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Lie-Isotopic Lifting of Unitary Symmetries and of Wigner's Theorem for Extended, Deformable Particles (*).

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Summary. – As is well known, the notions of symmetries and Wigner's theorem constitute some of the ultimate foundations of quantum mechanics. Nevertheless, the theory is crucially dependent on the simplest possible realization of Lie's theory, that via unitary linear or antilinear operators, which is characterized by enveloping associative algebras $\mathscr{E}$ of operators A, B ... with trivial associative product AB. A series of recent, mathematical and physical studies have established the existence of the Lie-isotopic reformulation of Lie's theory, which is based on enveloping algebras $\hat{\mathscr{E}}$ that are still associative, yet are realized via the less trivial associative-isotopic product $A*B = AgB$, where g is a suitable, fixed, operator. Furthermore, it has been proved that the Lie-isotopic theory can be consistently formulated on a Hilbert space, by providing realistic possibilities of achieving a generalization of quantum mechanics known under the name of «hadronic mechanics». In this paper, we present the notion of Lie-isotopic lifting of unitary linear and antilinear symmetries and of Wigner's theorem within the context of (the closed-exterior branch of) hadronic mechanics. The results are applied to the isotopic lifting of the operator formulation of the rotational symmetry. It is shown that the generalized symmetry can provide the invariance of all possible ellipsoidical deformations of spherical particles. This confirms the general lines of hadronic mechanics conjectured earlier, that space-time (and other) symmetries can be exact for extended particles, provided that they are expressed in a structurally more general way (isotopic-unitary), while the same symmetries can be violated when expressed via the simplest possible (unitary) realizations for pointlike approximations.

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As is well known, unitary linear and antilinear symmetries and Wigner's theorem [1-4] can be seen as the ultimate foundations of quantum mecanics, from which the dynamical structure and the rest of the theory follows.

Nevertheless, the results are crucially dependent on the simplest possible character of the assumed realization of Lie's theory, that in terms of the conventional enveloping associative algebras $\mathscr{E}$ of operators A, B, ... with trivial associative product AB.

Recent studies have shown the existence of the so-called *Lie-isotopic theory* [5-18], which is essentially characterized by enveloping algebras $\hat{\mathscr{E}}$ that are still associative,

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(1) H. WEYL: *Gruppentheorie und Quantenmechanik* (Leipzig, 1928).

(2) J. VON HEUMANN: *Grundlagen der Quantenmechanik* (Berlin, 1931).

(3) E. P. WIGNER: *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra*, (New York, N. Y., 1959).

(4) For additional early treatments of Wigner's theorem, see, *e.g.*, U. UHLHORN: *Ark. Fys.*, **23**, 307 (1962) and V. BARGMANN: *J. Math. Phys.* (*N. Y.*), **5**, 862 (1964). For more recent treatments, see, *e.g.*, L. FONDA and G. C. GHIRARDI: *Symmetry Principles in Quantum Physics* (New York, N. Y., 1970).

(5) The notion of *isotopy of Abstract Algebras* is rather old, and dates back to the early stages of set theory, *e.g.*, via the so-called Latin Squares (see R. H. BRUCK: *A Survey of Binary Systems* (Berlin, 1958)). The notion was extensively applied to a number of algebras, most notably, to the (commutative and noncommutative) Jordan algebras (see, *e.g.* K. MCCRIMMON: *Pacific J. Math.*. **15**, 925 (1965)). Nevertheless, its application to Lie's theory remained ignored in mathematical circles for considerable time. The first application of the isotopy to Lie algebra appeared in the physical literature in memoir [6] where the terms *Lie-isotopic theory* were first proposed, jointly with the reformulation of the essential structural theorems (Poincaré-Birkhoff-Witt theorem on the basis of the isotopic envelope; Lie's first, second, and third theorem for the Lie-isotopic commutation rules; Baker-Campbell-Hausdorff theorem for the composition law of isotopic Lie groups, etc.). An updated review of the (rather limited) current knowledge appears in monograph [7]. An additional review within the context of the broader *Lie-admissible generalization of Lie's theory* appears in monograph [8]. The first physical applications were for symmetries and appeared in the original proposal [6]. More recent physical applications have been presented in ref. [9,10] (isotopic lifting of the rotations), ref. [11] (isotopic lifting of the Lorentz symmetry) and ref. [12] (isotopic lifting of gauge symmetries). Mathematical studies on the Lie-isotopic theory initiated at the yearly *Workshop on Lie-admissible Formulations* [13]. (see also Tomber's Bibliography [14].) For more specific mathematical studies we suggest ref. [15-18].

(6) R. M. SANTILLI: *Hadronic J.*, **1**, 223 (1978) (see also *Phys. Rev. D*, **20**, 555 (1979)).

(7) R. M. SANTILLI: *Foundations of Theoretical Mechanics*, II: *Birkhoffian Generalization of Hamiltonian Mechanics* (New York, N. Y., 1982).

(8) R. M. SANTILLI: *Lie-admissible Approach to Hadronic Structure*, II: *Coverings of the Galilei and Einstein Relativities?* (Nonantum, Mass., 1982).

(9) R. M. SANTILLI: *Lie-isotopic lifting of Lie symmetries*, I: *General Considerations*, I.B.R. Preprint DE-83-2 (1983), submitted for publication.

(10) R. M. SANTILLI: *Lie-isotopic lifting of Lie symmetries*, II: *Lifting of Rotations*, I.B.R. Preprint DE-83-3 (1983), submitted for publication.

(11) R. M. SANTILLI: *Lett. Nuovo Cimento*, **37**, 545 (1983).

(12) M. GASPERINI: *Lie-isotopic lifting of gauge theories*, University of Bologna preprint, in *Hadronic J.* (in press).

(13) *Proceedings of the Second Workshop on Lie-admissible Formulations*, Volume *A*: *Review Papers*, *Hadronic J.*, **2**, 1252 (1979); Volume *B*: *Research papers*, *Hadronic J.*, **3**, 1 (1979); *Proceedings of the Third Workshop on Lie-admissible Formulations*, Volume *A*: *Mathematics*, *Hadronic J.*, **4**, 183 (1981); Volume *B*: *Theoretical Physics*, *Hadronic J.*, **4**, 608 (1981); Volume *C*: *Experimental Physics and Bibliography*, *Hadronic J.*, **4**, 1166 (1981); *Proceedings of the First International Conference on Nonpotential Interactions and Their Lie-admissible Treatment*, Vol. *A*: *Invited papers*, *Hadronic J.*, **5**, 245 (1982); Vol. *B*: *Invited papers*, *Hadronic J.*, **5**, 679 (1982); Vol. *C*: *Contributed papers*, *Hadronic J.*. **5**, 1194 (1982); and Vol. *D*: *Contributed papers*, *Hadronic J.*, **5**, 1627 (1982).

(14) M. L. TOMBER *et al.*: *Hadronic J.*, **3**, 507 (1979); **4**, 1318 (1981) and **4**, 1444 (1981).

(15) G. BENKART, J. M. OSBORN and D. J. BRITTEN: *Hadronic J.*, **4**, 497 (1981).

(16) H. C. MYUNG: *Hadronic J.*, **5**, 771 (1982).

(17) J. M. OSBORN: *Hadronic J.*, **5**, 904 (1982).

(18) A. A. SAGLE: *Hadronic J.*, **5**, 1546 (1982).

although expressed via the less trivial product $A*B = AgB$, where g is a fixed operator verifying certain topological restrictions. The lifting $\mathscr{E} \to \hat{\mathscr{E}}$ evidently implies a generalization of the virtual totality of Lie's theory, from the structure of the Lie product, to the structure of the Lie group, to the representation theory, etc. More recently, the Lie-isotopic theory has resulted to admit a consistent formulation on Hilbert spaces, by therefore confirming the hopes, at least on theoretical grounds, for a generalization of quantum mechanics called « hadronic mechanics » [19-34].

In this paper, we shall present a conceivable isotopic lifting of unitary linear and antilinear symmetries and of Wigner's theorem. We shall then indicate the plausibility of the extension as representing the extended and therefore deformable character of physical particles. Deviations from conventional, quantum-mechanical, symmetries will then result to be mere expressions of the insufficiency of pointlike approximations. To avoid excessive expectations, we would like to stress that a considerable amount of additional, mathematical, theoretical and experimental work is needed to reach a

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(19) The proposal to construct a generalization of quantum mechanics under the name of *hadronic mechanics* was submitted in memoir [20] immediately following that on the Lie-isotopic (and Lie-admissible) theory [6]. The same memoir presented the *Lie-isotropic generalization of Heisenberg's equations* (for the exterior, closed treatment of systems of extended particles), that is at the foundation of this work, as well as their *Lie-admissible generalization* (for the complementary interior treatment of open constituents). A systematic study of this dual, complementary, Lie-isotopic and Lie-admissible generalizations of quantum mechanics was then initiated at the *Yearly Workshops on Lie-admissible Formulations* [13] (see also ref. [14]). The literature in the field is nowadays considerable and expanding. We limit here to the indication of ref. [21] (on the generalization of Heisenberg's uncertainty principle for the interior of hadronic matter), ref. [22] (on the application of hadronic mechanics to the deformation of extended particles), ref. [23] (on the use of hadronic mechanics for the construction of a nonpotential generalization of the contemporary scattering theory), ref. [24] (on the generalization of the Fock space), *et al.* References [25,26] are particularly relevant for this paper inasmuch as they reached a Hilbert space formulation of both, the Lie-isotopic/exterior and the Lie-admissible/interior treatments. Reference [27] succeeded in reaching an operator formulation of the classical Nambu's mechanics. Intriguingly, the underlying envelope results to be of a particular isotopic type [28]. Thus, *Kalnay's quantization of Nambu's mechanics* is intimately related to hadronic mechanics (and not to ordinary quantum mechanics). A number of additional contributions are in press at the *Proceedings of the First Workshop on Hadronic Mechanics* [29]. It should be stressed that the Lie-isotopic lifting of unitary symmetries submitted in this paper is a particular case of an expected, more general, *Lie-admissible formulation of symmetries for the representation*, this time, *of time rate of variations of physical quantities*. This further generalization, which appears to be permitted by the Lie-admissible bimodule [30], would provide the notion of symmetry for all known dissipative generalization of Schrödinger's equations [31] owing to the universality of the Lie-admissible formulation of nonunitary time evolutions [32] as well as the direct proofs of their Lie-admissible character [33,34]

(20) R. M. SANTILLI: *Hadronic J.*, **1**, 574 (1978).

(21) R. M. SANTILLI: *Hadronic. J.*, **4**, 642 (1981).

(22) G. EDER: *Hadronic J.*, **4**, 2018 (1981).

(23) R. MIGNANI: *Hadronic J.*, **5**, 1120 (1982).

(24) A. JANNUSSIS, G. BRODIMAS, D. SOURLAS, A. STRECLAS, P. SIAFARICAS, L. PAPALOOUCAS and N. TSANGAS: *Hadronic J.*, **5**, 1923 (1982).

(25) H. C. MYUNG and R. M. SANTILLI: *Hadronic J.*, **5**, 1277 (1982).

(26) H. C. MYUNG and R. M. SANTILLI: *Hadronic J.*, **5**, 1377 (1982).

(27) A. KALNAY: *Hadronic J.*, **6**, 1 (1983).

(28) A. KALNAY and R. M. SANTILLI: in preparation for ref. [29].

(29) J. FRONTEAU *et al.*, Editors, *Proceedings of the First Workshop on Hadronic Mechanics*, *Hadronic J.*, Vol. **6** (1983), to appear.

(30) R. M. SANTILLI: *Hadronic J.*, **3**, 440 (1979).

(31) P. CALDIROLA: *Dissipation in quantum theory* (40 *years of research*, University of Milano preprint, invited paper for Proceedings [29]) (see also the contribution in the same Proceedings by R. BONIFACIO).

(32) R. M. SANTILLI: *Hadronic J.*, **2**, 1460 (1979), p. 1820.

(33) A. JANNUSSIS, G. BRODIMAS, V. PAPATHEOU and H. IOANNIDOU: *Lett. Nuovo Cimento*, **38**, 181 (1983) and invited paper in Proceedings [29].

(34) R. MIGNANI: *Lett. Nuovo Cimento*, **38**, 169 (1983).

final assessment of the possibilities offered by hadronic mechanics in generalizing quantum-mechanical symmetries. This paper, therefore, should be considered as a mere indication of these possibilities.

Let T be an operator that preserves all the essential topological properties of a positive-definite constant. For the rudimentary objectives of this work, it is therefore sufficient to assume hereon that T is a nonsingular, (conventionally) Hermitian and positive-definite operator. More rigorous mathematical treatments (*e.g.*, via essential self-adjointness in a common, invariant, dense, domain, and other means) are not needed for this work, and will be left as possible future refinements.

Following ref. ([25]), we shall now introduce the isotopic lifting $\hat{\mathscr{H}}$ of the Hilbert space $\mathscr{H}$ of quantum mechanics, with vectors $\psi, \phi, \ldots$, inner product

$$(1) \qquad (\phi \hat{|} \psi) = (\phi|T|\psi) = (\phi|T\psi) = (T\phi|\psi) \in \mathbf{C}$$

and normalization $(\phi \hat{|} \phi) = 1$, where all symbols without the upper hat denote the corresponding quantity (or operation) in $\mathscr{H}$. Note that, under the assumed restrictions on T, $\hat{\mathscr{H}}$ is still a Hilbert space. Note also the possibility that T is a positive-definite function.

The first implication of the lifting $\mathscr{H} \to \hat{\mathscr{H}}$ is that of generalizing the conventional transition probabilities according to the expression

$$(2) \qquad \hat{W}_{1\to 2} = |(\phi_1 \hat{|} \phi_2)|^2 = |(\phi_1|T|\phi_2)|^2 \in \boldsymbol{R}\,.$$

Next, we consider the enveloping associative algebra $\mathscr{E}$ of quantum-mechanical (Hermitian) operators A, B on $\mathbf{C}$ whose composition is given by the simplest possible associative product AB. Following ref. ([6]), we now subject $\mathscr{E}$ to an isotopic lifting $\hat{\mathscr{E}}$ characterized by a new unit, say, $\hat{I}$, with inverse $g(t, r, p, \ldots) \neq T(t, r, p, \ldots)$. For reasons to be elaborated shortly, we assume that the isotopic element g (or, equivalently, the unit $\hat{I}$) is nonsingular and Hermitean, but not necessarily positive-definite ([11]). This implies that the fundamental enveloping associative algebra of operators of hadronic mechanics is given by the isotope $\hat{\mathscr{E}}$ with (associative) product $A * B = AgB$; and (left and right) unit $\hat{I}$, $\hat{I} * A = A * \hat{I} = A$, for all $A \in \hat{\mathscr{E}}$. Jointly, we shall consider the lifting $\hat{\mathbf{C}}$ of the field $\mathbf{C}$ of complex numbers with composition $\hat{\mathbf{C}} * \hat{\mathscr{E}} = \mathbf{C}\hat{\mathscr{E}}$, *e.g.*, $\hat{c} * A = c\hat{I}gA = cA$. The corresponding lifting of the inner product $(\phi \hat{|} \psi) \to (\phi \hat{|} \psi)\hat{I}$ will be ignored because inessential for the closed exterior treatment of this paper ([25]) (with the understanding that it remains essential for the complementary open interior formulation ([26])).

Finally, and following ref. ([25]), we shall assume that the action of the algebra $\hat{\mathscr{E}}$ on the space $\hat{\mathscr{H}}$ is characterizeh by the modular isotopic form $A * \psi = Ag\psi$, *i.e.* by the isotopy of $\hat{\mathscr{E}}$ (and not that of $\hat{\mathscr{H}}$). This is due to reasons of consistency of the theory that need not be reviewed here. Note that we consider the lifting $\hat{\mathscr{E}}$ as well as $\hat{\mathscr{H}}$, while ref. ([25]) deals with the more particular case of an isotope $\hat{\mathscr{E}}$ acting on a conventional space $\mathscr{H}$.

The isotopic generalization of the various operations of a Hilbert space (transpose, Hermitian, unitary, antiunitary, trace, determinant, etc.) was studied in detail in ref. ([25]). Their re-formulation for our case of dual isotopy is straightforward. We shall, therefore, restrict ourselves to reviewing only the operations needed for our work.

The operation of *isotopic, linear, Hermitean, adjoint* is defined by

$$(3) \qquad (A * \phi \hat{|} \psi) = (\phi \hat{|} A^{\hat{\dagger}} * \psi)$$

and can be written (Theorem 2.3, ref. ([25]), p. 1293)

$$(4) \qquad (Ag\phi|T\psi) = (\phi|gA^{\dagger}T\psi) = (\phi|TA^{\hat{\dagger}}g\psi) ,$$

by therefore holding if and only if

$$(5) \qquad A^{\hat{\dagger}} = T^{-1}gA^{\dagger}Tg^{-1} .$$

Evidently, an operator A is isotopic, linear, Hermitian when $A = A^{\hat{\dagger}}$.

Similarly, the operation of *isotopic, antilinear, Hermitean, adjoint* is defined by

$$(6) \qquad (A * \varphi \hat{|} \psi) = \overline{(\phi \hat{|} A^{\hat{\dagger}} * \phi)} ,$$

where the upper bar denotes complex conjugation, and where the use of reduction (4) is implicit. The proof of the following property is then trivial.

Proposition 1. Under the condition that the isotopic element of the envelope coincides with that of the Hilbert space (by verifying all the restrictions of the latter, $g = T$) the isotonic linear and antilinear, Hermitean, adjoints coincide with the conventional adjoints, *e.g.*, $A^{\hat{\dagger}} = A^{\dagger}$. Under these conditions, physical quantities that are observable in quantum mechanics (*e.g.*, linear Hermitian), remain observable in hadronic mechanics (*e.g.*, isotpic linear Hermitian).

As a result of the above property, the case $g = T$ is evidently relevant. Nevertheless, the more general case $g \neq T$ should not be precluded, for instance, when the lifting of the Lorentz symmetry is desired ([11]) jointly with the condition that $\mathscr{H}$ be a Hilbert space. We shall, therefore, continue our analysis for the more general case $g \neq T$.

Next, an operator $\hat{U}$ is called *isotopic, linear, unitary*, when it verifies the condition

$$(7) \qquad (\hat{U} * \phi \hat{|} \hat{U} * \psi) = (\phi \hat{|} \psi) ,$$

which can be written

$$(8) \qquad (\hat{U}g\phi|T\hat{U}g\psi) = (\phi|g\hat{U}^{\dagger}T\hat{U}g\psi) = (\phi\, T\psi)$$

thus holding if and only if the following conditions are verified:

$$(9a) \qquad \hat{U} * (\hat{c} * \psi) = \hat{c} * (\hat{U} * \psi) = c(\hat{U}g\psi)$$

$$(9b) \qquad \hat{U}^{\hat{\dagger}} * \hat{U} = \hat{U} * \hat{U}^{\hat{\dagger}} = \hat{I} ,$$

where we have used eqs. (5).

Note that, even when the theory admits one, single, but nontrivia lisotopic element $g = T$, the notion of linear unitarity remains generalized.

Similarly, an operator $\hat{\mathscr{T}}$ is called *isotopic, antilinear, unitary* when it verifies the condition

$$(10) \qquad (\hat{\mathscr{T}} * \phi \hat{|} \hat{\mathscr{T}} * \psi) = \overline{(\phi \hat{|} \psi)} ,$$

which can hold if and only if the following conditions are verified:

$$(11a) \qquad \hat{U} * (\hat{c} * \psi) = \bar{\hat{c}} * (\hat{U} * \psi) = \bar{c}(\hat{U}g\psi) ,$$

$$(11b) \qquad \hat{U}^{\hat{\dagger}} * \hat{U} = \hat{U} * \hat{U}^{\hat{\dagger}} = \hat{I} .$$

The isotopic, linear, unitary operators provide the following fundamental realization of the Lie-isotopic theory (6) (see also Theorem 2.17, ref. (25), p. 1305)

$$\hat{U} = \hat{U}_g(\theta) = \exp\left[(i\theta X)\right]|_{\hat{\mathscr{E}}} = \hat{I} + \frac{i\theta X}{1!} + \frac{(i\theta X)g(i\theta X)}{2!} + \ldots = \tag{12}$$

$$= \left[\exp\left[(iXg\theta)|_{\mathscr{E}}\right]\right]\hat{I} = \hat{I}\left[\exp\left[(i\theta gX)|_{\mathscr{E}}\right]\right], \qquad X = X^{\hat{\dagger}} .$$

The operators $\hat{U}$, therefore, constitute a *continuous, Lie-isotopic, transformation group* in one parameter, $\hat{G}(1)$. For the extension to m parameters see the quoted literature. Note that the conventional unitary transformation groups of quantum mechanics are a trivial particular case of their isotonic coverings. (To avoid possible misrepresentations of the following analysis, a sufficient knowledge of the Lie-isotopic theory will be tacitly assumed hereon.)

The isotopic, antilinear, unitary operators provide a realization of the discrete part of the Lie-isotopic theory, *e.g.*,

$$\hat{\mathscr{T}} = \hat{U} * \hat{K}, \quad \hat{K} = K\hat{I}, \quad \hat{U}^{\hat{\dagger}} = \hat{U}^{-\hat{I}}, \tag{13}$$

where K is the conventional operator of complex conjugation. At this point, the reader interested in acquiring familiarity with the new techniques may work out the isotopic lifting of the conventional theory of antilinear unitary operators, *e.g*, prove that (isotopic) squares of isotopic, antilinear, unitary operators are (isotopic) linear unitary.

According to postulate II of ref. (25), states of hadronic mechanics are represented by *isotopic unit vectors* $\hat{\varphi}, \hat{\psi}, \ldots$ of $\hat{\mathscr{H}}$. The association, however, is not one-to-one, as in the conventional case. To by-pass this difficulty, we, therefore, introduce *isotopic unit rays* $\hat{\Psi} = \hat{c} * \hat{\psi} = c\hat{\psi}$, where $\hat{\psi}$ is a fixed isotopic unit vector and c varies in $\mathbf{C}$ under the condition $|c| = 1$. Evidently, $|(\hat{\Phi}\hat{|}\hat{\Psi})| = |(\hat{\varphi}\hat{|}\hat{\psi})|$ by construction. The association of isotopic unit rays to states of hadronic mechanics demands the lifting of superselection rules and the decomposition of $\hat{\mathscr{H}}$ into orthogonal, coherent, subspaces This latter lifting is not essential for this paper, and, for brevity, its study will not be conducted at this time. Nehertheless, we shall tacitly assume that the isotopic Hilbert spaces considered verify all the restrictions needed for the conventional formulation of Wigner's theorem (*e.g.*, coherence).

The isotopic lifting of the notion of symmetry of quantum mechanics, as defined, for instance, in ref. (3), can then be formulated as follows.

Definition 1. Let $\hat{\mathscr{H}}$ and $\hat{\mathscr{H}}'$ be two (not necessarily different) isotopic Hilbert spaces characterized by the same (nonsingular, Hermitean, and positive-definite) isotopic element T. An *isotopic transformation* is a one-to-one map $\hat{G}: \hat{\mathscr{H}} \to \hat{\mathscr{H}}'$ of unit rays $\hat{\Psi} \in \hat{\mathscr{H}}$ onto unit rays $\hat{\Psi}' \in \hat{\mathscr{H}}'$ characterized by the modular isotopic action $\hat{\Psi}' = \hat{G} * \hat{\Psi} = \hat{G}g\hat{\Psi}$ with g being a second (nonsingular, Hermitian, but not necessarily positive-definite) isotopic element generally different than T. The transformations so defined become *isotopic symmetries* when they verify the additional condition of preserving the hadronic probabilities, $|(\hat{\Phi}'\hat{|}\hat{\Psi}')| = |(\hat{\Phi}\hat{|}\hat{\Psi})|$.

The formulation of the *isotopic lifting of the inverse Wigner's theorem* is now trivial. In fact, all possible isotopic, linear or antilinear, unitary transformations on $\mathscr{H}$ are isotopic symmetries. The proof of the direct theorem, however, is not trivial in the conventional case, and a similar situation occurs under isotopy.

The proof, however, is greatly facilated by the following property.

Proposition 2. Within the context of the exterior/Lie treatment of closed systems, fields, Hilbert spaces, enveloping algebras of operators, unitary linear and antilinear transformations, the notion of symmetries and other quantities of quantum and hadronic mechanics admit a single, unified, realization-free formulation.

This is due to the fact that the isotopes $\hat{R}$ and $\hat{C}$ are fields, the isotopes $\hat{\mathscr{H}}$ are Hilbert spaces, and the isotopes $\hat{\mathscr{E}}$ are enveloping associative algebras. The differences with the conventional forms R, C, $\mathscr{H}$ and G are, therefore, merely a question of differences in realizations and numerical values, but not differences in the topological structure. In the final analysis, this is precisely the spirit intended for the construction of the (closed exterior branch of) hadronic mechanics via an isotopy of quantum mechanics [20].

Once the preservation of all the essential topological characteristics is understood, one can see the existence of a consistent, step-by-step, generalization of the proofs of Wigner's theorem, as given, for instance, in ref. [4]. It is merely given by the liftings $C \to \hat{C}$, $\mathscr{H} \to \hat{\mathscr{H}}$, $C \times \mathscr{H} \to \hat{C} * \hat{\mathscr{H}}$, $G \to \hat{G}$, and $G \times \mathscr{H} \to \hat{G} * \hat{\mathscr{H}}$ under the topological conditions specified above, and it will not be repeated here for brevity.

Theorem 1. Let $\hat{G}$ be an isotopic symmetry as per definition 1. Then, all possible onto mappings $\hat{G}: \hat{\mathscr{H}} \to \hat{\mathscr{H}}'$ with $\hat{\psi}' = \hat{G} * \hat{\psi} \in \hat{\mathscr{H}}'$ for each $\hat{\psi} \in \hat{\mathscr{H}}$, are either isotopic linear unitary or isotopic antilinear unitary transformations.

At the risk of being pedantic, it should be stressed that, in the final analysis, theorem 1 is not a generalization of Wigner's theorem, as far as the mathematical viewpoint is concerned (and for this reason we prefer the term lifting used in ref. [9,12]). In fact, it is a mere broader realization of a known mathematical property. Yet, its physical implications are far reaching, as we hope to illustrate below.

As well known, quantum mechanics represents particles as points. Such an approximation is evidently excellent for the electromagnetic interactions at large and most of the weak Nevertheless, the same approximation may become excessive when used for the description of the short-range interactions of hadrons, clearly, because they are extended charge distributions (with a charge radius of the order of $1F = 10^{-13}$ cm). Thus, hadrons are expected to exhibit deformations under sufficiently intense, short-range, external fields [20]. This implies the possibility that the charge distribution of hadrons can either be spherical, $r^t 1 r = x1x + y1y + z1z = 1$, or ellipsoidical

$$r^t g r = x b_1^2 x + y b_2^2 y + z b_3^2 z = 1 \qquad (g_{kk} = b_k^2 = \text{const}) . \tag{14}$$

In particular, the elliposidical character can hold either for closed-isolated systems, or as a deformation of the spherical shape due to external fields. In this paper, we shall consider only the former possibility (exterior/Lie-isotopic treatment), and this justifies the assumption of the constancy of the metric elements in eq. (14). Its diagonal character follows from the assumed restrictions on the metric.

The classical form invariance of ellipsoids has been achieved in ref. [9,11] In this paper, we would like to summarize our studies for its «hadronization», that is for its isotopic Hilbert-space formulation. Predictably, the results will be applications of theorem 1.

The physical foundations of the desired hadronization are provided by *Birkhoffian mechanics* [7], which is precisely an isotopic lifting of the (classical) Hamiltonian mechanics (and whose knowledge is assumed hereon). Consider a classical, extended, spherical, particle with charge distribution of radius 1 and mass 1. Suppose that the particle moves freely in vacuum (empty space). Then, its rotational symmetry is given by the conventional SO_3 transformations in Euclidean space E_3; the unit is the conventional quantity $I = \delta^{-1}$, where $\delta = (\delta_{ij})$ is the Euclidean metric; and the system can be described by the free Hamiltonian $H_{\text{free}} = \frac{1}{2} p^2$ with canonical action

$$A = \int_{t_1}^{t_2} (p\, \mathrm{d}r - H_{\text{free}}\, \mathrm{d}t)\,, \qquad r \in E_3\,.$$

Suppose now that the particle, even though free, has an ellipsoidical charge distribution because of internal effects. One of its simplest possible descriptions is provided by the Pfaffian-Birkhoffian action

$$A^{\text{gen}} = \int_{t_1}^{t_2} P(p)\, \mathrm{d}r - B(p)\, \mathrm{d}t] = \int_{t_1}^{t_2} [pg\, \mathrm{d}r - \tfrac{1}{2}(pgpg)\, \mathrm{d}t] = \int_{t_1}^{t_2} [p * \mathrm{d}r - \tfrac{1}{2}(p * p) * \mathrm{d}t]\,,$$

where g is precisely the metric of ellipsoids (14), and $B = \frac{1}{2} pgpg$ is the Birkhoffian. Note that the transformation $(r, p) \to (r, P(p))$ is generally noncanonical. The underlying dynamical equations are therefore Birkhoff's and not Hamilton's equations. This implies the following Lie-isotopic generalization of the Hamilton-Jacobi equations (ref. (7), eq. (6.1.23), p. 207)

$$\frac{\partial A^{\text{gen}}}{\partial t} + \frac{1}{2}(pgp)\, g = 0\,, \qquad pg = \frac{\partial A^{\text{gen}}}{\partial r}\,, \qquad \frac{\partial A^{\text{gen}}}{\partial p} = 0\,. \tag{15}$$

The naive quantization applied to the operator $\hat{P}$, then yields the following isotopic generalization of Schrödinger's equations:

$$i\, \frac{\partial \hat{\varphi}}{\partial t} = \hat{H} g \hat{\varphi} = \hat{H} * \hat{\varphi} = \frac{1}{2}(\nabla\nabla)\, \hat{\varphi}\,, \qquad \hat{H} = \frac{1}{2} \hat{p} g \hat{p}\,, \tag{16a}$$

$$\hat{p} g \hat{\varphi} = \hat{p} * \hat{\varphi} = \frac{1}{i} \nabla \hat{\varphi}\,. \tag{16b}$$

Evidently, the metric g is positive-definite and, therefore, it can be the isotopic element not only of a Lie group, but also of a Hilbert space. We, therefore, assume that the isotopic elements of $\hat{\mathscr{E}}$ and of $\hat{\mathscr{H}}$ are identical, $g = T$. This assumption implies the preservation of the Hermiticity of the $\hat{r}$ and $\hat{p}$ operator, as well as of their polynomial forms when properly symmetrized in $\hat{\mathscr{E}}$. Proposition 2 applies, and quantities that are observable in the ordinary case remain observable under lifting.

The canonical commutation rules are now subjected to the isotopy

$$\begin{aligned}([\hat{a}^\mu \,\hat{,}\, \hat{a}^\nu]) * \hat{\varphi} &= (\hat{a}^\mu * \hat{a}^\nu - \hat{a}^\nu * \hat{a}^\mu) * \hat{\varphi} = i(\hat{\omega}^{\mu\nu}) * \hat{\varphi} = \\ &= i(\omega^{\mu\nu})\,\hat{\varphi} = i \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \hat{\varphi}\,, \qquad \hat{a} = (\hat{r}, \hat{p})\,, \quad \mu, \nu = 1, 2, \ldots, 6\,,\end{aligned} \tag{17}$$

which preserves the conventional values (expressed by the fundamental, Lie, cosymplectic structure $\omega^{\mu\nu}$), even though the Lie product is generalized.

The preservation of the values $\omega^{\mu\nu}$ implies the validity of the conventional uncertainty. In fact, the application of the generalized uncertainty principle proposed in ref. (21)

to the case at hand yields the rules

$$(18)\qquad \Delta r\,\Delta p \geqslant \left|(\hat\phi|*(\hat r * \hat p)*|\hat\phi)\right| \geqslant \tfrac{1}{2}\left|(\hat\phi|*[\hat r \hat{,} \hat p]*|\hat\phi)\right| = \tfrac{1}{2}\left|(\hat\phi|*|\hat\phi)\right| = \tfrac{1}{2}\qquad (\hbar = 1)\,.$$

Since we are considering the exterior, Lie-isotopic treatment, the preservation of conventional Hermiticity, commutation rules, and uncertainty is fully in line with hadronic mechanics (where possible deviations are essentially admitted for the interior of hadronic matter, as indicated below).

Once an explicit realization of the $\hat r$- and $\hat p$-operators has been achieved, the operator formulation of the isotopic theory of rotations becomes consequential. Recall that quantities of the type $r_j p_k$ in $\mathscr{E}$ have no meaning in hadronic hechanics (*e.g.*, because they do not act linearly on the Hilbert space), and must be properly formulated in $\hat{\mathscr{E}}$, *e.g.*, $r_j g p_k$. The hadronic expression of the angular-momentum operator is then given by the (isotopic Hermitean) form $\hat J_i = \varepsilon_{ijk}\hat r_j g \hat p_k\, p = \varepsilon_{ijk} r_j * p_k$. The following isotopic commutation rules then trivially follow from rules (17):

$$(19)\qquad \widehat{SO}_3 : [\hat J_i \hat{,}\ \hat J_j] = \hat J_i g \hat J_j - \hat J_j g \hat J_i = i\varepsilon_{ijk}\hat J_k$$

and they coincide with the conventional ones at the level of abstract, realization-free formulations as desired. Thus, for the assumed metric, the operator realization of the isotope $\widehat{SO}_3$ is locally isomorphic to the conventional form SO_3 in exactly the same way as it occurs for the classical case [10,11].

The matrix representation for the regular, (3×3) case identified in ref. [10] then holds:

$$(20)\qquad \hat J_1 = i b_2^{-1} b_3^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix},\qquad \hat J_2 = i b_1^{-1} b_3^{-1}\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix},$$
$$\hat J_3 = i b_1^{-1} b_2^{-1}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The exponentiation to finite transformations is also possible under the assumed conditions, resulting in the Lie-isotopic group of rotations [10]

$$(21)\qquad \widehat{SO}_3 : \hat R = \hat R_g(\theta) = \exp[i\hat J_1\theta_1]|_{\hat{\mathscr{E}}}\, g \exp[i\hat J_2\theta_2]|_{\hat{\mathscr{E}}}\, g \exp[i\hat J_3\theta_3]|_{\hat{\mathscr{E}}} =$$
$$= \left(\prod_{k=1}^{3}\exp[i\hat J_k g\theta_k]|_{\mathscr{E}}\right)\hat I = S_g(\theta)\,\hat I\,.$$

An example of isotopic rotations around the third axis is given by [10]

$$(22)\qquad r' = \hat R_g(\theta_3) * r = S_g(\theta_3) r = \begin{pmatrix}\cos\theta_3 & (b_2/b_1)\sin\theta_3 & 0\\ -(b_1/b_2)\sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}.$$

This confirms the achievement of the desired operator formulation of the form-invariance of all possible ellipsoidical deformations of spherical charge distributions, as the reader is encouraged to verify via the applications of the isotopic rotations to composition (14).

The second-order isotopic Casimir invariant is evidently given by $\hat{J}^{\hat{2}} = \sum_{k=1}^{3} \hat{J}_k g \hat{J}_k = -2\hat{I}$. Also, $\hat{J}^{\hat{2}}$ and $\hat{J}_3$ constitute a set of maximal, isotopically commuting and Hermitian operators and, as such, they can be diagonalized simultaneously. The repetition of the standard procedures for the identification of the eigenvalues, then yields the lifting

$$\hat{J}^{\hat{2}} * \varphi = [\hat{j} * (\hat{j} + \hat{I})] * \hat{\varphi} = j(j+1)\hat{\varphi}\,, \qquad j = 0, \tfrac{1}{2}, 1, \ldots, \tag{23a}$$

$$\hat{J}_3 * \hat{\varphi} = \hat{m} * \hat{\varphi} = m\hat{\varphi}\,, \qquad m = j, j-1, \ldots, -j\,, \tag{23b}$$

where the half-odd-integer values have been included mainly for completeness (since we are *not* treating the lifting of the SU_2 spin group).

We recover in this way a result first achieved in ref. [22], that the conventional value of the magnitude and of the third component of angular momentum can be the conventional ones; yet the conventional rotational symmetry is broken.

The existence of a consistent lifting of the remaining parts of the quantum-mechanical theory of rotations is then expected, such as: the construction of the isotopic form of spherical functions, the diagonalization of basis (20) via an appropriate isotopic unitary transformation; the composition of isotopic rotations; etc. Except a few additional remarks presented in fig. 1, these aspects will not be considered at this time.

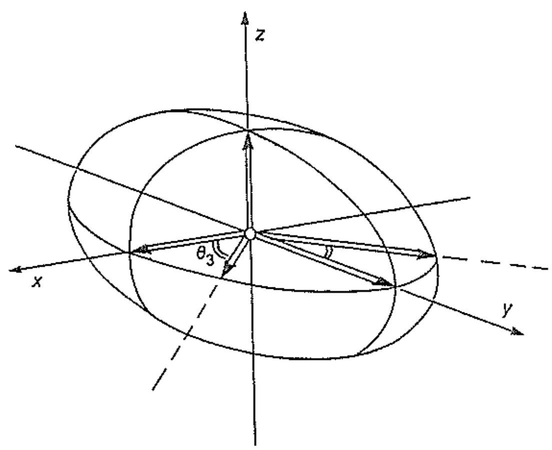

Fig. 1. – The figure presents a schematic view of the «hadronization» of angular momentum under the condition of preserving conventional magnitudes and third components. One can see the transition from the quantum-mechanical sphere $xx + yy + zz = [j(j+1)]^{\frac{1}{2}}$ to the hadronic ellipsoids $xb_1^2x + yb_2^2y + zb_3^2z = [j(j+1)]^{\frac{1}{2}}$, while the quantization of the third component is conventional. The conical sections for each projection m are, of course, elliptical. Note that the magnitude $\hat{J}^2$ is left invariant, while an isotopic rotation around the third axis (as depicted in the figure) is given by $\hat{J}_1' = \hat{J}_1 \cos\theta_3 + (b_2/b_1)\hat{J}_2 \sin\theta_3$, for which the axes are kept fixed as in the conventional case. If the unit vectors $\hat{\varphi}$ are rotated by an angle $-\theta_3$, then $\hat{J}_1' = \exp[-i\hat{J}_3 g\theta_3] J_1 \exp[i\theta_3 g\hat{J}_3] = \hat{J}_1\cos\theta_3 - \hat{J}_2 \sin\theta_3$, *i.e.* the transformation is conventional, trivially, because of the preservation of the commutation rules, eq. (19). Despite the assumed, simplest possible realization of the isotopy, the model indicates that, when the conventional symmetry under rotations is broken due to ellipsoidical deformations, the exact rotational symmetry can be regained via a suitable isotopic lifting. The physical implications are intriguing and deserving further study. For instance, it is conceivable that, for the ellipsoidical shape of this figure, there exist *different quanta of energy for different space orientations*, $\hat{\hbar} = \hbar b_k^{-2}$ (see text). This, and other aspects, will be investigated in separate papers in more detail.

It should be stressed that eigenvalues (23) are crucially dependent on the positive-definite character of the metric (which permits upper and lower bounds in the isotopic expectation values), as well as of the conventional value of the commutation rules (17) (which preserve the scale of the spectrum (23)).

A few concluding remarks are now in order. Recall that we have essentially studied an operator description of a free, nonspherical, charge distribution. Such a system is essentially *nonpointlike*, and, as such, *outside the technical possibilities of quantum mechanics in its current formulation.* At any rate, the conventional rotational symmetry is evidently and manifestly broken for the system considered.

In this note, we have showed that the description of the free, ellipsoidical, charge distribution can be achieved by preserving the abstract, realization-free structure of quantum mechanics, and by assuming instead a realization more general than the simplest possible one of current use. This note therefore confirms the hopes of hadronic mechanics achieving a description of extended and therefore deformed or deformable particles.

Most importantly, we have achieved a description of the ellipsoidical charge distribution in such a way that its center-of-mass motion is conformed to the available quantum-mechanical, experimental evidence. This is ensured by the validity of the conventional uncertainty, eq. (18), by the preservation of conventional values of angular momentum, eq. (23), and other properties. An aspect focused in this note is, therefore, the possibility that a number of hadrons currently believed to be perfectly spherical and possessing the conventional rotational symmetry SO_3, are in actuality ellipsoidical and verify the more general symmetry $\widehat{SO}_3$.

The study of this possibility evidently demands the calculation of the magnetic moments associated to ellipsoidical charge distributions. Predictably, these moments will be anomalous, by therefore confirming *the possibility that (at least some of the known) hadrons are not spherical.* As a matter of fact, it may well be that the established anomalous value of the magnetic moments of hadrons (and of nuclei) turns out to be a rather forceful argument favoring the broken character of the conventional rotational symmetry, and the validity of more general formulations.

Furthermore, the reader can readily see that the isotopic element T of the Hilbert space $\hat{\mathscr{H}}$ is a form of generalization of the inverse of Planck constant $\hbar$. In fact, by imposing $g = T = \hbar^{-1}$, hadronic mechanics recovers the conventional one identically. Without any claim to originality, the next issue raised by this paper is the possibility that *the exchanges of energy of ellipsoidical charge distributions are non-uniform in space, according to the distribution* $\hbar g_{kk}^{-1} = \hbar b_k^{-2}$.

This somewhat suggestive hypothesis implies the possibility that, in case, say, protons are not perfectly spherical, they have *different quantum of energies* $\hat{\hbar}_k = \hbar b_k^{-2}$ depending on the orientation in space of the emission-absorption.

The resolution of this hypothesis evidently demands specific, theoretical and experimental work. At this point we can only note that the isotope $\hat{\mathscr{H}}$ has been introduced following the lifting $\mathscr{E} \to \hat{\mathscr{E}}$ for the primary purpose of preserving the conventional observability (proposition 1). In fact, the symetry $\widehat{SO}_3$ can also be formulated in a conventional Hilbert space, in line with ref. [25]. The arguments supporting the lifting $\mathscr{H} \to \hat{\mathscr{H}}$ must, therefore, be of experimental nature, and cannot be of pure mathematical character.

At any rate, if future experimental studies will confirm the need for an isotopic lifting of the Hilbert space for extended, deformed-deformable particles, its isotopic element need not be identical with that of the envelope. This aspect will emerge quite naturally from the hadronization of the Lorentz-isotopic transformations of the preceding letter [11] to be presented elsewhere.

In essence, our studies indicate that, when conventional symmetries are broken because of the extended character of hadrons, their isotopic formulation may still be exact. Equivalently, we can say that space-time symmetries may be exact in nature, provided that they are realized in their structurally most general possible form, rather than in the simplest possible realization of current use in physics (*e.g.*, canonical or unitary). Contrary to impressions that may have been perceived by a number of researchers, hadronic mechanics is being constructed to preserve space-time symmetries, rather than breaking them, but, jointly, to relax excessively simplistic (often tacit) assumptions in their algebraic/group-theoretical structure.

Most importantly, the reader should keep in mind that the compatibility of the model with current experimental data on center-of-mass dynamics is crucially dependent on the constancy of the ellipsoidical shape. In fact, if such shape assumes a dependence on the local co-ordinates r, the value of the isotopic commutation rules is given by a general (noncanonical) Lie-cosymplectic structure $\hat{\Omega}^{\mu\nu}(t, \hat{a})$. In this case, we may have deviations from the conventional uncertainty. This latter aspect is not necessarily a drawback of the theory. In fact, under the conditions considered, the system becomes open. We are then referring to a novel physical arena for which no valuable experimental information on uncertainty exists at this time. As a matter of fact, one of the central aspects of hadronic mechanics is the indication of the possibility that, even though the center-of-mass of a closed hadronic system (say, a hadron or a nucleus) obeys conventional quantum-mechanical laws, the dynamics of its open, interior constituents can be fundamentally generalized.

The compliance with the conventional uncertainty of the free, deformed, particles considered in this note is, therefore, compatible with possible deviations for its open constituents.

To put it explicitly, the current experimental evidence favoring conventional uncertainties for the center-of-mass motion, say, of a proton under long-range, electromagnetic interactions (*e.g.*, when the proton moves in a particle accelerator) appears to be compatible with the generalized uncertainty pioneered by CALDIROLA and his associatives since 1939 ([31]) (and more recently formalized in the generalized principle of ref. ([21]), under the sole condition that the underlying, open, generalizations of Schrödinger's equations merely refer to the constituents of the proton, or, more generally, of closed systems of extended particles with contact/non-Hamiltonian internal forces ([7]).

These latter issues can be resolved experimentally, *e.g.*, via neutron interferometry. For instance, if short-range forces deform the charge distributions of nucleons, departures from their magnetic moments are expected to be consequential, with a deviation from the spin precession predicted by the quantum-mechanical descriptions for point-like approximations. After all, we should not forget that the analysis of ref. ([22]) has predicted about 1% deviation from the conventional rotational symmetry for neutrons within the intense fields of nuclei, and that the measures of experiments ([35]), even though preliminary, confirm such a value.

If these measures of rotational-asymmetry are confirmed by future tests, they will likely establish Caldirola's treatment of uncertainties for open, interior, hadronic, dynamics. This can be seen by the aspect indicated earlier that, for open conditions due to intense external fields, a dependence of deformations (14) on local variables implies generalized values of the fundamental (isotopic) commutation rules, and, inevitably, of Heisenberg's uncertainty (although still in a way possibly compatible with conventional uncertainties for the closed implementation of the system, when inclusive of the external source).

([35]) H. RAUCH: *Hadronic J.*, 5, 729 (1982).

A corresponding situation is expected to occur for the time reversal symmetry. In fact, the time symmetry of the center-of-mass trajectory of the deformed, but isolated system considered in this paper is expected to be compatible with the irreversibility of its open constituents, along the lines presented in the preceding note on hadronic mechanics (36). More detailed studies on this [rather intriguing and fundamental] problem will be presented elsewhere.

As a concluding note, we would like to recall that generalizations of the inner product of type (1) have been known since considerable time (37) (although for the case of non-Hermitean T). The novel aspects of the Birkhoffian and of the hadronic mechanics are therefore those related to the isotopic lifting of space-time (and other) symmetries, that is to the lifting of the envelope $\mathscr{E} \to \hat{\mathscr{E}}$. In this respect, it is amusing to recall that, as BRUCK put it in his memoir of 1958 (loc. cit (5), p. 56), these degrees of freedom « are so natural to creep in unnoticed ». Bruck's words turned out to be prophetic. In fact, the isotopic degrees of freedom of enveloping algebras and, consequently, of classical and operational space-time symmetries, have remained unnoticed in the virtual entirety of the theoretical physics literature of this century, even though their implications are manifestly deep.

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