

An Introduction to The Lie–Santilli Isotopic Theory

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Lie's theory in its current formulation is linear, local and canonical. As such, it is not applicable to a growing number of non-linear, non-local and non-canonical systems which have recently emerged in particle physics, superconductivity, astrophysics and other fields. In this paper, which is written by a physicist for mathematicians, we review and develop a generalization of Lie's theory proposed by the Italian–American physicist R. M. Santilli back in 1978 when at the Department of Mathematics of Harvard University and today called *Lie–Santilli isotopy*. The latter theory is based on the so-called *isotopies* which are non-linear, non-local and non-canonical maps of any given linear, local and canonical theory capable of reconstructing linearity, locality and canonicity in certain generalized spaces and fields. The emerging Lie–Santilli isotopy is remarkable because it preserves the abstract axioms of Lie's theory while being applicable to non-linear, non-local and non-canonical systems. After reviewing the foundations of the Lie–Santilli isogebras and isogroups, and introducing seemingly novel advances in their interconnections, we show that the Lie–Santilli isotopy provides the invariance of all infinitely possible (well-behaved), non-linear, non-local and non-canonical deformations of conventional Euclidean, Minkowskian or Riemannian invariants. We also show that the non-linear, non-local and non-canonical symmetry transformations of deformed invariants are easily computable from the linear, local and canonical symmetry transforms of the original invariants and the given deformation. We then briefly indicate a number of applications of the isotopy in various fields. Numerous rather fundamental and intriguing, open mathematical and physical problems are indicated during the course of our analysis.

1. Introduction

1.1. Limitations of Lie's theory

As it is well-known, Lie's theory has permitted outstanding achievements in various disciplines. Nevertheless, in its current conception [30] and realization (see, e.g. [13] for a physical treatment and [15] for a mathematical presentation), Lie's theory is *linear, local-differential and canonical-Hamiltonian*. As such, it possesses clear limitations.

An illustration is provided by the historical distinction introduced by Lagrange [29], Hamilton [14] and others between the *exterior dynamical problems* in vacuum and the *interior dynamical problems* within physical media. Exterior problems consist of particles which can be effectively approximated as being point-like while moving within the homogeneous and isotropic vacuum under action-at-a-distance interactions (such as a space-ship in a stationary orbit around Earth). The point-like character of particles permits the exact validity of conventional local-differential

topologies (e.g., the Zeeman topology in special relativity); the homogeneity and isotropy of space then allow the exact validity of the geometries underlying Lie's theory (such as the Riemannian geometry); and the action-at-a-distance interactions assures their representation via a potential with consequential canonical character.

Interior problems consists of extended, and therefore deformable particles moving within inhomogeneous and anisotropic physical media, with action-at-a-distance as well as contact-resistive interactions (such as a space-ship during re-entry in Earth's atmosphere). In the latter case the forces are of local-differential type (e.g., potential forces acting on the centre-of-mass of the particle) as well as of non-local-integral type (e.g., requiring an integral over the surface of the body), thus rendering inapplicable conventional local-differential topologies; the inhomogeneity and anisotropy of the medium imply the inapplicability of conventional geometries for their quantitative treatment; while contact-resistive interactions violate Helmholtz's conditions for the existence of a potential (the *conditions of variational selfadjointness* [49]), thus implying the non-canonical character of interior systems.

We can therefore say that Lie's theory in its conventional linear, local and canonical formulation is *exactly valid* for all exterior dynamical problems, while it is *inapplicable* (and not 'violated') for the more general interior dynamical problems on topological, geometrical, analytic and other grounds.

1.2. The need for a suitable generalization of Lie's theory

Lie's theory is currently applied to non-linear, non-local and non-canonical systems via their simplification into more treatable forms, e.g., via the expansion of non-local-integral terms into power series in the velocities and then the transformation of the system into a co-ordinate frame in which it admits a Hamiltonian via the Lie-Koenig Theorem or, equivalently, via a Darboux map [49].

However, assuming that a given non-local-integral interior system admits a local-differential non-Hamiltonian approximation, the transformations of a non-Hamiltonian system into a Hamiltonian form are necessarily (non-canonical and) non-linear. This implies the known fact that the Darboux chart is not realizable in actual experiment and, if used, it implies the necessary abandonment of conventional relativities, evidently because the transformed frames are highly non-inertial (see [49] for technical details). This establishes the need for a suitable generalization of Lie's theory which is applicable to local-differential non-Hamiltonian systems *in the co-ordinates of their experimental verification*. Only after achieving such a theory the use of co-ordinate transformations may acquire practical significance.

Moreover, non-linear, non-local and non-Hamiltonian interior systems cannot be, in general, consistently reduced or transformed into linear, local and Hamiltonian ones. An illustration exists in gravitation. The distinction between exterior and interior gravitational problems was in full use in the early part of this century (see, e.g., Schwarzschild's two papers, the first celebrated paper [74] on the exterior problem and the second little known paper [75] on the interior problem). The distinction was then kept in early well-written treatises in the field (see, e.g., [4, 38]). The distinction was then progressively abandoned up to the current treatment of all gravitational problems, whether interior or exterior, via the same local-differential Riemannian geometry.

The above trend was based on the belief that interior dynamical problems within physical media can be effectively reduced to a collection of exterior problems in

vacuum (e.g., the reduction of a space-ship during re-entry in our atmosphere to its elementary constituents moving in vacuum).

It is important for this paper to know that the above reduction is mathematically impossible. For instance, the so-called *No-Reduction Theorems* [54] prohibit the reduction of a macroscopic interior system (such as satellite during re-entry) with a *monotonically decreasing angular momentum*, to a finite collection of elementary particles each one with a *conserved angular momentum*, and *vice versa*.

On geometrical grounds, gravitational collapse and other interior gravitational problems are not composed of ideal points, but instead of a large number of extended and hyperdense particles (such as protons, neutrons and other particles) in conditions of total mutual penetration, as well as of compression in large numbers into small regions of space. This implies the emergence of a structure which is arbitrarily non-linear (in co-ordinates and velocities), non-local-integral (in various quantities) and non-Hamiltonian (variationally non-self-adjoint).

Additional insufficiencies of the current formulation of Lie's theory and of its underlying geometries exist for the characterization of antimatter, e.g., because of the lack of a suitable (e.g., antiautomorphic) map which permits the characterization of antimatter, first, at the classical-astrophysical level, and then at the level of its elementary constituents.

Similar occurrences have recently emerged in astrophysics, superconductivity, theoretical biology and other disciplines. These occurrences establish the need for a *generalization of the conventional Lie theory which is directly applicable* (i.e., applicable without approximation or transformations) to *non-linear, integro-differential and variationally non-self-adjoint systems* for the characterization of matter. The theory should then possess a suitable antiautomorphic map for the effective characterization of antimatter.

1.3. Santilli's isotopies and isodualities of Lie's theory

In a seminal memoir [47] written in 1978 when at the Department of Mathematics of Harvard University under support from the U.S. Department of Energy, the Italian-American scholar Ruggero Maria Santilli proposed a step-by-step generalization of the conventional Lie theory specifically conceived for non-linear, integro-differential and non-canonical equations. The generalized theory was subsequently studied by Santilli in Refs. [48-72], as well as by a number of mathematicians and theoreticians, and it is today called *Lie-Santilli isotopic theory or isotheory* (see papers [1, 2, 6, 11, 12, 16-23, 25, 32, 33, 35-37, 40-43], monographs [3, 24, 31, 76] and additional references quoted therein).

A main characteristic of the Lie-Santilli isotheory, which distinguishes it from all other possible generalizations, is its 'isotopic' character intended (from the Greek meaning of the word) as the capability of preserving the original Lie axioms. More specifically, Santilli's isotopies are maps of any given linear, local and canonical structure into its most general possible non-linear, non-local and non-canonical forms which are capable of reconstructing linearity, locality and canonicity in generalized isospaces and isofields within a fixed system of local co-ordinates.

The latter property is remarkable, mathematically and physically, inasmuch as it permits the preservation of the abstract Lie theory and the transition from exterior to

2.1. Isotopies and isodualities of the unit

Santilli's fundamental step from which all generalized formulations can be uniquely derived is the generalization of the unit I of the current formulation of Lie's theory into a quantity \hat{I} of the same dimension of I , but with unrestricted functional dependence of its elements in the local co-ordinates x , their derivatives of arbitrary order with respect to an independent variable $t, \dot{x}, \ddot{x}, \dots$ as well as any needed additional quantity [47, 49b, 61a],

$$I \rightarrow \hat{I} = \hat{I}(x, \dot{x}, \ddot{x}, \dots). \quad (2.1)$$

The isotopies [47] occur when \hat{I} preserves all the topological characteristics of I , such as nowhere-degeneracy, Hermiticity and positive-definiteness. The *genotopies* [47] occurs when \hat{I} is non-Hermitian (e.g., real-valued but non-symmetric), while the *hyperstructures* [73] occur when \hat{I} is a finite or infinite (and ordered or non-ordered) set of generally non-Hermitian quantities.

In conventional Lie's theory, the systems are identified via the sole knowledge of the Hamiltonian H . In the Lie-Santilli isothery, the identification of the systems requires the knowledge of two generally different quantities, the Hamiltonian H and the generalized unit \hat{I} .

Isotopic methods have resulted to be effective for the direct representation of closed-isolated systems of particles with conventional interactions represented with H plus internal non-local and non-Hamiltonian interactions represented with \hat{I} and time-reversal invariant centre-of-mass trajectories (from the property $\hat{I} = \hat{I}^\dagger$). The genotopic methods apply for the direct representation of open-non-conservative, non-local and non-Hamiltonian systems in irreversible conditions (from the property $\hat{I} \neq \hat{I}^\dagger$). The hyperstructural methods are significant for quantitative representations of the more complex biological systems.

Once the unit is generalized, there is the natural emergence of the map [52, 53, 59],

$$\hat{I} \rightarrow \hat{I}^\dagger = -\hat{I}, \quad (2.2)$$

called *isoduality* which provides an antiautomorphic image of all formulations based on \hat{I} . When properly formulated within the context of Hilbert spaces, the above map has resulted to be equivalent to charge conjugation [61b], thus permitting a representation of systems of antiparticles beginning at the classical level, the first known to this author, which then persists under quantization.

The above liftings were preliminarily classified by this author [22] into:

Class I. (generalized units that are smooth, bounded, non-degenerate, Hermitian and positive definite, characterizing the isotopies properly speaking);

Class II. (the same as Class I although \hat{I} is negative-definite, characterizing isodualities);

Class III. (the union of classes I and II);

Class IV. (Class III plus singular isounits); and

Class V. (Class IV plus unrestricted generalized units, e.g., realized via discontinuous functions, distributions, lattices, etc.).

All isotopic structures identified below also admit the same classification which will be omitted for brevity. In this paper we shall generally study isotopies of Classes I and II, at times treated in a unified way via those of Class III whenever no ambiguity arises. Santilli's isotopies of Classes IV and V are vastly unexplored at this writing.

interior problems via a more general *realization* of the same theory within the fixed frame of the experimenter.

Another main characteristic of the Lie-Santilli isothery is that of admitting a novel antiautomorphic map, called *isoduality*, which has resulted to be equivalent to charge conjugation, thus being effective for the first characterization on record of antimatter at the *classical* level with a consequential operator image.

It should be indicated that Santilli [47] submitted his isotopic theory as a *particular case* of a yet more general theory today called Santilli's *Lie-admissible theory* or *Lie-Santilli genotopic theory*, where the term *genotopic* is used (in its Greek meaning) to 'induce configuration', and interpreted in the sense of violating the original Lie axioms, yet *inducing* covering Lie-admissible axioms.

More recently, the Lie-Santilli isotopic and genotopic theories have resulted to be particular cases of yet more general formulations of *hyperstructural* type with a unit [73], thus resulting in a hierarchy of methods of increasing complexity for the representation of physical or biological systems with progressively more complex structures [61].

Finally, Santilli [52, 53, 59, 61] has shown that all preceding theories admit a novel anti-automorphic map he called *isoduality* particularly suited for the characterization of antimatter, which cannot be formulated in conventional mathematics because it requires a generalization of the basic unit.

This paper, written by a theoretical physicist, is devoted to the Lie-Santilli isothery. A study of the broader Lie-Santilli geno- and hypertheories are contemplated as future works.

In section 2 we outline the methodological foundations of the theory. The isotopies of Lie's theory are presented in section 3 jointly with new developments, such as a study of the transition form the Lie-Santilli isogroups to the corresponding isoalgebras. As an illustration of the capabilities of the isothery, we prove its 'direct universality' in gravitation, that is, the achievement of the symmetries of all possible gravitational metrics (universality), directly in the frame of the experimenter (direct universality). A number of fundamental open mathematical problems will be identified during the course of our analysis.

A comprehensive mathematical presentation of the Lie-Santilli isothery up to 1992 is available in monograph [76]. A historical perspective is available in monograph [31]. Recent mathematical studies on isomanifolds (today called *T sagas-Sourlas isomanifolds*) have been conducted in Ref. [77] which also provides a topological complement of the algebraic studies of this paper.

2. Isotopies and isodualities of numbers, fields, differential calculus, metric spaces, differential geometries, functional analysis, classical and quantum mechanics

Lie's theory is the embodiment of the virtual entirety of contemporary mathematics by encompassing: the theory of numbers; differential and exterior calculus; vector and metric spaces; geometry, algebra and topology; functional analysis; and other disciplines. Santilli's isotopies of Lie's theory require the isotopic lifting of *all* these mathematical methods. In this section we shall identify the basic isotopies and isodualities which are necessary for a correct formulation of the Lie-Santilli isothery, by referring to the quoted literature for more detailed treatments.

2.2. *Isotopies and isodualities of fields*

Lie's theory is constructed over ordinary fields $F(a, +, \times)$ hereon assumed to be of characteristic zero (the fields of real \mathfrak{R} complex C and quaternionic numbers \mathcal{Q}) with generic elements a , addition $a_1 + a_2$, multiplication $a_1 a_2 := a_1 \times a_2$, additive unit 0 , $a + 0 = 0 + a \equiv a$, and multiplicative unit 1 , $a \times 1 = 1 \times a \equiv a$, $\forall a, a_1, a_2 \in F$.

The Lie-Santilli isothory is based on a generalization of the very notion of numbers and, consequently of fields first introduced by Santilli at the *Conference on Differential Geometric Methods in Mathematical Physics* held in Clausthal, Germany, in 1980. A first rudimentary treatment appeared in Santilli's joint paper with the (mathematician) H.C. Myung [39] of 1982. Comprehensive studies were then conducted by Santilli in the following years (see paper [59] for a mathematical presentation and monographs [61] for extensive physical applications).

Consider a Class I lifting of the unit I of F , $I \rightarrow \hat{I}$, with \hat{I} being *outside* the original set, $\hat{I} \notin F$. In order for \hat{I} to be the left and right unit of the new theory, it is necessary to lift the conventional associative multiplication *ab into*, the so-called *isomultiplication* [47]

$$ab := a \times b \Rightarrow a * b := a \times \hat{T} \times a = a \hat{T} b, \quad \hat{T} = \text{fixed}, \quad (2.3)$$

where the quantity \hat{T} is called the *isotopic element*. Whenever $\hat{I} = \hat{T}^{-1}$, \hat{I} is the correct left and right unit of the theory, $\hat{I} * a = a * \hat{I} = a, \forall a \in F$, in which case (only) \hat{I} is called the *isounit*. In turn, the liftings $I \rightarrow \hat{I}$ and $\times \rightarrow *$, imply the generalization of fields into the Class I structures

$$\hat{F}_1 = \{(\hat{a}, +, *) | \hat{a} = a \times \hat{I}; a = n, c, q \in F; \times \rightarrow * = \times \hat{T} \times ; \hat{I} = \hat{T}^{-1}\}, \quad (2.4)$$

called *isofields*, with elements $\hat{a} \in \hat{F}$ called *isonumbers* [59]. It is instructive to verify that the above isofields satisfy all conventional axioms of ordinary fields as necessary for the lifting $F \rightarrow \hat{F}_1$ to be an isotopy (see [59] for details).

All conventional operations among numbers are evidently generalized in the transition from numbers to isonumbers. In fact, we have:

$$\begin{aligned} a + b &\rightarrow \hat{a} + \hat{b} = (a + b)\hat{I}, & a \times b &\rightarrow \hat{a} * \hat{b} = \hat{a} \times \hat{T} \times \hat{b} = (a \times b) \times \hat{I} = (ab)\hat{I}, \\ a^{-1} &\rightarrow \hat{a}^{-1} = a^{-1} \times \hat{I}, & a/b &= c \rightarrow \hat{a} \hat{I} / \hat{b} = \hat{c}, & \hat{c} &= c \times \hat{I} = c\hat{I}, \\ a^{1/2} &\rightarrow \hat{a}^{1/2} = a^{1/2} \hat{I}^{1/2}, \end{aligned}$$

etc. Thus, conventional squares $a^2 = a \times a = aa$ have no meaning under isotopy and must be lifted into the *isosquare* $\hat{a}^2 = \hat{a} * \hat{a} = a^2 \hat{I}$. The *isonorm* is

$$\hat{1} \hat{a} \hat{1} = (\hat{a} \times a)^{1/2} \times \hat{I} = |a| \times \hat{I} = |a| \times \hat{I} \in \hat{F}, \quad (2.5)$$

where \hat{a} denotes the conventional conjugation in F and $|a|$ is the conventional norm. Note that the *isonorm* is *positive-definite* (for isofields of Class I), as a necessary condition for isotopies.

The isotopic character of the lifting $1 \rightarrow \hat{I}$ is confirmed by the fact that the isounit \hat{I} verifies all axioms of 1,

$$\hat{I} * \hat{I} * \dots * \hat{I} \equiv \hat{I}, \quad \hat{I} \hat{I} \equiv \hat{I}, \quad \hat{I}^2 \equiv \hat{I}, \text{ etc.}$$

The *isodual isofields* [59] are the antihomomorphic image of $F(\hat{a}, +, *)$ induced by the map $\hat{I} \rightarrow \hat{I}^d = -\hat{I}$ and are given by the Class II structures

$$\begin{aligned} \hat{F}_1^d &= \{(\hat{a}^d, +, *^d) | \hat{a}^d = \hat{a} \times \hat{I}^d; a = n, c, q \in F; * \rightarrow *^d \\ &= \times \hat{T}^d \times \hat{I}^d = -\hat{T}, \hat{I}^d = -\hat{I}\}, \end{aligned} \quad (2.6)$$

in which the elements $\hat{a}^d = \hat{a} \times \hat{I}^d$ are called *isodual isonumbers*. For real numbers we have $n^d = -n$, for complex numbers we have $c^d = -\bar{c}$, where \bar{c} is the ordinary complex conjugate, and for quaternions in matrix representation we have $q^d = -q^\dagger$, where \dagger is the Hermitian conjugate. Note that the conjugation of a complex number is $(n + i \times m)^d = n^d + i^d \times m^d = -n + (-i)(-m) = -n + im$. The *isodual isosum* is given by $\hat{a}^d + \hat{b}^d = (\hat{a} + \hat{b}) \times \hat{I}^d$, while the *isodual isomultiplication* is given by [59]

$$\hat{a}^d *^d \hat{b}^d = \hat{a}^d \times \hat{T}^d \times \hat{b}^d = -\hat{a}^d \times \hat{T} \times \hat{b}^d = (\hat{a} \times \hat{b}) \times \hat{I}^d.$$

An important property is that the *norm of isodual isofields is negative-definite* because it is characterized by [59]

$$|\hat{a}^d \hat{I}^d| = |\hat{a}| \times \hat{I}^d = -|\hat{a}|. \quad (2.7)$$

The latter property has non-trivial implications. For instance, it implies that *physical quantities defined on an isodual isofield, such as time, energy, etc., are negative-definite*. For these reasons isodual theories provide a novel and intriguing characterization of antimatter [61].

Note also that, as a necessary condition for isotopies (isodualities) all isofields $F(\hat{a}, +, *)$ (isodual isofields $F(\hat{a}^d, +, *^d)$) are isomorphic (antiisomorphic) to the original field $F(a, +, *)$. The reader should be aware that the distinction between real, complex and quaternionic numbers is lost under isotopies because all possible numbers are unified by the isoreals owing to the freedom in the generalized unit [26].

Recall that the set of imaginary numbers *does not* constitute a field, evidently because not closed under the multiplication. On the contrary, Santilli's isofields $F(\hat{h}, +, *)$ with $\hat{h} = h \times i$; isounit $\hat{I} = i$ and n real do indeed verify the axioms for a field as one can readily verify. Note that the imaginary unit i is *isoseifdual*, i.e., invariant under isoduality, $i^d = -i \equiv i$.

We also recall [59] that the lifting $a \rightarrow \hat{a} = a \times \hat{I}$ is necessary for $F_1(\hat{a}, +, *)$ to preserve the axioms of $F(a, +, *)$ whenever the isounit \hat{I} is not an element of the original field. On the contrary, when $\hat{I} \in F(a, +, *)$ there is no need to lift the numbers and we shall write $F_1(a, +, *)$. In physical applications, the isounit is generally *outside* the original field and actually possesses a non-linear as well as integral dependence on the local variables and their derivatives. This implies that the 'numbers' used in the Lie-Santilli isothory generally have an *integral* structure.

As an example, the isounit used by Animalu [1] for the representation of the Cooper pair in superconductivity is given by

$$\hat{I} = I e^{iN} \int_{\psi_1 \psi_2} \psi_1 \psi_2 \psi_1 \psi_2, \quad (2.8)$$

where t represents time, N is a constant, and ψ_1 and ψ_2 are the wave functions of the two electrons of the Cooper pair in singlet coupling of their spin. *Animalu's isounit* (2.8) therefore represents the *non-local-integral* contributions due to the wave

overlapping of the two electrons in the Cooper pairs. Such contributions, since they are of contact type, are variationally non-self-adjoint and, therefore, they should be represented with anything possible, *except* the Hamiltonian. In Santilli's isotopies there are therefore represented with the isounit.

In particular, Animalu has shown that the lifting of the conventional Coulomb interactions characterized by isounit (2.8) produces an *attraction* among the *identical* electrons of the Cooper pair, as experimentally established in superconductivity. Note that when the overlapping of the wavepackets is no longer appreciable (e.g., at large mutual distances), the integral in the exponent of (2.8) is null and the isounit \hat{I} recovers the conventional unit I . Conventional fields $F(a, +, \times)$ are used for large distances among the electrons, while isofields $\hat{F}(\hat{a}, +, *)$ with isounit (2.8) are used when the wave-overlapping of the electrons is appreciable. Other examples of isounits will be provided later on.

We also recall Santilli's [59] more general *genofields*, which are characterized first by an isotopy of conventional fields, and then by an ordering of the isomultiplications, one to the right $\hat{a} > \hat{b} = \hat{a} \times \hat{R} \times \hat{b}$ and one to the left $\hat{a} < \hat{b} = \hat{a} \times \hat{S} \times \hat{b}$, $\hat{R} \neq \hat{S}$ which are different among themselves, yet they are commutative when the original field is commutative, $\hat{a} > \hat{b} = \hat{b} > \hat{a}$, $\hat{a} > \hat{b} = \hat{b} > \hat{a}$, $\hat{a} > \hat{b} \neq \hat{a} < \hat{b}$. In this case we have a *genounit to the right*, $\hat{I}^> = \hat{R}^{-1}$, and a *genounit to the left*, $\hat{I}^< = \hat{S}^{-1}$ which are usually interconnected via a conjugation, e.g., $\hat{I}^> = (\hat{I}^<)^{\dagger}$. The important property is that all abstract axioms of a field are verified *per each* ordered isomultiplication, thus yielding a *genofield to the right* $\hat{F}^>(\hat{a}^>, +, >)$ and a *genofield to the left* $\hat{F}^<(\hat{a}^<, +, <)$ which are at the foundation of Santilli's Lie-admissible theory [61]. The *hyperfields to the right and to the left* emerge when \hat{R} and \hat{S} are sets of generally non-Hermitian quantities [73].

We finally recall Santilli's [59] still more general liftings characterized by the generalization of the sum $+$ and related additive unit 0 , e.g., $+ \rightarrow \hat{+} = + \hat{R} +$, $0 = \hat{R} \neq 0$, $K \in F(a \hat{+} b = a + \hat{R} + b)$ called *pseudo-isotopies*, which *do not* preserve the axioms of a field (in fact, closure under the distributive law is not verified under the conventional \times or isotopic $*$ multiplication and the addition $\hat{+}$). Thus, *pseudo-isofields are not fields*. For these and other reasons (e.g., the general divergence of the exponentiation), applications in physics and biology are restricted to iso-, geno- and hyper-fields, while the pseudoiso- and pseudogeno- and pseudohyper-fields have a mere mathematical interest at this writing.

The care needed in inspecting and appraising the Lie-Santilli isothery can be pointed out from these introductory lines. In fact, familiar statements such as 'two multiplied by two equals four' are correct for the conventional Lie theory, but they have no mathematical meaning for the Lie-Santilli isothery because they lack the identification of the assumed unit and multiplication. In fact, for $\hat{I} = 3$, $\hat{2} * \hat{2} = 12$. Similarly, care must be expressed before claiming that a number is *prime* or not. In fact, Santilli [59] has shown that non-prime numbers can become prime under a proper selection of the unit.

Our current knowledge of Santilli's *theory of isonumbers* includes the lifting of all conventional numbers (real, complex and quaternionic numbers, plus the isotopies of octonions [59]) into the following four classes used in this paper: (A) *ordinary numbers* with unit I ; (B) *isonumbers* with isounits of Class I, $\hat{I} > 0$; (C) *isodual numbers* with isodual unit $\hat{I}^d = -1$; (D) *isodual isonumbers* with isodual isounits of Class II, $\hat{I}^d < 0$. In this paper we shall therefore have *four* different types of real numbers, complex

numbers and quaternions, at times unified in the isonumbers of Class III, excluding generalizations of Classes IV and V.

Despite the above advances, studies on the isonumber theory are their initiation and so much remains to be done. To begin, the entire conventional number theory (including all familiar theorems on factorization, primes, etc.) can be subjected to an isotopies of Classes I, II or III. Moreover, we have the birth of new numbers without counterpart in the current number theory, such as the isonumbers of Class IV (with singular isounits) and Class V (with distributions or discontinuous functions as isounits). The above liftings then admit antiautomorphic images under isoduality which are also absent in the conventional number theory. In turn, all the preceding generalizations can be subjected to further enlargements via the differentiation of the multiplications to the right and to the left, and then yet more general formulations via the multivalued hyperstructures.

One can begin to understand the vastity of the Lie-Santilli isothery as compared to the conventional formulation of Lie's theory by noting that the above hierarchy of fields implies a corresponding hierarchy of Lie-isotopic theories.

2.3. Isotopies and isodualities of the differential calculus.

The next important mathematical discovery by Santilli is an axiom-preserving integro-differential generalization of the conventional local-differential calculus called *isodifferential calculus*, first presented in a systematic way in the recent papers [71] although it is implicit in preceding works (e.g., [58, 61]).

Consider a set of functions $f(x), g(x), \dots$, on an N -dimensional space $S(x, \mathfrak{R})$ with local chart $x = \{x^k\}$, $k = 1, 2, \dots, N$, over the reals $\mathfrak{R}(n, +, \times)$. Let dx^k and $\partial_k = \partial/\partial x^k$ be the conventional differential and derivative on S , respectively.

Consider now the set of functions $f(x), g(x), \dots$, this time, on an N -dimensional isospace $\hat{S}(x, \hat{\mathfrak{R}})$, $x = \{x^k\}$, $k = 1, 2, \dots, N$, defined over the Class I isofield $\hat{\mathfrak{R}}(\hat{n}, +, *)$ (where we shall drop hereon the subscript I for simplicity), with N -dimensional positive-definite isounit $\hat{I} = \hat{I}^{\dagger} = (\hat{I}^{\dagger})^{-1} = (\hat{I}^{\dagger})^{-1} = (\hat{I}^{\dagger})^{-1} > 0$ whose elements possess a generally non-linear-integral dependence on all local quantities and their derivatives with respect to an independent variable t , $\hat{I}(x, \hat{x}, \ddot{x}, \dots)$. Santilli's *isodifferential calculus* is characterized by the *isodifferential*

$$\hat{d}x^k = \hat{I}^{\dagger} dx^k, \quad (2.9)$$

with corresponding *isoderivative*

$$\hat{\partial}_k = \frac{\hat{\partial}}{\hat{\partial}x^k} = \hat{I}^{\dagger} \frac{\partial}{\partial x^k} = \hat{I}^{\dagger} \partial_k. \quad (2.10)$$

under the condition that all conventional operations and properties of the ordinary differential calculus are lifted into their axiom-preserving isotopic form, e.g.,

$$\begin{aligned} \hat{d}f(x) &= \hat{\partial}_k f \times \hat{d}x^k = \hat{I}^{\dagger} \partial_k f \hat{I}^{\dagger} dx^k, \quad \hat{d}^2 x^k = \hat{d} \times \hat{d}x^k = \hat{I}^{\dagger} \hat{I}^{\dagger} dx^k dx^k, \\ \hat{\partial}_k^2 &= \hat{\partial}_k \times \hat{\partial}_k = \hat{I}^{\dagger} \hat{I}^{\dagger} \partial_k \partial_k, \text{ etc.}, \end{aligned}$$

where there is no sum over the repeated k -index.

A hidden condition is that, starting with a set of functions over an isofield $\hat{\mathfrak{R}}(\hat{n}, +, *)$ with isounit \hat{I} , the operations of isodifferentiation and isoderivatives must preserve

the original unit for consistency. This condition remains generally unidentified in the conventional calculus because the preservation of the unit follows from its constancy, $\partial_k I = 0$. For the case of a generalized unit with the same functional dependence as that of the functions, the condition of preservation of the unit must be added to the calculus to prevent the transition from the original set of functions defined with respect to \hat{I} to a new set of functions defined over a new unit \hat{I} .

As an example, the definition of the isodifferential

$$\hat{d}x^k = d(\hat{I}_i^k x^i) = (d\hat{I}_i^k) x^i + \hat{I}_i^k dx^i = \hat{I}_i^k dx^i + \hat{I}_i^k dx^i + \hat{I}_i^k x^m + \hat{I}_i^k$$

would imply the alteration of the isounit \hat{I} . In turn, the occurrence would have serious drawbacks in applications, such as lack of invariance of perturbative series.

Santilli's isodifferential calculus does verify the condition of preserving the basic isounit, although the question whether realizations (2.9) and (2.10) are unique has not been explored until now. Note also the mutual compatibility of isoforms (2.9) and (2.10).

The lifting of the integral calculus follows quite simply from the above isodifferential forms. We here limit ourselves to indicate that an *indefinite isointegral* defined as the operation inverse of the isodifferential is given by

$$\int \hat{d}x = \int \hat{I} \hat{I} dx = dx = x, \text{ i.e., } \int = \int \hat{I}. \tag{2.11}$$

Note that the isodifferential calculus is one of the simplest possible forms of *integro-differential calculus*, in the sense that each operation has a differential contribution characterized by d or ∂ and an integral component characterized by T or \hat{I} , respectively. Despite its simplicity, the isodifferential calculus has far reaching mathematical and physical implications. Mathematically, it permits a step-by-step generalization of conventional local-differential geometries into covering integro-differential geometries. Physically, the isocalculus permits a generalization of classical and quantum mechanics as well as of their interconnecting map (quantization), as outlined below.

The *isodual isodifferential calculus* is the antiautomorphic image of the preceding one characterized by the isodual isotopic element $\hat{I}^a = -\hat{I} < 0$ or isodual isounit $\hat{I}^a = -\hat{I} < 0$ and it is defined on the *isodual isospace* $S^a(\hat{x}, \hat{\mathfrak{R}}^a)$ defined over the Class II isodual isofield $\hat{\mathfrak{R}}^a$ (where, again, the subscript II has been dropped for simplicity). Note that the isodifferential calculus and its isodual can be unified into that of Class III.

The *genodifferential calculus* [61] occurs when the Hermiticity condition on the isounit is relaxed, $\hat{I} \neq \hat{I}^\dagger$. As such, the operation of differentiation itself acquires a structural ordering, namely, we have two different genoderivatives $\hat{\partial}^> f(x)$ and $f(x) \hat{\partial}$ defined for the corresponding units $\hat{I}^> = \hat{I}$ and $\hat{I}^< = \hat{I}^\dagger$ which are naturally set to represent the 'arrow of time'. This indicates that the genodifferential calculus is significant to represent *irreversible processes*. The *hyperdifferential calculus* has not been explored at this writing, to our best knowledge.

2.4. Isospaces, isogeometries and their isoduals

Santilli's third important discovery presented for the first time in paper [51] of 1983 (see also the recent paper [71] and the comprehensive treatment [61]) is the isotopic

lifting of conventional, N -dimensional, metric (or pseudo-metric) spaces and related geometries. Consider a metric space $S(x, g, \mathfrak{R})$ with local co-ordinates $x = \{x^k\}$ and (nowhere singular, real valued and symmetric) metric $g = (g_{ij})$ over the reals $\mathfrak{R}(n, +, \times)$. Its infinite class of isotopic images over the isoreals, called *isospaces* (hereon assumed for simplicity to be of Class I) is given by structures [51]

$$S(\hat{x}, \hat{g}, \hat{\mathfrak{R}}): \hat{g} = \hat{T} g, \hat{I} = \hat{T}^{-1}, \hat{x}^2 = (\hat{x}^i \hat{g} \hat{x}^j) \hat{I} \in \mathfrak{R}(\hat{n}, +, *) \tag{2.12}$$

$$\hat{x} = \{\hat{x}^k\} \equiv \{x^k\}, \hat{x}_k = \{\hat{g}_{ki} \hat{x}^i\} \neq x_k,$$

where $\hat{g} = Tg$ is called the *isometric*. The above liftings are necessary for compatibility with the isotopies of the unit $I \rightarrow \hat{I}$, of the product $x \rightarrow *$ and of the field $\mathfrak{R} \rightarrow \hat{\mathfrak{R}}$. From hereon we shall adopt Santilli's convention [71] of using symbols with a 'hat' to represent quantities computed in isospaces while ordinary symbols represent quantities computed in the original spaces.

Despite their simplicity, isospaces have far reaching implications. In fact, they imply that *the same abstract axioms of conventional spaces (such as the Euclidean, Minkowskian or Riemannian spaces) admit unrestricted functional dependence of the metric*. As an illustration, the conventional metric $g(x)$ of a Riemannian space $R(x, g, \mathfrak{R})$ is believed to be restricted to the sole dependence on the local co-ordinates x . Santilli has shown that the same Riemannian axioms permit an unrestricted functional dependence of the metric $\hat{g}(x, \hat{x}, \hat{x}, \dots)$. While Riemannian spaces $R(x, g, \mathfrak{R})$ are ideally suited for exterior gravitational problems, the *Riemann-Santilli isospaces* $\hat{\mathfrak{R}}(x, \hat{g}, \hat{\mathfrak{R}})$ are ideally suited for the treatment of *interior* gravitational problems with a non-linearity in the velocities, integral structure and variationally non-self-adjoint character (section 1).

This remarkable result is due to the construction of the isospaces via the deformation of the metric $g \rightarrow \hat{g} = Tg$ while jointly lifting the original unit in the inverse amount, $I \rightarrow \hat{I} = \hat{T}^{-1}$, under which isospaces $\hat{S}(\hat{x}, \hat{g}, \hat{\mathfrak{R}})$ (*isodual isospaces* $S^a(\hat{x}, \hat{g}^a, \hat{\mathfrak{R}}^a)$) are *locally isomorphic (antiisomorphic) to the original spaces* $S(x, g, \mathfrak{R})$. Additional salient properties of isospaces are the *preservation of the original dimensionality and of the original basis (except for renormalization factors)* [61a].

Via the use of the isotopies of fields, differential calculus and vector spaces, Santilli's has constructed step-by-step, non-local-integral isotopies and isodualities of conventional geometries on metric (or pseudo-metric) spaces. Their most salient application is the geometrization of interior physical media, that is, the geometrization of the departures from empty space caused by matter.

The isogeometries most important for physical applications are (see [61a, b] for details):

- (A) *Santilli's isoeuclidean geometry* of Class I on three-dimensional isospaces $\hat{E}(\hat{x}, \hat{\delta}, \hat{\mathfrak{R}})$, $\hat{\delta} = \hat{T} \delta = (\hat{T}^k \delta_k)$, $\delta = (\delta_{ij}) = \text{diag. } (1, 1, 1)$, over the isoreals $\mathfrak{R}(\hat{n}, +, *)$ with a 3×3 -dimensional isounit which, being positive-definite, can always be diagonalized into the form

$$\hat{I} = \text{diag. } (b_1^{-2}, b_2^{-2}, b_3^{-2}) > 0, \quad b_k = b_k(x, \hat{x}, \ddot{x}, \dots) > 0, \quad k = 1, 2, 3. \tag{2.14}$$

In this case the isometric $\hat{\delta}$ has an arbitrary functional dependence on local co-ordinates and their derivatives, $\hat{\delta}(x, \hat{x}, \ddot{x}, \dots)$. Yet the geometry is *isoflat*, that is, it verifies the axioms of flatness in isospace while its projection in the original space $E(x, \delta, \mathfrak{R})$ is evidently curved. An intriguing novel notion of the isoeuclidean geometry

is the *isosphere of Class I* $\hat{x}^k = (\hat{x}^i \hat{\delta}^j \hat{x}^k) \hat{I} = \hat{I}$ which is a perfect sphere in isoeuclidean space. Nevertheless, its projection in the original Euclidean space is given by all infinitely possible ellipsoids $x b_1^2 x + y_2 b_2^2 + z b_3^2 z = 1$ (where, according to the convention assumed earlier, \hat{x} is computed in \hat{E} and x in E). In fact, in isospace the original sphere with radius 1 is subjected to the deformations of its axes $1_k \rightarrow b_k^2$ while the corresponding units are deformed in the *inverse* amounts, $1_k \rightarrow b_k^{-2}$, thus preserving the perfectly spherical character. The *isosphere of Class III* unifies all compact and noncompact curves $\pm x b_1^2 x \pm y_2 b_2^2 y \pm z b_3^2 z \neq 0$ in isospace. The *isosphere of Class IV* unifies all compact and non-compact surfaces plus all cones $\pm x b_1^2 x \pm y_2 b_2^2 y \pm z b_3^2 z = 0$. The *isosphere of Class V* is an additional novel notion of a sphere with arbitrary unit (e.g., a lattice).

(B) *Santilli's isominkowskian geometry of Class I* on isospace $\hat{M}(x, \hat{\eta}, \hat{\mathfrak{R}})$, $\hat{\eta} = \hat{T} \eta, \eta = \text{diag.}(1, 1, 1, -1)$ over the isoreals with 4×4 -dimensional isounit reducible to the diagonal form

$$\hat{\Gamma} = \text{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) > 0, \quad b_\mu = b_\mu(x, \hat{x}, \ddot{x}, \dots) > 0, \quad \mu = 1, 2, 3, 4, \quad (2.15)$$

which represents *locally varying speeds of light* $c = c_0 b_4 = c_0/n_4$ where c_0 is the speed of light in vacuum and n_4 is the local index of refraction. As such, the isominkowskian geometry is particularly suited for the representation of light propagating within inhomogeneous and anisotropic physical media such as our atmosphere. An important notion of the isominkowskian geometry is the *isolight cone of Class I* [which is a perfect cone in isominkowski space but, when projected in the conventional Minkowski space, represents all infinitely possible deformed light cones $x n_1^{-2} x + y_2 n_2^{-2} y + z n_3^{-2} z - t c_0 n_4^{-2} t = 0$. In fact, each axis of the original light cone is deformed $1_\mu \rightarrow n_\mu^{-2}$, while the corresponding units are deformed of the inverse amount, $1_\mu \rightarrow n_\mu^2$, thus preserving the original perfect cone. The axiom-preserving character of the isotopies is such that the maximal causal speeds of the Minkowski and isominkowski spaces coincide and are given by the speed of light in vacuum c_0 .

(C) *Santilli's isoriemannian geometry* on isospaces $\hat{R}(\hat{x}, \hat{g}, \hat{\mathfrak{R}})$, $\hat{g} = \hat{T}g$ over isounit (2.14), which coincides with the conventional geometry at the abstract level. This implies that, unlike the isoeuclidean and isominkowskian geometries, the isoriemannian geometry is *isocurved*, that is, curved in isospace. As such, it permits the representation of interior gravitational problems with locally varying speeds of light, such as the bending of light within a physical medium with local speed $c = c_0/n_4 < c_0$, the contribution to cosmological redshift due to the decrease of the speed of light within astrophysical chromospheres, and other novel insights. An intriguing novel notion is that of *isogeodesics of Class I* which coincide in isospace with the geodesics in vacuum, but when projected in the original Riemannian space represents the actual non-geodesic trajectory of extended particles within physical media, such as that of a leaf in free fall in our atmosphere.

An isogeometry particularly important for the study of the Lie-Santilli theory is the *isosymplectic geometry* first presented in Ref. [57] (see also the more recent study [71]). Consider the conventional symplectic geometry (see, e.g. [34]) in canonical realization on the cotangent bundle $T^*E(x, \delta, \mathfrak{R})$, $\delta = \text{diag.}(1, 1, 1)$, with local chart $a = (a^\mu) = \{x^k, p_k, k = 1, 2, 3, \mu = 1, 2, 3, 4, 5, 6$. As well-known, the above geometry

is characterized by the canonical one-form

$$\theta = p_k dx^k, \quad (2.16)$$

with nowhere-degenerate, canonical, symplectic two-form

$$\omega = dp_k \wedge dx^k = \frac{1}{2} \omega_{\mu\nu} da^\mu \wedge da^\nu, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (2.17)$$

which is exact, $\omega = d\theta$, and therefore closed, $d\omega = d(d\theta) \equiv 0$ (Poincaré Lemma [34]). Corresponding higher-order forms are constructed accordingly.

Santilli's isosymplectic geometry [71] is defined on the *isoscotangent bundle* $T^* \hat{E}(\hat{x}, \hat{\delta}, \hat{\mathfrak{R}})$ with the local chart $\hat{a} = \{\hat{a}^\mu\} = \{\hat{x}^\mu, \hat{p}_k\}$, $\hat{x}^k \equiv x^k, \hat{p}_k \equiv p_k$, but now referred to the six-dimensional isounit given by the Kronecker product $\hat{I}_2 = \hat{I} \times \hat{I}$, resulting in the isodifferential forms $\hat{d}\hat{x}^k = \hat{I}_k^i dx^i, \hat{d}\hat{p}_k = \hat{I}_k^i dp_i, \hat{\delta}/\hat{\delta}\hat{x}^k = \hat{I}_k^i \partial/\partial x^i, \hat{\delta}/\hat{\delta}\hat{p}_k = \hat{I}_k^i \partial/\partial p_i$, etc. We then have the *one isoform*

$$\hat{\theta} = \hat{p}_k \hat{d}\hat{x}^k = p_k \hat{I}_k^i (x, p, \dots) dx^i, \quad (2.18)$$

and the *two-isoform*

$$\hat{\omega} = \hat{d}\hat{p}_k \wedge \hat{d}\hat{x}^k = \frac{1}{2} \omega_{\mu\nu} \hat{d}\hat{a}^\mu \wedge \hat{d}\hat{a}^\nu = \hat{I}_k^m (x, p, \dots) dp_m \wedge \hat{I}_n^k (x, p, \dots) dx^n \equiv \omega, \quad (2.19)$$

which is also nowhere degenerate as well as *isoeexact*, $\hat{\omega} = \hat{d}\hat{\theta}$, and therefore *isoclosed* in the *isocotangent bundle* (but not necessarily so in its projection on the original cotangent bundle), $\hat{d}\hat{\omega} = \hat{d}(d\hat{\theta}) \equiv 0$ (this is the isotopic Poincaré Lemma [57, 71]). Isoform (2.19) is then called *isosymplectic*. Higher-dimensional isoforms are then constructed accordingly.

An important geometric discovery which is permitted by the isosymplectic geometry is the following alternative of the Darboux theorem.

Theorem 2.1 (Santilli [61a, 71]). *Let $X(a)$ be a vector-field on a conventional tangent bundle and suppose that it is non-Hamiltonian in the point a , i.e., there exist no function $H(a)$ such that on a suitable neighborhood D of the chart a the following identities hold $\omega_{\mu\nu} X^\nu(a) da^\mu = dH(a)$. Then, there always exists an isotopy within the fixed local chart a under which the same vector field becomes Hamiltonian, i.e., the following identities hold in a neighbourhood D of a*

$$\hat{\omega}_{\mu\nu} X^\nu(a) \hat{d}\hat{a}^\mu = \hat{d}H(a)$$

in which case the vector-field is called *isohamiltonian*.

Recall from section 1.2 that a Darboux's transformation $a \rightarrow a' = a'(a) = (r'(r, p), p'(r, p))$ under which a vector-field becomes Hamiltonian cannot be generally used in physical applications because the transformed frame is highly nonlinear in the original co-ordinates, thus not realizable in actual experiment as well as highly non-inertial, thus incompatible with established relativities.

Santilli's motivation for the construction of the isosymplectic geometry is precisely to resolve these problematic aspects, by permitting a non-Hamiltonian vector-field to become Hamiltonian under the preservation of the fixed a -frame of the experimenter and merely changing instead the basic unit (thus the basic differentials) of the geometry.

Note that $\hat{\omega} \equiv \omega$ under the assumed conditions of $\hat{p}_k = p_k$ (see later on for differences) and this shows the 'hidden' character of the isotopies in the very structure of the conventional symplectic geometry. This also confirms that the *symplectic and isosymplectic geometries coincide at the abstract, realization-free level*, as established by the abstract identity of forms (2.15) and (2.18), or (2.17) and (2.19). Such an abstract identity is such that one can represent the isosymplectic geometry with the same symbols used for the co-ordinate free formulation of the symplectic geometry.

However, the two geometries admit *inequivalent realizations*. In fact, the symplectic geometry is strictly local-differential, does not admit nonlinearities in the velocities and it possesses a canonical structure. On the contrary, the isosymplectic geometry has an integro-differential structure (in the sense indicated earlier) and it is arbitrarily non-linear in the velocities.

All isogeometries indicated in this section admit intriguing isodual forms which can be easily identified by the reader via the rules of isodualities identified earlier. Regrettably, we have to refer the interested reader to monographs [61, 72] for details (see also paper [75] for topological aspects on isomanifolds).

At this writing, the isogeometries are minimally well-known for physical applications. Nevertheless, their mathematical study has yet to be initiated and a number of fundamental aspects remain open at this writing.

2.5. Isotopies and isodualities of functional analysis

As indicated earlier, the isotopies imply non-trivial generalizations of all mathematical structures of Lie's theory, inevitably leading to a generalization of functional analysis called by this author *functional isanalysis* [22].

The generalized discipline begins with the isotopy of continuity (whose knowledge is assumed when dealing with the technical aspects of section 3), and includes the isotopies of conventional square-integrable, Banach and Hilbert spaces, as well as the isotopies of all operations on them.

In particular, functional isanalysis includes a generalization of conventional special functions, distributions and transforms. For instance, the conventional Dirac delta distribution has no meaning under isotopy, mathematically, because of the loss of applicability of the conventional exponentiation and, physically, because particles are no longer point-like. The *isodirac distribution* is the reconstruction of the conventional distribution for an unrestricted unit permitting a direct treatment of the extended character of particles. The Fourier transform, Legendre polynomials, etc., also admit simple yet unique and unambiguous isotopies with important applications in various disciplines.

Regrettably, we are unable to review the above isotopies to prevent a prohibitive length of this paper, and refer the interested reader to [61a]. We shall merely identify in the next section only those isospecial functions which are necessary for an understanding of the Lie-Santilli isotheory. One should be aware that the elaboration of the Lie-Santilli isotheory via conventional functional analysis (e.g., the use of conventional trigonometry, logarithms, exponentiations, etc.) leads to inconsistencies which often remain undetected by the non-initiated reader.

2.6. Isotopies and isodualities of classical mechanics

As it is well-known (see, e.g., [13]), Lie's theory admits two fundamental realizations, one in classical and one in quantum mechanics, with interconnecting map given by the naive or symplectic quantization.

The preceding isotopies of fields, differential calculus, metric spaces, geometries, and functional analysis were used by Santilli for the construction of step-by-step isotopic generalizations of classical [72] and quantum [61] mechanics and their interconnecting maps. The new mechanics have been specifically conceived for the most general possible, non-linear, non-local and non-canonical, interior dynamical problems, as well as the fundamental classical and operator realizations of the Lie-Santilli isotheory. As a matter of fact, Santilli proposed the isotopies of Lie's theory precisely for quantitative treatment of the above generalized mechanics.

It is important to review at least the essential structural elements of the isotopic classical and operator mechanics because they provide the realizations of the Lie-Santilli isotheory most important for applications.

To conduct our outline, we shall keep using Santilli's notation of putting a 'hat' on all quantities belonging to isotopic formulations, while conventional symbols are used for quantities belonging to conventional formulations (see [71] for details). As it is well-known, conventional classical mechanics is formulated in the configuration space via the seven-dimensional space $E(t, \delta, \mathfrak{R}) \times E(x, \delta, \mathfrak{R}) \times E(v, \delta, \mathfrak{R})$ where t is time, $x = \{x^k\}$ represents the space co-ordinates and $v = \{v^k\}$ represents the velocities, the latter being independent from the former.

The isotopies of classical mechanics in configuration space require their formulation in the isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v}) = \hat{E}(\hat{t}, \hat{\delta}, \hat{\mathfrak{R}}) \times \hat{E}(\hat{x}, \hat{\delta}, \hat{\mathfrak{R}}) \times \hat{E}(\hat{v}, \hat{\delta}, \hat{\mathfrak{R}})$ characterized by the total isounit $\hat{I}_{tot} = \hat{I}_t \times \hat{I} \times \hat{I}$, where: $\hat{I}_t = T_t^{-1}$ is the (one-dimensional) *isounit of time* and $\hat{I} = \hat{I}^{-1}$ is the (three-dimensional) *isounit of space*. By assuming that the isotime is contravariant we have $\hat{t} \equiv t$, while for the space components we have the general rules

$$\begin{aligned}\hat{x} &= \{x^k\} \equiv \{\hat{x}^k\}, & \hat{x}_k &= \hat{d}_k x^i = \hat{T}_k^i \delta_{ij} x^j = \hat{T}_k^i x_i, & \hat{v} &= \{v^k\} \equiv \{\hat{v}^k\} \equiv \{dx^k/d\hat{t}\}, \\ \hat{v}_k &= \hat{d}_k v^i = \hat{T}_k^i v_i.\end{aligned}$$

The isodifferential calculus on $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ is then based on the following space and time isodifferentials and isoderivatives,

$$\hat{d}\hat{t} = \hat{I}_t dt, \quad \hat{d}\hat{x}^k = \hat{I}_t^k dx^k, \quad \hat{d}\hat{x}_k = \hat{T}_k^i dx_i, \quad \hat{d}\hat{v}^k = \hat{I}_t^k dv^k, \quad \hat{d}\hat{v}_k = \hat{T}_k^i dv_i, \quad (2.20a)$$

$$\hat{d}_i \hat{d}\hat{t} = \hat{T}_i dt/dt, \quad \hat{\partial}_i \hat{\partial} \hat{x}^k = \hat{T}_k^i \partial_i \partial x^k, \quad \hat{\partial}_i \hat{\partial} \hat{x}_k = \hat{T}_k^j \partial_i \partial x_j,$$

$$\hat{\partial}_i \hat{\partial} \hat{v}^k = \hat{T}_k^j \partial_i \partial v^j, \quad \hat{\partial}_i \hat{\partial} \hat{v}_k = \hat{T}_k^j \partial_i \partial v_j, \quad (2.20b)$$

with basic properties

$$\begin{aligned}\hat{\partial} \hat{x}^i / \hat{\partial} \hat{x}^j &= \delta_j^i, & \hat{\partial} \hat{x}_i / \hat{\partial} \hat{x}_j &= \delta_j^i, & \hat{\partial} \hat{x}^i / \hat{\partial} \hat{x}_j &= \hat{T}_j^i, & \hat{\partial} \hat{x}_i / \hat{\partial} \hat{x}^j &= \hat{T}_j^i, \\ \hat{\partial}(\hat{v}^i \hat{d}_i \hat{v}^j) / \hat{\partial} \hat{v}^k &= 2\hat{v}_k.\end{aligned} \quad (2.21)$$

We then have the following isotopies of classical mechanics:

(1) *Isonewtonian mechanics*. Newton's equations of motion $m dv_k/dt + \partial V/\partial x^k - F_k^{NSA} = 0$ on $S(t, x, v)$ over $\mathfrak{R}(n, +, \times)$ are lifted into the *Newton-Santilli equations* on the isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ first introduced in [71]

$$\hat{m} \frac{d\hat{v}_k}{d\hat{t}} + \frac{\partial \hat{V}(\hat{x})}{\partial \hat{x}^k} = 0, \tag{2.22}$$

which, when projected in the original space $S(t, x, v)$, assume the explicit form

$$\begin{aligned} \hat{m} \hat{I}_i \frac{d[\hat{I}_i^k(t, x, v, \dots) v_i]}{dt} + \hat{I}_i^k(t, x, v, \dots) \frac{\partial V(x)}{\partial x^k} \\ = \hat{I}_i^k [m dv_i/dt + \partial V(x)/\partial x^k + m \hat{I}_i^j (d\hat{I}_i^j/dt) v_j] = 0, \end{aligned} \tag{2.23}$$

where $\hat{m} \hat{I}_i = m$, and $\hat{m} = m \hat{I}$, is called the *isomass*.

As one can see, the Newton-Santilli equations permit the direct representation (i.e., representation in the fixed x -frame of the observer) of: (a) the actual, extended, non-spherical and deformable shape of the body considered; (b) non-local-integral interactions as permitted by the underlying integro-differential topology of $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ [77]; and (c) the representation of all possible non-potential forces $F_i^{NSA} = -m \hat{I}_i^j (d\hat{I}_i^j/dt) v_j$, via the *isogeometry* itself, (i.e., via the covariant form $\hat{v}_k = \hat{I}_i^k v_i$) in such a way that all forces F^{NSA} 'disappear' in expression (2.22) in isospace.

As a specific example, consider an originally spherical body of mass m which moves along the x -axis within a resistive medium (say, gas or liquid) by acquiring the ellipsoidal shape σ with semiaxes (a^2, b^2, c^2) . By ignoring potential forces for simplicity, suppose that the body experiences only a non-local-integral resistive force of the type $F_x^{NSA} = -\gamma v_x^2 \int_{\sigma} d\sigma \mathcal{F}(\sigma, \dots)$, where $\gamma > 0$ and \mathcal{F} is a suitable kernel. The above systems can be directly represented in isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$ via the Newton-Santilli equation

$$\begin{aligned} \hat{m} \frac{d\hat{v}_x}{d\hat{t}} = 0, \text{ i.e., } m d(\hat{I}_x^x v_x)/dt = \hat{I}_x^x [m dv_x/dt + m \hat{I}_x^z (d\hat{I}_x^z/dt) v_z] = 0, \\ \hat{m} = m, \hat{I}_i = 1, \hat{I}_x^z = \text{diag.}(a^{-2}, b^{-2}, c^{-2}) \exp \left\{ -\gamma t v_x \int_{\sigma} d\sigma \mathcal{F}(\sigma, \dots) \right\}. \end{aligned} \tag{2.24}$$

The interested reader can then construct a virtually endless number of other examples. Note that, by comparison, the conventional Newton's equations can only represent *point-like particles under local-differential interactions*. By recalling that the terms 'Newtonian mechanics' are referred to point-particles under local-differential interactions, the emerging new mechanics for extended-deformable particles under integro-differential interactions shall be referred to as the *Newton-Santilli isomechanics*.

(2) *Iso Lagrangian mechanics*. A conventional first-order Lagrangian $L(x, v) = \frac{1}{2} m v^k v_k + V(x)$ on configuration space $S(t, x, v)$ acquires the form $\hat{L}(\hat{x}, \hat{v}) = \frac{1}{2} \hat{m} \hat{v}^k \hat{v}_k + \hat{V}(\hat{x})$ in isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$. The isotopies of conventional variational principle of the *isoaction* $\hat{A} = \int_{\hat{t}_1}^{\hat{t}_2} \hat{L}(\hat{x}, \hat{v}) d\hat{t}$ (see [71] for details) then lead to the

Lagrange-Santilli equations on isospace $\hat{S}(\hat{t}, \hat{x}, \hat{v})$

$$\hat{d} \frac{\partial \hat{L}(\hat{x}, \hat{v})}{\partial \hat{t}} - \frac{\partial \hat{L}(\hat{x}, \hat{v})}{\partial \hat{x}^k} = 0, \tag{2.25}$$

which, under arbitrary but well behaved isolagrangians $\hat{L}(\hat{t}, \hat{x}, \hat{v})$, are *directly universal* for all possible isoequations (2.22). In fact, the above equations can be explicitly written on $S(t, x, v)$

$$\hat{I}_i \frac{d}{dt} \hat{I}_i^k \frac{\partial L(x, v)}{\partial v^k} - \frac{\partial L(x, v)}{\partial x^k} = 0 \tag{2.26}$$

by therefore coinciding with Equations (2.22). Note that the isolagrangian mechanics also permits the direct representation of extended, non-spherical and deformable bodies under conventional as well as non-local-integral nonpotential interactions with evident advances over the conventional formulation

(3) *Isohamiltonian mechanics*. The *isolegendre transform* is characterized by the invertible rules on a domain \hat{D} of the local variables [61b, 71],

$$\text{Det} \left(\frac{\partial^2 \hat{L}}{\partial \hat{v}^i \partial \hat{v}^j} \right) (\hat{D}) \neq 0, \hat{p}_k = \frac{\partial \hat{L}}{\partial \hat{v}^k} = m \hat{v}_k, \tag{2.27}$$

which are formulated on seven-dimensional isophase space $\hat{S}(\hat{t}, \hat{x}, \hat{p})$ with total isounit $\hat{I}_{\text{tot}} = \hat{I}_i \times \hat{I} \times \hat{I}$, yielding the *isocanonical action*

$$\hat{A} = \int_{\hat{t}_1}^{\hat{t}_2} (\hat{p}_k d\hat{x}^k - \hat{H} d\hat{t}) = \int_{\hat{t}_1}^{\hat{t}_2} dt [p_k \hat{I}_i^k(t, x, p, \dots) dx^i - H \hat{I}_i(t, x, p, \dots) dt], \tag{2.28}$$

with isohamiltonian $\hat{H} = \hat{p}_k \hat{p}^k/2m + \hat{V}(x)$. The use again of the isovariations then yields the *Hamilton-Santilli equations* [61b, 71],

$$\frac{d\hat{x}^k}{d\hat{t}} = \frac{\partial \hat{H}(\hat{x}, \hat{p})}{\partial \hat{p}_k}, \frac{d\hat{p}_k}{d\hat{t}} = -\frac{\partial \hat{H}(\hat{x}, \hat{p})}{\partial \hat{x}^k}, \tag{2.29}$$

The Hamilton-Jacobi equations are lifted into the *Hamilton-Jacobi-Santilli equations*

$$\hat{\sigma}_i \hat{A} + \hat{H} = 0, \hat{\sigma} \hat{A} / \hat{\sigma} \hat{x}^k - \hat{p}_k = 0, \hat{\sigma} \hat{A} / \hat{\sigma} \hat{p}_k = 0. \tag{2.30}$$

The conventional Poisson brackets, which are the realization in classical mechanics (CM) of the Lie product, are lifted into the *isopoisson brackets* first introduced in Ref. [47] (see also [71] for the explicit form below)

$$[A, B]_{\text{CM}} = \frac{\partial A}{\partial x^k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial x^j} = \frac{\partial A}{\partial x^k} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial x^k}, \tag{2.31}$$

which provide the desired *classical realization of the Lie-Santilli brackets*. In fact, it is easy to prove that the above brackets satisfy on isospace $\hat{S}(\hat{t}, \hat{x}, \hat{p})$ (but not in the original space) the Lie algebra axioms (see Refs. [61, 72] for a proof via the isotopies of the Poincaré Lemma of the symplectic geometry).

The exponentiated form of Hamilton's equation is a realization of a one-parameter Lie transformation group on $S(t, x, p)$. The exponentiated form of Eqs. (2.29) is

$$\hat{a}^{\nu\alpha} = \{e^{i\omega_{\mu\nu\lambda} \hat{h}^{\lambda\alpha}}\} \hat{a}^{\alpha}, \quad \hat{a}^{\alpha} = \{\hat{a}^{\alpha}\} = \{\hat{x}^{\alpha}, \hat{p}_{\alpha}\}, \quad \alpha, \mu, \nu = 1, 2, \dots, 6, \quad (2.32)$$

which when properly written in an isotopic form (see next subsection), provide a realization of a one-dimensional Lie-Santilli transformation group on $\hat{S}(\hat{t}, \hat{x}, \hat{p})$.

The emerging new mechanics is called *Hamilton-Santilli isomechanics*. Some of the advantages over the conventional Hamiltonian mechanics are now evident. To begin, the Hamilton-Santilli equations preserve all essential features of the Newton-Santilli equations, thus permitting the representation, beginning at the classical level, of extended-deformable bodies with local-differential-potential as well as non-local-integral-nonpotential interactions. As we shall see in the next section, these features are mainly the classical foundations for corresponding operator formulations.

An important analytic discovery is given by the following.

Theorem 2.2 (Santilli [61b, 71]). *The Hamilton-Santilli equations (2.29) are 'direct universal' in the Newton-Santilli mechanics, that is, they can represent all infinitely possible, analytic and regular, integro-differential, variationally non-self-adjoint first-order systems in a star-shaped region of their variables (universality), directly in the frame of the experimenter (direct universality).*

The above property (which is the analytic counterpart of Theorem 2.1) can easily be proved by noting that a well-behaved action of arbitrary order always admit an identical first-order isotopic reformulation (2.28). The theorem can be equivalently established via the proof of the direct universality of equations (2.29) for all possible Hamiltonians $\hat{H}(\hat{t}, \hat{x}, \hat{p})$ and isounits $\hat{I}(\hat{t}, \hat{x}, \hat{p}, \dots)$. By comparison, the conventional Hamiltonian mechanics can directly represent only a rather small number of conservative Newtonian systems, and the more general Birkhoffian mechanics [49b] is directly universal only for (well-behaved) local-differential systems.

To understand this paper, the reader should keep in mind the above direct universality because it establishes the corresponding direct universality of the Lie-Santilli isothory in classical mechanics with a corresponding direct universality for operator formulations indicated in the next subsection.

All formulations of this section admit isodual images on isospaces $\hat{S}^d(\hat{t}, \hat{x}, \hat{p})$ and $\hat{S}^d(\hat{t}, \hat{x}, \hat{p})$ over isodual isoreals $\hat{\mathfrak{R}}^d(\hat{h}^d, +, *^d)$ which have produced the first classical representation of antimatter [71] known to this author. In particular, the representation occurs via particles with negative-definite mass moving backward in time, although defined with respect to negative-definite units, thus resulting to be equivalent (although antiautomorphic) to particles with positive-definite mass moving forward in time when defined with respect to positive-definite units. As an example, the isodual Newton-Santilli equations are given by [71]

$$\hat{m}^d \hat{d}\hat{v}_k^d / \hat{d}\hat{t}^d + \hat{\delta} \hat{\mathcal{P}}^d(\hat{x}) / \hat{d}\hat{x}^d = 0$$

and characterize an antiparticle with mass $m^d = -m$ and time $t^d = -t$. A similar situation occurs for the isodual Lagrange-Santilli equations as well as for the isodual Hamilton-Santilli equations and the isodual Hamilton-Jacobi-Santilli equations which all represent antiparticles in isodual isospaces on isodual isofields.

2.7. *Isotopies and isodualities of quantum mechanics*

We now outline the operator realization of the Lie-Santilli isothory first identified in Ref. [48] and then studied in numerous subsequent papers (see Refs. [61b, 71] for the most recent accounts).

The isotopies of quantum mechanics were originally proposed by Santilli [48] under the name of *hadronic mechanics*, namely, a mechanics specifically built for strongly interacting particles called hadrons. Recall that quantum mechanics is strictly local and differential and has resulted to be *exactly valid* for electromagnetic and weak interactions, although there are historical doubts whether the same discipline can also be exact for the strong interactions, with the understanding that its approximate validity is unquestionable.

In fact, the charge radius of hadrons is of the same order of magnitude of the range of the strong interactions. Also, hadrons are some of the densest objects measured in laboratory until now. Therefore, the activation of the strong interactions requires the mutual penetration of these hyperdense particles, resulting in the historical expectation of non-linear, non-local-integral and non-Hamiltonian contributions whose quantitative treatment requires a suitable generalization of quantum mechanics.

Santilli [48] proposed the construction of the isotopies of quantum mechanics precisely for the treatment of the latter contributions in a form which preserves the original quantum mechanical axioms.

Let ξ be the enveloping associative operator algebra of quantum mechanics with elements A, B, \dots , unit I and conventional associative product $A \times B = AB$, and let \mathfrak{H} be a conventional Hilbert space with states $|\psi\rangle, |\phi\rangle, \dots$ and inner product $\langle \psi | \phi \rangle = \int d^3x \psi^\dagger(t, x) \phi(t, x)$ over the field complex numbers $C(c, +, \times)$.

By keeping the notation according to which quantities with a 'hat' are computed on isospaces over isofields while those without are computed on conventional spaces over conventional fields, hadronic mechanics is based on the following structures:

(1) The Class I lifting of the (space) unit $I \rightarrow \hat{I} = \hat{I}^{-1} > 0$ with consequential isofields of real $\hat{\mathfrak{R}}(\hat{h}, +, *)$ and complex isonumbers $\hat{C}(\hat{h}, +, *)$ (section 2.2);

(2) The corresponding lifting of the quantum mechanical representation spaces, such as the Euclidean $E(x, \delta, \mathfrak{R})$ or Minkowskian spaces $M(x, \eta, \mathfrak{R})$ into their isotopic form $\hat{E}(\hat{x}, \hat{\delta}, \hat{\mathfrak{R}})$ and $\hat{M}(\hat{x}, \hat{\eta}, \hat{\mathfrak{R}})$ (section 2.3)

(3) The lifting of the enveloping operator algebras ξ into the enveloping isosociative algebra $\hat{\xi}$ with the same original elements $\hat{A} = A, \hat{B} = B, \dots$ now equipped with the isounit \hat{I} and the isoassociative product $\hat{A} * \hat{B} = \hat{A} \hat{I} \hat{B}$, as well as the lifting of the Hilbert space \mathfrak{H} into the isohilbert space $\hat{\mathfrak{H}}$ with isostates $|\hat{\psi}\rangle, |\hat{\phi}\rangle, \dots$ and isoinner product

$$\hat{\mathfrak{H}}: \langle \hat{\phi} | \hat{\psi} \rangle = \langle \hat{\phi} | \hat{I} | \hat{\psi} \rangle \hat{I} \in \hat{C}(\hat{c}, +, *) \quad (2.33)$$

The fundamental dynamical equations of hadronic mechanics can be uniquely and unambiguously derived from the Hamilton-Santilli isomechanics via the isotopies of conventional or symplectic quantization. Recall that the naive quantization can be expressed via the mapping

$$A = \int_{J_1}^{J_2} (p dx^k - H dt) \rightarrow -i\hbar L_n \psi(t, x) \quad (2.34)$$

Such a mapping is now inapplicable to isoaction (2.28) because $\hat{A} \neq A$. But the basic unit of quantum mechanics $\hbar = 1$ is replaced under isotopies by the (space) isounit \hat{I} . The consistent application of the isotopies then yields the generalized mapping identified by Animalu and Santilli here presented for simplicity for the isounit independent from the local time and co-ordinates (but dependent on the velocities as essential for contact resistive forces, see [61b] for the general case and references)

$$\hat{A} = \int_{t_1}^{t_2} [\hat{p}_k \hat{d}x^k - \hat{H} \hat{d}t] \rightarrow -i \hat{I} \text{Ln} \hat{\psi}(\hat{t}, \hat{x}), \tag{2.35}$$

The above mapping is the naive isoquantization of the Hamilton-Jacob-Santilli equations (2.30) into the following fundamental dynamical equations of hadronic mechanics (see also Ref. [61b] for all references and details): the isoschrödinger equations for the linear momentum

$$-i \hat{\delta}_k \hat{\psi}(\hat{t}, \hat{x}) = -i T_k^i \hat{\delta}_i \psi(t, x) = \hat{p}_k * \psi(t, x) = \hat{p}_k \hat{T} \hat{\psi}(\hat{t}, \hat{x}), \tag{2.36}$$

with the related fundamental isocommutation rules

$$[\hat{p}_i, \hat{x}^j] = \hat{p}_i * \hat{x}^j - \hat{x}^j * \hat{p}_i = -\delta_i^j, \quad [\hat{p}_i, \hat{p}_j] = [\hat{x}^i, \hat{x}^j] \equiv 0 \tag{2.37}$$

(where we have used properties (2.21)), first identified by Santilli; the isoschrödinger equation for the energy

$$i \hat{\delta}_t \hat{\psi}(\hat{t}, \hat{x}) = i \hat{T}_t^i \hat{\delta}_i \psi(t, x) = \hat{H} * \hat{\psi}(\hat{t}, \hat{x}) = \hat{H} \hat{T} \hat{\psi}(\hat{t}, \hat{x}) = E \hat{\psi}(\hat{t}, \hat{x}), \tag{2.38}$$

$$\hat{H} = \hat{H}^\dagger, \quad E = E \hat{I} \in \mathfrak{H}(\hat{n}, +, *), \quad E \in \mathfrak{R}(n, +, \times), \tag{2.38}$$

first identified by Myung and Santilli and, independently, by Mignani, with the conventional differential calculus, and finalized by Santilli with the isoderivatives; and the Heisenberg-Santilli equation

$$i \hat{d}\hat{Q}/\hat{d}\hat{t} = [\hat{Q}, \hat{H}] = \hat{Q} * \hat{H} - \hat{H} * \hat{Q} = \hat{Q} \hat{T} \hat{H} - \hat{H} \hat{T} \hat{Q} \tag{2.39}$$

with integrated form

$$\hat{Q}(t) = e^{i\hat{H}\hat{T}t} \hat{Q}(0) e^{-i\hat{H}\hat{T}t}, \tag{2.40}$$

first identified by Santilli in the original proposal to build hadronic mechanics [48].

It should be recalled for subsequent need that the condition of isohermicity on an isohilbert space coincides with the conventional Hermiticity, $\hat{H}^\dagger = \hat{H}$. As a consequence, all operators which are Hermitian-observable in quantum mechanics remain so in hadronic mechanics. Also, unitary transforms on \mathfrak{H} , $UU^\dagger = U^\dagger = I$, are lifted under isotopies into the isounitary transformations

$$O * O^\dagger = O^\dagger * O = \hat{I}, \tag{2.41}$$

As a matter of fact, any conventionally non-unitary operator U , $UU^\dagger = \hat{I} \neq I$, on \mathfrak{H} always admits an identical isounitary form on \mathfrak{H} via the simple rule $U = \hat{O} \hat{T}^{1/2}$.

For the isotopies of the quantum mechanical axioms, isotopic laws and all other aspects we refer for brevity the interested reader to monograph [61b]. We here merely indicate that, from the positive-definiteness of the basic isounit \hat{I} , all distinctions between quantum and hadronic mechanics cease to exist at the abstract, realization-free level for which $\mathfrak{H} \approx \mathfrak{R}$, $C \approx \hat{C}$, $\xi \approx \hat{\xi}$, $E \approx \hat{E}$, $\mathfrak{H} \approx \hat{\mathfrak{H}}$, etc. This ultimate abstract

unity assures the correct axiomatic structure of hadronic mechanics to such an extent that criticisms on its structure may eventually result to be criticisms on quantum mechanics.

The fundamental operator realization of the Lie-Santilli isoproduct is then given by [48]

$$[\hat{A}, \hat{B}] = \hat{A} * \hat{B} - \hat{B} * \hat{A} = \hat{A} \hat{T} \hat{B} - \hat{B} \hat{T} \hat{A}, \tag{2.42}$$

which, as one can easily verify, satisfy the Lie axioms in both isospace and in conventional spaces. The fundamental operator realization of the isogroups is then given by equation (2.40) which, as we shall see in the next section, can be identically rewritten in terms of the isounitary transforms.

Note that the naive (or symplectic) isoquantization apply for all possible isoaction (2.28). By recalling the direct universality of the Hamilton-Santilli isomechanics, we can therefore see that hadronic mechanics is also directly universal for all possible (well-behaved), integro-differential, operator systems which are non-linear in the wave function and its derivatives [61b]. This property is remarkable inasmuch as it establishes the direct universality of the Lie-Santilli isotheory in its operator realization.

The advantages of hadronic over quantum mechanics are similar to those of the Hamilton-Santilli over the Hamiltonian mechanics. In fact, quantum mechanics can only represent (in first quantization) point-like particles under action-at-a-distance interactions. By comparison, hadronic mechanics can represent (in first isoquantization) the actual non-spherical shape of hadrons, their deformations as well as non-local-integral interactions due to mutual penetrations of the hadrons. The possibilities for broader applications in various disciplines are then evident.

The isodual Hamilton-Santilli isomechanics is mapped via naive isoquantization into the isodual hadronic mechanics which is based on: (1) the isodual isofields of isoreals $\hat{\mathfrak{R}}^d(\hat{n}^d, +, *^d)$ or isocomplex numbers $\hat{C}^d(\hat{c}^d, +, *^d)$ (section 2.2); (2) the isodual envelope $\hat{\xi}^d$ with isodual isounit $\hat{I}^d = -\hat{I}$, isodual elements $\hat{A}^d = -A$, $\hat{B}^d = -B$, etc., and isodual product $\hat{A}^d *^d \hat{B}^d = -\hat{A} \hat{T} \hat{B}$; the isodual isohilbert space $\hat{\mathfrak{H}}^d$ with isodual isostates $|\hat{\psi}\rangle^d = -\langle\hat{\psi}|$, etc, and isodual isoinner product $\langle\hat{\phi}|\hat{T}^d|\hat{\psi}\rangle^d$ over \hat{C}^d .

In particular, at this operator level, the isodual map has resulted to be equivalent to charge conjugation (see [61b] for brevity), although with a number of differences. For instance, charge conjugation maps a particle into an antiparticle in the same carrier space over the same field, while isoduality maps a particles in a given carrier space over a given field into a different carrier space over a different field (the isodual ones), charge conjugation changes the sign of the charge but preserves the sign of energy and time, while isoduality changes the signs of all physical characteristics, although they are now defined over a field of negative-definite norm; etc.

As an example, the isodual Heisenberg-Santilli equation is given by

$$i \hat{d}\hat{Q}^d/\hat{d}\hat{t}^d = \hat{Q}^d \hat{T}^d \hat{H}^d - \hat{H}^d \hat{T}^d \hat{Q}^d,$$

where we have used the isoselfduality of the imaginary quantity i (section 2.2).

2.8. Isolinearity, isolocality and isocanonicity

In section 1 we pointed out that the primary limitations of the contemporary formulation of Lie's theory are those of being linear, local and canonical. The classical

realizations identified earlier indicate rather clearly that the Lie–Santilli isothery is non-linear, non-local and non-canonical, as desired.

It is important to understand that such non-linearity, non-locality and non-canonicity occur only when the theory is projected in the original space over the original fields because the theory reconstructs linearity, locality and canonicity in isospace (see [61] for all details and references).

Let $S(x, F)$ be a conventional vector space with local co-ordinates x over a field F , and let $x' = A(w)x$ be a linear, local and canonical transformation on $S(x, F)$, $w \in F$. The lifting $S(x, F) \rightarrow \hat{S}(\hat{x}, \hat{F})$ requires a corresponding necessary isotopy of the transformations [47]

$$\hat{x}' = \hat{A}(\hat{w}) * \hat{x} = \hat{A}(\hat{w}) \hat{T} \hat{x}, \quad \hat{T} \text{ fixed}, \quad \hat{x} \in \hat{S}(\hat{x}, \hat{F}), \quad \hat{w} = w \hat{I} \in \hat{\mathfrak{F}}, \hat{I} = \hat{T}^{-1}, \quad (2.43)$$

called *isotransforms*, with *isodual isotransforms* $\hat{x}' = \hat{A}^d(\hat{w}^d * \hat{x}^d = -\hat{A}(\hat{w}) * \hat{x}$.

It is easy to see that the above isotransforms satisfy the condition of linearity in isospaces, called *isolinearity*

$$\hat{A} * (\hat{a} * \hat{x} + \hat{b} * \hat{y}) = \hat{a} * (\hat{A} * \hat{x}) + \hat{b} * (\hat{A} * \hat{y}), \quad \forall \hat{x}, \hat{y} \in \hat{S}(\hat{x}, \hat{F}), \quad \hat{a}, \hat{b} \in \hat{F}, \quad (2.44)$$

although their projection in the original space $S(x, F)$ are non-linear because $x' = \hat{A} T(x, \hat{x}, \dots)x$.

Theorem 2.3 (Santilli [61a]). *All possible (well-behaved) non-linear, classical or-operator systems of equations or of transformations always admit an identical isolinear form.*

The above property illustrates the primary mechanisms according to which the Lie–Santilli isothery applies to non-linear systems. In fact, as we shall see shortly, the latter theory is isilinear and, as such, it is capable of turning conventionally non-linear systems into identical forms which do verify the axioms of linearity in isospace, with evident advantages.

Isotransforms (2.39) are also *isolocal* in the sense that the theory formally deals with the local variables x while all non-local terms are embedded in the isounit, namely, all non-local-integral terms disappear at the abstract, realization-free level. Nevertheless, the theory is non-local when projected in the original space. Similarly, isotopic theories are *isocanonical* because they are derivable from the isoaction (2.28) which coincides at the abstract level with the canonical action.

3. Isotopies and isodualities of enveloping algebras, Lie algebras, Lie groups, symmetries, representation theory and their applications

As recalled in section 1, Lie's theory (see, e.g., [13, 15]) is centrally dependent on the basic n -dimensional unit $I = \text{diag.}(1, 1, \dots, 1)$ in all its major branches, such as enveloping algebras, Lie algebras, Lie groups, representation theory, etc. The main idea of the Lie–Santilli isothery [47, 49, 61, 72] is the reformulation of the entire conventional theory with respect to the most general possible, integro-differential isounit $\hat{I}(x, \hat{x}, \hat{x}, \dots)$.

One can therefore see from the outset the richness and novelty of the isotopic theory. In fact, it can be classified into five main classes as occurring for isofields,

isospaces, etc., and admits novel realizations and applications, e.g., in the construction of the symmetries of deformed line elements of metric spaces.

In this section we shall continue to use the notation according to which quantities with a 'hat' are computed on isospaces over isofields while conventional quantities are computed on conventional spaces over conventional fields.

3.1. Isotopies and isodualities of universal enveloping associative algebras

Let ξ be a universal enveloping associative algebra [15] over a field F (of characteristic zero) with generic elements A, B, C, \dots , trivial associative product AB and unit I . Their isotopes $\hat{\xi}$ were first introduced in [47] under the name of *universal isosociative enveloping algebras*. They coincide with ξ as vector spaces (i.e., $\hat{A} \equiv A, \hat{B} \equiv B$, etc.) but are equipped with the isoproduct so as to admit \hat{I} as the correct (right and left) unit

$$\hat{\xi}: \hat{A} * \hat{B} = \hat{A} \hat{T} \hat{B}, \quad \hat{T} \text{ fixed}, \quad \hat{I} * \hat{A} = \hat{A} * \hat{I} \equiv \hat{A} \equiv A \quad \forall \hat{A} \in \hat{\xi}, \quad \hat{I} = \hat{T}^{-1}. \quad (3.1)$$

Let $\xi = \xi(L)$ be the universal enveloping algebra of an N -dimensional Lie algebra L with ordered basis $\{X_k\}$, $k = 1, 2, \dots, N$, $[\xi(L)]^- \approx L$ over F , and let the infinite-dimensional basis of $\xi(L)$ be given by the Poincaré–Birkhoff–Witt theorem [15]. An important result achieved by Santilli in the original proposal [47] (see also [59, Vol. II, pp. 154–163]) is the following.

Theorem 3.1. *The cosets of \hat{I} and the standard, isotopically mapped monomials*

$$\hat{I}, \hat{X}_k \quad \hat{X}_i * \hat{X}_j \quad (i \leq j), \quad \hat{X}_i * \hat{X}_j * \hat{X}_k \quad (i \leq j \leq k), \dots \quad (3.2)$$

form a basis of the universal enveloping isosociative algebra $\hat{\xi}(L)$ of a Lie algebra L .

The above theorem is fundamental for the entire analysis of this paper. A first consequence is given by the following isotopies of the conventional exponentiation, called *isoexponentiation*, here expressed for $\hat{w} = w\hat{I} \in \hat{F}$, $\hat{X} \equiv X$,

$$e^{\hat{w} * \hat{X}} = \hat{I} + (i\hat{w} * \hat{X})/1! + (i\hat{w} * \hat{X}) * (i\hat{w} * \hat{X})/2! + \dots = \hat{I}(e^{wTX}) = \{e^{XTw}\} \hat{I}. \quad (3.3)$$

In turn, the notion of isoexponentiation permits the correct formulation of the isotransformations via expressions of the type $a' = \{\exp_{\hat{I}}(i\hat{w} * \hat{X})\} * a = \{\exp_{\hat{I}}(i\hat{w}TX)\} \hat{I} a = \{\exp_{\hat{I}}(i\hat{w}TX)\} a$. The quantity \hat{X} can first be a vector-field on an isomanifold with local chart a , thus providing a classical realization of the isothery. The quantity \hat{X} can also be a Hermitean operator on an isohilbert space, thus providing an operator realization of the isothery. In fact, it is easy to prove that, for $\hat{X} = X^\dagger$, the quantity $\hat{U} = \exp_{\hat{I}}(i\hat{w} * \hat{X})$ is an isounitary operator satisfying (2.4).

The implications of Theorem 3.1 also emerge at the level of functional isoisanalysis because all structures defined via the conventional exponentiation must be suitably lifted into a form compatible with Theorem 3.1. As an example, Fourier transforms are structurally dependent on the conventional exponentiation. As a result, they must

be lifted under isotopies into the expressions [23]

$$f(x) = (1/2\pi) \int_{-\infty}^{+\infty} g(k) * e^{ikx} dk, \quad g(k) = (1/2\pi) \int_{-\infty}^{+\infty} f(x) * e^{-ikx} dx, \quad (3.4)$$

with similar liftings for Laplace transforms, Dirac-delta distribution, etc., not reviewed here for brevity.

On physical grounds, Theorem 3.1 implies that the isotransform of a Gaussian in functional isoanalysis is given by [23]

$$f(x) = N * e^{-x^2/2a^2} = N e^{-x^2 \hat{T} / 2a^2} \rightarrow g(k) = N * e^{-k^2 \hat{T} a^2 / 2} = N e^{-k^2 \hat{T} a^2 / 2}. \quad (3.5)$$

As a result, the widths are of the type $\Delta x \approx a \hat{T}^{-1/2}$, $\Delta k \approx a^{-1} \hat{T}^{-1/2}$. It then follows that the isotopies imply the loss of the conventional uncertainties $\Delta x \Delta k \approx 1$ in favor of the local-interior *isouncertainties* [61b]

$$\Delta x \Delta k \approx \hat{T}, \quad (3.6)$$

although the isospectation values recover the conventional value, $\langle \hat{T} \rangle = \langle | \hat{T} \hat{T} \rangle / \langle | \hat{T} \rangle = 1$ which allows to recover in full conventional uncertainties for the exterior, centre-of-mass behaviour of hadrons [61b].

The *isodual isoenvolopes* $\hat{\xi}^d$ are characterized by the isodual basis $X_k^d = -X_k$ defined with respect to the isodual isounits $\hat{1}^d = -\hat{1}$ and isodual isotopic element $\hat{F}^d = -\hat{F}$ over the isodual isofields \hat{F}^d . The *isodual isoexponentiation* is then given by

$$e^{\hat{F}^d * x} = \hat{1}^d \{ e^{iwx} \} = -e^{iwx} \quad (3.7)$$

and plays an important role for the characterization of antiparticles via isodual isosymmetries, with negative-definite energy and moving backward in time.

It is easy to see that Theorem 3.1 holds, as originally formulated [47], for Hermitian isounit of undefined signature now called of Class III, thus unifying isovolopes $\hat{\xi}$ and their isoduals $\hat{\xi}^d$. In fact, the theorem was conceived to unify with one single envelope simple compact and non-compact algebras of the same dimension N . A first illustration was provided in [47] for the case of the Lie algebra $so(3)$ of the rotational group $SO(3)$ with generators $X_k, k = 1, 2, 3$, in their fundamental three-dimensional representation, according to which $[\hat{\xi}_i^2(so(3))]^- \approx so(3)$ for $\hat{I} = I = \text{diag.}(1, 1, 1)$, $[\hat{\xi}_i^2(so(3))]^- \approx so(2.1)$ for $\hat{I} = \text{diag.}(1, 1, -1)$ with more general realizations for more general forms of the isounits (see section 3.5 for more details). In the subsequent paper [51] Santilli illustrated how the isoenvolope $\hat{\xi}(so(4))$ unifies all possible simple, compact and noncompact six-dimensional Lie algebras, $so(4), so(3.1), so(2.2)$, as well as all their infinitely possible isotopes (see section 3.6 for more details). The possibility whether the preceding unifications holds for all possible simple Lie algebras of the same dimension was formulated by Santilli as a *conjecture* [61b]. Appendix 8.A) which has remained unexplored until now to our knowledge.

Note that the isotopy $\hat{\xi} \rightarrow \hat{\xi}^d$ is not a conventional map because the local coordinates x , the infinitesimal generators X_k and the parameters w_k are not changed by assumption. Only the underlying unit and related associative product are changed.

The non-triviality of the isothory is first illustrated by the emergence of the non-linear-non-local isotopic element \hat{T} directly in the exponent of isoexponentiation (3.3), thus ensuring the desired generalization. Also, in their operator realizations, the

general transformation of the Lie into the Lie-Santilli isothory is given by a non-unitary transformations for which

$$I \rightarrow \hat{I} = UIU^\dagger, \quad AB \rightarrow UABU^\dagger = A'\hat{T}B', \quad U(AB - BA)U^\dagger = A'\hat{T}B' - B'\hat{T}A', \quad (3.8a)$$

$$UU^\dagger = \hat{I} \neq I, \quad \hat{F} = (UU^\dagger)^{-1}, \quad \hat{I} = \hat{T}^{-1}, \quad \hat{I} = \hat{T}, \quad \hat{T} = \hat{T}^\dagger, \quad (3.8b)$$

$$A' = UAU^\dagger, \quad B' = UBU^\dagger,$$

where one should note not only the emergence of the correct isotopic structure, but even that with the correct Hermiticity of \hat{I} and \hat{T} . Once an isotopic structure is reached via non-unitary transforms, it remains form-invariant under the isounitary realization of non-unitary transforms [61b]. In fact, under a further non-unitary isounitary transforms we have the invariance rules

$$O * \hat{I} * O^\dagger = \hat{I}, \quad O * \hat{A} * \hat{B} * O^\dagger = \hat{A} * \hat{B},$$

$$O * (\hat{A} * \hat{B} - \hat{B} * \hat{A}) * O^\dagger = A' * \hat{B}' - \hat{B}' * \hat{A}',$$

which establishes the form-invariance, first, of the fundamental isounit, and then of the isothory.

The lack of equivalence of the two theories is further illustrated by the inequivalence between conventional eigenvalue equations, $H|b\rangle = E|b\rangle$, $H = H^\dagger$, $E \in \mathfrak{R}(n, +, \times)$, and their isotopic form in the same Hamiltonian

$$\hat{H} * |\hat{b}\rangle = \hat{H} \hat{T} |\hat{b}\rangle = \hat{E} * |\hat{b}\rangle \equiv E' |\hat{b}\rangle, \quad \hat{H} = H = H^\dagger, \quad E' \neq E,$$

with consequential *different eigenvalues for the same operator H* (see section 3.5 for an example). From the above occurrences it is easy to see that *the weights of the Lie and Lie-Santilli theories are different*, thus confirming the inequivalence of the two theories.

3.2. Isotopies and isodualities of Lie algebras

A (finite-dimensional) isospace \hat{L} over the isofield \hat{F} of isoreal $\mathfrak{R}(\hat{n}, +, *)$ or isocomplex numbers $\hat{C}(\hat{c}, +, *)$ with isotopic element \hat{T} and isounit $\hat{1} = \hat{T}^{-1}$ is called a *Lie-Santilli isospace* over \hat{F} (see [47, 49, 61, 72] for original studies and monographs [3, 24, 31, 76] with quoted papers for independent studies), when there is a composition $[\hat{A}, \hat{B}]$ in \hat{L} , called *isocommutator*, which is isilinear (i.e., satisfies condition (2.44)) and such that for all $\hat{A}, \hat{B}, \hat{C} \in \hat{L}$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}], \quad [\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0, \quad (3.9a)$$

$$[\hat{A} * \hat{B}, \hat{C}] = \hat{A} * [\hat{B}, \hat{C}] + [\hat{A}, \hat{C}] * \hat{B}. \quad (3.9b)$$

The isospace are said to be: *isoreal (isocomplex)* when $\hat{F} = \mathfrak{R}(\hat{F} = \hat{C})$, and *isobelian* when $[\hat{A}, \hat{B}] = 0, \forall \hat{A}, \hat{B} \in \hat{L}$. A subset \hat{L}_0 of \hat{L} is said to be an *isosubalgebra* of \hat{L} when $[\hat{L}_0, \hat{L}_0] \subset \hat{L}_0$ and an *isoidéal* when $[\hat{L}, \hat{L}_0] \subset \hat{L}_0$. A maximal isoidéal which verifies the property $[\hat{L}, \hat{L}_0] = 0$ is called the *isocenter* of \hat{L} . For the isotopies of conventional notions, theorems and properties of Lie algebras, one may see monograph [76].

We recall the isotopic generalizations of the celebrated Lie's First, Second and Third Theorems introduced in the original proposal [47], but which we do not review here for brevity (see [49b, 61b, 76]). For instance, the Lie-Santilli second theorem reads

$$[\hat{X}_i, \hat{X}_j] = \hat{X}_i * \hat{X}_j - \hat{X}_j * \hat{X}_i = \hat{X}_i \hat{F}(\hat{x}, \dots) \hat{X}_j - \hat{X}_j \hat{F}(\hat{x}, \dots) \hat{X}_i = \hat{C}_{ij}^k(\hat{x}, \dots) * \hat{X}_k, \tag{3.10}$$

where the \hat{X} 's are vector-fields on an isomanifold with local chart \hat{x} , or operators on a isohilbert spaces, and the \hat{C} 's are called the structure functions because they generally have an explicit dependence on the local co-ordinates (see the example of section 3.5) restricted by certain conditions of the Lie-Santilli Third Theorem.

Let L be an N -dimensional Lie algebra with conventional commutation rules and structure constants C_{ij}^k on a space $S(x, F)$ with local co-ordinates x over a field F , and let \tilde{L} be (homomorphic to) the antisymmetric algebra $[\xi(L)]^-$ attached to the associative envelope $\xi(L)$. Then \tilde{L} can be equivalently defined as (homomorphic to) the antisymmetric algebra $[\tilde{\xi}(L)]^-$ attached to the isoassociative envelope $\tilde{\xi}(L)$ [47, 49, 76]. In this way, an infinite number of isoalgebras \tilde{L} , depending on all possible isounits \tilde{I} , can be constructed via the isotopies of one single Lie algebra L . It is easy to prove the following result.

Theorem 3.2 [61a]. The isotopies $L \rightarrow \tilde{L}$ of an N -dimensional Lie algebra L preserve the original dimensionality.

In fact, the basis $e_k, k = 1, 2, \dots, N$ of a Lie algebra L is not changed under isotopy, except for renormalization factors denoted \hat{e}_k . Let the commutation rules of L be given by $[e_i, e_j] = C_{ij}^k e_k$. The isocommutation rules of the isotopes \tilde{L} are then given by

$$[\hat{e}_i, \hat{e}_j] = \hat{e}_i \hat{F} \hat{e}_j - \hat{e}_j \hat{F} \hat{e}_i = \hat{C}_{ij}^k(x, \dots) \hat{e}_k, \quad \hat{C} = \hat{C} \hat{F}. \tag{3.11}$$

One can then see in this way the necessity of lifting the structure $\langle \text{constants} \rangle$ into structure $\langle \text{functions} \rangle$, as correctly predicted by the Lie-Santilli Second Theorem [47]. The structure theory of the above isoalgebras is still unexplored to a considerable extent. In the following we shall show that the main lines of the conventional structure of Lie theory do indeed admit a consistent isotopic lifting. To begin, we here introduce the general isoinverse and isocomplex Lie-Santilli algebras denoted $G\tilde{L}(n, \hat{C})$ as the vector isospaces of all $n \times n$ complex matrices over \hat{C} . It is easy to see that they are closed under isocommutators as in the conventional case. The isocenter of $G\tilde{L}(n, \hat{C})$ is then given by $\hat{a} * \hat{I}, \forall \hat{a} \in \mathfrak{H}$. The subset of all complex $n \times n$ matrices with null trace is also closed under isocommutators. We shall call it the special, complex, isoinverse isoalgebra and denote it with $S\tilde{L}(n, \hat{C})$. The subset of all antisymmetric $n \times n$ real matrices $X, X' = -X, \dots$, is also closed under isocommutators, it is called the isorthogonal algebra, and it is denoted with $\hat{O}(n)$.

By proceeding along similar lines, we classify all classical, non-exceptional, Lie-Santilli algebras over an isofield of characteristic zero into the isotopes of the conventional forms, denoted with $\hat{A}_n, \hat{B}_n, \hat{C}_n$ and \hat{D}_n , each one admitting realizations of Classes I-V (of which only Classes I-III are studied herein). In fact, $\hat{A}_{n-1} = S\tilde{L}(n, \hat{C})$; $\hat{B}_n = \hat{O}(2n+1, \hat{C})$; $\hat{C}_n = S\tilde{P}(n, \hat{C})$; and $\hat{D}_n = \hat{O}(2n, \hat{C})$. One can begin to see in this way the richness of the isotopic theory as compared to the conventional theory.

The notions of homomorphism, automorphism and isomorphism of two isoalgebras \tilde{L} and \tilde{L}' , as well as of simplicity and semisimplicity are the conventional ones.

Similarly, all properties of Lie algebras based on the addition, such as the direct and semidirect sums, carry over to the isotopic context unchanged (because of the preservation of the additive unit 0).

An isoderivation \hat{D} of an isoalgebra \tilde{L} is an isolinear mapping of \tilde{L} into itself satisfying the property

$$\hat{D}([\hat{A}, \hat{B}]) = [\hat{D}(\hat{A}), \hat{B}] + [\hat{A}, \hat{D}(\hat{B})] \quad \forall \hat{A}, \hat{B} \in \tilde{L}. \tag{3.12}$$

If two maps \hat{D}_1 and \hat{D}_2 are isoderivations, then $\hat{a} * \hat{D}_1 + \hat{b} * \hat{D}_2$ is also an isoderivation, and the isocommutators of \hat{D}_1 and \hat{D}_2 is also an isoderivation. Thus, the set of all isoderivations forms a Lie-Santilli isoalgebra as in the conventional case.

The isolinear map $\hat{a}\hat{d}(\tilde{L})$ of \tilde{L} into itself defined by

$$\hat{a}\hat{d} \hat{A}(\hat{B}) = [\hat{A}, \hat{B}], \quad \forall \hat{A}, \hat{B} \in \tilde{L} \tag{3.13}$$

is called the isoadjoint map. It is an isoderivation, as one can prove via the iso-Jacobi identity. The set of all $\hat{a}\hat{d}(\hat{A})$ is therefore an isolinear isoalgebra, called isoadjoint algebra and denoted \hat{L}_* . It also results to be an isideal of the algebra of all isoderivations as in the conventional case.

Let $\hat{L}^{(0)} = \tilde{L}$. Then $\hat{L}^{(1)} = [\hat{L}^{(0)}, \hat{L}^{(0)}]$, $\hat{L}^{(2)} = [\hat{L}^{(1)}, \hat{L}^{(1)}]$, etc., are also isideals of \tilde{L} . \tilde{L} is then called isosoluble if, for some positive integer n , $\hat{L}^{(n)} \equiv 0$. Consider also the sequence

$$\hat{L}^{(0)} = L, \quad \hat{L}^{(1)} = [\hat{L}^{(0)}, \tilde{L}], \quad \hat{L}^{(2)} = [\hat{L}^{(1)}, \tilde{L}], \text{ etc.},$$

Then \tilde{L} is said to be isonilpotent if, for some positive integer n , $\hat{L}^{(n)} \equiv 0$. One can then see that, as in the conventional case, an isonilpotent algebra is also isosoluble, but the converse is not necessarily true.

Let the isotrace of a matrix be given by the element of the isofield [61]

$$\text{Tr} \hat{A} = (\text{Tr } A) \hat{I} \in \hat{F}, \tag{3.14}$$

where $\text{Tr } A$ is the conventional trace. Then

$$\text{Tr}(\hat{A} * \hat{B}) = (\text{Tr } \hat{A}) * (\text{Tr } \hat{B}), \quad \text{Tr}(\hat{B} \hat{A} \hat{B}^{-1}) = \text{Tr } \hat{A}.$$

Thus, the $\text{Tr } \hat{A}$ preserves the axioms of $\text{Tr } A$, by therefore being a correct isotopy. Then the isoscalar product

$$(\hat{A}, \hat{B}) = \text{Tr}[(\hat{a}\hat{d} \hat{A}) * (\hat{a}\hat{d} \hat{B})] \tag{3.15}$$

is here called the isokilling form. It is easy to see that (\hat{A}, \hat{B}) is symmetric, bilinear, and verifies the property $(\hat{a}\hat{d} \hat{X}(\hat{Y}), \hat{Z}) + (\hat{Y}, \hat{a}\hat{d} \hat{X}(\hat{Z})) = 0$, thus being a correct, axiom-preserving isotopy of the conventional Killing form.

Let $e_k, k = 1, 2, \dots, N$, be the basis of L with one-to-one invertible map $e_k \rightarrow \hat{e}_k$ to the basis of \tilde{L} . Generic elements in \tilde{L} can then be written in terms of local co-ordinates $\hat{x}, \hat{y}, \hat{z}, \hat{A} = \hat{x}^i \hat{e}_i$ and $\hat{B} = \hat{y}^j \hat{e}_j$, and $\hat{C} = \hat{z}^k \hat{e}_k = [\hat{A}, \hat{B}] = \hat{x}^i \hat{y}^j [\hat{e}_i, \hat{e}_j] = \hat{x}^i \hat{y}^j \hat{C}_{ij}^k \hat{e}_k$. Thus,

$$[\hat{a}\hat{d} \hat{A}(\hat{B})]^* = [\hat{A}, \hat{B}]^* = \hat{C}_{ij}^k \hat{x}^i \hat{y}^j. \tag{3.16}$$

We now introduce the isocartan tensor \hat{g}_{ij} of an isoalgebra \tilde{L} via the definition $(\hat{A}, \hat{B}) = \hat{g}_{ij} \hat{x}^i \hat{y}^j$ yielding

$$\hat{g}_{ij}(\hat{x}, \dots) = \hat{C}_{ij}^k \hat{C}_{jk}^i. \tag{3.17}$$