomic mechanics and of the virtual totality of nuclear forces used until now) are fundamentally insufficient to achieve the objective considered because trivially compatible with established relativities.

The problem is then reduced to the identification of the explicit form of the nonselfadjoint forces responsible for mutation (5.2.20). It should be recalled that the most effective class of these forces, that producing an essentially nonselfadjoint breaking, is such involved on technical grounds to be beyond our capabilities for any effective treatment at this time. We are therefore forced to assume simpler forms of the SU(2)-spin symmetry breaking forces, those of nonessentially nonselfadjoint type. The first class of these weaker breakings, those producing a canonical breaking, also produce substantial technical complexities. The reason is quite simple. We have indicated earlier that the spin-orbit couplings are expected to have in our model of hadronic structure a substantially greater value than that of atomic or nuclear mechanics. In these mechanics, the couplings considered are realized via terms which are additive to the free term. In our hadronic mechanics the same couplings are realized instead via terms which are additive but also and most importantly multiplicative to the free terms. The costant mutation term used until now in this section then comes to life. Indeed, the hadronic realization of the spin-orbit couplings demands their appearance in the mutation terms, i.e.,

$$H^{can} = -\rho \frac{\hbar^2}{2\mu} + V \quad , \quad \rho = const \longrightarrow \rho(\underline{s}\cdot\underline{M}). \qquad (5.2.23)$$

In turn, this implies at least a third order generalization of Schrodinger's equation (when ρ is linearly dependent on $\underline{s}\cdot\underline{M}$), with higher orders being fully admitted e.g, when

$$\rho = e^{\alpha\, \underline{s}\cdot\underline{M}} = 1 + \frac{\alpha}{1!}\underline{s}\cdot\underline{M} + \frac{\alpha^2}{2!}(\underline{s}\cdot\underline{M})^2 + \dots, \quad \alpha \gtrsim 0. \qquad (5.2.24)$$

The study of these equations would render the length of this paper prohibitive. As a

consequence, we are regrettably forced to ignore spin-orbit couplings in our model.

Despite that, we still remain with a significant class of forces, those representative of what we have called untin now the "phase condition" for the spin of coupled eletons. A study of this situation indicates that such condition can be quantitatively treated via the spin-spin interactions of atomic and nuclear mechanics, but now embedded into our hadronic mechanics, that is, realized via terms which multiply the free term. An inspection of the structure of the Hamiltonian representation of nonselfadjoint forces, indicates that this is indeed a necessary condition for these forces to depend on the spin-spin couplings.

This produces a substantial technical simplification over the hadronic spin-orbit couplings which is still non-trivial, in the sense that it takes in due effect the crucial role of the spin of the constituents while it preserve the linearity and 2^{nd} order nature of the wave equation.

A further inspection of the Hamiltonian representation of nonselfadjoint systems then indicates that, for the nonselfadjoint forces to be at least quadratically dependent on the velocities (as the simplest possible generalization of the Lorentz force), the mutation terms must depend on the coordinates.

But there is still another technical difficulty to overcome. The forces under consideration produce such an effective breaking of the SU(2)-spin to create the problem of the very representation of the "spin" under these conditions. To overcome these difficulties, we must exclude the canonical breaking (both simmetries and conservation laws are lost) and use instead the weakest possible form of breaking, the semicanonical breaking (the symmetry persists, but the conservation laws are lost -- see Table 4.11). This has the effect that the entire formalism of SU(2) can indeed be constructed according to conventional patterns including and most importantly the Pauli matrices, but it possesses only a canonical meaning . The physical quantities are then provided by our Lie-admissible mutation. The net effect is that we are allowed to preserve the use of Pauli matrices in the realization of our "phase condition" for the spins, or our hadronic spin-spin interactions, which are then realized via their presence in the mutation terms

$$\rho = \rho(\sigma_i^k), \quad i = x, y, z, \quad k = 1, 2. \qquad (5.2.25)$$

It is understood that the Pauli matrices now lose their direct physical significance as representative of the spin 1/2. This is a necessary condition of physical consistency in the sense that the preservation of the conventional, direct physical significance, more generally, of the canonical quantities implies the absence of nonselfadjoint forces, as outlined in Section 3.

In conclusion, the simplest possible time evolution equation for our model $\pi^{\pm} = |\varepsilon^+, \varepsilon^{\pm}, \varepsilon^-|$ is characterized by a mutation of that for the model $\mu^{\pm} = |\varepsilon^+, \varepsilon^{\pm}, \varepsilon^-|$ via a semicanonical breaking of the SU(2)-spin and only spin- spin hadronic couplings, e.g., of the type (for the relative motion)

$$i\hbar \frac{\partial}{\partial t} \Psi = \Big\{ -\rho\big(\alpha\, \underset{\sim}{\sigma}_1 \cdot \sigma_3 + \beta\, \underset{\sim}{\sigma}_1 \cdot \underset{\sim}{\sigma}_2 + \dots\big) \frac{\hbar^2}{m_e} \Delta_1 \qquad (5.2.26)$$

$$- \rho'\big(\alpha'\, \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{\sigma}_3 + \beta'\, \underset{\sim}{\sigma}_2 \cdot \underset{\sim}{\sigma}_1 + \dots\big) \frac{\hbar^2}{m_e} \Delta_2 + V^{\text{Coul} + \text{Hulthén}} \Big\} \Psi,$$

$$\alpha > \beta, \quad \alpha' > \beta',$$

where we have ignored a possible dependence of the mutation term in the coordinates.

By assuming that the eigenfunctions are separable in the intrinsic components

$$\Psi = \psi(\underset{\sim}{z}_1, \underset{\sim}{z}_2)\, \eta(s_1, s_2, \dots), \qquad (5.2.27)$$

we reach a decoupling of the spin part of the equation from the space part in the sense that the spin part can be independently treated with Lie-admissible techniques, while the space part reacquires the costant mutation term of the muonic model under the assumption of eigenvalue equations of the type (hereon tacitly implemented)

$$\rho\, \eta = \text{const}\, \eta, \quad \rho' \eta = \text{const}\, \eta. \qquad (5.2.28)$$

At this point the physical Hamiltonian must be computed as for the π^0 and $\mu^{\pm}$ particles. By repeating the same approach we finally reach the system

$$b^{-1}_{\pi^{\pm}} = 10^{-13}\, \text{cm} = b^{-1}_{\pi^0}, \qquad (5.2.29a)$$

$$K_1 \left[1 - (K_2 - 1)^2\right] = \frac{1}{2\hbar c} \left(E^{tot}_{\pi^{\pm}} b^{-1}_{\pi^{\pm}}\right) = 0.25 \times 10^{-11} \left(E^{tot}_{\pi^{\pm}} b^{-1}_{\pi^{\pm}}\right) \qquad (5.2.29b)$$

$$= 3.49 \times 10^{-3},$$

$$\frac{(K_2 - 1)^6}{K_1} = \frac{(4.8)^2 (137)^2}{4 \times \pi \times 3 \times 10^{10}} \left(\tau^{-1}_{\pi^{\pm}} b^{-1}_{\pi^{\pm}}\right) = 1.14 \times 10^{-4} \left(\tau^{-1}_{\pi^{\pm}} b^{-1}_{\pi^{\pm}}\right) \qquad (5.2.29c)$$

$$= 1.14 \times 10^{-9},$$

with solution

$$K_2 = 1 + \varepsilon, \quad \varepsilon = \sqrt[6]{3.97} \times 10^{-11/6}, \qquad (5.2.30a)$$

$$K_1 = 3.49 \times 10^{-3}. \qquad (5.2.30b)$$

Since K_2 is again close to 1, we conclude that the the model again provides a spectrum suppressing mutation of the parahelium, this time with an additional semicanonical breaking of the SU(2) symmetry.

On a comparative basis between the models $\mu^{\pm} = |\varepsilon^+, \varepsilon^{\pm}, \varepsilon^-|$ and $\pi^{\pm} = |\varepsilon^+, \varepsilon^{\pm}, \varepsilon^-|$, the former demands a small (but finite) mutation of the intrinsic characteristics of the electrons (to account for the anomalous magnetic moment of the muon as well as the decay modes), while the latter demands a nontrivial, finite mutation of the same characteristics. A part from this quantitative difference, the methodological context is the same.

In conclusion, under the assumption that the $\pi^{\pm}$ constituents are subjected to a radical departure from established relativity and quantum mechanical laws, the proposed structure model of these particles allows:

(1) the identification of the constituents with physical particles, by therefore avoiding the assumption of yet unidentified constituents and of a yet unsettled confinement process; and

(2) a quantitative interpretation of all the intrinsic characteristics, that is,

- (a) the mass;
- (b) the mean life;
- (c) the charge and charge radius;
- (d) the spin;
- (e) the spin parity (which can be identified with that of the central eletons).

Notice that the model, independently from Hypothesis 5.2.1, produces a very low binding energy and an upper bound for the mass of the constituents, this time of the order of 69 MeV. This implies that the constituents are expected to be free and produced in the spontaneous decays. Hypothesis 5.2.1 is then consequential.

A few remarks in relation to the decay modes are here in order. The primary decay of the $\pi^{\pm}$ particles,

$$\pi \rightarrow \mu \ \nu, \tag{5.2.31}$$

is here interpreted as the most promising indication of physical consistency of the model. Indeed, on account of mutation (5.2.20), the only admissible decay of the central eletons is given by

$$\mathcal{E}^{\pm} \left| \begin{array}{l} q = \pm e \\ M = 0.5\ MeV \\ S = 1 \end{array} \right. \longrightarrow e^{\pm} \ \nu . \tag{5.2.31}$$

This reproduces the primary decay (5.2.31) in its entirety, by preserving a restricted three-body structure. Equivalently, neutrinos emerge in our model with a rather intriguing and new meaning: they can potentially represent a direct measure of the departure from established relativity and quantum mechanical laws. Indeed, according to one of our central contention, the notion of mutation of the intrinsic characteristics of the electrons can exist only when these particles are bounded at very small distances, yielding an unstable state (the stable particles are the ordinary electrons and positrons). When these mutated particles exhit a hadronic medium, they must reacquire the conventionally quantized value of their intrinsic characteristics. Under the assumption that the mass and charge are not mutated, the reacquisition of these intrinsic characteristics can only occur via the emission of neutrinos. This yields decay (5.2.31) for the case of a large mutation of the spin and decay (5.2.16) for the case of small mutation.

It should be stressed that the central eletons $\mathcal{E}^{\pm}$ of our model are not interpreted as a bound state of an electron and a neutrino under hadronic laws. Such a model would be inconstistent with our methods, to the best of our knowledge at this time, because the neutrino is massless and ordinary interactions (selfadjoint) for this particle are already weak. The very applicability of a strong nonselfadjoint force is then highly questionable. Instead, neutrinos are interpreted as being created in decay process (5.2.31).

TABLE 5.3: OUTLINE OF THE EXTENSION OF THE MODEL TO OTHER MESONS.

In the transition from the pions to other mesons such as

$$K^{\pm}_{493\ MeV}\ ,\ (K^0_S\ ,\ K^0_L)_{497\ MeV}\ ;\ \eta_{548\ MeV}\ ; \tag{5.3.1}$$

our model demands more and more departures from established laws. More specifically, in addition to the departure from established relativities already present at the level of the pions, the model demands the additional breaking of the gauge symmetry, of course, at the level of the constituents only and in such a way to be consistent with the conservation of the total charge for a meson as a whole under ut most electromagnetic interactions.

As by now familiar, the central idea of the model is that the constituents are lightly bound and, thus, they are produced free in the spontaneous decays, irrespective of the hadron considered. The intent is to achieve a consistent identification of the hadronic constituents with physical particles. If established laws are incompatible with this interpretation, rather than abandoning the model in favor of these laws, we abandon the laws instead and search for compatible covering laws. In essence, we argue that currently available quantum mechanical laws were conceived for the interpretation of the atomic phenomenology available at the time of its inception, the atomic spectra. The corresponding hadronic notions are the decay modes and their fractions. We therefore search for covering laws specifically conceived for this new layer of the physical reality and the new data to be interpreted. After all, there is no a priori reason why laws which have proved so effective for the interpretation of the atomic structure should be necessarily valid also for the hadronic structure. Also, if the atomic laws are assumed as valid for the hadronic structure too, the direct consequence is that the hadronic constituents cannot be interpreted as physical particles simply produced in their spontaneous decays. The quark models and their problematic aspects are then consequential. This leads to the crucial alternative identified in this paper: violation (preservation) of established laws and consequential interpretation (lack of interpretation) of the hadronic constituents with physical particles.

An inspection of the decay modes of mesons (5.3.1) indicates that these hadrons do not decay into electrons and massless particles (except rare events). We therefore conclude that these particles are not directly constituted by an eletonic bound state of electrons and positrons.

Independently from that, the identification of the constituents with the massive particles produced in the decay modes with lowest fraction (tunnel effect as per π°, $\mu^{\pm}$ and $\pi^{\pm}$) becomes highly ambiguous, assuming that it can indeed be practically realized. This is due to the variety of the decay modes with low fractions, such as in the case

$$
K^{\pm}: \quad
\begin{array}{ll|ll}
\mu\nu & 63.61 \pm 0.16\% & e\nu & 1.54 \pm 0.09 \times 10^{-5} \\
\pi\pi^{0} & 21.05 \pm 0.14\% & e\nu\gamma & 1.62 \pm 0.47 \times 10^{-5} \\
\pi\pi^{-}\pi^{0} & 5.59 \pm 0.03\% & \pi\pi^{0}\gamma & 2.71 \pm 0.19 \times 10^{-4} \\
\pi\pi^{0}\pi^{0} & 1.73 \pm 0.05\% & \pi\pi^{+}\pi^{-}\gamma & 1.0 \pm 0.4 \times 10^{-4} \\
\mu\pi^{0}\nu & 3.20 \pm 0.09\% & \mu\pi^{0}\nu\gamma & < 6 \times 10^{-6} \\
e\pi^{0}\nu & 4.82 \pm 0.05\% & e\pi^{0}\nu\gamma & 3.7 \times 10^{-4} \\
\mu\nu\gamma & 5.8 \times 10^{-3} & \pi e^{+}e^{-} & 2.6 \pm 0.5 \times 10^{-7} \\
e\pi^{0}\pi^{0}\nu & 1.8 \times 10^{-5} & \pi^{\mp}e^{\pm}e^{\pm} & < 1.5 \times 10^{-5} \\
\pi\pi^{\mp}e^{\pm}\nu & 3.7 \times 10^{-5} & \pi\mu^{+}\mu^{-} & < 2.4 \times 10^{-6} \\
\pi\pi^{\pm}e^{\mp}\nu & < 5 \times 10^{-7} & \pi\gamma\gamma & < 3.5 \times 10^{-5} \\
\pi\pi^{\mp}\mu^{\pm}\nu & 0.9 \pm 0.4 \times 10^{-5} & \pi\gamma\gamma\gamma & < 3.0 \times 10^{-4} \\
\pi\pi^{\pm}\mu^{\mp}\nu & < 3 \times 10^{-6} & \pi\nu\bar{\nu} & < 0.6 \times 10^{-6} \\
 & & \pi\gamma & < 4 \times 10^{-6} \\
 & & e\mu^{\pm}\pi^{\mp} & < 2.8 \times 10^{-8} \\
 & & e\pi^{\pm}\mu^{\mp} & < 1.4 \times 10^{-8} \\
 & & \mu\nu\nu\nu & < 6 \times 10^{-6}
\end{array}
\tag{5.3.2}
$$

Still independently from that, if a structure model of heavier mesons is attempted according to a generalization of our model $\pi^{\circ} = |\varepsilon^{+}, \varepsilon^{-}|$ and $\pi^{\pm} = |\varepsilon^{+}, \varepsilon^{\pm}, \varepsilon^{-}|$ of atomic inspiration, that is, via an increase in the number of constituents, stability arguments will likely produce the breaking of one of the fundamental characteristics of hadrons: the lack of sensible increase of their charge radius with mass.

This situation suggests that our structure model for the pions does not admit a direct generalization to hevier mesons, and calls for a reinspection of the situation aiming at the identification of the physical laws capable of producing a quantitative interpretation of the "hadronic spectra" (5.3. 2) without sacrificing our uncompromisable requirement of

physical consistency: the identification of the constituents with physical particles.

An epistemological outline of this new layer of hadronic reality is the following. One of the central hypothesis presented in this paper is that, while the Lorentz force is unique in its analytic structure, the strong nonselfadjoint forces are not unique. Instead, there exhists a hierarchy of these forces of increasing complexities and methodological implications. This hierarchy has been identified in Section 3 by using purely algebraic consideration related to the structure of the proposed Lie-admissible relativity:

1 - when such covering relativity is strictly Lie in algebraic character and used for the hadronic structure, it implies the simplest possible form of strong hadronic forces, that derivable from a potential (the relativity is then the conventional one), while the hadronic constituents are approximated as point-like particles;

2 - when such covering relativity is partially Lie in algebraic character, we have then the first nontrivial representation of the hadronic constituents as extended particles and strong nonselfadjoint forces, although of restricted type;

3 - when such covering relativity is strictly Lie-admissible in algebraic character, we then have the highest possible characterization of the hadronic constituents as extended particles which is admitted by local theories.

Each of the primary classes 1, 2 and 3 then admits numerous subclasses for a virtually endless variety of strong forces.

This hierarchy of strong nonselfadjoint forces must now be subjected to confrontation with physical data. The central problem is the following. If our model for the pionic structure does not admit a direct generalization to the heavier mesons, our problem then consists in identifying the physical laws which need a further generalization in the transition from pions to other mesons.

First, let us recall that the manifestation of the indicated hierarchy of strong forces is already present in our pionic model. Indeed, our model for the π^{o} preserve the intrinsic characteristics of the electrons and calls for the simplest possible forces producing a generalized structure of a Hamiltonian (that via a constant mutation term). In the transition to the $\pi^{\pm}$ there was the mandatory need of increasing the complexity of the strong forces to achieve a mutation of the spin.

Neverthless, we note that in all models of Tables 5.1 and 5.2, the gauge symmetry was exact at both the structure level and at the level of a state as a whole, and the breaking of symmetries for the structure problem was restricted to the space-time symmetries of relativity character only. This is due to the fact, as indicated, that the charge of the eletons in these models is conventionally quantized and of constant value $\pm e$.

The contention of this table is that in the transition from the pions to heavier mesons, besides the breaking of conventional relativities, there is the additional breaking of the gauge symmetry, also according to our hierarchy (semicanonical, canonical and essentially nonself-adjoint breakings).

Our first argument is the same as that for conventional relativities. If the gauge symmetry is imposed at the level of the constituents (we here refer to field theoretical realizations), this implies a direct restriction on the class of admissible forces. For the case of the structure of the π^o we indicated that the imposition of conventional relativities for the structure equations implies a dynamical equivalence with the positronium structure, that is, the lack of an effective departure from the electromagnetic setting. Along the same lines, we now argue that the condition of preservation of the gauge symmetry for the structure of heavier mesons also implies a dynamical equivalence with the electromagnetic interactions. Indeed, as is well known, the gauge symmetry of the electromagnetic interactions is one of the fundamental characterizationsof their physical structure. The preservation of the same symmetry also for the strong interations then necessarily implies a form of dynamical equivalence among these interactions, contrary to their profound physical differences as manifested in nature.

Our second argument is that the identification of the constituents of the heavier mesons with physical particles demands their identification, according to our model, with a mutated form of pions and muons. The secret for the possible physical consistency of such identification, however, rests on the admitted mutation. If the charge of these constituents is conserved as in ordinary particles, a study of the problem indicates a number of difficulties in the quantitative representation of the decay modes. If, on the contrary, the charge of these constituents is also mutated, jointly with any needed intrinsic characteristics, the situation is profoundly different. But this necessarily implies the breaking of gauge symmetry.

An explicit illustration is here in order. Consider, for instance, the primary decays of K^o_S

$$K^o_S : \quad \begin{array}{ll} \pi^+\pi^- & 68.67\ \%, \\ \pi^o\pi^o & 31.33\ \%, \\ \mu^+\mu^- & < 3.2\times 10^{-7}. \end{array} \qquad (5.3.3)$$

The most straighforward structure model capable of complying with these decays is that based on an eletonic bound state of (mutated) pions, i.e.,

$$K^o_S = \left| \varepsilon^+_{\pi^+}, \varepsilon^-_{\pi^-} \right|, \quad \lim_{F_{NSA}\to 0} \varepsilon^{\pm}_{\pi} = \pi^{\pm}. \qquad (5.3.4)$$

Under the condition that all intrinsic characteristics of the pions are mutated, model (5.3. 4) is fully equivalent to the following models

$$K^o_S = \left| \varepsilon^+_{\pi^+}, \varepsilon^-_{\pi^-} \right| = \left| \varepsilon^+_{\pi^o}, \varepsilon^-_{\pi^o} \right| = \left| \varepsilon^+_{\mu^+}, \varepsilon^-_{\mu^-} \right|. \qquad (5.3.4)$$

Explicitly, under the indicated mutation, there is no dynamical difference between the mutated forms of π^o , $\pi^{\pm}$ and $\mu^{\pm}$. Their physical differences occur only when these particles are detected under electromagnetic interactions, while, under the action of the most general admissible forms of the strong nonselfadjoint forces, all these particles are equivalent. The model then offers hopes for a quantitative interpretation of the decays without sacrificing the physical nature of the constituents. As a matter of fact, the ratios

$$\frac{K^o_S \to \pi^+\pi^-}{K^o_S \to \pi^o\pi^o} \Big/ K^o_S \to \mu^+\mu^- \qquad (5.3.5)$$

can be interpreted as a measure of the statistical distrubution of the eletonic pair into $|\varepsilon^{+}_{\pi^{+}}, \varepsilon^{-}_{\pi^{+}}|$, $|\varepsilon^{+}_{\pi^{o}}, \varepsilon^{-}_{\pi^{o}}|$ and $|\varepsilon^{+}_{\mu^{+}}, \varepsilon^{-}_{\mu^{-}}|$.

Most importantly, the model allows the characterization of the structure of K^0_S without increasing the charge radius because the orbit of eletons is not dependent on the mass of the constituent.

Equally importantly, the model is a two-body system (it would be four-body for a direct generalization of π^o) and, as such, solvable (at least in principle). Its explicit construction is left to the interested researcher. The only technical difficulty which is needed in addition to the model for the $\pi^{\pm}$ is the identification of still broader forces which guarantee the breaking of the gauge symmetry. If such symmetry is preserved, the proposed model of the structure of the K^o_S becomes manifestly inconsistent with experimental data. We shall also leave to the interested reader the extension of such a model for K^0_S to the other mesons (5.3.1).

A few concluding remarks are here in order. The attentif reader has by now noticed our virtually complete silence on the weak interactions. This has been dictated primarely by an intended scientific caution. The physical effectiveness of the unified gauge theories of weak and electromagnetic interactions (only) is unequivocal and not in question. What is in question is whether the physical results of this discipline, particularly those related to the decays of meson customarily called "weak", can be in the final analysis subjected to a still deeper treatment. In essence, our precaution is suggested by the intent of gaining some knowledge on a virtually endless variety of strong nonselfadjoint forces before attempting a physically effective identification, within such a broader context, of the weak interactions in their present interpretation. Also, there is no a priori reason why, the current classification of interactions into "strong", "electromagnetic", "weak" and "gravitational" will truly persists after the achieving of the final solution of the problem of the hadronic structure.

For instance, the studies by R.M.SANTILLI of ref.[70] which lead, via purely gravitational considerations, to the structure model of the π^o here reviewed, were centered on the shift of the conventional ideas from the search of a "unified theory", to that of the "identification of the gravitational field of the π^o with its structure fields". Although substantially qualitative (lacking an established quantum mechanical treatment of gravity), the model indicated that such an identification, complemented with the notion of mutation here developed in more details,

can indeed account for a quantitative characterization of the exterior gravitational field of the π^o which is assumed as the hadronic image of a massive object with the same characteristics (null total electromagnetic phenomenology, such as zero total charge, zero electric and magnetic moments, zero radiation, etc.). The extension of the model for the case of nonnull, total electromagnetic phenomenology was then trivial (via the simple increase of the number of charged constituents).

From a different profile, the studies of ref.[70] were intented to attempt the shift from the search for a unified theory (which, in any case has resiste the efforts of more then one generation of fully qualified scientists) to the study of the "origin" of the gravitational field, of course, at the level of the structure of the lighest known hadron, the π^o. The result of this study is that, pending the verification by interested researchers, there is no need of interpreting the gravitational field as physically distinct from other fields because Einstein's equations for the exterior problem can be quantitatively reproduced via the sole use of the electromagnetic and short range structure forces provided that

(a) the constituents are very light, massive and charged particles in a high dynamical conditions (which lead to the hypothesis that the constituents of the π^o are the ordinary electrons and positrons);

(b) the use of mass terms in the interior problem is abandoned and the problem of structure is instead confronted;

(c) the constituents and their fields are subjected to a mutation due to the very small distances (which has been then elaborated in this paper and in refs.[22,23]).

In this way, rather than being physically superimposed, the gravitational field emerged as a direct consequence of the structure of the π^o, by remouving the very existence of the problem of unification. This is the intended meaning of the terms "origin of the gravitational field".

The point which we intended to make in relation to the weak interactions is that the conceptual, technical and methodological implications of the final solution of the problem of the hadronic structure are expected to be so deep, to likely call for the same reinspection of the terms "strong interactions", let alone those of "weak interactions".

There is one aspect of the problem of gauge symmetries which should be clearly stated. Permit me to reinstate that the physical effectiveness of the unified gauge theories of weak and strong interactions is unquestionable. Instead, we would like here to consider the recent proliferations of attempts of constructing a unified gauge theory of weak, electromagnetic and strong interactions. Our studies cast shadows on these latter attempts. First of all, there simply is no experimental or theoretical evidence available at this time that the gauge symmetry is verified at the level of the hadronic constituents. Secondly, if our argument will eventually prove to be consistent with physical reality, the gauge symmetry must be necessarily abandoned to adequately differentiate the electromagnetic and the strong interactions. In conclusion, studies on the preservation of the gauge symmetry in strong interactions should continue. But jointly, fundamentally different approaches should be initiated in order to achieve a treatment of the problem which is commensurate to its physical relevance.

By combining all these aspects (including that of weak interactions), one of the objectives of this paper was to indicate that, despite valuable progresses, the crucial decay chain

$$K^{+} \longrightarrow \pi^{+} \longrightarrow \mu^{+} \longrightarrow e^{-} \qquad (5.3.4)$$

is still fundamentally unsolved as of now* . Our contention is that it will remain fundamentally unsolved until the problem of the identification of the constituents of these particles is not solved in an incontrovertible form. In the final analysis, chain (5.3.4) may well result to hold the secret of all interactions and the disciplines which will eventually emerge as applicable, with particular reference to the strong, but also to the weak and gravitational interactions.

* This is the reason why chain (5.3.4) has been selected for the front page of this Journal.

TABLE 5.4: THE COMPATIBILITY OF THE PROPOSED STRUCTURE MODEL OF HADRONS IN GENERALIZED FORMULATIONS WITH THE ESTABLISHED UNITARY MODELS OF HADRONIC CLASSIFICATION AND RELATED CONVENTIONAL FORMULATIONS.

The model of hadron structure proposed in this paper appears to be fully compatible with the established unitary models of classification. The first reason is that of methodological nature. As we all known, the established unitary models of classification hare based on the use of conventional relativity and quantum mechanical laws. Even though our structure model is based on generalized formulations for the description of the hadronic constituents, the methodological compatibility is provided by the fact that conventional formulations for the behaviour of a hadron as a whole are recovered by construction in our model. The second reason of compatibility is due to the fact that, once all the intrinsic characteristics of each individual hadron are recovered, the use of "chemical numbers" for a Mendeléyev-type classification of hadrons (isospin, hypercharge, etc.) is recovered in its entirety. For instance, even though the π^{0} and the $\pi^{\pm}$ particles have a different structure in our model, the notion of isotopic triplet of pions is recovered in its entirety at the level of classification. But perhaps the most important reason of compatibility is that which motivated this study: the possible emergence of two different but compatible approaches to hadrons, one for the classification and one for the structure, in exactly the same way as it occurred at the atomic level. Indeed, the established unitary models of hadronic classification in which a group of physically different hadrons is combined into a unitary multiplet, by no means, prohibit the possible existence of a compatible model of structure in which each individual hadron of the multiplet is treated as an individual entity. This is precisely the objective which has been attempted in this paper.

For the sake of clarity, permit me to stress that we consider this compatibility truly crucial. As a matter of fact, such compatibility can be also interpreted as a necessary condition for physical consistency for any new model of structure. This is due to the fact that the physical effectiveness of the established unitary models of hadron classification is simply unequivocal. No structure model of hadrons can achieve a meaningful degree of plausibility unless the unitary classification is recovered in its entirety. Therefore, the objective of this paper was that of attempting the identification of the arena of physical relevance of unitary models and, under no circumstance, should be interpreted as suggesting the removal of unitary Lie groups in hadron physics.

An intriguing question is whether there exists a degree of compatibility between our model and the quark models also at the structure level. A study of this issue suggests its treatment according to the following distinct layers of hadronic reality.

(A) The case of light hadrons, such as the so-called "stable" mesons. It appears that our structure model and the quark models are fundamentally incompatible at this level. It is sufficient in this respect to note that our model calls for an essential increase in the number of constituents in going from π to $\pi^{\pm}$ and to other mesons, while for the quark models all these particles necessarily have the same number of constituents, a quark and an antiquark (we here ignore the sea of gluons).

(B) The case of heavy hadrons, such as the charmed particles. At this level the situation is different and not readily solvable in terms of simple arguments. As such, it demands a specific study. In essence, the concept of quark constitutes a generalization of that of physically detected particles in the sense that it implies the transition from an integrally charged to a fractionally charged state. Our concept of eleton constitutes a further generalization to an arbitrary, nonconserved value of the charge as well as of all other intrinsic characteristics. Now, by its very conception, our hadronic mechanics has been proposed to produce arbitrarily assigned physical characteristics of a bound state of these particles. The aspect which is here pertinent is whether a bound state of eletons can also produce a fractionally charged state. The answer is that, if this state is a physical particle as experimentally detected under electromagnetic interactions, such fractional total charge is prohibited. However, if we consider a cluster of eleton within hadronic matter, a fractional value of the charge becomes fully admissible. The net effect is that a hadronic bound state of eletons can literally construct a quark while within a hadron. This would be in line with the idea that quarks are composite states, which is appearing with more frequency in recent literature.[10] Such a view has some intriguing features. First of all it removes the very existence of the problem of confinement because our eletons, by construction, do not exist under electromagnetic interactions only. Thus, when a cluster of eletons with fractional charge leaves the hadronic medium it recovers conventionally quantized integrally charged particles. Secondly, the regularities of the charmonium spectrum are impressive. It may well be that the constituents of the charmonium cluster into states with fractional charge and, in any case, this possibility cannot be excluded

until the contrary is experimentally proved. In conclusion, the possibility that our model can be used to construct a quark model for the charmed particles cannot be excluded at this time. As a matter of fact, studies to this effect are encouraged because such an approach would allow the identification of the constituents of charmed particles with physical particles, only in a different dynamical state, by therefore removing the very existence of the problem of confinement. A few aspects, however, deserve an elaboration.

(a) Chromodynamics is fundamentally dependent on established physical laws. An eletonic construction of a quark model for charmed particles would imply the restriction of the physical relevance of the charmed models to that of only a first approximation of physical reality in which the constituents are treated as point-like. The eletonic approach and related generalized formulations would therefore constitute a sublayer of the conventional charmed structure in which the constituents are considered as physical extended particles, with consequential implementation of our line of study (nonselfadjoint forces and related generalized formulations).

(b) A central difference between an eleton and a quark is not given by the charge (because an eleton can have an arbitrary charge, including a fractional value as a particular case). It is given instead by the fact that an eleton is fundamentally independent, by construction, on unitary quantum numbers, while a quark is centrally dependent on these numbers. Thus, an eletonic sublayer of a charmed structure can be achieved, provided that the eletons cluster into states with the necessary regularities to apply unitary quantum numbers in much of the lines according to which an isotopic triplet is recovered by our eletonic model.

(c) It is often believed that heavier particles, and the charmed particles in particular, have a simpler structure than that of light particles, such as mesons. We disagree. The occurrences at the atomic and nuclear level have indicated a substantial increase of the complexity of these structures with mass, without however prohibiting the existence of certain regularities beyond a mass value. There is no a priori reason why exactly the same situation will not eventually result also for the hadronic structure. The contrary argument presented by charmed models is inconclusive and will remain inconclusive until an incontrovertible solution of the problem of the constituents or of confinement is achieved. This is a crucial point for researchers interested in attempting the construction of a sublayer of charmed models. According to our model, the sole physical constituents of hadrons are the electrons and positrons, even though appearing in a direct way for light hadrons and in an indirect way (that is, via eletonic bound states of the same) in heavier hadrons. The net effect is that the number of physical constituents generally demands an increase with mass in our model. If this number is small (as for mesons), there simply is no way to cluster these constituents for the objective of "constructing" a quark within hadronic matter. We therefore recover in this way the irreconciliable incompatibility between our eletonic model and the quark models for light hadrons indicated earlier. It is only when the number of constituents is sufficiently high that our methods permit a possible attempt of reaching compatibility according to points (a) and (b) above. In conclusion, the possible construction of a sublayer of the charmed models is centrally dependent on a sufficiently high number of eletonic constituents and on a structure whose complexity is not even remotely reminiscent of the simplicity of the proposed structure of the π^{o}.

There is one crucial implication which emerges from these remarks which should be clearly stated. The possible experimental detection of isolated, fractionally charged particles under no circumstance should be interpreted as the final evidence of the validity of the quark models. The reason is due to the fact that fractional charges can be admitted also by structure models which are fundamentally different than the quark models, beginning at the level of the applicable laws. For instance, in our model based on covering relativity and quantum mechanical laws, clusters of eletons, as pointed out in this table, can have an arbitrary charge, including a fractional charge.

TABLE 5.5: CONCLUDING REMARKS ON THE FUTURE ORIENTATION OF EXPERIMENTAL HIGH ENERGY PHYSICS WHICH IS NEEDED TO PROVIDE A PHYSICALLY EFFECTIVE SELECTION AMONG AN EVER INCREASING NUMBER OF HADRONIC MODELS.

This paper can also be considered as a manifestation of the considerable uneasiness experienced by a segment of our community of basic studies in regard to the current status of hadron physics. The reason is due to the ever increasing proliferation of models combined with our inability to perform a clear selection on clear physical grounds.

The only way to fullfill the increasing need for such a physically effective selection is the experimental way. We therefore believe that it is time to conduct an assessement of current experimental trends in hadron physics and an identification of the future orientation which is needed to achieve the objective considered.

As we all know, experimental hadron physics has been dominated until now by the search for new particles and the identification of their physical characteristics. Our contention is that, until this trend is continued, the problem of the hadronic structure will likely remain fundamentally unresolved.

In the simplest possible form, our argument is the following. An anamnesis of the last decade of experiments in hadron physics and of their relationship with theoretical studies along unitary models indicates a symbiotic condition of research between these two profiles with initial achievements of unequivocal physical relevance, such as the prediction and discovery of the Ω^- particle. Subsequent experimental discoveries have indicated the inability of the originally conceived SU(3) model to comply with physical reality. As a result, the model was subjected to initial implementations. Still subsequent experimental data indicated the need of additional implementations of the original SU(3) model most notably, that via color, and related increase in the assumption of different quarks. The most recent experimental data indicate the possible need of further implementations with new degrees of freedom and new quarks, as indicated in Section 2. In conclusion, the quarks models have virtually proved their capabilities to accomodate new particles via suitable implementations. The crucial point is that in doing so the basic problematic aspects of the quark models increase, rather than decrease, and the problem of the hadronic structure remains fundamentally unsolved. This is due, in our view, to the fact that these trends have been unable to resolve until now the truly fundamental aspect of hadron physics; the unequivocal identification of the hadronic constituents with physical particles, and the related issues, such as the nature of the forces and the applicable laws. We then argue that an increase in the number of different quarks without direct experimental support constitutes an increase in the problematic aspects which confirms the fundamentally unsolved nature of the problem.

A most significative case is that of the more recent discovery of the J/ψ particle. There is do doubt that this discovery constitutes a substantial contribution to human knowledge. However, this discovery, jointly with the preceding and subsequent discoveries of new particles, has not produced the final solution of the problem of hadron structure.

We here submit that experimental efforts toward the identification of new particles and their properties should indeed continue, because of selfevident physical values. But jointly, new experimental trends should be initiated aming at physical data which have a more direct significance for the problem of structure.

This paper will achieved its objective if it succeeds in pointing out the possible dichotomy of classification and structure of hadrons, and in presenting an initial identification of the aspects which appear to be crucial from a structure viewpoint. According to this view, the sole identification of a new particle is a sole contribution for the classification. The experimental profile which is needed for the structure problem is fundamentally different than that. Some of the most relevant aspects have been studied during the course of our analysis and they essentially are the following.

(A) The experimental finalization of the problem whether the hadronic constituents are point-like or extended particles;

(B) The experimental finalization of the problem whether the forces between the hadronic constituents are derivable from a potential or not; and, last but not least,

(C) The experimental finalization of the problem whether the relativity and quantum mechanical laws which have proved so effective for the atomic structure are valid or invalid for the hadronic structure.

In conclusion, our contention is that, until the problem of the physical nature of the constituents, the analytic character of their forces and the applicable laws have not been experimentally resolved, the problem of the hadronic structure will remain unresolved. The direct relevance of problems (A), (B) and (C) for the structure of hadrons is selfevident. In simplistic terms, if the hadronic constituents will result to be point-like, with selfadjoint forces and obeying established laws, the quark models on the hadronic structure are likely to

emerge as the only models which are conceivable on grounds of our current knowledge. On the contrary, if the hadronic constituents will emerge as being extended particles, with nonselfadjoint forces and obeying covering laws, the concept of quark as the constituent of hadronic matter is likely to be ruled out in a final form.

Intriguingly, our studies suggest the beginning of experiments at the nuclear level, with a reinspection of the nuclear forces to see whether very small contributions from nonselfadjoint forces are admitted, and with the finalization of the issue whether very small deviations from Pauli's exclusion principle can be experimentally established or this principle is exactly valid in nuclear physics.*

In the transition to the hadronic level the technical difficulties are expected to increase considerably, to the point of rendering premature even the proposal of specific experiments for the central issues, that is, validity or invalidity of Einstein's special relativity and Pauli's exclusion principle. After all, we are here referring to the transition from the experimental detection of a hadron as a whole to that of its constituents, which is clearly no easy experimental task. The final element, however, remains of human rather than technical nature: the decision whether the complexity of the technological problems to be confronted is reason for withdrewal or not.

Irrespective of any technical consideration, there is an aestetical aspect which should not be overlooked, which goes beyond the mere identification of the hadronic constituents with physical particles, and which stresses the need of conducting in due time the experiments along the indicated lines. Our Lie-admissible approach to hadron structure indicates a possible reduction of the physical universe to only three stable particles: electrons, photons and neutrinos.

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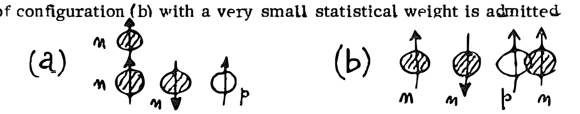
* Pauli's exclusion principle can be experimentally tested in a variety of ways in nuclear physics along the lines indicated in this paper. On purely indicative grounds, one can for instance either reinspect available data or conduct specific experiments with $_7N^{16}$ to see whether the last four nucleons admit only configuration (a) below or the presence of configuration (b) with a very small statistical weight is admitted.



The experimenter should however keep in mind that the latter occurrence is not necessarily expected to be universal, and might instead occur only in specific cases. Thus, the study of a number of sufficiently diversified nuclei appears recommendable.

REFERENCES AND FOOTNOTES

1. The problem whether Einstein's special relativity is valid or invalid at small distances has already been considered by a number of authors. See, for instance, D.Y. KIM, Lettere al Nuovo Cimento 2, 591 (1975), and quoted papers.

2. The problem whether Pauli's exclusion principle is valid or invalid for the hadronic constituents has been considered by R.M.SANTILLI in "Need of subjecting the validity of Pauli's exclusion principle within a hadron to an experimental verification", Harvard preprint HUTP-77|AO85 and "Need of subjecting the validity of Einstein's special relativity within a hadron to an experimental verification", Harvard preprint HUTP-77 |AO86 (December 1977, unpublished). This paper consists of an expanded and combined presentation of the contents of these two unpublished notes.

3. See, for instance, R. MOORE, Niels Bohr: the Man, His Science and the World they changed .

4. For an exquisite account see L. FERMI, Atoms in the Family .

5. See, for instance, K. WILDERMUTH and Y.C.TANG, A unified theory of the nucleus Academic Press, N.Y. (1977).

6. M.GELL-MANN, Phys. Lett. 8, 214 (1964) and G. ZWEIG, CERN reports 8182/TH 401 and 8419/TH 412 (1964).

7. For a review of the early stages of the quark models see, for instance, J.J.J.KOKKEDEE, The quark model, Benjamin, N.Y. (1969). For a recent account see, for instance, F.E. CLOSE, Introduction to quarks and partons, Academic Press, London (in press).

8. O.W.GREENBERG and C.A.NELSON, Phys. Reports 32C, 69 (1977).

9. For a recent account of the search for quarks see, for instance, L. W. JONES, Rev. Mod. Phys. 49 , 717 (1977).

10. For a recent review of confinement and other aspects see, for instance, O.W.GREENBERG, "Quarks", Ann. Rev. Nucl. and Part. Science 28 (1978- in press).

11. For a recent review of confinement attempted via lattive models see, for instance, S. D. DRELL, in Quark spectroscopy and hadron dynamics, SLAC publication (1977).

12. For a recent attempt of confinement via singular gauge transformations see G. 't HOOFT, Uthrecht preprint (1977).

13. For an attempt of confinement via a pressure characterized by a bag constant (the so-called MIT bag model) see, for instance, T. DE GRAND, R.L.JAFFE, K. JOHNSON and J. KISKIS, Phys. Rev. D12, 2060 (1975).

14. For an attempt of confinement using spontaneously broken symmetry via a scalar field (the so-called SLAC bag model) see, for instance, W.A.BARDEEN, M.S.CHANOWITZ and S.D.DRELL, Phys. Rev. D11, 1094 (1975).

15. For an attempt of confinement via vacuum degeneracy in nonabelian gauge theories see, for instance, C. CALLAN, R. DASHEN and D. GROSS Phys. Lett. 63B, 334 (1976).

16. For an attempt of confinement via the use of instantons and merons see, for instance, J. GLIMM and A. JAFFE, Phys. Rev. Letters 40, 277 (1978).

17. For an attempt of confinement using octonionic representations of the Poincaré group see M. GÜNAYDIN and F. GÜRSEY, Phys. Rev., D9, 3387 (1974).

18. For an attempt of confinement via the use of a Jordan-admissible algebra see G. DOMOKOS and S. KŐVESI-DOMOKOS, John Hopkins preprint JHU-HET 7801 (1978).

19. For an attempt of confinement via phase transition processes see, for instance, W. MARCIANO and H. PAGELS, Physics Reports (1978- in press).

20. According to the Encyclopedia Britannica (Vol. 17, page 420), the question what is a hadron "would be provided by a quantitative theory which as yet does not exist."

21. R.M.SANTILLI, Foundations of Theoretical Mechanics, Volume I: The Inverse Problem in Newtonian Mechanics (initially released as MIT-CTP preprint no. 606, March 1977) and Volume II: Generalizations of the Inverse Problem in Newtonian Mechanics (initially released as MIT-CTP preprint 607, March 1977), Speringer-Verlag, Heidelberg, in press. See also, Volume III: The Inverse Problem in Field Theory, MIT-CTP preprint 608, March 1978; Annals of Physics 103, 354 (1977); 103, 409 (1977); 105, 227 (1978); and MIT-CTP preprints 609 and 610 (1977).

22. R.M. SANTILLI, Lie-admissible Approach to the Hadronic Structure, Volume I: Nonapplicability of the Galilei and Einstein relativities?, Volume II: Coverings of the Galilei and Einstein relativities ?, Volume III: Identification of the hadronic constituents with physical particles?, Hadronic Press, Nonantum, Ma 02195, in press.

23. R.M.SANTILLI, Hadronic Journal 1. 223 (1978).

24. E.EICHTEN and K. GOTTFRIED, Phys. Letters 66B, 286 (1977) and W. CELMASTER, H. GEORGI and M. MACHACEK, Phys. Rev. D17, 874 and 879 (1978), and quoted papers.

25. I. PRIGOGINE, F. MAYNÉ, C. GEORGE and M. de HAAN, Proc. Natl. Acad. Sci. (U.S.A.) 74, 4152 (1977) and I. PRIGOGINE, in preparation for the Hadronic Journal.*

26. We have here used the terms "charge volume" and we shall continue to do so with the understanding that it is a conceptual abstraction of the notion of extended particle. The quantitative formulation of this notion of extended particle while within a hadronic structure is, by far, nontrivial. For instance, the reader is discouraged to use the notion of "wave packet" until the discipline which is applicable under forces non-derivable from a potential is identified.

27. R.M.SANTILLI, Nuovo Cimento 51A, 570 (1967) and Supplemento al Nuovo Cimento 6, 1225 (1968).

28. It might be of some usefulness for the interested reader to recall the following theorems of the Inverse Problem.

THEOREM 1[21-I, 22-I, 23]: A necessary and sufficient condition for the Newtonian system

$$\left[\ddot{q}_k - f_k(t,q,\dot{q})\right]^{C^2} = 0, \quad k = 1,2,\ldots,3N, \quad (m_k = 1) \qquad (1)$$

to be selfadjoint in a region R of points $(t,q,\dot{q})$ is that all acting forces are linear in the velocities, i.e., the system is of the form

$$\left[\ddot{q}_k - \rho_{ki}(t,q)\dot{q}^i - \sigma_k(t,q)\right]^{C^2} = 0, \qquad (2)$$

and all the following conditions of selfadjointness

* NOTE ADDED IN PROOF: See Cl. GEORGE, F. HENIN, F. MAYNÉ and I. PRIGOGINE, Hadronic Journal 1, 520 (1978), in this issue.

$$\rho_{ki} + \rho_{ik} = 0,$$

$$\frac{\partial \rho_{ij}}{\partial q^k} + \frac{\partial \rho_{jk}}{\partial q^i} + \frac{\partial \rho_{ki}}{\partial q^j} = 0, \qquad (3)$$

$$\frac{\partial \rho_{ij}}{\partial t} = \frac{\partial \sigma_i}{\partial q^j} - \frac{\partial \sigma_j}{\partial q^i},$$

are identically verified in the subregion $R' \subset R$ of points (t,q).

When the forces are variationally selfadjoint (and well defined in a star-shaped region of the variables) they are derivable from a potential, that is, Lagrangian representation (4.3.4a) exists and a Lagrangian can be computed via the techniques of the Inverse Problem. For a new method for the construction of a Lagrangian see E. ENGELS, Hadronic Journal 1, 465 (1978), in this issue. The Lorentz force does satisfy Theorem 1. In particular, and most significantly, the Lorentz force is linear in the velocities. However, this class of force is highly restrictive in Newtonian mechanics. This leads to the following

THEOREM 2[21-I, 22-II, 23]: A necessary and sufficient condition for the Newtonian system

$$\left[A_{ki}(t,q,\dot{q})\ddot{q}^i + B_k(t,q,\dot{q})\right]^{C^2, R} = 0 \qquad (4)$$

to be selfadjoint in a region R of points $(t,q,\dot{q})$ is that all the following conditions

$$A_{ij} = A_{ji}, \quad \frac{\partial A_{ik}}{\partial \dot{q}^j} = \frac{\partial A_{jk}}{\partial \dot{q}^i},$$

$$\frac{\partial B_i}{\partial \dot{q}^j} + \frac{\partial B_j}{\partial \dot{q}^i} = 2\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\} A_{ij}, \qquad (5)$$

$$\frac{\partial B_i}{\partial q^j} - \frac{\partial B_j}{\partial q^i} = \frac{1}{2}\left\{\frac{\partial}{\partial t} + \dot{q}^k \frac{\partial}{\partial q^k}\right\}\left(\frac{\partial B_i}{\partial \dot{q}^j} - \frac{\partial B_j}{\partial \dot{q}^i}\right),$$

are identically verified.

The above conditions do not imply linearily of the A - and B-terms of the equations of motion in the velocities (as well as in the coordinates) and, thus, this broader approach allows the analytic representation of broader forces.

The aspect which is relevant for the analysis of this paper is that the conditions of variational selfadjointness are the integrability conditions for the existence of a Lagrangian representation (after a number of technical aspects are taken into account[21,23]).

The transition of the variational approach to a relativistic equation in Minkowski space is expressed by the following

THEOREM 3[22-I]: A necessary and sufficient condition for the system

$$\left[m_0 \ddot{x}_\mu - K_\mu(s, x, \dot{x})\right]^{C^2} = 0, \quad \dot{x} = \frac{dx}{ds}, \ \ddot{x} = \frac{d^2x}{ds^2}, \ \mu = 0, 1, 2, 3, \tag{6}$$

to be selfadjoint in a region R of points $(s, x, \dot{x})$ is that the Minkowski force is linear in the velocity, i.e.,

$$K_\mu = \rho_{\mu\nu}(s, x)\dot{x}^\nu + \sigma_\mu(s, x) \tag{7}$$

and all the following conditions of variational selfadjointness

$$\begin{aligned} &\rho_{\mu\nu} + \rho_{\nu\mu} = 0, \\ &\frac{\partial \rho_{\mu\nu}}{\partial x^\tau} + \frac{\partial \rho_{\nu\tau}}{\partial x^\mu} + \frac{\partial \rho_{\tau\mu}}{\partial x^\nu} = 0, \\ &\frac{\partial \rho_{\mu\nu}}{\partial s} = \frac{\partial \sigma_\mu}{\partial x^\nu} - \frac{\partial \sigma_\nu}{\partial x^\mu}, \end{aligned} \tag{8}$$

are identically verified in a subregion $R' \subset R$ with points (s, x).

The interested reader is uged to verify that the relativistic formulation of the Lorentz force

$$K_\mu = e F_{\mu\nu} \dot{x}^\nu \tag{9}$$

does indeed verify the conditions of Theorem 3. Within the context of currently used relativistic models Theorem 3 is fully sufficient. Within the context of this paper Theorem 3 is highly insufficient, because we are primarily interested in studying forces which are nonlinear in the velocity to attempt a nontrivial departure from the electromagnetic interactions.

This situation suggests the study of broader systems in Minkowski space according to the following relativistic extension of Theorem 2.

THEOREM 4[22-I]: A necessary and sufficient condition for the system

$$\left[A_{\mu\nu}(s, x, \dot{x})\ddot{x}^\nu + B_\mu(s, x, \dot{x})\right]^{C^2, R} = 0 \tag{10}$$

to be selfadjoint in the region R with points $(s, x, \dot{x})$ is that all the following conditions

$$\begin{aligned} &A_{\mu\nu} = A_{\nu\mu}, \quad \frac{\partial A_{\mu\nu}}{\partial \dot{x}^\tau} = \frac{\partial A_{\tau\nu}}{\partial \dot{x}^\mu}, \\ &\frac{\partial B_\mu}{\partial \dot{x}^\nu} + \frac{\partial B_\nu}{\partial \dot{x}^\mu} = 2\left\{\frac{\partial}{\partial s} + \dot{x}^\tau \frac{\partial}{\partial x^\tau}\right\} A_{\mu\nu}, \\ &\frac{\partial B_\mu}{\partial x^\nu} - \frac{\partial B_\nu}{\partial x^\mu} = \frac{1}{2}\left\{\frac{\partial}{\partial s} + \dot{x}^\tau \frac{\partial}{\partial x^\tau}\right\}\left(\frac{\partial B_\mu}{\partial \dot{x}^\nu} - \frac{\partial B_\nu}{\partial \dot{x}^\mu}\right), \end{aligned} \tag{11}$$

are identically verified in R.

The transition from Eqs. (6) to Eqs. (10) removes the restriction of linearity in the velocity, but does not guarantee the existence of a Lagrangian or a Hamiltonian. It merely identify the simplest possible generalizations of the Lorentz force which still allow the use of the canonical formalism via the identification of a Hamiltonian.

The reader should be aware that the analysis of this paper is particularly devoted to generalizations of the Lorentz force which violate the integrability conditions for the existence of a Lagrangian (or an action functional) in the coordinate frame of their experimental identification. It is only when the study of the forces considered is brought up to this level that the problematic aspects of Einstein's special relativity become conspicuous (Table 4.20).

For completeness, let me recall the field theoretical versions of the above theorems for the case of second-order partial differential equations (either in Minkowski or in Euclidean space)

THEOREM 5[21-III, 22-I]: A necessary and sufficient condition for the semilinear system of second-order partial differential equations

$$\left[(\Box + m_k^2)\phi_k - f_k(x^\alpha, \phi^j, \phi^j_\alpha)\right]^{C^2} = 0, \quad \phi^j_\alpha = \frac{\partial \phi^j}{\partial x^\alpha}, \tag{12}$$

to be selfadjoint in a region R of the variables $(x^\alpha, \phi^j, \phi^j_\alpha)$ is that the couplings are linear in the partial derivatives, i.e.,

$$f_k = \rho^{\mu}_{ki}(x^{\alpha}, \phi^{i})\,\phi^{i}_{;\mu} + \sigma_k(x^{\alpha}, \phi^{i}) \qquad (13)$$

and all the following conditions

$$\rho^{\mu}_{ki} + \rho^{\mu}_{ik} = 0\,, \qquad (14)$$

$$\rho^{\mu}_{ki}{}^{;}{}_{j} + \rho^{\mu}_{ij}{}^{;}{}_{k} + \rho^{\mu}_{jk}{}^{;}{}_{i} = 0\,, \quad \rho^{\mu}_{ki}{}^{;}{}_{j} = \frac{\partial \rho^{\mu}_{ki}}{\partial \phi^{j}}\,,$$

$$\rho^{\mu}_{ki}{}^{;}{}_{\mu} = \sigma_k{}^{;}{}_{i} - \sigma_i{}^{;}{}_{k}\,, \quad \rho^{\mu}_{ki;\mu} = \partial_\mu \rho^{\mu}_{ki}\,,$$

are identically verified in the subregion R with poin ts (x^{α}, ϕ^{i}).

The interested reader is urged to verify that not only the electromagnetic couplings of conventional use, but all coupli ngs of the (Abelian or non-Abelian) gauge theories verify the conditions of Theorem 5 when the field equations (rather than the Lagrange's equations) are properly written, e.g., in the form for the external field case[21-III]

$$\left[\begin{pmatrix} (\Box + m^2)\,\bar{e} \\ (\Box + m^2)\,e \end{pmatrix}_{SA} - \begin{pmatrix} e^2 A_\mu A^\mu \bar{e} - 2ie\, A_\mu \bar{e}^{;\mu} \\ e^2 A_\mu A^\mu e + 2ie\, A_\mu e^{;\mu} \end{pmatrix}_{SA}\right]_{SA} = 0\,. \qquad (15)$$

In conclusion, the statement that all couplings of gauge theories are variationally selfadjoint is the technical language of the Inverse Problem for the representation of the known property that all these theories admit a Lagrangian density and such density has the trivial structure

$$\mathcal{L}_{tot} = \mathcal{L}_{free}(\phi^{i}_{;\alpha}) + \mathcal{L}_{int}(\phi^{i}, \phi^{i}_{;\alpha})\,. \qquad (16)$$

QUADRATIC DEPEND. LINEAR DEPEND.

Again, in this paper we are interested in couplings which are nonselfadjoint, to attempt an analytical differentiation between the electromagnetic and the strong interactions, as a conceivable prerequisite to achieve a genuine dynamical differentiation between these interactions. As a result, we are solely interested in couplings which break the selfadjointness property of the electromagnetic couplings as well as, more generally, of all Abelian and non-Abelian gauge couplings. This calls for a generalization of Theorem 5 which can be formul ated as follows.

21-III, 22-I

THEOREM 6: A necessary and sufficient condition for the quasilinear second-order system of partial differential equations

$$\left[A^{\mu\nu}_{ki}(x^{\alpha}, \phi^{j}, \phi^{j}_{;\alpha})\,\phi^{i}_{;\mu\nu} + B_k(x^{\alpha}, \phi^{j}, \phi^{j}_{;\alpha})\right]_{C^2, R} = 0 \qquad (17)$$

to be selfadjoint in a region of the variables $(x^{\alpha}, \phi^{j}, \phi^{j}_{;\alpha})$ is that all the following conditions

$$\left\{\begin{array}{l} A^{\mu\nu}_{ki} = A^{\nu\mu}_{ik}\,, \\ A^{\nu\alpha;\mu}_{ki\;\;\;\;j} + A^{\nu\alpha;\mu}_{ij\;\;\;\;k} = A^{\mu\nu;\alpha}_{ik\;\;\;\;j}\,, \\ A^{\alpha\beta;\mu;\nu}_{ki\;\;\;j\;\;e} = A^{\alpha\beta;\mu;\nu}_{ji\;\;\;k\;\;e}\,, \\ B_k{}^{;\mu}{}_{i} + B_i{}^{;\mu}{}_{k} = 2\left\{\partial_\nu + \phi^{j}_{;\nu}\dfrac{\partial}{\partial \phi^{j}}\right\} A^{\mu\nu}_{ki}\,, \\ B_k{}^{;}{}_{i} - B_i{}^{;}{}_{k} = \dfrac{1}{2}\left\{\partial_\nu + \phi^{j}_{;\nu}\dfrac{\partial}{\partial \phi^{j}}\right\}\left(B_k{}^{;\nu}{}_{i} - B_i{}^{;\nu}{}_{k}\right), \end{array}\right. \qquad (18)$$

$$A^{\mu\nu;\alpha}_{ki\;\;\;\;j} = \partial A^{\mu\nu}_{ki} / \partial \phi^{j}_{;\alpha}\,, \text{ etc.},$$

$$A^{\overline{\mu\nu;\alpha}}_{ki\;\;\;j} = A^{\mu\nu;\alpha}_{ki\;\;\;j} + A^{\mu\alpha;\nu}_{ki\;\;\;j}, \text{ etc.},$$

$$A^{\alpha\beta;\mu;\nu}_{\underline{ki\;\;j\;\;e}} = A^{\alpha\beta;\mu;\nu}_{ki\;\;j\;\;e} + A^{\alpha\beta;\mu;\nu}_{ke\;\;j\;\;i}, \text{ etc.},$$

are identically verified in R.

The above theorem does indeed achieve the desired result, the removal of the linearity in the partial derivative of the electromagnetic and, more generally, gauge couplings, by therefore allowing the construction of Lagrangian representations of interactions which are genuinely more general than those interactions (here studied for the strong interactions). The crucial point is, again, that Theorem 6 is still restrictive. It merely allows the identification of a class of generalization of gauge theories for which conventional analytic techniques are still applicable. This paper is devoted instead to the most general class of local, class C^∞ and nonselfadjoint couplings $F_k(x^\alpha, \phi^i, \phi^i_{;\alpha})$. Such class violates Theorem 6, by therefore rendering inapplicable the entire machinery of conventional analytic approaches to field theories. It is only when the analysis is brought up to this level that the problem of the relativiti laws which are applicable to the systems

$$\left[\left(m_k \ddot{z}_k - f_{\mu k}\right)_{SA} - F_{\mu k}\right]_{NSA} = 0,$$

$$\left[\left(m_0 \ddot{x}_\mu - K_\mu\right)_{SA} - F_\mu\right]_{NSA} = 0, \qquad (19)$$

$$\left\{\left[\left(\Box + m_k^2\right)\phi_k - f_k\right]_{SA} - F_k\right\}_{NSA} = 0,$$

can be meaningfully treated.

Later on, we shall also be involved in a generalization of Dirac's equation for nonselfadjoint couplings. It is therefore of some significance to recall the corresponding setting for first-order field equations.

22-I

THEOREM 7: A necessary and sufficient condition for the quasi-linear system of first order partial differential equations

$$F_k = \left[C^{\mu}_{ki}(x^\alpha, \phi^i)\,\phi^i_{;\mu} + D_k(x^\alpha, \phi^i)\right]^{C^2, R} = 0, \qquad (20)$$

to be selfadjoint in a region R of points (x^α, ϕ^i), is that the following conditions

$$\begin{cases} F_{i;}{}^{;\mu}_{j} + F_{j;}{}^{;\mu}_{i} = 0, \\ F_{i;}{}_{j} - F_{j;}{}_{i} = d_\mu F_{i;}{}^{;\mu}_{j}, \end{cases} \qquad (21)$$

or their equivalent forms

$$\begin{cases} C^{\mu}_{ij} + C^{\mu}_{ji} = 0, \\ C^{\mu}_{ij;k} + C^{\mu}_{jk;i} + C^{\mu}_{ki;j} = 0, \\ C^{\mu}_{ij;\mu} = D_{i;j} - D_{j;i}, \end{cases} \qquad (22)$$

are all identically verified in R.

Again, the reader is urged to verify that Theorem 7 is verified not only by Dirac's equations (in a trivial way), but more generally by the virtual totality of first order field equations currently available in the literature, including ∞-dimensional equations. In conclusion, the variational selfadjointness of the forces or couplings of physical systems constitutes a property which is rather universally verified by the virtual entirety of available models, with only few exceptions known to me. This is trivially due to the fact that all physical models considered until now, with truly few exceptions, admit a Lagrangian. The realization of the strong interactions we are interested in, is instead that for which a Lagrangian does not exist by assumption (in the assumed local coordinates).

The extension of the above analysis to the case of systems in a Riemannian manifold is considerably more involved on technical grounds, but entirely equivalent on conceptual grounds and, as such, its presentation is omitted.

In very simplictic terms, when the strong interactions are implemented into a Riemannian manifold and assumed as being nonselfadjoint to dynamically differentiate them from the selfadjoint electromagnetic interactions, they imply the lack of derivability of the system from a variational principle, as well as numerous incompatibilities with Einstein's general theory for the interior problem only, such as the lack of a meaningful characterization of curvature as geodesic deviation, lack of conservation laws, etc. In conclusion, the methods of the Inverse Problem in Euclidean space, Minkowski space, field theory or Riemannian manifolds (of which we have here recalled only the very first step) constitute effective methodological tools for the quantitative treatment of the central contention of this paper, that there exist forces or couplings which violate Galilei's, Einstein's special and Einstein's general relativity for the interior problem (only). The objective of this paper is to conduct an analysis of the dynamical effects of such powerful forces or couplings and assess the possibility whether they can be consistently interpreted as representative of the strong interactions.

29. T. LEVI CIVITA, Prace Mat.-Fiz. 17, 1 (1906). See also T. LEVI-CIVITA and U. AMALDI, Lezioni di Meccanica Razionale, Zanichelli, Bologna (1927), Vol. 2, part 2.

30. S. SHANMUGADHASAN, Jour. Math. Phys. 14, 677 (1973).

31. See, for instance, H. GOLDSHMIDT, Ann. Math. 86, 246 (1967) and Jour. Diff. Geometry 1, 269 (1967); D. C. SPENCER, Bull. Amer. Math. Soc. 75, 179 (1969); and J. GASQUI, Jour. Diff. Geometry 10, 61 (1975).

32. See, for instance, in addition to conventional treatises in quantum mechanics, L. VAN HOVE, Acad. Roy. Belg. Cl. Sci., Mem. 6 (1951); I. SEGAL, Mathematical problems of Relativistic Physics, Amer. Math. Soc., Providence, R.I. (1963); R. PROSSNER, Jour. Math. Phys. 5, 701 (1964); C. B. KOSTANT, Quantization and Unitary Representations, Lectures in Mathematics Vol. 170, Springer-Verlag, Heidelberg (1970); J. SOURIAU, Structures des Systemes Dynamiques, Dunod, Paris (1970); and V. GUILLEMIN and S. STERNBERG, Geometric Asymptotics, Amer. Math. Soc., Providence, R.I. (1977).

33. The fact that our Galilei-admissible relativity group generally admit only nonlinear representation is considered particularly intriguing for the line of study of this paper. The reason is that it constitutes an additional critical element for the differentiation between the electromagnetic and the strong interactions in the sense that fields under the former interactions admit the conventional linear covariance law under the Poincaré (Lie) group, while fields under the latter interactions would admit a covering nonlinear covariance law under our Lie-admissible relativity group. This aspect, which demands a field theoretical analysis, will be investigated in some subsequent paper.

34. See, for instance, C.W. MISNER, K. S. THORNE and J.A.WHEELER, Gravitation, Freeman, San Francisco (1970).

35. K. R. SYMON, Mechanics, Addison-Wesley, Reading, Ma (1960); D.A, WELLS, Lagrangian Dynamics, McGraw-Hill, N.Y. (1967); and L.D.LANDAU and E.M. LIFSHITZ, Mechanics, Addison-Wesley, Reading, Ma (1960).

36. Of course, non-Hermitian operators are not new in conventional quantum mechanics. See, for instance, J.M.LEVY-LEBLOND, "Who is afraid of nonhermitian operators ? A quantum description of angle and phase", Ann. Phys. 101, 319 (1976). For a study of non-Hermitian operators see, for instance, C. FOIAS and Sz. NAGY, Harmonic Analysis of Operators on Hilbert Space, North Holland, Amsterdam (1970) and J. T SCHWARTZ and N. DUMFORD, Linear operators, Volumes I, II (particularly) and III, Wiley-Interscience, N.Y. (1964-71). As an incidental note, the terms "selfadjoint and nonselfadjoint operators" will not be used in this paper to avoid possible confusion with the terms" selfadjoint and nonselfadjoint forces". Clearly the former meaning is in the operational sense while the latter is in the variational sense (with rather significant interrelations-see ref.[21], Vol. I). We have preserved the latter terms because the variational approach to selfadjointness was born considerably earlier than that of the corresponding operational approach.

37. J.D.IMMELE, K.K.KAN and J.J. GRIFFIN, "Special examples of quantized friction", Maryland preprint 75-049 (1975).; K.K.HAN and J.J.GRIFFIN, Phys. Lett. 50B, 241 (1974) and Proc. Intern. Conf. on Reactions between Complex Nuclei, North-Holland, Amsterdam (1974). It should be stressed that these authors represents dissipative quantum mechanical

systems via a <u>nonlinear</u> generalization of Schrodinger's equations of the type

$$i \frac{\partial}{\partial t} \psi(t, r) = \left\{ -\frac{1}{2m} \Delta + V(r) - \eta \frac{i}{2m} \ln \frac{\psi(t,r)}{\psi^*(t,r)} \right\} \psi(t,r), \quad \hbar = 1,$$

also for the case of linear velocity damping. In contrast, our representation of the same system is strictly <u>linear</u> . In the final analysis, this is allowed by the methods of the Inverse Problem[21] for the construction of a Hamiltonian representation for nonconservative systems.

38. There exists a distinction between nonconservative and nonselfadjoint force which is significant to indicate. In essence a nonconservative force may also be derivable from a potential. This is the case of the trivial force $F = F(t)$ which can be formally derived from the potential $V = F(t)\, r$. A nonselfadjoint force, instead, is <u>always</u> nonderivable from a potential.

39. See, for instance. P. HAVAS, Jour. Math. Phys. <u>16</u>, 1461 (1975) and F. CANTRIJN, Jour. Phys. A: Math. Gen. <u>10</u>, 491 (1977).

40. D.G.CURRIE and E.J.SALETAN, Jour. Math. Phys. <u>7</u>, 967 (1966).

41. G. MARMO and E.J. SALETAN, "q-equivalent particle Hamiltonians III: the two dimensional quantum oscillator", Naples preprint (1978).

42. R.M.SANTILLI, Indiana Conference of 1968, see the proceedings: <u>Analytic Methods of Mathematical Physics</u>, R.P.GILBERT and R.G.NEWTON, Editors, Gordon and Breach N.Y.(1970).

43. A.A.ALBERT, Trans. Amer. Math. Soc. <u>64</u>, 552 (1948).

44 A digression of algebraic character is here in order. For a vector space with elements x, y, z, ... over a field F with elements α, β, γ, ... equipped with a bilinear composition law xy to characterize a (nonassociative) <u>algebra</u>, it must verify the distributive and scalar rules, in which case a bilinear structure is truly ensured. If either the distributive (left or right or both) or the scalar rules are violated , we do not

have an "algebra" as commonly understood. Rules (4.14.13) have the effect of guaranteeing that the linear vector space U with elements A, B, C, ... over a field F, when equipped with product (4.14.11) is indeed an <u>algebra</u>. The Lie-admissible nature of such algebra is a second property. The reason for this digression is that the distributive and scalar rules are often written in the form

$$\begin{aligned} (x+y)z &= xz + yz, \\ z(x+y) &= zx + zy, \\ \alpha(xy) &= (\alpha x)y + x(\alpha y), \end{aligned} \qquad \text{for all } x, y, z \in U, \ \alpha \in F \qquad (1)$$

See, for instance, R.D.SCHAFER, <u>An Introduction to Nonassociative Algebras</u>, Academic Press, N.Y. (1966). Now, the scalar rules as given above, when written in terms of our product (A, B), is <u>violated</u>, i.e.,

$$(\alpha, (A,B)) \neq ((\alpha, A), B) \neq (A, (\alpha, B)), \qquad (2)$$

while the form (4.14.13c) of the scalar rule is indeed verified. The point which we would like to make is that rules (4.14.13) are sufficient to render a vector space U with product (4.14.11) an algebra, that is, they are sufficient to ensure bilinearity. I would like to express my appreciation to H.C.MYUNG for invaluable assistance on this point. The violation of the scalar rule in the form (2) above has however intriguing algebraic properties, as pointed out to me by Professor Myung. If $\mathcal{A}$ is an associative algebra with elements A, B, C, ... and product AB, and U is the new algebra in terms of the same elements of $\mathcal{A}$ but now characterized by the product (4.14.11), the violation of the scalar rule (2) above implies that, when $\mathcal{A}$ has an identity, this identity is no longer in the center of its extension U.

As additional remarks, it is here appropriate to indicate that product (4.14.11) violates the differential rules,

$$(A, BC) \neq (A,B)C + B(A,C); \quad (AB, C) \neq (A, C)B + A(B,C), \qquad (3)$$

as well as the power-associative rules

$$\begin{aligned} ((A,A),A) &\neq (A,(A,A)), \\ ((A,A),(A,A)) &\neq (((A,A),A),A) \end{aligned} \qquad (4)$$

These occurrences have no algebraic implications (in the sense of prohibiting a consistent algebraic treatment of our general Lie-admissible algebras). However, they do have physical implications which will be pointed out at some later time. By recalling that the Lie algebras satisfy the differential rule and are power-associative (we are here interested only to the case of algebras and fields of characteristic zero), these latter properties can be considered as algebraic characterization of nonselfadjoint forces. Almost needless to say, we are here referring to product (4.14.11) in the given general form. When some of its numerous subcases are considered (see later on the content of Section 3), the differential rule and the power-associativity may be verified. Thus, product (4.14.11) appears to be one of the most truly general products of a Lie-admissible algebra which is inclusive as a particular case of all other physically significant (nonassociative and associative) Lie-admissible algebras (product (4.14.11) trivially recovers the conventional associative product of quantum mechanics for R = 1 and S = 0). This is here pointed out so that the reader will expect the selection of simper, particularized, forms of product (4.14.11) for initial practical applications.

As a last remark and somewhat intriguing the loss of the differential rule appears to be a pure "quantum mechanical" effect in the sense that the classical form of the most general Lie-admissible product (Table 4.3)

$$(A,B)_{cl} = \frac{\partial A}{\partial b^\mu} S^{\mu\nu}(t,b) \frac{\partial B}{\partial b^\nu} , \quad S^{\mu\nu} = \frac{\partial b^\mu}{\partial R_\nu} , \tag{5}$$

does indeed satisfy the scalar rule

$$(A,BC)_{cl} = (A,B)_{cl} C + B (A,C)_{cl} ; \; (AB,C)_{cl} = (A,C)_{cl} B + B (A,C)_{cl} . \tag{6}$$

45. R.M.SANTILLI and G.SOLIANI, "Statistics and Parastatistics Formal Unification", University of Torino internal note (1967), unpublished.

46. See, for instance, N, JACOBSON, Lie algebras, Interscience Publishers, N.Y. (1962).

47. C. N. KTORIDES, Jour. Math. Phys. 16, 2130 (1975).

48. H.C.MYUNG, Canad. Jour. Math. 23, 270 (1971); Proc. Amer. Math. Soc. 31, 95 (1972); Trans. Amer. Math. Soc. 167, 79 (1972); Canad. Jour. Math. 26, 1192 (1974); Proc. Amer. Math. Soc. 59, 6 (1976); and Hadronic Journal 1, 169 (1978).

49. R. J. DUFFIN, Arch. Rational Mech. Anal. 9, 309 (1962).

50. It should be here indicated that in the following diagram of Aspect I we have implemented a broader notion of genotopy and isotopy in the sense that we also admit a change in the realization of the elements of the algebra (these notions, as introduced in the simplest possible form in Table 4.14, implies the preservation of the underlying vector space).

51. W.S.HELLMAN and C.G.HOOD, Phys. Rev. D5, 1552 (1972) and C.G.HOOD, "On the quantization of a class of nonlinear Lagrangians", Thesis, Department of Physics, Boston University, Boston, Ma (1971).

52. C.N.KTORIDES, Hadronic Journal 1, 194 (1978).

53. P. JORDAN, Nachr. Ges. Wiss. Göttingen, 569-575 (1932) and 209-214 (1933); and Z. Phys. 80, 285 (1933).

54. P. JORDAN, J. von NEUMANN and E. WIGNER, Ann. Math. 36, 29 (1934).

55. It is significant at this point to recall that the validity of quantum mechanics beyond the originally conceived physical context, the atomic structure, was questioned by numerous authoritative physicists during the first part of this century. It has been only lately that this speculative analysis, which, after all, is at the foundation of genuine physical progress, has been abandoned and quantum mechanics has reached the currently assumed "universal applicability" in the miscroscopic word. Besides the statement by P. JORDAN, J. von NEUMANN and E. WIGNER, it is here appropriate also to quote ENRICO FERMI when, in relation to strong interactions and their short range, he states

> ".... there are some doubts as to whether the usual concepts of geometry hold for such small region of space."
>
> ENRICO FERMI, Nuclear Physics, The University of Chicago Press, Chicago, (1949), page 111.

Intriguingly, it has been indicated to me that ENRICO FERMI suggested the main line of study of this paper, realization of strong interactions via forces non-derivable from a potential, before I was born. Regrettably, I have been unable to identify a specific source confirming this proposal. Any communication by collegues who are aware of a reference to this proposal by ENRICO FERMI would be greatly appreciated. Equally appreciated would be the indication of similar proposals by physicists during the first part of this century.

56. See, for instance, A.A.ALBERT, Editor, Studies in Modern Algebras, Prentice-Hall, Englewood Cliff, N.J. (1963); H. BROWN and M. KOECHER, Jordan-Algebren, Springer-Verlag, Berlin (1966); and N. JACOBSON, Structure and Representation of Jordan Algebras, Amer. Math. Soc., Providence, R.I. (1968).

57. A. PAIS, Phys. Rev. Letters 7, 291 (1961).

58. H.H.GOLDISTINE and L.P. HORWITZ, Proc. Nat. Acad. 48, 1[illegible]34 (1962) and Ann. Math. 154, 1 (1964).

59. A.GAMBA, in High Energy Physics and Elementary Particles, A. SALAM, Editor, IAEA, Vienna (1965).

60. L.P.HORWITZ and L.C. BIEDENHARN, Helv. Phys. Acta 38, [illegible]85 (1965).

61. M. GÜNAYDIN and F. GÜRSEY, Jour. Math. Phys. 14, 1651 (1973).

62. M.GÜNAYDIN, "Moufan plane and octonionic quantum mechanics", Univ. of Geneva preprint UGVA-DPT/12-154 (1978).

63. R.M.SANTILLI, "On a possible Lie-admissible covering of Einstein's special relativity for strong interactions as not derivable from a potential", in preparation for the Hadronic Journal.

64. If the nonselfadjoint field equations (4.20.3) verify a Lie-admissible relativity, the familiar Lorentz covariance rule (4.20.4) is prohibited because linear. See in this respect footnote[33]. We hope to study this aspect in more details in the forthcoming paper[63].

65. We have stressed in footnote[55] that a critical attitude toward experimentally unverified knowledge is a vital prerequisite for genuine progress in basic research. We can also express the spirit of Table 4.20 by conducting a critical inspection of the following statement by C.W. MISNER, K. S. THORNE and J. A. WHEELER in their excellent treatese on gravitation (ref. [34], page 304):

> "Of all the foundations of physics, none is more firmly established than special relativity; and of all the lessons of special relativity none stand out with greater force than these. (1) Spacetime, far from being stratified, is homogeneous and isotropic throughout any region small enough ("local region") that gravitational tide-producing effects ("spacetime curvature") are negligible. (2) No local experiment whatsoever can distinguish one local inertial frame from another. (3) The speed of light is the same in every local inertial frame. (4) It is not possible to give frame-independent meaning to the separation in time ("no Newtonian stratification"). (5) Between every event and every nearby event there exists a frame-independent, coordinate independent spacetime interval ("Riemannian geometry")'. (6) Spacetime is always and everywhere locally Lorentz in character ("local Lorentz character of this Riemannian geometry")."

I here humbly submit that the above statement be complemented with the following

> (7) The problem of the validity within a hadron of properties (I) through (6) recalled by Misner, Thorne and Wheeler is open at this time on both theoretical and experimental grounds.

66. W. PAULI, Z. Phys. 31, 765 (1925). See also Science 103, 213 (1946).

67. As indicated earlier, we contemplate to study the quantum field theoretical profile of nonselfadjoint systems in a separate paper. Neverthless, it is of some significance to indicate that the spin-statistics theorem can be expected to be generally inapplicable from purely classical considerations. Indeed, if the systems considered generally violate the Lorentz covariance, there is no a priory reason to suspect that such covariance is regained upon quantization. The expectation then follows from the role of Lorentz covariance in the proof of the spin-statistics theorem. Almost needless to say, the inapplicability of Lorentz covariance implies that of the Wightman axioms. All these departures from familiar approaches may appear disturbing to some reader. I personally consider them most significant to stimulate the search of broader approaches. In few words, the methods of the Inverse Problem identify one arena of unequivocal applicability of conventional classical, quantum mechanical and quantum field theoretical disciplines, that of essentially selfadjoint systems. This includes not only electromagnetic interactions, but also the recent unified gauge theories of weak and electromagnetic interactions, as well as the systems which have been proved by the constructive field theory as verifying the Wightman axioms. The point is that it is extremely unlike that the physical universe can be entirely described by local, essentially selfadjoint forces. In any case, it is our duty to identify broader forces and the methods for their treatment. Only in this case we will be in a position to produce a quantitative

resolution of the problem of the forces governing the physical universe. When the issue is seen from this profile, it suggests the following research attitude . If the strong interactions in general and the strong hadronic forces in particular are realized in terms of essentially nonselfadjoint couplings or forces, they demand the abandonment of the virtual entirely of our current theoretical knowledge and the courageous search for covering disciplines specifically conceived for the considered layer of systems, in exactly the same way as it occurred for the electromagnetic interactions in general and the problem of the atomic structure in particular. It is this courageous research attitude which may result in basic advancements in due time. The alternative research attitude, that of preserving current knowledge as much as possible, is clearly unproductive as far as basic research is concerned.

68. We however expect a unity as far the constituents is concerned, in the sense that the constituents of mesons may result to be the same as those of baryons, only embedded in still more complex dynamical conditions and thus, demanding more adequate methods.

69. E. NOWAK, J. SUCHER and C.H. WOO, Phys. Rev. D16, 2874 (1977) and quoted papers.

70. R.M.SANTILLI, Ann. Phys. 83, 108 (1974).

71. L.I.SHIFF, H. SNYDER and J. WEINBERG, Phys. Rev. 57, 315 (1940).

72. L. ROSENFELD, Nuclear Forces, North-Holland, Amsterdam (1948), Vol. I, page 77; ans S. FLÜGGE, Practical Quantum Mechanics, Vol. I, page 177, Springer-Verlag, Heidelberg (1971).

73. E. CORINALDESI and F. STROCCHI, Relativistic Wave Mechanics, North-Holland, Amsterdam (1963), page 121.

74. See, for instance, B. G. WYBOURNE, Classical Groups for Physicists, Wiley, N.Y, (1974), Chapter 21.

75. J. H. FIELD, E. PICASSO and F. COMBLEY, " Tests of fun damental physical theories from measurements on free charged leptons", Soviet Phys. Uspekhi, in press.

ERRATA - CORRIGE

R.M.SANTILLI, "On a possible Lie-admissible covering of the Galilei relativity in Newtonian Mechanics for nonconservative and Galilei form-noninvariant systems", HADRONIC JOURNAL 1, 223-423 (1978).

ERRATA	CORRIGE
- Eq. (3.4.11)	- add minum to last term,
- Eqs. (3.7.7), (3.7.16) and (3.7.28)	- add ordering $i \leq j \leq k$,
- page 358, third line of last paragraph "vered at the limit of nonconservative forces."	- "vered at the limit of null nonconservative forces."
- Eq. (3.8.21) $\lim_{F_{ka}\to 0} \hat{S}_2 \equiv \omega_2 = \omega_{\mu\nu}\, db^\mu \cdot db^\nu$	$\lim_{F_{ka}\to 0} \hat{S}_2 \equiv \omega_2 = \omega_{\mu\nu}\, db^\mu \wedge db^\nu$
- page 371, last line of Thoerem 3.8.1 "reduces $\hat{S}'_2$ to the fundamental symplectic-admissible form $\hat{S}_2$ in b."	- " reduces $\hat{S}'_2$ to the fundamental symplectic-admissible form $\hat{S}_2$ in b up to local isotopy."
- Eq (4.14c)	- change T into R
- Eq. (4.43)	

$$\left\{\left[(\Box+\mu^2)\varphi(x) - j^\mu\left(\varphi,\frac{\partial\varphi}{\partial x^\alpha}\right)\right]_{SA}^{C^\infty,R} - J^\mu\left(x^\alpha,\varphi,\frac{\partial\varphi}{\partial x^\alpha}\right)\right\}_{NSA}^{C^\infty,R}=0\,; \quad \left\{\left[(\Box+\mu^2)\varphi(x) - j\left(\varphi,\frac{\partial\varphi}{\partial x^\alpha}\right)\right]_{SA}^{C^\infty,R} - J\left(x^\alpha,\varphi,\frac{\partial\varphi}{\partial x^\alpha}\right)\right\}_{NSA}^{C^\infty,R}=0\,;$$

ERRATA	CORRIGE
- page 402, 17-th line "tools for the study of forces (or currents) ....."	- "tools for the study of forces (or couplings)... "