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Lie-isotopic liftings of Lie symmetries. II. Lifting of rotations

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As is well known, perfectly spherical objects with invariant $xx + yy + zz = 1$ can be deformed into the ellipsoids $xa^{-2}x + yb^{-2}y + zc^{-2}z = 1$, with consequent manifest loss of the symmetry under rotations. We show that the Lie-isotopic lifting of Lie symmetries presented in the preceding paper permits a generalization of the contemporary theory of rotations into a form which provides the invariance of all possible deformations of the sphere, while being able to recover the original theory identically, whenever the perfectly spherical form is regained. The resultant lifting of rotations turns out to be applicable also to the motion of extended particles within inhomogeneous and unisotropic media. A number of classical applications are indicated, together with the identification of intriguing open problems for the possible extension of the theory to particle physics.

I. INTRODUCTION

As is well known, when absolute rigidity is relaxed to admit the deformations of the real world,[1,2] perfectly spherical objects in Euclidean space,

$$r^t r = xx + yy + zz = 1, \qquad (1)$$

can be deformed into ellipsoids

$$r^t g r = x b_1^2 x + y b_2^2 y + z b_3^2 z = 1, \qquad (2)$$

with the consequent manifest loss of the symmetry under rotations.

Similarly, when the motion of extended objects occurs within inhomogeneous and unisotropic material media, the Euclidean invariant (1) is generalized to a form of the type

$$r^t g r = r^i g_{ij}(t, r, \dot{r}, \ldots) r^j, \qquad (3)$$

where, in general, the metric tensor has a dependence on time, coordinates, velocities, and a number of additional physical quantities (such as temperature, density, pressure, etc.).

In this paper, we shall propose a generalization of the theory of rotations which provides the invariance of all possible deformations of the sphere, Eq. (2), while recovering the conventional theory identically whenever the original structure (1) is resumed. We shall then show that the generalized theory also provides the invariance of the generalized metric of Eq. (3).

These objectives will be achieved via the use of the Lie-isotopic lifting of Lie symmetries presented in the preceding paper[3] of this series (hereinafter referred as I), with particular reference to Theorem 2.1 of that paper. All notation, definitions, and results of Paper I will be tacitly assumed.

II. LIE-ISOTOPIC LIFTING OF ROTATIONS

The Lie-isotopic lifting of Lie symmetries proposed in Section 2 of I permits a step-by-step generalization of the theory of rotations for the achievement of the form invariance of all possible deformations of the sphere. Some of the main lines of this program can be presented as follows.

Our basic space is the conventional Euclidean space in three dimensions, $E(3, \delta, \mathbb{R})$, with local coordinates

$$r = \{r^k\} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad k = 1, 2, 3,$$

and composition

$$r^t r = r^i \delta_{ij} r^j = xx + yy + zz. \qquad (4)$$

The continuous component SO(3) of the metric-preserving group[4] O(3) is given by the familiar form

$$\mathrm{SO}(3): \quad R(\theta) = e^{J_1\theta_1}|_{\mathscr{E}} e^{J_2\theta_2}|_{\mathscr{E}} e^{J_3\theta_3}|_{\mathscr{E}}, \qquad (5)$$

verifying the conditions

$$R^t R = R R^t = I, \tag{6a}$$

$$R^t = R^{-I}, \tag{6b}$$

$$\det R = 1, \tag{6c}$$

where the θ's are Euler's angles; the skew—Hermitean generators are given by

$$J_1 = J_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$J_2 = J_{31} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$J_3 = J_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J_k^t = -J_k; \tag{7}$$

and the infinite series leading to the exponentiations (7a) and (7b) are computed in the universal enveloping algebra $\mathscr{E}$ with conventional associative product of matrices and unit

$$\mathscr{E}: \quad J_i J_j = \text{associative product}, \tag{8a}$$

$$I J_i = J_i I = J_i,$$

$$I = \operatorname{diag}(+1, +1, +1) \tag{8b}$$

The attached Lie algebra is characterized by the familiar commutation rules

$$\mathrm{SO}(3): \quad [J_i, J_j] = J_i J_j - J_j J_i = -\epsilon_{ijk} J_k,$$

$$i, j, k = 1,2,3, \tag{9}$$

while the second-order Casimir invariant is given by

$$\mathbb{J}^2 = \sum_{k=1}^{3} J_k J_k = -2I. \tag{10}$$

The discrete part of O(3) is characterized by the inversion

$$Pr = -r, \qquad P = \operatorname{diag}(-1, -1, -1),$$

$$\det P = -1, \tag{11}$$

which, as well known, commutes with all elements of SO(3). We shall keep in mind that O(3) is not connected and that, since the reflections do not contain the identity, they constitute a group only when combined with SO(3).

We now introduce arbitrary, nonsingular, symmetric, and sufficiently smooth metrics over $\mathbb{R}$:

$$g = (g_{ij}) = (g_{ij}(t, r, \dot{r}, \ldots)) \tag{12}$$

with composition law

$$r^t * r = r^t g r = r^i g_{ij} r^j \tag{13}$$

characterizing the isotopic liftings $\hat{E}(n, g, \hat{\mathbb{R}})$ of $E(n, \delta, \mathbb{R})$, according to the specifications of Sec. II of Paper I.

We are interested in identifying the Lie-isotopic liftings $\hat{\mathrm{O}}(3)$ of the group of rotations O(3), that is, the set of transformations

$$X' = \hat{R}(\theta) * X = \hat{R}(\theta) g X, \tag{14}$$

which verify the conditions for constituting a Lie-isotopic group, including the isotopic rules

$$\hat{R}(0) = \hat{I} = g^{-1}, \tag{15a}$$

$$\hat{R}(\theta_2) * \hat{R}(\theta_2) = \hat{R}(\theta_1 + \theta_2), \tag{15b}$$

$$\hat{R}(\theta) * \hat{R}(-\theta) = \hat{I}, \tag{15c}$$

while leaving invariant composition (11), i.e.,

$$r'^t * r' = r^t * \hat{R}^t * \hat{R} * r = r^t * r. \tag{16}$$

As indicated in Sec. II of Paper I, the latter property holds if the elements $\hat{R}(\theta) \in \hat{\mathrm{O}}(3)$ verify the isotopic-orthogonality conditions

$$\hat{R}^t * \hat{R} = \hat{R} * \hat{R}^t = \hat{I}, \tag{17}$$

which can be expressed in terms of the inverse operation with respect to the new unit $\hat{I}$,

$$\hat{R}^t = \hat{R}^{-\hat{I}}, \tag{18}$$

and imply the following generalization of the condition (6c):

$$(\det \hat{R})^2 = (\det \hat{I})^2, \tag{19}$$

or equivalently (because of the symmetric character of g),

$$[\det(\hat{R} g)]^2 = 1. \tag{20}$$

The desired liftings $\hat{\mathrm{O}}(3)$ of O(3) can be explicitly constructed for each given metric g via the methods of Paper I (hereby tacitly assumed). The first recommendable step is the isotopic lifting $\hat{\mathscr{E}}$ of the envelope $\mathscr{E}$. This is essentially achieved via the associativity-preserving generalization of the product J_iJ_j of $\mathscr{E}$ (associative isotopy),

$$\hat{\mathscr{E}}: \quad J_i * J_j = J_i g J_j, \qquad g \text{ fixed}, \tag{21}$$

with consequent generalization of the unit

$$\hat{I} = g^{-1}, \qquad \hat{I} * J_i = J_i * \hat{I} = J_i \tag{22}$$

and of the methodology of enveloping algebras (Poincaré-Birkhoff-Witt theorem, etc.).

The Lie-isotopic groups $\hat{\mathrm{O}}(3)$ are then constructed in such a way to admit the inverse of the metric as the new, generalized unit, that is, as the Casimir invariant of order zero. The preservation of the new separation (13) is then ensured by construction.

The continuous component of $\hat{\mathrm{O}}(3)$, say, $\widehat{\mathrm{SO}}(3)$, is characterized by the reformulation of the expansion (5) in the new envelope $\hat{\mathscr{E}}$ according to Eq. (45) of Paper I, i.e.,

$$\widehat{\mathrm{SO}}(3): \quad \hat{R}(\theta) = e^{J_1\theta_1}|_{\hat{\mathscr{E}}} * e^{J_2\theta_2}|_{\hat{\mathscr{E}}} * e^{J_3\theta_3}|_{\hat{\mathscr{E}}}, \tag{23}$$

and can be equivalently formulated in the old envelope $\mathscr{E}$ for computational convenience, resulting in the factorization of the isotopic unit

$$\widehat{\mathrm{SO}}(3): \quad \hat{R}(\theta) = \left(e^{J_1 g\theta_1}|_{\mathscr{E}} e^{J_2 g\theta_2}|_{\mathscr{E}} e^{J_3 g\theta_3}|_{\mathscr{E}}\right)\hat{I}$$
$$= \left(\prod_{k=1}^{3} e^{J_k g\theta_k}\right)\hat{I} \stackrel{\text{def}}{=} S_g(\theta)\hat{I}, \tag{24}$$

where the reduced elements verify the rules (59) of Paper I, i.e.,

$$S^t g S = g. \tag{25}$$

Note that, from the rule $R^tR = I$ and $[R^t, R] = 0$, we have the isotopic rules $\hat{R}^t * \hat{R} = \hat{I}$ and $[\hat{R}^t, \hat{R}] = 0$, from which Eqs. (17) follow. For the case of factorization $\hat{R} = S\hat{I}$ as in Eq. (24), we have condition (25) as a consequence of (17). However, in general, $[S^t, S] \neq 0$ and $S^tgS \neq SgS^t$. Also, $\det S = 1$, but $S^tS \neq I$. The reader interested in learning about Lie isotopy is encouraged to verify these (and other) properties.

The discrete component of $\hat{\mathrm{O}}(3)$ can be characterized by the isotopic inversions (61) of Paper I, i.e.,

$$\hat{P} * r = -r, \qquad \hat{P} = P\hat{I}, \tag{26}$$

where P is the conventional inversion (11).

One readily verifies that the isotopic inversions alone do not constitute a Lie-isotopic (or an ordinary) group. However, the set of all possible combinations of isotopic rotations (23) and inversions (26) does form a Lie-isotopic group, as the reader is encouraged to verify.

Note that the isotopes $\hat{\mathrm{O}}(3)$ can be explicitly constructed for each given metric g, as indicated earlier. In fact, the only unknown of Eqs. (23) and (26) is precisely the assumed metric g. Note also that the invariance of the generalized separation (13) is achieved for all possible metrics of the admitted class, including generally nondiagonal metrics.

To simplify our analysis, we restrict ourselves from here on to the canonical reference frames, that is, the frame for which the metric is diagonal, and we shall write

$$r^t * r = r^t g r = x g_{11} x + y g_{22} y + z g_{33} z \tag{27}$$

It should be noted that the reduction to a diagonal form can always be achieved for all metrics of the class admitted via a similarity transformation, as is well known in the theory of metric spaces. The reader should however keep in mind that the diagonal character of the metric, holding in the canonical frame, is not generally preserved in other frames, irrespective of whether they are inertial or not.

Despite these physical limitations, the canonical frame provides a great simplification of the computations. In particular, it permits the speedy identification of the Lie-isotopic algebra via the rule (49) of Paper I, i.e.,

$$[X_i, X_j]_{\hat{\mathscr{E}}} \stackrel{\text{def}}{=} [X_i \hat{,} X_j]$$
$$= X_i g X_j - X_j g X_i$$
$$= [X_i, X_j]g + X_i[g, X_j] + X_j[X_i, g]$$
$$= g[X_i, X_j] + [X_i, g]X_j + [g, X_j]X_i, \tag{28}$$

which yields the desired commutation rules

$$\widehat{SO}(3): \quad [J_i \hat{,} J_j] = J_i g J_j - J_j g J_i = \hat{C}_{ij}^k * J_k, \qquad i, j, k = 1,2,3, \tag{29}$$

where

$$\hat{C}_{ij}^k = -\epsilon_{ijk} g_{kk}(t, r, \dot{r}, \dots)\hat{I}. \tag{30}$$

One can see in this way the generalization of the "structure constants" of the standard formulation of Lie's theory into "structure functions," as correctly predicted by the isotopic generalization of Lie's second theorem (see Refs. 3 and 12 of Paper I).

The commutativity between isotopic inversions and rotations holds in the canonical frame, owing to the identities

$$[J_k \hat{,} \hat{P}] = [J_k, P] = 0, \qquad k = 1,2,3. \tag{31}$$

Under the conditions specified above, the isotopic inversions therefore constitute a discrete, invariant subgroup of $\hat{O}(3)$. The decomposition of $\hat{O}(3)$ into a continuous and a discrete component can then be done essentially along the conventional lines.

The corresponding decomposition for the case of nondiagonal metrics demands additional, specific studies that will not be conducted at this time. This is due to the fact that the topological structure of $\hat{O}(3)$ is expected to be considerably broader than that of O(3). The relationship between discrete and continuous transformations for arbitrary, generally nondiagonal metrics is therefore expected to depend on delicate, yet unexplored properties (e.g., of cohomological nature).

Our next problem is the classification of all possible Lie algebras $\widehat{SO}(3)$ characterized by all possible metrics of the class admitted (regular, diagonal, and sufficiently smooth, but not necessarily positive or negative definite).

First, it is evident from rules (2.26) that the isotopes have no proper invariant subalgebra. The algebras $\widehat{SO}(3)$ are therefore simple in the conventional (abstract) sense.

Second, to identify the compact or noncompact character of the isotopes, we consider an arbitrary element $X = a_1 J_1 + a_2 J_2 + a_3 J_3$. The Killing form of $\widehat{SO}(3)$ can be written

$$K(X, X) = \operatorname{tr}(\hat{ad} X)^2 = \begin{vmatrix} 0 & -a_3 g_{22} & a_2 g_{33} \\ a_3 g_{11} & 0 & -a_1 g_{33} \\ -a_2 g_{11} & a_1 g_{22} & 0 \end{vmatrix}^2 = -2\left(a_1^2 g_{22} g_{33} + a_2^2 g_{11} g_{33} + a_3^2 g_{11} g_{22}\right). \tag{32}$$

One can readily see that the above form is negative definite, and the isotopes $\widehat{SO}(3)$ are compact, for all elements g_{kk} possessing the same sign (whether positive or negative). The isotopes are noncompact whenever the elements g_{11}, g_{22}, and g_{33} have different signs.

Since the metric elements are functions of the local variables, $g_{kk} = g_{kk}(t, r, \dot{r}, \dots)$, their sign cannot in general be globally defined. As a consequence, we must assume an additional local restriction for the achievement of a first classification of $\widehat{SO}(3)$. More particularly, we shall assume sufficient topological restrictions on the functions g_{kk} to preserve the sign of their value in the neighborhood of the considered point $(t, r, \dot{r}, \dots)$ of their variables.

Under these restrictions, all possible isotopes $\widehat{SO}(3)$ are characterized by all possible invariants

$$r^t g r = \pm x b_1^2 x \pm y b_2^2 y \pm z b_3^2 z. \tag{33}$$

It is then easy to see that the only compact Lie-isotopic algebras are the following two:

$$\widehat{SO_1}(3): \quad \operatorname{sign} g = (+, +, +), \tag{34a}$$

$$\widehat{SO_2}(3): \quad \operatorname{sign} g = (-, -, -), \tag{34b}$$

while all the remaining six algebras are noncompact, according to the classification

$$\widehat{SO_3}(3): \quad \operatorname{sign} g = (+, +, -), \tag{35a}$$

$$\widehat{SO_4}(3): \quad \operatorname{sign} g = (+, -, +), \tag{35b}$$

$$\widehat{SO_5}(3): \quad \operatorname{sign} g = (-, +, +), \tag{35c}$$

$$\widehat{SO_6}(3): \quad \operatorname{sign} g = (-, -, +), \tag{35d}$$

$$\widehat{SO_7}(3): \quad \operatorname{sign} g = (-, +, -), \tag{35e}$$

$$\widehat{SO_8}(3): \quad \operatorname{sign} g = (+, -, -). \tag{35f}$$

To identify the type of algebras, we introduce the following redefinition of the generators

$$\tilde{J}_1 = b_2^{-1} b_3^{-1} J_1, \qquad \tilde{J}_2 = b_1^{-1} b_3^{-1} J_2, \qquad \tilde{J}_3 = b_1^{-1} b_2^{-1} J_3. \tag{36}$$

The Lie-isotopic commutation rules (29) for the compact algebras (34) then become

$\widehat{\mathrm{SO}_1}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = \tilde{J}_3, \qquad [\tilde{J}_2 \hat{,} \tilde{J}_3] = \tilde{J}_1, \qquad [\tilde{J}_3 \hat{,} \tilde{J}_1] = \tilde{J}_2, \tag{37a}$$

$\widehat{\mathrm{SO}_2}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = -\tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = -\tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = -\tilde{J}_2. \tag{37b}$$

The second-order isotopic Casimir invariants are then given by

$$\tilde{J}^{\hat{2}}_{(\alpha)} = \sum_{k=1}^{3} \tilde{J}_k g_{(\alpha)} \tilde{J}_k = -2\hat{I}_{(\alpha)}, \qquad \alpha = 1, 2. \tag{38}$$

Comparison of Eqs. (37) and (38) with (9) and (10), respectively, then leads to the following result.

Proposition 2.1. All compact isotopes $\widehat{\mathrm{SO}}(3)$ are locally isomorphic to the SO(3) algebra, and they occur for positive or negative definite metrics.

Under the assumed topological restrictions on the metric, the Lie-isotopic algebras are integrable to their corresponding groups. The exponentials (24) therefore exist and characterize well-defined, finite isotopic rotations.

Numerous examples can be explicitly computed. As an illustration, we consider a compact isotopic rotation around the third axis for the case of the isotope $\widehat{\mathrm{SO}_1}(3)$. Trivial calculations then yield the group element

$$\hat{R}(\theta_3) = S_g(\theta_3)\hat{I} = \begin{pmatrix} \cos\theta_3 & \frac{b_2}{b_1}\sin\theta_3 & 0 \\ -\frac{b_1}{b_2}\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{I}, \tag{39}$$

with underlying transformations

$$r' = \hat{R}(\theta_3) * r = S_g(\theta_3) r = \begin{pmatrix} x\cos\theta_3 + y\frac{b_2}{b_1}\sin\theta_3 \\ -x\frac{b_1}{b_2}\sin\theta_3 + y\cos\theta_3 \\ z \end{pmatrix}, \tag{40}$$

which leave invariant the hyperboloids

$$\begin{aligned} r'^{t} g_{(1)} r' &= x' b_1^2 x' + y' b_2^2 y' + z' b_3^2 z' \\ &= x b_1^2 x + y b_2^2 y + z b_3^2 z \\ &= r^{t} g_{(1)} r, \end{aligned} \tag{41}$$

as the reader is encouraged to verify.

Note that the isotopic commutation rules of $\widehat{\mathrm{SO}_1}(3)$ and those of the conventional algebra SO(3) coincide at the abstract level of realization-free treatment of rotations. The same situation occurs for all other aspects of the theory, such as enveloping algebra, Casimir invariants, etc. A similar, formal unification can also be reached between the full orthogonal-isotopic group $\widehat{\mathrm{O}_1}(3)$ and the conventional one O(3).

Our main results of this section can then be expressed as follows.

Theorem 2.1. The groups of isometries of all possible ellipsoidical deformations of the sphere,

$$r^{t} g_{(1)} r = x b_1^2 x + y b_2^2 y + z b_3^2 z = 1,$$
$$b_k = b_k(t, r, \dot{r}, \ldots), \tag{42}$$

here denoted $\widehat{\mathrm{O}_1}(3)$, verify the following properties:

(A) The groups $\widehat{\mathrm{O}_1}(3)$ are all locally isomorphic to O(3) when isotopically realized in such a way that their units $\hat{I}_{(1)}$ are the inverse of the metrics $g_{(1)}$ of ellipsoids (42).

(B) The groups $\widehat{\mathrm{O}_1}(3)$ consist of infinitely many different (but isomorphic) realizations, corresponding to the infinite possibilities of explicit, local forms of the units $\hat{I}_{(1)}$ (or, equivalently, of the metrics $g_{(1)}$).

(C) The groups $\widehat{O_1}(3)$ constitute "isotopic coverings" of O(3), in the sense that

- (C-a) the groups $\widehat{O_1}(3)$ are constructed via methods structurally more general than those of O(3);
- (C-b) the groups $\widehat{O_1}(3)$ apply for physical conditions broader than those of O(3); and
- (C-c) all groups $\widehat{O_1}(3)$ recover O(3) identically whenever ellipsoids (42) reduce to the sphere.

The nontriviality of the notion of isotopic covering can be illustrated via the following important property.

Corollary 2.1.A. While the action of O(3) on local coordinates is linear, i.e.,

$$r' = R(\theta)r, \tag{43}$$

that of its isotopic coverings $\widehat{O_1}(3)$ is generally nonlinear, i.e.,

$$\begin{aligned} r' &= \hat{R}(\theta) * r = S_g(\theta)r \\ &= S(t, r, \dot{r}, \ldots; \theta)r. \end{aligned} \tag{44}$$

An illustration of this occurrence is given by transformations (40). In fact, the nonlinearity occurs because the elements b_k entering into the transformations are generally dependent on the local coordinates.

The nonlinearity of the action of $\widehat{O_1}(3)$ constitutes the primary reason for our hopes of the physical relevance of the theory for strong interactions, as we shall indicate in the next paper.

We pass now to the study of the noncompact forms, which, besides being useful for achieving a classification of all possible isotopic images of rotations, constitute the foundations of the next paper on the Lie-isotopic lifting of special relativity.

For the case of the noncompact algebras (35), isotopic rules (29) become

$\widehat{SO_3}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = -\tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = \tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = \tilde{J}_2, \tag{45a}$$

$\widehat{SO_4}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = \tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = \tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = -\tilde{J}_2, \tag{45b}$$

$\widehat{SO_5}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = \tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = -\tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = \tilde{J}_2, \tag{45c}$$

$\widehat{SO_6}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = \tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = -\tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = -\tilde{J}_2, \tag{45d}$$

$\widehat{SO_7}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = -\tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = -\tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = \tilde{J}_2, \tag{45e}$$

$\widehat{SO_8}(3)$:

$$[\tilde{J}_1 \hat{,} \tilde{J}_2] = -\tilde{J}_3, \quad [\tilde{J}_2 \hat{,} \tilde{J}_3] = \tilde{J}_1, \quad [\tilde{J}_3 \hat{,} \tilde{J}_1] = -\tilde{J}_2, \tag{45f}$$

while the second-order Casimir invariants preserve the form (38), i.e.,

$$\tilde{J}^{\hat{2}}_{(\alpha)} = \sum_k \tilde{J}_k g_{(\alpha)} \tilde{J}_k = -2\hat{I}_{(\alpha)}, \qquad \alpha = 3, 4, \ldots, 8. \tag{46}$$

The following result then follows.

Proposition 2. All noncompact isotopes $\widehat{SO}(3)$ are locally isomorphic to the SO(2.1) algebra, and they occur for (diagonal) metrics whose elements have different signs.

Under the assumed restrictions, the noncompact isotopic algebras are also integrable to their corresponding groups. The exponentials (24) therefore exist, although the range of the parameters now becomes infinite.

Again, numerous examples of noncompact isotopic rotations can be explicitly constructed for all cases (35). As an illustration, we consider a "rotation" around the third axis for the case of the isotope $\widehat{SO_3}(3)$. Then, trivial calculations yield the group element

$$\hat{R}(\theta_3) = \begin{pmatrix} \cosh\theta_3 & -\dfrac{b_2}{b_1}\sinh\theta_3 & 0 \\ -\dfrac{b_1}{b_2}\sinh\theta_3 & \cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \hat{I} \tag{47}$$

with underlying isotopic transformations

$$r' = \hat{R}(\theta_3)r = S_g(\theta_3)r = \begin{pmatrix} x\cosh\theta_3 - y\dfrac{b_2}{b_1}\sinh\theta_3 \\ -x\dfrac{b_1}{b_2}\sinh\theta_3 + y\cosh\theta_3 \\ z \end{pmatrix}, \qquad (48)$$

which, this time, leave invariant the hyperbolic form

$$\begin{aligned} r'^t g_{(3)} r' &= x'b_1^2x' + y'b_2^2y' - z'b_3^2z' \\ &= xb_1^2x + yb_2^2y - zb_3^2z \\ &= r^t g_{(3)} r. \end{aligned} \qquad (49)$$

Again, the noncompact isotopes are indistinguishable from SO(2.1) at the level of abstract, realization-free formulations. This aspect, including the corresponding formulation of Theorem 2.1, will be studied in the next paper for the more general case of the Lorentz group.

In summary, the isotopic lifting of Lie algebras do not produce new Lie algebras, because (as stressed in Sec. III of Paper I) all Lie algebras over a field of characteristic zero are already known. Thus, the Lie-isotopic theory merely provides new realizations of known algebras. The results of Propositions 1 and 2 were therefore predictable from the simplicity of the algebra (29). In fact, all simple, three-dimensional Lie algebras over the reals are known and are given either by SO(3) or by SO(2.1) (or by algebras isomorphic to them)

To complete our classification, we need additional information on Lie-isotopic algebras whose metrics have opposite signs.

Definition 1. Let G be an isotopic algebra characterized by (diagonal) metrics with elements g_{kk}. *The isotopic dual* G^d *of* G *is the algebra characterized by the* (diagonal) metric with elements $g_{kk}^d = -g_{kk}$, $k = 1, 2, \ldots, n$.

It is then easy to prove the following result.

Proposition 3. Isotopically dual Lie algebras are locally isomorphic.

Note that the proposition above includes the case when one of the algebras is conventional. We discover in this way that SO(3) has an image that cannot be identified via the simplest possible Lie product $AB - BA$ of current use, and demands instead the use of a more general product, such as $AgB - BgA$.

In fact, besides its conventional form, SO(3) can be realized via the isotopic dual, according to the expressions

$$\begin{pmatrix} \mathrm{SO}(3): \\ [J_i \hat{,} J_j] \\ \quad = J_i g J_j - J_j g J_i \\ \quad = -\epsilon_{ijk} J_k, \\ g = \mathrm{diag}(+1, +1, +1), \\ J_k^t = -J_k \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathrm{SO}^d(3): \\ [J_i \hat{,} J_j] \\ \quad = J_i g J_j - J_j g J_i \\ \quad = +\epsilon_{ijk} J_k, \\ g = \mathrm{diag}(-1, -1, -1), \\ J_k^t = -J_k \end{pmatrix}. \qquad (50)$$

At the level of the full orthogonal group O(3), this essentially implies the interchange of the identity I with the total inversion $\hat{I} = -I$, the latter becoming the identity of the isotopic dual. It is then easy to see that the basic algebras (50) and the eight isotopes (37) and (45) can be divided into two sets interconnected by isotopic duality.

Until now we have considered isotopes characterized by metrics with locally definite topological characters, resulting in locally definite compact or noncompact groups. To complete our classification, we should indicate the existence of isotopes that can smoothly transform compact algebras into noncompact ones, and vice versa. *Evidently, this topic is technically involved (and yet unexplored); it therefore* demands specific, detailed investigations. We shall thus content ourselves with the mere indication of the existence of this additional class of isotopies.

For this purpose it is more effective to return to the original basis J_k of Eqs. (7), to the original isotopic rules (29), and to the generic separation (27), with diagonal metric elements g_{kk}. An isotopic rotation around the third axis can be readily computed from the exponentiations (24), resulting in the expression

$$S_g(\theta) = \begin{pmatrix} \cos\left(\theta_3 g_{11}^{1/2} g_{22}^{1/2}\right) & g_{22}\left(g_{11}g_{22}\right)^{-1/2}\sin\left(\theta_3 g_{11}^{1/2} g_{22}^{1/2}\right) & 0 \\ -g_{11}\left(g_{11}g_{22}\right)^{-1/2}\sin\left(\theta_3 g_{11}^{1/2} g_{22}^{1/2}\right) & \cos\left(\theta_3 g_{11}^{1/2} g_{22}^{1/2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (51)$$

It is easy to see that the above transformations do not have, in general, a globally defined compact or noncompact character. In particular, they can be isomorphic to SO(3) for given values of the local variables and to SO(2.1) for others. Thus they can continuously interconnect compact and noncompact structures. Evidently, this is the most general possible isotopic lifting of rotations, which includes as particular cases all other forms considered so far.

To illustrate, assume that the elements g_{11} and g_{33} have the value $+1$, while the element g_{22} is given by a function of the local variables $t, r, \dot{r}, \ldots$ that interconnects smoothly the values $+1$ and -1. It is then easy to see that, for the case $g_{11} = g_{22} = g_{33} = +1$, the transformations (51) reduce to the familiar, compact rotations

$$S_g(\theta_3) = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{52}$$

while for $g_{11} = -g_{22} = g_{33} = 1$, the transformations (51) reduce to the equally familiar, but this time noncompact, Lorentz transformations

$$S_g(\theta_3) = \begin{pmatrix} \cosh\theta_3 & -\sinh\theta_3 & 0 \\ -\sinh\theta_3 & \cosh\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{53}$$

TABLE I. A preliminary classification of the isotopes of the group of rotations characterized by generalized, diagonal units $\hat{I} = g^{-1}$.

$\widehat{O}_0(3)$	$g = (+1, +1, +1)$	$g = (-1, -1, -1)$	$\widehat{O}_0^d(3)$
$\widehat{O}_1(3)$	$g = (+b_1^2, +b_2^2, +b_3^3)$	$g = (-b_1^2, -b_2^2, -b_3^2)$	$\widehat{O}_1^d(3)$
$\widehat{O}_3(3)$	$g = (+b_1^2, +b_2^2, -b_3^2)$	$g = (-b_1^2, -b_2^2, +b_3^2)$	$\widehat{O}_3^d(3)$
$\widehat{O}_4(3)$	$g = (+b_1^2, -b_2^2, +b_3^2)$	$g = (-b_1^2, +b_2^2, -b_3^2)$	$\widehat{O}_4^d(3)$
$\widehat{O}_5(3)$	$g = (-b_1^2, +b_2^2, +b_3^2)$	$g = (+b_1^2, -b_2^2, -b_3^2)$	$\widehat{O}_5^d(3)$
$\hat{O}(3)$: $g = (g_{11}, g_{22}, g_{33})$			

The basic element $\widehat{O}_0(3)$ is the conventional group of rotations realized in terms of the 3×3 unit I. Its isotopic dual $\widehat{O}_3^d(3)$ is constructed in such a way to admit the unit $\hat{I} = -1$ opposite to that of $O_0(3)$. The isotope $\widehat{O}_1(3)$ achieves the form invariance of all possible ellipsoidical deformations of the sphere, and it is locally isomorphic to $\widehat{O}_0(3)$, although it characterizes an infinite family of isotopic coverings of $\widehat{O}_0(3)$ (theorem 2.1). The subsequent isotopes $\widehat{O}_3(3)$, $\widehat{O}_4(3)$, and $\widehat{O}_5(3)$ and their duals are noncompact and locally isomorphic to the Lorentz group in two (2 + 1) dimensions. The last and most general isotope is $\hat{O}(3)$. The unit is still diagonal, but the signs of its elements cannot be globally defined. Therefore, $\hat{O}(3)$ does not possess a globally defined compact or noncompact character; it thus achieves a unified form of all simple, nonisomorphic three-dimensional Lie groups over the reals, O(3) and O(2.1). It is evident that the family of isotopes $\hat{O}(3)$ contains all the preceding ones, including their dual forms. all units (or metrics) have been restricted in these papers to be nonsingular, symmetric, and sufficiently smooth. As such, they can be all diagonalized, resulting in the classification above. Nevertheless, specific studies of the isotopic liftings of the group of rotations for nondiagonal units (or metrics) are recommended here. This is because if a unit is diagonal for the observer at rest with respect to the canonical frame, the same unit is generally nondiagonal for other frames, whether they are inertial or not. In turn, this may have important implications for the relationship between discrete and continuous transformations as well as other aspects.

The generalization to metrics (33) is self-evident and will be studied in details in Paper III. A summary view of our results with additional comments is provided in Table I. Note the *lack* of consideration for the *trivial isotopy*

$$\widehat{\mathrm{SO}}(3): \quad [\hat{J}_i \hat{,} \hat{J}_j] = -\epsilon_{ijk}\hat{J}_k, \quad \hat{J}_k = J_k g^{-1}, \quad J_k \in \mathrm{SO}(3), \tag{54}$$

which does not provide the invariance of the ellipsoidical deformations of the sphere, as indicated in the closing remarks of Sec. II of Paper I. On the contrary, the realization

$$\mathrm{SO}(3): \quad [K_i, K_j] = -\epsilon_{ijk}K_k,$$

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{b_3}{b_2} \\ -\dfrac{b_2}{b_3} & 0 & 0 \end{pmatrix},$$

$$K_2 = \begin{pmatrix} 0 & 0 & -\dfrac{b_3}{b_1} \\ 0 & 0 & 0 \\ \dfrac{b_1}{b_3} & 0 & 0 \end{pmatrix},$$

$$K_3 = \begin{pmatrix} 0 & \dfrac{b_2}{b_1} & 0 \\ -\dfrac{b_1}{b_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{54}$$

even though conventional in structure (that is, realized via the conventional associative envelope without any isotopy), reaches the basic invariance property (25), as the reader is encouraged to verify.

Here it is important to understand that by no means can the results under consideration be uniquely derived via the Lie-isotopic theory. In fact, the structure (54) indicates the possibility of recovering the form invariance of the ellipsoidical deformations of the sphere via the *conventionally realized* O(3). The Lie-isotopic liftings of Lie symmetries have been merely submitted in this paper on grounds of their pragmatic effectiveness in constructing the covering symmetry when a given, conventionally realized Lie symmetry is broken, while admitting the latter as a particular case whenever the original physical conditions are regained.

III. PRELIMINARY APPLICATIONS IN CLASSICAL MECHANICS

In order to identify physical applications of the Lie-isotopic theory of rotations, it is desirable to identify first its dynamical foundations. This in turn can be done more effectively in the arena of our best intuitions, Newtonian mechanics; we then pass to abstract, operator-type applications to particle physics.

The recently achieved Birkhoffian generalization of (classical) Hamiltonian mechanics (Paper I, Ref. 12) provides the desired dynamical setting quite naturally. In fact, the generalized mechanics was constructed via a joint analytic, geometric, and algebraic isotopy of the conventional mechanics. In particular, the Birkhoffian and Hamiltonian mechanics lose any distinction at the level of abstract, coordinate-free formulations, being characterized by different realizations of the same abstract structures. Most importantly for the content of this paper, the local realization of the relativity underlying Hamiltonian mechanics, Galilei's relativity, is not applicable to the Birkhoffian mechanics, not only because of profound physical differences, but also because of irreconcilable differences in the mathematical structures. For these reasons, a generalization of Galilei's relativity which is directly applicable to Birkhoffian mechanics has been submitted in Ref. 12 of Paper I under the name of *Galilei-isotopic relativity*.

In this section, we shall indicate that the Lie-isotopic theory of rotations worked out in this paper is the rotational component of the Galilei-isotopic relativity. The applicability of Birkhoffian mechanics and related laws then follows. In turn, this provides the classical dynamical foundations needed for operator formulations.

For the reader's convenience, as well as for notational needs, let us recall the main lines of the analytic, geometric, and algebraic isotopies of the Hamiltonian into the Birkhoffian mechanics.

(1) *Analytic lifting* of the canonical (Hamilton's) action principle

$$\begin{aligned} A(t,r) &= \int_{t_1}^{t_2} dt\,[p_k \dot{r}_k - H(t,r,p)] \\ &\stackrel{\text{def}}{=} \int_{t_1}^{t_2} [R^0_\mu(a)\,da^\mu - H(t,a)\,dt] \\ &\stackrel{\text{def}}{=} \int_{t_1}^{t_2} \tilde{R}^0_\mu(\tilde{a})\,\tilde{a}^\mu \\ &= \int_{t_1}^{t_2} \tilde{R}^0_1(\tilde{a}), \end{aligned} \tag{55a}$$

$$a=(r,p),\quad R^0=(p,0),$$

$$\mu=1,2,\ldots,6N, \tag{55b}$$

$$\tilde{a}=(t,r,p),\quad \tilde{R}^0=(-H,p,0),$$

$$\mu=0,1,2,\ldots,6N, \tag{55c}$$

into the most general possible first-order (Pfaffian) principle

$$\begin{aligned}A^{\text{gen}}(t,r,p) &= \int_{t_1}^{t_2} dt\left[P_k(t,r,p)\dot{r}_k + Q_k(t,r,p)\dot{p}_k - B(t,r,p)\right]\\ &\stackrel{\text{def}}{=} \int_{t_1}^{t_2} R_\mu(t,a)\,da^\mu - B(t,a)\,dt\\ &\stackrel{\text{def}}{=} \int_{t_1}^{t_2} \tilde{R}_\mu(\tilde{a})\,d\tilde{a}^\mu\\ &\stackrel{\text{def}}{=} \int_{t_1}^{t_2} \tilde{R}_1(\tilde{a}).\end{aligned} \tag{56}$$

(2) *Geometric lifting* of the exact, canonical, contact two-form on $\mathbb{R}\times T^*M$:

$$\tilde{\omega}_2(\tilde{a}) = d\tilde{R}_1^0(\tilde{a}) = \frac{1}{2}\left(\frac{\partial \tilde{R}_\nu^0}{\partial \tilde{a}^\mu} - \frac{\partial \tilde{R}_\mu^0}{\partial \tilde{a}^\nu}\right) d\tilde{a}^\mu \wedge d\tilde{a}^\nu$$

$$= \tfrac{1}{2}\tilde{\omega}_{\mu\nu}(\tilde{a})\,d\tilde{a}^\mu \wedge d\tilde{a}^\nu, \tag{57a}$$

$$(\tilde{\omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial H}{\partial a^\mu} \\ \hline -\dfrac{\partial H}{\partial a^\mu} & \omega_{\mu\nu}\end{array}\right) \tag{57b}$$

with symplectic subform

$$(\omega_{\mu\nu}) = \left(\frac{\partial R_\nu^0}{\partial a^\mu} - \frac{\partial R_\mu^0}{\partial a^\nu}\right)$$

$$= \left(\begin{array}{c|c} 0 & -I \\ \hline I & 0\end{array}\right), \tag{58a}$$

$$\det(\omega_{\mu\nu}) = 1, \tag{58b}$$

into the most general possible, local, exact, contact two-forms

$$\tilde{\Omega}_2(\tilde{a}) = d\tilde{R}_1(\tilde{a})$$

$$= \frac{1}{2}\left(\frac{\partial \tilde{R}_\nu}{\partial \tilde{a}^\mu} - \frac{\partial \tilde{R}_\mu}{\partial \tilde{a}^\nu}\right) d\tilde{a}^\mu \wedge d\tilde{a}^\nu \tag{59a}$$

$$= \tfrac{1}{2}\tilde{\Omega}_{\mu\nu}(\tilde{a})\,d\tilde{a}^\mu \wedge d\tilde{a}^\nu$$

$$(\tilde{\Omega}_{\mu\nu}) = \left(\begin{array}{c|c} 0 & \dfrac{\partial B}{\partial a^\mu} - \dfrac{\partial R_\mu}{\partial t} \\ \hline -\dfrac{\partial B}{\partial a^\mu} + \dfrac{\partial R_\mu}{\partial t} & \Omega_{\mu\nu}\end{array}\right) \tag{59b}$$

with symplectic subforms

$$(\Omega_{\mu\nu}) = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)$$

$$= \left(\begin{array}{c|c} \left(\dfrac{\partial P_j}{\partial r_i} - \dfrac{\partial P_i}{\partial r_j}\right) & \left(\dfrac{\partial Q_j}{\partial r_i} - \dfrac{\partial P_i}{\partial p_j}\right) \\ \hline \left(\dfrac{\partial P_i}{\partial p_j} - \dfrac{\partial Q_j}{\partial r_i}\right) & \left(\dfrac{\partial Q_j}{\partial p_i} - \dfrac{\partial \varphi_i}{\partial p_j}\right)\end{array}\right), \tag{60a}$$

$$\det(\Omega_{\mu\nu}) \neq 0. \tag{60b}$$

(3) *Algebraic lifting* of the conventional, regular, canonical realization of the Lie product (the Poisson brackets),

$$[A,B] = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial B}{\partial a^\nu}$$

$$= \frac{\partial A}{\partial r_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k}\frac{\partial A}{\partial p_k}, \tag{61a}$$

$$\omega^{\mu\nu} = \left(\|\omega_{\alpha\beta}\|^{-1}\right)^{\mu\nu}, \tag{61b}$$

into the most general possible local and regular realization of the Lie-algebra product in Newtonian mechanics (the Birkhoffian brackets),

$$[A\,\hat{;}\,B] = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu}, \tag{62a}$$

$$\Omega^{\mu\nu} = \left(\|\Omega_{\alpha\beta}\|^{-1}\right)^{\mu\nu}. \tag{62b}$$

Hamilton's equations (without external terms), as emerging from the principle (55) in their covariant-symplectic version

$$\omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial H(t,a)}{\partial a^{\mu}} = 0, \qquad \mu = 1,2,\ldots,6N, \tag{63}$$

or in their equivalent, contravariant-Lie version

$$\dot{a}^{\mu} = \omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = [a^{\mu}, H], \tag{64a}$$

$$[a^{\mu}, a^{\nu}] = \omega^{\mu\nu}, \tag{64b}$$

are then generalized into *Birkhoff's equations* in their corresponding covariant-symplectic version originating from the principle (56),

$$\Omega_{\mu\nu}(a)\dot{a}^{\nu} - \frac{\partial B(t,a)}{\partial a^{\mu}} = 0, \tag{65}$$

or in their equivalent contravariant-Lie version

$$\dot{a}^{\mu} = \Omega^{\mu\nu}\frac{\partial B}{\partial a^{\nu}} = [a^{\mu}\hat{,} B], \tag{66a}$$

$$[a^{\mu}\hat{,} a^{\nu}] = \Omega^{\mu\nu}(a), \tag{66b}$$

where we have considered only the semiautonomous case (see Ref. 12 of Paper I for the full nonautonomous case).

As is well known, the function H of Eqs. (64) is called the Hamiltonian, and generally represents the total energy. The function B of Eqs. (66) is called instead the *Birkhoffian* to stress its differences from the Hamiltonian, particularly in view of the novel degrees of freedom, called *Birkhoffian gauges*,

$$R_{\mu} \to R_{\mu}^{\dagger} = R_{\mu} + \frac{\partial G(t,a)}{\partial a^{\mu}},$$

$$B \to B^{\dagger} = B - \frac{\partial G(t,a)}{\partial t}. \tag{67}$$

The understanding is that, under certain restrictions worked out in Ref. 12 of Paper I and here tacitly assumed, the Birkhoffian can indeed represent the total energy $B = T + V = E_{\text{tot}}$.

Along similar lines, the conventional Hamilton-Jacobi equations

$$\frac{\partial A}{\partial t} + H(t,a) = 0, \tag{68a}$$

$$(R_{\mu}^{0}) = \begin{pmatrix} p_k \\ 0 \end{pmatrix} = \left(\frac{\partial A}{\partial a^{\mu}}\right) = \begin{pmatrix} \dfrac{\partial A}{\partial r_k} \\ 0 \end{pmatrix}, \tag{68b}$$

are generalized into the Birkhoffian form [Ref. 12 of Paper I, Eqs. (6.1.24), p. 207]

$$\frac{\partial A^{\text{gen}}}{\partial t} + B(t,a) = 0, \tag{69a}$$

$$(R_{\mu}) = \begin{pmatrix} P_k(t,r) \\ Q_k(t,r) \end{pmatrix} = \left(\frac{\partial A^{\text{gen}}}{\partial a^{\mu}}\right) = \begin{pmatrix} \dfrac{\partial A^{\text{gen}}}{\partial r_k} \\ \dfrac{\partial A^{\text{gen}}}{\partial p_k} \end{pmatrix} \tag{69b}$$

under the condition

$$\det\left(\frac{\partial R_{\mu}}{\partial a^{\nu}}\right) \neq 0, \tag{70}$$

which is always realizable because of the gauges (67).

Each and every aspect of Hamiltonian mechanics (transformation theory, symmetries, first integrals, etc.) then admits a consistent Birkhoffian generalization. For brevity, these aspects are not reviewed here. We merely restrict ourself to the indication that the conventional canonical realization of the *Galilei symmetry*

$$G(3.1): \quad a' = \left[\prod_{k=1}^{10} \exp\left(\theta_k \omega^{\alpha\beta}\frac{\partial X_k}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)\right] a, \tag{71a}$$

$$X = \{E_{\text{tot}}, \mathbb{p}, \mathbb{J} = \mathbb{r} \times \mathbb{p}, m\mathbb{r}\}, \tag{71b}$$

$$\theta = \{t_0; \mathbb{r}_0; \theta_1, \theta_2, \theta_3; \mathbb{v}_0\}, \tag{71c}$$

is generalized into the *Galilei-isotopic symmetry* [Ref. 12 of Paper I, Eq. (6.3.60), p. 245]

$$\hat{G}(3.1): \quad a' = \left[\prod_{k=1}^{10} \exp\left(\theta_k \Omega^{\alpha\beta}\frac{\partial X_k}{\partial a^{\beta}}\frac{\partial}{\partial a^{\alpha}}\right)\right] a \tag{72}$$

where, in line with the basic idea of Lie isotopy (Paper I), the generators, the parameters, and the local variable are the conventional ones (and we have ignored scalar extensions for simplicity). Note that the time component of Eq. (72) is exactly the exponentiated form of Birkhoff's equations, much as in the Hamiltonian case.

The generality of the Birkhoffian over the Hamiltonian mechanics is evident. In fact, the latter is contained as a particular case of the former whenever the functions $(R_\mu) = (P, Q)$ assume the canonical form $(R_\mu^0) = (p, 0)$.

We should also recall that the Birkhoffian mechanics achieves the so-called "direct universality" in mechanics, that is, the capability of representing in the local frame of the observer all systems of ordinary differential equations in first-order form that verify certain topological conditions (locality, regularity, and analyticity). On the contrary, Hamiltonian mechanics can represent only a subset of Newton's equations in the frame of the observer (the so-called nonessentially non-self-adjoint systems).

A primary physical result emerging from the lifting of the Hamiltonian into the Birkhoffian mechanics is the identification of a new class of interactions called closed, variationally non-self-adjoint interactions. We are referring to systems of extended particles which are closed in the conventional sense of verifying all conservation laws of total quantities (see Ref. 12 of Paper I, pp. 235–237):

$$\dot{X}_k = \frac{\partial X_k}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial E_{tot}}{\partial a^\nu} = [X_k \hat{;} E_{tot}] = 0, \qquad (73)$$

$$B = E_{tot}$$

Nevertheless, the internal forces are partially of non-Hamiltonian type, due to the extended character of the constituents and the consequent existence of contact forces for which the notion of potential energy has no physical meaning.

In short, the theory of isotopic lifting, in general, and the Lie-isotopy theory, in particular, when realized in classical mechanics, permit a representation of the extended character of particles via contact forces that are beyond the technical capabilities of Hamiltonian mechanics. In particular, the non-Hamiltonian forces are represented by the generalized structure of the Lie product, therefore opening up a new horizon of possible, intriguing advances in the problem of interactions.

Once the foundations of the Birkhoffian mechanics are known, the identification of the dynamics underlying the Lie-isotopic theory of rotations is straightforward.

For simplicity but without loss of generality, let us consider the case of one, free, extended particle in Euclidean space $E(3, \delta, \mathbb{R})$, and the trivial canonical action

$$A(t, r) = \int_{t_1}^{t_2} dt\, [\mathbf{p}\cdot\dot{\mathbf{r}} - \tfrac{1}{2}\mathbf{p}\cdot\mathbf{p}]$$

$$= \int_{t_1}^{t_2} dt\, [p_k \dot{r}_k - H], \qquad m = 1. \qquad (74)$$

Suppose that, at a given value of time, the particle experiences only contact non-Hamiltonian forces due to its extended character (e.g., because of penetration within a resistive, generally anisotropic and inhomogeneous, material medium). Suppose that these physical conditions can be represented via the isotopic lifting $\hat{E}(3, g, \hat{\mathbb{R}})$ of the Euclidean space (Paper I), i.e., via the generalization of the action (74) into the form

$$A^{gen}(t, r) = \int_{t_1}^{t_2} dt\, (\mathbf{p} * \dot{\mathbf{r}} - \tfrac{1}{2}\mathbf{p} * \mathbf{p})$$

$$= \int_{t_1}^{t_2} dt\, \left[p_k \cdot g_{ij}\dot{r}_j - \tfrac{1}{2}p_i g_{ij} p_j\right],$$

$$g = g(t, r, \dot{r}, \ldots), \qquad (75)$$

which is manifestly of Birkhoffian non-Hamiltonian type with identifications

$$P_k(t, r, p) = p_i g_{ik}, \qquad B = \tfrac{1}{2}p_i g_{ij} p_j. \qquad (76)$$

The non-Hamiltonian character of the theory can be technically established via the property that the equations of motion underlying the action (56) generally violate the integrability conditions for the existence of a Hamiltonian in the r-frame considered (see e.g. Ref. 5, Theorem 3.12.1, p. 176). The inapplicability of the Hamiltonian mechanics implies, in particular, the inapplicability of the Poisson brackets for the Lie characterization of both the time evolution and the theory of rotations.

The direct applicability of Birkhoffian mechanics has the immediate advantage of permitting the identification of the generalized Lie product for both the time evolution and the applicable theory of rotations. For the limited objectives of this paper, it is sufficient to restrict ourselves to the case of a diagonal metric g with constant elements

$$g = \text{diag}(b_1^2, b_2^2, b_3^2), \qquad b_k = \text{const.} \qquad (77)$$

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Use of Eqs. (60) and (62b) then readily yields the Lie-isotopic tensor

$$(\Omega^{\mu\nu}) = \left(\left\| \begin{array}{c|c} 0 & -\dfrac{\partial P_i}{\partial p_j} \\ \hline \dfrac{\partial P_i}{\partial p_j} & 0 \end{array} \right\|^{-1} \right)$$

$$= \left(\begin{array}{c|c} 0 & -g^{-1} \\ \hline g^{-1} & 0 \end{array} \right) \tag{78}$$

with generalized brackets

$$[A\hat{,} B] = \frac{\partial A}{\partial r_i} g_{ij} \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r_i} g_{ij} \frac{\partial A}{\partial p_j}. \tag{79}$$

Simple calculations then establish the following Newtonian realization of the isotope $\widehat{SO_1}(3)$ of rotations:

$$\widehat{SO_1}(3): \quad [J_i\hat{,} J_j] = \epsilon_{ijk} b_k^{-2} J_k, \tag{80}$$

with redefinition according to Eq. (37a):

$$\widehat{SO_1}(3): \quad [\tilde{J}_i\hat{,} \tilde{J}_j] = \epsilon_{ijk} \tilde{J}_k, \tag{81a}$$

$$\tilde{J}_1 = b_2 b_3 J_1, \qquad \tilde{J}_2 = b_1 b_3 J_2, \qquad \tilde{J}_3 = b_1 b_2 J_3, \tag{81b}$$

and group form of the symbolic type

$$\widehat{SO_1}(3): \quad a' = \left[\prod_{k=1}^{3} \exp\left(\theta_k \Omega^{\mu\nu} \frac{\partial \tilde{J}_k}{\partial a^\nu} \frac{\partial}{\partial a^\mu} \right) \right] a, \tag{82}$$

with a corresponding reduction to a form of type (24).

The achievement of the desired objective is then confirmed by illustrative examples. For instance, an isotopic rotation around the third axis with generator J_3 can be computed via exponentials (82), yielding the transformations

$$r' = \begin{pmatrix} x\cos(\theta_3 b_1 b_2) - y\dfrac{b_2}{b_1}\sin(\theta_3 b_1 b_2) \\ x\dfrac{b_1}{b_2}\sin(\theta_3 b_1 b_2) + y\cos(\theta_3 b_1 b_2) \\ z \end{pmatrix}, \tag{83}$$

with additional transformations of the type (39) for the generator $\tilde{J}_3$.

The achievement of the form invariance of the Pfaffian action (75) is then consequential. Action-at-a-distance forces can be trivially incorporated in the theory via additive potentials in the Birkhoffian $B = \frac{1}{2}p^2$, provided that they are properly written in $\hat{E}(3, g, \hat{\mathbb{R}})$, e.g., with "squares" of the type (2).

As an application, we shall now present a generalization of Euler's theorem (on the displacement of rigid bodies) to the case of elastic bodies. As the reader recalls,[6] Euler's theorem essentially states that *the general displacement of a rigid body with one point fixed is a continuous rotation around some axis.*

Suppose that the object is an elastic sphere of radius 1, and that the fixed point is the origin of the reference frame. In the absence of deformation, the displacements of the objects are given by time-dependent transformations $R = R(t) \in SO(3)$. At time $t = 0$ one can assume

$$R(0) = I = \mathrm{diag}(+1, +1, +1). \tag{84}$$

At subsequent times t, the rotations are such that their eigenvalues are the elements of the conventional 3×3 unit I, i.e., there exists an eigenvector $\mathbf{a}$ of $R(t)$ which preserves its components in the rotated system:

$$\mathbf{a}' = R(t)\mathbf{a} = \mathbf{a}, \tag{85}$$

or, equivalently, rotations verify the eigenvalue equations

$$[R(t) - I]\mathbf{a} = 0 \tag{86}$$

with secular determinant

$$\det(R - I) = 0. \tag{87}$$

Suppose now that at time $t = t_0$ the sphere experiences an infinitesimal deformation into the ellipsoid with

$$r^t g r = x(1+\epsilon_1)x + y(1+\epsilon_2)y + z(1+\epsilon_3)2 = 1. \tag{88}$$

It is easy to see that the displacement can now be described via a compact isotopic rotation $\hat{R}(t) \in \widehat{SO_1}(3)$, beginning with the identification

$$\hat{R}(\epsilon) = \hat{I} = g^{-1}. \tag{89}$$

It is also easy to prove that the eigenvalue equation for the rigid motion, Eq. (86), admits the isotopic generalization

$$[\hat{R}(t) - \hat{I}] * a = [S_g(t) - I]a = 0 \qquad (90)$$

with isotopic-secular determinant

$$\det(\hat{R} - \hat{I}) = \det(S - I) = 0, \qquad (91)$$

where we have used the decomposition of Eq. (24), $\hat{R} = S_g\hat{I}$, and Theorem 2.19 of Ref. 15 of Paper I, p. 1310.

In fact, from Eq. (19), $\det \hat{R}(t) = \det \hat{I}$. A step-by-step generalization of the conventional proof (see, e.g., Ref. 6, pp. 119–123) then leads to the following result.

Lemma 3.1. The isotopic eigenvalues of the compact-isotopic rotations of type 1 are the elements of the (diagonal) generalized unit $\hat{I} = g^{-1}$.

Thus, much as in the conventional case, the compact-isotopic rotations admit an eigenvector that preserves its components in the transformed system. By recalling that the transformations considered here can only be continuous, the extensions to the case of finite deformations and to nonspherical objects are straightforward, yielding the following result.

Theorem 3.1 (Isotopic lifting of Euler's theorem). The general displacement of an elastic body with one point fixed is a compact isotopic rotation of type 1 around some fixed axis.

Numerous additional applications to the dynamics of extended, elastic, and deformable bodies are possible. Here, we limit ourselves to the indication that the isotopes of O(3) seem to be naturally set for the description of deformations, with the understanding that the theory generally demands the use of nondiagonal metrics. In fact, all metrics of the theory of elasticity are permitted by the isotopic theory of rotations described in this paper.

An additional class of physical applications is the motion of extended objects within generally inhomogeneous and anisotropic material media. In effect, the description of the displacement of elastic bodies (Theorem 3.1) and that of the motion on material media are complementary to each other, in the sense that they can both be reduced to suitable isotopic liftings of the Euclidean space.

To illustrate this possibility, consider a (classical) particle moving in a region of empty space for which the Euclidean geometry applies. Suppose now that the region considered is filled with intense radiation originating from a distant and constant source, assumed to be at infinity. It is evident that, under these novel physical conditions, the particle cannot be considered as moving in empty space. The new medium of propagation is space filled with radiation. Depending on the physical characteristics of the particle (size, charge, electric and magnetic moments, etc.), the new medium will directly affect the trajectory of the particle, that is, its dynamical evolution. In particular, the new medium is homogeneous but manifestly anisotropic, in the sense that the distribution of radiative energy is uniform, but the medium has a preferred orientation in space given by the direction of propagation of the background radiation.

Clearly, the Euclidean geometry is merely approximated for these broader physical conditions. The selection of an appropriate isotopic lifting is then relevant. We select the Finsler space[7] with composition

$$r^t g r = r^i f(r, u)\delta_{ij} r^j, \qquad (92a)$$

$$f(r, u) = \frac{(r \cdot u)^2}{(r \cdot r)^2}, \qquad (92b)$$

where u is a unit vector ($u^2 = u_k u_k = 1$), here assumed along the direction of the radiation.

It is evident that the Finsler space with composition (92) characterizes an isotope $\hat{E}(3, g, \hat{\mathbb{R}})$ of $E(3, \delta, \mathbb{R})$. As a result, the symmetry $\hat{O}(3)$ applies (including isotopic reflections). The reader should be aware that the symmetry O(3) is broken for the composition (92) because of its inability to preserve the preferred direction in space. The achievement of this preservation via the covering symmetry $\hat{O}(3)$ is instead ensured by the invariance of the metric under isotopic rotations [Eq. (60) of Paper I), i.e.,

$$e^{-\theta g J} g e^{J g \theta} = g. \qquad (93)$$

It is also clear that, in the transition from the Euclidean to the Finsler space, we have the transition from a flat, homogeneous, and isotropic geometry to a curved, homogeneous, and anisotropic one. Numerous intriguing properties then follow. Here we limit ourselves to the indication that, owing to the particular metric of Eq. (92), the conventional Casimir $J^2 = J_k J_k$ is preserved by the isotopic rotations,

$$\mathbb{J}^{2\prime} = \hat{R} * \left(\sum_{k=1}^{3} J_k J_k \right) * \hat{R}^t = \mathbb{J}^2, \tag{94}$$

as the interested reader is encouraged to verify. This result indicates that the angular momentum can be conserved also for motion within anisotropic media in which the conventional rotational symmetry is broken.

We recover in this way a result already known in analytic mechanics (Paper I, Ref. 12). We are referring here to the fact that the conservation of angular momentum by no means necessarily implies the symmetry under the conventional rotation group. In fact, angular momentum conservation can be also characterized by isotopic symmetries.

The generalization of the model to an inhomogeneous form is possible, and occurs, for instance, when the energy distribution of the background sea of radiation is not homogeneous. This is the case when the intensity of the radiation varies in space and time, in which case the metric (92) is generalized to forms of the type

$$g_{ij} = f(r, u)\, d_{ij}(t, r, \dot{r}, \ldots), \tag{95}$$

where the inhomogeneity and anisotropy are differentiated and represented by the respective terms $f(u, r)$ and $d_{ij}(t, r, \dot{r}, \ldots)$.

Note that, with sufficient care, the applications of the model (95) can also be extended to treat the motion of Newtonian systems within a resistive medium with density varying in space and time, and with a preferred direction in space.

Numerous additional applications are conceivable, but for brevity they are not considered here.

We conclude this paper with a few remarks on the structure of the Newtonian realization of $\hat{O}(3)$, which are relevant for the construction of corresponding operator forms.

The attentive reader has certainly noted a rather fundamental difference between the isotopic form of the Lie product in Newtonian mechanics, Eq. (80), and that of the abstract treatment, Eq. (29). In the latter case, the metric g is the isotopic element resulting in the Lie product $AgB - BgA$, while in the former case we see the appearance of the inverse g^{-1} in the Lie product. A similar difference occurs in the exponentiations, which are characterized by the metric g in the abstract case and g^{-1} in its Newtonian realization. Despite these differences, the isotopic structure functions, the eigenvalues of the isotopic Casimirs, and the explicit forms of the isotopic rotations are the same in both the abstract and the Newtonian cases.

The reasons for these differences have been identified in Ref. 8 (pp. 1330–1334). They are due to the fact that the envelope of the conventional, abstract Lie product is associative,

$$[A, B]_{\mathscr{E}} = AB - BA; \tag{96}$$

$$\mathscr{E}: \quad AB = \text{associative product}$$

$$(AB)C = A(BC),$$

while the envelope of the Poisson brackets is nonassociative, i.e., we can write the Poisson brackets as the attached brackets of an algebra $\mathscr{U}$:

$$[A, B]_{\mathscr{U}} = (A, B) - (B, A); \tag{97}$$

$$\mathscr{U}: \quad (A, B) = \frac{\partial A}{\partial r_k} \frac{\partial B}{\partial p_k}$$

$$= \text{nonassociative product},$$

$$((A, B), C) \neq (A, (B, C)),$$

which is a nonassociative Lie-admissible algebra.

In the transition to the isotopic forms, the associative and nonassociative characters persist. In fact, in the former case we have

$$[\hat{A}, B]_{\mathscr{E}} = AgB - BgA; \tag{98}$$

$$\hat{\mathscr{E}}: \quad AgB = \text{associative product},$$

$$(AgB)gC = Ag(BgC),$$

while in the latter case we have

$$[A \hat{,} B]_{\mathscr{U}} = (A \hat{,} B) - (B \hat{,} A); \tag{99}$$

$$\mathscr{U}: \quad (A \hat{,} B) = \frac{\partial A}{\partial r_i} g_{ij}^{-1} \frac{\partial B}{\partial p_j}$$

$$= \text{nonassociative product},$$

$$((A \hat{,} B) \hat{,} C) \neq (A \hat{,} (B \hat{,} C).$$

As a result, the element $\hat{I} = g^{-1}$ is the unit of the algebra $\hat{\mathscr{E}}$ but not of $\hat{\mathscr{U}}$. The possibility of using the element g^{-1} as the isotopic element of the Lie product (97) then follows.

The implications of these facts are far reaching for virtually all levels of the theory. In fact, we learn that the most adequate formulation of the Lie-isotopic generalization of rotations is that along the lines of the original proposal of Ref. 9, via a *nonassociative* Lie-admissible envelope $\mathscr{U}$, rather than the simpler form presented in Sec. II. This reformulation will not be studied at this time; the interested reader may consult theoretical papers[9–14] and experimental papers[15–19] on related issues.

In conclusion, Lie's theory has been presented historically in mathematical and physical treatments via an associative Lie-admissible envelope $\mathscr{E}$. However, Newtonian mechanics teaches us that the realizations of both the conventional and the isotopic forms of Lie's theory that occur in practical cases are those via nonassociative Lie-admissible forms. The nonassociative character of the envelope then has rather profound (and mostly unexplored) implications in the transition to the operator forms. In fact, to avoid possible inconsistencies, the preservation of such nonassociative character is desirable in the operator treatments of particles physics, much as Jordan's teaching has proposed (Paper I, Ref. 4). It is for this and other reasons that more recent treatises on Lie's theory[20] stress the need for the nonassociative character of the envelope.

The fundamental character of the nonassociative Lie-admissible algebras can therefore be seen in the structure of the conventional Poisson brackets. Thus it is expected to persist at all subsequent levels: classical or operator, physical or mathematical.

As a concluding remark, we would like to indicate that by no means is the Lie-isotopic theory of rotations submitted here as the only possibility of representing extended particles. In fact, a number of additional possibilities have been identified in the literature, most notably, Kálnay's approach via the use of intervals[21] and Prugovecki's studies via stochastic techniques.[22] Each of these approaches has its own preferred features. For instance, the Lie-isotopic approach has been conceived, specifically, for the treatment of the deformation of extended particles; Kalnay's approach is particularly tailored for quantum-mechanical measures; Prugovecki's approach is particularly suited for extended (perfectly spherical) particles under electromagnetic interactions. Despite these differences, the interrelations among these (and other) approaches to extended particles are quite intriguing, as we hope to indicate in a future paper.

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