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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory
II. Direct Analytic Representations of Tensorial Field Equations*

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By using a characterization of the concept of analytic representation and a variational approach to self-adjointness introduced in a preceding paper, we prove a theorem, according to which a necessary and sufficient condition for a class $\mathscr{C}^2$, regular, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the Lagrange equations in a region R of its variables is that the system is self-adjoint in R. We point out as a first corollary that if the ordering requirement is removed from the definition of analytic representation, then the condition of self-adjointness of the field equations is only sufficient for the existence of a Lagrangian density. We then provide as a second corollary a methodology for the computation of the Lagrangian density for the representation of self-adjoint quasi-linear tensorial field equations. This methodology is also particularized for ordinary semilinear systems of tensorial field equations through a third corollary. The above results are interpreted from the viewpoint of interactions. We first recover, through a fourth corollary, the conventional structure of the total Lagrangian density $\mathscr{L}_{\text{Tot}} = \sum_{1}^{n}{}_{a} \mathscr{L}^{(a)}_{\text{Free}} + \mathscr{L}_{\text{Int}}$ for the semilinear form of the field equations, and then introduce through a fifth corollary a generalized structure of the type $\mathscr{L}_{\text{Tot}} = \sum_{1}^{n}{}_{a} \mathscr{L}^{(a)}_{\text{Int,I}} \mathscr{L}^{(a)}_{\text{Free}} + \mathscr{L}_{\text{Int,II}}$ for the representations of the field equations in the quasi-linear form. Therefore, our analysis seems to indicate that a general form of representing interacting fields is characterized by $(n+1)$-interaction terms in the Lagrangian: n multiplicative terms and one additive term to the Lagrangian for the free fields.

1. INTRODUCTION

In a preceding paper [1], we have studied class $\mathscr{C}^2$, regular, Lorentz-covariant, tensorial field equations in (a) the *nonlinear form*:

$$F_{a_1}(x_\alpha, \phi^a, \phi^{a;}{}_{\alpha}, \phi^{a;}{}_{\alpha\beta}) = 0,$$
$$a_1, a = 1, 2, \ldots, n, \qquad \alpha, \beta = 0, 1, 2, 3,$$
$$\phi^{a;}{}_{\alpha} \equiv \frac{\partial \phi^a}{\partial x^\alpha}, \qquad \phi^{a;}{}_{\alpha\beta} \equiv \frac{\partial^2 \phi^a}{\partial x^\alpha \, \partial x^\beta}; \tag{1.1}$$

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(b) the *quasi-linear form*:

$$A^{\mu\nu}_{a_1 a_2}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)\, \phi^{a_2;}{}_{\mu\nu} + B_{a_1}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha) = 0,$$
$$a_1, a_2, a = 1, 2, \ldots, n, \qquad \alpha, \mu, \nu = 0, 1, 2, 3; \tag{1.2}$$

(c) the *semilinear form*:

$$g^{\mu\nu}\phi_{a_1}{}^{;}{}_{\mu\nu} - \rho_{a_1 a_2}^{\mu}(x_\alpha, \phi^a)\, \phi^{a_2;}{}_\mu - \sigma_{a_1}(x_\alpha, \phi^a) = 0,$$
$$(g^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} = \Box\phi_a, \tag{1.3}$$
$$a_1, a_2, a = 1, 2, \ldots, n, \qquad \mu, \nu, \alpha = 0, 1, 2, 3;$$

and we have identified the necessary and sufficient conditions for the above forms to be *self-adjoint* [1, Theorems 6.1, 6.2, 6.3], namely, to be such that their systems of equations of variation coincide with the adjoint systems for all admissible variations.

Such necessary and sufficient conditions result in being certain systems of quasi-linear overdetermined [2] systems of partial differential equations, which we have termed *conditions of self-adjointness*, and which are given

(a) for the *nonlinear form*, by

$$\begin{cases} F_{a_1}{}^{;\mu\nu}{}_{a_2} = F_{a_2;a_2}^{;\mu\nu} = F_{a_1}{}^{;\nu\mu}{}_{a_2}, & (1.4a) \\ F_{a_1}{}^{;\mu}{}_{a_2} + F_{a_2}{}^{;\mu}{}_{a_2} = 2d_\nu F_{a_2}{}^{;\mu\nu}{}_{a_1}, & (1.4b) \\ F_{a_1}{}^{;}{}_{a_2} - F_{a_2}{}^{;}{}_{a_1} = \tfrac{1}{2}d_\mu(F_{a_1}{}^{;\mu}{}_{a_2} - F_{a_2}{}^{;\mu}{}_{a_1}), & (1.4c) \end{cases}$$
$$a_1, a_2 = 1, 2, \ldots, n, \qquad \mu, \nu = 0, 1, 2, 3;$$

$$F_{a_1}{}^{;\mu\nu}{}_{a_2} \equiv (\partial F_{a_1}/\partial\phi^{a_2;}{}_{\mu\nu}), \tag{1.4d}$$

$$F_{a_1}{}^{;\mu}{}_{a_2} \equiv (\partial F_{a_1}/\partial\phi^{a_2;}{}_\mu), \tag{1.4e}$$

$$F_{a_1}{}^{;}{}_{a_2} \equiv (\partial F_{a_1}/\partial\phi^{a_2}), \tag{1.4f}$$

$$d_\mu \equiv \partial_\mu + \phi^{a;}{}_\mu(\partial/\partial\phi^a) + \phi^{a;}{}_{\mu\alpha}(\partial/\partial\phi^{a;}{}_\alpha) + \phi^{a;}{}_{\mu\alpha\beta}(\partial/\partial\phi^{a;}{}_{\alpha\beta}); \tag{1.4g}$$

(b) for the *quasi-linear form*, by

$$\left\{\begin{aligned} &A^{\mu\ \nu}_{a_1a_2} = A^{\nu\ \mu}_{a_2a_1} = A^{\nu\ \mu}_{a_1a_2}, && (1.5a)\\ &A^{\nu\ \alpha\ ;\mu}_{a_1a_3\ a_2} + A^{\nu\ \alpha\ ;\mu}_{a_2a_3\ a_1} = A^{\mu\ \overline{\nu\ ;\alpha}}_{a_2a_1\ a_3}, && (1.5b)\\ &A^{\overline{\alpha\ \beta}\ \overline{;\mu\ ;\nu}}_{a_1a_4\ \underline{a_2\ a_3}} = A^{\overline{\alpha\ \beta}\ \overline{;\mu\ ;\nu}}_{a_2a_4\ \underline{a_1\ a_3}}, && (1.5c)\\ &B_{a_1}{}^{;\mu}{}_{a_2} + B_{a_2}{}^{;\mu}{}_{a_1} = 2\{\partial_\nu + \phi^{a_3;}{}_\nu(\partial/\partial\phi^{a_3})\}\, A^{\mu\ \nu}_{a_1a_2}, && (1.5d)\\ &B_{a_1}{}^{:}{}_{a_2} - B_{a_2}{}^{:}{}_{a_1} = \tfrac{1}{2}\{\partial_\mu + \phi^{a_3;}{}_\mu(\partial/\partial\phi^{a_3})\}(B_{a_1}{}^{;\mu}{}_{a_2} - B_{a_2}{}^{;\mu}{}_{a_1}), && (1.5e)\end{aligned}\right.$$

$$a_1, a_2, a_3, a_4 = 1, 2,\ldots, n, \qquad \mu, \nu = 0, 1, 2, 3;$$

$$A^{\mu\ \nu\ ;\alpha}_{a_1a_2\ a_3} \equiv \frac{\partial A^{\mu\ \nu}_{a_1a_2}}{\partial\phi^{a_3;}{}_\alpha}, \qquad A^{\mu\ \nu\ ;\alpha\ ;\beta}_{a_1a_2\ a_3\ a_4} = \frac{\partial^2 A^{\mu\ \nu}_{a_1a_2}}{\partial\phi^{a_4;}{}_\beta\,\partial\phi^{a_3;}{}_\alpha}, \tag{1.5f}$$

$$B_{a_1}{}^{;\mu}{}_{a_2} \equiv \frac{\partial B_{a_1}}{\partial\phi^{a_2;}{}_\mu}, \qquad B_{a_1}{}^{:}{}_{a_2} \equiv \frac{\partial B_{a_1}}{\partial\phi^{a_2}}, \tag{1.5g}$$

where the horizontal bar denotes symmetrization of the indicated indices, e.g.,

$$A^{\mu\ \overline{\nu\ ;\alpha}}_{a_1a_2\ a_3} \equiv A^{\mu\ \nu\ ;\alpha}_{a_1a_2\ a_3} + A^{\mu\ \alpha\ ;\nu}_{a_1a_2\ a_3}, \tag{1.6a}$$

$$A^{\mu\ \nu\ ;\alpha\ ;\beta}_{a_1\underline{a_2\ a_3\ a_4}} \equiv A^{\mu\ \nu\ ;\alpha\ ;\beta}_{a_1a_2\ a_3\ a_4} + B^{\mu\ \nu\ ;\alpha\ ;\beta}_{a_1a_4\ a_3\ a_2}; \tag{1.6b}$$

(c) for the *semilinear form*, by

$$\left\{\begin{aligned} &\rho^{\mu}_{a_1a_2} + \rho^{\mu}_{a_2a_1} = 0, && (1.7a)\\ &\rho^{\mu}_{a_1a_2}{}^{:}{}_{a_3} + \rho^{\mu}_{a_3a_1}{}^{:}{}_{a_2} + \rho^{\mu}_{a_2a_3}{}^{:}{}_{a_1} = 0, && (1.7b)\\ &\partial_\mu\rho^{\mu}_{a_1a_2} = \sigma_{a_1}{}^{:}{}_{a_2} - \sigma_{a_2}{}^{:}{}_{a_1}, && (1.7c)\end{aligned}\right.$$

$$a_1, a_2 = 1, 2,\ldots, n, \qquad \mu = 0, 1, 2, 3;$$

$$\rho^{\mu}_{a_1a_2}{}^{:}{}_{a_3} \equiv (\partial\rho^{\mu}_{a_1a_2}/\partial\phi^{a_3}) \tag{1.7d}$$

$$\sigma_{a_1}{}^{:}{}_{a_2} \equiv (\partial\sigma_1/\partial\phi^{a_2}) \tag{1.7e}$$

In [1] we studied the conventional *Lagrange equations* for classical field theories

$$\begin{aligned}\mathscr{L}_{a_1}(\phi) &= d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a_1;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^{a_1}} \\ &= \frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_\mu \, \partial \phi^{a_2;}{}_\nu} \phi^{a_2;}{}_{\mu\nu} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_\mu \, \partial \phi^{a_2}} \phi^{a_2;}{}_\mu + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_\mu \, \partial x^\mu} - \frac{\partial \mathscr{L}}{\partial \phi^{a_1}} \qquad (1.8a) \\ &= \tfrac{1}{2}(\mathscr{L}^{;\mu\,;\nu}_{a_1\,a_2} + \mathscr{L}^{\nu\,;\mu}_{a_1\,a_2})\, \phi^{a_2;}{}_{\mu\nu} + \mathscr{L}^{;\mu\,;}_{a_1\,a_2} \phi^{a_2;}{}_\mu + \mathscr{L}^{;\mu\,;}_{a_1\,\mu} - \mathscr{L}^{;}{}_{a_1} = 0,\end{aligned}$$

$$\mathscr{L}^{;}{}_\mu = \frac{\partial \mathscr{L}}{\partial x^\mu}, \qquad \mathscr{L}^{;}{}_a = \frac{\partial \mathscr{L}}{\partial \phi^a}, \qquad \mathscr{L}^{;\mu}_a = \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_\mu}, \qquad \text{etc.}, \qquad (1.8b)$$

$$a_1, a_2 = 1, 2, ..., n, \qquad \mu, \nu = 0, 1, 2, 3$$

in a *Lagrangian density* $\mathscr{L} = \mathscr{L}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ and proved that, for class $\mathscr{C}^4$ and regular Lagrangians, they are always self-adjoint [1, Theorem 7.1].

Since the Lagrange *operator* $\{d_\mu \partial/\partial\phi^{a;}{}_\mu - \partial/\partial\phi^a\}$ is self-adjoint in the conventional sense used in the theory of linear operators [3], [1]'s analysis essentially provides a variational approach to self-adjointness.

Under the assumed continuity and regularity conditions, the Lagrange equations can then be written in terms of the symbolic notation

$$[\mathscr{L}_{a_1}(\phi)]^{\mathscr{C}^2, R}_{\mathrm{SA}} = 0. \qquad (1.9)$$

Equations (1.1), (1.2), and (1.3) are termed *self-adjoint* when *all* the corresponding conditions of self-adjointness, (1.4), (1.5), and (1.7), are satisfied. Then we shall symbolically write

$$\begin{aligned}[F_{a_1}]^{\mathscr{C}^2, R}_{\mathrm{SA}} &= 0, \\ [A^{\mu\,\nu}_{a_1 a_2} \phi^{a_2;}{}_{\mu\nu} + B_{a_1}]^{\mathscr{C}^2, R}_{\mathrm{SA}} &= 0, \qquad (1.10) \\ [g^{\mu\nu} \phi_{a_1}{}^{;}{}_{\mu\nu} - \rho^\mu_{a_1 a_2} \phi^{a_2;}{}_\mu - \sigma_{a_1}]^{\mathscr{C}^1}_{\mathrm{SA}} &= 0,\end{aligned}$$

where we have reduced the continuity assumptions of Eq. (1.3) because conditions $\rho^\mu_{a_1 a_2}, \sigma_{a_1} \in \mathscr{C}^2$ are, in this case, redundant.

Equations (1.1), (1.2), and (1.3) are termed *non-self-adjoint* when at least *one* of the corresponding conditions of self-adjointness (1.4), (1.5), and (1.7) is violated. We shall symbolically write, in this case,

$$\begin{aligned}[F_{a_1}]^{\mathscr{C}^2, R}_{\mathrm{NSA}} &= 0, \\ [A^{\mu\,\nu}_{a_1 a_2} \phi^{a_2;}{}_{\mu\nu} + B_{a_1}]^{\mathscr{C}^2, R}_{\mathrm{NSA}} &= 0, \qquad (1.11) \\ [g^{\mu\nu} \phi_{a_1}{}^{;}{}_{\mu\nu} - \rho^\mu_{a_1 a_2} \phi^{a_2;}{}_\mu - \sigma_{a_1}]^{\mathscr{C}^1}_{\mathrm{NSA}} &= 0.\end{aligned}$$

In [2] we introduced a definition of analytic representation of a class $\mathscr{C}^2$, regular, covariant, tensorial system of field equations $F_{a_1}(\phi) = 0$ in terms of the Lagrange equations $\mathscr{L}_{a_1}(\phi) = 0$, which occurs when there exists a class $\mathscr{C}^2$ regular $n \times n$ matrix (h) with elements $h_{a_1}{}^{a_2} = h_{a_1}{}^{a_2}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ such that

$$[\mathscr{L}_{a_1}(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R} \equiv \{h_{a_1}{}^{a_2}[F_{a_2}(\phi)]^{\mathscr{C}^2,R}\}^{\mathscr{C}^2,R} = 0, \qquad a_1, a_2 = 1, 2, \ldots, n. \quad (1.12)$$

The above definition was then specialized into *ordered direct analytic representations*

$$[\mathscr{L}_{a_1}(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R} \equiv [F_{a_1}(\phi)]^{\mathscr{C}^2,R} = 0, \qquad a_1 = 1, 2, \ldots, n, \quad (1.13)$$

and *ordered indirect analytic representations*

$$[\mathscr{L}_{a_1}(\phi)]_{\mathrm{SA}}^{\mathscr{C}^2,R} \equiv \{h_{a_1}{}^{a_2}[F_{a_2}(\phi)]^{\mathscr{C}^2,R}\}^{\mathscr{C}^2,R}$$
$$h_{a_1}{}^{a_2} \neq \delta_{a_1}{}^{a_2}, \qquad a_1, a_2 = 1, 2, \ldots, n. \quad (1.14)$$

A variational analysis of the above analytic representations, when they existed, was also conducted in [1] with the result that the concept of ordering ensured the identity not only of the Lagrange equations with the field equations but also of the corresponding equations of variation and related adjoint systems, e.g., for case (1.13)

Lagrange equations	$\mathscr{L}_{a_1}(\phi) \equiv F_{a_1} \phi)$	field equations;
Jacobi equations	$\Omega_{a_1}(\eta) \equiv M_{a_1}(\eta)$	equations of variation;
adjoint system of the Jacobi equations	$\tilde{\Omega}_{a_1}(\tilde{\eta}) \equiv \tilde{M}_{a_1}(\tilde{\eta})$	adjoint system of the equations of variation.

(1.15)

To avoid possible misinterpretation, let us also recall from [1]'s analysis that any quasi-linear system can always be transformed into an equivalent system for which the symmetry properties

$$A^{\mu\ \nu}_{a_1 a_2} = A^{\nu\ \mu}_{a_1 a_2} \quad (1.16)$$

are verified. Indeed, if this is not the case, we can always write, from the symmetry properties $\phi^{a;}{}_{\mu\nu} = \phi^{a;}{}_{\nu\mu}$,

$$A^{\mu\ \nu}_{a_1 a_2}\phi^{a_2;}{}_{\mu\nu} + B_{a_1} \equiv \tfrac{1}{2}(A^{\mu\ \nu}_{a_1 a_2} + A^{\nu\ \mu}_{a_1 a_2})\, \phi^{a_2;}{}_{\mu\nu} + \tfrac{1}{2}(A^{\mu\ \nu}_{a_1 a_2} - A^{\nu\ \mu}_{a_1 a_2})\, \phi^{a_2;}{}_{\mu\nu} + B_{a_1}$$
$$\equiv \tfrac{1}{2}(A^{\mu\ \nu}_{a_1 a_2} + A^{\nu\ \mu}_{a_1 a_2})\, \phi^{a})^{;}{}_{\mu\nu} + B_{a_1} = 0, \quad (1.17)$$

in which case symmetries (1.16) hold for the last form of Eqs. (1.17), i.e., for the redefined terms $A'^{\mu\ \nu}_{a_1 a_2} = \frac{1}{2}(A^{\mu\ \nu}_{a_1 a_2} + A^{\mu\ \nu}_{a_1 a_2})$.

In the present paper we shall tacitly assume that all considered quasi-linear systems satisfy symmetry properties (1.16). This implies in particular that the conditions of self-adjointness (1.5) will be referred to systems obeying such properties.

The above symmetrization procedure also applies to the Lagrange equations. Indeed, if one uses symmetry properties (1.16) for the Lagrange equations in the third form of Eqs. (1.8), the relations

$$\frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_{\mu}\, \partial \phi^{a_2;}{}_{\nu}} = \frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_{\nu}\, \partial \phi^{a_2;}{}_{\mu}}, \qquad \mu, \nu = 0, 1, 2, 3, \qquad a_1, a_2 = 1, 2, \ldots, n \tag{1.18}$$

might result. The point which we would like to recall from [1] is that within the context of our analysis, which is ultimately based on an arbitrary structure of the Lagrangian density, properties (1.18) *are not* implied by the continuity assumption $\mathscr{L} \in \mathscr{C}^4$, and they are in general *erroneous*.

Again, symmetry properties (1.16) must be applied to the Lagrange equations in their symmetrized form, i.e., the last form of Eqs. (1.8), in which case they trivially hold for the terms

$$A'^{\mu\ \nu}_{a_1 a_2} = \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_{\mu}\, \partial \phi^{a_2;}{}_{\nu}} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a_1;}{}_{\nu}\, \partial \phi^{a_2;}{}_{\mu}}\right), \tag{1.19}$$

irrespective of any continuity condition of the Lagrangian.

In the present paper we shall tacitly assume that the Lagrange equations are written in the above indicated symmetrized form and that the conditions of self-adjointness (1.5) are always referred to and identically verified for class $\mathscr{C}^2$ and regular Lagrange equations in this symmetrized form.

The objectives of this paper are to identify the necessary and sufficient conditions for a given class $\mathscr{C}^2$, regular, Lorentz covariant, tensorial, quasi-linear system of field equations to admit an ordered direct analytic representation in terms of the Lagrange equations; to provide a method for the construction of a Lagrangian density, when it exists, from given field equations; and to explore the "structure" of the Lagrangian capable of representing tensorial fields with arbitrary forms of coupling.

The case of ordered *indirect* analytic representations is treated in subsequent paper III. We plan to study the same problems for other type of field equations (e.g., spinorial or degenerate) as well as to explore some initial significance of the underlying methodology in Field Theory at a later time.

2. A Theorem on the Existence of a Lagrangian Density for Ordered Direct Analytic Representations

Let us recall that quasi-linear systems (1.2) and the Lagrange equations (1.8) in the Lagrangian densities $\mathscr{L}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ are defined in a *region* R of the variables x_α, ϕ^a, and $\phi^{a;}{}_\alpha$ $(a = 1, 2,..., n, \alpha = 0, 1, 2, 3)$, where the dependence of these equations in the terms $\phi^{a;}{}_{\mu\nu}$ is ignored due to their linearity. Here the term "region" means an open and connected point set of the values of the indicated variables.

The condition that field equations (1.2) are of (at least) *class* $\mathscr{C}^2$ can thus be reduced to the condition that the terms $A^{\mu_1\mu_2}_{a_1a_2}$ and B_{a_1} possess continuous partial derivatives with respect to all of their variables $(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ in the considered region R.

It should be indicated that within this context the variables ϕ^a are not necessarily the solutions of system (1.2). As a matter of fact, our approach to the problem of identifying the necessary and sufficient conditions for the existence of a Lagrangian *does not* demand the knowledge of the solutions of the field equations. This aspect is essential in view of the generally nonlinear nature of the considered field equations.

A region R of the variables $(x_a, \phi^a, \phi^{a;}{}_\alpha)$ is here termed a *domain* when it is perfect, internally connected, and each of its points is a point of accumulation of interior points. Then, if R is a region, $\bar{R} = R \cup \partial R$ (where ∂R is the boundary of R) is a domain.

In principle, a domain of definition of Eqs. (1.2) and (1.8) can be arbitrarily selected and, since it is closed, it may consist of the entire set of possible values of the variables x_α, ϕ^a, and $\phi^{a;}{}_\alpha$, thus including the points at infinity. This domain is, however, redundant for the problem under consideration. Besides, the behavior of the conditions of self-adjointness at infinity is quite delicate to handle.

This raises the question, which is an effective region of definition of field equations (1.2) for the problem of the existence of their Lagrangian representation.

The answer to this question is provided in the Appendices, particularly Appendix B. A region R of the variables $(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ is termed a *star-shaped region* and denoted with R^* when it contains, jointly with a given open and connected set of points $(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$, all points $(x_\alpha, \tau\phi^a, \tau\phi^{a;}{}_\alpha)$ for $0 \leqslant \tau \leqslant 1$. Notice that we assume no restriction on the behavior of the Minkowski coordinates x_α, for reasons which will appear self-evident later on. Notice also that all star-shaped regions contain the (local) origin $\phi^a = 0$, $\phi^{a;}{}_\alpha = 0$, $a = 1, 2,..., n$, $\alpha = 0, 1, 2, 3$. Again, if R^* is a star-shaped region, $\bar{R}^* = R^* \cup \partial R^*$ is a domain.

Our analysis of the problem of the existence of a Lagrangian will be conducted on a star-shaped rather than a conventional region. The reason is that such regions R^* are needed for the formulation of the Converse of the Poincaré Lemma and its generalization given in Appendix B, which, as is well known, constitute effective tools for the study, in general, of all integrability conditions. In view of their redundancy (as well as the delicate nature of their technical implications) we shall tacitly assume that all considered star-shaped domains *do not* contain points at infinity.

Our minimal region of definition of Eqs. (1.2) for the problem of the existence of their Lagrangian representation will then be a star-shaped domain $\bar{R}^*$ of the variables $(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)$ whose boundary ∂R^* consists of the unit "circle" around the origin of the variables ϕ^a and $\phi^{a;}{}_\alpha$, together with an arbitrary (but bounded) region in the Minkowski coordinates. Under this assumption the distinction between the regions R and R^* becomes purely formal.

Our analysis of the integrability conditions for the existence of a Lagrangian will be conducted within the framework of the ordinary calculus of differential forms in the local coordinates ϕ^a and its extension to the case of local coordinates $\phi^{a;}{}_\alpha$, which are outlined, for the reader's convenience, in Appendix A and B.

This raises the question, of which is an effective form of the conditions of self-adjointness (1.5) within the framework of differentiable manifolds with local coordinates ϕ^a and $\phi^{a;}{}_\alpha$.

This question is explored in Appendix C, resulting in the set of integrability conditions

$$\delta^{a_1 a_2 \mu_1 \mu_2}_{b_1 b_2 \nu_1 \nu_2} A^{\nu_1 \nu_2}_{a_1 a_2} = 0, \tag{2.1a}$$

$$\delta^{a_1 a_2 a_3 \mu_1 \mu_2 \mu_3}_{b_1 b_2 b_3 \nu_1 \nu_2 \nu_3} A^{\nu_1 \nu_2 ; \nu_3}_{a_1 a_2 \ a_3} = 0, \tag{2.1b}$$

$$\delta^{a_1 a_2 a_3 a_4 \mu_1 \mu_2 \mu_3 \mu_4}_{b_1 b_2 b_3 b_4 \nu_1 \nu_2 \nu_3 \nu_4} A^{\nu_1 \nu_2 ; \nu_3 ; \nu_4}_{a_1 a_2 \ a_3 \ a_4} = 0, \tag{2.1c}$$

$$\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} = 0, \tag{2.1d}$$

$$B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3} = 2(A^{\mu\ \nu;}_{a_1 a_3\ a_2} - A^{\mu\ \nu;}_{a_2 a_3\ a_1}), \tag{2.1e}$$

$$b_1, \ldots, b_4 = 1, 2, \ldots, n,$$

$$\mu_1, \ldots, \mu_4, \mu, \nu = 0, 1, 2, 3,$$

where $A^{\mu_1 \mu_2}_{a_1 a_2}$ and B_{a_1} are characterized by the given system of field equations (1.2) and

$$\delta^{a_1 \cdots a_\mu\ \mu_1 \cdots \mu_\mu}_{b_1 \cdots b_\mu\ \nu_1 \cdots \nu_\mu} \equiv \begin{vmatrix} \delta^{a_1}_{b_1} & \cdots & \delta^{a_1}_{b_n} \\ & \cdots & \\ \delta^{a_n}_{b_1} & \cdots & \delta^{a_n}_{b_n} \end{vmatrix} \times \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \cdots & \delta^{\mu_1}_{\nu_n} \\ & \cdots & \\ \delta^{\mu_n}_{\nu_1} & \cdots & \delta^{\mu_n}_{\nu_n} \end{vmatrix}. \tag{2.2}$$

Let us state, using Appendix C, that Eqs. (2.1) are implied by the conditions of self-adjointness (1.5) in the sense that when *all* conditions (1.5) hold in a given star-shaped region R^* of the variables $(x_\alpha\,;\, \phi^a, \phi^{a;}{}_\alpha)$ then all Eqs. (2.1) are identically verified in R^*. In essence, Eqs. (2.1) are a suitably selected linear combination of Eqs. (1.5).

In the following, we shall freely use either conditions (1.5) or (2.1), depending on the case at hand.

We are now equipped to formulate and prove

THEOREM 2.1 [4]. *Necessary and sufficient condition for a Lorentz covariant, tensorial, quasi-linear system of field equations*

$$A^{\mu_1\mu_2}_{a_1a_2}(x_\mu\,, \phi^a, \phi^{a;}{}_\mu)\, \phi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1}(x_\mu\,, \phi^a, \phi^{a;}{}_\mu) = 0,$$
$$a, a_1\,, a_2 = 1, 2,\ldots, n, \qquad \mu, \mu_1\,, \mu_2 = 0, 1, 2, 3, \tag{2.3}$$

which is defined, of (at least) class $\mathscr{C}^2$ *and is regular in a star-shaped region* R^* *of the variables* $(x_\mu\,, \phi^a, \phi^{a;}{}_\mu)$, *to admit an ordered direct analytic representation in terms of the Lagrange equations in* R^*

$$d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a_1;}{}_\mu} - \frac{\partial \mathscr{L}}{\partial \phi^{a_1}} \equiv A^{\mu_1\mu_2}_{a_1a_2}\phi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1} = 0 \tag{2.4}$$

is that the system of field equations is self-adjoint in every bounded domain in the interior of R^*.

Proof. Since the system of field equations is of (at least) class $\mathscr{C}^2$ and is regular in R^*, a necessary condition for the existence of analytic representation (2.4) is that the Lagrangian $\mathscr{L}$ be of (at least) class $\mathscr{C}^4$ and be regular in R^*. Then, Theorem 7.1 of [1] applies and the Lagrange equations are self-adjoint in R^*. The condition of self-adjointness of the system of field equations is then *necessary* for the existence of the ordered identification (2.4) in view of the self-adjointness of their lhs.

To prove *sufficiency*, we shall show that under the conditions of self-adjointness of the system of field equations in R^* a Lagrangian always exists.

From the condition of regularity it follows that a general structure of the Lagrangian density is given by

$$\mathscr{L}(x_\mu\,, \phi^a, \phi^{a;}{}_\mu)$$
$$= K(x_\mu\,, \phi^a, \phi^{a;}{}_\mu) + D_a{}^\mu(x_\mu\,, \phi^a)\, \phi^{a;}{}_\mu + C(x_\mu\,, \phi^a), \tag{2.5}$$

where the "kinetic" density K is nonlinear in the derivative terms and all the densities K, $D_a{}^\mu$, and C are of (at least) class $\mathscr{C}^4$ in the star-shaped region of their respective variables.

By substituting form (2.5) in identifications (2.4) and from the quasi-linear structure of the Lagrange equations (1.8) we first reach the two sets of identities,

$$\tfrac{1}{2}(K^{;\mu_1;\mu_2}_{a_1\ a_2} + K^{;\mu_2;\mu_1}_{a_1\ a_2}) \equiv A^{\mu_1\mu_2}_{a_1a_2}, \tag{2.6a}$$

$$(D^{\mu\ ;}_{a_1\ a_2} - D^{\mu\ ;}_{a_2\ a_1})\,\phi^{a_2;}{}_{\mu} + (D^{\mu\ ;}_{a_1\ \mu} - C^{;}{}_{a_1})$$
$$\equiv B_{a_1} + K^{;}{}_{a_1} - K^{;\mu\ ;}_{a_1\ \mu} - (K^{;\mu\ ;}_{a_1\ a_2})\,\phi^{a_2;}{}_{\mu}, \tag{2.6b}$$
$$a_1, a_2 = 1, 2,\ldots, n, \qquad \mu, \mu_1, \mu_2 = 0, 1, 2, 3,$$

which must individually hold for a Lagrangian to exist, where we are again using, for convenience, the derivative notation of type (1.8b).

Equations (2.6a) constitute an independent system of conditions for the existence of the K density.

The assumption that such equations are solved first, allows us to consider all terms in the K density of Eqs. (2.6b) as known. For this reason they are written in the rhs jointly with the assigned B terms.

By writing Eqs. (2.6b) in the a_1 and a_2 indices, by differentiating with respect to $\phi^{a_2;}{}_{\mu}$ and $\phi^{a_1;}{}_{\mu}$, respectively, and by subtracting we reach the equations (see Appendix C for more details)

$$D^{\mu\ ;}_{a_1\ a_2} - D^{\mu\ ;}_{a_2\ a_1} \equiv \tfrac{1}{2}(B_{a_1}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_1}) + (K^{;}{}_{a_1}{}^{;\mu}_{a_2} - K^{;\mu\ ;}_{a_1\ a_2}),$$
$$a_1, a_2 = 1, 2,\ldots, n, \qquad \mu = 0, 1, 2, 3, \tag{2.7}$$

which constitute, for a given K, an independent set of conditions for the existence of the D densities.

By assuming that such D densities exist and are computed, we now substitute Eqs. (2.7) in (2.6b) by reaching in this way the equations

$$C^{;}{}_{a_1} \equiv D^{\mu\ ;}_{a_1\ \mu} - B_{a_1} - K^{;}{}_{a_1} + K^{;\mu\ ;}_{a_1\ \mu} + [K^{;\mu\ ;}_{a_2\ a_1} + \tfrac{1}{2}(B_{a_2}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_1})]\,\phi^{a_2;}{}_{\mu},$$
$$a_1, a_2 = 1, 2,\ldots, n, \qquad \mu = 0, 1, 2, 3, \tag{2.8}$$

which constitute, for given K and D^{μ}_{a}, an independent set of conditions for the existence of the C density.

The combined set of conditions (2.6a), (2.7), and (2.8), i.e.,

$$\tfrac{1}{2}(K^{;\mu_1;\mu_2}_{a_1\ a_2} + K^{;\mu_2;\mu_1}_{a_1\ a_2}) \equiv A^{\mu_1\mu_2}_{a_1a_2}, \tag{2.9a}$$

$$D^{\mu\ ;}_{a_1\ a_2} - D^{\mu\ ;}_{a_2\ a_1} \equiv \tfrac{1}{2}(B_{a_1}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_2}) + (K^{;}{}_{a_1}{}^{;\mu}_{a_2} - K^{;}{}_{a_2}{}^{;\mu}_{a_1}), \tag{2.9b}$$

$$C^{;}{}_{a_1} \equiv D^{\mu\ ;}_{a_1\ \mu} - B_{a_1} - K^{;}{}_{a_1} + K^{;\mu\ ;}_{a_1\ \mu} + [K^{;\mu\ ;}_{a_2\ a_1} + \tfrac{1}{2}(B_{a_1}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_1})]\,\phi^{a_2;}{}_{\mu}, \tag{2.9c}$$
$$a_1, a_2 = 1, 2,\ldots, n, \qquad \mu, \mu_1, \mu_2 = 0, 1, 2, 3$$

constitutes a generally overdetermined system of partial differential equations in the $(4n+2)$ unknown densities K, D^{μ}_{a}, and C which characterize a Lagrangian according to Eqs. (2.5).

Our proof of sufficiency will consist in showing that *the conditions of self-adjointness* (1.5) *or* (2.1) *are the integrability conditions of Eqs.* (2.9).

1. *Integrability Conditions of Eqs.* (2.9a)

Introduce the quantities

$$T^{\mu}_{a} \equiv K^{;\mu}_{a} \tag{2.10}$$

and consider the system of first-order partial differential equations

$$T^{\mu_1;\mu_2}_{a_1\ a_2} - A^{\mu_1\mu_2}_{a_1a_2} = 0 \tag{2.11}$$

with underlying (2, 2) form (see Appendix A)

$$T^{(2,2)} = (T^{\mu_1;\mu_2}_{a_1\ a_2} - A^{\mu_1\mu_2}_{a_1a_2})\, d\phi^{a_1;}{}_{\mu_1} \wedge d\phi^{a_2;}{}_{\mu_2}. \tag{2.12}$$

But, from the Converse of the Generalized Poincaré Lemma (see Appendix B), the conditions of self-adjointness (2.1b) are the integrability conditions of Eqs. (2.12). Thus, under the assumptions of the theorem, a solution of Eqs. (2.11) exists. However, this solution is not necessarily consistent with Eqs. (2.9a), due to the lack of symmetrization. This demands that, together with Eqs. (2.11), the equations

$$T^{\mu_1;\mu_2}_{a_1\ a_2} - A^{\mu_2\mu_1}_{a_1a_2} = 0 \tag{2.13}$$

hold with underlying (2, 2) form

$$T'^{(2,2)} = (T^{\mu_1;\mu_2}_{a_1\ a_2} - A^{\mu_2\mu_1}_{a_1a_2})\, d\phi^{a_2;}{}_{\mu_1} \wedge d\phi^{a_2;}{}_{\mu_2}. \tag{2.14}$$

The consistency condition of Eqs. (2.11) and (2.13) reads

$$(A^{\mu_1\mu_2;\mu_3}_{a_1a_2\ a_3} - A^{\mu_1\mu_3;\mu_2}_{a_1a_3\ a_2})\, d\phi^{a_1;}{}_{\mu_1} \wedge d\phi^{a_2;}{}_{\mu_2} \wedge d\phi^{a_3;}{}_{\mu_3} = 0. \tag{2.15}$$

Conditions of self-adjointness (2.1a) and (2.1c) then guarantee the validity of Eqs. (2.15).

It follows that a solution of the equations

$$T^{\mu_1;\mu_2}_{a_1\ a_2} - \tfrac{1}{2}(A^{\mu_1\mu_2}_{a_1a_2} + A^{\mu_2\mu_1}_{a_2a_1}) = 0 \tag{2.16}$$

exists and is given, from Eq. (B.30), by

$$T^{\mu_1}_{a_1} = 2\left(\int_0^1 d\tau\ \tau A^{\mu_1\mu_2}_{a_1a_2}(\tau\phi^{a;}{}_{\mu})\right)\ \phi^{a_2;}{}_{\mu_2}\,. \tag{2.17}$$

The second step is to consider Eqs. (2.10), i.e.,

$$K^{;\mu_1}_{a_1} - T^{\mu_1}_{a_1} = K^{;\mu_1}_{a_1} - 2\left[\int_0^1 d\tau\ \tau A^{\mu_1\mu_2}_{a_1a_2}(\tau\phi^{a;}{}_{\mu})\right]\ \phi^{a_2;}{}_{\mu_2} = 0. \tag{2.18}$$

The related integrability conditions are

$$\delta^{a_1a_2\mu_1\mu_2}_{b_1b_2\nu_1\nu_2}T^{\nu_1;\,\nu_2}_{a_1\ a_2} = 2\int_0^1 d\tau\ \tau\delta^{a_1a_2\mu_1\mu_2}_{b_1b_2\nu_1\nu_2}A^{\nu_1\nu_2}_{a_1a_2}(\tau\phi^{a;}{}_{\mu})$$
$$+\left[\int_0^1 d\tau\ \tau^2\delta^{a_1a_2a_3\mu_1\mu_2\mu_3}_{b_1b_2b_3\nu_1\nu_2\nu_3}A^{\nu_1\nu_2;\,\nu_3}_{a_1a_2\ a_3}(\tau\phi^{a;}{}_{\mu})\right]\ \phi^{b_3;}{}_{\mu_3} = 0, \tag{2.19}$$

and they identically hold in view of Eqs. (2.1a) and (2.1b).

Therefore, under the assumptions of the theorem, a solution of Eqs. (2.16) exists and is given by

$$K(x_\mu\,,\,\phi^a,\,\phi^{a;}{}_{\mu}) = 2\phi^{a_1;}{}_{\mu_1}\int_0^1 d\tau'\left[\phi^{a_2;}{}_{\mu_2}\int_0^1 d\tau\ \tau A^{\mu_1\mu_2}_{a_1a_2}(x_\mu\,,\,\phi^a,\,\tau\phi^{a;}{}_{\mu})\right](\tau'\phi^{a;}{}_{\mu}), \tag{2.20}$$

where the square bracket indicates that the function of derivative terms resulting after integration with respect to τ must be computed along $\tau'\phi^{a;}{}_{\mu}$ prior to the integration with respect to τ'.

This completes the first part of our proof and shows that *the conditions of self-adjointness* (2.1a), (2.1b), *and* (2.1c) *are the integrability conditions of* Eqs. (2.9a) [6].

2. *Integrability Conditions of Eqs.* (2.9b) *and* (2.9c)

We consider now Eqs. (2.9b) and (2.9c), which we write

$$D^{\mu\,;}_{a_1\ a_2} - D^{\mu\,;}_{a_2\ a_1} - Z^{\mu}_{a_1a_2} = 0, \tag{2.21a}$$

$$C^{;}{}_{a_1} - W_{a_1} = 0, \tag{2.21b}$$

with a self-explanatory definition of the terms $Z^{\mu}_{a_1a_2}$ and W_{a_1}. The underlying differential froms are the collection of $(\mu, 2)$, $\mu = 0, 1, 2, 3$ and 1 forms, respectively (see Appendix A),

$$Z^{(\mu,2)} = Z^{\mu}_{a_1a_2}\,d\phi^{a_1}\wedge d\phi^{a_2} \tag{2.22a}$$

$$W^{(1)} = W_{a_1}\,d\phi^{a_1}, \tag{2.22b}$$

with integrability conditions

$$\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} Z^{\mu}_{a_1 a_2}{}^{;}{}_{a_3} = 0, \tag{2.23a}$$

$$\delta^{a_1 a_2}_{b_1 b_2} W_{a_1}{}^{;}{}_{a_2} = 0, \tag{2.23b}$$

respectively.

By using the explicit form of the $Z^{\mu}_{a_1 a_2}$ and W_{a_1} terms from the rhs of Eqs. (2.9b) and (2.9c), respectively, conditions (2.23) can be written (see Appendix C for more details)

$$\begin{aligned} \delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} &= 0, \\ \tfrac{1}{2}(\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3})\, \phi^{b_3;}{}_{\mu} &= 0, \end{aligned} \tag{2.24}$$

and they identically hold in view of Eqs. (2.1d).

Therefore, under the conditions of self-adjointness the solutions of Eqs. (2.21) exist and are given by

$$\begin{aligned} D^{\mu}_{a_1}(x_{\alpha}, \phi^{a}) &= \phi^{a_2} \int_0^1 d\tau\, \tau Z^{\mu}_{a_1 a_2}(x_{\alpha}, \tau\phi^{a}), \\ C(x_{\alpha}, \phi^{a}) &= \phi^{a_2} \int_0^1 d\tau\, W_{a_1}(x_{\alpha}, \tau\phi^{a}). \end{aligned} \tag{2.25}$$

This completes the second part of our proof and shows that *the conditions of self-adjointness* (2.1d) *are the integrability conditions of Eqs.* (2.9b) *and* (2.9c).

3. *Consistency of Eqs.* (2.9)

To complete our proof, we must first show that, for consistency, the rhs of Eqs. (2.9b) and (2.9c) is independent of derivative terms.

By differentiating Eqs. (2.9b) and (2.9c) with respect to $\phi^{a_3;}{}_{\nu}$ we reach the respective conditions

$$\begin{aligned} (B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3}) - 2(A^{\mu\ \nu\ ;}_{a_1 a_3}{}_{a_2} - A^{\mu\ \nu\ ;}_{a_2 a_3}{}_{a_1}) &= 0, \\ [(B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_3}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_1}) - 2(A^{\mu\ \nu\ ;}_{a_1 a_2}{}_{a_3} - A^{\mu\ \nu\ ;}_{a_2 a_3}{}_{a_1})]\, \phi^{a_3;}{}_{\nu} &= 0, \end{aligned} \tag{2.26}$$

which clearly hold in view of Eqs. (2.1e).

This proves that conditions of self-adjointness (2.1e) *guarantee the independence of the rhs of Eqs.* (2.9a) *and* (2.9b) *from the derivative terms.*

Our proof will be completed by showing, for consistency, that Eqs. (2.9) are compatible among themselves.

Since Eqs. (2.9a) must be solved first, the problem of compatibility of Eqs. (2.9) can be reduced to the proof that Eqs. (2.9b) and (2.9c), under identifications (2.9a), are compatible among themselves.

Let us rewrite these equations in the form

$$D^{\mu}_{a_1}{}^{;}{}_{a_2} - D^{\mu}_{a_2}{}^{;}{}_{a_1} = Z^{\mu}_{a_1 a_2}, \tag{2.27a}$$

$$D^{\mu}_{a_1}{}^{;}{}_{\mu} - C^{;}{}_{a_1} = W'{}_{a_1}, \tag{2.27b}$$

where $W'{}_{a_2} = D^{\mu}_{a_1}{}^{;}{}_{\mu} - W_a$.

After partial differentiation with respect to x^μ and ϕ^{a_2} respectively, we can write

$$\begin{aligned} D^{\mu}_{a_1}{}^{;}{}_{a_2}{}^{;}{}_{\mu} &= Z^{\mu}_{a_1 a_2}{}^{;}{}_{\mu} + D^{\mu}_{a_2}{}^{;}{}_{a_1}{}^{;}{}_{\mu}, \\ D^{\mu}_{a_1}{}^{;}{}_{\mu}{}^{;}{}_{a_2} &= W'{}_{a_1}{}^{;}{}_{a_2} + C^{;}{}_{a_1}{}^{;}{}_{a_2}. \end{aligned} \tag{2.28}$$

Thus, the *necessary* conditions for the consistency of Eqs. (2.27) are

$$Z^{\mu}_{a_1 a_2}{}^{;}{}_{\mu} = W'{}_{a_1}{}^{;}{}_{a_2} - W'{}_{a_2}{}^{;}{}_{a_1}, \tag{2.29}$$

where we have used Eqs. (2.27b).

To prove that conditions (2.29) are also *sufficient* for the consistency of Eqs. (2.27), consider such equations for fixed values of the indices $a_1 = a_1{}^0$ and $a_2 = a_2{}^0$ ($\neq a_1{}^0$). Then the existence theorems for linear partial differential equations [8] apply (in view of the continuity properties of the $Z^{\mu}_{a_1 a_2}$ and $W'{}_{a_1}$ functions) and a solution $D^{\mu}_{a_1{}^0}$, $D^{\mu}_{a_2{}^0}$, and C exists.

We now substitute such solutions into Eqs. (2.27) in the form

$$\begin{aligned} D^{\mu}_{a_2}{}^{;}{}_{a_1{}^0} &= D^{\mu}_{a_1{}^0}{}^{;}{}_{a_2} - Z^{\mu}_{a_1{}^0}{}^{;}{}_{a_2}, \\ D^{\mu}_{a_2}{}^{;}{}_{\mu} &= W'{}_{a_2} + C^{;}{}_{a_2}. \end{aligned} \tag{2.30}$$

Such equations are compatible provided that

$$D^{\mu}_{a_1{}^0}{}^{;}{}_{a_2}{}^{;}{}_{\mu} - Z^{\mu}_{a_1{}^0 a_2}{}^{;}{}_{\mu} = W'{}_{a_2}{}^{;}{}_{a_1{}^0} + C^{;}{}_{a_2}{}^{;}{}_{a_1{}^0}. \tag{2.31}$$

But the above conditions reduce to Eqs. (2.29) after use of Eqs. (2.27b).

Thus, Eqs. (2.29) are the necessary and sufficient conditions for the consistency of Eqs. (2.27).

We must now inspect Eqs. (2.29) by using the explicit form of the $Z^{\mu}_{a_1 a_2}$ and $W'{}_{a_1}$ terms. From the rhs of Eqs. (2.9b) and (2.9c) (and by recalling that $W'{}_{a_1} = D^{\mu}_{a_1}{}^{;}{}_{\mu} - W_a$), Eqs. (2.27) become

$$\begin{aligned} &B_{a_1}{}^{;}{}_{a_2} - B_{a_2}{}^{;}{}_{a_1} - \tfrac{1}{2}(B_{a_2}{}^{;\mu}{}_{a_1} - B_{a_2}{}^{;\mu}{}_{a_1})^{;}{}_{\mu} \\ &\quad + [(B_{a_2}{}^{;\mu}{}_{a_3} - B_{a_3}{}^{;\mu}{}_{a_2})^{;}{}_{a_1} + (B_{a_3}{}^{;\mu}{}_{a_1} - B_{a_1}{}^{;\mu}{}_{a_3})^{;}{}_{a_2}]\, \phi^{a_3;}{}_{\mu} = 0, \end{aligned} \tag{2.32}$$

which, by using conditions of self-adjointness (1.5d), can be written

$$\tfrac{1}{2}(\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3})\ \phi^{b_3;}{}_{\mu} = 0, \tag{2.33}$$

and they identically hold in view of conditions (2.1d).

This completes the third part of our proof and shows that *Eqs.* (2.1d) *are not only the integrability conditions for Eqs.* (2.9b) *and* (2.9c), *but also the necessary and sufficient conditions for their consistency.*

Thus, when the conditions of self-adjointness (1.5) or (2.1) hold, Eqs. (2.9) always admit solutions K, $D_a{}^\mu$, and C and a Lagrangian (2.5) always exists. Q.E.D.

A few comments are now in order. Theorem 2.1 and its proof clearly indicate the effectiveness of our variational approach to self-adjointness for the problem of the existence of a Lagrangian density in classical Field Theory. Let us recall from Appendix C that *all* conditions of self-adjointness (1.5) enter into the construction of the integrability conditions (2.1). Therefore, without redundancy, all conditions of self-adjointness enter into the proof of the theorem.

This latter remark must be kept in mind for practical applications. Indeed, if the assigned system of field equations violates only *one* of the conditions of self-adjointness, then, according to our terminology, it is non-self-adjoint and a Lagrangian for the ordered direct identification (2.4) *does not* exist.

In this case, however, one can seek for an ordered "indirect" analytic representation. This aspect will be investigated in subsequent paper III.

The significance of the concept of "ordering" in the statement and proof of the theorem demands some elaboration.

Let us remark that the conditions of self-adjointness for the case of *ordinary* differential equations were rather generally considered to be both necessary and sufficient for the existence of a Lagrangian for a "direct analytic representation" without any reference as far as the ordering is concerned [5].

This author, however, identified [5] (apparently for the first time) a counterexample to the above position concerning the *necessity* of the conditions of self-adjointness.

This counterexample essentially consists of the identification of the variational properties of the Morse–Feshbach method [9] for representing certain nonconservative systems. Explicitly, the system of second-order ordinary differential equations (see [1, Appendix C] for more details)

$$\begin{aligned}\begin{pmatrix} m\ddot{q}_1 + b\dot{q}_1 + kq_1 \\ m\ddot{q}_2 - b\dot{q}_2 + kq_2 \end{pmatrix} &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= (c_{a_1 a_2}\ddot{q}_2) + (b_{a_1 a_2}\dot{q}_{a_2}) + (a_{a_1 a_2} q_{a_2}) = 0 \end{aligned} \tag{2.34}$$

is *non-self-adjoint* because it violates the conditions of self-adjointness

$$\begin{aligned} c_{a_1a_2} - c_{a_2a_1} &= 0, \\ b_{a_1a_2} + b_{a_2a_1} &= 2(d/dt)\, c_{a_1a_2}, \\ a_{a_1a_2} - a_{a_2a_1} &= \tfrac{1}{2}(d/dt)(b_{a_1a_2} - b_{a_2a_1}) \end{aligned} \tag{2.35}$$

in the $b_{a_1a_2}$ terms. Nevertheless, a Lagrangian for their analytic representation exists and is given by [9]

$$L = m\dot{q}_1\dot{q}_2 + \tfrac{1}{2}b(q_1\dot{q}_2 - \dot{q}_1q_2) - kq_1q_2\,. \tag{2.36}$$

The solution of the above rather puzzling situation [5] is easily found by noting that the Lagrange equations in the Lagrangian (2.36) *do not* reproduce Eqs. (2.34), but rather the same system in the inverted order, i.e.,

$$\begin{aligned} \begin{pmatrix} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_1} - \dfrac{\partial L}{\partial q_1} \\ \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_2} - \dfrac{\partial L}{\partial q_2} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m\ddot{q}_1 + b\dot{q}_1 + kq_1 \\ m\ddot{q}_2 - b\dot{q}_2 + kq_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ &= (c'_{a_1a_2}\ddot{q}_2) + (b'_{a_1a_2}\dot{q}_{a_2}) + (a'_{a_1a_2}q_{a_2}) = 0. \end{aligned} \tag{2.37}$$

A simple inspection then indicates that the equations of motion in the inverted order (2.37) satisfy all Eqs. (2.35), and thus, *are* self-adjoint. Incidentally, this also confirms the self-adjointness of the Lagrange equations in the Lagrangian (2.36).

To summarize, for the case of the equations of motion under consideration, the permutation of the *order* in which these equations are assigned changes the system from non-self-adjoint to self-adjoint or vice versa.

In the transition to Field Theory the above framework remains conceptually unchanged. Indeed, by performing the transition

$$\begin{aligned} &m^{1/2}q_1 \to \varphi, \qquad m^{1/2}q_2 \to \bar{\varphi}, \\ &b \to 2mieA_\mu^e(x), \qquad k \to m(m_0^2 - e^2A_\mu^eA^{e\mu}), \end{aligned} \tag{2.38}$$

the Morse–Feshbach Lagrangian (2.36) becomes that of the complex scalar field in interaction with an (external) electromagnetic field

$$\begin{aligned} L \to \mathscr{L} = \bar{\varphi}^{;}{}_{\mu}\varphi^{;\mu} - m_0^2\bar{\varphi}\varphi + e^2A^e{}_\mu A^{e\mu}\bar{\varphi}\varphi \\ - ie(\bar{\varphi}^{;}{}_\mu\varphi - \bar{\varphi}\varphi^{;}{}_\mu)\, A^{e\mu}, \end{aligned} \tag{2.39}$$

with underlying field equations

$$
\begin{aligned}
(\Box + m_0^2 - e^2 A^e{}_\mu A^{e\mu})\,\varphi - 2ieA^{e\mu}\varphi^{;}{}_\mu &= 0,\\
(\Box + m_0^2 - e^2 A^e{}_\mu A^{e\mu})\,\bar\varphi + 2ieA^{e\mu}\bar\varphi^{;}{}_\mu &= 0.
\end{aligned}
\tag{2.40}
$$

Again, system (2.40) in the ordering ($\phi^{a_1} = \varphi$, $\phi^{a_2} = \bar\varphi$) is *non-self-adjoint* (see [1, Appendix C]. However, if the same system is written in the reverse ordering produced by a permutation of the indices, then it is *self-adjoint*. And indeed, as the reader can verify with a simple inspection and as is the case in general for all Lagrangians with gauge invariant terms of the type $\bar\varphi\varphi$, the Lagrange equations in Lagrangian (2.39) produce the field equation (2.41) in their inverted ordering, i.e., in their self-adjoint form.

The above remarks illustrate the breakdown of the necessity of the condition of self-adjointness when the ordering of the field equations is ignored. Theorem 2.1 restores the necessity of the conditions of self-adjointness through a properly selected definition of the analytic representations they are referred to, namely, that of an "ordered direct analytic representation" with underlying structure (1.15).

This state of affairs can be summarized with

COROLLARY 2.1A. *If the ordering criterion is relaxed in Theorem* 2.1 *the condition of self-adjointness of the field equations is only sufficient for the existence of an analytic representation.*

Notice that the methodology which underlies the formulation and proof of Theorem 2.1 is purely variational in nature. In this respect the following remarks are in order.

Let us recall from [1] that the concept of self-adjointness for systems of *ordinary* differential equations originated within the framework of the so-called *Inverse Problem of the Calculus of Variations*. In our extension of this framework to Field Theory the extremal aspect of the problem has been ignored.

Basically, the single-integral path functionals

$$A(q) = \int_{t_1}^{t_2} dt\, L(t, q, \dot q) \tag{2.41}$$

consitute well-defined variational problems, provided that the Legendre (in essence, our condition of regularity) as well as other conditions are verified. Then, the inverse problem is well defined too. Intensive investigations were conducted in this respect but, regrettably, only up to the first part of this century [1, 5].

In the transition to Lorentz-covariant Field Theories, the above framework is considerably altered because the extremal problem for *hyperbolic* multiple integral path functions

$$A(\phi) = \int_R d^4x \, \mathscr{L}(x_\mu, \phi^a, \phi^{a}{}_{;\mu}) \tag{2.42}$$

is generally vacuous despite the use of boundary conditions [10].

This is, in ultimate analysis, a consequence of the Lorentz covariance of the theory, which imposes the hyperbolic nature of the problem; this hyperbolic nature implies, in view of the indefinite nature of the underlying metric, the impossibility of satisfying the Legendre condition despite the verification of the regularity condition; and this, in turn, implies the lack, in general, of either a maximum or a minimum for the functional (2.42).

Incidentally, this might be a reason for the lack of investigation, to the best knowledge of this author, on the inverse problem of hyperbolic multiple integral path functionals.

Of course, the above difficulties are absent for a Field Theory on a Euclidean space. Indeed, in this case the definite nature of the underlying metric does imply the possibility of satisfying the Legendre condition. Then, the conventional extremal problem as well as its inverse are well defined.

The point which we would like to stress here is that the extremal aspect of the problem is immaterial for the methodology to identify a Lagrangian. Therefore, for the case of Lorentz-covariant Field Theories, even though the extremala spect is problematic, the question of the existence of a Lagrangian is well defined. This is in line with our presentation and proof of Theorem 2.1, where we have ignored the extremal aspect and used only the variational techniques for the identification of the conditions of self-adjointness.

To indicate the variational nature of our treatment, let us remark that the conventional techniques nowadays used in Field Theory such as Hamilton's Principle, Noether Theorem, etc., are, from the viewpoint of the Calculus of Variations, only first-order techniques. This is due to the fact that they arise within the context of the first-order variations of the action functional.

Another point which we would like to stress is that, to the best knowledge of this author, such first-order techniques are insufficient to provide the necessary methodology for the existence of a Lagrangian. And indeed, our proof of Theorem 2.1 demands the use of both first- *and* second-order variational techniques.

More specifically, with reference to the explicit structure (1.15) of an analytic representation, the first line arises within the framework of first-order techniques. This is, in essence, the derivation of the Lagrange equations and related identifications (2.4) from Hamilton's Principle. The other lines of structure (1.15) can be derived only within the framework of second-order variations which, in the Calculus of Variations, are related to the so-called "accessory extremal problem."

Therefore, our analysis is ultimately variational in nature because the concept of self-adjointness demands the use of first- and second-order variations.

Alternatively, and by also referring to structure (1.15), we can say that, together with the use of the Lagrange equations, our treatment demands the use of the related Jacobi equations which, again, are of second-order variational nature.

It should be noted, incidentally, that the joint use of Lagrange and the related Jacobi equations might also be of some significance for other aspects of Field Theory, particularly in relation to nonlinear theories. Indeed, when the solutions of the Lagrange equations cannot be computed, the solutions of the related Jacobi equations *can* always be computed (under the necessary continuity and regularity requirements) because they are always linear irrespective of the linearity or nonlinearity of the Lagrange equations.

Our analysis and the above remarks seem to indicate that, despite a rather general belief to the contrary, the methodology of the Calculus of Variations at large might have a rather profound impact in Theoretical Physics which goes considerably beyond the framework of Hamilton's Principle and its applications (e.g., the Noether Theorem).

As a final point, we would like to stress that our proof of Theorem 2.1 is the simplest that this author has been able to formulate and, as such, it makes the most economical use possible of the methodology of the calculus of variations. What we want to recall here is that such methodology is rather vast indeed, and it includes tools such as [10–11] the Weierstrass' function, the formulation in terms of Hilbert's invariant integral, Weyl's theory, Charthéodory theory, etc., and topics of geometrical nature [12]. It is not inconceivable that several variational methods may have direct significance for the problem of the existence of the Lagrangian as well as for other aspects of Field Theories (e.g., the study of nonlinear theories, or, more generally, the study of arbitrary forms of Lorentz-covariant couplings). These profiles are here left to the imagination of the individual reader. For an analysis of some alternative methods for the inverse problem of single integral path functionals see [5].

3. A Method for the Construction of a Lagrangian Density and an Analysis of Its Structure

Our proof of Theorem 2.1 provides not only the system of partial differential equations (2.9) for the construction of a Lagrangian density, when it exists, but also its solution.

This result, in essence, originates from the use of the calculus of differential forms in general, and the Converse of the Poincaré Lemma in particular.

And indeed it is a matter of a simple restatement of the proof of Theorem 2.1 to reach

COROLLARY 2.1B. *A Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, quasi-linear systems of field equations*

$$A^{\mu_1\mu_2}_{a_1a_2}(x_\mu, \phi^a, \phi^{a;}{}_\mu)\, \phi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1}(x_\mu, \phi^a, \phi^{a;}{}_\mu) = 0,$$
$$a, a_1, a_2 = 1, 2,..., n, \qquad \mu, \mu_1, \mu_2 = 0, 1, 2, 3, \tag{3.1}$$

which are of (at least) class $\mathscr{C}^2$, are regular, and are self-adjoint in a star-shaped region R^* of points $(x_\mu, \phi^a, \phi^{a;}{}_\mu)$, is given by

$$\mathscr{L}(x_\mu, \phi^a, \phi^{a;}{}_\mu) = K(x_\mu, \phi^a, \phi^{a;}{}_\mu) + D^{\mu_1}_{a_1}(x_\mu, \phi^a)\, \phi^{a_1;}{}_{\mu_1} + C(x_\mu, \phi^a), \tag{3.2}$$

where the $(4n + 2)$ *densities* K, $D^{\mu_1}_{a_1}$, *and* C *are a solution of the linear, generally overdetermined system of partial differential equations*

$$\tfrac{1}{2}(K^{;\mu_1;\mu_2}_{\ a_1\ a_2} + K^{;\mu_2;\mu_1}_{\ a_1\ a_2}) = A^{\mu_1\mu_2}_{a_1a_2}(x_\mu, \phi^a, \phi^{a;\mu}{}_{\mu}), \tag{3.3a}$$

$$D^{\mu_1;}_{a_1\ a_2} - D^{\mu_1;}_{a_2\ a_1} = \tfrac{1}{2}(B_{a_1\ a_2}^{\ ;\mu_1} - B_{a_2\ a_1}^{\ ;\mu_1}) + (K^{;\ ;\mu_1}_{\ a_1\ a_2} - K^{;\ ;\mu_1}_{\ a_2\ a_1})$$
$$\equiv Z^{\mu_1}_{a_1a_2}(x_\mu, \phi^a), \tag{3.3b}$$

$$C^{;}{}_{a_1} = D^{\mu_1;}_{a_1\ \mu_1} - B_{a_1} - K^{;}{}_{a_1} - K^{;\mu_1;}_{a_1\ \mu_1} + [K^{;\ ;\mu}_{\ a_1\ a_2} + \tfrac{1}{2}(B_{a_1\ a_2}^{\ ;\mu_1} - B_{a_2\ a_1}^{\ ;\mu_1})]\, \phi^{a_2;}{}_{\mu_1}$$
$$\equiv W_{a_1}(x_\mu, \phi^a), \tag{3.3c}$$
$$a, a_1, a_2 = 1, 2,..., n, \qquad \mu, \mu_1, \mu_2 = 0, 1, 2, 3.$$

given by

$$K = 2\phi^{a_1;}{}_{\mu_1} \int_0^1 d\tau' \left[\phi^{a_2;}{}_{\mu_2} \int_0^1 d\tau\, \tau A^{\mu_1\mu_2}_{a_1a_2}(x_\mu, \phi^a, \tau\phi^{a;}{}_\mu)\right] (\tau'\phi^{a;}{}_\mu),$$
$$D^{\mu_1}_{a_1} = \phi^{a_2} \int_0^1 d\tau\, \tau Z^{\mu_1}_{a_1a_2}(x_\mu, \tau\phi^a), \tag{3.4}$$
$$C = \phi^{a_1} \int_0^1 d\tau\, W_{a_1}(x_\mu, \tau\phi^a).$$

A few comments are now in order prior to illustrating the above corollary with some examples. First of all let us remark that, under the assumed continuity and regularity conditions, there is no need to verify the consistency of system (3.3) when the assigned system of field equations is self-adjoint in R^*. And indeed, the proof of sufficiency of Theorem 2.1 is precisely centered on the property that the conditions of self-adjointness *are* the integrability conditions of Eqs. (3.3).

Therefore, for practical applications, one must verify the continuity, regularity, and self-adjointness requirements of the assigned system of field equations in a star-shaped region R^*. When such requirements are verified, a solution of Eqs. (3.3) exists.

Notice that no knowledge of the solutions of the field equations is required for the construction of the Lagrangian according to the method of Corollary 2.1B.

To avoid possible misinterpretation, let us indicate that Eqs. (3.3) must be solved in the assigned order, namely, one must first solve Eqs. (3.3a); the knowledge of the "kinetic" density K as well as of the assigned terms B_a then allows the solution of Eqs. (3.3b) in the $D_a{}^\mu$ densities; and, finally, the knowledge of the K and $D_a{}^\mu$ densities allows the solution of Eqs. (3.3) in the C density.

Notice also that the integrals of solutions (3.4) are insensitive to the dependence of the integrands other than those indicated. For instance, the $A^{\mu_1\mu_2}_{a_1a_2}$ terms of integrals (3.1a) depend, in general, on the x_μ and ϕ^a variables as well as the derivative terms $\phi^{a;}{}_\mu$. Nevertheless, the inclusion of the τ variable is done as indicated, in the derivative terms only. Furthermore, as indicated in the proof of Theorem 2.1, the bracket of Eq. (3.4a) indicates that the function of $\phi^{a;}{}_\mu$ resulting from the integration with respect to τ must be computed along $\tau'\phi^{a;}{}_\mu$ prior to the integration with respect to τ'.

The reader should keep in mind, as indicated in Appendix C, that if K is a particular solution of Eqs. (3.3a), its general solution is precisely given by the structure (3.2) of the Lagrangian. This point can also be derived from the "degrees of freedom" of primitive forms of type (B. 12).

The reader should also keep in mind that the solutions (3.4) are local in nature, as it is the case, in general, for all applications of the Converse of the Poincaré Lemma [7].

We now come to a crucial as well as delicate point of our method for computing a Lagrangian. This is constituted by the fact that the solutions (3.4) apply iff their integrals are well defined. In turn, this point is intimately linked to the requirement that the field equations are well defined in a star-shaped rather than an ordinary region.

Before commenting on this point, for the sake of clarity let us note, from the Appendices and from Section 2, that on practical grounds, one can ignore the distinction between ordinary and star-shaped regions and work on the "minimal domain" $\bar{R}_{\text{Min}}$ whose boundary is constituted by unit circles around the origin in the "plane" of local coordinates $(\phi^a, \phi^{a;}{}_\mu)$. A requirement of Theorem 2 and of Corollary 2.1B is then that the field equations are well defined at least in such domain $\bar{R}_{\text{Min}}$. Alternatively and on more pragmatic grounds, one can simply verify that the field equations are well defined for all values $0 \leqslant \phi^a \leqslant 1$ and $0 \leqslant \phi^{a;}{}_\mu \leqslant 1$ ($a = 1, 2, \ldots, n$, $\mu = 0, 1, 2, 3$). If this is the case, then integrals (3.4) are well defined, too.

Now, the above requirements can clearly be violated in practical applications. This is the case when the field equations incorporate terms such as $\log \phi$, $\operatorname{cosec} \phi$, etc.

But, within the context of solutions (3.4), the variables ϕ^a and $\phi^{a;}{}_{\mu}$ are local in nature. This allows the redefinition of these variables aiming at the removal of the divergences in R_{Min}. For $\operatorname{cosec} \phi$ a trivial redefinition is $\phi' = \phi + \text{const}$, in which case one can use Eqs. (3.3) and solutions (3.4) in the redefined rather than the original fields.

In conclusion, and as we shall illustrate later on, solutions (3.4) generally hold, up to redefinition of the local variables, for Field Theories in Minkowski space.

Despite that, the reader should be alerted that counterexamples are conceivable. Furthermore, the extension of the method of Corollary 2.1B to Field Theories in a Riemannian or pseudo-Riemannian manifold demands considerable care, particularly when the local coordinates are the elements $g^{\mu\nu}$ of the metric tensor and their covariant derivatives.

It should be indicated that, to the best knowledge of this author, the case when integrals of type (3.4) fail to exist is not yet known within the context of the calculus of differential forms.

We shall therefore not enter into this aspect at this time and content ourselves with the obtained solutions (3.4).

Another point which the reader should keep in mind is that Eqs. (3.4) ultimately constitute only *one* method of solving Eqs. (3.3), and other methods are conceivable. Therefore, if solutions (3.4) fail to hold, this does not prohibit the possibility of solving Eqs. (3.3) with methods other than that of the Converse of the Poincaré Lemma and its generalization as presented in Appendix B.

To clarify this point, let us first note that the system of differential equations (3.3) for the characterization of a Lagrangian exists irrespective of the type of region of definition of the field equations. This is the spirit of their derivation in the first part of the proof of Theorem 2.1. Therefore, if solutions (3.4) do not exist, this *does not* necessarily imply that system (3.3) is also not defined in a non-star-shaped region (e.g., $0 < \phi^a, \phi^{a;}{}_{\mu} < 1$, or $1 < \phi^a, \phi^a{}_{,\mu}$).

On similar grounds, the conditions of self-adjointness, as derived in [1], *do not* necessarily need a star-shaped region to be well defined.

The possibility referred to above is thus constituted by the case when the integrals of solutions (3.4) do not exist, but system (3.3) and the conditions of self-adjointness are well defined in a non-star-shaped region. It is under such circumstances that other methods of integrating Eqs. (3.3) are conceivable.

Stated in somewhat different terms, if counterexamples to the universality of solutions (3.4) up to redefinition of the local variables for all (tensorial) Field Theories in Minkowski space do exist, they *do not* necessarily imply the breakdown of system (3.3) and of self-adjointness in some ordinary region of the variables and, therefore, a Lagrangian may still exist.

In any case, this point demands specific supplementary investigation. It is for this reason that we have formulated Theorem 2.1 and Corollary 2.1B, as a precautionary measure, on a star-shaped rather than an ordinary region.

The case of nonlinear systems of field equations (1.1) is excluded by Theorem 2.1 in line with the fact that the most general system of field equations which can be represented with the Lagrange equations is, from their structure (1.8), of the quasi-linear type (1.2). We shall therefore ignore from now on the nonlinear form (1.1).

The semilinear form (1.3) of the field equations is, on the contrary, significant. This is due to the fact, already stressed in [1] that the great majority of tensorial field equations considered until now are of the semilinear form

$$\square\phi_{a_1} - f_{a_1}(x_\mu, \phi^a, \phi^{a;}{}_\mu) = 0. \tag{3.5}$$

However, the necessary and sufficient conditions for Eqs. (3.5) to be self-adjoint are that they are linear in the derivative terms $\phi^{a;}{}_\mu$, i.e., they are of form (1.3), and all conditions of self-adjointness (1.7) are satisfied [1, Theorem 6.3]. Therefore, we shall now restrict our analysis to semilinear systems of the reduced form (1.3).

The problem of the existence of an ordered direct analytic representation for systems (1.3) is clearly a particular case of Theorem 2.1 with

$$A^{\mu_1\mu_2}_{a_1a_2} = \delta_{a_1a_2} \otimes g^{\mu_1\mu_2}, \tag{3.6a}$$

$$B_{a_1} = -\rho^{\mu_1}_{a_1a_2}(x_\mu, \phi^a)\, \phi^{a_2;}{}_{\mu_1} - \sigma(x_\mu, \phi^a). \tag{3.6b}$$

It is, however, an instructive exercise for the interested reader to again prove Theorem 2.1 for identifications (3.6). This proof is considerably simpler, because Eqs. (3.3a) reduce, in this case, to

$$\tfrac{1}{2}(\bar{K}^{;\mu_1;\mu_2}_{a_1\ a_2} + \bar{K}^{;\mu_2;\mu_1}_{a_1\ a_2}) = \delta_{a_1a_2} \otimes g^{\mu_1\mu_2}. \tag{3.7}$$

From Eq. (3.4a) we then recover the well-known kinetic term

$$\begin{aligned} \bar{K} &= 2\phi^{a_1;}{}_{\mu_1} \int_0^1 d\tau' \left[\phi^{a_2;}{}_{\mu_2} \int_0^1 d\tau\ \tau\delta_{a_1a_2} \otimes g^{\mu_1\mu_2}\right] (\tau'\phi^{a;}{}_\mu) \\ &= 2\phi^{a_1;}{}_{\mu_1} \int_0^1 d\tau' \left[\phi_{a_1}{}^{;\mu_1} \int_0^1 d\tau\ \tau\right] (\tau'\phi^{a;}{}_\mu) \\ &= 2\phi^{a_1;}{}_{\mu_1} \int_0^1 d\tau'\ \tfrac{1}{2}\tau'\phi_{a_1}{}^{;\mu_1} \\ &= \tfrac{1}{2}\phi^{a_1;}{}_{\mu_1}\phi_{a_1}{}^{;\mu_1}. \end{aligned} \tag{3.8}$$

Then the conditions of self-adjointness (1.7a) and (1.7b) coincide with the integrability and consistency conditions of Eqs. (3.3b) and (3.3c) (parts 2 and 3 of the proof of Theorem 2.1).

For the reader's convenience, the following corollary summarizes the methodology to compute a Lagrangian in this simpler case.

COROLLARY 2.1C. *A Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, semilinear systems of field equations*

$$g^{\mu_1\mu_2}\phi_{a_1;\mu_1\mu_2} - \rho^{\mu_1}_{a_1a_2}(x_\mu\,;\phi^a)\,\phi^{a_2;}{}_{\mu_1} - \sigma_{a_1}(x_\mu\,,\phi^a) = 0,$$
$$a, a_1, a_2 = 1, 2,\ldots, n, \qquad \mu, \mu_1, \mu_2 = 0, 1, 2, 3, \tag{3.9}$$

which are of (at least) class $\mathscr{C}^1$ and self-adjoint in a star-shaped region R^* of points $(x_\mu\,, \phi^a)$ is given by

$$\mathscr{L}(x_\mu\,;\phi^a;\phi^{a;}{}_\mu) = \tfrac{1}{2}\phi^{a;}{}_\mu\phi_a{}^{;\mu} + D^{\mu_1}_{a_1}(x_\mu\,,\phi^a)\,\phi^{a_1;}{}_{\mu_1} + C(x_\mu\,,\phi^a), \tag{3.10}$$

where the $(4n+1)$ densities $D^{\mu_1}_{a_1}$ and C are a solution of the linear, generally overdetermined system of partial differential equations

$$\begin{aligned} D^{\mu\,;}_{a_1\,a_2} - D^{\mu\,;}_{a_2\,a_1} &= -\rho^{\mu}_{a_1a_2}\,,\\ C^{;}{}_{a_1} &= D^{\mu_1;}_{a_1\,\mu_2} - \sigma_{a_1}\,, \end{aligned} \tag{3.11}$$

given by

$$\begin{aligned} D^{\mu_1}_{a_1} &= \phi^{a_2}\int_0^1 d\tau\;\tau\rho^{\mu_1}_{a_1a_2}(x_\mu\,,\tau\phi^a),\\ C &= \phi^{a_1}\int_0^1 d\tau\;[D^{\mu_1;}_{a_1\,\mu_1} + \sigma_{a_1}](x_\mu\,,\tau\phi^a). \end{aligned} \tag{3.12}$$

Notice that now the minimal continuity conditions are that the terms $\rho^{\mu_1}_{a_1a_2}$ and σ_{a_1} are of class $\mathscr{C}^1$ rather than of class $\mathscr{C}^2$ as in Corollary 2.1B. This is due to the fact that the conditions of self-adjointness (1.7) imply only first-order partial derivatives. Also, the condition of regularity has been dropped because Eqs. (3.9) are always everywhere regular.

Again we must stress the point that while system (3.11) and conditions of self-adjointness (1.7) apply irrespective of the type of region of definition of system (3.9), the solutions (3.12) apply only when the related integrals exist. Counterexamples when the integrals of Eqs. (3.12) do not exist might be conceivable, but in this case other methods of integration of Eqs. (3.11) are equally conceivable.

Let us also recall from [1] that the conditions of self-adjointness (1.7a) and (1.7b) imply that the $\rho^{\mu_1}_{a_1 a_2}$ term has the curl of structure

$$-\rho^{\mu}_{a_1 a_2} = \Gamma^{\mu_1;}_{a_1\ a_2} - \Gamma^{\mu_1;}_{a_2\ a_1}, \tag{3.13}$$

where the $4n$ terms $\Gamma^{\mu}_{a_1}$ are, in general, functions of (x_μ, ϕ^a).

Under structure (3.13), i.e., for the system

$$g^{\mu_1\mu_2}\phi_{a_1}{}^{;}{}_{\mu_1\mu_2} + [\Gamma^{\mu_1}_{a_1}(x_\mu, \phi^a)^{;}{}_{a_2} - \Gamma^{\mu_1}_{a_2}(x_\mu, \phi^a)^{;}{}_{a_1}]\,\phi^{a_2;}{}_{\mu_1} - \sigma_{a_1}(x_\mu, \phi^a) = 0, \tag{3.14}$$

the conditions of self-adjointness (1.7) reduce only to Eq. (1.7c); the densities $D_a{}^\mu$ coincide with $\Gamma_a{}^\mu$, i.e., the solutions (3.12a) read (see Appendix B)

$$D^{\mu_1}_{a_1} = \int_0^1 d\tau\,(d/d\tau)(\tau\Gamma^{\mu_1}_{a_1}) = \Gamma^{\mu_1}_{a_1}, \tag{3.15}$$

and the Lagrangian (3.10) takes the form

$$\mathscr{L}(x_\mu, \phi^a, \phi^{a;}{}_\mu) = \tfrac{1}{2}\phi^{a;}{}_\mu\phi_a{}^{;\mu} + \Gamma_a{}^\mu(x_\alpha, \phi^b)\,\phi^{a;}{}_\mu + C(x_\alpha, \phi^b), \tag{3.16}$$

where the density C is given by

$$C = \phi^a \int_0^1 d\tau\,[\Gamma^{\mu;}_{a\ \mu} + \sigma_a](x_\alpha, \tau\phi^b). \tag{3.17}$$

As a final remark, let us stress that the Lagrangian computed with the above method, when it exists, *is not* unique. This is for several reasons, including the "degrees of freedom" of type (B.12) of the primitive forms of the Converse of the Poincare Lemma and of its generalization.

It is for this reason that throughout our analysis we have always referred to the existence of *a* Lagrangian rather than *the* Lagrangian.

The study of the "degrees of freedom" of a Lagrangian for a given system satisfying the requirements of Theorem 2.1 is contemplated as a subsequent step.

A few examples are now in order. First, let us consider the simple case of the real scalar free field

$$(\Box + m^2)\,\varphi \equiv g^{\mu_1\mu_2}\varphi^{;}{}_{\mu_1\mu_2} + m^2\varphi = 0. \tag{3.18}$$

As we know from [1], this system is self-adjoint. By inspection we then see that all requirements of Theorem 2.1, or Corollary 2.1C, are satisfied. Equation (3.11) reduces to

$$C^{;}{}_{a_1} = -m^2\varphi. \tag{3.19}$$

The solution (3.12b) becomes

$$C = -\varphi \int_0^1 d\tau\, m^2\tau\varphi = \tfrac{1}{2}m^2\varphi^2, \tag{3.20}$$

and the familiar Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\varphi^{;}{}_{\mu}\varphi^{;\mu} - m^2\varphi^2) \tag{3.21}$$

is then recovered from Eq. (3.10).

The extension to the case of self-coupled fields

$$\Box\varphi + m^2\varphi + \lambda\varphi^3 = 0 \tag{3.22}$$

is trivial. Indeed, Eqs. (3.20) become

$$\begin{aligned} C &= -\varphi \int_0^1 d\tau\, [m^2(\tau\varphi) + \lambda(\tau\varphi)^3] \\ &= -\tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{4}\lambda\varphi^4, \end{aligned} \tag{3.23}$$

with the familiar Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\varphi^{;\mu}\varphi^{;}{}_{\mu} - \tfrac{1}{2}m^2\varphi^2 - \tfrac{1}{4}\lambda\varphi^4. \tag{3.24}$$

For the Sine–Gordon equations

$$\Box\varphi + \sin\varphi = 0, \tag{3.25}$$

which is also self-adjoint, we have

$$\begin{aligned} C &= -\varphi \int_0^1 d\tau\, \sin\tau\varphi = [\cos\tau\varphi]_{\tau=0}^{\tau=1} \\ &= \cos\varphi - 1, \end{aligned} \tag{3.26}$$

and the known Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\varphi^{;\mu}\varphi^{;}{}_{\mu} + [\cos\varphi - 1] \tag{3.27}$$

is thus recovered.

In more than one dimension one of the most significant cases is that of the gauge invariant theories. Consider, in this respect, Eqs. (2.40) in the self-adjoint form

$$\begin{aligned} &\begin{pmatrix} 0 & g^{\mu_1\mu_2} \\ g^{\mu_1\mu_2} & 0 \end{pmatrix}\begin{pmatrix} \varphi^{;}{}_{\mu_1\mu_2} \\ \bar{\varphi}^{;}{}_{\mu_1\mu_2} \end{pmatrix} + \begin{pmatrix} 0 & 2ieA^{\mu} \\ -2ieA & 0 \end{pmatrix}\begin{pmatrix} \varphi^{;}{}_{\mu} \\ \bar{\varphi}^{;}{}_{\mu} \end{pmatrix} + \begin{pmatrix} 0 & (m^2 - e^2A_{\mu}A^{\mu}) \\ (m^2 - e^2A_{\mu}A^{\mu}) & 0 \end{pmatrix}\begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} \\ &\quad = (A^{\mu_1\mu_2}_{a_1a_2}\phi^{a_2;}{}_{\mu_1\mu_2}) + (B_{a_1}) = 0. \end{aligned} \tag{3.28}$$

For the kinetic term we then have, from Eq. (3.4a),

$$K = 4\bar{\varphi}^{;}{}_{\mu_1} \int_0^1 d\tau' \left[\varphi^{;\mu_1} \int_0^1 d\tau\ \tau\right] (\tau\varphi^{;\mu_1})$$
$$= \bar{\varphi}^{;}{}_{\mu_1}\varphi^{;\mu_1}. \tag{3.29}$$

Equations (3.3b) become

$$D^{\mu;}_{\varphi\,\bar{\varphi}} = 2ieA^{\mu} \equiv Z^{\mu}_{\varphi\bar{\varphi}},$$
$$D^{\mu;}_{\bar{\varphi}\,\varphi} = -2ieA^{\mu} \equiv -Z^{\mu}_{\varphi\bar{\varphi}}, \tag{3.30}$$

with solutions, from Eq. (3.4b),

$$D^{\mu}_{\varphi} = \bar{\varphi} \int_0^1 \tau Z^{\mu}_{\varphi\bar{\varphi}}\, d\tau = ieA^{\mu}\bar{\varphi},$$
$$D^{\mu}_{\bar{\varphi}} = -\varphi \int_0^1 \tau Z^{\mu}_{\varphi\bar{\varphi}}\, d\tau = -ieA^{\mu}\varphi. \tag{3.31}$$

Equations (3.3c) now read

$$\begin{pmatrix} C^{;}_{\varphi} \\ C^{;}_{\bar{\varphi}} \end{pmatrix} = \begin{pmatrix} -B_{\varphi} \\ -B_{\bar{\varphi}} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} (B_{\varphi\ \bar{\varphi}}^{\ ;\mu})\, \bar{\varphi}^{;}{}_{\mu} - (B_{\bar{\varphi}\ \varphi}^{\ ;\mu})\, \bar{\varphi}^{;}{}_{\mu} \\ (B_{\bar{\varphi}\ \varphi}^{\ ;\mu})\, \varphi^{;}{}_{\mu} - (B_{\varphi\ \bar{\varphi}}^{\ ;\mu})\, \varphi^{;} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & (-m^2 + e^2A_{\mu}A^{\mu}) \\ (-m^2 + e^2A_{\mu}A^{\mu}) & 0 \end{pmatrix} \begin{pmatrix} \varphi \\ \bar{\varphi} \end{pmatrix} = \begin{pmatrix} (-m^2 + e^2A_{\mu}A^{\mu})\, \bar{\varphi} \\ (-m^2 + e^2A_{\mu}A^{\mu})\, \varphi \end{pmatrix} \equiv \begin{pmatrix} W_{\varphi} \\ W_{\bar{\varphi}} \end{pmatrix}. \tag{3.32}$$

Notice the lack of dependence of the rhs of the above expression from derivative terms in accordance with part 3 of Theorem 2.1.

The solution (3.4c) then becomes

$$C = \varphi \int_0^1 d\tau\ W_{\varphi}(\tau\bar{\varphi}) + \bar{\varphi} \int_0^1 d\tau\ W_{\bar{\varphi}}(\tau\varphi)$$
$$= (-m^2 + e^2A^{\mu}A_{\mu})\, \varphi\bar{\varphi}. \tag{3.33}$$

The densities (3.29), (3.31), and (3.33) then reproduce the known Lagrangian (2.39) through structure (3.2).

An element emerging from our analysis which is significant for the problem of the coupling of tensorial fields is that the structure of the field equations for gauge invariant theories *is not* of the semilinear form (3.5) but rather of the more general quasi-linear form (3.1) with the terms $A^{\mu_1\mu_2}_{a_1\ a_2}$ incorporating a permutation of the Latin indices. This point will be further elaborated later on.

Another significant example in more than one dimension is given by the (self-adjoint) chiral equations

$$G_{a_1a_2}(\varphi)\, g^{\mu_1\mu_2}\varphi^{a_2;}{}_{\mu_1\mu_2} + \frac{\partial G_{a_1a_3}(\varphi^a)}{\partial\varphi^{a_2}}\, \varphi^{a_2;}{}_{\mu}\varphi^{a_3;\mu}$$
$$- \frac{1}{2}\,\frac{\partial G_{a_2a_3}(\varphi^a)}{\partial\varphi^{a_1}}\, \varphi^{a_2;}{}_{\mu}\varphi^{a_3;\mu} + \frac{\partial R(\varphi)}{\partial\varphi^{a_1}} = A^{\mu_1\mu_2}_{a_1a_2}\varphi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1} = 0,$$
$$G_{a_1a_2} = G_{a_2a_1}. \tag{3.34}$$

Then, solution (3.4a) becomes

$$K = 2G_{a_1a_2}\varphi^{a_1;}{}_{\mu}\int_0^1 d\tau' \left[\varphi^{a_2;\mu}\int_0^1 d\tau\, \tau\right](\tau'\phi^{a_2;\mu})$$
$$= \tfrac{1}{2}\varphi^{a_1;}{}_{\mu}G_{a_1a_2}(\varphi^a)\,\varphi^{a_2;\mu}. \tag{3.35}$$

Equations (3.3b) yield

$$D^{\mu\,;}_{a_1\;a_2} - D^{\mu\,;}_{a_2\;a_1} \equiv 0, \tag{3.36}$$

with a particular solution $D_a{}^\mu = 0$. Equations (3.3c) are trivial. One then recovers from the kinetic term (3.35) the familiar chiral Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\varphi^{a_1;}{}_{\mu}G_{a_1a_2}(\varphi^a)\,\varphi^{a_2;\mu} + R(\varphi^a). \tag{3.37}$$

Let us stress again that the field equations in this case too are of the quasi-linear form (3.1), and *not* of the semilinear form (3.5).

This completes the illustration of our method of computing a Lagrangian. The interested reader can work out other examples with similar procedures.

We must now reinspect the above results from the viewpoint of interactions.

The problem in which we are interested at this time is the following: What is a general form of modification of the Lagrangian density for free tensorial fields capable of representing the same fields in interaction when expressed in a class $\mathscr{C}^2$, regular, Lorentz-covariant, tensorial, and self-adjoint, but otherwise arbitrary *quasi-linear* form? Or, alternatively, which is a general form of coupling tensorial fields in a way compatible with Theorem 2.1?

In this paper we have established the necessary methodology to answer this question, at least on formal grounds.

An analysis of the Newtonian counterpart of this problem is conducted in [5]. A few remarks within this framework are useful to illustrate certain points.

First, let us review the conventional approach to the problem. Consider, for simplicity, an unconstrained Newtonian system of particles of unit masses with generalized coordinates $q_1, \ldots, q_n$ $(n = 3N)$ representing a collection of conventional (e.g., Cartesian) coordinates. Let $\sum_{1\,a}^{n} L^{(a)}_{\text{Free}} = \sum_{1\,a}^{n} \tfrac{1}{2}(\dot{q}_a)^2$ be the Lagrangian of this system in the absence of external forces. Then we can say that this system of particles is in interaction when the Lagrangian contains the free term $\sum_1^n{}_a L_{\text{Free}}^{(a)}$ and a nontrivial *additive* interaction term L_{Int}, i.e.,

$$L_{\text{Tot}}(q, \dot{q}) = \sum_{1}^{n}{}_a L_{\text{Free}}^{(a)}(\dot{q}) + L_{\text{Int}}(q, \dot{q}). \tag{3.38}$$

It should be indicated that the term L_{Int} is nontrivial when the Lagrange equations in $\sum_1^n{}_a L_{\text{Free}}^{(a)}$ and those in L_{Tot} *are not* equivalent. This ensures the existence of a *modification* of the actual path due to the acting forces and eliminates possible equivalence (i.e., gauge) transformations of $\sum_1^n{}_a L_{\text{Free}}^{(a)}$ induced by L_{Int}.

In more conventional notation one writes $\sum_1^n{}_a L_{\text{Free}}^{(a)} = T =$ Kinetic Energy and $-L_{\text{Int}} = U =$ potential energy.

The extension of the above well-known concept to Field Theory is straightforward and equally well known. Its derivation within the context of Theorem 2.1 reads

COROLLARY 2.1D. *A total Lagrangian density for the ordered direct analytic representation of Lorentz-covariant, tensorial, semilinear systems of coupled field equations*

$$(\square + m^2(a_1))\, \phi_{a_1} = [\Gamma^{\mu}_{a_1}(\phi^a)^{;}{}_{a_2} - \Gamma^{\mu}_{a_2}(\phi^a)^{;}{}_{a_1}]\, \phi^{a_2;}{}_{\mu} + \Lambda_{a_1}(\phi^a), \tag{3.39}$$

which are of (at least) class $\mathscr{C}^1$ *and are self-adjoint in a star-shaped region* R^* *of points* ϕ^a *is given by*

$$\begin{aligned}&\mathscr{L}_{\text{Tot}}(\phi^1,\ldots, \phi^n, \phi^{1;}{}_{\mu},\ldots, \phi^{n;}{}_{\mu})\\ &\quad= \sum_{1}^{n}{}_a \mathscr{L}_{\text{Free}}^{(a)}(\phi^a, \phi^{a;}{}_{\mu}) + \mathscr{L}_{\text{Int}}(\phi^1,\ldots, \phi^n, \phi^{1;}{}_{\mu},\ldots, \phi^{n;}{}_{\mu}),\end{aligned} \tag{3.40}$$

where

$$\mathscr{L}_{\text{Free}}^{(a)} = \tfrac{1}{2}(\phi^{a;}{}_{\mu}\phi_a{}^{;\mu} - m^2(a)\, \phi^a\phi_a), \tag{3.41a}$$

$$\mathscr{L}_{\text{Int}} = \Gamma^{\mu}_{a}(\phi^b)\, \phi^{a;}{}_{\mu} + \Xi(\phi^b). \tag{3.41b}$$

there is no summation with respect to the a index (only) in Eqs. (3.39) *and* (3.41a), *and the density* Ξ *is given by*

$$\Xi = \phi^a \int_0^1 d\tau\, [\Gamma^{\mu;}_{a\ \mu} + \Lambda_a](\tau\phi). \tag{3.42}$$

Again, whenever the Lagrange equations in $\sum_1^n{}_a \mathscr{L}_{\text{Free}}^{(a)}$ and $\mathscr{L}_{\text{Tot}}$ *are not* equivalent, the term $\mathscr{L}_{\text{Int}}$ of the conventional structure (3.40) represents a bona fide interaction or coupling of the considered fields.

One of the implications of our analysis is that the conventional structure (3.40) of the total Lagrangian density is not exhaustive and more general structures are conceivable.

It is at his point that an inspection of the Newtonian framework, which is ultimately the arena of most of our intuitions, may be effective.

Consider again an unconstrained Newtonian system of *n free* particles of unit masses in the coordinates $q_1, \ldots, q_n$. The equations of motions are, trivially,

$$\ddot{q}_a = 0, \qquad a = 1, 2, \ldots, n. \tag{3.43}$$

A general form of coupling the above system is constituted by the superposition of at least three different types of couplings, i.e., (I) coordinate couplings, (II) velocity couplings, (III) acceleration couplings.

To stress the physical significance of the use of these couplings it is sufficient to note that if some of them are ignored, the considered equations of motion may only be an approximation of the physical reality.

An example is needed to clarify this point. Consider, as a first step, only linear couplings. Then the type I couplings applied to Eqs. (3.43) produce the familiar form of coupled oscillators, i.e.,

$$\ddot{q}_{a_1} + c_{a_1a_2}q_{a_2} = 0, \qquad c_{a_1a_2} = \text{const}, \qquad a_1, a_2 = 1, 2, \ldots, n. \tag{3.44}$$

This *conservative* system is, however, insufficient to represent an actual system in our environment, due to the inevitable presence of dissipative forces. One then introduces type II couplings, obtaining in this way the familiar form of the system of coupled and damped oscillators

$$\ddot{q}_{a_1} + b_{a_1a_2}\dot{q}_{a_2} + c_{a_1a_2}q_{a_2} = 0, \qquad b_{a_1a_2} = \text{const}. \tag{3.45}$$

This implementation, however, is still insufficient because, as is well known in the theory of coupled oscillators, type III couplings also occur. One then obtains the familiar form of the linear equations of motion for coupled oscillators

$$\begin{aligned} &a_{a_1a_2}\ddot{q}_{a_2} + b_{a_1a_2}\dot{q}_{a_2} + c_{a_1a_2}q_{a_2} = 0, \\ &a_{a_1a_2} = \text{const}, \qquad \det(a_{a_1a_2}) \neq 0. \end{aligned} \tag{3.46}$$

Notice that the "acceleration couplings" occur because the off-diagonal as well as the diagonal terms of the matrix $(a_{a_1a_2})$ are generally nonnull.

Equations (3.46) still constitute an approximation of the physical reality due to the linear nature of the considered couplings as well as the constancy of the elements $a_{a_1a_2}$, $b_{a_1a_2}$, and $c_{a_1a_2}$. And indeed, Eqs. (3.46), as is well known, are customarily valid only for the case of small oscillations.

The removal of the linearity of couplings I and II and of the constancy of the coefficients then brings Eqs. (3.46) into the fundamental form of Newton's equations of motion

$$A_{a_1a_2}(t, q, \dot{q})\, \ddot{q}_{a_2} + B_{a_1}(t, q, \dot{q}) = 0, \tag{3.47}$$

where external forces, which are essential to preserve the motion for the desired period of time, can now be included in the B terms.

Even though they are somewhat hidden, the three indicated classes of couplings are still present in the general structure (3.47). And indeed, the coordinate and velocity couplings are represented by the q and $\dot{q}$ dependence, respectively, of the $A_{a_1a_2}$ and B terms, while the acceleration couplings are represented by the nonnull values of the off-diagonal elements of the matrix $(A_{a_1a_2})$. The central difference with Eqs. (3.46) is that for the general form (3.47) the coordinates and velocity couplings are not necessarily linear. However, the acceleration couplings are always linear in the accelerations in order to preserve the Newtonian structure of the equations of motion. For an elaboration of this point see [5].

The point which we would like to stress is that, irrespective of our interpretation and classification of the forms of couplings, when an accurate description of the physical reality is needed, structures of type (3.44) and (3.45) must be abandoned and the fundamental form (3.47) of the equations of motions must be adopted.

At this point one can argue that, under the condition of regularity, i.e.,

$$\det(A_{a_1a_2}) \neq 0, \tag{3.48}$$

Eqs. (3.4) can always be reduced to the semilinear form

$$\ddot{q}_a - f_a(t, q, \dot{q}) = 0, \tag{3.49}$$

where the implicit functions f_a are trivially given by

$$f_a = -A_{ab}^{-1} B_b\,, \qquad (A_{ab}^{-1}) \equiv (A_{ab})^{-1}. \tag{3.50}$$

Thus, the acceleration couplings are not essential to represent the motion.

It is precisely in this respect that our analysis of the necessary and sufficient conditions for the existence of a Lagrangian becomes crucial.

And indeed, the statement that class $\mathscr{C}^2$ regular systems (3.47) can always be reduced to form (3.49) is true from the Theorem on Implicit Functions. Similarly, the statement that Eqs. (3.49) without acceleration couplings can equivalently represent the motion is equally true.

However, within the framework of a Lagrangian representation of the equations of motion the situation is considerably different. Indeed, Newton's equations of motion in the form (3.49) *are* non-self-adjoint (unless trivial forms of couplings are considered) and therefore a Lagrangian for their ordered direct analytic representation *does not*, in general, exist.

This leaves, as one possibility, the study of the equivalence transformations of Eqs. (3.49) which preserve continuity, regularity, and linearity in the acceleration and which induce a self-adjoint structure, i.e., transformations of the type [5]

$$\{h_{a_1 a_2}(t, q, \dot{q})[\ddot{q}_{a_2} - f_a]^{\mathscr{C}^2, R}_{\text{NSA}}\}^{\mathscr{C}^2, R}_{\text{SA}} = 0. \tag{3.51}$$

If one equivalence transformation of this type exists, then a Lagrangian exists.

The point which we want to stress is that the ultimate effect of the "integrating factors" $h_{a_1 a_2}$ of Eqs. (3.51) is that of reproducing equations of motion of type (3.47) *with* acceleration couplings.

In conclusion, while the acceleration couplings are usually not essential to represent the motion under arbitrary Newtonian forces within the context of the theory of ordinary differential equations, they are vital within the context of their Lagrangian representation. And indeed, if they are ignored within this latter context, the net effect is a considerable simplification of the form of couplings. In turn, this ultimately reduces the type of admissible forces to those derivable from an additive potential function.

From the viewpoint of the structure of the Lagrangian capable of representing interacting Newtonian systems, the elimination of acceleration couplings inevitably implies the use of the conventional structure (3.38). The point here is, again, that the Lagrange equations in Lagrangian (3.38) are indeed of type (3.49). However, only a special subclass of Eqs. (3.49) admits a direct Lagrangian representation and when it does, it often constitutes only an approximation of the physical reality.

These remarks suggest the problem initially posed, namely, which is a general form of modification of the Lagrangian for the free motion capable of representing interacting systems with arbitrary forms of coupling, or, equivalently, with arbitrary Newtonian forces.

A systematic analysis of the Newtonian counterpart of this problem is conducted in [5]. In essence, when a Newtonian system is interacting according to a combination of couplings I, II, and III, the total Lagrangian contains $(n + 1)$ interacting terms, n multiplicative terms, and one additive term to the Lagrangian for the free motion according to the structure

$$L^{\text{Gen}}_{\text{Tot}}(t, q, \dot{q}) = \sum_{1}^{n}{}_a L^{(a)}_{\text{Int,I}}(t, q, \dot{q})\, L^{(a)}_{\text{Free}}(\dot{q}) + L_{\text{Int,II}}(t, q, \dot{q}). \tag{3.52}$$

Alternatively, whenever some of the acting forces are not derivable from a potential function, the motion can be represented with multiplicative as well as additive interaction terms to the kinetic energy according to structure (3.52).

Again, the $(n+1)$ interaction terms are nontrivial when the Lagrange equations in $\sum_1^n{}_a L_{\text{Free}}^{(a)}$ and $L_{\text{Tot}}^{\text{Gen}}$ *are not* equivalent.

As a trivial example in one dimension, the system with damping

$$\ddot{q} + b\dot{q} = 0, \qquad b = \text{const} \tag{3.53}$$

does not admit a direct analytic representation, and a Lagrangian of type (3.38) *does not* exist because the force is not derivable from a potential. And indeed, Eq. (3.53) is non-self-adjoint.

However, a Lagrangian representation for the equivalent system

$$e^{bt}(\ddot{q} + b\dot{q}) = 0 \tag{3.54}$$

exists and is given by the generalized structure (3.52), i.e.,

$$\begin{aligned} L_{\text{Tot}}^{\text{Gen}} &= L_{\text{Int,I}} L_{\text{Free}}, \\ L_{\text{Free}} &= \tfrac{1}{2}\dot{q}^2, \qquad L_{\text{Int,I}} = e^{bt}. \end{aligned} \tag{3.55}$$

Indeed, the equation of motion in the equivalent form (3.54) is self-adjoint.

More generally, the conventional Lagrangian structure (3.38) induces equations of motion of type (3.49) while the generalize structure (3.52) induces the general form (3.47) of Newton's equations of motion.

In ultimate analysis the presence of the multiplicative interaction term in structure (3.52) is intimately linked to the presence of the acceleration couplings of Eqs. (3.47). It is precisely this feature which allows the Lagrangian representation of forces which are not necessarily derivable from a potential.

In relation to the corresponding problem in Field Theory we shall here limit ourselves to only a few remarks. We want also to stress that such remarks must be considered of a conjectural nature. This is due to the apparaently limited knowledge we have at this time of the physical significance of more general forms of couplings as well as of the technical implications of the problem of their quantization.

On formal grounds, all the remarks made above within a Newtonian context directly extend to (classical) Field Theories.

The conventional structure (3.40) of the total Lagrangian density is, in ultimate analysis, a direct consequence of the restriction of the field equations to a semilinear self-adjoint form. This is equivalent to the assumption that the acting forces of all admissible couplings must be derivable from an (additive) interaction term.

If, however, this restriction is lifted, the total Lagrangian density with the conventional structure (3.40) *fails* to be acceptable. This point can be seen in several ways. First of all, one can easily see that structure (3.40) *cannot* represent the field equations in their general form, i.e., the quasi-linear form (3.1) which is, ultimately, the field theoretical version of the general Newtonian form (3.47).

Alternatively, if one reduces the quasi-linear form to a semilinear one by using the Implicit Function Theorem, then this new form is non-self-adjoint (unless the couplings are, again, trivial) and, as such, a Lagrangian for their ordered direct analytic representation does not exist.

In this case one must seek for an indirect representation of the type

$$\left[d_\mu \frac{\partial \mathscr{L}}{\partial \phi^{a_1;}{}_{\mu}} - \frac{\partial \mathscr{L}}{\partial \phi^{a_1}}\right]_{\text{SA}}^{\mathscr{C}^2, R} \equiv \{h_{a_1}{}^{a_2}(x_\alpha, \phi^a, \phi^{a;}{}_\alpha)[g^{\mu_1\mu_2}\phi_{a_2}{}^{;}{}_{\mu_1\mu_2} - f_a]_{\text{NSA}}^{\mathscr{C}^2, R}\}_{\text{SA}}^{\mathscr{C}^2, R} = 0. \tag{3.56}$$

These representations will be studied in paper III.

Again, the net effect of the "integrating factors" $h_{a_1}{}^{a_2}$ is that of restoring the quasi-linear form (3.1), i.e., of restoring the "acceleration couplings."

The above situations can also be seen directly from the conditions of self-adjointness. Indeed, the conditions of self-adjointness for the semilinear form (3.39) admit a nonlinearity in the field ϕ^a, but they prohibit a nonlinearity in their derivatives $\phi^{a;}{}_\mu$, by thus restricting the forms of coupling. On the contrary, the conditions of self-adjointness of the quasi-linear form (3.1) *do allow* a nonlinearity in both the field components and their derivatives as stressed in [1].

Thus, the quasi-linear form (3.1) of the field equations allows any class $\mathscr{C}^2$ and regular combination of (I) generally nonlinear couplings in the field components ϕ^a, (II) generally nonlinear couplings in the field "velocities" $\phi^{a;}{}_\mu$, (III) linear couplings in the field "acceleration" $\phi^{a;}{}_{\mu\nu}$. This is precisely equivalent, on conceptual grounds, to the corresponding Newtonian framework.

On physical grounds, however, a problematic aspect appears. It is well known that the class of Newtonian forces is, in general, nonconservative and that only one of their subclasses is derivable from a potential. Therefore, the physical significance of the inclusion of nonconservative forces in a Lagrangian representation, at a Newtonian level, is self-evident.

The corresponding situation within the field theoretical framework is, however, different. Indeed, the physical significance of the field theoretical interactions whose Newtonian limits are nonconservative, to the best knowledge of this author, is not known at this time and even the necessary methodology to study their implications (e.g., for symmetries and conservation laws) is not fully explored.

In this respect it has been surprising and gratifying to this author to see that the ultimate nature of the interactions in gauge invariant theories is precisely of this more general type. This is clearly exhibited by the analysis of Section 2, where the Newtonian limit of the gauge equations (2.40) is given by Eqs. (2.34). It is easy to identify in this system the combination of two forces, i.e.,

(I) a conservative force $\mathbf{F}^{C} \equiv (kq_1, kq_2)$ with corresponding linear, type I coupling and (II) a nonconservative force $\mathbf{F}^{NC} \equiv (b\dot{q}_1, -b\dot{q}_2)$ with corresponding linear, type II coupling.

It is at this point rather tempting to say that, perhaps, the success of the gauge theories in the unification of the electromagnetic and weak interaction is also due to these more general forms of admissible couplings.

It is equally tempting to say that the failure until now of the unified gauge theories to incorporate strong interactions might be due to the need for further generalizations of the forms of couplings.

Notice in this respect that, despite the generalized form of couplings, the "acceleration couplings" are still absent in the gauge theories. This is transparently exhibited by the equations of motion in the self-adjoint form (3.28) and it is due to the fact that the diagonal element of the $(A^{\mu_1\mu_2}_{a_1a_2})$ matrix of Eqs. (3.28) are null. Nevertheless, as stressed earlier such equations are of a simple but true quasi-linear type (3.1). And indeed, the corresponding *semilinear* form (2.40) is non-self-adjoint.

It then follows that the physical significance of coupling types I and II in Field Theory within the context of the quasi-linear form of the field equations might be acceptable on the grounds of the structure of the gauge invariant theories and the success of their applications.

The last question which remains is, what is the physical significance of the inclusion in Field Theory of the third form of couplings, namely, that of the "acceleration couplings." The answer to this question is provided by the chiral type of field theories. Indeed, the field equations of type (3.34) do represent "acceleration" (as well as nonlinear "velocity" and "coordinate") couplings unless the matrix $(G_{a_1a_2})$ is trivial, i.e., it is the identity or a permutation as in the gauge theories.

And indeed, the chiral Lagrangian (3.37) exhibits interaction terms which are "multiplicative" as well as "additive" to the free term. This is precisely along the lines of the generalized structure (3.52) of the total Lagrangian for an arbitrary Newtonian system.

With an open mind to the above remarks, we are now equipped to derive a generalized form of the Lagrangian in (classical) Field Theory within the context of Theorem 2.1

COROLLARY 2.1E. *A general structure of the total Lagrangian density for an ordered analytic representation of Lorentz-covariant, tensorial, quasi-linear system of coupled field equations*

$$A^{\mu_1\mu_2}_{a_1a_2}(x_\mu, \phi^a, \phi^{a;}{}_\mu)\, \phi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1}(x_\mu, \phi^a, \phi^{a;}{}_\mu) = 0, \qquad a_1 = 1, 2, \dots, n, \quad (3.57)$$

which are of (at least) class $\mathscr{C}^2$, *regular, and self-adjoint in a star-shaped* *region R^* of points $(x_\mu, \phi^a, \phi^{a;}{}_\mu)$ is characterized by $(n+1)$ interaction terms: n multiplicative and one additive term to the Lagrangian density for the free fields, according to the structure*

$$\mathscr{L}^{\text{Gen}}_{\text{Tot}}(x_\alpha, \phi^b, \phi^{b;}{}_\alpha) = \sum_{1}^{n}{}_a \mathscr{L}^{(a)}_{\text{Int,I}}(x_\alpha, \phi^b, \phi^{b;}{}_\alpha)\, \mathscr{L}^{(a)}_{\text{Free}}(\phi^a, \phi^{a;}{}_\alpha) + \mathscr{L}_{\text{Int,II}}(x_\alpha, \phi^b, \phi^{b;}{}_\alpha), \quad (3.58)$$

where the terms $\mathscr{L}^{(a)}_{\text{Free}}$, $\mathscr{L}^{(a)}_{\text{Int,I}}$, and $\mathscr{L}_{\text{Int,II}}$ admit the decompositions

$$\begin{aligned}
\mathscr{L}^{(a)}_{\text{Free}} &= \tfrac{1}{2}(\phi_a{}^{;\mu}\phi^{a;}{}_\mu - m^2(a)\,\phi_a\phi^a),\\
\mathscr{L}^{(a)}_{\text{Int,I}} &= K^{(a)}_{\text{I}}(x_\alpha, \phi^b, \phi^{b;}{}_\alpha) + D^{(a)\mu}_{\text{I}}{}_{a_1}(x_\alpha, \phi^b)\,\phi^{a_1;}{}_\mu + C^{(a)}_{\text{I}}(x_\alpha, \phi^b), \quad (3.59)\\
\mathscr{L}_{\text{Int,II}} &= K_{\text{II}}(x_\alpha, \phi^b, \phi^{b;}{}_\alpha) + D_{\text{II}}{}^\mu_{a_1}(x_\alpha, \phi^b)\,\phi^{a_1;}{}_\mu + C_{\text{II}}(x_\alpha, \phi^b),
\end{aligned}$$

and they can be expressed in terms of the solutions of Eqs. (3.3) *by means of the identifications*

$$\begin{aligned}
K &= \sum_{1}^{n}{}_a \tfrac{1}{2}K^{(a)}_{\text{I}}\phi_a{}^{;\mu}\phi^{a;}{}_\mu - \sum_{1}^{n}{}_a K^{(a)}_{\text{I}}m^2(a)\,\phi^a\phi_a\\
&\quad + \sum_{1}^{n}{}_{a,b} \tfrac{1}{2}D^{(a)\mu}_{\text{I}}{}_b\phi^{b;}{}_\mu\phi_a{}^{;\alpha}\phi^{a;}{}_\alpha + \sum_{1}^{n}{}_a \tfrac{1}{2}C^{(a)}_{\text{I}}\phi_a{}^{;\alpha}\phi^{a;}{}_\alpha + K_{\text{II}},\\
D^\mu_a &= -\sum_{1}^{n}{}_b \tfrac{1}{2}m^2(b)\,\phi^b\phi_b D^{(b)\mu}_{\text{I}}{}_a + D_{\text{II}}{}^\mu_a,\\
C &= -\sum_{1}^{n}{}_a \tfrac{1}{2}m^2(a)\,\phi^a\phi_a C^{(a)}_{\text{I}} + C_{\text{II}}.
\end{aligned} \quad (3.60)$$

It should be indicated that the generalized structure (3.58) is different than the chiral structure (3.37). The relationship between these two structures demands the study of the equivalent forms of writing structure (3.58). This problem will be investigated in a subsequent paper. Within this context we shall also show the property indicated in [1], namely, that the generalized structure (3.58) can also represent, in an equivalent way, *semilinear* systems, or, alternatively, that the systems which admit the *conventional* structure (3.40) of the total Lagrangian can equivalently be represented, under certain circumstances, with the *generalized* structure (3.58). This should reinforce the representational capabilities of structure (3.58).

A point which we would like here to stress is that the structure of the total Lagrangian (3.58), when it holds, is not necessarily unique, and other forms are conceivable. Essentially, structure (3.58) is *one* practical way of representing interacting systems with arbitrary forms of couplings as specified above.

As a purely formal example, consider the single, regular, tensorial field equation

$$\begin{aligned} F(\varphi) \equiv & [\tfrac{1}{2}G_1^2(\varphi^{;}{}_{\mu}\varphi^{;\mu} - m^2\varphi^2 + G_2\varphi^4) + 2G_1]\, \varphi^{;\mu}\varphi^{;\nu}\varphi^{;}{}_{\mu\nu} \\ & + [1 + \tfrac{1}{2}G_1(\varphi^{;\mu}\varphi^{;}{}_{\mu} - m^2\varphi + G_2\varphi^4)]\, \Box\varphi \\ & + G_1(-m^2\varphi + 2G_2\varphi^3)\, \varphi^{;\mu}C^{;}{}_{\mu} + m^2\varphi - 2G_2\varphi^3 = 0. \end{aligned} \tag{3.61}$$

The above equation is non-self-adjoint because it violates condition (1.5d). However, if we multiply the equation by the factor term $\exp(G_1\varphi^{;\mu}\varphi^{;}{}_{\mu})$, the new equation becomes self-adjoint. Our Theorem 2.1 can now be applied to this equivalent system. Corollary 2.1B and 2.1E then give the Lagrangian density of the generalized type (3.58), i.e.,

$$\begin{aligned} \mathscr{L}_{\text{Tot}} &= \mathscr{L}_{\text{Int,I}}\mathscr{L}_{\text{Free}} + \mathscr{L}_{\text{Int,II}}, \\ \mathscr{L}_{\text{Free}} &= \tfrac{1}{2}(\varphi^{;\mu}\varphi^{;}{}_{\mu} - m^2\varphi^2), \\ \mathscr{L}_{\text{Int,I}} &= \exp(G_1\varphi^{;\mu}\varphi^{;}{}_{\mu}), \\ \mathscr{L}_{\text{Int,II}} &= \tfrac{1}{2}G_2\varphi^4 \exp(G_1\varphi^{;}{}_{\mu}\varphi^{;\mu}). \end{aligned} \tag{3.62}$$

Notice that the equation is covariant and such that

$$\lim_{G_1, G_2 \to 0} F(\varphi) = (\Box + m^2)\, \varphi = 0. \tag{3.63}$$

Notice also that the reduction of Eq. (3.61) to the conventional form (3.9) *is not* trivial due to the presence of the terms with the factor $\varphi^{;\mu}\varphi^{;\nu}\varphi^{;}{}_{\mu\nu}$. This aspect, which was pointed out in [1], seems to indicate that the quasi-linear form of the field equations, unlike the Newtonian case, is a bona fide generalization of the semilinear form.

At this time we shall leave open for the interested reader the formulation of other examples.

It must be stressed, as indicated earlier, that the generalized structure (3.58) of the total Lagrangian density is derived on entirely formal grounds and presented on a purely conjectural basis.

This is due to the fact that the actual use in quantum field theory of this generalized structure with nontrivial multiplicative interaction terms demands the inspection of a rather large variety of methodological aspects, such as, the problem of the symmetrization, the problem of the equivalence of the (quantum-mechanical) Lagrange and Hamiltonian formulations, the problem of renormalization, the behavior of the Feynman diagrams, etc. Besides, all these problems, to the best knowledge of this author, have been only partially explored or not explored at all at this time.

It is, however, significant to note that, within the framework of Newtonian Mechanics, the structure of a Lagrangian capable of representing systems with arbitrary Newtonian forces must necessarily be of the generalized type (3.52) (or of some equivalent type). The analysis of this paper indicates that, within the framework of classical field theory, generalized structures of the Lagrangian density are also admissible. This seems to suggest that, despite the several technical problems which are still open, a generalized structure of the Lagrangian could be significant for quantum field theories, too.

APPENDIX A: Elements of the Calculus of Differential Forms on Minkowski Space

Our Theorem 2.1 on the necessary and sufficient conditions for the existence of a Lagrangian density for quasi-linear systems of tensorial fields is ultimately based on the property that the conditions of self-adjointness (1.5) are the integrability conditions for the overdetermined system of partial differential equation (3.3) in the unknown Lagrangian $\mathscr{L}$.

As is well known, one of the most effective frameworks for studying the integrability conditions is the calculus of differential forms in general, and the Converse of the Poincaré Lemma in particular [7].

An inspection of Eqs. (3.3) indicates the presence of partial derivatives with respect to a set of n fields $\phi^a(x_\alpha)$, $(a = 1, 2,..., n, \alpha = 0, 1, 2, 3)$, as well as their derivatives on the Minkowski coordinates $\phi^{a;}{}_\mu = \partial\phi^a/\partial x^\mu$ $(a = 1, 2,..., n,\ \mu = 0, 1, 2, 3)$.

The handling of this system within the framework of the calculus of differential forms demands the interpretation of both the fields ϕ^a and their derivatives $\phi^{a;}{}_\mu$ as local coordinates of an n-dimensional and a $4n$-dimensional differentiable manifold m_n and m_{4n} respectively.

The methodology for the Converse of the Poincaré Lemma for the manifold m_n with local coordinates ϕ^a is fully established [7]. However, its extension to the manifold m_{4n} with local coordinates $\phi^{a;}{}_\mu$ is nontrivial due to the appearance of the additional Minkowski index μ and, to the best knowledge of this author, has not been worked out until now.

In this Appendix we shall first review, for the reader's convenience, some basic aspects of the calculus of ordinary differential forms in the local coordinates ϕ^a, according to the presentation by Lovelock and Rund [7], and then explore their generalization to the case of local coordinates $\phi^{a;}{}_\mu$. For notational convenience we shall ignore the nature of the terms $\phi^{a;}{}_\mu$ and simply write

$$\phi_\mu{}^a \equiv \phi^{a;}{}_\mu = (\partial\phi^a/\partial x^\mu). \tag{A.1}$$

Let m_n be a differentiable manifold with local coordinates ϕ^a ($a = 1, 2, \ldots, n$). A generic tensor on m_n with r-contravariant and s-covariant indices will be written $T^{a_1 \cdots a_r}_{b_1 \cdots b_s}$ and termed a tensor of type (r, s).

The (0, 0)- tensor (i.e., scalar)

$$A^{(1)} = A_a d\phi^a \tag{A.2}$$

induced by the contraction of an infinitesimal displacement $d\phi^a$ of type (1, 0) and a (0, 1) tensor A_a is termed a *Pfaffian* or 1-*form*.

The *addition* of 1 forms is done with the conventional rule, e.g.,

$$A^{(1)} + B^{(1)} = (A_a + B_a)\, d\phi^a. \tag{A.3}$$

The multiplication of 1 forms, however, demands a new concept, termed *exterior product* and often denoted with the symbol $\wedge$, which preserves the distributive law of ordinary multiplication but obeys the anticommutativity rather than the commutativity law, according to the scheme

$$\begin{aligned} A^{(1)} \wedge B^{(1)} &= (A_a^{(1)}\, d\phi^a) \wedge (B_b^{(1)}\, d\phi^b) = A_a B_b\, d\phi^a \wedge d\phi^b \\ &= -A_a B_b\, d\phi^b \wedge d\phi^a = \tfrac{1}{2}(A_a B_b - B_b A_a)\, d\phi^a \wedge d\phi^b. \end{aligned} \tag{A.4}$$

The structure emerging from the above product is termed a 2 *form*. Repeated use of the exterior product then induces the so-called *p forms*, which, in general, are scalars characterized by a $(0, p)$ tensor $A_{a_1 \cdots a_p}$ contracted with the antisymmetric $(p, 0)$ tensor $d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$, i.e.,

$$A^{(p)} \equiv A_{a_1 \cdots a_p}\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}. \tag{A.5}$$

The ordinary partial derivatives $\partial/\partial\phi^a$ are now generalized into the so-called *exterior derivatives* which, for the case of *p forms* $A^{(p)}$ are often written $dA^{(p)}$ and characterize a scalar $(p + 1)$ form according to the rule

$$dA^{(p)} \equiv \frac{\partial A_{a_1 \cdots a_p}}{\partial \phi^b}\, d\phi^b \wedge d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}. \tag{A.6}$$

Similarly, the exterior derivative $d(dA^{(p)})$ of the $(p + 1)$ form $dA^{(p)}$ is the $(p + 2)$ form

$$d(dA^{(p)}) = \frac{\partial^2 A_{a_1 \cdots a_p}}{\partial \phi^c\, \partial \phi^b}\, d\phi^c \wedge d\phi^b \wedge d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}. \tag{A.7}$$

As we shall see better later on, the methodology which underlies the integrability conditions is based on the necessary and sufficient conditions for cerain p forms and their exterior derivatives to be identically null. This objective is achieved by introducing the so-called *generalized Kronecker delta*, which is a tensor of type (p, p) defined by the $(p \times p)$ determinant

$$\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} \equiv \begin{vmatrix} \delta^{a_1}_{b_1} & \cdots & \delta^{a_1}_{b_n} \\ & \cdots & \\ \delta^{a_n}_{b_1} & \cdots & \delta^{a_n}_{b_n} \end{vmatrix}. \tag{A.8}$$

For $p = 1$ one recovers the ordinary Kronecker delta $\delta^{a_1}_{b_1}$. However, for $p = 2$ one has the generalized form

$$\delta^{a_1 a_2}_{b_1 b_2} = \delta^{a_1}_{b_1}\delta^{a_2}_{b_2} - \delta^{a_1}_{b_2}\delta^{a_2}_{b_1}, \tag{A.9}$$

and similarly for $p = 3, 4, \ldots, n$. For $p > n$ the generalized δ is identically null because at least some of the indices must in this case coincide.

It is useful here to recall the "normalization property" [7]

$$\delta^{a_1 \cdots a_p}_{a_1 \cdots a_p} = \frac{n!}{(n-p)!}, \tag{A.10}$$

where summation of repeated indices is understood, and the "realization" in terms of the contravariant and covariant Levi–Civita tensor [7]

$$\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} = \epsilon_{b_1 \cdots b_p} \epsilon^{a_1 \cdots a_p}. \tag{A.11}$$

A central property of the tensor $\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p}$ is that it is antisymmetric under the exchange of any two of either the contravariant or the covariant indices. This renders it an effective tool to represent the structure of the p forms and their exterior derivatives.

Indeed the following property identically holds:

$$d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p} \equiv (1/p!)\, \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p}\, d\phi^{b_1} \wedge \cdots \wedge d\phi^{b_p}. \tag{A.12}$$

As a result, the p form (A.5), its first exterior derivative (A.6), and its second exterior derivative (A.7) can be written, respectively,

$$A^{(p)} \equiv (1/p!)\, \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} A_{a_1 \cdots a_p}\, d\phi^{b_1} \wedge \cdots \wedge d\phi^{b_p}, \tag{A.13a}$$

$$dA^{(p)} \equiv \frac{(-1)^p}{(p+1)!}\, \delta^{a_1 \cdots a_{p+1}}_{b_1 \cdots b_{p+1}} \frac{\partial A_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}}\, d\phi^{b_1} \wedge \cdots \wedge d\phi^{b_{p+1}}, \tag{A.13b}$$

$$d(dA^{(p)}) \equiv \frac{(-1)^p\,(-1)^{p+1}}{(p+1)!\,(p+2)!}\, \delta^{a_1 \cdots a_{p+2}}_{b_1 \cdots b_{p+2}} \frac{\partial^2 A_{a_1 \cdots a_p}}{\partial \phi^{a_{p+2}}\, \partial \phi^{a_{p+1}}}\, d\phi^{b_1} \wedge \cdots \wedge d\phi^{b_{p+2}}. \tag{A.13c}$$

It then follows that the necessary and sufficient condition for a p form $A^{(p)}$ or its exterior derivative $dA^{(p)}$ to be identically null is that the corresponding conditions

$$\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} A_{a_1 \cdots a_p} = 0, \tag{A.14a}$$

$$\delta^{a_1 \cdots a_{p+1}}_{b_1 \cdots b_{p+1}} (\partial A_{a_1 \cdots a_p} / \partial \phi^{a_{p+1}}) = 0, \tag{A.14b}$$

$$b_1, \ldots, b_{p+1} = 1, 2, \ldots, n$$

individually hold. It also follows that the second order exterior derivative $d(dA^{(p)})$ i.e., Eq. (A.13c), is always identically null provided that the $(0, p)$ tensor $A_{a_1 \cdots a_p}$ is of at least class $\mathscr{C}^2$ in the local coordinates ϕ^a. This is due to the symmetry of the two partial derivatives in $\phi^{a_{p+1}}$ and $\phi^{a_{p+2}}$ of Eq. (A.13c).

Conditions (A.14) will play a crucial role in our analysis. It should be indicated that those conditions are the generalization to an order p of the known property that the necessary and sufficient condition for a $(0, 2)$ tensor $A_{a_1 a_2}$ to possess an identically null contraction with the total antisymmetric tensor $d\phi^{a_1} \wedge d\phi^{a_2}$ is that it is totally symmetric, i.e., the 2 form

$$\begin{aligned} A^{(2)} &= A_{a_1 a_2}\, d\phi^{a_1} \wedge d\phi^{a_2} = \tfrac{1}{2}(A_{a_1 a_2} - A_{a_2 a_1})\, d\phi^{a_1} \wedge d\phi^{a_2} \\ &= \tfrac{1}{2} \delta^{a_1 a_2}_{b_1 b_2} A_{a_1 a_2}\, d\phi^{b_1} \wedge d\phi^{b_2} = \sum_{a_1 < a_2} (A_{a_1 a_2} - A_{a_2 a_1})\, d\phi^{a_1} \wedge d\phi^{a_2} \end{aligned} \tag{A.15}$$

is identically null iff

$$\delta^{a_1 a_2}_{b_1 b_2} A_{a_1 a_2} \equiv A_{a_1 a_2} - A_{a_2 a_1} = 0. \tag{A.16}$$

This property can be alternatively seen from the linear independence of the terms $d\phi^{a_1} \wedge d\phi^{a_2}$ for $a_1 < a_2$.

We shall now explore the generalization of the above framework to the case of local coordinates $\phi_\mu{}^a \equiv \partial \phi^a / \partial x^\mu$.

Consider a $4n$-dimensional differentiable manifold m_{4n} with local coordinates $\phi_\mu{}^a$ $(a = 1, 2, \ldots, n, \mu = 0, 1, 2, 3)$. A generic tensor in m_{4n} will now possess r_1 contravariant indices $a_1, \ldots, a_{r_1}$ and s_1 covariant indices $b_1, \ldots, b_{s_1}$, as well as r_2 contravariant indices $\mu_1, \ldots, \mu_{r_2}$ and s_2 covariant indices $\nu_1, \ldots, \nu_{s_2}$. Such a tensor will be written $T^{a_1 \cdots a_{r_1} \mu_1 \cdots \mu_{r_2}}_{b_1 \cdots b_{s_1} \nu_1 \cdots \nu_{s_2}}$ and termed a tensor of type $(r_1, r_2; s_1, s_2)$. The above distinction is indicated by the fact that the metric tensors for raising and lowering the Latin and Greek indices *do not* in general coincide. For instance, if one considers the case of a scalar field $\varphi(x)$ and its complex conjugated $\bar{\varphi}(x)$ then $n = 2$, $\phi^1 \equiv \varphi = \delta^{1a} \phi_a$ and $\phi^2 \equiv \bar{\varphi} = \delta^{2a} \phi_a$, while $\phi_\mu{}^1 \equiv \varphi^{\centerdot}{}_\mu = g_{\mu\nu} \phi^{1\nu}$ and $\phi_\mu{}^2 \equiv \bar{\varphi}^{\centerdot}{}_\mu = g_{\mu\nu} \phi^{2\nu}$; namely, the metric tensor for the Latin indices is, in this case, the Kronecker delta δ_{ab}, while that for the Greek indices is the Mindowski metric $g^{\mu\nu}$. It should, however, be indicated that the case when the metric tensors for both the

Latin and Greek indices coincide is not excluded. This is the case for the electromagnetic field $\phi^\mu \equiv A^\mu(x)$ and its derivatives $\phi_\nu{}^\mu \equiv A^{\mu;}{}_\nu$ or for any vector field on a Minkowski space.

Finally, it should be stressed that our analysis is restricted to field theories on a Minkowski space and other types of field theories, e.g., those on a pseudo-Rimannian manifold, are left open for the interested reader.

Within the framework of $4n$-dimensional manifolds m_{4n} with local coordinates $\phi_\mu{}^a$, the first possibility of constructing a scalar, i.e., a (0, 0, 0, 0) tensor is through the contraction of a (0, 1; 1, 0) tensor $A_a{}^\mu$ with the (1, 0; 0, 1) displacement $d\phi_\mu{}^a$. We reach in this way the generalized (1, 1) form

$$A^{(1,1)} \equiv A_a{}^\mu d\phi_\mu{}^a. \tag{A.17}$$

The equivalent case of type $A_\mu{}^a d\phi_a{}^\mu$ will hereafter be ignored. The addition of two (1, 1) forms can agains be performed with the ordinary rule. The *exterior product* of two generalized (1, 1) forms, however, now demands the identification of the interplay between the Latin and Greek indices in the product $d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2}$. This interplay can be seen from the expression

$$\begin{aligned} A^{(1,1)} \wedge B^{(1,1)} &= (A^{\mu_1}_{a_1}\, d\phi^{a_1}_{\mu_1}) \wedge (B^{\mu_2}_{a_2}\, d\phi^{a_2}_{\mu_2}) \\ &= A^{\mu_1}_{a_1} B^{\mu_2}_{a_2}\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \\ &= \tfrac{1}{2}(A^{\mu_1}_{a_1} B^{\mu_2}_{a_2} + A^{\mu_2}_{a_2} B^{\mu_1}_{a_1})\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \\ &= \tfrac{1}{2}(A^{\mu_1}_{a_1} B^{\mu_2}_{a_2} - A^{\mu_2}_{a_1} B^{\mu_1}_{a_2})\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \\ &= \tfrac{1}{2}(A^{\mu_1}_{a_1} B^{\mu_2}_{a_2} - A^{\mu_1}_{a_2} B^{\mu_2}_{a_1})\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \\ &= \tfrac{1}{4}(A^{\mu_1}_{a_1} B^{\mu_2}_{a_2} - A^{\mu_2}_{a_1} B^{\mu_1}_{a_2} - B^{\mu_1}_{a_2} B^{\mu_2}_{a_1} + A^{\mu_2}_{a_2} B^{\mu_1}_{a_1})\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2}, \end{aligned} \tag{A.18}$$

or, alternatively, by the self-explanatory properties

$$d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} = -d\phi^{a_1}_{\mu_2} \wedge d\phi^{a_2}_{\mu_1} = -d\phi^{a_2}_{\mu_1} \wedge d\phi^{a_1}_{\mu_2} = +d\phi^{a_2}_{\mu_2} \wedge d\phi^{a_1}_{\mu_1}, \tag{A.19}$$

where the last expression holds in view of the even number of involved permutations.

Structure (A.18) or (A.19) suggests the use of the following *extension of the generalized Kronecker delta* for the (2 + 2)-dimensional case

$$\delta^{a_1 a_2 \mu_1 \mu_2}_{b_1 b_2 \nu_1 \nu_2} \equiv \delta^{a_1 a_2}_{b_1 b_2} \delta^{\mu_1 \mu_2}_{\nu_1 \nu_2} = \begin{vmatrix} \delta^{a_1}_{b_1} \delta^{a_1}_{b_2} \\ \\ \delta^{a_2}_{b_1} \delta^{a_2}_{b_2} \end{vmatrix} \otimes \begin{vmatrix} \delta^{\mu_1}_{\nu_1} \delta^{\mu_1}_{\nu_2} \\ \\ \delta^{\mu_2}_{\nu_1} \delta^{\mu_2}_{\nu_2} \end{vmatrix} \tag{A.20}$$

with "normalization" $\delta^{a_1 a_2}_{a_1 a_2} \delta^{\mu_1 \mu_2}_{\mu_1 \mu_2} = 2!2! = 4$.

Expression (A.18) can then be written

$$A^{(1,1)} \wedge B^{(1,1)} = \tfrac{1}{4}\delta^{a_1 a_2 \mu_1 \mu_2}_{b_1 b_2 \nu_1 \nu_2} A^{\nu_1}_{a_1} B^{\nu_2}_{a_2}\, d\phi^{b_1}_{\mu_1} \wedge d\phi^{b_2}_{\mu_2} \tag{A.21}$$

and interpreted as a generalized (2, 2) form.

The extension to arbitrary (p, p) forms

$$A^{(p,p)} \equiv A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p} \tag{A.22}$$

is then immediate. Indeed, we can first introduce the tensor

$$\delta^{a_1 \cdots a_{p_1} \mu_1 \cdots \mu_{p_2}}_{b_1 \cdots b_{p_1} \nu_1 \cdots \nu_{p_2}} \equiv \delta^{a_1 \cdots a_{p_1}}_{b_1 \cdots b_{p_1}} \otimes \delta^{\mu_1 \cdots \mu_{p_2}}_{\nu_1 \cdots \nu_{p_2}}$$

$$= \epsilon_{b_1 \cdots b_{p_1}} \epsilon_{\nu_1 \cdots \nu_{p_2}} \epsilon^{a_1 \cdots a_{p_1}} \epsilon^{\mu_1 \cdots \mu_{p_2}} \tag{A.23}$$

with "normalization"

$$\delta^{a_1 \cdots a_{p_1} \mu_1 \cdots \mu_{p_2}}_{a_1 \cdots a_{p_1} \mu_1 \cdots \mu_{p_2}} = \frac{(n!)^2}{(n - p_1)!\,(n - p_2)!}\,. \tag{A.24}$$

Then the (p, p) form (A.22) can be written

$$A^{(p,p)} \equiv \frac{1}{(p!)^2}\, \delta^{a_1 \cdots a_p \mu_1 \cdots \mu_p}_{b_1 \cdots b_p \nu_1 \cdots \nu_p} A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}\, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_p}_{\mu_p}\,. \tag{A.25}$$

The necessary and sufficient condition for such a (p, p) form to be identically null is that the following conditions on the $(0, p; p, 0)$ tensor

$$\delta^{a_1 \cdots a_p \mu_1 \cdots \mu_p}_{b_1 \cdots b_p \nu_1 \cdots \nu_p} A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p} = 0,$$

$$b_1, \ldots, b_p = 1, 2, \ldots, n, \qquad \mu_1, \ldots, \mu_p = 0, 1, 2, 3 \tag{A.26a}$$

be identically verified.

The generalization of the exterior derivative (A.6) can now be written as the $(p + 1, p + 1)$ form

$$dA^{(p,p)} \equiv \frac{\partial A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}}{\partial \phi^a_\mu}\, d\phi^a_\mu \wedge d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}$$

$$= \frac{1}{[(p + 1)!]^2}\, \delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \frac{\partial A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}}}\, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_{p+1}}_{\mu_{p+1}}\,. \tag{A.26b}$$

Notice the disappearance of the factor $(-1)^p$, which is due to the fact that the permutation of $d\phi^{a_i}_{\mu_i}$ with $d\phi^{a_j}_{\mu_j}$ is now always *even*.

The necessary and sufficient condition for derivative (A.26) to be identically null is that

$$\delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \frac{\partial A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}}} = 0,$$

$$b_1, \ldots, b_{p+1} = 1, 2, \ldots, n, \qquad \mu_1, \ldots, \mu_{p+1} = 0, 1, 2, 3 \qquad \text{(A.27)}$$

be identically verified.

The second exterior derivative $d(dA^{(p,p)})$ can now be written

$$\begin{aligned} d(dA^{(p,p)}) &\equiv \frac{\partial^2 A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}}{\partial \phi^c_\beta \, \partial \phi^b_\alpha} d\phi^c_\beta \wedge d\phi^b_\alpha \wedge d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p} \\ &= \frac{1}{[(p+1)!]^2 \, [(p+2)!]^2} \delta^{a_1 \cdots a_{p+2} \mu_1 \cdots \mu_{p+2}}_{b_1 \cdots b_{p+2} \nu_1 \cdots \nu_{p+2}} \\ &\quad \times \frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+2}}_{\nu_{p+2}} \, \partial \phi^{a_{p+2}}_{\nu_{p+1}}} d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_p}_{\mu_p}, \end{aligned} \qquad \text{(A.28)}$$

where the disappearance of the factor $(-1)^p (-1)^{p+1} = -1$ should be pointed out.

The first nontrivial implication of the extension of the calculus of differential forms under consideration is due to the fact that, unlike the ordinary case of form (A.13c), the second-order exterior derivative $d(A^{(p,p)})$ of a (p, p) form $A^{(p,p)}$ *is not* identically null despite the continuity assumption $A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}(\phi_\mu{}^a) \in \mathscr{C}^2(R_{\phi^a_\mu})$. This is due to the fact that this continuity assumption implies that

$$\frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+2}}_{\nu_{p+2}} \, \partial \phi^{a_{p+1}}_{\nu_{p+1}}} = \frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}} \, \partial \phi^{a_{p+2}}_{\nu_{p+2}}}, \qquad \text{(A.29)}$$

but, in general,

$$\frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+2}}_{\nu_{p+2}} \, \partial \phi^{a_{p+1}}_{\nu_{p+1}}} \neq \frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}_{\nu_{p+2}} \, \partial \phi^{a_{p+2}}_{\nu_{p+1}}}. \qquad \text{(A.30)}$$

Therefore, the necessary and sufficient condition for a second-order exterior derivative $d(dA^{(p,p)})$ to be identically null is that the conditions

$$\begin{gathered} \delta^{a_1 \cdots a_{p+2} \mu_1 \cdots \mu_{p+2}}_{b_1 \cdots b_{p+2} \nu_1 \cdots \nu_{p+2}} \frac{\partial^2 A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+2}}_{\nu_{p+2}} \, \partial \phi^{a_{p+1}}_{\nu_{p+1}}} = 0, \\ b_1, \ldots, b_{p+2} = 1, 2, \ldots, n, \\ \mu_1, \ldots, \mu_{p+2} = 0, 1, 2, 3 \end{gathered} \qquad \text{(A.31)}$$

identically hold. It is an instructive exercise for the interested reader to verify explicitly for the cases $p = 1$ and $p = 2$ that Eqs. (A.31) *are not identities* (as for the ordinary case) but *conditions* on the tensor $A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}$.

As we shall see in Appendix B, the above findings have nontrivial implications for the Poincaré Lemma and its Converse.

Another extension of the ordinary case which we shall need is characterized by (μ, p) forms of the type

$$B^{(\mu, p)} = B^{\mu}_{a_1 \cdots a_p}(\phi^a)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}, \qquad \mu = 0, 1, 2, 3. \tag{A.32}$$

Such forms can be considered as a collection of ordinary forms (A.5) for $\mu = 0, 1, 2, 3$ and, as a result, the underlying methodology is trivially equivalent to that of the ordinary case.

One could also conceive a further generalization to a manifold m_{4n+n} with local coordinates $\phi_\mu{}^a$ and ϕ^b $(a, b = 1, 2,..., n,\ \mu = 0, 1, 2, 3)$. This generalization, however, is not needed for the proof of Theorem 2.1 and, as such, it will be left open for the interested reader.

APPENDIX B. The Poincaré Lemma, Its Converse, and Their Generalizations

Let m_n be an n-dimensional differentiable manifold with local coordinates ϕ^a $(a = 1, 2,..., n)$. By following Lovelock and Rund [7] again, we recall that a generic p form on m_n

$$A^{(p)}(\phi^a) \equiv A_{a_1 \cdots a_p}(\phi^a)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p} \tag{B.1}$$

is termed *exact* when there exists a $(p-1)$ form $B^{(p-1)}$ whose exterior derivative coincides with $A^{(p)}$, i.e.,

$$A^{(p)} \equiv dB^{(p-1)} = \frac{(-1)^{p-1}}{p!}\, \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} \frac{\partial B_{a_1 \cdots a_{p-1}}}{\partial \phi^{a_p}}\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}. \tag{B.2}$$

Form (B.1) is termed *closed* when its exterior derivative is identically null, i.e.,

$$dA^{(p)} \equiv \frac{(-1)^p}{(p+1)!}\, \delta^{a_1 \cdots a_{p+1}}_{b_1 \cdots b_{p+1}} \frac{\partial A_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}}\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_{p+1}} = 0,$$
$$b_1, ..., b_{p+1} = 1, 2, ..., n. \tag{B.3}$$

The conventional Poincaré Lemma simply states the property pointed out in Appendix A, according to which the second-order exterior derivative of a p form with class $\mathscr{C}^2$ $(0, p)$ tensors is identically null; i.e., POINCARÉ LEMMA. *Every p form $A_{a_1 \cdots a_p} d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$ on an n-dimensional differentiable manifold m_n with local coordinates ϕ^a ($a = 1, 2, \ldots, n$) which is exact and whose (0, p) tensor $A_{a_1 \cdots a_p}(\phi^a)$ is of at least class $\mathscr{C}^1$ in a region R_n of points ϕ^a, is closed in m_n.*

Alternatively, we can write

$$dA^{(p)} \equiv d(dB^{(p-1)}) = 0. \tag{B.4}$$

The extension to a collection of p forms of the (μ, p) type (see Appendix A), i.e.,

$$A^{(\mu,p)} \equiv A^{\mu}_{a_1 \cdots a_p}(\phi^a)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}, \qquad \mu = 0, 1, 2, 3, \tag{B.5}$$

is then immediate. Indeed, we trivially have that, when the assumptions of the Poincaré Lemma are valid for each element of the collection $A^{(\mu,p)}$, $\mu = 0, 1, 2, 3$, then the collection is closed.

What is particularly significant for the problem of the integrability conditions is the converse of the Poincaré Lemma rather than the lemma per se.

By also following Lovelock and Rund [7], let us review the conventional case of a manifold m_n with local coordinates ϕ^a. A region $R_n{}^*$ on m_a is termed *star-shaped* when, together with a given open and connected set of points ϕ^a, all points $\phi'^a \equiv \tau\phi^a$, $0 \leqslant \tau \leqslant 1$, are also contained in $R_n{}^*$. Notice that this region contains the (local) origin $\phi^a = 0$.

We now assume that all considered p forms (B.1) are defined and of at least class $\mathscr{C}^1$ on a star-shaped region $R_n{}^*$, and introduce the operations

$$\begin{aligned} DA^{(p)} &\equiv \sum_{1}^{p}{}_r (-1)^{r-1} \left[\int_0^1 d\tau\, \tau^{p-1} A_{a_1 \cdots a_p}(\tau\phi^a)\right] \phi^{a_r} \\ &\quad \times d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_{r-1}} \wedge d\phi^{a_{r+1}} \wedge \cdots \wedge d\phi^{a_p} \\ &= \frac{1}{(p+1)!} \left[\int_0^1 d\tau\, \tau^{p-1} A_{a_1 \cdots a_p}(\tau\phi^a)\right] \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} \phi^{b_1}\, d\phi^{b_2} \wedge \cdots \wedge d\phi^{b_p}, \end{aligned} \tag{B.6a}$$

$$\begin{aligned} d(DA^{(p)}) &\equiv \sum_{1}^{p}{}_r (-1)^{r-1} \left[\int_0^1 d\tau\, \tau^{p} \frac{\partial A_{a_1 \cdots a_p}(\tau\phi^a)}{\partial\phi^b}\right] \phi^{a_r} \\ &\quad \times d\phi^b\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_{r-1}} \wedge d\phi^{a_{r+1}} \wedge \cdots \wedge d\phi^{a_p} \\ &\quad + p \left[\int_0^1 d\tau\, \tau^{p-1} A_{a_1 \cdots a_p}(\tau\phi^a)\right] d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}, \end{aligned} \tag{B.6b}$$

$$D(dA^{(p)}) \equiv -\sum_{1}^{p}{}_{r} (-1)^{r-1} \left[\int_0^1 d\tau\, \tau^p \frac{\partial A_{a_1\cdots a_p}(\tau\phi^a)}{\partial \phi^b}\right] \phi^{a_r}$$
$$\times\, d\phi^b \wedge d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_{r-1}} \wedge d\phi^{a_{r+1}} \wedge \cdots \wedge d\phi^{a_p}$$
$$+ \left[\int_0^1 d\tau\, \tau^p \frac{\partial A_{a_1\cdots a_p}(\tau\phi^a)}{\partial \phi^b}\right] \phi^b\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}. \tag{B.6c}$$

Then, for any p form which is defined and of at least class $\mathscr{C}^1$ in R^* the following identity holds:

$$d(DA^{(p)}) + D(dA^{(p)}) = \left\{\int_0^1 d\tau \frac{d}{d\tau} [\tau^p A_{a_1\cdots a_p}(\tau\phi^a)]\right\} d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$$
$$= A_{a_1\cdots a_p}\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}, \tag{B.7}$$

namely,

$$A^{(p)} \equiv d(DA^{(p)}) + D(dA^{(p)}). \tag{B.8}$$

Suppose now that the $A^{(p)}$ form is closed i.e., $dA^{(p)} = 0$. Then identity (B.8) reduces to

$$A^{(p)} = d(DA^{(p)}). \tag{B.9}$$

This established the existence of a $(p - 1)$ form

$$B^{(p-1)} \equiv DA^{(p)}, \tag{B.10}$$

which we shall term the "primitive form" of $A^{(p)}$ and which is explicitly given by Eqs. (B.6a) whenever $A^{(p)}$ is closed. This proves the

CONVERSE OF THE POINCARÉ LEMMA. *Every p form $A_{a_1\cdots a_2}(\phi^a)\, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$ on an n-dimensional differentiable manifold m_n with local coordinates ϕ^a $(a = 1, 2,\ldots, n)$ which is closed, defined, and of at least class $\mathscr{C}^1$ in a star-shaped region R_n^* of points ϕ^a is exact on R_n^*.*

On a comparative basis with respect to the Poincaré Lemma, notice the appearance of a new condition, namely, that the p form *must be defined on a star-shaped region.* This restricts the validity of the above Converse of the Poincaré Lemma to only those tensors $A_{a_1\cdots a_p}$ for which the integrals of Eqs. (B.6) are well defined.

The *integrability condition* for the existence of the $(p - 1)$ form (B.10) is that (whenever $A^{(p)}$ is defined and of at least class $\mathscr{C}^1$ in R_n^*) the form $A^{(p)}$ is closed, i.e., that condition (B.3) holds, or, alternatively, that all the conditions

$$\delta^{a_1\cdots a_{p+1}}_{b_1\cdots b_{p+1}} \frac{\partial A_{a_1\cdots a_p}}{\partial \phi^{a_{p+1}}} = 0, \qquad b_1, \ldots, b_{p+1} = 1, 2, \ldots, n \tag{B.11}$$

are everywhere identically verified in R_n^*.

It should be stressed that the $(p-1)$ form (B.10), when it exists, is not unique. And indeed the substitution

$$B^{(p-1)} \to B'^{(p-1)} = B^{(p-1)} + dC^{(p-2)} \tag{B.12}$$

leaves rules (B.9) unchanged. This property is significant for the problem of the structure of the Lagrangian density.

The extension of the Converse of the Poincaré Lemma to a collection of p forms of type (B.5) is immediate. Indeed, when the conditions of the converse hold for all elements of the collection $A^{(\mu,p)}$, $\mu = 0, 1, 2, 3$, then the collection is exact. Notice that in this case the integrability conditions (B.11) are replaced by

$$\delta^{a_1 \cdots a_{p+1}}_{b_1 \cdots b_{p+1}} \frac{\partial A^{\mu}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}} = 0, \qquad b_1, \ldots, b_{p+1} = 1, 2, \ldots, n, \quad \mu = 0, 1, 2, 3. \tag{B.13}$$

We must now study the generalization of the above methodology to the case of differentiable manifolds m_{4n} with local coordinates $\phi_\mu{}^a$ $(a = 1, 2, \ldots, n, n = 0, 1, 3)$. Indeed, as we shall see in the proof of Theorem 2.1, this generalization is essential to study the integrability conditions of the "kinetic term" of the Lagrangian which, as is well known, depend on the derivative terms $\phi_\mu{}^a \equiv \partial\phi^a/\partial x^\mu$.

Consider in this respect the (p, p) form (see Appendix A)

$$A^{(p,p)} \equiv A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\phi^a_\mu)\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}. \tag{B.14}$$

We shall again term this form *exact* when there exists a $(p-1)$ form whose exterior derivative coincides with $A^{(p,p)}$, i.e.,

$$A^{(p,p)} \equiv dB^{(p-1,p-1)} = \frac{1}{(p!)^2} \delta^{a_1 \cdots a_p \mu_1 \cdots \mu_p}_{b_1 \cdots b_p \nu_1 \cdots \nu_p} \frac{\partial B^{\nu_1 \cdots \nu_{p-1}}_{a_1 \cdots a_{p-1}}}{\partial \phi^{a_p}_{\nu_p}}\, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_p}_{\mu_p}. \tag{B.15}$$

Similarly, the (p, p) form (B.14) will be again termed *closed* when its exterior derivative is identically null, i.e.,

$$dA^{(p,p)} \equiv \frac{1}{[(p+1)!]^2} \delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \frac{\partial A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}}}\, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_{p+1}}_{\mu_{p+1}} = 0. \tag{B.16}$$

However, as pointed out in Appendix A, the second-order exterior derivative of the (p, p) forms *is not* necessarily null and, thus, the Conventional Poincaré Lemma breaks down for the above defined differentiable manifold m_{4n}.

And indeed we can state that every exact (p, p) form (A.5) on m_{4n} whose $(0, p; p, 0)$ tensor $A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}(\phi_\nu{}^a)$ is of at least class $\mathscr{C}^1$ in a region R_{4n} of points $\phi_\nu{}^a$ *is not* necessarily closed in R_{4n}.

We can, however, classify the exact (p, p) forms (B.14) as *strictly exact* and *weakly exact* depending on whether the $B^{(p-1,p-1)}$ form of Eq. (B.15) does or does not possess an identically null second-order exterior derivative, i.e.,

$$d(dB^{(p-1)}) \equiv \frac{1}{[(p+1)!]^2 (p!)^2} \delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \times \frac{\partial^2 B^{\nu_1 \cdots \nu_{p-1}}_{a_1 \cdots a_{p-1}}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}} \, \partial \phi^{a_p}_{\nu_p}} \, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_{p+1}}_{\mu_{p+1}} \lesseqgtr \begin{matrix} = 0, \\ \neq 0, \end{matrix} \tag{B.17}$$

namely,

$$\delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \frac{\partial^2 B^{\nu_1 \cdots \nu_{p-1}}_{a_1 \cdots a_{p-1}}}{\partial \phi^{a_{p+1}}_{\nu_{p+1}} \, \partial \phi^{a_p}_{\nu_p}} \lesseqgtr \begin{matrix} = 0, \\ \neq 0, \end{matrix}$$
$$b_1, \ldots, b_{p+1} = 1, 2, \ldots, n, \qquad \nu_1, \ldots, \nu_{p+1} = 0, 1, 2, 3. \tag{B.18}$$

We are now equipped to formulate the

GENERALIZED POINCARÉ LEMMA. *Every (p, p) form $A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\phi_\mu{}^a) \, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}$ on a $4n$-dimensional differentiable manifold m_{4n} with local coordinates $\phi_\mu{}^a$ $(a = 1, 2, \ldots, n, \mu = 0, 1, 2, 3)$, which is strictly exact and of at least class $\mathscr{C}^1$ in a region R_{4n} of points $\phi_\mu{}^a$, is closed in R_{4n}.*

Alternatively, we can write, under the equality sign of relation (B.18),

$$dA^{(p,p)} \equiv d(dB^{(p-1,p-1)}) = 0. \tag{B.19}$$

We now explore the converse of the above generalized lemma.

Consider in this respect the (p, p) form (B.14). A region R_{4n} on m_{4n} will be termed *star-shaped* and denoted with R^*_{4n} when it contains, jointly with the set of points $\phi_\mu{}^a$, all points $\phi'^a_\mu \equiv \tau \phi_\mu{}^a$, $0 \leqslant \tau \leqslant 1$.

We now restrict all considered (p, p) forms to be defined and of (at least) class $\mathscr{C}^1$ in R^*_{4n} and introduce the operation

$$DA^{(p,p)} \equiv \sum_{1}^{p}{}_r \left[\int_0^1 d\tau \, \tau^{p-1} A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau \phi^a_\mu) \right] \times \phi^{a_r}_{\mu_r} \, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_{r-1}}_{\mu_{r-1}} \wedge d\phi^{a_{r+1}}_{\mu_{r+1}} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}. \tag{B.20}$$

Notice on a comparative basis with respect to the ordinary case, i.e., operation (B.6a), the lack of the factor $(-1)^{r-1}$ in the above expression. This is due to the fact that, unlike the ordinary case, the displacements $d\phi^{a_i}_{\mu_i}$ and $d\phi^{a_j}_{\mu_j}$ can be permuted without changing the sign of the form. As a result, operation (B.20) can be expressed in a form equivalent to that of the second term of Eq. (B.6a), i.e.,

$$DA^{(p,p)} = \frac{1}{[(p+1)!]^2} \left[\int_0^1 d\tau\, \tau^{p-1} A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)\right] \delta^{a_1 \cdots a_p \mu_1 \cdots \mu_p}_{b_1 \cdots b_p \nu_1 \cdots \nu_p} \times \phi^{b_1}_{\mu_1}\, d\phi^{b_2}_{\mu_2} \wedge \cdots \wedge d\phi^{b_p}_{\mu_p}. \tag{B.21}$$

The above remark is significant because at the limit when the Greek indices are ignored the primitive tensors of corresponding forms in m_n and m_{4n} can be made to coincide up to numerical factors.

Another property which is shared by operations (B.6a) and (B.20) is that when the given form $A^{(p,p)}$ is null, so is $DA^{(p,p)}$. This can be easily seen from Eq. (B.21), which is null iff

$$\delta^{a_1 \cdots a_p \mu_1 \cdots \mu_p}_{b_1 \cdots b_p \nu_1 \cdots \nu_p} A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p} = 0,$$
$$b_1, \ldots, b_p = 1, 2, \ldots, n, \qquad \mu_1, \ldots, \mu_p = 0, 1, 2, 3. \tag{B.22}$$

As we saw in Section 2, this is precisely the case for the equation in the "kinetic term" of the Lagrangian density.

We now compute the exterior derivative of operation (B.20) yielding, from definition (A.26), the expression

$$\begin{aligned} d(DA^{(p,p)}) &= \sum_{1}^{p}{}_r \left[\int_0^1 d\tau\, \tau^p \frac{\partial A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)}{\partial \phi^{a_{p+1}}_{\mu_{p+1}}}\right] \\ &\quad \times \phi^{a_2}_{\mu_2}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_{r-1}}_{\mu_{r-1}} \wedge d\phi^{a_{r+1}}_{\mu_{r+1}} \wedge \cdots \wedge d\phi^{a_{p+1}}_{\mu_{p+1}} \\ &\quad + p \left[\int_0^1 d\tau\, \tau^{p-1} A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)\right] d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p} \\ &= \frac{1}{(p!)^2} \left[\int_0^1 d\tau\, \tau^p \delta^{a_1 \cdots a_{p+1} \mu_1 \cdots \mu_{p+1}}_{b_1 \cdots b_{p+1} \nu_1 \cdots \nu_{p+1}} \frac{\partial A^{\nu_1 \cdots \nu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)}{\partial \phi^{a_{p+1}}_{\nu_{p+1}}}\right] \\ &\quad \times \phi^{b_1}_{\mu_1}\, d\phi^{b_2}_{\mu_2} \wedge \cdots \wedge d\phi^{b_{p+1}}_{\mu_{p+1}} \\ &\quad + \left[\int_0^1 d\tau\, p\tau^{p+1} A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)\right] d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}, \end{aligned} \tag{B.23}$$

where we have used the properties

$$\sum_{1}^{p}{}_r\, d\phi^{a_2}_{\mu_2} \wedge d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_{r-1}}_{\mu_{r-1}} \wedge d\phi^{a_{r+1}}_{\mu_{r+1}} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p} = p\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}, \tag{B.24a}$$

$$\frac{\partial}{\partial \phi^b_\alpha} A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu) = \frac{\partial A^{\mu_1 \cdots \mu_p}_{a_1 \cdots a_p}(\tau\phi^a_\mu)}{\partial \phi^c_\beta} \frac{\partial(\tau\phi^c_\beta)}{\partial \phi^b_\alpha}. \tag{B.24b}$$

We now consider the D operation on a $(p+1, p+1)$ form, i.e.,

$$
\begin{aligned}
DB^{(p+1,p+1)} &= \sum_{1}^{p+1}{}_r \left[\int_0^1 d\tau\ \tau^p B^{\mu_1\cdots\mu_{p+1}}_{a_1\cdots a_{p+1}}(\tau\phi^a_\mu)\right] \\
&\quad \times \phi^{a_2}_{\mu_2}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_{r-1}}_{\mu_{r-1}} \wedge d\phi^{a_{r+1}}_{\mu_{r+1}} \wedge \cdots \wedge d\phi^{a_{p+1}}_{\mu_{p+1}} \\
&= \sum_{1}^{p}{}_r \left[\int_0^1 d\tau\ \tau^p B^{\mu_1\cdots\mu_{p+1}}_{a_1\cdots a_{p+1}}(\tau\phi^a_\mu)\right] \\
&\quad \times \phi^{a_r}_{\mu_r}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_{r-1}}_{\mu_{r-1}} \wedge d\phi^{a_{r+1}}_{\mu_{r+1}} \wedge \cdots \wedge d\phi^{a_{p+1}}_{\mu_{p+1}} \\
&\quad + \left[\int_0^1 d\tau\ \tau^p B^{\mu_1\cdots\mu_{p+1}}_{a_1\cdots a_{p+1}}(\tau\phi^a_\mu)\right] \phi^{a_{p+1}}_{\mu_{p+1}}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}. \qquad \text{(B.25)}
\end{aligned}
$$

By assuming $B^{(p+1,p+1)} \equiv dA^{(p,p)}$, i.e.,

$$
B^{\mu_1\cdots\mu_{p+1}}_{a_1\cdots a_{p+1}} \equiv \frac{\partial A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}}{\partial \phi^{a_{p+1}}_{\mu_{p+1}}}, \qquad \text{(B.26)}
$$

we finally reach the expression

$$
\begin{aligned}
& d(DA^{(p,p)}) + D(dA^{(p,p)}) \\
&= \frac{2}{p!}\left[\int_0^1 d\tau\ \tau^p \delta^{a_1\cdots a_{p+1}\mu_1\cdots\mu_{p+1}}_{b_1\cdots b_{p+1}\nu_1\cdots\nu_{p+1}} \frac{\partial A^{\nu_1\cdots\nu_p}_{a_1\cdots a_p}(\tau\phi^a_\mu)}{\partial\phi^{a_{p+1}}_{\nu_{p+1}}}\right] \\
&\quad \times \phi^{b_1}_{\mu_1}\, d\phi^{b_2}_{\mu_2} \wedge \cdots \wedge d\phi^{b_{p+1}}_{\mu_{p+1}} \\
&\quad + \left\{\int_0^1 d\tau \left[p\tau^{p-1} A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}(\tau\phi^a_\mu) + \tau^p \frac{\partial A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}(\tau\phi^a_\mu)}{\partial\phi^{a_{p+1}}_{\mu_{p+1}}}\ \phi^{a_{p+1}}_{\mu_{p+1}}\right]\right\} \\
&\quad \times d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}. \qquad \text{(B.27)}
\end{aligned}
$$

We now assume that the form $A^{(p,p)}$ is closed in R^*_{4n}, i.e., $dA^{(p,p)} = 0$, or, explicitly,

$$
\begin{aligned}
& \delta^{a_1\cdots a_{p+1}\mu_1\cdots\mu_{p+1}}_{b_1\cdots b_{p+1}\nu_1\cdots\nu_{p+1}} \frac{\partial A^{\nu_1\cdots\nu_p}_{a_1\cdots a_p}}{\partial\phi^{a_{p+1}}_{\nu_{p+1}}} = 0, \\
& b_1, \ldots, b_{p+1} = 1, 2, \ldots, n, \qquad \mu_1, \ldots, \mu_{p+1} = 0, 1, 2, 3. \qquad \text{(B.28)}
\end{aligned}
$$

Then expression (B.27) reduces to

$$
\begin{aligned}
d(DA^{(p,p)}) &\equiv \left\{\int_0^1 d\tau \frac{d}{d\tau}\left[\tau^p A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}(\tau\phi^a_\mu)\right]\right\} d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p} \\
&= A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}. \qquad \text{(B.29)}
\end{aligned}
$$

As a result, under the above assumptions there exists a primitive form

$$B^{(p-1,p-1)} \equiv DA^{(p,p)} \tag{B.30}$$

such that

$$A^{(p,p)} \equiv dB^{(p-1,p-1)}, \tag{B.31}$$

and, thus, the form $A^{(p,p)}$ is exact.

We have therefore proved the

CONVERSE OF THE GENERALIZED POINCARÉ LEMMA. *Every* (p,p) *form* $A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}(\phi_\mu{}^a)\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_p}_{\mu_p}$ *on a* 4*n-dimensional differentiable manifold* m_{4n} *with local coordinates* $\phi_\mu{}^a$ $(a = 1, 2,\ldots, n, \mu = 0, 1, 2, 3)$ *which is closed, defined, and of at least class* $\mathscr{C}^1$ *in a star-shaped region* R^*_{4n} *of points* $\phi_\mu{}^a$ *is exact on* R^*_{4n}.

Let us stress for the sake of clarity that the conditions for the existence of form (B.30) are:

(1) The $A^{(p,p)}$ forms are defined on a star-shaped region R^*_{4n}. In particular, the integral appearing in Eq. (B.20) must be well defined.

(2) The $A^{(p,p)}$ forms, or more specifically, their coefficients $A^{\mu_1\cdots\mu_p}_{a_1\cdots a_p}$ must be of at least class $\mathscr{C}^1$ on R^*_{4n},

(3) The $A^{(p,p)}$ forms must satisfy the closure condition, namely, the *integrability conditions* (B.28) must be everywhere identically satisfied in R^*_{4n}.

A comparison between the Generalized Poincaré Lemma and its Converse is significant. The former demands that the forms be strictly exact to be closed, while for the latter, when the forms are closed, they are exact irrespective of whether they are strictly or weakly exact.

The Converse of the Poincaré Lemma and its generalization, as indicated earlier, will be crucial for the proof of Theorem 2.1. In essence, this methodology provides the necessary and sufficient conditions for the existence of primitive forms. In practical applications these conditions are the *integrability conditions* for generally overdetermined systems of partial differential equations. And indeed, as we saw in Sections 2 and 3, this is exactly the case for the problem of the existence of a Lagrangian density in Field Theory.

There is, however, another aspect which is significant for practical applications. The methodology of this Appendix provides not only the integrability conditions, but also an explicit *solution* of the underlying system of partial differential equations.

Indeed, given a p form (B.1) satisfying the integrability conditions (B.11), the rule (B.10) provides a solution for the primitive $(p-1)$ form. Similarly, given a (p,p) form (B.14) satisfying the integrability conditions (B.28), the rule (B.31) provides a solution for the primitive $(p-1, p-1)$ form. The reader should keep in mind that such solutions are, of course, of a local nature.

As we shall see, this methodology is significant for the problem of identifying a Lagrangian density, when it exists, from a given system of field equations.

APPENDIX C: The Conditions of Self-Adjointness within the Framework of the Calculus of Differential Forms in Minkowsky Space

In this Appendix we shall reformulate the conditions of self-adjointness for quasi-linear systems, i.e., Eqs. (1.5), in a form which is more suitable for the proof of Theorem 2.1, namely, a form which more directly expresses their significance as the integrability conditions for the existence of a Lagrangian density.

An ordered direct analytic representation of quasi-linear systems

$$d_{\mu_1} \frac{\partial \mathscr{L}}{\partial \phi^{a_1;}{}_{\mu_1}} - \frac{\partial \mathscr{L}}{\partial \phi^{a_1}} \equiv A^{\mu_1\mu_2}_{a_1a_2} \phi^{a_2;}{}_{\mu_1\mu_2} + B_{a_1} = 0, \qquad a_1 = 1, 2, \ldots, n \quad \text{(C.1)}$$

demands the validity of the separate identities

$$(\mathscr{L}^{;\mu_1;\mu_2}_{\ a_1\ a_2})\, \phi^{a_2;}{}_{\mu_1\mu_2} \equiv (A^{\mu_1\mu_2}_{a_1a_2})\, \phi^{a_2;}{}_{\mu_1\mu_2}, \qquad \text{(C.2a)}$$

$$(\mathscr{L}^{;\mu\ ;}_{\ a_1\ a_2})\, \phi^{a_2;}{}_{\mu} + \mathscr{L}^{;\mu\ ;}_{\ a_1\ \mu} - \mathscr{L}^{;}{}_{a_1} \equiv B_{a_1}, \qquad \text{(C.2b)}$$

$$\mathscr{L}^{;}{}_{\mu} \equiv \frac{\partial \mathscr{L}}{\partial x^\mu}, \qquad \mathscr{L}^{;}{}_{a_1} \equiv \frac{\partial \mathscr{L}}{\partial \phi^{a_1}}, \qquad \mathscr{L}^{;\mu}_{\ a} \equiv \frac{\partial \mathscr{L}}{\partial \phi^{a;}{}_{\mu}}, \qquad \text{etc.} \qquad \text{(C.2c)}$$

Let $K(x_\mu, \phi^a, \phi^{a;}{}_{\mu})$ be a particular solution of Eqs. (C.2a). Then conditions of self-adjointness (1.5a) imply that identifications (C.2a) can be written

$$\tfrac{1}{4}(K^{;\mu_1;\mu_2}_{\ a_1\ a_2} + K^{;\mu_2;\mu_1}_{\ a_1\ a_2} + K^{;\mu_1;\mu_2}_{\ a_2\ a_1} + K^{;\mu_2;\mu_1}_{\ a_2\ a_1}) = A^{\mu_1\mu_2}_{a_1a_2}. \qquad \text{(C.3)}$$

This is an indication that the (1, 1) form

$$A^{(1,1)} = A^{\mu_1\mu_2}_{a_1a_2}\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \qquad \text{(C.4)}$$

is identically null. Indeed, from conditions (1.5a) we have (see also Appendix A)

$$\delta^{a_1a_2\mu_1\mu_2}_{b_1b_2\nu_1\nu_2} A^{\nu_1\nu_2}_{a_1a_2} = A^{\mu_1\mu_2}_{b_1b_2} - A^{\mu_2\mu_1}_{b_1b_2} - A^{\mu_1\mu_2}_{b_2b_1} + A^{\mu_2\mu_1}_{b_2b_1} = 0,$$
$$b_1, b_2 = 1, 2, \ldots, n, \qquad \mu_1, \mu_2 = 0, 1, 2, 3. \qquad \text{(C.5)}$$

We consider now the exterior derivative of the (1, 1) form (C.4), i.e.,

$$dA^{(1,1)} = A^{\mu_1\mu_2;\mu_3}_{a_1a_2\ a_3}\, d\phi^{a_1}_{\mu_1} \wedge d\phi^{a_2}_{\mu_2} \wedge d\phi^{a_3}_{\mu_3}$$

$$= (1/36)\, \delta^{a_1a_2a_3\mu_1\mu_2\mu_3}_{b_1b_2b_3\nu_1\nu_2\nu_3} A^{\nu_1\nu_2;\nu_3}_{a_1a_2\ a_3}\, d\phi^{b_1}_{\mu_1} \wedge d\phi^{b_2}_{\mu_2} \cdot d\phi^{b_3}_{\mu_3}, \tag{C.6a}$$

$$A^{\nu_1\nu_2;\nu_3}_{a_1a_2\ a_3} \equiv \partial A^{\nu_1\nu_2}_{a_1a_2}/\partial\phi^{a_3}_{\nu_3}. \tag{C.6b}$$

Conditions of self-adjointness (1.5a) and (1.5b) imply that the above exterior derivative is also identically null, i.e.,

$$\delta^{a_1a_2a_3\mu_1\mu_2\mu_3}_{b_1b_2b_3\nu_1\nu_2\nu_3} A^{\nu_1\nu_2;\nu_3}_{a_1a_2\ a_3} \equiv \begin{vmatrix} \delta^{a_1}_{b_1} & \delta^{a_1}_{b_2} & \delta^{a_1}_{b_3} \\ \delta^{a_2}_{b_1} & \delta^{a_2}_{b_2} & \delta^{a_2}_{b_3} \\ \delta^{a_3}_{b_1} & \delta^{a_3}_{b_2} & \delta^{a_3}_{b_3} \end{vmatrix} \otimes \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \delta^{\mu_1}_{\nu_2} & \delta^{\mu_1}_{\nu_3} \\ \delta^{\mu_2}_{\nu_1} & \delta^{\mu_2}_{\nu_2} & \delta^{\mu_2}_{\nu_3} \\ \delta^{\mu_3}_{\nu_1} & \delta^{\mu_3}_{\nu_2} & \delta^{\mu_3}_{\nu_3} \end{vmatrix} A^{\nu_1\nu_2;\nu_3}_{a_1a_2\ a_3} = 0,$$

$$b_1, b_2, b_3 = 1, 2,\ldots, n, \qquad \mu_1, \mu_2, \mu_3 = 0, 1, 2, 3. \tag{C.7}$$

The next step is to consider the exterior derivative of $dA^{(1,1)}$, i.e.,

$$d(dA^{(p,p)}) \equiv A^{\mu_1\mu_2;\mu_3;\mu_4}_{a_1a_2\ a_3\ a_4}\, d\phi^{a_1}_{\mu_1} \wedge \cdots \wedge d\phi^{a_4}_{\mu_4}$$

$$= \frac{1}{(3!)^2\,(4!)^2}\, \delta^{a_1\cdots a_4\mu_1\cdots\mu_4}_{b_1\cdots b_4\nu_1\cdots\nu_4} A^{\nu_1\nu_2;\nu_3;\nu_4}_{a_1a_2\ a_3\ a_4}\, d\phi^{b_1}_{\mu_1} \wedge \cdots \wedge d\phi^{b_4}_{\mu_4} \tag{C.8}$$

which, as stressed in Appendix A and B, is not necessarily null.

As the reader can verify with tedious but straightforward calculations, the conditions of self-adjointness (1.5a), (1.5b), *and* (1.5c) imply that exterior derivative (C.7) is also identically null, i.e.,

$$\delta^{a_1\cdots a_4\mu_1\cdots\mu_4}_{b_1\cdots b_4\nu_1\cdots\nu_4} A^{\nu_1\nu_2;\nu_3;\nu_4}_{a_1a_2\ a_3\ a_4} = 0,$$

$$b_1,\ldots, b_4 = 1, 2,\ldots, n, \qquad \mu_1,\ldots, \mu_4 = 0, 1, 2, 3. \tag{C.9}$$

Within the context of the calculus of differential forms on Minkowski space we shall consider Eqs. (C.5), (C.7), and (C.9), rather than Eqs. (1.5a), (1.5b), and (1.5c), as the conditions of self-adjointness on the terms $A^{\mu_1\mu_2}_{a_1a_2}$.

Let K be a particular solution of Eqs. (C.3). Then the general solution, from a property of type (B.12), can be written

$$\mathscr{L}(x_\mu, \phi^a\ \phi^{a;}{}_\mu) = K(x_\mu, \phi^a, \phi^{a;}{}_\mu) + D^\alpha_{a_1}(x_\mu, \phi^a)\, \phi^{a_1;}{}_\alpha + C(x_\mu, \phi^a), \tag{C.10}$$

where the densities $D^{\mu_1}_{a_1}$ and C are, at this point, unknown.

By substituting into Eqs. (C.2b) we have

$$(D^{\mu\,;}_{a_1\;a_2} - D^{\mu\,;}_{a_2\;a_1})\,\phi^{a_2;}{}_{\mu} + (D^{\mu\,;}_{a_1\;\mu} - C^{;}{}_{a_1})$$
$$\equiv B_{a_1} + K^{;}{}_{a_1} - K^{;\mu\,;}_{a_1\;\mu} - (K^{;\mu\,;}_{a_1\;a_2})\,\phi^{a_2;}{}_{\mu}, \qquad \text{(C.11)}$$

where we have written all terms involving the density K in the rhs because they are assumed, at this point, to be known from the solution of Eqs. (C.3).

By writing Eqs. (C.11) in the a_1 and a_2 indices, by differentiating with respect to $\phi^{a_2;}{}_{\mu}$ and $\phi^{a_1;}{}_{\mu}$ respectively, and by adding and subtracting we reach the expressions

$$B_{a_1}{}^{;\mu_1}{}_{a_2} - B_{a_2}{}^{;\mu_1}{}_{a_1} = 2\left\{\partial_{\mu_2} + \phi^{a_3;}{}_{\mu_2}\frac{\partial}{\partial\phi^{a_3}}\right\} A^{\mu_1\mu_2}_{a_1a_2}, \qquad \text{(C.12a)}$$

$$D^{\mu\,;}_{a_1\;a_2} - D^{\mu\,;}_{a_2\;a_1} = \tfrac{1}{2}(B_{a_1}{}^{;\mu}{}_{a_2} - B_{a_2}{}^{;\mu}{}_{a_1}) + (K^{;}{}_{a_1}{}^{;\mu}{}_{a_2} - K^{;}{}_{a_2}{}^{;\mu}{}_{a_1})\,\phi^{a_2;}{}_{\mu}$$
$$\equiv Z^{\mu}_{a_1a_2}, \qquad \text{(C.12b)}$$

where we have used identifications (C.3).

Equations (C.12a) coincide with conditions of self-adjointness (1.5d). Equations (C.12b) constitute, in addition to Eqs. (C.3), a second independent set of conditions for the existence of Lagrangian (C.9). It should be indicated that Eqs. (C.12b) can also be more directly reached by differentiating Eqs. (C.11) with respect to $\phi^{a_2;}{}_{\mu}$ and by using conditions (1.5d). Notice that system (C.12b) in the unknowns $D_a{}^{\mu}$ is, in general, overdetermined.

We consider now the $(\mu, 2)$ form (see Appendix A)

$$Z^{(\mu,2)} \equiv Z^{\mu}_{a_1a_2}\, d\phi^{a_1} \wedge d\phi^{a_2}. \qquad \text{(C.13)}$$

The closure condition (see Appendix B) reads

$$dZ^{(\mu,2)} \equiv \frac{1}{3!}\,\delta^{a_1a_2a_3}_{b_1b_2b_3} Z^{\mu}_{a_1a_2}{}^{;}{}_{a_3}\, d\phi^{b_1} \wedge d\phi^{b_2} \wedge d\phi^{b_3}, \qquad \text{(C.14)}$$

and it holds identically iff

$$\delta^{a_1a_2a_3}_{b_1b_2b_3} Z^{\mu}_{a_1a_2}{}^{;}{}_{a_3} = 0, \qquad b_1, b_2, b_3 = 1, 2,\ldots, n. \qquad \text{(C.15)}$$

But the term $Z^{\mu}_{a_1a_2}$ is totally antisymmetric in the indices a_1 and a_2. Therefore, Eqs. (C.15) reduce to

$$Z^{\mu}_{b_1b_2}{}^{;}{}_{b_3} + Z^{\mu}_{b_3b_1}{}^{;}{}_{b_2} + Z^{\mu}_{b_2b_3}{}^{;}{}_{b_1} = 0,$$
$$b_1, b_2, b_3 = 1, 2,\ldots, n, \qquad \mu = 0, 1, 2, 3. \qquad \text{(C.16)}$$

By substituting the value of $z^{\mu}_{a_1 a_2}$ from Eqs. (C.12b) we reach the conditions on the B terms

$$\tfrac{1}{2}[(B_{a_2}{}^{;\mu}_{a_3} - B_{a_3}{}^{;\mu}_{a_2})^{;}{}_{a_1} + (B_{a_3}{}^{;\mu}_{a_1} - B_{a_1}{}^{;\mu}_{a_3})^{;}{}_{a_2} + (B_{a_1}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_1})^{;}{}_{a_3}] = 0 \quad \text{(C.17)}$$

(where we have used the continuity properties of the K density) which must be identically satisfied for the $(\mu, 2)$ form (C.13) to be closed. Notice that the above conditions can be equivalently written

$$\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} = 0. \quad \text{(C.18)}$$

Consider in this respect the conditions of self-adjointness (1.5e) for the pairs of indices (a_3, a_2), (a_1, a_3), and (a_2, a_1). By differentiating with respect to $\phi^{a_1;}{}_{\mu_1}$, $\phi^{a_2;}{}_{\mu}$, and $\phi^{a_3;}{}_{\mu}$, respectively, and by adding up, we reach the expression

$$\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} = \frac{1}{3}\left\{\partial_\nu + \phi^{a;}{}_{\nu} \frac{\partial}{\partial \phi^{a}}\right\} \delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_3}. \quad \text{(C.19)}$$

Now, the above equations can be written, from the antisymmetry in the (ν, μ) indices,

$$\begin{aligned}
&\frac{1}{3}\left\{\partial_\nu + \phi^{a;}{}_{\nu} \frac{\partial}{\partial \phi^{a}}\right\} \delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_3} \\
&\quad = \frac{1}{6}\left\{\partial_\nu + \phi^{a;}{}_{\nu} \frac{\partial}{\partial \phi^{a}}\right\} [(B_{a_1}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_3} - B_{a_2}{}^{;\nu}{}_{a_1}{}^{;\mu}{}_{a_3}) - (B_{a_1}{}^{;\nu}{}_{a_3}{}^{;\mu}{}_{a_2} - B_{a_3}{}^{;\nu}{}_{a_1}{}^{;\mu}{}_{a_2}) \\
&\qquad + (B_{a_2}{}^{;\nu}{}_{a_3}{}^{;\mu}{}_{a_1} - B_{a_3}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_1}) - (B_{a_1}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_3} - B_{a_1}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_3}) \\
&\qquad + (B_{a_3}{}^{;\nu}{}_{a_1}{}^{;\mu}{}_{a_2} - B_{a_1}{}^{;\nu}{}_{a_3}{}^{;\mu}{}_{a_2}) - (B_{a_3}{}^{;\nu}{}_{a_2}{}^{;\mu}{}_{a_1} - B_{a_2}{}^{;\nu}{}_{a_3}{}^{;\mu}{}_{a_1})].
\end{aligned} \quad \text{(C.20)}$$

But, after the proper use of conditions (1.5d), we obtain

$$\begin{aligned}
\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;\alpha}{}_{a_2}{}^{;\mu}{}_{a_3} = 2 \left\{\partial_\beta + \phi^{a;}{}_{\beta} \frac{\partial}{\partial \phi^{a}}\right\} &[A^{\alpha\ \beta\ ;\mu}_{a_1 a_2\ a_3} + A^{\alpha\ \beta\ ;\mu}_{a_3 a_1\ a_2} + A^{\alpha\ \beta\ ;\mu}_{a_2 a_3\ a_1} \\
&- A^{\mu\ \beta\ ;\alpha}_{a_1 a_2\ a_3} - A^{\mu\ \beta\ ;\alpha}_{a_3 a_1\ a_2} - A^{\mu\ \beta\ ;\alpha}_{a_2 a_3\ a_1}] = 0,
\end{aligned} \quad \text{(C.21)}$$

where the term within brackets on the rhs is null from conditions (1.5b). Thus, the closure (C.18), under the condition of self-adjointness, identically holds.

We now substitute Eqs. (C.12b) into (C.11) by obtaining in this way the equations in the C density

$$\begin{aligned}
C^{;}{}_{a_1} = {} & D^{\mu}_{a_1}{}^{;}{}_{\mu} - B_{a_1} - K^{;}{}_{a_1} + K^{;\mu}_{a_1}{}^{;}{}_{\mu} \\
& + [K^{;\mu}_{a_2}{}^{;}{}_{a_1} + \tfrac{1}{2}(B_{a_1}{}^{;\mu}_{a_2} - B_{a_2}{}^{;\mu}_{a_1})]\, \phi^{a_2;}{}_{\mu},
\end{aligned} \quad \text{(C.22)}$$

where the solutions $D_a{}^{\mu}$ of Eqs. (C.12b) are at this point assumed to be known.

This allows us to define the ordinary 1 form

$$C^{(1)} \equiv C^{;}{}_{a_2}\, d\phi^{a_1}, \tag{C.23}$$

whose closure conditions simply read

$$\delta^{a_1 a_2}_{b_1 b_2} C^{;}{}_{a_2}{}^{;}{}_{a_2} = C^{;}{}_{b_1}{}^{;}{}_{b_2} - C^{;}{}_{b_2}{}^{;}{}_{b_1} = 0, \qquad b_1, b_2 = 1, 2, \ldots, n. \tag{C.24}$$

By using Eqs. (C.22) and (C.12b) the above conditions become

$$\begin{aligned} \delta^{a_1 a_2}_{b_1 b_2} C^{;}{}_{a_1}{}^{;}{}_{a_2} &\equiv (B_{a_2}{}^{;}{}_{a_1} - B_{a_1}{}^{;}{}_{a_2}) + \tfrac{1}{2}(B_{a_1}{}^{;\mu}{}_{a_2} - B_{a_2}{}^{;\mu}{}_{a_1})^{;}{}_{\mu} \\ &+ \tfrac{1}{2}(B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} - B_{a_3}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_1} - B_{a_2}{}^{;}{}_{a_1}{}^{;\mu}{}_{a_3} + B_{a_3}{}^{;}{}_{a_1}{}^{;\mu}{}_{a_2})\, \phi^{a_3;}{}_{\mu}, \end{aligned} \tag{C.25}$$

and, from conditions (1.5e), can be written

$$\delta^{a_1 a_2}_{b_1 b_2} C^{;}{}_{a_1}{}^{;}{}_{a_2} \equiv \tfrac{1}{2}(\delta^{a_1 a_2 a_3}_{b_1 b_2 b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3})\, \phi^{b_3;}{}_{\mu}, \tag{C.26}$$

and thus, it identically holds from Eqs. (C.18).

Notice that the closure conditions for Eqs. (C.12b) and (C.22) are equivalent.

Finally, for consistency, the rhs of Eqs. (C.12b) and (C.22) must be independent of the derivative terms. By differentiating these equations with respect to $\phi^{a_3;}{}_{\nu}$ we reach the respective conditions

$$\begin{aligned} (B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3}) + 2(A^{\mu\ \nu;}_{a_2 a_3}{}_{a_1} - A^{\mu\ \nu;}_{a_1 a_3}{}_{a_2}) &= 0, \\ [(B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_3}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_1}) + 2(A^{\mu\ \nu;}_{a_2 a_3}{}_{a_1} - A^{\mu\ \nu;}_{a_1 a_2}{}_{a_3})]\, \phi^{a_3;}{}_{\nu} &= 0, \end{aligned} \tag{C.27}$$

which are clearly equivalent and which must hold in view of the conditions of self-adjointness (1.5).

Consider in this respect identity (C.21), i.e.,

$$B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} + B_{a_3}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_2} + B_{a_2}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_1} - B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3} - B_{a_3}{}^{;\mu}{}_{a_2}{}^{;\mu}{}_{a_1} - B_{a_1}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_2} = 0. \tag{C.28}$$

Then, by using the relations originating from conditions (1.5d),

$$\begin{aligned} B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} &= B_{a_3}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_2} + 2\left\{\partial_\alpha + \phi^{a;}{}_{\alpha}\frac{\partial}{\partial\phi^a}\right\} \\ &\quad\times (A^{\mu\ \alpha\ ;\nu}_{a_1 a_2}{}_{a_3} - A^{\alpha\ \nu\ ;\mu}_{a_3 a_2}{}_{a_1}) + 2(A^{\mu\ \nu;}_{a_1 a_2}{}_{a_3} - A^{\mu\ \nu;}_{a_2 a_3}{}_{a_1}), \\ B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3} &= B_{a_1}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_2} + 2\left\{\partial_\alpha + \phi^{a;}{}_{\alpha}\frac{\partial}{\partial\phi^a}\right\} \\ &\quad\times (A^{\nu\ \alpha\ ;\mu}_{a_3 a_2}{}_{a_1} - A^{\mu\ \alpha\ ;\nu}_{a_1 a_3}{}_{a_2}) + 2(A^{\mu\ \nu;}_{a_3 a_2}{}_{a_1} - A^{\mu\ \nu;}_{a_1 a_3}{}_{a_2}), \end{aligned} \tag{C.29}$$

and by permuting the indices we can write

$$
\begin{aligned}
\delta^{a_1a_2a_3}_{b_1b_2b_3} B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} &\equiv 3(B_{a_3}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_2} - B_{a_1}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_2}) + 2\left\{\partial_\alpha + \phi^{a;}{}_\alpha \frac{\partial}{\partial \phi^a}\right\} \\
&\quad \times [A^{\mu\alpha;\nu}_{a_1a_2\ a_3} - A^{\alpha\nu;\mu}_{a_3a_2\ a_1} - A^{\nu\alpha;\mu}_{a_3a_2\ a_1} + A^{\mu\alpha;\nu}_{a_1a_3\ a_2} \\
&\quad + A^{\nu\alpha;\mu}_{a_1a_2\ a_3} - A^{\mu\alpha;\nu}_{a_3a_1\ a_2} - A^{\mu\alpha;\nu}_{a_3a_2\ a_1} + A^{\alpha\nu;\mu}_{a_1a_2\ a_3}] \\
&\quad + 2(A^{\mu\nu;}_{a_1a_2\ a_3} - A^{\mu\nu;}_{a_2a_3\ a_1} - A^{\mu\nu;}_{a_3a_2\ a_1} + A^{\mu\nu;}_{a_1a_3\ a_2} \\
&\quad + A^{\mu\nu;}_{a_1a_2\ a_3} - A^{\mu\nu;}_{a_3a_2\ a_1} - A^{\mu\nu;}_{a_3a_2\ a_1} + A^{\mu\nu;}_{a_2a_1\ a_3}) \\
&\equiv 3[(B_{a_3}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_2} - B_{a_1}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_2}) + 2(A^{\mu\nu;}_{a_1a_2\ a_3} - A^{\mu\nu;}_{a_2a_3\ a_1})] \\
&\quad + 2\left\{\partial_\alpha + \phi^{a;}{}_\alpha \frac{\partial}{\partial \phi^a}\right\} [A^{\mu\alpha;\nu}_{a_1a_2\ a_3} + A^{\nu\alpha;\mu}_{a_1a_2\ a_3} + A^{\alpha\nu;\mu}_{a_1a_2\ a_3} \\
&\quad - A^{\alpha\nu;\mu}_{a_3a_2\ a_1} - A^{\nu\alpha;\mu}_{a_3a_2\ a_1} - A^{\mu\alpha;\nu}_{a_3a_2\ a_1}] \\
&\equiv 3[(B_{a_3}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_2} - B_{a_1}{}^{;\mu}{}_{a_3}{}^{;\nu}{}_{a_2}) + 2(A^{\mu\nu;}_{a_1a_2\ a_3} - A^{\mu\nu;}_{a_2a_3\ a_1})] = 0,
\end{aligned} \tag{C.30}
$$

where we have used conditions (1.5a), (1.5b), and (C.21). Thus, consistency conditions (C.27) identically hold under the assumption of self-adjointness.

We reach in this way the following set of independent *integrability conditions for the existence of a Lagrangian*:

$$
\begin{aligned}
\delta^{a_1a_2\mu_1\mu_2}_{b_1b_2\nu_1\nu_2} A^{\nu_1\nu_2}_{a_1a_2} &= 0, \\
\delta^{a_1a_2a_3\mu_1\mu_2\mu_3}_{b_1b_2b_3\nu_1\nu_2\nu_3} A^{\nu_1\nu_2;\nu_3}_{a_1a_2\ a_3} &= 0, \\
\delta^{a_1a_2a_3a_4\mu_1\mu_2\mu_3\mu_4}_{b_1b_2b_3b_4\nu_1\nu_2\nu_3\nu_4} A^{\nu_1\nu_2;\nu_3;\nu_4}_{a_1a_2\ a_3\ a_4} &= 0, \\
\delta^{a_1a_2a_3}_{b_1b_2b_3} B_{a_1}{}^{;}{}_{a_2}{}^{;\mu}{}_{a_3} &= 0, \\
B_{a_1}{}^{;\mu}{}_{a_2}{}^{;\nu}{}_{a_3} - B_{a_2}{}^{;\mu}{}_{a_1}{}^{;\nu}{}_{a_3} + 2(A^{\mu\nu;}_{a_2a_3\ a_1} - A^{\mu\nu;}_{a_1a_3\ a_2}) &= 0,
\end{aligned} \tag{C.31}
$$

which identically hold whenever all conditions of self-adjointness (1.5) are satisfied.

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2. For overdetermined quasi-linear systems of partial differential equations see: D. C. SPENCER, *Bull. Amer. Math. Soc.* **75** (1969), 179. See also: H. GOLDSCHMIDT, *Ann. of Math.* **86** (1967), 246 and *J. Differential Geometry* **1** (1967), 269; J. GASQUI, *J. Differential Geometry* **10** (1975), 61. This author is indebted to D. C. Spencer for calling to his attention these papers.
3. M. REED AND B. SIMON, "Methods of Modern Mathematical Physics, Vol. 2: Fourier Analysis, Self-Adjointness," Academic Press, New York, 1975.
4. This theorem, to the best knowledge of this author, is introduced here for the first time. We have, however, made heavy use of the methodology which underlies the inverse problem of the calculus of variations for single integrals which, in essence, provides the Newtonian counterpart of Theorem 2.1, with particular reference to D. R. DAVIS, *Bull. Amer. Math. Soc.* **35** (1929), 371. For an analysis of this problem from a Newtonian profile see the forthcoming monographs [5].
5. R. M. SANTILLI, The Inverse Problem in Newtonian Mechanics, to appear.
   This monograph contains the *Newtonian* formulation and proof of Theorem 2.1, which we have there termed the *fundamental analytic theorem for configuration space formulations.* The monograph also contains a list of all relevant references in the inverse problem of the calculus of variations known to this author. For a list of the most significant references on this problem see [1].
6. A comparative analysis of this framework in the local variables $\phi^{a};_{\mu}$ with the ordinary framework, say, in the local variables x^a, $a = 1, 2,..., n$, is here significant. Consider, for instance, the stress tensor characterized by the equations

$$S_{a_1a_2} = \tfrac{1}{2}\,[(\partial U_{a_1}/\partial x^{a_2}) + (\partial U_{a_2}/\partial x^{a_1})].$$

   Then, under the assumption that the S tensor is of at least class $\mathscr{C}^2$, the integrability conditions are

$$S_{a_1a_2} = S_{a_2a_1}$$

$$\frac{\partial^2 S_{a_1a_2}}{\partial x^{a_4}\partial x^{a_3}} - \frac{\partial^2 S_{a_3a_2}}{\partial x^{a_4}\partial x^{a_1}} - \frac{\partial^2 S_{a_1a_4}}{\partial x^{a_2}\partial x^{a_3}} + \frac{\partial^2 S_{a_3a_4}}{\partial x^{a_2}\partial x^{a_1}} = 0.$$

   See in this respect [7, Chap. 5], particularly Problem 5.16, p. 178. The reader should, however, be alerted that within the context of the above ordinary case the second-order exterior derivative of a p form is always identically null under the assumed continuity conditions. This is, in essence the ordinary Poincaré Lemma (see [7, p. 141] or Appendix B). Within the context of Theorem 2.1 the situation is significantly different because, despite corresponding continuity assumptions, the second-order exterior derivative of a generalized (p, p) form *is not* identically null unless the form is strictly exact. This is what we have called in Appendix B the Generalized Poincaré Lemma. If one follows the same pattern of the ordinary case, then the integrability conditions of Eq. (2.16) become a *linear combination* of conditions (2.1c) together with conditions (2.1a).

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