# ORIGIN, PROBLEMATIC ASPECTS AND INVARIANT FORMULATION OF CLASSICAL AND OPERATOR DEFORMATIONS

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In this paper we study three aspects of generalized classical and operator theories, herein generically called *deformations*, which do not appear to have propagated in the rather vast literature in the field: (1) the first known studies on classical and operator deformations; (2) their rather serious physical and mathematical shortcomings due to lack of invariance when conventionally formulated; and (3) the ongoing efforts for the achievement of invariant formulations preserving the axiomatic consistency of the original theories. We begin by recalling the mathematical beauty, axiomatic consistency and experimental verifications of the special relativity at both classical and quantum levels, and its main axiomatic properties: universal invariance of the fundamental units of space and time; preservation of hermiticity-observability at all times; uniqueness and invariance of numerical predictions; and other known properties. We then review the first known, generally ignored, classical and operator deformations. We then study the generally ignored problematic aspects of classical and operator deformations in their current formulation which include: lack of invariance of the fundamental units of space and times with consequential inapplicability to real measurements; loss of observability in time; lack of uniqueness and invariance of numerical predictions; violation of causality and probability laws; and, above all, violation of Einstein's special relativity. We finally outline the generally ignored ongoing efforts for the resolutions of the above shortcomings, and show that they require the necessary use of *new mathematics* specifically constructed for the task. We finally present a systematic study for the identical reformulation of existing classical and operator deformations in an invariant form.

*Keywords*: Classical and quantum deformations, Lie-isotopic and Lie-admissible theories, isomathematics and genomathematics.

## 1. Statement of the Problem

The first part of this century will undoubtedly be considered in the history of physics as signaling the triumph of the special relativity[1] in both its classical and quantum versions because of its mathematical beauty, axiomatic consistency and experimental verifications.

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Quite likely, the second part of this century will also pass to the history of physics as being characterized by numerous attempts at broadening the mathematical, physical and experimental, as well as nonrelativistic and relativistic foundations of the special relativity. The latter theories are generically known under the name of (classical and operator) *deformations*, and we shall preserve such a terminology in this study.

This paper is devoted to the differences in axiomatic structure between the special relativity and the various generalized theories. In particular, we shall show that, as currently formulated on conventional spaces over fields, deformations have an undeniable mathematical beauty but, in their current formulation, are afflicted by a number of rather serious problematic aspects of *physical* consistency, because they lose the main properties of the special relativity, such as: invariance of the fundamental units of space and time; preservation in time of hermiticity-observability; invariant probabilities; causality; and other known features.

The literature of this century contains numerous generalizations-deformations, first, of the classical setting underlying the special relativity, Hamiltonian mechanics, and then of its operator foundations, (nonrelativistic or relativistic) quantum mechanics, which require an individual inspection.

The first objective of this paper is to recall the origin of what are today called classical deformations which are generally ignored in the vast literature in the field, and which can be traced back to the founders of analytic mechanics, such as Lagrange,[2] Hamilton,[3] Jacobi,[4] and others. In fact, Lagrange and Hamilton presented their celebrated equations *with external terms* which were then removed in this century, resulting in what are often called the "truncated Lagrange and Hamilton equations." Similarly, Jacobi [*loc. cit.*] did not prove his celebrated theorem for the analytic equations of the contemporary literature, but rather for the original ones with external terms.

Despite the successes of special relativity (which mostly motivated the removal of the external terms in the analytic equations), the legacy of Lagrange and Hamilton has persisted and actually increased with the passing of time. In essence, the special relativity is exactly valid under the conditions of its original conception, which were historically referred to as those of the *exterior dynamical problem*, here denoting particles which can be well approximated as being pointlike when moving in the homogeneous and isotropic vacuum under action-a-distance/potential interactions. Typical examples of exterior dynamical problems are a spaceship in a stationary orbit around earth or an electron in an atomic cloud.

By contrast, in Lagrange's and Hamilton's view there exist conditions under which one sole quantity, today called Lagrangian or Hamiltonian, cannot represent the entire physical reality and, for this reason, they added the external terms to their celebrated equations. The latter conditions were historically referred as those of the *interior dynamical problem*, here denoting particles which cannot be approximated as being pointlike when moving within a generally inhomogeneous and anisotropic physical medium. Typical examples of interior dynamical problems are a spaceship

during re-entry in our atmosphere, or a neutron in the core of a neutron star, which experience both action-a-distance, potential-Hamiltonian as well as contact nonpotential–non-Hamiltonian interactions due to the motion of an extended object (whether a spaceship or a neutron) within a physical medium.

Lagrange's and Hamilton's historical view is therefore amply sufficient to provide physical motivations for the study of generalized theories.

However, the addition of the external terms in the dynamical equations has profound mathematical and physical implications because it implies the loss of the entire Lie theory in favor of a covering formulation known as *Lie-admissible theory*, whose algebraic axioms were identified by Albert[5] in 1948 and whose applicability to Hamilton's equation with external term was identified by Santilli[6(a)] in l967 (see Refs. 7(a) and 7(b) for a general presentation up to 1983,[7(e),7(f)] for a recent study, and Ref. 8 for independent accounts).

An important feature of Lie-admissible formulations is their *direct universality*,[7(b)] i.e. their capability to represent all possible Newtonian systems (universality) in the frame of the experimenter (direct universality). In fact, the Hamiltonian represents all potential interactions, while the external terms represent all non-Hamiltonian forces and effects. By comparison, conventional Hamiltonian mechanics can represent only a rather limited class of Newtonian systems in the frame of the observer.[7(c)]

We shall exclude throughout our analysis the use of Darboux's transforms and only use *direct representations*, i.e. representations in the fixed coordinates of the observers. The main reason is that, since they map non-Hamiltonian into Hamiltonian systems, Darboux's transformations are highly *nonlinear* in all variables. As such, *Darboux's transforms imply the loss of the inertial character of the reference frames with consequential loss of Galilei's and Einstein's relativities.* Only *after* non-Hamiltonian systems have been represented in the *fixed* inertial frame of the observer, may the transformation theory acquire a physical value, precisely as it is the case for conventional conservative systems.

It is evident that the transition from Lie's theory to its Lie-admissible covering implies structural departures from the physical foundations of the special relativity. Even though mathematically appealing, this creates the rather serious problems of identifying a new covering relativity, proving its axiomatic consistency and then establishing it experimentally.

Another objective of this paper is to recall the complementary line of generalized classical formulations initiated by (G. D.) Birkhoff[9] in 1927. His main point was also the expected insufficiency of one single quantity, the Hamiltonian, to represent the entire physical reality. Rather than adding the external terms, Birkhoff's considered the most general possible, first-order, Pfaffian, action functional.

This implied a broadening of the brackets of the Hamiltonian time evolution into the so-called *generalized Lie brackets*, as studied by numerous authors (see, e.g. Ref. 10). Birkhoff's approach was studied in details by Santilli[6(d),7(c),7(d)] who showed that the brackets characterized by Birkhoff's equations imply a step-by-step,

axiom-preserving generalization of Lie's theory in all its branches (enveloping algebras, Lie algebras, Lie groups, representation theory, etc.).

The latter theory was submitted under the name of *Lie-isotopic theory*[6(d),7(d)] where the term "isotopic" is used in its Greek meaning of being "axiom-preserving." The emerging new mechanics was then submitted under the name of *Birkhoffian mechanics*.[7(d)] The more general mechanics of Lie-admissible type was submitted under the name of *Birkhoff-admissible mechanics*[6(d),7(b)] to denote the fact that the Lie-isotopic theory is a particular case of the Lie-admissible version.

In particular, Lie-isotopic formulations also resulted to be directly universal although for the more restricted class of well behaved, local-differential and analytic systems in a star-shaped regular point of their variables.[7(d)]

It is evident that, despite the preservation of the Lie character, a departure from the *canonical* realization of space–time symmetries implies an inevitable departure from the special relativity, thus creating again the problems of identifying a covering relativity, proving its axiomatic consistency and then establishing it experimentally.

Note since these introductory lines that, in view of the totally antisymmetric character of the product, Lie-isotopic theories admit conventional total conservation laws under generalized internal forces represented precisely by the generalized brackets (see systematic studies[7]). On the contrary, since their product is neither totally antisymmetric nor totally symmetric, the covering Lie-admissible theories are particularly set to represent open nonconservative systems under unrestricted external forces.[7]

In summary, as a result of their direct universality, all possible, well behaved, unconstrained, classical deformations can be classified into:

(I)  Deformations preserving the Lie character of Hamiltonian mechanics, in which case they can studied via one of the various realizations of the Lie-isotopic theory; or

(II)  Deformations abandoning the Lie character of Hamiltonian mechanics in favor of a covering of the Lie and Lie-isotopic theories, in which case they can be studied via the Lie-admissible theory.

(III)  Deformations of the still broader multivalued type currently under study by a restricted class of experts, which will not be studied in this paper for brevity (see later on Refs. 28 for details).

The above classification is important because it permits the study of axiomatic profiles in a unified way, rather than for individually for a seemingly disparate variety of deformations.

Additional types of generalized theories, such as the antiautomorphic images of Classes (I), (II), (III) currently under study for a classical treatment of antimatter, will not be considered at this time for brevity (see Refs. 7(e) and 7(f)). The extension of the results of this papers to Classes (I), (II), (III) and their antiautomorphic images to the case with subsidiary constraints is left to the interested reader.

Another objective of this paper is the identification of the origin of what are today called operator deformations which can be traced back to the inception of quantum mechanics itself, such as the theories relaxing the linearity of quantum mechanics (see the historical accounts in Refs. 12), or relaxing the potential character,[13] or relaxing the local structure,[7(d)] or relaxing the algebraic structure via external collision terms.[14]

The first known deformations of the Lie product $[A, B] = A \times B - B \times A$ of quantum mechanics was identified by Santilli[6(a)] back in 1967 via the expression $(A, B) = p \times A \times B - q \times B \times A$ where $p$ and $q$ are non-null parameters. The $q$-deformations with product $(A, B) = A \times B - q \times B \times A$ introduced by Biedenharn[15(a)] and Macfarlane[15(b)] in 1989, are an evident particular case Santilli's $(p, q)$-deformations. The latter $q$-deformations were then studies by a large number of authors (see the representative list[16]) although without the quotation of the origination of the deformations in Ref. 6(a).

The above studies were then follows by a large variety of operator deformations. Without any claim of completeness due to their shear number, we here mention: the deformations under the somewhat misleading name of "quantum groups;"[17] the $k$-deformations (which are a particular relativistic version of quantum groups);[18] the so-called "star theories"[19] whose product is, as we shall see, the basic isoassociative product of the Lie-isotopic theories; theories with nonassociative envelopes;[20] the so-called "squeezed states theories;"[21] a nonunitary statistical mechanics by Prigogine and his associates;[22] the Ellis–Mavromatos–Nanopoulos model of black hole dynamics with Santilli's Lie-admissible structure;[23] noncanonical time theories;[24] supersymmetric theories;[25] Kac–Moody superalgebras;[26] and others.

As we shall see, all the above generalized operator deformations can also be classified depending on their algebraic character. In fact, all $(p, q)$- and $q$-deformations evidently abandon the Lie character of quantum mechanics in favor of a generalized algebra which, since it is not totally antisymmetric or symmetric, it also results to be of Lie-admissible type as originally proposed by Santilli.[6(a)] However, quantum groups,[17] generalized statistical formulations[22] and other theories preserve the Lie character of the underlying algebra, although expressed in a generalized form, in which case they can be considered as a particular class of Lie-isotopic theories.

We should note that other theories, such as the nonlinear models[12] *appear* to have a conventional Lie algebra structure in their brackets, while at a deeper inspection such a structure results to be generalized, as evidently expected from the strictly *linear* character of Lie's theory when compared to the *nonlinear* character of the models here considered.

In regards to the generalized operator formulations it is therefore sufficient for us to consider only the classes of Lie-isotopic and Lie-admissible generalizations, because the latter have also resulted to be directly universal in operator settings.[7(f)] The understanding is that, again, the former are a particular case of the latter.

To avoid trivial cases, we shall solely consider classical (operator) deformations *outside the class of equivalence of Hamiltonian (quantum) mechanics*. Also, we shall

solely consider deformations as currently treated, that is, on conventional spaces over fields.

To avoid possible misrepresentations, we shall use the generic term "operator" rather than "quantum" deformations, because we are dealing with theories outside the class of equivalence of quantum mechanics.

The above definition of deformations evidently includes current gravitational theories. No study of the problematic aspects of available deformations can therefore be considered as sufficiently exhaustive without a consideration of classical and operator theories of gravity.

In summary, the first objective of this paper is to review the origin of classical and operator deformations and their unified treatments via Santilli's Lie-isotopic and Lie-admissible formulations[6,7] which are significant in the study of deformations because they unify seemingly disparate approaches, yet are generally ignored in the vast literature in the field.

The second objective of this paper is to point out that, even though with an unquestionable mathematical beauty, all possible classical and quantum deformations as currently treated are afflicted by rather serious problems of physical consistency. These problematic aspects have been studied in Refs. 27 and 28(b) mainly for the operator version. To our best knowledge, this is the first systematic study of the problematic aspects of deformations beginning at the classical level and then passing to their operator counterpart.

It is at this point where the mathematical beauty, axiomatic consistency and experimental validity of the special relativity emerge in their full light. A fundamental quantity of the special relativity is the four-dimensional unit

$$I = \mathrm{Diag}(\{1,1,1\},1),  \tag{1.1}$$

which represents in a dimensionless form the basic units of space $\{1,1,1\}$ (e.g. 1 cm, 1 cm, 1 cm for the three Euclidean axes), as well as the basic unit of time (e.g. 1 sec).

A pillar of the axiomatic consistency of the special relativity at both classical and quantum levels is *the universal invariance of the basic space and time units* (1.1), where the term "universal" stands to indicate invariance under all possible space–time symmetries as well as dynamical equations.

In fact, quantity (1.1) is the fundamental unit of the Minkowski space and of its basic Poincaré symmetry. As such, unit (1.1) is the unit of the universal enveloping associative algebra of the acting space–time symmetry, which is the *definition of unit* tacitly implied hereon.

The invariance of the basic unit is not a mere mathematical curiosity because it carries fundamental physical implications. In fact, it first implies lack of ambiguities in the physical applications and experimental verifications of the theory, evidently because the basic units used in measurements say, (1 cm, 1 cm, 1 cm, 1 sec) are universal invariants. The same invariance has then implications at all axiomatic and physical levels.

A primary objective of this paper is to study the problematic aspect of deformations implied by the following theorem whose classical proof will be presented in Sec. 3 and its operator counterpart in Sec. 4.

**Theorem 1.1.** *All possible classical and operator deformations, here defined as being outside the class of equivalence of conventional theories yet defined on conventional spaces over conventional fields, do not possess invariant units of space, time, energy, etc.*

As a result, the numerical applications and experimental verifications of deformed theories, whether classical or operator, are in question because of the lack of invariance of the basic units used for the measurements themselves.

A subsequent objective of this paper is to show that the lack of invariance of the basic units implies additional rather serious problematic aspects, again, of physical character.

As an example, one may attempt to bypass the problematic aspects of Theorem 1.1 by assuming that the rest of the universe is deformed jointly with that of the basic units, thus implying valid measurements.

Such a position is evidently questionable for the measurements of far away objects which, as such, are independent from local dynamics.

Independently from that, the above position is insufficient to resolve the shortcomings, because the lack of invariance of the unit has additional, rather serious implications. For example, it implies: the loss of the base field with evidently disastrous axiomatic consequences; the lack of preservation in time of the hermiticity with consequential lack of physically acceptable observables; the lack of uniqueness as well as invariance of the numerical predictions; the loss of invariant probabilities; the violation of causality; and, above all, the violation of the axioms of the special relativity.

These problematic aspects are sufficiently serious, first, to warrant their collegial awareness, and then to require systematic studies for their resolutions.

Note that we have used the terms "problematic aspects," rather than "inconsistencies," because we do not claim at all that theories[12–26] are physically inconsistent. We only insist that their problematic aspect should be addressed in the only possible scientific way, via publications.

As a matter of fact, our third objective is to indicate the ongoing efforts on the invariant formulation of Lie-admissible and Lie-isotopic theories which also appear to be largely ignored in the vast literature on deformations.

In fact, in the memoirs[28] we present mathematical and physical studies for an apparent first solution of the above problematic aspects which consists of generalized classical and operator theories constructed under the fundamental requirement of preserving the axiomatic structure of the special relativity, thus including universally invariant basic units, the preservation of hermiticity-observability at all times, uniqueness and invariance of the numerical predictions, etc.

Other resolutions of the problematic aspects are also possible, and their study is here encouraged, but under the physically uncompromisable condition of possessing invariant basic units.

A summary of the content of this paper is presented in note.[27(g)]

## 2. The Notions of Lie-Admissibility and Lie-Isotopy

### 2.1. *The first notion of Lie-admissibility*

In 1948 Albert[5] introduced the *first notion of Jordan admissible and Lie-admissible algebras* as generally nonassociative algebras $U$ with elements $a, b, c$, and abstract product $ab$ which are such that the attached algebras $U^+$ and $U^-$, which are the same vector spaces as $U$ equipped with the products $\{a, b\}_U = ab + ba$ and $[a, b]_U = ab - ba$, are Jordan and Lie algebras, respectively. Albert then studied the algebra with product

$$(A, B) = p \times A \times B + (1 - p) \times B \times A, \tag{2.1}$$

where $p$ is a parameter, $A, B$ are matrices or operators (hereon assumed to be Hermitian), and $A \times B$ is the associative product. It is easy to see that the above product is indeed jointly Jordan- and Lie-admissible because $\{A, B\}_U = A \times B + B \times A$ and $[A, B]_U = (2p - 1) \times (A \times B - B \times A)$.

Note that for $p = 0$ product (2.1) becomes that of a *commutative Jordan algebra*, but there exist no (finite) value of $p$ under which product (2.1) recovers the Lie product. As a result, product (2.1) cannot be used for possible coverings of current physical theories. In fact, Albert [*loc. cit.*] was primarily interested in the *Jordan*, rather than in the Lie content of nonassociative algebras (see Ref. 5 for more details).

### 2.2. *The second and third notions of Lie-admissibility*

In view of the above occurrence, in 1967 Santilli[6(a)] introduced the *second notion of Lie-admissibility* which is Albert's first notion [*loc. cit.*], plus the condition that the algebras $U$ admit Lie algebras in their classification or, equivalently, that the generalized Lie product admits the conventional one as a particular case.

Santilli[6(a)-6(c)] therefore introduced the algebra with product

$$(A, B) = p \times A \times B - q \times B \times A, \tag{2.2}$$

with related time evolution in the infinitesimal and finite forms ($\hbar = 1$)

$$i \times \frac{dA}{dt} = p \times A \times H - q \times H \times A, \tag{2.3a}$$

$$A(t) = e^{i \times q \times t \times H} \times A(0) \times e^{-i \times p \times t \times H}, \tag{2.3b}$$

where: $p$ and $q$ are non-null parameters with non-null values $p \pm q$; $A$, $B$ are Hermitian operator (or matrices), and $A \times B$ is also the associative product. It is easy to

see that product (2.2) is Lie- and Jordan-admissible and admits the Lie and Jordan products as particular (nondegenerate) cases.

The second notion of Lie-admissibility[6(a)] also resulted to be insufficient for physical applications because, as we shall see shortly, the parameters $p$ and $q$ become *operators* under the time evolution of the theory. Santilli[6(b),6(e)] therefore introduced the *third notion of Lie-admissibility* (also called *general Lie-admissibility*) which is the second notion plus the condition that the algebras $U$ admit *Lie-isotopic* (rather then Lie) algebras in their classification (see below).

The latter notion was realized via the *general Lie-admissible product* (first introduced in Ref. 6(e), p. 719)

$$(A, B) = A \times P \times B - B \times Q \times A, \tag{2.4}$$

and time evolution in infinitesimal and finite forms (Ref. 6(e), pp. 741, 742)

$$i \times \frac{dA}{dt} = A \times P \times H - H \times Q \times A, \tag{2.5a}$$

$$A(t) = e^{i \times H \times Q \times t} \times A(0) \times e^{-i \times t \times P \times H}, \tag{2.5b}$$

where $P$, $Q$ and $P \pm Q$ are nonsingular, generally non-Hermitian operators with nonsingular values $P \pm Q$ admitting of the parametric values $p$ and $q$ as particular cases. The conventional Heisenberg's equations are evidently recovered for $P = Q = 1$.

Note that the $P$ and $Q$ operators must be sandwiched in between the elements $A$ and $B$ to characterize an algebra as commonly understood in mathematics. In fact, the script $P \times A \times B - Q \times B \times A$ with $P$, $Q$ fixed, is acceptable when $P$ and $Q$ are parameters, but it would *not* characterize an algebra for $P$ and $Q$ operators because of the violation of the right distributive and scalar laws (see Ref. 7(d) for details).

### 2.3. *The notion of Lie-isotopy*

A fundamental property of the general Lie-admissible algebras $U$ identified in Refs. 6(d) and 6(e) is that their attached antisymmetric algebras $U^-$ *are not* characterized by the traditional Lie product $[A, B] = A \times B - B \times A$, but rather by the product (first introduced in Ref. 6(e), p. 725)

$$[A \hat{,} B]_U = A \hat{\times} B = B \hat{\times} A = A \times T \times B - B \times T \times A, \tag{2.6}$$
$$T = P + Q = T^\dagger,$$

called *Lie-isotopic*, because verifying the Lie axioms although in a more general way. The product $A \hat{\times} B = A \times T \times B$ is called *isoassociative* because more general then the conventional associative product $A \times B$, yet preserving associativity, $A \hat{\times} (B \hat{\times} C) \equiv (A \hat{\times} B) \hat{\times} C$.[6(e)]

According to the above results, the *nonassociative* algebra $U$ with product $(A, B)$, Eq. (2.4), can be replaced with an algebra $\hat{\xi}$ with *isoassociative* product

$A \hat{\times} B = A \times T \times B$, in the characterization of the attached antisymmetric algebra[6(e),7(d)]

$$[A \hat{,} B]_U = (A, B) - (B, A) \equiv [A \hat{,} B]_{\hat{\xi}} = A \hat{\times} B - B \hat{\times} A. \qquad (2.7)$$

The latter property permitted the construction of the *Lie-isotopic theory*,[6(d),6(e),7(d)] i.e. a step-by-step axiom-preserving lifting of the conventional formulation of Lie theory in terms of the isoassociative product $A \hat{\times} B$, including the lifting of numbers, spaces, enveloping algebras, Lie algebras, Lie groups, Lie symmetries, transformation and representation theory, etc. The emerging new theory is today called *Lie–Santilli isotheory*.[8,30,31]

As a particular case of the broader Lie-admissible formulations, Santilli[6(e),7(d),7(f)] therefore studied the Lie-isotopic time evolution in infinitesimal and finite forms for $T = T^{\dagger}$ (first introduced in Ref. 6(e), p. 752)

$$i \times \frac{dA}{dt} = [A \hat{,} H]_{\hat{\xi}} = A \hat{\times} H - H \hat{\times} A = A \times T \times H - H \times T \times A, \quad (2.8a)$$

$$A(t) = e^{i \times H \times T \times t} \times A(0) \times e^{-i \times t \times T \times H}, \qquad (2.8b)$$

which admit conventional quantum equations for $T = 1$.

Note for future needs that the Lie-admissible product (2.4) the (Lie-isotopic product (2.7)) are the most general possible nonantisymmetric (antisymmetric) product, respectively. This property is at the foundation of the unified treatment of deformations as we shall see. For additional details we refer the interested reader to monographs.[7(e),7(f)]

## 3. Origin and Problematic Aspects of Classical Deformations

### 3.1. *Birkhoffian mechanics*

No operator theory has sufficient depth without well-defined classical foundations. For this reasons, Santilli studied the classical counterparts of the preceding theories, as reported in monographs.[7] In essence, the classical action underlying Lie-isotopic theories resulted to be the most general possible, first-order, Pfaffian action in phase space[7(d)]

$$A = \int_{t_1}^{t_2} dt \left[ R_{\mu}(b) \frac{db^{\mu}}{dt} + H(t, b) \right],$$

$$b = \{b^{\mu}\} = \{r^k, p_k\}, \qquad R = \{R_{\mu}\} = \{A_k(r, p), B^k(r, p)\}, \qquad (3.1)$$

$$\mu = 1, 2, \ldots, 6, \qquad k = 1, 2, 3,$$

whose variations yield *Birkhoff's equations*[9] in the covariant and contravariant forms (see Ref. 7(d) for all historical notes and references)

$$\Omega_{\mu\nu}(b) \frac{db^{\nu}}{dt} = \frac{\partial H(t, b)}{\partial b^{\mu}}, \qquad (3.2a)$$

$$\frac{db^{\mu}}{dt} = \Omega^{\mu\nu}(b) \frac{\partial H(t, b)}{\partial b^{\nu}}, \qquad (3.2b)$$

with (nowhere degenerate) covariant and contravariant tensors

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial b^\mu} - \frac{\partial R_\mu}{\partial b^\nu} \,, \tag{3.3a}$$

$$\Omega^{\mu\nu}(b) = (|\Omega_{\alpha\beta}|^{-1})^{\mu\nu} \,. \tag{3.3b}$$

The ensuing mechanics, called *Birkhoffian mechanics* in Ref. 7(d), was said to be isotopic because it preserves the main axioms of conventional Hamiltonian mechanics although realized in their most general possible form. We are here referring to: (1) derivability from the most general possible first order action (analytic isotopy); (2) characterization by the most general possible, regular symplectic structure in local coordinates (geometric isotopy),

$$\Omega = \Omega_{\mu\nu}(b)db^\mu \wedge db^\nu \,; \tag{3.4}$$

and (3) characterization by the most general possible regular (unconstrained) brackets verifying the Lie axioms (algebraic isotopy)

$$[A, B]^* = \frac{\partial A}{\partial b^\mu} \Omega^{\mu\nu}(b) \frac{\partial B}{\partial b^\nu} \,. \tag{3.5}$$

Conventional classical Hamiltonian mechanics is admitted as a particular case at all levels for $R = R^0 = (p, 0)$, as one can easily verify.

One may consult Ref. 7(d) for additional aspects, including: the unified treatment via *the conditions of variational selfadjointness*; the isotopies of Lie's theory; the proof of the "direct universality" of the mechanics for local-differential and analytic systems; and other aspects.

Since Eqs. (2.8) and (3.5) have the same generalized (unconstrained and regular) Lie structures, the latter were introduced in Refs. 6(e) and 7(d) as the classical counterpart of the former, an assumption subsequently confirmed by specific studies.[7(f)]

## 3.2. *Hamilton-admissible and Birkhoff-admissible mechanics*

References 6(b), 6(d), 7(a) and 7(b) were devoted to the study of the classical counterpart of Lie-admissible equations (2.5). Conventional Newtonian forces are divided into variationally *self-adjoint* (SA) and *non-self-adjoint forces* (NSA),[7(c)] $F_k(t, b) = F_k^{\text{SA}} + F_k^{\text{NSA}}$. The SA forces are represented in terms of a conventional potential $U(t, b)$ via the techniques of the inverse problem [*loc. cit.*]. The NSA forces are represented via the algebraic tensor of the theory, according to the equations first introduced in Ref. 6(d)

$$\frac{db^\nu}{dt} - S^{\mu\nu}(t, b)\frac{\partial H(t, b)}{\partial b^\mu} \equiv \frac{m\, dv_k}{dt} - F_k^{\text{SA}}(t, b) - F_k^{\text{NSA}}(t, b) \,, \tag{3.6a}$$

$$(S^{\mu\nu}) = (\omega^{\mu\nu}) + (s^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (F^{\text{NSA}}/(\partial H/\partial p)) \end{pmatrix} \,, \tag{3.6b}$$

where $\omega^{\mu\nu}$ is the familiar canonical Lie tensor and $S^{\mu\nu}$ is a Lie-admissible tensor because

$$S^{\mu\nu}(t,b) - S^{\nu\mu}(t,b) = 2\omega^{\mu\nu} \,. \tag{3.7}$$

Consequently, the brackets of the time evolution

$$\frac{dA}{dt} = (A,H) = \frac{\partial A}{\partial b^{\mu}} S^{\mu\nu}(t,b) \frac{\partial H}{\partial b^{\nu}} \,, \tag{3.8}$$

are of Lie-admissible type,

$$(A,B) - (B,A) = 2[A,B] \,, \tag{3.9}$$

(with a trivial character of this type because the factor 2 is constant) with a compatible lifting of the symplectic two-form (3.4) called *symplectic-admissible*.[7(b)]

The emerging mechanics was called in Ref. 7(b) *Hamilton-admissible mechanics* when the attached antisymmetric tensor is Lie (as in Eqs. (3.6)) or *Birkhoff-admissible mechanics* when the attached antisymmetric tensor is the Birkhoffian one.

Note the simple direct universality of the Hamilton-admissible mechanics (without any need to go to the broader Birkhoff-admissible case) for all possible Newtonian systems, owing to general algebraic solution (3.6b). This simple direct universality should be compared with the rather complex direct universality of Birkhoff's equations (3.4).[7(d)]

It is important to know that Lie-admissible equations (3.7) were constructed along the original Hamilton's equations, those with *external terms* here denoted $F_k^{\text{NSA}}$. In fact, the number of independent functions in the external terms $F_k^{\text{NSA}}$ and that in the Lie-admissible tensor $S^{\mu\nu}$ coincide.

Reformulation (3.6) is requested by the fact that the brackets of Hamilton's equations with external terms violate the conditions to form any algebra, let alone Lie algebras, thus preventing the construction of a covering of conventional Hamiltonian mechanics. On the contrary, brackets (3.8), first of all, verify all conditions to characterize an algebra, and, second, that algebra results to be Lie-admissible, i.e. a covering of the algebraic structure of conventional Hamiltonian mechanics.

Note also that *the (autonomous) Lie-isotopic equations* (3.4) *are structurally reversible*, that is, they are reversible for reversible Hamiltonians. On the contrary, *Lie-admissible equations* (3.6) *are structurally irreversible*, that is, they are irreversible even for reversible Hamiltonians. These main characteristics will persist at the operator level of the next section.

Therefore, the Lie-admissible equations are particularly suited for an *axiomatization of irreversibility*, that is, its representation via the *structure* of the theory, rather than the addition of symmetry breaking terms in a time-symmetric Lagrangian or Hamiltonian.