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Necessary and Sufficient Conditions for the Existence of a Lagrangian in Field Theory. I. Variational Approach to Self-adjointness for Tensorial Field Equations*

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In this paper we identify some of the most significant references on the inverse problem of the calculus of variations for single integrals and initiate the study of the generalization of the underlying methodology to classical field theories. We first classify Lorentz-covariant tensorial field equations into nonlinear, quasi-linear, and semilinear forms, and then introduce their systems of equations of variation and adjoint systems. The necessary and sufficient conditions for the self-adjointness of class $\mathscr{C}^2$, regular, tensorial, nonlinear, quasi-linear and semilinear forms are worked out. We study the Lagrange equations, their system of equations of variations (Jacobi equations) and their adjoint system by proving that, for class $\mathscr{C}^4$ and regular Lagrangian densities, they are always self-adjoint. We then introduce a concept of analytic representation which occurs when the Lagrange equations coincide with the field equations up to equivalence transformations and refine the definition by particularizing it as direct or indirect and ordered or nonordered. Some of the conventional cases of tensorial fields are considered and we prove, in particular, that the conventional representation of the complex scalar field in interaction with the electromagnetic field is of the ordered indirect type. For the objective of identifying our program we recall the two classes of equivalence transformations of the Lagrangian densities which are primarily used nowadays, namely, the Lorentz (coordinate) transformations and the gauge transformations (transformations of fields within a fixed coordinate system), and postulate the existence of a third class, which we term isotopic transformations of the Lagrangian density and which consist of equivalence transformations within a fixed coordinate system and gauge. We finally outline the objectives of our program, which essentially consist of the identification of the necessary and sufficient conditions for the existence of a Lagrangian in field theories and their first application to the transformation theory within the framework of our variational approach to self-adjointness.

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1. Introduction

The *inverse problem of the calculus of variations* can be formulated as follows. Given a family of ∞^{2n} paths $y_k(x)$, $k = 1, 2, \ldots, n$, solutions of the system of ordinary second-order differential equations

$$F_k(x, y, y', y'') = 0, \qquad y' = dy/dx, \qquad y'' = d^2y/dx^2, \tag{1.1}$$

determine whether these paths can be identified with the totality of extremals of a variational problem

$$J(y) = \int_{x_1}^{x_2} dx\, L(x, y, y'). \tag{1.2}$$

A physical significance of this problem within the framework of Newtonian mechanics is that its solution provides the means to ascertain whether, given a system of Newton's equations of motion in configuration space [1]

$$F_k = A_{ki}(t, q, \dot{q})\, \ddot{q}_i + B_k(t, q, \dot{q}) = 0, \qquad \dot{q} = dq/dt, \qquad \ddot{q} = d^2q/dt^2, \tag{1.3}$$

a Lagrangian $L(t, q, \dot{q})$ for the analytic representation [2]

$$L_k = (d/dt)(\partial L/\partial \dot{q}_k) - (\partial L/\partial q_k) \equiv A_{ki}\ddot{q}_i + B_k = 0, \qquad k = 1, 2, \ldots, n \tag{1.4}$$

exists and, in case of an affirmative answer, it provides the methodology for its computation.

The present status of the investigations on this problem within the framework of Newtonian mechanics is reviewed and somewhat expanded in a forthcoming monograph by this author [3].

This problem for the case $n = 1$ was first solved by Darboux in 1894 [4], and subsequently expanded to higher derivatives by Hirsch in 1898 [5] and Bohem in 1900 [6]. The case $n > 1$ has been studied by several authors, including Helmholtz in 1887 [7], Mayer in 1896 [8], Königsberger in 1901 [9], Hamel in 1903 [10], Kürshack in 1906 [11], Davis in 1928–1929 [12], and Douglas in 1941 [13].

These authors essentially introduced what are customarily called the *conditions of self-adjointness*, namely, the (necessary and sufficient) conditions for the equations of variation of system (1.3) to coincide with the adjoint system. Then they proved for the case of regular [14] systems that such conditions of self-adjointness are sufficient for the existence of a Lagrangian for representation (1.4).

The ultimate reason for such effectiveness of the condition of self-adjointness rests on the fact that the Lagrange equations in regular class $\mathscr{C}^4$ Lagrangians are self-afjoint. This property was apparently established for the first time by Jacobi in 1837 [15]. And indeed, the equations of variations of the Lagrange equations, or, equivalently of the Euler equations of a variational problem, are customarily called the *Jacobi equation* in the current literature in the Calculus of Variations [16–23]. It is a matter of elementary calculations to see that the Jacobi equations, under the assumed regularity and continuity conditions, coincide with the adjoint system. This establishes the self-adjointness of the lhs of identifications (1.4). Under a suitable characterization of the type of analytic representation they are referred to [3], the conditions of self-adjointness of the equations of motion then became the necessary and sufficient conditions for the existence of a Lagrangian.

Since the Douglas' investigations of 1941, the inverse problem on the calculus of variations has remained virtually ignored in both mathematical and physical literature with very few exceptions known to this author, among them the study by Dedecker of 1949–1950 [25] and Havas of 1957 [26]. This is rather regrettable because the implications of the methodology which underlies the problem are significant.

First of all, the conditions of self-adjointness are insensitive as to whether the acting forces are derivable from a potential or not. Therefore, starting from the conventional structure of the Lagrangian for a conservative (unconstrained) Newtonian interacting system

$$L_{\text{Tot}} = \sum_{1}^{n}{}_{k}\, L_{\text{Free}}^{(k)} + L_{\text{Int}}\,, \tag{1.5}$$

where $L_{\text{Free}}^{(k)}$ is the kinetic energy of the particle k and $V = -L_{\text{Int}}$ is the potential function, the methodology which underlies the inverse problem of the calculus of variations gives the conditions for the existence of a Lagrangian also when the forces are not derivable from a potential. In this case, however, an "additive" interaction term L_{Int} to the kinetic energy is insufficient by itself to represent the motion. A structure of the Lagrangian capable of representing this broader class of Newtonian systems is given by [3],

$$L_{\text{Tot}} = \sum_{1}^{n}{}_{k}\, L_{\text{Int,I}}^{(k)} L_{\text{Free}}^{(k)} + L_{\text{Int,II}}\,, \tag{1.6}$$

namely, it is characterized by $(n+1)$ interaction terms, n multiplicative terms $L_{\text{Int,I}}^{(k)}$, and one additive term $L_{\text{Int,II}}$. Trivially, at the limit when all multiplicative interaction terms reduce to unity, the generalized concept of Newtonian interactions according to the Lagrangian (1.6) reduces to the conventional concept of Lagrangian (1.5).

This aspect, however, does not exhaust the significance of the conditions of self-adjointness. For instance, within the framework of the transformation theory for Newtonian systems, such conditions allow the identification of new types of equivalence transformations of the Lagrangian. In essence, there exist infinite varieties of systems of differential equations capable of representing the same path, i.e., all admitting the same general solution. One family of such equivalent equations of motion can be characterized by

$$h_{ki}(A_{ij}\ddot{q}_j + B_i) = A^*_{kj}\,\ddot{q}_j + B_k{}^* = 0, \qquad h_{ki} = h_{ki}(t, q, \dot{q}), \qquad \det(h_{ij}) \neq 0. \tag{1.7}$$

The conditions of self-adjointness establish that for *all* elements of the family (1.7) of equations of motion which are self-adjoint (but not necessarily identical), the Lagrangian exists. Thus, if a given system admits the analytic representation (1.4) and a self-adjoint equivalence transformation (1.7), then the new representation

$$(d/dt)(\partial L^*/\partial \dot{q}_k) - (\partial L^*/\partial q_k) \equiv A^*_{kj}\ddot{q}_j + B_k{}^* = 0, \qquad k = 1, 2,..., n \tag{1.8}$$

exists. The transition $L \to L^*$ so constructed is an equivalence transformation of the Lagrangian which *cannot* be derived either with a point transformation $L(t, q, \dot{q}) \to L'(t, q', \dot{q}')$ (because it occurs within a fixed coordinate system) or with a Newtonian "gauge" transformation $L \to L^+ = L + \dot{G}(t, q)$ (because the Lagrange equations in L and L^* do not coincide, unless the mapping is trivial).

As a consequence, the equivalence transformations $L(t, q, \dot{q}) \to L^*(t, q, \dot{q})$ constructed with the above use of the conditions of self-adjointness constitute a third identifiable layer of the transformation theory in configuration space, besides the conventional point and gauge transformations. Such transformations $L \to L^*$ have been termed by this author, for certain algebraic reasons, *isotopic transformations* [3]. It should be indicated here that these transformations result in being the extension to arbitrary (finite) dimension n of the equivalence transformations of the Lagrangian considered by Currie and Saletan in 1966 [27] for the case $n = 1$, without the methodology of the inverse problem of the calculus of variations.

From the viewpoint of the generalized concept of Newtonian interactions according to Lagrangian (1.6), the isotopic transformation essentially represent the "degrees of freedom" of the $(n + 1)$ interaction terms within a fixed path. Indeed, the transition $L \to L^*$ can be written

$$L_{\text{Tot}} = \sum_{1}^{n}{}_{k} L^{(k)}_{\text{Int,I}} L^{(k)}_{\text{Free}} + L_{\text{Int,II}} \to L^*_{\text{Tot}} = \sum_{1}^{n}{}_{k} L^{*(k)}_{\text{Int,I}} L^{(k)}_{\text{Free}} + L^*_{\text{Int,II}}. \tag{1.9}$$

Needless to say, isotopic mappings (1.9) also apply when the original Lagrangian L_{Tot} has the conventional structure (1.5). This, however, implies the transition from conventional to generalized Lagrangian structures for *conservative* Newtonian systems, i.e.,

$$L_{\text{Tot}} = \sum_{1}^{n}{}_k L_{\text{Free}}^{(k)} + L_{\text{Int}} \rightarrow L_{\text{Tot}}^{*} = \sum_{1}^{n}{}_k L_{\text{Int.I}}^{*(k)} L_{\text{Free}}^{(k)} + L_{\text{Int.II}}^{*} . \tag{1.10}$$

A trivial example for the one-dimensional harmonic oscillator $\ddot{x} + x = 0$ ($m = 1$; $k = 1$; $x \neq 0$; $\dot{x} \neq 0$) is given by [3]

$$L_{\text{Free}} = \tfrac{1}{2}\dot{x}^2;$$

$$\text{Structure (1.5):} \begin{cases} L_{\text{Int}} = -\tfrac{1}{2}x^2, \\ L_{\text{Tot}} = L_{\text{Free}} + L_{\text{Int}} = \tfrac{1}{2}(\dot{x}^2 - x^2); \end{cases}$$

$$\text{Structure (1.6):} \begin{cases} L_{\text{Int.I}}^{*} = \tfrac{1}{3}\dot{x}\cos t + x \sin t, \\ L_{\text{Int.II}}^{*} = -x^2\dot{x}\cos t, \\ L_{\text{Tot}}^{*} = L_{\text{Int.I}}^{*} L_{\text{Free}} + L_{\text{Int.II}}^{*} \\ \qquad = \tfrac{1}{6}\dot{x}^3 \cos t + \tfrac{1}{2}x\dot{x}^2 \sin t - x^2\dot{x}\cos t; \end{cases}$$

$$\text{Isotopic Mapping:} \qquad L_{\text{Tot}} \rightarrow L_{\text{Tot}}^{*} . \tag{1.11}$$

This point is indicated to emphasize the fact that, within the framework of Newtonian mechanics, the Lagrangian for the representation of *conservative* interacting systems must not necessarily have the conventional structure (1.5) because equivalent generalized structures (1.6) are also admissible. Therefore, the generalized concept of Newtonian interactions according to Lagrangian (1.6) has a twofold significance. First, it allows the representation of a broader class of Newtonian interacting systems, and second it is significant for the transformation theory of both nonconservative and conservative systems.

The significance of the conditions of self-adjointness for canonical formulations of Newtonian systems [3] is even more intriguing than that for configuration space formulations. Indeed, such conditions, besides providing the methodology for the identification of a Hamiltonian without any prior knowledge of a Lagrangian, possess relevant algebraic and geometrical meanings. Furthermore, the phase-space image of the above indicated transformations $L \rightarrow L^*$ can be expressed in terms of an invertible Lie algebra axiom-preserving mapping of the Poisson bracket [3] which, in the theory of abstract algebras is termed an "isotopic mapping" of the product. This indicates the reason for the selected terminology.

In these papers we shall initiate the identification of the generalization of the methodology of the inverse problem of the calculus of variations to the case of classical field theory and attempt a preliminary analysis of its possible significance. Such a problem essentially corresponds to the inverse problem for the case of multiple integrals. Apparently, this problem too has remained virtually ignored in both the mathematical and physical literature. However, due to the vast number of references which exist in the several branches of the calculus of variations, as well as of field theory, this author makes no claim to originality.

2. Notation

We shall denote with $x = \{x^\mu\} \equiv (x^0, x^1, x^2, x^3)$ generic points of the *Minkowski space* $M_{3,1}$ with metric tensor $g^{\mu\nu} : g^{00} = -g^{kk} = 1;\ g^{\mu\nu} = 0,\ \mu \neq \nu;\ k = 1, 2, 3;$ $\mu, \nu = 0, 1, 2, 3$. A generic point set in $M_{3,1}$ will be denoted with R_x. The closure $\bar{R}_x$ of R_x is the union $R_x \cup \partial R_x$, where ∂R_x is the boundary of R_x. We shall say that R_x is a *region* when it is open and connected and it is a *domain* when it is perfect, internally connected, and each of its points is a point of accumulation of interior points. Then if R_x is a region, $\bar{R}_x$ is a domain.

We shall say that a set of (classical) fields $\Phi(x) = \{\phi_a(x)\}$, $a = 1, 2,..., n$, defined in a common region R_x is of class $\mathscr{C}^m$ in R_x and write $\Phi \in \mathscr{C}^m(R_x)$, $m = 0, 1, 2,...,$ ∞, when each component ϕ_a possesses continuous (Holder) derivatives in all coordinates x^μ, $\mu = 0, 1, 2, 3$, up to and including the order m in every bounded domain contained in R_x. Partial derivatives of $\phi_a(x)$ with respect to x^μ will be denoted with the symbols

$$\begin{aligned} &\phi_a{}^{;}{}_\mu \equiv \frac{\partial \phi_a}{\partial x^\mu}, \qquad \phi^{a;}{}_\mu \equiv \frac{\partial \phi^a}{\partial x^\mu}, \qquad \phi_a{}^{;}{}_{\mu\nu} \equiv \frac{\partial^2 \phi_a}{\partial x^\mu\, \partial x^\nu}, \qquad \text{etc.;} \\ &\phi_a{}^{;\mu} \equiv \frac{\partial \phi_a}{\partial x_\mu}, \qquad \phi^{a;\mu} \equiv \frac{\partial \phi^a}{\partial x_\mu}, \qquad \phi^{a;\mu\nu} \equiv \frac{\partial^2 \phi_a}{\partial x^\mu\, \partial x^\nu}, \qquad \text{etc.;} \\ &\qquad x_\mu = g_{\mu\nu} x^\nu, \qquad x^\mu = g^{\mu\nu} x_\nu . \end{aligned} \tag{2.1}$$

Throughout this paper we shall only consider sets of fields $\Phi(x) = \{\phi_a(x)\}$ which can be partitioned into subsets $\Phi_1(x) = \{\phi_{a_1}(x)\},..., \Phi_r(x) = \{\phi_{a_r}(x)\}$, $a_i = 1,$ $2,..., r_i$; $i = 1, 2,..., r$; $r_1 + r_2 + \cdots + r_r = n$, each one transforming covariantly under a reducible or irreducible tensorial representation of the Poincaré group

$$\Phi_i(x) \xrightarrow{P(\Lambda, a)} \Phi_i'(x) = \Phi_i(\Lambda^{-1}(x - a)). \tag{2.2}$$

For instance, $\{\phi_a(x)\}$, $a = 1, 2,..., 6$, can represent the set constituted by the electromagnetic potentials $A_\mu(x)$, $\mu = 0, 1, 2, 3$, a complex scalar field $\varphi(x)$ and its *complex conjugate* field $\bar{\varphi}(x)$, i.e., $\{\phi_a\} = \{A_\mu ; \varphi; \bar{\varphi}\}$. The case of spinorial fields will be considered in a subsequent paper.

A set of functions $F = \{F_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha, \phi^{c;}{}_{\alpha\beta})\}$ $(a, c = 1, 2,..., n)$ will be termed of class $\mathscr{C}^m$ in a region R of the variables x_α, ϕ^c, $\phi^{c;}{}_\mu$, and $\phi^{c;}{}_{\mu\nu}$ when all of its elements F_a possess continuous partial derivatives with respect to all its variables up to and including the order m in every bounded domain contained in R. Here the region R is the Kronecker product $R_{x_\alpha} \otimes R_{\phi^c} \otimes R_{\phi^{c;}{}_\mu} \otimes R_{\phi^{c;}{}_{\mu\nu}}$. *Partial derivatives* of F_a with respect to the Minkowski coordinates will be denoted with

$$
\begin{aligned}
\partial_\mu F_a &= \frac{\partial F_a}{\partial x^\mu}, \qquad \partial_\mu \partial_\nu F_a = \frac{\partial^2 F_a}{\partial x^\mu \, \partial x^\nu}, \qquad \text{etc.,} \\
\partial^\mu F_a &= \frac{\partial F_a}{\partial x_\mu}, \qquad \partial_\mu \partial_\nu F_a = \frac{\partial^2 F_a}{\partial x_\mu \, \partial x_\nu}, \qquad \text{etc.,}
\end{aligned}
\tag{2.3}
$$

while for *total derivatives* we shall use the symbol

$$
\begin{aligned}
d_\alpha F_a &= \left\{ \partial_\alpha + \phi^{c;}{}_\alpha \frac{\partial}{\partial \phi^c} + \phi^{c;}{}_{\alpha\beta} \frac{\partial}{\partial \phi^{c;}{}_\beta} + \phi^{c;}{}_{\alpha\beta\gamma} \frac{\partial}{\partial \phi^{c;}{}_{\beta\gamma}} \right\} F_a \\
&= \left\{ \partial_\alpha + \phi_c{}^{;}{}_\alpha \frac{\partial}{\partial \phi_c} + \phi_c{}^{;}{}_{\alpha\beta} \frac{\partial}{\partial \phi_c{}^{;}{}_\beta} + \phi_c{}^{;}{}_{\alpha\beta\gamma} \frac{\partial}{\partial \phi_c{}^{;}{}_{\beta\gamma}} \right\} F_a, \\
d^\alpha F_a &= \left\{ \partial^\alpha + \phi^{c;\alpha} \frac{\partial}{\partial \phi^c} + \phi^{c;\alpha}{}_\beta \frac{\partial}{\partial \phi^{c;}{}_\beta} + \phi^{c;\alpha}{}_{\beta\gamma} \frac{\partial}{\partial \phi^{c;}{}_{\beta\gamma}} \right\} F_a,
\end{aligned}
\tag{2.4}
$$

and similarly for the case of total derivatives of higher order.

Throughout this paper, unless otherwise stated, we shall tacitly assume that all considered functions $F_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha, \phi^{c;}{}_{\alpha\beta})$ transform covariantly under the Lorentz group. We shall, however, preserve for completeness a possible explicit dependence on the coordinates.

3. Field Equations

We shall initially be concerned with (covariant) systems of n second-order partial differential equations of the type

$$
F_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha, \phi^{c;}{}_{\alpha\beta}) = 0, \qquad a = 1, 2,..., n,
\tag{3.1}
$$

which are not necessarily linear in any of their variables x_α, ϕ^c, $\phi^{c;}{}_\alpha$, and $\phi^{c;}{}_{\alpha\beta}$. We shall often refer to Eqs. (3.1) as the *nonlinear form* of the field equations. For reasons which will be self-evident later on, we shall only consider systems (3.1) whose functions F_a are of at least class $\mathscr{C}^2$ in a region R_{NL} of their variables. The *functional determinant* of Eqs. (3.1) is given by

$$d_{NL} = d_{NL}(R_{NL}) = | \partial F_a / \partial \phi^{b;}{}_{\mu\nu} | = | d^{\mu\nu}_{ab} |, \tag{3.2}$$

and its elements $d^{\mu\nu}_{ab}$ are also defined in R_{NL}. The most significant possibilities occur when the determinant d_{NL} is either identically null everywhere in $\bar{R}_{NL}$ or is not.

In the former (latter) case, by analogy with a terminology often used in the corresponding Newtonian cases [14], we shall say that system (3.1) is *regular* (*degenerate*). Notice that the regularity of a system does not exclude the existence of isolated zeros of its functional determinant [28]. If the rank of the functional matrix is zero, then system (3.1) reduces to a system of first-order partial differential equations. In this case we shall term system (3.1) *totally degenerate*. The analysis of this paper will be restricted to systems (3.1) which are not only of at least class $\mathscr{C}^2$ in a region R_{NL} of their variables, but also regular everywhere in it. Such systems will be indicated with the symbol

$$(F_a)^{\mathscr{C}^2, R} = 0. \tag{3.3}$$

As is well known, the Lagrange equations for continuous systems are always linear in the second-order partial derivatives $\phi^{c;}{}_{\mu\nu}$. Therefore, one of the most significant subclasses of systems (3.1) can be written

$$\begin{gathered} F_a = A^{\mu\nu}_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = 0, \\ a, b, c = 1, 2, \ldots, n, \qquad \mu, \nu, \alpha = 0, 1, 2, 3. \end{gathered} \tag{3.4}$$

Notice that the system is not necessarily linear in x_α, ϕ^c, and/or $\phi^{c;}{}_\alpha$. We shall refer to Eqs. (3.4) as the *quasi-linear* [29] *form* of the field equations.

The continuity properties of the functions F_a can now be assumed in a region R_{QL} of the variables x_α, ϕ^c, and $\phi^{c;}{}_\alpha$ only. The functional determinant of Eqs. (3.4) is now $d_{QL} = | A^{\mu\nu}_{ab} |(R_{QL})$.

A point of particular significance for our analysis is that the terms $A^{\mu\nu}_{ab}$ of Eqs. (3.4) can always be assumed to be symmetrical in the (μ, ν) indices, i.e.,

$$A^{\mu\nu}_{ab} \equiv A^{\nu\mu}_{ab}. \tag{3.5}$$

This is due to the fact that these indices are contracted with the totally symmetric derivatives $\phi^{b;}{}_{\mu\nu} = \partial_\mu \partial_\nu \phi^b = \phi^{b;}{}_{\nu\mu}$. And indeed, if the terms $A^{\mu\nu}_{ab}$ do not satisfy such properties, they can always be written $A^{\mu\nu}_{ab} = \frac{1}{2}(A^{\mu\nu}_{ab} + A^{\nu\mu}_{ab}) - \frac{1}{2}(A^{\mu\nu}_{ab} - A^{\nu\mu}_{ab})$. But the contraction of the antisymmetric parts with the symmetric terms $\phi^{b;}{}_{\mu\nu}$ is null. Thus, properties (3.5) always hold up to redefinition.

Throughout our analysis we shall always assume properties (3.5).

System (3.4) will be again termed regular (degenerate) in R_{QL} when determinant $|A_{ab}^{\mu\nu}|$ is everywhere non-null (null) in $\bar{R}_{QL}$. Similarly, we shall say that system (3.4) is totally degenerate when $A_{ab}^{\mu\nu} \equiv 0$ for all values of the indices, in which case it reduces to the generally nonlinear system of first-order equations $B_a = 0$. In line with assumptions (3.3) we shall consider only systems (3.4) which are of at least class $\mathscr{C}^2$ and regular in R_{QL}, i.e., systems of the type

$$(A_{ab}^{\mu\nu}\phi^{b;}{}_{\mu\nu} + B_a)^{\mathscr{C}^2, R} = 0. \tag{3.6}$$

The almost totality of free or interacting tensorial field equations nowadays considered constitutes the following subclass of Eqs. (3.4):

$$\begin{gathered} F_a = A_{ab}^{\mu\nu}\phi^{b;}{}_{\mu\nu} + B_a \equiv g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - f_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = 0, \\ g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} = \Box\phi_a, \end{gathered} \tag{3.7}$$

namely, it can be obtained from Eqs. (3.4) through the particularization

$$A_a{}^{b\mu\nu} = \delta_a{}^b \otimes g^{\mu\nu}, \qquad B_a = -f_a. \tag{3.8}$$

This is the case, for instance, for the equations of the electromagnetic potential, i.e., $\Box A_\mu = 0$, or the complex scalar fields in interaction with the electromagnetic field; etc. Equations of type (3.7) are called, in the theory of partial differential equations the *canonical form* [29] of system (3.4), and the transition from Eqs. (3.4) to (3.8) is called, the *reduction of the general quasi-linear form to the canonical form*.

This terminology is, however, confusing for our framework, particularly when the Hamiltonian formulation is considered. We shall therefore refer to Eqs. (3.7) as the *semilinear form* of the equations of motion. The reader should, however, be aware of the fact that what are customarily referred to as semilinear forms in the theory of partial differential equations are expressions of the type

$$g(x_\alpha)^{\mu\nu}\,\phi_a{}^{;}{}_{\mu\nu} - f_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha) = 0, \tag{3.9}$$

namely, when the coefficients $A_a{}^{b\mu\nu} = \delta_a{}^b \otimes g^{\mu\nu}(x_\alpha)$ of the quasi-linear form depend on the coordinate x_α only [29].

Nevertheless, equations of type (3.9) appear in a natural way when considering field theory in curved spaces. As a matter of fact, Eqs. (3.7) can sometimes be considered as the limit of Eqs. (3.9) to the (flat) Minkowski space. It is in this sense that we shall use the terminology "quasi-linear form" for Eqs. (3.7), too.

Notice that the reduction from the quasi-linear form to the semilinear form is always possible for $A_{ab}^{\mu\nu} = A_{ab} \otimes g^{\mu\nu}$, $|A_{ab}| \neq 0$. Indeed, in this case the inverse matrix $(A_{ab})^{-1} \equiv (A^{-1}{}_{ab})$ exists everywhere in R_{QL}. Then the implicit functions f_a of Eqs. (3.8) are given by

$$f_a = -A_a^{-1b}B_b. \tag{3.10}$$

Notice also that the hyperbolic character of the field equations under consideration is transparently exhibited by the quasi- and semi-linear form. This character will be tacitly assumed for all other types of considered field equations.

Our analysis of field equations in the semilinear form will be restricted to only those systems which are of at least class $\mathscr{C}^1$ in a region R_{SL} of the variables x_α, ϕ^c, and $\phi^{c;}{}_\alpha$ (this will later on correspond to Lagrangian densities which are of at least class $\mathscr{C}^3$ in R_{SL}), and we shall write

$$(g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - f_a)^{\mathscr{C}^2} = 0. \tag{3.11}$$

Notice that the condition of regularity is redundant for the semilinear form because the functional determinant $d_{\text{SL}} = | g^{\mu\nu} |$ is everywhere non-null. Therefore, semilinear forms of the field equations are always everywhere regular.

Another type of field equation we shall consider can be written

$$g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} + \alpha^\mu_{ab}(x_\alpha)\, \phi^{b;}{}_\mu + \beta_a(x_\alpha)\, \phi^b = 0, \tag{3.12}$$

namely, it is linear in ϕ^c, $\phi^{c;}{}_\alpha$, and $\phi^{c;}{}_{\alpha\beta}$. For this reason we shall refer to it as the *linear form* of the field equations. Again, the minimal continuity properties will be that all terms α^μ_{ab} and β_a are at least of class $\mathscr{C}^1$ in a region R_x'.

4. Equations of Variation

Consider the $(n + 4)$-dimensional region R_{n+4} with points $\{\phi^c, x_\alpha\}$, $a = 1, 2,\ldots, n$, $\alpha = 0, 1, 2, 3$. The equations $\phi^c = \phi^c(x)$ determine a hypersurface $R_n \subset R_{n+4}$. We shall assume that $\phi^c \in \mathscr{C}^2(R_n)$. Then $\phi^{c;}{}_\mu$ and $\phi^{c;}{}_{\mu\nu}$ exist and are continuous in R_n. We shall further assume that the matrix $(\phi^{c;}{}_\alpha)$ has rank n and, thus, R_n has dimension n.

As is customary in the calculus of variations for multiple integrals [23] we now consider an ∞^1-parameter family of fields $\Phi(x, w) = \{\phi^c(x, w)\}$ with $\phi^c \in \mathscr{C}^2(R_n)$, where w is a free parameter such that $| w | < \epsilon$ (or $w \in O_\epsilon$) and $\phi^c(x, w)|_{w=0} = \phi^c(x)$, This situation can be interpreted by saying that the hypersurface R_n depends on a free parameter, $R_n = R_n(w)$, and the x's can be regarded as Gaussian coordinates on $R_n(w)$ [23]. Consider two neighboring hypersurfaces $R_n(w)$ and $R_n(0)$. The variation of the fields in the transition from $R_n(0)$ to $R_n(w)$ at a fixed value of the coordinates is given by

$$\delta\phi^c = w\eta^c(x), \qquad \eta^c = (\partial\phi^c/\partial w)|_{w=0}, \qquad w \in O_\epsilon. \tag{4.1}$$

The functions $\eta^c(x)$ so defined will be termed the *variations* of the field $\phi^c(x)$. Clearly such variations possess the same continuity properties of the fields. Under the assumptions $\phi^c \in \mathscr{C}^2(R_n(w))$ we can therefore extend the above procedure and define the variations $\eta^{c;}{}_{\mu}$ and $\eta^{c;}{}_{\mu\nu}$ through the equations

$$\begin{aligned} \delta\phi^{c;}{}_{\mu} &= w\eta^{c;}{}_{\mu}, \qquad \eta^{c;}{}_{\mu} = (\partial\phi^{c;}{}_{\mu}/\partial w)|_{w=0}, \\ \delta\phi^{c;}{}_{\mu\nu} &= w\eta^{c;}{}_{\mu\nu}, \qquad \eta^{c;}{}_{\mu\nu} = (\partial\phi^{c;}{}_{\mu\nu}/\partial w)|_{w=0}. \end{aligned} \tag{4.2}$$

Then, $\eta^{c;}{}_{\mu} \in \mathscr{C}^1$ and $\eta^{c;}{}_{\mu\nu} \in \mathscr{C}^0$.

We shall term the *family of all admissible variations* of the fields $\phi^c(x)$ the set of all variations $\eta^c(x)$ which possess the same continuity properties of the ϕ's. The above parametric method of the construction of the variations $\eta^c(x)$ is introduced to simply indicate the possibility of constructing different elements of the family of admissible variations and, as such, it should not be considered either exhaustive or unique.

We are now equipped to introduce the first tool of central significance for our analysis, namely, the system of *equations of variation* of a given set of (tensorial) field equations.

Consider the nonlinear form

$$[F_a(x_\alpha, \phi^c, \phi^{c;}{}_{\alpha}, \phi^{c;}{}_{\alpha\beta})]^{\mathscr{C}^2, R} = 0, \tag{4.3}$$

computed along an ∞^1 family of class $\mathscr{C}^2$ fields $\phi^c(x, w)$. Then the system in question can be introduced through the equations

$$\begin{aligned} &M_a(x_\alpha, \eta^c, \eta^{c;}{}_{\alpha}, \eta^{c;}{}_{\alpha\beta}) \\ &\qquad = (dF_a/dw)|_{w=0} = a_{ab}(x_\alpha)\,\eta^b + b^{\mu}_{ab}(x_\alpha)\,\eta^{b;}{}_{\mu} + c^{\mu\nu}_{ab}(x_\alpha)\,\eta^{b;}{}_{\mu\nu} = 0, \end{aligned} \tag{4.4}$$

where all terms

$$a_{ab} = (\partial F_a/\partial\phi^b)|_{w=0}, \qquad b^{\mu}_{ab} = (\partial F_a/\partial\phi^{b;}{}_{\mu})|_{w=0}, \qquad c^{\mu\nu}_{ab} = (\partial F_a/\partial\phi^{b;}{}_{\mu\nu}) \tag{4.5}$$

now possess only a dependence on the coordinates x_α because the functions F_a are computed along assigned fields $\phi^c(x, w)$. For simplicity of notation we shall denote the forms (4.4) with the symbol $M_a(\eta)$.

A comparative analysis of systems (4.3) and (4.4) indicates that the system of equations of variation always constitutes a *linear* (homogeneous) system of partial differential equations in η^b, $\eta^{b;}{}_{\mu}$, and $\eta^{b;}{}_{\mu\nu}$, irrespective of whether the original system is linear or not.

The equations of variation for other forms of the field equations (Section 3) can be constructed accordingly.

5. Adjoint Systems of Variations

Consider the differential form $M_a(\eta)$ of variations (4.4) and let τ be the family of all admissible variations. A system $\tilde{M}_a(\tilde{\eta})$ with $\tilde{\eta} \in \tau$, $\tilde{\eta} \neq \eta$ is termed the *adjoint system* of variations $M_a(\eta)$ when there exists a current density $J^\mu(x_\alpha, \eta^c, \eta^{c;}{}_\alpha, \tilde{\eta}^c; \tilde{\eta}^{c;}{}_\alpha)$ such that

$$\tilde{\eta}^a M_a(\eta) - \eta^a \tilde{M}_a(\tilde{\eta}) = \partial_\mu J^\mu \tag{5.1}$$

for all admissible variations.

Apparently, condition (5.1) for the case of only one independent variable $x = t$ has been introduced for the first time by Lagrange [30]. We shall therefore term it the *Lagrange identity.*

We shall term system $\tilde{M}_a(\tilde{\eta})$ a *quasi-adjoint system* of variations $M_a(\eta)$ when the following integral version of Eqs. (5.1) holds:

$$\int_{R_x} d^4x \, [\tilde{\eta}^a M_a(\eta) - \eta^a \tilde{M}_a(\tilde{\eta})] = \int_{\partial R_x} d\sigma_\mu(x) \, J^\mu. \tag{5.2}$$

The above expression is the *Green formula* of the theory of partial differential equations [29].

It is evident that conditions (5.1) and (5.2) are *not* equivalent. Indeed, one can have (e.g., for nonlocal theories) systems $\tilde{M}_a(\tilde{\eta})$ which do not satisfy the Lagrange condition, but they are such that

$$\tilde{\eta}^a M_a(\eta) - \eta^a \tilde{M}_a(\tilde{\eta}) = \partial_\mu J^\mu + D, \tag{5.3}$$

where the additive density D is such that [31]

$$\int_{R_x} d^4x \, (\partial_\mu J^\mu + D) = \int_{\partial R_x} d\sigma_\mu(x) \, J^\mu. \tag{5.4}$$

In this case $\tilde{M}_a$ is *not* the adjoint system of M_a according to our definition. However, if Eqs. (5.3) and (5.4) hold with $D \neq 0$, then the system $\tilde{M}_a$ is quasi-adjoint.

This paper's analysis will be based on the use of the concept of adjoint systems with related Lagrange identities, and therefore we shall devote only marginal attention at this time to the concept of quasi-adjoint systems and related Green identities.

To identitfy the structure of the adjoint system $\tilde{M}_a$ we now write

$$\begin{aligned} \tilde{\eta}^a M_a = {} & \eta^b[\tilde{\eta}^a a_{ab} - \partial_\mu(\tilde{\eta}^a b^\mu_{ab}) + \partial_\nu\partial_\mu(\tilde{\eta}^a c^{\nu\mu}_{ab})] \\ & + \partial_\mu\{\tilde{\eta}^a b^\mu_{ab}\eta^b + \tilde{\eta}^a c^{\mu\nu}_{ab}\eta^{b;}{}_\nu - [\partial_\nu(\tilde{\eta}^a c^{\nu\mu}_{ab})]\,\eta^b\}. \end{aligned} \tag{5.5}$$

Thus, an explicit form of the adjoint system is

$$\tilde{M}_b(\tilde{\eta}) = \tilde{\eta}^a a_{ab} - \partial_\mu(\tilde{\eta}^a b^\mu_{ab}) + \partial_\nu \partial_\mu(\tilde{\eta}^a c^{\nu\mu}_{ab}), \tag{5.6}$$

while the current density of the Lagrange identity is

$$J^\mu(\eta, \tilde{\eta}) = \tilde{\eta}^a b^\mu_{ab} \eta^b + \tilde{\eta}^a c^{\mu\nu}_{ab} \eta^b{}_{,\nu} - [\partial_\nu(\tilde{\eta}^a c^{\nu\mu}_{ab})]\, \eta^b. \tag{5.7}$$

It is possible to prove that form (5.6) of the adjoint system is indeed unique under the assumed continuity and regularity conditions. Suppose that there are two adjoints $\tilde{M}_a$ and $\tilde{M}_a'$ and, therefore, two currents J^μ and J'^μ for the same system M_a. By subtracting the corresponding identities (5.1) and by integrating we reach the equation

$$\int_{R_x} d^4x\, \eta^a(\tilde{M}_a - \tilde{M}_a') = \int_{\partial R_x} d\sigma_\mu(x)(J^\mu - J'^\mu). \tag{5.8}$$

But this expression must be independent of points in the interior of the $(\eta, \tilde{\eta})$ space. This can be so iff $\tilde{M}_a \equiv \tilde{M}'_a$.

Notice that the quasi-adjoint system is not necessarily unique. This indicates another reason for preferring the Lagrange over the Green identity.

6. Conditions of Self-Adjointness

A system of differential forms M_a is termed *self-adjoint* when it coincides with the adjoint system $\tilde{M}_a$ for all admissible variations, i.e.,

$$M_a(\eta) \equiv \tilde{M}_a(\eta) \qquad \text{for all } \eta \in \tau. \tag{6.1}$$

The conditions of self-adjointness can be obtained by simply imposing the identity between forms (4.4) and (5.6), i.e.,

$$a_{ab}\eta^b + b^\mu_{ab}\eta^{b;}{}_\mu + c^{\mu\nu}_{ab}\eta^{b;}{}_{\mu\nu} \equiv \eta^b a_{ba} - \partial_\mu(\eta^b b^\mu_{ba}) + \partial_\nu\partial_\mu(\eta^b c^{\nu\mu}_{ba}). \tag{6.2}$$

In this way we reach the conditions which we term the *conditions of self-adjointness of the equations of variation*:

$$c^{\mu\nu}_{ab} = c^{\nu\mu}_{ba} = c^{\nu\mu}_{ab} = c^{\mu\nu}_{ba}, \tag{6.3a}$$

$$b^\mu_{ab} + b^\mu_{ba} = \partial_\nu c^{\nu\mu}_{ba} + \partial_\nu c^{\mu\nu}_{ba} = 2\partial_\nu c^{\mu\nu}_{ba}, \tag{6.3b}$$

$$a_{ab} - a_{ba} = \partial_\nu\partial_\mu c^{\nu\mu}_{ba} - \partial_\mu b^\mu_{ba}, \tag{6.3c}$$

where the last two identities of Eqs. (6.3a) hold under the same redefinition as for Eqs. (3.5).

We must inspect the possible conservation of the current J^μ of the Lagrange identity (5.1) under the condition of self-adjointness. The term $\partial_\mu J^\mu$, in view of Eqs. (5.7) can be written

$$\partial_\mu J^\mu = \tilde{\eta}^a[b^\mu_{ab}\eta^{b;}{}_\mu + c^{\mu\nu}_{ab}\eta^{b;}{}_{\mu\nu} + (\partial_\mu b^\mu_{ab})\,\eta^b - (\partial_\nu\partial_\mu c^{\nu\mu}_{ab})\,\eta^b] + \tilde{\eta}^{a;}{}_\mu[b^\mu_{ab}\eta^b - (\partial_\nu c^{\nu\mu}_{ab})\,\eta^b - (\partial_\nu c^{\mu\nu}_{ab})\,\eta^b] + \tilde{\eta}^{a;}{}_{\nu\mu}c^{\nu\mu}_{ab}\eta^b. \tag{6.4}$$

Suppose that one studies the possible self-adjointness of the system of *equations* $M_a = 0$ rather than of the differential *forms* M_a. Then the admissible variations η and $\tilde{\eta}$ are restricted to the solutions η_0 and $\tilde{\eta}_0$ of the system of equations of variation and its adjoint system, respectively. By using Eqs. (4.4) we first write

$$\partial_\mu J^\mu |_{\substack{\eta=\eta_0\\ \tilde{\eta}=\tilde{\eta}_0}} = [\tilde{\eta}^a_0(-a_{ab} + \partial_\mu b^\mu_{ab} - \partial_\nu\partial_\mu c^{\nu\mu}_{ab}) + (b^\mu_{ab} - \partial_\nu c^{\nu\mu}_{ab} - \partial_\nu c^{\mu\nu}_{ab}) - \tilde{\eta}^{a;}_0{}_{\nu\mu}c^{\nu\mu}_{ab}]\,\eta^b_0, \tag{6.5}$$

and by using the conditions of self-adjointness (6.3) we finally obtain the conservation law

$$\partial_\mu J\ |_{\substack{\eta=\eta_0\\ \tilde{\eta}=\tilde{\eta}_0}} = -(a_{ba}\tilde{\eta}^a_0 + b^\mu_{ba}\tilde{\eta}^{b;}_0{}_\mu + c^{\mu\nu}_{ba}\tilde{\eta}^{b;}_0{}_{\mu\nu})\,\eta^a = 0. \tag{6.6}$$

As an alternative to definition (6.1) we can therefore say that a system of differential equations $M_a = 0$ is termed *self-adjoint* when the current density J^μ of the Lagrange identity (5.1) is conserved along the solutions of the system, i.e.,

$$\partial_\mu J^\mu(\eta, \tilde{\eta})|_{\eta=\tilde{\eta}=\eta_0} = 0. \tag{6.7}$$

Notice that definitions (6.1) and (6.7) are equivalent. Indeed, *all* conditions of self-adjointness (6.3) enter into condition (6.7). Let us also recall that, under our assumptions, Eqs. (6.7) are the necessary and sufficient conditions for the surface independence [32]

$$[\delta/\delta\sigma(x)] \int_{\sigma=\partial R_x=\text{space-like}} d\sigma_\mu(x)\, J^\mu(\eta_0, \tilde{\eta}_0) = 0. \tag{6.8}$$

Therefore, conservation law (6.8) is also equivalent to Eq. (6.7).

A system of field equations is termed *self-adjoint* when its system of equations of variation is self-adjoint.

We shall now give the necessary and sufficient conditions of self-adjointness for the various forms of field equations considered in Section 3.

First of all, by using Eqs. (6.3) and (4.5) together with the uniqueness of the adjoint system we have

THEOREM 6.1. *Necessary and sufficient condition for class $\mathscr{C}^2$, regular, nonlinear, tensorial field equations*

$$[F_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha, \phi^{c;}{}_{\alpha\beta})]^{\mathscr{C}^2, R} = 0, \qquad a = 1, 2, \ldots, n \tag{6.9}$$

to be self-adjoint in a region R_{NL} of their variables is that all the following conditions

$$\frac{\partial F_a}{\partial \phi^{b;}{}_{\mu\nu}} = \frac{\partial F_b}{\partial \phi^{a;}{}_{\nu\mu}} = \frac{\partial F_a}{\partial \phi^{b;}{}_{\nu\mu}} = \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu\nu}}, \tag{6.10a}$$

$$\frac{\partial F_a}{\partial \phi^{b;}{}_{\mu}} + \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu}} = d_\nu \left(\frac{\partial F_b}{\partial \phi^{a;}{}_{\nu\mu}} + \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu\nu}}\right) = 2 d_\nu \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu\nu}}, \tag{6.10b}$$

$$\frac{\partial F_a}{\partial \phi^{b}} - \frac{\partial F_b}{\partial \phi^{a}} = d_\mu d_\nu \frac{\partial F_b}{\partial \phi^{a;}{}_{\nu\mu}} - d_\mu \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu}} = \frac{1}{2} d_\mu \left(\frac{\partial F_a}{\partial \phi^{b;}{}_{\mu}} - \frac{\partial F_b}{\partial \phi^{a;}{}_{\mu}}\right),$$
$$a, b = 1, 2, \ldots, n, \qquad \mu, \nu = 0, 1, 2, 3 \tag{6.10c}$$

are satisfied in every bounded domain in the interior of R_{NL}.

Notice that conditions (6.10) imply second-order derivatives. This illustrates the reason for our restriction to systems which are of at least class $\mathscr{C}^2$.

Equations (6.10) will be referred to here as the *conditions of self-adjointness of the nonlinear form* of the field equations. When *all* these conditions hold we shall symbolically write

$$(F_a)_{\mathrm{SA}}^{\mathscr{C}^2, R} = 0. \tag{6.11}$$

When at least *one* of Eqs. (6.10) is violated we shall say that the nonlinear form is *non-self-adjoint* and we shall write

$$(F_a)_{\mathrm{NSA}}^{\mathscr{C}^2, R} = 0. \tag{6.12}$$

Notice that brackets of the type $(\ )_{\mathrm{SA}}^{\mathscr{C}^m, R}$ are a *notation* and not an operation. Notice also that for the case under consideration, namely, (unconstrained) class $\mathscr{C}^2$ and regular field equations, to verify the self-adjointness it is sufficient to check identities (6.1) or (6.7) or (6.8) for only one admissible variation or to show that Eqs. (6.10) hold as identities between functions. The reader should be aware of the fact that for field theories with subsidiary constraints this framework is considerably altered because the constraints generally impose restrictions on the class of admissible variations [3]. This aspect, however, will not be considered at this time.

Let us note at this point that the nonlinear form of the field equations, even though it is the most general form for the class of field equations under consideration (i.e., of the second-order type), it ultimately possesses only a formal significance for our program. This is due to the fact, as we shall see better in Section 7, that the largest form of tensorial field equations which can be represented in terms of the Lagrange equations is the *quasi-linear* form.

The conditions of self-adjointness for the quasi-linear form of the field equations are therefore crucial for our program. It is a matter of simple calculations, which are given for the reader's convenience in Appendix A, to prove

THEOREM 6.2. *Necessary and sufficient conditions for class $\mathscr{C}^2$, regular, Lorentz-covariant, tensorial, quasi-linear systems of field equations*

$$[A^{\mu\nu}_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^2, R} = 0, \tag{6.13a}$$

$$A^{\mu\nu}_{ab} \equiv A^{\nu\mu}_{ab}, \tag{6.13b}$$

$$a, b, c = 1, 2, \ldots, n, \qquad \mu, \nu, \alpha = 0, 1, 2, 3,$$

to be self-adjoint in a region R_{QL} of their variables are that all the following conditions

$$A^{\mu\nu}_{ab} = A^{\nu\mu}_{ba} = A^{\nu\mu}_{ab}, \tag{6.14a}$$

$$A^{\nu\alpha;\alpha}_{ac\ \ b} + A^{\nu\alpha;\mu}_{bc\ \ a} = A^{\mu\overline{\nu;\alpha}}_{ba\ \ c}, \tag{6.14b}$$

$$A^{\overline{\alpha}\overline{\beta;\mu};\nu}_{a\underline{d\ b\ c}} = A^{\overline{\alpha}\overline{\beta;\mu};\nu}_{b\underline{da\ \ c}}, \tag{6.14c}$$

$$B_a{}^{;\mu}_{\ b} + B_b{}^{;\mu}_{\ a} = 2\{\partial_\nu + \phi^{c;}{}_\nu(\partial/\partial\phi^c)\}\, A^{\mu\nu}_{ab}, \tag{6.14d}$$

$$B_a{}^{;}{}_b - B_b{}^{;}{}_a = \tfrac{1}{2}\{\partial_\nu + \phi^{c;}{}_\nu(\partial/\partial\phi^c)\}(B_a{}^{;\nu}_{\ b} - B_b{}^{;\nu}_{\ a}); \tag{6.14e}$$

$$A^{\mu\nu;\alpha}_{ab\ \ c} \equiv (\partial A^{\mu\nu}_{ab}/\partial\phi^{c;}{}_\alpha), \qquad B_a{}^{;}{}_b \equiv (\partial B_a/\partial\phi^b), \qquad \text{etc.}, \tag{6.14f}$$

$$A^{\mu\overline{\nu;\alpha}}_{ab\ \ c} \equiv A^{\mu\nu;\alpha}_{ab\ \ c} + A^{\mu\alpha;\nu}_{ab\ \ c}, \tag{6.14g}$$

$$A^{\mu\nu;\alpha;\beta}_{a\underline{b\ c\ d}} \equiv A^{\mu\nu;\alpha;\beta}_{ab\ c\ d} + A^{\mu\nu;\alpha;\beta}_{ad\ c\ b}, \tag{6.14h}$$

$$a, b, c, d = 1, 2, \ldots, n, \qquad \mu, \nu, \alpha, \beta = 0, 1, 2, 3,$$

are verified in every bounded domain in the interior of R_{QL}.

Notice that the horizontal bars in Eqs. (6.14) denote symmetrization of the indicated indices as expressed by Eqs. (6.14g) and (6.14h). The symbolic derivative notation, as in Eqs. (6.14f) is here introduced for notational convenience. However, for the sake of clarity, we shall revert to the conventional notation whenever possible. Notice that Eqs. (6.14) contain partial derivatives up to and including the second order. This indicates the reason why our minimal continuity conditions are that Eqs. (6.13) be of class C^2. Notice also that symmetry properties (6.13b) hold under redefinition, as pointed out in Section 3.

We shall term Eqs. (6.14) the *conditions of self-adjointness of the quasi-linear form of field equations.* Quasi-linear systems will be termed self-adjoint when *all* conditions (6.14) hold, and we shall then symbolically write

$$(A_{ab}^{\mu\nu}\phi^{b;}{}_{\mu\nu} + B_a)_{\mathrm{SA}}^{\mathscr{C}^2, R} = 0. \tag{6.15}$$

If at least *one* of conditions (6.14) is violated we shall term the system *non-self-adjoint* and write

$$(A_{ab}^{\mu\nu}\phi^{b;}{}_{\mu\nu} + B_a)_{\mathrm{NSA}}^{\mathscr{C}^2, R} = 0. \tag{6.16}$$

As indicated earlier, the almost totality of field equations considered nowadays is of the semilinear rather than of the quasi-linear type. It is therefore useful for our analysis to work out the conditions of self-adjointness specifically for the semilinear form.

It is again a matter of simple calculations (given in Appendix B) to prove

THEOREM 6.3. *Necessary and sufficient conditions for class $\mathscr{C}^1$, semilinear, tensorial field equations*

$$[g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - f_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^1} = 0 \tag{6.17}$$

to be self-adjoint in a region R_{SL} of their variables are that the system is linear in the partial derivatives $\phi^c{}_{,\alpha}$, i.e., it is of the type

$$[g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - \rho_{ab}^{\mu}(x_\alpha, \phi^c)\, \phi^{b;}{}_\mu - \sigma_a(x_\alpha, \phi^c)]^{\mathscr{C}^1} = 0, \tag{6.18}$$

and all the conditions of self-adjointness

$$\rho_{ab}^{\mu} + \rho_{ba}^{\mu} = 0, \tag{6.19a}$$

$$(\partial\rho_{ab}^{\mu}/\partial\phi^c) + (\partial\rho_{bc}^{\mu}/\partial\phi^a) + (\partial\rho_{ca}^{\mu}/\partial\phi^b) = 0, \tag{6.19b}$$

$$\partial_\mu\rho_{ab}^{\mu} = (\partial\sigma_a/\partial\phi^b) - (\partial\sigma\,/\partial\phi^a), \tag{6.19c}$$

$$a, b, c = 1, 2,..., n, \qquad \mu = 0, 1, 2, 3,$$

are everywhere satisfied in every bounded domain in the interior of the subregion R'_{SL} of R_{SL} *with points* (x_α, ϕ^c).

Notice that conditions (6.19) imply only first-order partial derivatives and therefore the minimal continuity condition is that the semilinear form be of class $\mathscr{C}^1$ in R'_{SL}. Notice also that the derivatives of Eq. (6.19c) are partial. Therefore, if the terms ρ^μ_{ab} do not depend explicitly on the coordinates x_α, i.e., Eqs. (6.18) are of the type

$$[g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} - \rho^\mu_{ab}(\phi^c)\,\phi^{b;}{}_\mu - \sigma_a(x_\alpha, \phi^c)]^{\mathscr{C}^1} = 0, \tag{6.20}$$

then the conditions of self-adjointness become

$$\rho^\mu_{ab} + \rho^\mu_{ba} = 0, \tag{6.21a}$$

$$(\partial\rho^\mu_{ab}/\partial\phi^c) + (\partial\rho^\mu_{bc}/\partial\phi^a) + (\partial\rho^\mu_{ca}/\partial\phi^b) = 0, \tag{6.21b}$$

$$(\partial c_a/\partial\phi^b) - (\partial c_b/\partial\phi^a) = 0, \tag{6.21c}$$

$$a, b, c = 1, 2, \ldots, n, \qquad \mu = 0, 1, 2, 3.$$

When conditions (6.19) hold, the semilinear form (6.18) can be further specialized. Indeed, antisymmetry condition (6.19a) combined with the recursive condition (6.19b) implies that the factor coefficients ρ^μ_{ab} of the derivative terms $\phi^{b;}{}_\mu$ are the curl of some function, say $\Gamma_a{}^\mu(x_\alpha, \phi^c)$, i.e.,

$$\rho^\mu_{ab}(x_\alpha, \phi^c) = (\partial\Gamma_a{}^\mu/\partial\phi^b) - (\partial\Gamma_b{}^\mu/\partial\phi^a). \tag{6.22}$$

Suppose also that the additive term σ_a admits the decomposition

$$\sigma_a(x_\alpha, \phi^c) = -m^2_{(a)}\phi_a + \Lambda_a(x_\alpha, \phi^c), \tag{6.23}$$

where the m's are constants and there is *no summation* on the indices of the term $m^2_{(a)}\phi_a$. Then Eqs. (6.18) can be written in the form

$$(\Box + m^2_{(a)})\,\phi_a = \left(\frac{\partial\Gamma_a{}^\mu(x_\alpha, \phi^c)}{\partial\phi^b} - \frac{\partial\Gamma_b{}^\mu(x_\alpha, \phi^c)}{\partial\phi^a}\right)\phi^{b;}{}_\mu + \Lambda_a(x_\alpha, \phi^c), \tag{6.24}$$

which we term the *conventional form* of the field equations. Notice that conditions (6.19a) and (6.19b) are identically verified for this form and, as such, they can be ignored.

We are now equipped to study the self-adjointness or non-self-adjointness of some of the known cases of free or interacting tensorial fields.

COROLLARY 6.3A. *The equations for free tensorial fields in the conventional form*

$$(\Box + m^2_{(a)})\, \phi_a = 0, \qquad a = 1, 2,..., n \tag{6.25}$$

are always self-adjoint.

This is the case for instance, for the electromagnetic potential $\{\phi_a\} = \{A_\mu\}$, $m_a = 0$, the real scalar field $\phi = \varphi$, the complex scalar field $\{\phi_1, \phi_2\} = \{\varphi, \bar{\varphi}\}$, $m_1 = m_2$, or the complex vector field $\{\phi_a\} = \{\psi^\mu, \bar{\psi}^\mu\}$, $m_\mu = m$, $\mu = 0, 1, 2, 3$.

The "conventional" form of Eqs. (6.25) must be stressed. Indeed, if such equations are written, for instance, in the "unconventional" form

$$(1/\phi^a)(\Box\phi_a) + m^2_{(a)} = 0 \tag{6.26}$$

(no summation on repeated indices), then they are no longer self-adjoint. Indeed, the transition from Eqs. (6.25) to Eqs. (6.26) is in actuality a transition from the semilinear to the quasi-linear form. Therefore, the condition of self-adjointness (6.14) rather than (6.19) must now be used. By putting $A^{\mu\nu}_{ab} = \delta_{ab} \otimes g^{\mu\nu}/\phi_a$ and $B_a = m^2_{(a)}$, the only surviving condition originating from Eqs. (6.14) is given by

$$2\{\partial_\nu + \phi^c{}_{,\nu}(\partial/\partial\phi^c)\}\, A^{\mu\nu}_{ab} = 2[\partial_\nu(\phi^a)^{-1} - \phi^a{}_{,\nu}(\phi^a)^{-2}] = 0,$$

and it is clearly violated by the solutions $\phi_a(x)$ of the field equations (6.25).

Notice the insensitivity of the self-adjointness of Eqs. (6.25) for the behavior of the fields under discrete transformations. Thus, Eqs. (6.25) for $\{\phi_a\} = \{\varphi, \bar{\varphi}\}$ are self-adjoint irrespective of whether the represented field is scalar or pseudoscalar. Similarly for other cases. The case of internal symmetries or supersymmetries, however, demands a specific treatment and it will not be considered at this time.

COROLLARY 6.3B. *Necessary and sufficient condition for the equations of interacting tensorial fields without derivative couplings in the conventional form*

$$(\Box + m^2_{(a)})\, \phi_a = \Lambda_a(x_\alpha, \phi^c), \qquad a = 1, 2,..., n \tag{6.27}$$

to be self-adjoint in a region R'_{SL} of points $(x_\alpha\,;\, \phi^c)$ is that all the following conditions

$$(\partial\Lambda_a/\partial\phi_b) - (\partial\Lambda^a/\partial\phi^a) = 0, \qquad a, b = 1, 2,..., n \tag{6.28}$$

are verified in every bounded domain in the interior of R'_{SL}.

In particular, self-coupled models of the type

$$(\Box + m^2)\, \phi = \Lambda(\phi), \tag{6.29}$$

where $\Lambda(\phi)$ is any class $\mathscr{C}^1$ function of its argument are always self-adjoint. This is the case, for instance, of the ϕ^4 theories with spontaneous breaking of the $\phi \leftrightarrow -\phi$ symmetry [33]

$$(\Box + m^2)\, \phi = (\partial \Xi / \partial \phi),\ \Xi = (m^4/2g^2) + (g^2/2)\, \phi^4. \tag{6.30}$$

But, again, if the field equation is written in some "unconventional" form they are not necessarily self-adjoint.

Notice that the conditions of self-adjointness (6.28) are insensitive to a possible explicit dependence of the Λ terms in the coordinates.

COROLLARY 6.3C. *Necessary and sufficient conditions for the equations of interacting tensorial fields with derivative couplings in the conventional form*

$$(\Box + m^2_{(a)})\, \phi_a = j_a(x_\alpha\,,\, \phi^c,\, \phi^{c;}{}_\alpha) \tag{6.31}$$

to be self-adjoint in a region R_{SL} of points $(x_\alpha\,,\, \phi^c,\, \phi^{c;}{}_\alpha)$ are that the j terms possess a structure of the type

$$j_a = \left[\frac{\partial \Gamma_a{}^\mu(x_\alpha\,,\, \phi^c)}{\partial \phi^b} - \frac{\partial \Gamma_b{}^\mu(x_\alpha\,,\, \phi^c)}{\partial \phi^a}\right] \phi^{b;}{}_\mu + \Lambda_a(x_\alpha\,,\, \phi^c), \tag{6.32}$$

and all the following conditions

$$\partial_\mu \left(\frac{\partial \Gamma_a{}^\mu}{\partial \phi^b} - \frac{\partial \Gamma_b{}^\mu}{\partial \phi^a}\right) = \frac{\partial \Lambda_a}{\partial \phi^b} - \frac{\partial \Lambda_b}{\partial \phi^a} \tag{6.33}$$

are verified in every bounded domain in the interior of the subregion $R'_{\mathrm{SL}} \subset R_{\mathrm{SL}}$ with points $(x_\alpha\,,\, \phi^c)$.

Thus, when derivative couplings $\phi^{b;}{}_\mu$ occur, some necessary (but not sufficient) conditions to satisfy the requirement of self-adjointness are that they must occur linearly and their factor term must be the curl of some functions $\Gamma_a{}^\mu$.

For instance, the field equations

$$\begin{aligned} (\Box + m^2)\, \varphi &= i a^\mu \varphi^{;}{}_\mu\,, \\ (\Box + m^2)\, \bar{\varphi} &= -i a^\mu \bar{\varphi}^{;}{}_\mu \end{aligned} \tag{6.34}$$

for some constant four-vector a^μ are self-adjoint because there exist the functions

$$\Gamma_1{}^\mu = \Gamma_2{}^\mu = (i/2)\, a^\mu(\bar{\varphi} - \varphi) \tag{6.35}$$

such that

$$ia^\mu = \frac{\partial \Gamma_1^\mu}{\partial \bar{\varphi}} - \frac{\partial \Gamma_2^\mu}{\partial \varphi}, \qquad -ia^\mu = \frac{\partial \Gamma_2^\mu}{\partial \varphi} - \frac{\partial \Gamma_1^\mu}{\partial \bar{\varphi}}, \tag{6.36}$$

and Eqs. (6.33) hold.

The reader should be aware of the fact that the conditions of self-adjointness (6.14) for the quasi-linear form *do not* demand linearity of the derivative couplings.

A most important example of semilinear systems is constituted by the complex scalar field in interaction with the electromagnetic field with the conventional form of the field equations

$$\begin{aligned} \Box A_\mu &= ie[(\bar{\varphi}^{\cdot}{}_\mu + ieA_\mu\bar{\varphi})\,\varphi - \bar{\varphi}(\varphi^{\cdot}{}_\mu - ieA_\mu\varphi)], \\ (\Box + m^2)\,\varphi &= e^2 A_\mu A^\mu \varphi + 2ieA^\mu \varphi^{\cdot}{}_\mu\,, \\ (\Box + m^2)\,\bar{\varphi} &= e^2 A_\mu A^\mu \bar{\varphi} - 2ieA^\mu \bar{\varphi}^{\cdot}{}_\mu\,. \end{aligned} \tag{6.37}$$

It is easy to see that this system *is not* self-adjoint, because conditions (6.33) are generally violated despite the linearity of the derivative couplings (see also Appendix C). This case of "self-adjointness breaking" will play a significant role in our analysis.

Table I summarizes our findings.

We now briefly discuss a possible interplay between the quasi-linear and the semilinear forms. Here a digression into Newton's equations in configuration space may be instructive. The Newtonian equivalent of the semilinear form (6.17) is the system of second-order ordinary differential equations

$$\ddot{q}_k - f_k(t, q, \dot{q}) = 0, \qquad k = 1, 2,..., n. \tag{6.38}$$

An example is given by the system of coupled, linearly damped, and forced oscillators

$$\ddot{q}_k + b_{ki}(t)\,\dot{q}_i + a_{ki}(t)\,q_i - g_k(t) = 0. \tag{6.39}$$

Similarly, the Newtonian equivalent of the quasi-linear form (6.13) is given by the fundamental form of the equations of motion

$$A_{ki}(t, q, \dot{q})\,\ddot{q}_i + B_k(t, q, \dot{q}) = 0, \tag{6.40a}$$

$$A_{ki}\,, B_k \in \mathscr{C}^2, \qquad |\,A_{ij}\,| \neq 0. \tag{6.40b}$$

A significant example is given by the broader system of coupled oscillators

$$\begin{gathered} c_{ki}(t)\,\ddot{q}_i + b_{ki}(t)\,\dot{q}_i + a_{ki}(t)\,q_i - g_k(t) = 0, \\ |\,c_{ij}\,| \neq 0. \end{gathered} \tag{6.41}$$

TABLE I

Some Examples of Self-adjoint (SA) and Non-Self-adjoint (NSA) Tensorial Field Equations[a]

Case	Field equations $F_a(\phi) = 0$	Equations of variation $M_a(\eta) = 0$	Adjoint system $\tilde{M}_a(\tilde{\eta}) = 0$	SA	NSA
1	$\Box A_\mu = 0$	$\Box \eta_\mu{}^A = 0$	$\Box \tilde{\eta}_\mu{}^A = 0$	×	
2	$(\Box + m^2)\varphi = 0$	$(\Box + m^2)\eta^\varphi = 0$	$(\Box + m^2)\tilde{\eta}^\varphi = 0$	×	
3	$(\Box + m^2)\varphi + (\lambda/n)\varphi^n = 0$	$(\Box + m^2 + \lambda\varphi^{n-1})\eta^\varphi = 0$	$(\Box + m^2 + \lambda\varphi^{n-1})\tilde{\eta}^\varphi = 0$	×	
4	$(\Box + m^2)\varphi + (\lambda/n)\varphi^n$ $+\frac{1}{2}G\varphi;^\mu\varphi;_\mu = 0$	$(\Box + m^2 + \lambda\varphi^{n-1})\eta^\varphi$ $+G\varphi;^\mu\eta^\varphi;_\mu = 0$	$(\Box + m^2 + \lambda\varphi^{n-1})\tilde{\eta}^\varphi$ $-G\varphi;^\mu\tilde{\eta}^\varphi;_\mu = 0$		×
5	$(\Box + m^2)\psi_\mu = 0$	$(\Box + m^2)\eta_\mu{}^\psi = 0$	$(\Box + m^2)\tilde{\eta}_\mu{}^\psi = 0$	×	
6	$(\Box + m_\varphi{}^2)\varphi$ $+G_1\psi^\mu\varphi;_\mu = 0$ $(\Box + m_\psi{}^2)\psi_\mu$ $+G_2\varphi;_\mu$	$(\Box + m_\varphi{}^2)\eta^\varphi$ $+G_1(\psi^\mu\eta.^\varphi{}_\mu + \varphi;^\mu\eta_\mu{}^\psi) = 0$ $(\Box + m_\psi{}^2)\eta_\mu{}^\psi$ $+G_2\eta^\varphi;_\mu = 0$	$(\Box + m_\varphi{}^2 - G_1\psi^\mu,_\mu)\tilde{\eta}^\varphi$ $-G_1\psi^\mu\tilde{\eta};^\varphi{}_\mu - G_2\tilde{\eta}_\mu{}^\psi;^\mu = 0$ $(\Box + m_\psi{}^2)\tilde{\eta}_\mu{}^\psi$ $+G_1\varphi;_\mu\eta^\varphi = 0$		×
7	$-\Box A_\mu + ie[(\bar{\varphi};_\mu + ieA_\mu\bar{\varphi})\varphi$ $-\bar{\varphi}(\varphi;_\mu - ieA_\mu\varphi)] = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\varphi$ $-2ieA^\mu\varphi;_\mu = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\bar{\varphi}$ $+2ieA^\mu\bar{\varphi};_\mu = 0$	$-(\Box + 2e^2\bar{\varphi}\varphi)\eta_\mu{}^A$ $+ie[(\bar{\varphi};_\mu + 2ieA_\mu\bar{\varphi})\eta^\varphi$ $-\eta^{\bar{\varphi}}(\varphi;_\mu - 2ieA_\mu\varphi)]$ $-ie(\bar{\varphi}\eta^\varphi;_\mu - \varphi\eta^{\bar{\varphi}};_\mu) = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\eta^\varphi$ $-2ie(\varphi;^\mu - ieA^\mu\varphi)\eta_\mu{}^A$ $-2ieA^\mu\eta^\varphi;_\mu - ie\varphi\eta_\mu{}^A;^\mu = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\eta^{\bar{\varphi}}$ $+2ie(\bar{\varphi};^\mu + ieA^\mu\bar{\varphi})\eta_\mu{}^A$ $+2ieA^\mu\eta^\varphi;_\mu + ie\bar{\varphi}\eta_\mu{}^A;^\mu = 0$	$-(\Box + 2e^2\bar{\varphi}\varphi)\tilde{\eta}_\mu{}^A$ $+ie[(\bar{\varphi};_\mu + ieA_\mu\bar{\varphi})\tilde{\eta}^\varphi$ $-(\varphi;_\mu - ieA_\psi\varphi)\tilde{\eta}^\varphi] = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\tilde{\eta}^\varphi$ $+2ie(\bar{\varphi};^\mu + ieA^\mu\bar{\varphi})\tilde{\eta}_\mu{}^A$ $+2ieA^\mu\tilde{\eta}^\varphi;_\mu + ie\bar{\varphi}\tilde{\eta}_\mu{}^A;^\mu = 0$ $(\Box + m^2 - e^2A_\mu A^\mu)\tilde{\eta}^{\bar{\varphi}}$ $-2ie(\varphi;^\mu - ieA^\mu\varphi)\tilde{\eta}_\mu{}^A$ $-2ieA^\mu\tilde{\eta}^{\bar{\varphi}};_\mu - ie\varphi\tilde{\eta}_\mu{}^A;^\mu = 0$		×

[a] The equations of variation are computed by using Eqs. (4.4), and the adjoint system is computed by using Eqs. (5.6). Notice that all equations of variation and adjoint systems are linear. For instance, case 3, which is a self-coupled nonlinear model, possesses a linear equation of variation. This system is also self-adjoint. The subsequent example 4 constitutes a simple example of "self-adjointness-breaking" due to the presence of nonlinear derivative couplings. However, system 6, which does possess linear derivative couplings is also non-self-adjoint. The most conspicuous example of "self-adjointness-breaking" is constituted by example 7, namely, by the complex scalar field in interaction with the electromagnetic field. Notice the modification of the coupling terms from the field equations to the equations of variation.

Clearly, system (6.41) is more general than system (6.29) in view of the *acceleration couplings* which occur whenever $c_{ki} \neq N_i \delta_{ki}$, $N_i = \text{const.}$

However, system (6.41) can be reduced to a system of type (6.39) when it is regular, i.e., det $(c_{ij}) \neq 0$. Indeed, we can trivially write the system equivalent to (6.41),

$$\begin{gathered} \ddot{q}_k + b'_{ki}(t)\, \dot{q}_i + a'_{ki}(t)\, q_i - g'_k(t) = 0, \\ b'_{ki} = c^{-1}_{kj} b_{ji}, \qquad a'_{ki} = c^{-1}_{kj} a_{ji}, \\ g'_k = c^{-1}_{kj} g_j . \end{gathered} \tag{6.42}$$

More generally, when the conditions of the implicit function theorem apply, Eqs. (6.40) can be reduced, at least locally, to Eqs. (6.38) with the implicit functions given, trivially, by

$$f_k = -A^{-1}_{kj} B_j . \tag{6.43}$$

We can thus say that, under our continuity and regularity assumptions (6.40b), forms (6.38) and (6.40) are equivalent in the sense that each of these forms can be transformed into the other, and vice versa, through equivalence transforms.

As we have shown in [3], this equivalence of forms (6.38) and (6.40), even though it is trivial within the framework of ordinary differential equations, is not trivial for the problem of the existence of a Lagrangian, as well as for that of equivalent Lagrangians. This is due to the fact that the transition from Eqs. (6.38) to (6.40), and vice versa, does not in general preserve the self-adjointness or non-self-adjointness. Therefore, if a Lagrangian exists for one of these forms it does not necessarily exist for the other. Furthermore, there may exist different (but equivalent) self-adjoint forms, in which case there exist different Lagrangians all representing the same system.

The corresponding situation in field theory is considerably (although not entirely) equivalent to the above Newtonian framework. First of all, one can always transform the semilinear into a quasi-linear form through an equivalence mapping of the type

$$\begin{aligned} &\{h_a{}^b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)[g^{\mu\nu}\phi_b{}^;{}_{\mu\nu} - f_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^2, R}\}^{\mathscr{C}^2, R} \\ &\qquad = [A_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, g^{\mu\nu}\phi^{b;}{}_{\mu\nu} + B_\alpha(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^2, R} = 0. \end{aligned} \tag{6.44}$$

This, in essence, corresponds to the introduction of "acceleration couplings" similar in concept to those of system (6.41) in a way which preserves the equivalence of the old and the new systems.

It is then a matter of simple inspection to see that if the original semilinear system is self-adjoint (or non-self-adjoint), this is not necessarily the case for the equivalent quasi-linear form, and vice versa. This property will have significant implications for our analysis.

The inverse transition (from a quasi-linear to a semilinear form), however, does not appear to necessarily follow the Newtonian pattern. This is due to the fact that the terms $A^{\mu\nu}_{ab}$ *do not* generally admit the factorization

$$A^{\mu\nu}_{ab} = A_{ab} \otimes g^{\mu\nu}, \tag{6.45}$$

and therefore the inverse reduction of (6.44) is not in general trivial. For instance, when $n = 5$ and $\{\phi_a\} = \{\varphi, \psi^\mu\}$, where φ and ψ^μ are scalar and vector fields, respectively, one might conceive coupled equations of the type

$$\begin{gathered}
\psi^\alpha \varphi^{;}{}_\alpha(\Box + m_\varphi{}^2)\, \varphi + g_1\{\psi^\mu, \varphi^{;\nu}\}\, \varphi^{;}{}_{\mu\nu} = 0, \\
g_2 \varphi^{;}{}_\alpha\{\psi^\mu, \psi^\nu\}\, \varphi^{;}{}_{\mu\nu} + (\Box + m_\psi)\, \psi_\alpha = 0, \\
(A^{\mu\nu}_{ab}) = \begin{pmatrix} (\psi^\alpha \varphi^{;}{}_\alpha g^{\mu\nu} + g_1\{\psi^\mu, \varphi^{;\nu}\}) & (0) \\ (g_2 \varphi^{;}{}_\alpha\{\psi^\mu, \psi^\nu\}) & (g^{\mu\nu})_\alpha \end{pmatrix}, \\
\{A, B\} = AB + BA,
\end{gathered} \tag{6.46}$$

which reduce to free-field equations at the limit $g_1, g_2 \to 0$. In this case the (20×20) matrix $(A^{\mu\nu}_{ab})$ does not admit factorization (6.45) and the reduction to a semilinear form is not trivial.

Needless to say, system (6.46), although Lorentz covariant, *is not* "coupled" according to currently used rules. Nevertheless, the example is significant to illustrate that what we have termed the "quasi-linear form" of the field equations is not necessarily related in a trivial way to the conventional semilinear form and, as such, it might constitute a true generalization of conceivable ways of coupling tensorial fields.

As a final point we want to stress that this quasi-linear form *has not* been constructed ad hoc. It simply constitutes the most general form of tensorial field equations which can be represented in terms of Lagrange equations.

7. Lagrange Equations

We now consider the conventional *Lagrange equations*

$$\begin{gathered}
\mathscr{L}_a(\phi) = d_\mu(\partial\mathscr{L}/\partial\phi^{a;}{}_\mu) - (\partial\mathscr{L}/\partial\phi^a) = 0, \\
a = 1, 2, \ldots, n, \qquad \mu = 0, 1, 2, 3,
\end{gathered} \tag{7.1}$$

where $\mathscr{L} = \mathscr{L}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ is the *Lagrangian density* customarily defined in a region R_{LE} of its variables x_α, ϕ^c, and $\phi^{c;}{}_\alpha$. Equations (7.1) are the Euler equations of the hyperbolic multiple integral variational problem [34]

$$A(\phi) = \int_{R_x} d^4x\ \mathscr{L}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha), \tag{7.2}$$

and they can be explicitly written

$$\mathscr{L}_a(\phi) \equiv \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu}\, \phi^{b;}{}_{\mu\nu} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^b}\, \phi^{b;}{}_\mu + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial x^\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a} = 0 \tag{7.3}$$

by transparently exhibiting in this way their quasi-linear nature. When Eqs. (7.1) are linear, e.g., of the type

$$\mathscr{L}_a(\phi) = g^{\mu\nu}\phi_a{}^{;}{}_{\mu\nu} + \alpha^\mu_{ab}(x)\, \phi^{b;}{}_\mu + \beta_{ab}(x)\, \phi^b = 0, \tag{7.4}$$

we shall write them in the customary form

$$\mathscr{L}_a(\phi) = \partial_\mu(\partial\mathscr{L}/\partial\phi^{a;}{}_\mu) - (\partial\mathscr{L}/\partial\phi^a) = 0. \tag{7.5}$$

The reader should, however, be aware that form (7.5) is in general *erroneous* for our framework. We shall elaborate on this point later on in this section.

We shall say that Eqs. (7.1) are *regular* (*degenerate*) when their functional determinant (Hessian)

$$d_{\mathrm{LE}} = d_{\mathrm{LE}}(R_{\mathrm{LE}}) = \left| \frac{\partial \mathscr{L}_a}{\partial \phi^{b;}{}_{\mu\nu}} \right| = \left| \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu} \right| \tag{7.6}$$

is everywhere non-null (null) in a region R_{LE} of its variables $(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ [35]. We shall also say that Eqs. (7.1) are of class $\mathscr{C}^m$ when the Lagrangian density $\mathscr{L}$ is of class $\mathscr{C}^{m+2}$, $m = 0, 1, 2,..., \infty$ in R_{LE}, and vice versa.

We now restrict our analysis to Lagrange equations which are of at least class $\mathscr{C}^2$ and regular in R_{LE}, i.e.,

$$[\mathscr{L}_a(\phi)]^{\mathscr{C}^2, R} = 0. \tag{7.7}$$

A central aspect of the problem of the identification of the representational capabilities of Eqs. (7.7) is the study of their variational properties. According to the methodology of Section 4, suppose that Eqs. (7.1) are computed along an ∞^1-parameter family $\Phi(x; w) = \{\phi^c(x; w)\}$, $w \in O_\epsilon$ of solutions which is of at least class $\mathscr{C}^2$ in $R_n(w)$. The equations of variation of Eqs. (7.7) along this family are given by

$$\begin{aligned} \Omega_a(\eta) \equiv (d\mathscr{L}_a/dw)|_{w=0} = \partial\mu\, (\partial\Omega/\partial\eta^{a;}{}_\mu) - (\partial\Omega/\partial\eta^a) = 0,\\ a = 1, 2,..., n, \qquad \mu = 0, 1, 2, 3, \end{aligned} \tag{7.8}$$

where

$$\Omega(x_\alpha, \eta^c, \eta^{c;}{}_\alpha) = \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^{b;}{}_\nu}\,\eta^{a;}{}_\mu\eta^{b;}{}_\nu + 2\,\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^b}\,\eta^{a;}{}_\mu\eta^b + \frac{\partial^2 \mathscr{L}}{\partial\phi^a\,\partial\phi^b}\,\eta^a\eta^b\right)_{w=0}. \tag{7.9}$$

Equations (7.8) are customarily called in the calculus of variations the *Jacobi equations* and we shall preserve this terminology for our field theoretical framework too. More explicitly, Eqs. (7.8) can be written

$$\begin{aligned}
\Omega_a(\eta) &= \Big\{d_\mu\left(\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^{b;}{}_\nu}\,\eta^{b;}{}_\nu + \frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^b}\,\eta^{b;}{}_\nu\right) \\
&\quad - \left(\frac{\partial^2 \mathscr{L}}{\partial\phi^a\,\partial\phi^{b;}{}_\mu}\,\eta^{b;}{}_\mu + \frac{\partial^2 \mathscr{L}}{\partial\phi^a\,\partial\phi^b}\,\eta^b\right)\Big\}_{w=0} \\
&= \left[d_\mu\left(\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^b}\right) - \frac{\partial^2 \mathscr{L}}{\partial\phi^a\,\partial\phi^b}\right]_{w=0}\eta^b \\
&\quad + \left[d_\mu\left(\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^{b;}{}_\nu}\right) + \frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\nu\,\partial\phi^b} - \frac{\partial^2 \mathscr{L}}{\partial\phi^a\,\partial\phi^{b;}{}_\nu}\right]_{w=0}\eta^{b;}{}_\nu \\
&\quad + \left[\frac{\partial^2 \mathscr{L}}{\partial\phi^{a;}{}_\mu\,\partial\phi^{b;}{}_\nu}\right]_{w=0}\eta^{b;}{}_{\mu\nu},
\end{aligned} \tag{7.10}$$

and they are the Euler equations of the "accessory minimum problem"

$$A'(\eta) = \int_{R_x} d^4x\ \Omega(x_\alpha, \eta^c, \eta^{c;}{}_\alpha). \tag{7.11}$$

Notice that the value of the functional determinant of Eqs. (7.1), when computed along a family of solutions, coincides with the functional determinant of Eqs. (7.8). Therefore, when the Lagrange equations are regular, so are the Jacobi equations (and vice versa). More generally, the properties

$$[\Omega_a(\eta)]^{\mathscr{C}^1,R} = 0 \tag{7.12}$$

are a direct consequence of assumptions (7.7).

Equations (7.8) are strikingly similar in structure to the Lagrange equations. Nevertheless, Eqs. (7.7) and (7.8) are *not* equivalent, namely, they admit different families of solutions, unless Eqs. (7.7) are linear homogeneous, i.e., of type (7.4). Indeed, the Jacobi equations of system (7.4) are

$$\Omega_a(\eta) = g^{\mu\nu}\eta_a{}^{;}{}_{\mu\nu} + \alpha^\mu_{ab}(x)\,\eta^{b;}{}_\mu + \beta_{ab}(x)\,\eta^b = 0, \tag{7.13}$$

and, trivially, the families of solutions of Eqs. (7.4) and (7.13) coincide. In essence, the Lagrange equations are generally quasi-linear while the Jacobi equations are always linear and, thus, these two systems are not necessarily equivalent.

The *adjoint system of variations of the Lagrange equations*, from definition (5.6), is given by

$$\begin{aligned}\tilde{\Omega}_a(\tilde{\eta}) &= \tilde{\eta}^b \left[d_\mu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\mu\, \partial \phi^a}\right) - \frac{\partial^2 \mathscr{L}}{\partial \phi^b\, \partial \phi^a}\right]_{w=0} \\ &\quad - d_\nu \left\{\tilde{\eta}^b \left[d_\mu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\mu\, \partial \phi^{a;}{}_\nu}\right) + \frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\nu\, \partial \phi^a} - \frac{\partial^2 \mathscr{L}}{\partial \phi^b\, \partial \phi^{a;}{}_\nu}\right]_{w=0}\right\} \\ &\quad + d_\nu d_\mu \left[\tilde{\eta}^b \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\mu\, \partial \phi^{a;}{}_\nu}\right)_{w=0}\right] \\ &= \tilde{\eta}^b \left[d_\mu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^b\, \partial \phi^{a;}{}_\mu}\right) - \frac{\partial^2 \mathscr{L}}{\partial \phi^b\, \partial \phi^a}\right]_{w=0} \\ &\quad + \tilde{\eta}^{b;}{}_\nu \left[d_\mu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\mu\, \partial \phi^{a;}{}_\mu}\right) + \frac{\partial^2 \mathscr{L}}{\partial \phi^b\, \partial \phi^{a;}{}_\nu} - \frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\nu\, \partial \phi^a}\right]_{w=0} \\ &\quad + \tilde{\eta}^{b;}{}_{\mu\nu} \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{b;}{}_\mu\, \partial \phi^{a;}{}_\nu}\right)_{w=0}.\end{aligned} \tag{7.14}$$

The following theorem can now be trivially proved.

THEOREM 7.1. *Class $\mathscr{C}^2$ regular Lagrange equations for tensorial fields are always self-adjoint in their region of definition.*

Notice that the identity of the Jacobi equations (7.8) with the adjoint system (7.14), i.e.,

$$\Omega_a(\eta) \equiv \tilde{\Omega}_a(\eta), \tag{7.15}$$

reduces to the identities

$$\begin{aligned} d_\mu(\partial^2 \mathscr{L}/\partial \phi^{a;}{}_\mu\, \partial \phi^b) &\equiv d_\mu(\partial^2 \mathscr{L}/\partial \phi^b\, \partial \phi^{a;}{}_\mu), \\ (\partial^2 \mathscr{L}/\partial \phi^a\, \partial \phi^b) &\equiv (\partial^2 \mathscr{L}/\partial \phi^b\, \partial \phi^a), \\ (\partial^2 \mathscr{L}/\partial \phi^{a;}{}_\nu\, \partial \phi^b) &\equiv (\partial^2 \mathscr{L}/\partial \phi^b\, \partial \phi^{a;}{}_\nu), \\ (\partial^2 \mathscr{L}/\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu) &\equiv (\partial^2 \mathscr{L}/\partial \phi^{b;}{}_\nu\, \partial \phi^{a;}{}_\mu), \end{aligned} \tag{7.16}$$

which are always satisfied in view of the assumed continuity properties of the Lagrangian density.

Theorem 7.1 can be proved in several other ways. For instance, although it is more laborious, one can prove the equations' self-adjointness by showing that all conditions (6.10) or (6.14) are verified for Eqs. (7.7). This is left as an exercise for the interested reader. For comments on this point see the end of this section.

Notice that the assumption $\mathscr{L} \in \mathscr{C}^3(R_{\text{LE}})$ is sufficient to establish the continuity of the Jacobi equations but it is generally insufficient for the continuity of the adjoint system in view of the fourth-order derivatives appearing in Eqs. (7.14).

We must now inspect the structure of the conserved current density associated with the Lagrange identity (5.1). From definition (5.7) and Eqs. (7.10) we can write

$$\begin{aligned} J^\mu(\eta, \tilde{\eta}) &= \tilde{\eta}^a b_{ab}^\mu \eta^b + \tilde{\eta}^a c_{ac}^{\mu\nu} \eta^{b;}{}_\nu - [\partial_\nu(\tilde{\eta}^a c_{ab}^{\nu\mu})]\, \eta^b \\ &= \tilde{\eta}^a \left[d_\nu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu\, \partial \phi^{b;}{}_\mu} \right) + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^b} - \frac{\partial^2 \mathscr{L}}{\partial \phi^a\, \partial \phi^{b;}{}_\mu} \right]_{w=0} \eta^b \\ &\quad + \tilde{\eta}^a \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu} \right)_{w=0} \eta^{b;}{}_\nu - \left\{ d_\nu \left[\tilde{\eta}^a \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu\, \partial \phi^{b;}{}_\mu} \right) \right] \right\}_{w=0} \eta^b \\ &= \tilde{\eta}^a \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^b} - \frac{\partial^2 \mathscr{L}}{\partial \phi^a\, \partial \phi^{b;}{}_\mu} \right)_{w=0} \eta^b \\ &\quad + \tilde{\eta}^a \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu\, \partial \phi^{b;}{}_\nu} \right)_{w=0} \eta^{b;}{}_\nu - \tilde{\eta}^{a;}{}_\nu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu\, \partial \phi^{b;}{}_\mu} \right)_{w=0} \eta^b \\ &\quad + \tilde{\eta}^a \left[d_\nu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu\, \partial \phi^{b;}{}_\mu} \right) - d_\nu \left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu\, \partial \phi^{b;}{}_\mu} \right) \right]_{w=0} \eta^b. \end{aligned} \tag{7.17}$$

The reader should be alerted that a proper symmetrization of the indices of the above expressions is here tacitly assumed. We shall comment on this point later on in this section.

When the terms within brackets are computed along a solution of the Lagrange equations and the current J^μ is restricted along the solutions of the Jacobi equations, we have $\eta \equiv \tilde{\eta}$ and $J^\mu \equiv 0, \quad \mu = 0, 1, 2, 3.$

It should be noted that Theorem 7.1 is a *variational* formulation of the known property that the Lagrange *operator*

$$\mathscr{L}_a \equiv d_\mu(\partial/\partial \phi^{a;}{}_\mu) - (\partial/\partial \phi^a) \tag{7.18}$$

is self-adjoint in the conventional sense of the theory of linear operator [36]. This also indicates the reason for preserving the term "self-adjointness" in our variational treatment. We would also like to indicate that, according to our best knowledge at this time, the variational approach to self-adjointness appears to be more effective than the operational approach for the problem of identifying the conditions under which a Lagrangian exists in classical field theories, as we shall see better in the subsequent papers. It is for this reason that we have preferred the former over the latter approach.

A few comments on the structure of the Lagrange equations are in order. First of all let us note that, as the reader can verify with a simple inspection, in the currently available textbooks in field theory the Lagrange equations are written with *partial* derivatives ∂_μ, as in form (7.5), while in all textbooks in the calculus of variations for multiple integrals such equations are written with *total* derivatives d_μ, as in form (7.1) or (7.3). In the opinion of this author the former approach is incorrect while the latter is correct. This is due to the fact that the proper handling of the variational techniques which underlie the derivation of such equations from an action principle demands the use of the *total* rather than partial derivatives. Besides, if Eqs. (7.5) were correct, then their Newtonian counterpart should be given by the form

$$\frac{\partial}{\partial t}\frac{\partial L(t, q, \dot{q})}{\partial \dot{q}_k} - \frac{\partial L(t, q, \dot{q})}{\partial q_k} = 0, \qquad k = 1, 2,..., n, \tag{7.19}$$

which is known to be erroneous, while the correct form is that of Eqs. (1.4).

The above remark, however, must be put in its proper perspective. Indeed, for the vast majority of Lagrangian densities used nowadays, which are essentially constituted by a kinetic term plus an additive term, as in the form

$$\mathscr{L} = \tfrac{1}{2}\phi^{a;}{}_\alpha \phi_a{}^{;b} + B(\phi), \tag{7.20}$$

Eqs. (7.3) and (7.5) produce identical equations of motion, in which case they are equivalent for all practical purposes. This, however, is no longer the case for more general models, such as chiral Lagrangians of the type

$$\mathscr{L} = \tfrac{1}{2}\phi_a{}^{;}{}_\alpha G^{ab}(\phi)\, \phi_b{}^{;\alpha} + B(\phi). \tag{7.21}$$

The analysis of these papers relates to Lagrangian densities which, besides satisfying the assumed continuity, regularity, and Lorentz-invariance conditions, have an arbitrary functional dependence in their arguments. As a result, the proper form of the Lagrange equations is (7.1) or (7.3).

It should be indicated that these Lagrangians, in general, satisfy the properties

$$(\partial^2\mathscr{L}/\partial\phi^{a;}{}_\mu\partial\phi^{b;}{}_\nu) \neq (\partial^2\mathscr{L}/\partial\phi^{a;}{}_\nu\partial\phi^{b;}{}_\mu) \tag{7.22}$$

without being in conflict with the continuity assumptions, i.e.,

$$(\partial^2\mathscr{L}/\partial\phi^{a;}{}_\mu\partial\phi^{b;}{}_\nu) \equiv (\partial^2\mathscr{L}/\partial\phi^{b;}{}_\nu\partial\phi^{a;}{}_\mu) \tag{7.23}$$

This fact demands few comments in relation to our variational approach to self-adjointness. Indeed, properties (7.22) are in apparent violation of the conditions of self-adjointness (6.10a) or (6.14a). This apparent contradiction is, however, easily resolved by recalling, from Section 3, that the terms $\partial^2\mathscr{L}/\partial\phi^{a;}{}_\mu\partial\phi^{b;}{}_\nu$ are contracted with the totally symmetric derivatives $\phi^{b;}{}_{\mu\nu} \equiv \phi^{b;}{}_{\nu\mu}$. As a result, Eqs. (7.3) can be equivalently written

$$\mathscr{L}_a(\phi) \equiv \frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu \, \partial \phi^{b;}{}_\nu} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu \, \partial \phi^{b;}{}_\mu}\right) \phi^{b;}{}_{\mu\nu} + \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu \, \partial \phi^b} \phi^{b;}{}_\mu$$

$$+ \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu \, \partial x^\mu} - \frac{\partial \mathscr{L}}{\partial \phi^a} = 0 \tag{7.24}$$

in view of the identities

$$\frac{1}{2}\left(\frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\mu \, \partial \phi^{b;}{}_\nu} - \frac{\partial^2 \mathscr{L}}{\partial \phi^{a;}{}_\nu \, \partial \phi^{b;}{}_\mu}\right) \phi^{b;}{}_{\mu\nu} \equiv 0. \tag{7.25}$$

Of course the same symmetrization applies for the the Jacobi equations (7.10). The conditions of self-adjointness (6.10a), i.e.,

$$\frac{\partial \mathscr{L}_a}{\partial \phi^{b;}{}_{\mu\nu}} = \frac{\partial \mathscr{L}_b}{\partial \phi^{a;}{}_{\mu\nu}} = \frac{\partial \mathscr{L}_a}{\partial \phi^{b;}{}_{\nu\mu}} = \frac{\partial \mathscr{L}_b}{\partial \phi^{a;}{}_{\mu\nu}} \tag{7.26}$$

are therefore identically verified for Eqs. (7.24) in view of the assumed continuity conditions of the Lagrangian density.

Throughout these papers we shall tacitly assume that, whenever properties (7.22) hold, the explicit form of the Lagrange equations is given by Eqs. (7.24).

It might be of some significance to indicate the criterion we have followed in deriving and presenting the conditions of self-adjointness (6.10). In essence, such conditions were derived (by having in mind the structure of the Lagrange equations) first for the general case (7.22), and then the symmetry of those terms in the greek indices was imposed. As a result, the first identities in the lhs of Eqs. (6.10), i.e.,

$$\frac{\partial F_a}{\partial \phi^b{}_{,\mu\nu}} = \frac{\partial F_b}{\partial \phi^a{}_{,\nu\mu}}, \tag{7.27a}$$

$$\frac{\partial F_a}{\partial \phi^b{}_{,\mu}} + \frac{\partial F_b}{\partial \phi^a{}_{,\mu}} = d_\nu \left(\frac{\partial F_b}{\partial \phi^a{}_{,\nu\mu}} + \frac{\partial F_b}{\partial \phi^a{}_{,\mu\nu}}\right), \tag{7.27b}$$

$$\frac{\partial F_a}{\partial \phi^b} - \frac{\partial F_b}{\partial \phi^a} = \frac{1}{2} d_\mu \left(\frac{\partial F_a}{\partial \phi^b{}_{,\mu}} - \frac{\partial F_b}{\partial \phi^a{}_{,\mu}}\right), \tag{7.27c}$$

apply for the Lagrange equations in the form (7.3). The full set of conditions of self-adjointness (6.10) must, however, be applied to the Lagrange equations in their actual form (7.24).

This produces an alternative to our proof of Theorem 7.1, namely, that class $\mathscr{C}^2$, regular Lagrange equations are self-adjoint.

A further alternative proof of this crucial property can be achieved by using the conditions of self-adjointness of the quasi-linear form, i.e., Eqs. (6.14). In this case, however, the use of the actual form (7.24) of the Lagrange equations is advisable.

8. Analytic Representations

The main objective of these papers is to assign a given (covariant, tensorial) system of quasi-linear field equations and then study the conditions under which a Lagrangian density capable of "representing" that system exists. For this objective it is essential to clarify the concept of an "analytic representation," namely, a representation of the system in terms of the Lagrange equations.

In principle, we can say that given a quasi-linear system of field equations

$$[A^{\mu\nu}_{ab}(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)\, \phi^{b;}{}_{\mu\nu} + B_a(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)]^{\mathscr{C}^2,R} = 0, \tag{8.1}$$

its "analytic representation" in terms of the Lagrange equations

$$[\mathscr{L}_a(\phi)]^{\mathscr{C}^2,R}_{\text{SA}} = 0 \tag{8.2}$$

exists when the general solutions [37] of systems (8.1) and (8.2) coincide. In practice, however, the above definition can predictably be faced with severe difficulties, e.g., when the equations are nonlinear or when they admit singular solutions.

In order to circumvent these difficulties we shall say that Eqs. (8.1) admit an *analytic representation* in terms of Eqs. (8.2) in a region R_{LE} of points $(x_\alpha, \phi^c, \phi^c{}_\alpha)$ when there exist n^2 functions $h_a{}^b = h_a{}^b(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$ which are of at least class $\mathscr{C}^2$ and whose matrix $(h_a{}^b)$ is everywhere regular in R_{LE}, such that

$$[\mathscr{L}_a(\phi)]^{\mathscr{C}^2,R}_{\text{SA}} \equiv [h_a{}^b(A^{\mu\nu}_{bc}\phi^{c;}{}_{\mu\nu} + B_b)^{\mathscr{C}^2,R}]^{\mathscr{C}^2,R} = 0, \qquad a = 1, 2,..., n. \tag{8.3}$$

Alternatively, we can say that an analytic representation exists when the Lagrange equations coincide with the field equations up to an equivalence transform, i.e.,

$$\{g_a{}^b[\mathscr{L}_b(\phi)]^{\mathscr{C}^2,R}_{\text{SA}}\}^{\mathscr{C}^2,R} \equiv [A^{\mu\nu}_{ac}\phi^{c;}{}_{\mu\nu} + B_a]^{\mathscr{C}^2,R} = 0, \qquad a = 1, 2,..., n. \tag{8.4}$$

Clearly, definitions (8.3) and (8.4) are equivalent. Since both matrices $(h_a{}^b)$ and $(g_a{}^b)$ are everywhere regular in R_{LE} by assumption, their inverses, say $(h^{-1}{}_a{}^b)$ and $(g^{-1}{}_a{}^b)$, exist. When representation (8.3) exists for given $\mathscr{L}$ and $h_a{}^b$, then representation (8.4) exists too, trivially, with $g_a{}^b = h^{-1}{}_a{}^b$ and vice versa. Notice that we *exclude* a dependence of the *factor functions* $h_a{}^b$ or $g_a{}^b$ on the second-order derivatives, although we *do* assume a possible nontrivial dependence in $(x_\alpha, \phi^c, \phi^{c;}{}_\alpha)$.