R. M. Santilli
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An Introduction to Lie-Admissible Algebras (*).

R. M. Santilli (**)
Center for Theoretical Studies, University of Miami - Coral Gables, Fla.

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Contents. —

1. – Introduction.

The central role of *Lie algebras* in particle physics is well known. Starting from fundamental mathematical tools such as the Lie algebras of the Poincaré group, the rotational group and the isospin group, the interest for Lie algebras received a determining impulse with the celebrated Racah lectures at Princeton in 1951. Finally in 1959 there has been the beginning of the use of the Lie algebras of the unitary compact groups whose importance for hadron physics is today well known.

Recently a large number of Lie algebras of the orthogonal, unitary, simplectic, special linear compact and noncompact Lie groups have been investigated either as invariance algebras of the Hamiltonian or as symmetry algebras in the global sense.

Thus we are near to a clarification on the one hand of the meaning of a large number of Lie algebras for particle physics, and on the other hand of the limits of validity of Lie algebras for particle physics, particularly in connection with some interacting or decaying regions. In the framework of the above investigations algebraic structures more general than Lie algebras and

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(**) On leave of absence from the Istituto di Fisica dell'Università, Torino. Present address: Boston University, Physics Department, Boston, Mass.

their enveloping associative algebras, but possessing a well-defined Lie algebra content, may have a direct physical interest.

Lie algebras are generally introduced in particle physics in terms of the product $[a, b] = a\cdot b - b\cdot a$, where $a\cdot b$ is an associative product. However, according to ALBERT (1), a Lie algebra can also be introduced in terms of the product $[a, b] = ab - ba$, where ab is the product of a nonassociative algebra. More exactly, any algebra U with product ab is called *a Lie-admissible algebra* if the attached algebra U^-, which is the same vector space as U but with the new product $[a, b] = ab - ba$, is a Lie algebra.

The present notes are devoted to an elementary introduction to the theory of the Lie-admissible algebras with the purpose of stimulating the interest of physicists in this class of mathematical tools.

2. – The « fundamental » algebras.

In connection with the Lie-admissibility properties the algebras which admit realizations as mutations of an associative (*) algebra play an essential role. We assume these algebras as « fundamental ». They are:

1) *Lie algebras*, which are (nonassociative) anticommutative algebras satisfying the identities

$$a^2 = 0\,, \tag{2.1}$$

$$(ab)c + (bc)a + (ca)b = 0\,. \tag{2.2}$$

2) (*Commutative*) *Jordan algebras*, which are (nonassociative) algebras satisfying the relations (2-4)

$$ab = ba\,, \tag{2.3}$$

$$(a^2b)a = a^2(ba)\,. \tag{2.4}$$

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(1) A. A. ALBERT: *Trans. Amer. Math. Soc.*, **64**, 552 (1948).

(*) We define a *ring* R to be an additive Abelian group possessing a multiplication satisfying the distributive law $(a+b)c=ac+bc$, $a(b+c)=ab+ac$, for every $a, b, c\in R$. We define on *algebra* U to be a vector space over a field F possessing a bilinear operation of multiplication verifying the identities $(\alpha a)(\beta b) = (\beta a)(\alpha b) = \alpha\beta(ab)$, $(a+b)c = ac + bc$, $a(b+c) = ab + ac$ for every $a, b, c\in U$ and $\alpha, \beta\in F$. If in addition the associative law of multiplication holds, *i.e.* $(ab)c = a(bc)$ for every $a, b, c\in R(U)$, then $R(U)$ is called an *associative ring* (*algebra*). The amount by which three elements fail to obey the associative law of multiplication is called the associator and is denoted by $[a, b, c] = (ab)c - a(bc)$.

(2) H. BRAUN and M. KOECHER: *Jordan-Algebren* (Berlin, 1966).

(3) L. J. PAIGE: *Jordan Algebras*, MAA Studies in Math., vol. **2** (1963).

(4) R. D. SCHAFER: *An Introduction to Nonassociative Algebras* (New York, 1966).

3) *Noncommutative Jordan algebras*, which are (nonassociative) algebras defined by the relations [2-4]

$$(ab)a = a(ba)\,, \tag{2.5}$$

$$(a^2b) = a^2(ba)\,. \tag{2.6}$$

In connection with the explicit forms of the product let us consider an associative algebra A, with product $a\cdot b$, over a field F of characteristic (*) other than two. It is then possible to construct a new algebra, usually denoted with A^-, by means of the anticommutative product

$$[a, b] = a\cdot b - b\cdot a\,. \tag{2.7}$$

Clearly A^- is a Lie algebra in the usual form used by physicists. A fundamental result in the theory of Lie algebras is the Poincaré-Birkhoff-Witt theorem which states that every Lie algebra is isomorphic to a subalgebra of some algebra A^-. There is no analogue of the Poincaré-Birkhoff-Witt theorem for the remaining «fundamental» algebras. Consequently inequivalent forms of the product are possible for both the commutative and noncommutative Jordan algebras.

Another algebra which can be constructed by means of the associative product is characterized by

$$\tfrac{1}{2}\{a, b\} = \tfrac{1}{2}(a\cdot b + b\cdot a)\,. \tag{2.8}$$

This product is commutative and characterizes a class of Jordan algebras. Indeed the (commutative) Jordan algebras which are isomorphic to a subalgebra of some algebra A^+ are called *special Jordan algebras*; the remainder which are not isomorphic to some A^+ are called *exceptional Jordan algebras*.

In the same way it is possible to introduce always in terms of the associative product the following bilinear form [1]:

$$ab = \lambda a\cdot b + (1-\lambda) b\cdot a = \lambda[a, b] + b\cdot a\,, \tag{2.9}$$

(*) Let Z be the additive group of integers. The cyclic additive subgroup $Z(a)$ of a ring R is the set of elements (ma) with $a\in R$ and $m\in Z$. The mapping $m\to ma$ is an homomorphism of Z into R and is an isomorphism if $ma\neq na$ for all a and $m\neq n$. Every element a of a ring R generates a cyclic additive subgroup $Z(a)$. If every such subgroup has finite order m_a then the least common multiple of (m_a) is called the *characteristic* of the ring. If such finite number does not exist, then the ring is said to have *characteristic zero* (for instance the field of real numbers has characteristic zero). The above definition applies equivalently to fields and algebras too.

where λ is a free scalar belonging to the field, which characterizes the *λ-mutations* $A(\lambda)$ of A. Clearly $A(1)$ is isomorphic to A; $A(0)$ is anti-isomorphic to A; $A(\frac{1}{2})$ is isomorphic to A^+.

An algebra U over a field of characteristic other than two is called a *quasi-associative algebra* if there exists a scalar extension (*) R of F and an element $\lambda \in R$ such that $U_R = A(\lambda)$ where A is an associative algebra over R [1]. The quasi-associative algebras constitute the basic algebra of the noncommutative Jordan algebras and they represent an interesting example of (nonassociative) Lie-admissible algebras, as we shall see in Sect. 3. However there is no finite value of λ in the product (2.9) able to reproduce the commutator (2.7), which lessens the possibilities of physical interest.

In this connection a more general bilinear form in terms of the associative product is given by [5,6]

$$(a, b) = \lambda a \cdot b + \mu b \cdot a = \varrho[a, b] + \sigma\{a, b\}, \tag{2.10}$$

where $\lambda = \sigma + \varrho$ and $\mu = \sigma - \varrho$ are free scalars belonging to the field, which constitutes the basic product of the *(λ, μ)-mutations* $A(\lambda, \mu)$ of A. Clearly $A(1, 0)$ is isomorphic to A; $A(0, 1)$ is anti-isomorphic to A; $A(1, -1)$ is isomorphic to A^-; $A(\frac{1}{2}, \frac{1}{2})$ is isomorphic to A^+; and $A(\lambda, 1-\lambda)$ is isomorphic to $A(\lambda)$.

The $A(\lambda, \mu)$ algebras are equivalent to $A(\lambda)$ for any finite $\lambda \neq -\mu$. Indeed, by putting $\tau = \lambda + \mu$, $\lambda = \lambda'\tau$, $\mu = \mu'\tau$, where $\lambda' + \mu' = 1$, we can write

$$(a, b) = \lambda'\tau a \cdot b + \mu'\tau b \cdot a = \lambda' a^* b + (1 - \lambda') b^* a, \tag{2.11}$$

that is the (λ, μ)-mutations of A are isomorphic to the λ'-mutations of the isotopic algebra (**) A^* with product $a^* b = \tau a \cdot b$. Consequently for any $\lambda \neq -\mu$ the essential results on the theory of quasiassociative algebras U_R, such as for example the structure theorems, the construction of the matrix representations, the multiplication table, etc., can be applied to the $A(\lambda, \mu)$ algebra over R [5].

(*) The *scalar extension* U_R of an algebra U is the Kronecker product $U_R = R \otimes_F U$ of an arbitrary extension R of the field F and U itself.

(5) R. M. SANTILLI: *Nuovo Cimento*, **51** A, 570 (1967).

(6) R. M. SANTILLI and G. SOLIANI: *Statistic and parastatistic formal unification*, to appear.

(**) Let A be an associative algebra with multiplication $a \cdot b$ and let c be an invertible element of A. It is then possible to introduce a new multiplication given by $a^* b = a \cdot c \cdot b$. This multiplication essentially leaves the algebraic structure unchanged since the associative law of multiplication is preserved. The algebra induced by the multiplication $a^* b$ is called an *isotope* A^* of A. As a particular case we may have $c = \alpha 1$, with $\alpha \in F$ and $\alpha \neq 0$. Then the new multiplication $a^* b$ is simply α times the original product $a \cdot b$.

The $A(\lambda, \mu)$ algebras are some of the most interesting examples of (nonassociative) Lie-admissible algebras because, in addition to their Lie-admissibility content as for the $A(\lambda)$ algebras, they are able to transform directly into Lie algebras when anticommutativity of the product is requested (*).

In connection with the problem of the classification of the simple algebras (**) we note:

1) *Lie algebras.* – The Cartan classification of the complex simple Lie algebras and the construction of the corresponding real forms by means of inner and outer involutive automorphisms is well known. However the above classification refers to algebras of characteristic zero. In addition other simple Lie algebras of characteristic p exist. Indeed:

i) *For characteristic zero* we have the well-known simple algebras
Classical algebras: A, B, C, D;
Exceptional algebras: G_2, F_4, E_6, E_7, E_8.

ii) *For characteristic* p we list the following simple algebras
p-dimensional algebras by WITT;
p^n-dimensional algebras by ZASSENHAUS;
np^n-dimensional algebras by JACOBSON;
rp^n-dimensional algebras by KAPLANSKY;
$(n-1)(p^n-1)$-dimensional algebras by FRANZ;
T_n, V_m, L_0 and L_δ algebras by ALBERT and FRANZ;
$L(T, \delta, f)$ algebras by BLOCK.

(*) See footnote (*) on p. 1232.

(**) Let us recall that an algebra U is called *simple* if 0 and U are the only ideals of U and $U^2 = UU \neq 0$. An algebra U is called *semi-simple* if the radical is the zero ideal. For the definition of the radical the associativity or nonassociativity of the algebra plays an essential role. Indeed for finite-dimensional associative algebras the radical is defined as the unique maximal nilpotential ideal. For (nonassociative) algebras the concept of nilpotency gives rise to difficulties. Thus for finite-dimensional Lie and (commutative) Jordan algebras the radical is defined as the unique maximal solvable ideal (3). However for finite-dimensional noncommutative Jordan algebras the radical is defined as the maximal nilideal (that is an ideal composed by nilpotent elements) (7). This rules out Lie algebras in the classification of the noncommutative Jordan algebras (see Sect. 2) since all Lie algebras are nilalgebras of degree 2. For exhaustive discussions in connection with the radical of (nonassociative) algebras see papers (8-14).

(7) K. McCRIMMON: *Pacific Journ. Math.*, **15**, 187 (1965); *Proc. Amer. Math. Soc.* **17**, 1455 (1966); *Trans. Amer. Math. Soc.*, **121**, 187 (1966).

(8) A. A. ALBERT: *Bull. Amer. Math. Soc.*, **48**, 891 (1942).

(9) R. DUBISH and S. PERLIS: *Amer. Journ. Math.*, **70**, 540 (1948).

(10) W. E. JENNER: *Proc. Amer. Math. Soc.*, **1**, 348 (1950).

(11) A. MALGEV: *Dokl. Akad. Nauk SSSR*, **36**, 42 (1942).

(12) R. REE: *Proc. Amer. Math. Soc.*, **9**, 886 (1958).

(13) M. F. SMILEY: *Proc. Amer. Math. Soc.*, **2**, 138 (1951).

(14) S. A. AMITSUR: *Amer. Journ. Math.*, **74**, 774 (1952); **76**, 100, 126 (1954).

For an exhaustive discussion on simple Lie algebras of characteristic p see, for example, ref ([15]).

2) (*Commutative*) *Jordan algebras.* In this case the central simple algebras (*) of characteristic p are equivalent to those of characteristic 0 if $p > 2$. Furthermore it has been shown for (commutative) Jordan algebras that the scalar extension of a central simple algebra is a reduced (**) simple algebra, and *vice versa* any reduced simple algebra is central simple. Consequently the classification can be performed for reduced simple (commutative) Jordan algebras J_n^N of degree (**) n and dimension N over a field F, which we assume of characteristic zero, according to ([2-4]).

a) *Degree* $n = 1$: In this case we have that $J = eF$, where e is the primitive identity element of F.

b) *Degree* $n = 2$: Let $B_m(F)$ $(m > 2)$ be an m-dimensional vector space of symmetric bilinear nondegenerate forms $|a, b|$ with values in F, possessing an element e such that $|e, e| = 1$ and satisfying the relations: $|a, b| = |b, a|$; $|a+b, c| = |a, c| + |b, c|$; $|a, b+c| = |a, b| + |a, c|$; $\alpha|a, b| = |a, \alpha b| = |\alpha a, b|$ for any $a, b, c \in B_m(F)$ and $\alpha \in F$. Furthermore let U be the direct sum $U = F1 \oplus^m \oplus^m B_m(F)$. Then the algebras J_2^N can be introduced as special Jordan algebras isomorphic to some subalgebra of U^+ of dimension $N = m+1$.

([15]) G. B. SELIGMAN: *Modular Lie Algebras* (Berlin, 1967); *Nat. Acad. Sci.* (*Nat. Res. Council*) *Pub.*, **502**, 24 (1957).

(*) The *centre* C of an associative algebra A consists of all the elements $c \in C$ for which $ac = ca$ with any $a \in A$. If A is simple, then C is a field (but not necessarily the ground field) and A can be considered as an algebra over its centre. For the definition of the centre of a (nonassociative) algebra U not only commutativity but also associativity is requested between the elements of the centre and the elements of the algebra. The *centroid* $C(U)$ of U is the set of linear transformations t of the algebra $L(U)$ of all linear transformations of U for which $(ab)t = a(bt) = (at)b$ for any $a, b \in U$. A (nonassociative) algebra U over a field F is called *central* if the centroid coincides with the field. It has been shown that any simple algebra over F is central simple if and only if all the scalar extensions are simple. Furthermore every simple algebra U over F is central simple when considered as an algebra over its centroid. Thus the classification of the simple algebras can be reduced to the classification of all the central simple algebras.

(**) An element e of an algebra U is called *idempotent* when $e^2 = ee = e$ $(\neq 0)$. An idempotent e is called *principal* when U does not contain idempotents orthogonal to e. An idempotent e is called *primitive* when U does not contain orthogonal idempotents e_1 and e_2 such that $e = e_1 + e_2$. A primitive idempotent e is called *absolutely primitive* if it remains primitive in the scalar extensions U_R of U. An element 1 of U is called *left* (*right*) *identity* if $1a = a$ $(a1 = a)$ for any element a. An identity is an element 1 for which both $1a = a$ and $a1 = a$ hold for any a. The *degree* of an algebra is the number of pairwise orthogonal absolutely primitive idempotents in the decomposition of the identity. A *reduced algebra* is an algebra possessing an identity which can be decomposed in terms of absolutely primitive orthogonal idempotents.

c) Degree $n > 2$: In this case J_n^N is isomorphic to the Jordan algebra $J(U_n, I)$ of elements of an alternative (*) algebra U_n of $n \times n$ matrices which are self-adjoint with respect to a canonical involution (**). Then we have the following classes:

A) the algebras of the *I*-symmetric matrices whose elements are real numbers;

B) the algebras of the *I*-Hermitian matrices whose elements are complex numbers;

C) the algebras of the *I*-Hermitian matrices whose elements are quaternions;

D) the algebras of the *I*-Hermitian matrices whose elements are octonions (Cayley numbers). In this case the only possible value of n is 3 and $N = 8$.

The dimension N is given by $N = n + n(n-1)r/2$, where $r = 1, 2, 4, 8$ corresponding respectively to the classes *A*, *B*, *C*, *D*. We note that U_n is always an associative algebra with the only exception of the class *D*. Thus the algebras of the classes *A*, *B*, and *C* are special and the algebras of the class *D* are exceptional.

For specific investigations on (commutative) Jordan algebras see ref [16-46].

(*) An *alternative algebra* is an algebra satisfying the identities $(aa)b = a(ab)$, $(ab)b = a(bb)$, for any element a, b, called respectively left and right alternative laws. We recall for instance that the algebra of the Cayley numbers (also called octonions) is an alternative (nonassociative) algebra.

(**) An *involution* of an algebra U is antiautomorphic mapping $a \to \underline{a}$ on U such that $\underline{ab} = \underline{b}\,\underline{a}$ and $\underline{\underline{a}} = a$, that is the antiautomorphism is of degree two. An involution of the $n \times n$ matrix algebra D_n whose elements (belonging to D) are real numbers or complex numbers or quaternions or Cayley numbers, is called a *standard involution* when the operation of conjugate transpose is induced, *i.e.* $(a_{ij}) \to (\underline{a}_{ij})$. An involution of D_n is called a *canonical involution* when $a \to T^{-1} a' T$, where a' is given by a standard involution and T is a diagonal matrix whose elements are in the field.

(16) P. Jordan, J. von Neumann and E. Wigner: *Ann. Math.*, **35**, 29 (1934).
(17) A. A. Albert: *Trans. Amer. Math. Soc.*, **59**, 524 (1946).
(18) A. A. Albert: *Ann. Math.*, **48**, 1, 546 (1947); *Proc. Nat. Acad. Sci.*, **36**, 372 (1950); *Ann. Math.*, **67**, 1 (1958); *Journ. Math. Mech.*, **8**, 331 (1959); *Proc. Nat. Acad. Sci.*, **50**, 446 (1963).
(19) A. A. Albert and N. Jacobson: *Ann. Math.*, **66**, 400 (1957).
(20) A. A. Albert and L. J. Paige: *Trans. Amer. Math. Soc.*, **93**, 20 (1959).
(21) V. G. Askinuze: *Ukrain Math. Žurn.*, **3**, 381 (1951).
(22) G. Birkhoff and P. M. Whitman: *Trans. Amer. Math. Soc.*, **65**, 116 (1949).
(23) P. Civin and B. Yood: *Pacific Journ. Math.*, **15**, 775 (1965).
(24) P. M. Cohn: *Canad. Journ. Math.*, **6**, 253 (1954); *Proc. London Math. Soc.*, **9**, 503 (1959).

3) *Noncommutative Jordan algebras.* The central simple algebras U over F such that U^+ is a central simple (commutative) Jordan algebra have been classified according to (4)

A) the central simple (commutative) Jordan algebras;

B) the flexible quadratic algebras (*) with nondegenerate norm;

C) the central simple algebras of quasi-associative type.

(*) An algebra U is called a *flexible algebra* if the following identity $(ab)a = a(ba)$ holds for any $a, b \in U$. Then $((a+b)a)(a+b) = (a+b)(c(a+b))$ by which $(ab)c + (cb)a = a(bc) + c(ba)$ or (1) $[a, b, c] = -[b, c, a]$. In the framework of the Lie-admissible algebras flexibility has a particular importance. Indeed, by noting that all the « fundamental » algebras are flexible, flexibility can be considered as a generalization of both the (2.1) and (2.3) axioms. Furthermore, as we shall see in Sect. 3, the defining identities of the Lie-admissible algebras reduce to the corresponding ones of the Lie algebras when anticommutativity is requested. Consequently the Lie-admissible flexible algebras characterized by the relations $(ab)a = a(ba)$, $[a, b, c] + [b, c, a] + [c, a, b] = 0$, can be interpreted as a generalization of the Lie algebras to algebras whose product is neither totally antisymmetric not totally symmetric. Indeed, when anticommutativity is requested, the Lie-admissibility condition reduces to the Jacobi identity and the Lie-admissible algebras reduce to the Lie algebras, as it occurs for the $A(\lambda, \mu)$ algebras. The investigations on a possible physical meaning of the transition from a Lie algebra to a Lie-admissible flexible algebra $A(\lambda, \mu)$ in terms of dissipativity are in progress (47) on the basis of the Duffin analytical dynamics formulation for dissipative systems (48). An algebra U is called a *quadratic algebra* if U possesses an identity 1 over a field F and is such that for any $a \in U$: $a^2 + t(a)a + n(a)1 = 0$ where $t(a)$, $n(a) \in F$.

(25) M. Hall: *Proc. Amer. Math. Soc.*, **7**, 990 (1956).
(26) L. R. Harper: *Proc. Nat. Acad. Sci.*, **42**, 137 (1956).
(27) B. Harris: *Pacific Journ. Math.*, **8**, 757 (1958); **9**, 495 (1959).
(28) I. N. Herstein: *Amer. Journ. Math.*, **77**, 279 (1955); *Trans. Amer. Math. Soc.*, **81**, 331 (1956); *Amer. Journ. Math.*, **78**, 629 (1956); *Bull. Amer. Math. Soc.*, **67**, 517 (1961).
(29) Ch. Hertneck: *Math. Ann.*, **146**, 433 (1962).
(30) F. D. Jacobson and N. Jacobson: *Trans. Amer. Math. Soc.*, **65**, 141 (1949).
(31) N. Jacobson: *Amer. Journ. Math.*, **70**, 317 (1948); *Ann. Math.*, **50**, 866 (1949); *Amer. Journ. Math.*, **71**, 149 (1949); *Trans. Amer. Math. Soc.*, **70**, 509 (1951); *Amer. Math. Soc.*, **2**, 37 (1952); *Proc. Amer. Math. Soc.*, **3**, 973 (1952); *Proc. Intern. Congress of Math. Amsterdam*, vol. **3**, p. 28 (1954); *Osaka Math. Journ.*, **6**, 1 (1954); *Proc. Nat. Acad. Sci.*, **42**, 140 (1956); *Math. Ann.*, **136**, 375 (1958); *Math.*, **201**, 178 (1959); **204**, 74 (1960); **207**, 61 (1961); *Arch. Math.*, **13**, 241 (1962); *Proc. Nat. Acad. Sci.*, **48**, 1154 (1962).
(32) N. Jacobson and L. J. Paige: *Journ. Math. Mech.*, **6**, 895 (1957).
(33) N. Jacobson and C. E. Rickart: *Trans. Amer. Math. Soc.*, **69**, 479 (1950); **72**, 310 (1952).
(34) G. K. Kalisch: *Trans. Amer. Math. Soc.*, **61**, 482 (1947).
(35) M. Koecher: *Math. Phys. Kl.*, **2** A, 67 (1958); *Bull. Amer. Math. Soc.*, **68** 374 (1962).

In connection with the simple algebras of characteristic p (where U^+ is no longer a central simple Jordan algebra) we note as an example the nodal algebras (*) characterized by the product [49]

$$ab = a \circ b + \frac{1}{2}\sum_{ij} \frac{\partial a}{\partial x_i} \circ \frac{\partial b}{\partial x_j} \circ c_{ij} ,$$

where $a \circ b = \frac{1}{2}(ab+ba)$, $c_{ij} = -c_{ji}$, at least one c_{ij} possesses an inverse, a and b belong to a nilpotent polynomial ring $F(x_1, \ldots, x_n)$. For specific references on noncommutative Jordan algebras and connected problems of flexibility, trace admissibility (**) and power associativity (***), interesting for any Lie-admissible algebra, see papers [7,50-62].

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(36) I. G. MacDonald: *Proc. London Math. Soc.*, **10**, 395 (1960).

(37) K. McCrimmon: *Bull. Amer. Math. Soc.*, **70**, 702 (1964).

(38) K. Meyberg: *Math. Zeits.*, **89**, 52 (1965).

(39) W. H. Mills: *Pacific Journ. Math.*, **1**, 255 (1951).

(40) A. J. Penico: *Trans. Amer. Math. Soc.*, **70**, 404 (1951).

(41) R. D. Schafer: *Amer. Journ. Math.*, **70**, 82 (1948); *Proc. Amer. Math. Soc*,. **2**, 290 (1951); *Amer. Math. Soc.*, **84**, 426, (1957).

(42) M. F. Smiley: *Proc. Amer. Math. Soc.*, **8**, 668 (1957); *Portugal. Math.*, **20**, 147 (1961).

(43) T. A. Springer: *Proc. Nederl. Akad. Wetensch.*, A **62**, 254 (1959); A **63**, 414 (1960).

(44) E. Störmer: *Trans. Amer. Math. Soc.*, **120**, 438 (1965).

(45) J. Tits: *Proc. Nederl. Akad. Wetensch.*, A **65**, 530 (1962).

(46) D. M. Topping: *Memor. Amer. Math. Soc.*, **53**, (1965).

(47) R. M. Santilli: *Haag theorem and Lie-admissible algebras*, contributed paper at the *Indiana Symposium on Analytic Methods in Mathematical Physics, Bloomington, Ind., June* 3-6, 1968; *Dissipativity and Lie-admissible algebras*, Coral Gables preprint CTS M 67 2.

(48) R. J. Duffin: *Arch. Rational Mech. Anal.*, **9**, 309 (1962).

(*) A *nodal* noncommutative Jordan algebra U is a finite-dimensional algebra possessing the element 1, which can be decomposed according to: $U = F1 \oplus N$, where N is nilpotent but not a subalgebra of U.

(49) L. A. Kokoris: *Canad. Journ. Math.*, **12**, 488 (1960).

(**) An algebra U over a field F is called *trace admissible* if there is a bilinear form $n(a, b)$ with arguments in U and values in F such that $n(a, b) = n(b, a)$; $n(ab, c) = n(a, bc)$; $n(a, b) = 0$ if ab is nilpotent; and $n(e, e) \neq 0$ if e is an idempotent ($\neq 0$).

(***) An algebra U is called *power associative* if for all elements $a \in U$ and $m, n = 1, 2, 3, \ldots$, $a^m a^n = a^{m+n}$, where recursively $a^1 = a$, $a^2 = a^1 a \ldots$, $a^{k+1} = a^k a$. Then the following identities are satisfied: $a^2 a = a a^2$, $a^2 a^2 = (a^2 a) a$. Conversely the above identities imply that U is power associative if some restrictions on the characteristic are introduced, for instance a characteristic zero (or $\neq 2, 3, 5$) is assumed.

(50) A. A. Albert: *Proc. Nat. Acad. Sci.*, **35**, 317 (1949); *Summa Brasiliensis Math.*, **2**, 183 (1951); *Amer. Math. Soc., Proc. Intern. Congress Math. Cambridge*, **2**, 2 (1952); *Proc. Amer. Math. Soc.*, **9**, 928 (1958).

(51) J. D. Leadley and R. W. Ritchie: *Proc. Amer. Math. Soc.*, **11**, 399 (1960).

(52) E. Fleinfeld and L. A. Kokoris: *Proc. Amer. Math. Soc.*, **13**, 891 (1962).

(53) J. Knopfmacher: *Quater. Journ. Math. Oxford*, **13**, 264 (1962).

An interesting connection exists between the « fundamental » algebras and particularly between the exceptional Lie algebras and the exceptional (commutative) Jordan algebras, and between the Lie algebras of characteristic $p \neq 0$ and the nodal noncommutative Jordan algebras. See, for instance, in this connection ref. (4).

As historical remark let us recall that the Jordan algebras were introduced by a physicist, JORDAN, in the early 1930's. A paper by JORDAN, VON NEUMANN and WIGNER in 1934 introduces these algebras with the original name of « r-number algebras » (16). The name « Jordan algebras » was introduced by ALBERT in 1946 (17). The noncommutative Jordan algebras were introduced by SCHAFER in paper (63) on the basis of the theory of the quasi-associative algebras previously introduced by ALBERT in paper (1).

The (commutative) Jordan algebras were introduced essentially for quantum mechanical purposes. However at the present no evidence for a large physical contribution in quantum mechanics has appeared, although the algebras stimulated a new direction of algebraic studies in whose framework the essential tools for the characterization of the Lie-admissible algebras have been developed. A possible reason for the above disappointment in physical applications may be the lack of a nontrivial Lie-admissibility content. Indeed the only anticommutative algebras which can be constructed by a mutation of a commutative algebra are the zero algebras (*).

The situation is different for the Lie-admissible algebras because of their Lie algebra content. Indeed at least in principle at the present time the investigations on possible physical applications of the Lie-admissible algebras seem to be interesting, for instance, for regions where the Lie algebra invariance no longer holds, as for some interpolating field. Then, the possibility of a

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(*) A *zero algebra* is a nilpotent algebra of degree two. Let us recall that the nilpotency of a (nonassociative) algebra requires that all the possible products of k elements of the algebra are zero.

direct transformation of a Lie-admissible algebra into a Lie algebra (as for the $A(\lambda, \mu)$ algebras) seems to be interesting in connection with the asymptotic conditions connecting an interpolating field with a free field ([47]). Furthermore, Lie-admissible algebras may be also interesting as an alternative to the enveloping associative algebras for the construction of a Lie algebra in terms of the product $[a, b] = ab - ba$, where ab is nonassociative. The assumption of a Lie-admissible mutation algebra for nonassociative extension implies the introduction of free parameters in the structure constants of a Lie algebra (as we shall see in Sect. 5), that which may be interesting for an algebraic interpretation of renormalization procedures of field vectors or currents ([47]).

3. – Lie-admissible algebras.

Corresponding to any algebra U with product ab it is possible to define an anticommutative algebra U^- which is the same vector space as U but with the new product

$$[a, b]_{U^-} = ab - ba . \tag{3.1}$$

An algebra U is called *Lie-admissible* if the algebra U^- is a Lie algebra. An algebra U is called *L-simple* (*L-semi-simple*) if the algebra U^- is a simple (semi-simple) Lie algebra.

It is easy to see that all the algebras of Fig. 1 are Lie-admissible. Indeed:

1) If U is an *associative algebra* A, then the product (3.1) coincide with (2.7) and A^- is a Lie algebra in its more usual form. Thus the associative algebras constitute a basic class of Lie-admissible algebras.

2) If U is a *Lie algebra* L with product $ab = a\cdot b - b\cdot a$, then L^- is still a Lie algebra isomorphic to the isotopic algebra A^{*-} with product

$$[a, b]_{L^-} = [a, b]_{A^{*-}} = a^*b - b^*a = 2(a\cdot b - b\cdot a) . \tag{3.2}$$

Hence Lie algebras are always Lie-admissible algebras.

3) If U is a *special Jordan algebra* J, then J^- is a zero algebra because

$$[a, b]_{J^-} = ab - ba = 0 . \tag{3.3}$$

More generally any commutative algebra is trivially Lie-admissible.

4) If U is a *mutation algebra* $A(\lambda)$, then $[A(\lambda)]^-$ is isomorphic to the isotopic algebra A^{*-} with product

$$[a, b]_{[A(\lambda)]^-} = [a, b]_{A^{*-}} = a^*b - b^*a = (2\lambda - 1)(a\cdot b - b\cdot a) . \tag{3.4}$$

5) If U is a *mutation algebra* $A(\lambda, \mu)$, then $[A(\lambda, \mu)]^-$ is isomorphic to the isotopic algebra A^{*-} with product

$$(3.5) \qquad [a, b]_{[A(\lambda,\mu)]^-} = [a, b]_{A^{*-}} = a^* b - b^* a = (\lambda - \mu)(a \cdot b - b \cdot a).$$

As a consequence of the above properties we note that, in addition to the usual meaning of the attached algebra A^- of an associative algebra, a Lie algebra can also be introduced as the attached algebra $[A(\lambda)]^-$ or $[A(\lambda, \mu)]^-$ of the (nonassociative) Lie-admissible algebras $A(\lambda)$ and $A(\lambda, \mu)$.

We note that the α-mutation $A\big((\lambda)\alpha\big)$ of $A(\lambda)$ is characterized by the product

$$(3.6) \qquad ab = (1 - \lambda - \alpha + 2\lambda\alpha)\, a \cdot b + (\lambda + \alpha - 2\lambda\alpha)\, b \cdot a.$$

Hence $A\big((\lambda)\alpha\big)$ can be considered as a Ω-mutation $A(\Omega)$ with $\Omega = 1 - \lambda - \alpha + 2\lambda\alpha$. If $\lambda \neq \frac{1}{2}$, A can be recovered by $A(\lambda)$ by means of an α-mutation with $\alpha = \lambda/(2\lambda - 1)$ $(\alpha \neq 0, \frac{1}{2}, 1)$. If $\lambda = \frac{1}{2}$ we cannot recover A since all mutations of a commutative algebra leave the algebra unchanged.

Similarly the (α, β)-mutations $A\big((\lambda, \mu)\alpha, \beta\big)$ of $A(\lambda, \mu)$ is characterized by the product

$$(3.7) \qquad (a, b) = (\alpha\lambda + \beta\mu)\, a \cdot b + (\alpha\mu + \beta\lambda)\, b \cdot a.$$

Hence $A\big((\lambda, \mu)\alpha, \beta\big)$ can be condidered as an (Ω, ω)-mutation $A(\Omega, \omega)$ of A with $\Omega = \alpha\lambda + \beta\mu$ and $\omega = \alpha\mu + \beta\lambda$. If $\lambda \neq \pm\mu$ then A can be recovered from $A(\lambda, \mu)$ by means of an (α, β)-mutation where $\alpha = \lambda/(\lambda^2 - \mu^2)$ and $\beta = \mu/(\mu^2 - \lambda^2)$. If $\lambda = \pm\mu$ we cannot recover A since the (λ, μ)-mutation of both a commutative and an anticommutative algebra leaves the algebraic structure unchanged.

Hence the algebras L, J, $A(\lambda)$ and $A(\lambda, \mu)$ can be all introduced as mutations of an associative algebra A. Conversely A can be recovered by means of mutations of $A(\lambda)$ (for $\lambda \neq \frac{1}{2}$) and $A(\lambda, \mu)$ (for $\lambda \neq \pm\mu$) but not by means of mutations of L and J. Similarly L and J can be constructed by means of mutations of A, $A(\lambda)$ and $A(\lambda, \mu)$, but the mutations of L and J do not produce new algebraic structures (see Fig. 1).

In order to derive the condition for Lie admissibility for any algebra U we note that U^- is anticommutative by construction. Thus U is Lie-admissible if and only if U^- satisfies the Jacobi identity, that is the following relation holds:

$$(3.8) \qquad [a, b, c] + [b, c, a] + [c, a, b] = [c, b, a] + [b, a, c] + [a, c, b]$$

for any $a, b, c \in U$, where $[a, b, c] = (ab)c - a(bc)$ is the associator (*).

(*) See footnote (*) on p. 1226.

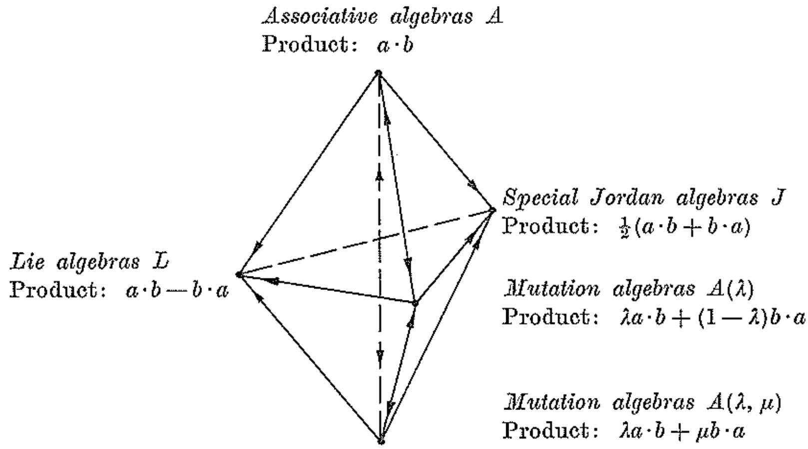

Fig. 1. – The «hexahedron» of algebras constituted by the associative algebras and their mutations: Lie algebras, special Jordan algebras, $A(\lambda)$ algebras and $A(\lambda, \mu)$ algebras. These algebras are characterized by the common properties of flexibility, Lie-admissibility and Jordan-admissibility. The arrows between two algebras denote the possibility of constructing one algebra by means of a mutation of the other.

If U is flexible, then (*) $[a, b, c] = -[c, b, a]$ and condition (3.8) reduces to the expression

$$[a, b, c] + [b, c, a] + [c, a, b] = 0 . \tag{3.9}$$

If U is anticommutative (any anticommutative algebra is flexible) condition (3.9) reduces to the Jacobi identity itself

$$(ab)c + (bc)a + (ca)b = 0 , \tag{3.9'}$$

and we have proved the following (*)

Lemma 1. Any anticommutative Lie-admissible algebra is a Lie algebra.

Finally if U is commutative (any commutative algebra is flexible) condition (3.9) is trivially satisfied since $(ab)c+(bc)a+(ca)b \equiv a(bc)+b(ca)+c(ab)$.

In terms of right and left multiplications R_a and L_a (**) relation (3.8) can be written

$$R_{ab-ba} - L_{ab-ba} = (R_a - L_a)(R_b - L_b) - (R_b - L_b)(R_a - L_a) . \tag{3.10}$$

If we require that U be flexible, then $R_{ab} - L_{ba} = R_a R_b - L_a L_b$, and $R_b L_a -$

(*) See footnote (*) on p. 1232.

(**) The multiplication algebra $M(U)$ of any algebra U is the enveloping associative algebra of $R(U) \cup L(U)$ where $R(U)$ and $L(U)$ are the sets of right and left multiplications R_a: $x \to xa = aR_a$, L_a: $x \to ax = L_a x$, the mapping $a \to R_a$ and $a \to L_a$ being

$-L_a R_b = L_b R_a - R_a L_b$ by which, using (3.10) we get

$$(3.11)\qquad \begin{cases} R_{ab-ba} = R_a(R_b - L_b) - (R_b - L_b)R_a\,, \\ L_{ab-ba} = (L_b - R_b)L_a - L_a(L_b - R_b)\,. \end{cases}$$

Conversely relations (3.11) imply flexibility, that is $R_a L_a = L_a R_a$ (*).

Theorem 1 (ALBERT (1)). An algebra U is a flexible Lie-admissible algebra if one of conditions (3.11) holds for every element $a, b, \in U$.

We note that conditions (3.11) can be respectively written

$$(3.12)\qquad \begin{cases} [a, b, c] - [a, c, b] + [c, a, b] = 0\,, \\ [a, b, c] - [a, c, b] - [b, a, c] = 0\,. \end{cases}$$

Hence they are equivalent to (3.9) in virtue of the flexibility property $[a, b, c] = -[c, b, a]$.

Let us also note that an imbedding of Lie algebras L into (nonassociative) Lie-admissible algebras U has been recently proposed according to (5)

$$(3.13)\qquad L \xrightarrow{\text{Isomorpism}} U^- \xrightarrow{\text{Imbedding}} U\,,$$

that is by requiring that U^- is isomorphic to L by construction. Clearly U cannot be a commutative algebra since in this case the attached algebra U^- is a zero algebra. If U is an anticommutative algebra then it is a Lie algebra by Lemma 1. Thus the nontrivial (nonassociative) Lie-admissible extensions U must be neither commutative nor anticommutative. Indeed it has been shown (5) that the only simple power associative and trace admissible extensions U of degree $n > 2$ are the algebras of quasi-associative type. Consequently the $A(\lambda, \mu)$ algebras, in addition to their Lie-admissibility content and possibility of direct transformation into a Lie algebra are some of the most interesting examples of extension U for the imbedding (3.13).

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linear mappings and R_a and L_a being linear transformations of the vector space U for all $a \in U$. For example the relations $\alpha(ab) = (\alpha a)b = a(\alpha b)$ and $a(b+c) = ab + ac$ correspond to $\alpha R_b = R_{\alpha b}$ and $R_{b+c} = R_b + R_c$. If U is an associative algebra A then the identity $(ab)c = a(bc)$ implies that $aR_bR_c = aR_{bc}$, $L_aL_bc = L_{ab}c$, $R_{bc} = R_bR_c$, $L_{ab} = L_aL_b$. Consequently the mappings $a \to R_a$ and $a \to L_a$ are homomorphisms of A into $M(A)$. These homomorphisms become isomorphisms if A possesses the unit element since in this case $R_a = R_b$ implies $a = b$. If the algebra is commutative than $R_a = L_a$ and one mapping only can be used. Let us note that the Jacobi identity (2.2) can be written $L_{ab} + R_aR_b + L_bR_a = 0$, the Jordan identity (2.4) becomes $L_{aa}R_a = R_aL_{aa}$, and the flexibility law can be expressed by the commutativity law $L_aR_a = R_aL_a$.

(*) See the footnote (**) on the preceding page.

4. – Jordan-admissible algebras.

Corresponding to any algebra U with product ab it is possible te define, as for U^-, a commutative algebra U^+ which is the same vector space as U but with the new product

$$\tfrac{1}{2}\{a, b\}_{U^+} = \tfrac{1}{2}(ab + ba) . \tag{4.1}$$

Clearly Lie algebras L have a trivial content of the attached algebras L^+ on account of the anticommutativity of their product. On the contrary nontrivial Lie-admissible algebras have a well-defined content of U^+ because, as we have seen, they are neither commutative nor anticommutative. Consequently for the characterization of the nontrivial Lie-admissible algebras U the determination of both U^- and U^+ is essential.

In this connection the most interesting case occurs when U^+ is a (commutative) Jordan algebra. An algebra U is said to be *Jordan admissible* if U^+ is a (commutative) Jordan algebra. An algebra U is said to be *J-simple* (*J–semi-simple*) if U^+ is a simple (semi-simple) Jordan algebra.

All algebras of Fig. 1 are Jordan admissible. Indeed:

1) If U is an *associative algebra* A, then the product (4.1) reduces to (2.8) and A^+ is a special Jordan algebra. Thus associative algebras constitute a basic class of Jordan-admissible algebras.

2) If U is a *Lie algebra* L, then L^+ is a zero algebra since

$$\tfrac{1}{2}\{a, b\}_{L^+} = \tfrac{1}{2}(ab + ba) = 0 . \tag{4.2}$$

More generally any anticommutative algebra is trivially Jordan admissible.

3) If U is a *special Jordan algebra* J, then J^+ is still a special Jordan algebra isomorphic to the isotopic algebra A^{*+} with product

$$\tfrac{1}{2}\{a, b\}_{J^+} = \tfrac{1}{2}\{a, b\}_{A^{*+}} = \tfrac{1}{2}(a^*b + b^*a) = a\cdot b + b\cdot a . \tag{4.3}$$

4) If U is a *mutation algebra* $A(\lambda)$, then $[A(\lambda)]^+$ coincides with A^+, since

$$\tfrac{1}{2}\{a, b\}_{[A(\lambda)]^+} = \tfrac{1}{2}\{a, b\}_{A^+} = \tfrac{1}{2}(a\cdot b + b\cdot a) . \tag{4.4}$$

5) If U is a *mutation algebra* $A(\lambda, \mu)$, then $[A(\lambda, \mu)]^+$ is isomorphic to the isotopic algebra A^{*+} with product

$$\tfrac{1}{2}\{a, b\}_{[A(\lambda,\mu)]^+} = \tfrac{1}{2}\{a, b\}_{A^{*+}} = \tfrac{1}{2}(a^*b + b^*a) = \frac{\lambda+\mu}{2}(a\cdot b + b\cdot a) . \tag{4.5}$$

In connection with the Jordan admissibility conditions we note that U^+ is commutative by construction. Hence U is Jordan admissible if and only if U^+ satisfies the Jordan identity, that is the following relation holds

$$(a^2b)a + a(ba^2) + (ba^2)a + a(a^2b) = a^2(ba) + (ab)a^2 + a^2(ab) + (ba)a^2 . \tag{4.6}$$

If we require that U is flexible, then $aa^2 = a^2a$, $(a^2b)a - a^2(ba) = a(ba^2) - (ab)a^2$, $a(a^2b) - (aa^2)b = (ba^2)a - b(a^2a)$, and $a^2(ab) - (a^2a)b = (ba)a^2 - b(aa^2)$, by which relation (4.6) becomes

$$(a^2b)a + a(a^2b) = a^2(ba) + a^2(ab) . \tag{4.7}$$

If we require that U is commutative, then the above relation reduces to one of the following equivalent forms:

$$\begin{cases} (a^2b)a = a^2(ba) , & a(a^2b) = a^2(ab) , \\ a(ba^2) = (ab)a^2 , & (ba)a^2 = (ba^2)a , \end{cases} \tag{4.8}$$

by which we have proved the following:

Lemma 1. Any commutative Jordan admissible algebra is a (commutative) Jordan algebra.

Finally if U is anticommutative then all the above relations are trivially satisfied since the square value of any element is zero.

In terms of right and left multiplications (*) (4.6) becomes

$$(R_a + L_a)(R_{aa} + L_{aa}) = (R_{aa} + L_{aa})(R_a + L_a) . \tag{4.9}$$

Similarly (4.7) can be written

$$R_aL_{aa} - L_{aa}R_a = L_{aa}L_a - L_aL_{aa} \tag{4.10}$$

while conditions (4.8) correspond to the commutativity of R_a and L_a with R_{aa} and L_{aa}

$$\begin{cases} L_{aa}R_a = R_aL_{aa} , & L_{aa}L_a = L_aL_{aa} , \\ R_{aa}L_a = L_aR_{aa} , & R_{aa}R_a = R_aR_{aa} . \end{cases} \tag{4.10$'$}$$

Let us recall that for flexible algebras all relations (4.8) are equivalent [1]. Consequently the condition for Jordan admissibility of flexible algebras is

(*) See footnote (**) on p. 1237.

that one of conditions (4.8) holds. In this case the algebra is in addition power associative (*) if some restrictions on the characteristic of the field are introduced, since flexibility implies that $a^2a = aa^2$ and one of the conditions (4.8) implies that $a^2a^2 = (a^2a)a$. Indeed:

Theorem 1 (ALBERT (1)): A flexible algebra U over a field of characteristic $\neq 2, 3, 5$ is Jordan admissible and power associative if one of conditions (4.8) holds for every $a, b \in U$.

More particularly let us recall that any flexible Jordan-admissible algebra is by definition a noncommutative Jordan algebra (4).

5. – Some properties of the Lie-admissible algebras.

One of the most interesting features of the Lie-admissible algebras is that some of the methodological procedures used in the theory of Lie algebras can be extended to the Lie-admissible algebras if a supplementary condition is introduced, for instance flexibility or trace admissibility or L-semi-simplicity is requested. Few investigations have been done in this connection.

Let us consider a nontrivial Lie-admissible algebra U, with elements X_ϱ, X_σ, ... and product $X_\varrho X_\sigma$, over a field F. We note that $X_\varrho X_\sigma$ can be written

$$X_\varrho X_\sigma = \tfrac{1}{2}[X_\varrho, X_\sigma]_{U^-} + \tfrac{1}{2}\{X_\varrho, X_\sigma\}_{U^+}. \tag{5.1}$$

Consequently U can be given by

$$U = U^{*-} @\, U^+, \tag{5.2}$$

where the operation @ means that an element $Z = X_\varrho X_\sigma$ of U can be obtained as a sum of a product of X_ϱ and X_σ in U^{*-} and U^+, while U^{*-} is the isotopic algebra with the product $[X_\varrho, X_\sigma]_{U^{*-}} = X^*_\varrho X_\sigma - X^*_\sigma X_\varrho = \tfrac{1}{2}(X_\varrho X_\sigma - X_\sigma X_\varrho)$.

Let us discuss first the U^- content by supposing that U is L-semi-simple. Furthermore let the commutation rules of U^- be given by

$$[X_\varrho, X_\sigma]_{U^-} = C'^{\tau}_{\varrho\sigma} X_\tau, \tag{5.3}$$

where the structure constants $C'^{\tau}_{\varrho\sigma}$ satisfy the Cartan condition $\mathrm{Det}\,|g_{\varrho\sigma}| = \mathrm{Det}\,|C'^{\mu}_{\varrho\lambda} C'^{\lambda}_{\sigma\mu}| \neq 0$. Corresponding to two elements $A = \alpha^\mu X_\mu$ and $X = \beta^\varrho X_\varrho$ where $\alpha^\mu, \beta^\varrho \in F$, the eigenvalue problem for getting the standard form of the

(*) See footnote (***) on p. 1233.

generators holds

$$[A, X]_{U^-} = \varrho X\,, \qquad \varrho \in F\,. \tag{5.4}$$

Consequently all the essential results on semi-simple Lie algebras occur as in the usual way. For instance the number of independent elements $H_1, H_2, \ldots, H_r$ which commute with each other is equal to the multiplicity of the null root, *i.e.* the rank, and the standard form of the commutation rules of U^- can be introduced

$$\begin{cases} [H_i, H_j]_{U^-} = 0\,, & [E_\alpha, E_\beta]_{U^-} = C'^{\alpha+\beta}_{\alpha\beta} E_{\alpha+\beta} (\alpha+\beta \neq 0)\,, \\ [H_i, E_\alpha]_{U^-} = \alpha_i E_\beta\,, & [E_\alpha, E_{-\alpha}]_{U^-} = \alpha^i H_i\,. \end{cases} \tag{5.5}$$

If U is an associative algebra, the form of the above procedure coincides with the usual one. However the same results occur also, but in a more general form, if U is nonassociative, provided that U is L–semi-simple.

If we assume that the Lie-admissible algebra U (not necessarily L–semi-simple) is a mutation algebra $A(\lambda, \mu)$, and we denote with $C^\tau_{\varrho\sigma}$ the structure constants of the Lie algebra A^- isomorphic to U^-, then the connection between $C'^\tau_{\varrho\sigma}$ and $C^\tau_{\varrho\sigma}$ is characterized by the relation

$$C'^\tau_{\varrho\sigma} = (\lambda - \mu)\, C^\tau_{\varrho\sigma}\,. \tag{5.6}$$

This means that if we construct a Lie algebra by means of a Lie-admissible algebra $A(\lambda, \mu)$ instead of A the free parameters λ and μ appear directly in the structure constants. Furthermore if the following supplementary condition is introduced ([5])

$$\lambda^2 + \mu^2 = 2\,, \tag{5.7}$$

relation (5.6) can be written

$$C'^\tau_{\varrho\sigma} = 2 \cos\alpha\, C^\tau_{\varrho\sigma}\,. \tag{5.8}$$

Thus for the «angle» $\alpha = 0$ the Lie-admissible algebra $A(\lambda, \mu)$ reduces to the Lie algebra A^-, and for $\alpha = \frac{1}{2}\pi$ $A(\lambda, \mu)$ reduces to the trivially Lie-admissible algebra A^+.

Let us consider the case now when the Lie-admissible algebra U is flexible and power associative. If B is a subspace of U, we denote with U^B the commutator space of U, *i.e.* the set of elements $X_\varrho \in U$ such that $[X_\varrho, X^B_\sigma] = 0$ for any $X^B_\sigma \in B$. Furthermore let (X_ϱ) be the set of all the scalar multiples of X_ϱ, and the characteristic be always equal to zero.

Lemma 1 (WEINER (64)). If U is a flexible, power associative and Lie-admissible algebra, then $U^{(X_\varrho)} \subseteq U^{(X_\varrho^2)}$ for any $X_\varrho \in U$.

Proof. By definition of $U^B[X_\varrho, X_\sigma] = 0$ for any $X_\sigma \in U$. Then by the Lie-admissibility condition (3.9) we can write $[X_\varrho, X_\sigma^2] = (X_\sigma X_\varrho)X_\sigma + (X_\varrho X_\sigma)X_\sigma - - X_\sigma(X_\sigma X_\varrho) - X_\sigma(X_\varrho X_\sigma)$. Furthermore $[X_\varrho, X_\sigma^2] = 0$ from the flexibility condition. Thus $X_\sigma \in U^{(X_\varrho^2)}$.

Theorem 1 (WEINER (64)). If U is a flexible power associative and Lie-admissible algebra and B is a subspace of U, then B is a subalgebra of U if B^- is a subalgebra of U^- and $X_\varrho^2 \in B$ for every $X_\varrho \in B$.

Proof. Since B^- is a subalgebra of U^-, $[X_\varrho, X_\sigma] \in B$ for any X_ϱ, $X_\sigma \in B$. Moreover $\{X_\varrho, X_\sigma\} \in B$ for any $X_\varrho, X_\sigma \in B$ since $\{X_\varrho, X_\sigma\} = (X_\varrho + X_\sigma)^2 - - X_\varrho^2 - X_\sigma^2$ and X_ϱ^2, $X_\sigma^2 \in B$. Thus $X_\varrho X_\sigma = \frac{1}{2}[X_\varrho, X_\sigma] + \frac{1}{2}\{X_\varrho, X_\sigma\} \in B$ and B is a subalgebra of U.

We note also that $C = U^B$ is a subalgebra of U for any subspace B of U. Indeed $C^- = (U^B)^-$ is a subalgebra of U^- since for any X_ϱ, $X_\sigma \in U$ and $X_\delta \in U^B$ by the Jacobi identity we have $[[X_\varrho, X_\sigma], X_\delta] = [X_\varrho, [X_\sigma, X_\delta]] + [X_\sigma, [X_\delta, X_\varrho]]$ and by the property $[X_\sigma, X_\delta] = [X_\delta, X_\varrho] = 0$ we have $[[X_\varrho, X_\sigma], X_\delta] = 0$. Furthermore if $X_\delta \in C$ by Lemma 1 $X_\delta^2 \in C$. Hence Theorem 1 applies to C.

We consider now the case when the Lie-admissible algebra U is only flexible and not necessarily power associative, always of characteristic zero.

Lemma 2. The set of transformations $D_{X_\varrho} = R_{X_\varrho} - L_{X_\varrho}$ of a flexible Lie-admissible algebra U for any $X_\varrho \in U$ is a derivation algebra of U.

Proof. Relations (3.11) can be written for any X_ϱ, $X_\sigma \in U$

$$R_{[X_\varrho, X_\sigma]} = [R_{X_\varrho}, R_{X_\sigma} - L_{X_\sigma}] \quad \text{and} \quad L_{[X_\varrho, X_\sigma]} = [L_{X_\varrho}, R_{X_\sigma} - L_{X_\sigma}],$$

by which

$$R_{X_\varrho D_{X_\sigma}} = [R_{X_\varrho}, D_{X_\sigma}], \quad L_{X_\varrho D_{X_\sigma}} = [L_{X_\varrho}, D_{X_\sigma}]. \tag{5.9}$$

The above relations constitute an equivalent for for defining the derivation property $(X_\varrho X_\sigma)D = (X_\varrho D)X_\sigma + X_\varrho(X_\sigma D)$.

Let us define an element X_ψ of U to belong to the characteristic root α of the derivation D_{X_ω} for $X_\omega \in U$ if

$$X_\psi(D_{X_\omega} - \alpha 1)^p = 0 \tag{5.10}$$

for some integer p.

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(64) L. M. WEINER: *Revista Mat. Fis. Teor. Tucuman*, **11**, 10 (1957).

Lemma 3 (LAUFER-TOMBER [65]). The elements $X_\varrho X_\sigma$ and $X_\sigma X_\varrho$ of a flexible Lie-admissible algebra U over an algebraically closed field F belong to the root $\alpha+\beta$, where $\alpha+\beta$ is a root of D_{X_τ} or zero, if for X_ϱ, X_σ, $X_\tau \in U$ and $\alpha, \beta \in F$, α and β are roots of D_{X_τ} and X_ϱ, X_σ belong respectively to α and β.

Proof. By the property of the derivations we can write

$$X_\varrho X_\sigma(D_{X_\tau} - (\alpha+\beta)1) = X_\varrho(D_{X_\tau} - \alpha 1)X_\sigma + X_\varrho(X_\sigma(D_{X_\tau} - \beta 1)) .$$

Consequently the proof by induction used for Lie algebras can be used also for Lie-admissible algebras.

Theorem 2 (LAUFER-TOMBER [65]). A flexible L–semi-simple algebra U over an algebraically closed field F is a direct sum of simple flexible Lie-admissible algebras.

Proof. Since U^- is semi-simple it can be decomposed in terms of simple Lie algebras L_γ ($\gamma = 1, 2, \ldots, n$) according to the direct sum $U^- = \sum_\gamma^\oplus L_\gamma$. Correspondingly there is the decomposition of U according to the vector space direct sum of subspaces $U = U_1 + \ldots + U_n$ such that U_γ and L_γ are the same additive groups.

Let us first prove that U_γ is a subalgebra of U for any $\gamma = 1, \ldots, n$. We write the product $X_{\gamma_1} X_{\gamma_2}$ for X_{γ_1}, $X_{\gamma_2} \in U_\gamma$ according to $X_{\gamma_1} X_{\gamma_2} = \sum_\alpha X_\alpha$ where $X_\alpha \in U_\alpha$. If $X_\beta \neq 0$ there is an element $X'_\beta \in L_\beta$ such that $[X_\beta, X'_\beta] \neq 0$ since L_β is simple. Then in account of the semi-simplicity of U^- we can write

$$(X_{\gamma_1} X_{\gamma_2}) D_{X'_\beta} = \Big(\sum_\alpha X_\alpha\Big) D_{X'_\beta} = (X_{\gamma_1} D_{X'_\beta}) X_{\gamma_2} + X_{\gamma_1}(X_{\gamma_2} D_{X'_\beta}) = X_\beta D_{X'_\beta} .$$

Furthermore for $\beta \neq \gamma$ $(X_{\gamma_1} X_{\gamma_2}) D_{X'_\beta} = 0$. Hence for any $\beta \neq \gamma$ we have $X_\beta D_{X'_\beta} = [X_\beta, X'_\beta] = 0$ by which $X_\beta = 0$, $X_{\gamma_1} X_{\gamma_2} \in U_\gamma$ and U_γ is a subalgebra of U.

Let us now show that each U_γ is simple and that U is a direct sum of the U_γ. We consider the elements $X_\alpha \in U_\alpha$ and $X_\beta \in U_\beta$ with $\alpha \neq \beta$ and we suppose that X_α belongs to a root of a Cartan subalgebra $\mathscr{H}_\alpha \subset U_\alpha$ of L_α. Then by definition of root there is an element H_α and a nonnull scalar $\varrho \in F$ such that $[H_\alpha, X_\alpha] = X_\alpha D_{H_\alpha} = \varrho X_\alpha$. We write $X_\alpha X_\beta = \sum_\gamma X_\gamma$. Then by applying D_{H_α} we have

$$(X_\alpha X_\beta) D_{H_\alpha} = (X_\alpha D_{H_\alpha}) X_\beta + X_\alpha(X_\beta D_{H_\alpha}) = \varrho X_\alpha X_\beta = [X_\alpha, H_\alpha] \in U_\alpha .$$

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(65) P. J. LAUFER and M. L. TOMBER: *Canad. Jour. Math.*, **14**, 287 (1962).

Similarly

$$(X_\alpha X_\beta) D_{X_{-\alpha}} = [X_\alpha, X_{-\alpha}] X_\beta = H_\alpha X_\beta \in U_\alpha ,$$

where $X_{\pm\alpha}$ belong to the corresponding roots $\pm\alpha$ and $X_\beta \in U_\beta$. Consequently each U_α is an ideal of U and is simple. Then U is the direct sum $U = U_1 \oplus \oplus U_2 \oplus \ldots \oplus U_n$.

Equivalent results were obtained in paper (64) with the supplementary request of power associativity.

In connection with the U^+ content of U let us assume that U is power associative over a field F of characteristic $\neq 2, 3, 5$. The linearization of the of the identities $aa^2 = a^2a$ and $a^2a^2 = (a^2a)a$ in U gives rise to the respective relations

$$(5.11) \qquad (X_\alpha X_\beta + X_\beta X_\alpha) X_\gamma + (X_\beta X_\gamma + X_\gamma X_\beta) X_\alpha + (X_\gamma X_\alpha + X_\alpha X_\gamma) X_\beta = \\ = X_\gamma (X_\alpha X_\beta + X_\beta X_\alpha) + X_\alpha (X_\beta X_\gamma + X_\gamma X_\beta) + X_\beta (X_\gamma X_\alpha + X_\alpha X_\gamma) ,$$

$$(5.12) \qquad \sum_6 \mathrm{Perm}\,(X_\alpha X_\beta + X_\beta X_\alpha)(X_\gamma X_\delta + X_\delta X_\gamma) = \\ = \sum_4 \mathrm{Perm}\left[\sum_3 \mathrm{Perm}\,(X_\alpha X_\beta + X_\beta X_\alpha) X_\gamma\right] X_\delta ,$$

corresponding to any $X_\alpha, X_\beta, X_\gamma, X_\delta \in U$. The above relations can be written

$$(5.13) \qquad R_{X_\alpha X_\beta + X_\beta X_\alpha} - L_{X_\alpha X_\beta + X_\beta X_\alpha} = \\ = (R_{X_\alpha} + L_{X_\alpha})(R_{X_\beta} - L_{X_\beta}) + (R_{X_\beta} - L_{X_\beta})(R_{X_\alpha} - L_{X_\alpha}) ,$$

$$(5.14) \qquad L_{(X_\alpha X_\beta + X_\beta X_\alpha) X_\gamma} + L_{(X_\beta X_\gamma + X_\gamma X_\beta) X_\alpha} + L_{(X_\gamma X_\alpha + X_\alpha X_\gamma) X_\beta} = \\ = (R_{X_\alpha} + L_{X_\alpha})(R_{X_\beta X_\gamma + X_\gamma X_\beta} + L_{X_\beta X_\gamma + X_\gamma X_\beta} - R_{X_\beta} R_{X_\gamma} - R_{X_\gamma} R_{X_\beta}) + \\ + (R_{X_\beta} + L_{X_\beta})(R_{X_\alpha X_\gamma + X_\gamma X_\alpha} + L_{X_\alpha X_\gamma + X_\gamma X_\alpha} - R_{X_\alpha} R_{X_\gamma} - R_{X_\gamma} R_{X_\alpha}) - \\ - (R_{X_\gamma} L_{X_\alpha X_\beta + X_\beta X_\alpha} + R_{X_\beta} L_{X_\alpha X_\gamma + X_\gamma X_\alpha} + R_{X_\alpha} L_{X_\beta X_\gamma + X_\gamma X_\beta}) .$$

By putting $X_\alpha = X_\beta$ in (5.13) and $X_\alpha = X_\beta = X_\gamma$ in (5.14) we have

$$(5.15) \qquad R_{X_\alpha X_\alpha} + L_{X_\alpha X_\alpha} = (R_{X_\alpha} + L_{X_\alpha})(R_{X_\alpha} - L_{X_\alpha}) ,$$

$$(5.16) \qquad L_{(X_\alpha X_\alpha) X_\alpha} = (R_{X_\alpha} + L_{X_\alpha})(R_{X_\alpha X_\alpha} + L_{X_\alpha X_\alpha} - R_{X_\alpha} R_{X_\alpha}) - L_{X_\alpha X_\alpha} R_{X_\alpha} .$$

Finally, by considering the case of an idempotent $X = e$ we have

$$(5.17) \qquad (R_e - L_e)(R_e + L_e - 1) = 0 ,$$

$$(5.18) \qquad (R_e + L_e)^2 - (R_e + L_e) R_e^2 - L_e (R_e + 1) = 0 .$$

We note that the above relations are not able to establish characteristic equations for R_e and L_e. Consequently the characteristic values of R_e and L_e remain arbitrary in any nontrivial Lie-admissible algebra. However if we consider U^+ which is commutative by definition, then $R_e = L_e$ corresponding to the idempotent $e \in U^+$ and by (5.18) we can write $R_e(2R_e - 1)(R_e - 1) = 0$, by which we see that the characteristic roots of R_e are 0, $\frac{1}{2}$ and 1.

Under the above assumptions the U^+ content of U can be decomposed by means of the Pierce decomposition [2-4] with respect to the idempotent e according to

$$U^+ = U_0^+(e) \oplus U_{\frac{1}{2}}^+(e) \oplus U_1^+(e)\,, \tag{5.19}$$

where $U_i^+(e)$ $(i = 0, \frac{1}{2}, 1)$ are invariant subspaces with respect to R_e defined by

$$U_i^+(e) = (X_\varrho | X_\varrho e = ie)\,, \qquad X_\varrho \in U,\; i = 0, \tfrac{1}{2}, 1\,, \tag{5.20}$$

where now $X_\varrho e$ is the product in U^+, and $U_0^+(e)$, $U_1^+(e)$ are zero or orthogonal subalgebras of U^+ satisfying the inclusion relations

$$U_0^+(e)\, U_1^+(e) = 0\,, \tag{5.21}$$

$$U_0^+(e)\, U_0^+(e) = U_0^+(e)\,, \tag{5.22}$$

$$U_1^+(e)\, U_1^+(e) = U_1^+(e)\,, \tag{5.23}$$

$$U_0^+(e)\, U_{\frac{1}{2}}^+(e) \subset U_0^+(e) + U_1^+(e)\,, \tag{5.24}$$

$$U_1^+(e)\, U_{\frac{1}{2}}^+(e) \subset U_1^+(e) + U_{\frac{1}{2}}^+(e)\,, \tag{5.25}$$

$$U_{\frac{1}{2}}^+(e)\, U_{\frac{1}{2}}^+(e) \subset U_0^+(e) + U_{\frac{1}{2}}^+(e)\,. \tag{5.26}$$

We now consider the case when the attached algebra U^+ of U is a (commutative) Jordan algebra. Then instead of (5.24) and (5.25) we have

$$U_0^+(e)\, U_{\frac{1}{2}}^+(e) \subset U_{\frac{1}{2}}^+(e)\,, \tag{5.27}$$

$$U_1^+(e)\, U_{\frac{1}{2}}^+(e) \subset U_{\frac{1}{2}}^+(e)\,. \tag{5.28}$$

Furthermore if the following closure relation holds for U^+

$$\{X_\varrho, X_\sigma\} = D'^{\tau}_{\varrho\sigma} X_\tau\,, \tag{5.29}$$

then by (5.1), (5.3) and (5.29) we can write

$$X_\varrho X_\sigma = R^{\tau}_{\varrho\sigma} X_\tau = \tfrac{1}{2}(C'^{\tau}_{\varrho\sigma} + D'^{\tau}_{\varrho\sigma}) X_\tau\,. \tag{5.30}$$

where $R^{\tau}_{\varrho\sigma}$ are the «structure constants» of U. Let us note for instance that the closure relations (5.30) occur corresponding to the fundamental representations of the U_n Lie algebras, which are closed under both commutators and anticommutators, when a Lie-admissible mutation algebra is assumed for extension U.

Finally if U_n^+ is a reduced (commutative) Jordan algebra of degree n we have the decomposition of U_n^+ with respect to the set of idempotents (e_i) $(i = 1, ..., n)$ with $e = \sum_i e_i$

$$U_n^+ = \sum_i{}^{\oplus} U_{ii}^+ \oplus \sum_{ij}{}^{\oplus} U_{ij}^+ , \tag{5.31}$$

where the subspaces U_{ii}^+ and U_{ij}^+ are defined by

$$U_{ii}^+ = \left(X_\varrho | X_\varrho e_i = X_\varrho\right), \qquad U_{ij}^+ = \left(X_\varrho | X_\varrho e_i = X_\varrho e_j = \tfrac{1}{2} X_\varrho\right) \tag{5.32}$$

and are related to $U_i^+(e)$ by the relations

$$U_{ii}^+ = U_i^+(e_i), \qquad \sum_{\substack{ij \\ i\neq j}}{}^{\oplus} U_{ij}^+ = U_{\frac{1}{2}}^+(e_i), \qquad \sum_{\substack{jk \\ j,k\neq i}}{}^{\oplus} U_{jk}^+ = U_0^+(e_i). \tag{5.33}$$

For specific investigations when the U^+ content of U is a (commutative) Jordan algebra we refer to papers [16-46].

Since idempotents in U^+ are idempotents in U, decomposition (5.19) can be directly extended to the Lie-admissible algebra U according to

$$U = U_0(e) \oplus U_{\frac{1}{2}}(e) \oplus U_1(e), \tag{5.34}$$

where the subspaces $U_i(e)$ are defined by

$$U_i(e) = \left(X_\varrho | X_\varrho e + e X_\varrho = 2i X_\varrho\right), \tag{5.35}$$

with $X_\varrho e$ now defining the product of U.

We note that the Pierce decomposition can be introduced for a nontrivial Lie-admissible algebra but not for a Lie algebra since in this case the same concept of idempotent is meaningless.

Finally let us briefly discuss the case when the Lie-admissible algebra U is a noncommutative Jordan algebra. As we have seen the latter represent the only possible simple power associative and trace admissible extensions U, hence they constitute a case of central interest. We note that the Pierce decomposition (5.34) can be extended to the set of idempotent (e_i) $(i = 1, ..., n)$

with $e = \sum_i e_i$ according to (*)

$$U = \sum_i^{\oplus} U_{ii} \oplus \sum_{\substack{ij \\ i\neq j}}^{\oplus} U_{ij}, \tag{5.36}$$

where

$$U_{00} = (X_\varrho | eX_\varrho + X_\varrho e = 0), \tag{5.37}$$

$$U_{0i} = (X_\varrho | e_i X_\varrho + X_\varrho e_i = eX_\varrho + X_\varrho e = X_\varrho) = U_{i0}, \tag{5.38}$$

$$U_{ii} = (X_\varrho | e_i X_\varrho + X_\varrho e_i = eX_\varrho + X_\varrho e = 2X_\varrho), \tag{5.39}$$

$$U_{ij} = (X_\varrho | e_i X_\varrho + X_\varrho e_i = e_j X_\varrho + X_\varrho e_j = X_\varrho) = U_{ji}, \tag{5.40}$$

for any $i \neq j \neq 0$ while the inclusion properties becomes

$$U_{ii} U_{ii} \subset U_{ii}, \tag{5.41}$$

$$U_{ii} U_{ij} + U_{ij} U_{ii} \subset U_{ij}, \tag{5.42}$$

$$U_{ij} U_{jk} + U_{jk} U_{ij} \subset U_{ik}, \tag{5.43}$$

$$U_{ij} U_{ij} \subset U_{ii} + U_{ij} + U_{jj}, \tag{5.44}$$

with all other products zero and $U_{00} = U_{0i} = 0$ if e is the identity 1.

For supplementary specific papers on noncommutative Jordan algebras see ref ([7,50-62]).

As a concluding remark let us quote without proof the following:

Theorem 3 (LAUFER-TOMBER ([65])). A flexible power-associative L-simple algebra U over an algebraically closed field of characteristic zero is a simple Lie algebra isomorphic to U^-.

(*) Let us consider as an example the imbedding (3.13) of the SU_3 Lie algebra of the Gell-Mann λ_k-matrices (fundamental representations of SU_3) into the $A_3^0(\lambda, \mu)$ Lie-admissible algebra of degree 3, dimension 9 and elements $a = (a_{ij})$ $(i, j = 1, 2, 3)$. Then the $[A_3^0(\lambda, \mu)]^+$ content can be decomposed with respect to the idempotents e_j $(j = 1, 2, 3)$ according to $[A_3^0(\lambda, \mu)]_i^+(e_j) = (a | ae_j = ie_j;\ i = 0, \frac{1}{2}, 1;\ j = 1, 2, 3)$ where for instance corresponding to e_1 the elements a are explicitly given by for $i = 0$: $a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$; for $i = \frac{1}{2}$: $a = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}$; for $i = 1$: $a = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and similarly for e_2 and e_3, in such a way that decompositions (5.31) and (5.36) hold respectively for $[A_3^0(\lambda, \mu)]^+$ and $A_3^0(\lambda, \mu)$.