

NUCLEAR REALIZATION OF HADRONIC MECHANICS  
AND THE EXACT REPRESENTATION OF  
NUCLEAR MAGNETIC MOMENTS

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**Summary.** We present a new realization of relativistic hadronic mechanics specifically constructed for nuclear physics, following the achievement of sufficient maturity for mathematical content in memoir [6], axiomatic formulation in memoir [6m], and generalized symmetries in memoir [7e]. We show that the proposed formulation permits: 1) the representation of nucleons as extended, nonspherical, and deformable charge distributions with alterable intrinsic magnetic moments yet conventional angular momentum and spin; 2) a new conception of the nuclear force with nonlinear, nonlocal and nonunitary terms due to (generally small) mutual penetrations of the hyperdense nucleons in the nuclear structure; 3) the reconstruction of linearity, locality and unitarity on certain generalized spaces over generalized fields, thus coinciding at the realization-free level with relativistic nuclear formulations; 4) the interpretation as a "completion" of relativistic nuclear formulations much along the historical argument by Einstein, Podolsky and Rosen; and 5) the preservation of all conventional quantum axioms and physical laws, with particular reference to the Minkowskian geometry, the Poincaré symmetry and the special relativity. We then show that the proposed nuclear formulation permits the apparently first exact, numerical representation of total magnetic moments of few-body nuclei which, despite efforts conducted through this century, have escaped an exact representation via relativistic quantum mechanics. The finalization of the  $4\pi$  neutron interferometric experiment is suggested for the direct verification of the apparent ability of the intrinsic magnetic moment of nucleons, provided that neutrons propagate not in vacuum but within sufficiently intense fields. A number of intriguing implications, applications and developments are pointed out.

# 1 Open character of total nuclear magnetic moments.

## 1.1 Deuteron magnetic moment

One of the fundamental, still unsolved problems of contemporary nuclear physics is the *exact* representation of the total magnetic moments of nuclei, particularly those of few-body nuclei such as the deuteron, tritium and helium.

As an example, the experimental value of the deuteron magnetic moment is given by

$$\mu_D^{\text{exp}} = 0.857406, \quad (1.1)$$

while its representation via nonrelativistic quantum mechanics (QM) on D-states yields the familiar expression (see, e.g., [1]).

$$\mu_{\text{QM}}^{\text{theor}} = g_N + g_p = 0.880, \quad (1.2)$$

(where  $g_p$  and  $g_n$  are the gyromagnetic factors of protons and neutrons, respectively) which is about 2.6 % off *in excess* of the experimental value.

It is known that the use of all possible corrections via relativistic quantum mechanics (RQM) on a mixture of S-, D- and P-states can reduce the above deviation down to about 1 % as established by the recent studies [2], thus reaching an excellent approximation of experimental value (1.1).

The issue addressed in this paper is the realization of RQM which does finally allow the achievement of an *exact* representation of the experimental value (1.1).

The open problem is created by the fact that, on one side, all possible relativistic corrections have been exhausted while, on the other side, the remaining 1 % deviation cannot be eliminated via quark theories because, unlike the corresponding case in the atomic structure, the quark orbits are very small, and their polarization yields corrections to the total magnetic moment of nucleons expected to be of the order of  $10^{-3}$ .

A similar situation exists for representation of the total magnetic moment of the tritium, helium and other few-body nuclei.

## 1.2 The historical hypothesis on the alterability of the intrinsic magnetic moments of nucleons

The most plausible explanation of the above occurrence was formulated by the Founding Fathers of nuclear physics in the late 1940's immediately after the identification of the numerical value (1.1). For instance, in p. 31 of [1] one can read: "It is possible that the *intrinsic magnetism of nucleon is different when it is in close proximity to another nucleon.*"

Recall that nucleons are not point-like, but have an extended charge distribution with the radius of about 1 fm ( $10^{-13}$  cm). Since perfectly rigid bodies do not exist in the universe, it is plausible to expect that such a distribution can be deformed under sufficient external forces. But the deformation of a charged and spinning sphere implies a necessary alteration of its intrinsic magnetic moment, as established by classical electrodynamics.

The above "historical hypothesis" (as referred to hereon) therefore assumes that, when a proton and a neutron are coupled into the deuteron or other nuclear structures, their charge distributions are altered by the nuclear force, resulting in an alteration of their conventional *intrinsic* magnetic moments as measured in vacuum. In turn the assumption of a departure from standard quantum values of the magnetic moments of nucleons readily permits an exact representation of the total magnetic moment of few-body nuclei, as we shall see in Sect. 4.

Note for subsequent needs that the representation of  $\mu_D^{\text{exp}}$  requires a *decrease* of the intrinsic magnetic moments of nucleons, and that such a decrease can only occur for a *prolate* deformation of nucleons referred to their spin axis.

Note also that the above historical hypothesis is *model independent*, i.e., it applies independently from any assumed structure of the nucleons, and consists of the geometric deformation of their charge distributions whatever the constituents are.

As one can see, the historical hypothesis essentially expresses the impossibility for the charge distribution of nucleons to be perfectly rigid.

### 1.3 Experimental evidence in support of the historical hypothesis

Since no exact representation of  $\mu_D^{exp}$  has been achieved via conventional intrinsic magnetic moments of nucleons following about three-quarters of a century of attempts,  $\mu_D^{exp}$  may well result to constitute direct experimental evidence on the alterability of the intrinsic magnetic moments of nucleons in the transition from motion in vacuum to motion within a nuclear structure.

THE HISTORICAL HYPOTHESIS ON THE ALTERABILITY OF THE INTRINSIC MAGNETIC MOMENTS OF NUCLEONS

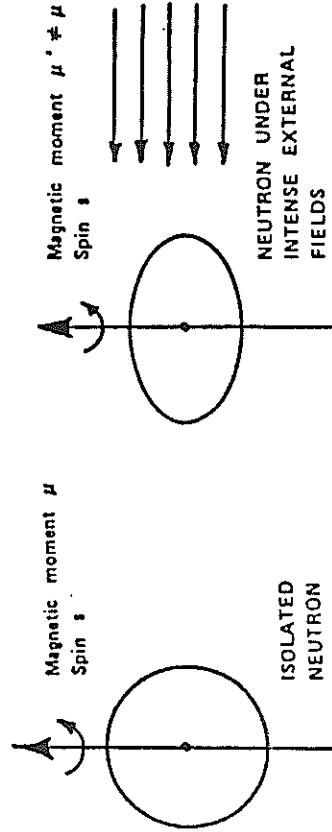


FIGURE 1. A schematic view of the historical hypothesis on the deformability of the charge distribution of nucleons with consequential alteration of their intrinsic magnetic moments which constitutes the main topic of study of this paper. As we hope to illustrate later on, quantitative studies of the above hypothesis may stimulate intriguing advances in nuclear physics, such as a novel notion of nuclear constituent and related symmetries.

More generally, all total nuclear magnetic moments may well result to be experimental evidence of the alterability of the intrinsic magnetic moments of nucleons. In fact, total nuclear magnetic moments do not follow QM

rules, and are instead contained within minimal and maximal values (called *Schmidt's limits*) [1]. The historical hypothesis was motivated precisely by these QM deviations.

Additional, although preliminary, direct experiments on the alterability of the intrinsic magnetic moments of nucleons were conducted from 1975 to 1979 by H. Rauch and his associates [3] via interferometric measures of the  $4\pi$  spinorial symmetry of the neutron. The measures are performed via a familiar perfect crystal which splits a thermal neutron beam into two branches which are then coherently recombined. In one (or both) branches experiments [3] placed an electromagnet calibrated at 7,496 G which, for the conventional value of the intrinsic magnetic moment of the neutron, would yield an exact multiple of two complete ( $4\pi = 720^\circ$ ) spin flips, as requested by the Fermi-Dirac character of the neutron and as necessary for a coherent recombination.

In order to improve accuracy, the experimenters filled up the electromagnet gap with Mu-metal sheets which reduce stray fields [3]. While crossing the electromagnet gap, the neutron beam is exposed to the field of 7,496 G as well as to the intense electric and magnetic fields in the vicinity of Mu-metal nuclei. The best interferometric measures date back to 1979 with re-elaboration done in 1981 [3e], and are given by

$$\theta = 715.37^\circ \pm 3.8^\circ, \quad \theta_{\min} = 712.07^\circ, \quad \theta_{\max} = 719.67^\circ. \quad (1.3)$$

As such, they do not contain  $720^\circ$  in the minimal and maximal values. However, the deviation is smaller than the error and, therefore, the above measures are inconclusive.

Similar measures were conducted in 1975 by S.A. Werner and his associates [3f] although also with unsettled results. To our best knowledge, no additional interferometric measures have been done for the  $4\pi$  spinorial symmetry of the neutron since 1979, thus indicating the need for final tests which are now permitted in view of the technological advances and improved accuracy occurred since the late 1970's.

Despite the above unsettled character, measures (1.3) are significant, as shown in theoretical studies [4]. In fact, as a realization of Albert's [4a] abstract notion of Lie-admissibility, Santilli [4b] introduced back in 1967 the apparently first (p, q)-deformations of quantum structures with product (A, B) = pAB - qBA, where p, q are non-null parameters and pA, AB, etc.,

are ordinary associative products. Subsequently, at meeting *Differential Geometric Methods in Theoretical Physics* held at the University of Clausthal, Germany, back in 1980, Santilli [4c] introduced the apparently first quantum deformation of the SU(2) algebra with specific application to a first interpretation of interferometric measures (1.3).

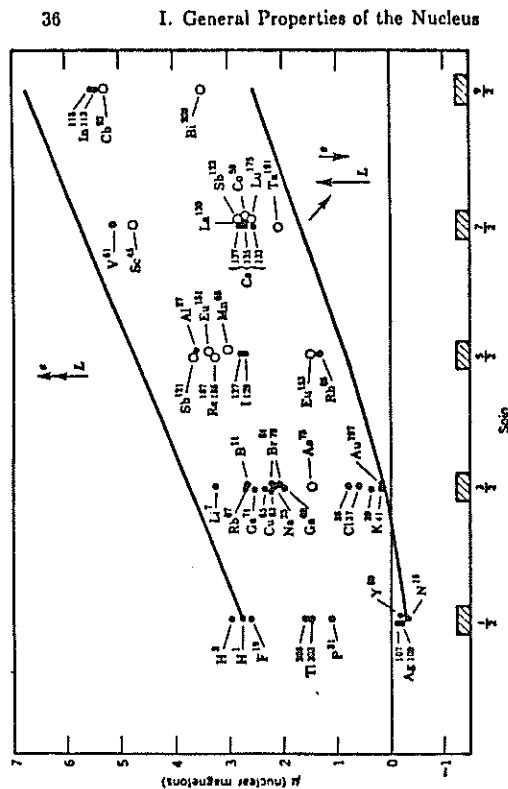


FIGURE 2. An illustration that the total nuclear magnetic moments do not follow QM predictions, but are within minimal and maximal values (Schmidt's limits) reproduced in this figure from [1] which motivated the historical hypothesis of Fig. 1. Additional insufficiencies also exist for other nuclear aspects. Needless to say, QM provides an excellent approximation of nuclear data. We are therefore referring to deviations which are generally small, yet they have rather important implications, as indicated in Sects 4

and 5.

The latter studies pointed out that all median angles measured in tests [3] (with the electromagnet gap filled up with Mu-metal sheets) have resulted to be smaller than the expected  $720^\circ$ , an occurrence was called in Ref. [4c] *angle slow-down effect*. This apparent effect is significant inasmuch as it requires a decrease of the standard magnetic moment of the neutron for the (polarized) conditions of the experimental set up, which is precisely in line with the decrease of the same magnetic moments needed for the interpretation of  $\mu_D^{exp}$ , as recalled earlier.

By using the preceding studies [4b-4c], Eder [4d] showed in 1981 that the alteration of the charge distribution of the neutron caused by the intense electric and magnetic fields in the vicinity of Mu-metal nuclei could indeed yield "spin fluctuations" with about 1 % deviation of the intrinsic magnetic moment which is precisely the order of magnitude needed for the resolution of the historical problem of total nuclear magnetic moments. Eder [loc. cit.] also showed that the strong interactions of Mu-metal nuclei do not have a significant contribution because their sectional area along the thermal neutron beam is very small.

The electric and magnetic fields in the vicinity of Mu-metal nuclei are known and result to be (in average) of the order of 20,000 G. The biggest unknown is the deformability of the charge distribution of neutrons under known external fields, which can only be established from  $\mu_D^{exp}$  for the deuteron conditions at this stage of our knowledge, as done in Sect. 4, or via interferometric measures for more general conditions.

As noted earlier,  $4\pi$ -interferometric tests can only measure the deformability of neutrons under the intense electric and magnetic fields of the Mu-metal (or other heavy) nuclei, but not under the strong nuclear forces as occurring in the structure of the deuteron.

However, it is known from classical electrodynamics that a small deformation of a spinning and charged sphere can yield a relatively large change of its magnetic moment. Also, the deformability of the charge distributions of nucleons in the deuteron structure may eventually be due to the electric and magnetic fields of the nucleons themselves. Intense electric and magnetic fields of large, many-body nuclei could therefore approximate sufficiently well

the electric and magnetic fields of the two-body deuteron.

The above aspects, combined with the resolution of the still open historical hypothesis as well as with its implications pointed out in Sect 5, are sufficient to warrant the study of novel methods for the exact representation of  $\mu_D^{exp}$ , as well as the finalization of interferometric measures on the  $4\pi$ -spinorial symmetry of the neutron.

## 2 Representation of the hystorical hypothesis

### 2.1 Need to preserve established quantum axioms and physical laws

The current, conventional realization of QM is fully established as being exactly valid for the so-called *exterior dynamical problems*, here referred to as particles moving in vacuum under action-at-a-distance/potential interactions at sufficiently large mutual distances to allow an effective point-like approximation of their wavepackets and/or charge distributions, as occurring in the atomic structure and electroweak interactions at large. In fact, QM provided an exact representation of all experimental data available for the systems considered.

By comparison, the same realization of QM does not appear to be exactly valid for the so-called *interior dynamical problems*, here referred to particles whose wavepackets or charge distributions cannot be effectively approximated as being point-like because moving at small mutual distances of the order of their size (1 fm), as occurring in the structure of nuclei (as well as of hadrons and stars not considered in this paper).

In fact, the conventional realization of QM has not provided an exact representation of total nuclear magnetic moments as well as of other nuclear data [1,2] and does not permit a quantitative representation of the historical hypothesis on the deformability of the charge distribution of nucleons (evidently, from the rotational symmetry and its inherent perfect rigidity).

After studying the issue since the time of Ref. [4c], this author would like to submit the viewpoint that no achievement of an exact representation of total nuclear magnetic moments can be considered to be valid on ground of

our current knowledge unless it preserves in their entirety conventional quantum axioms and physical laws, such as Pauli's exclusion principle, Heisenberg uncertainties principle, etc.

This expectation is based on a number of aspects. First, no deviation from quantum mechanical laws has been conclusively measured to date, to our best knowledge. Second, as indicated earlier, the deformations of the charge distribution of nucleons is expected to be due to the intense electric and magnetic fields in the nuclear structure, and we have no reason to expect deviations from quantum laws under electromagnetic interactions. Finally, the exact representation presented in this paper does indeed preserve conventional quantum laws, thus voiding the need for their violation.

The preservation of quantum axioms is evidently a consequence of the preservation of quantum laws, because any structural alteration of the latter would imply deviations from the former, and viceversa.

The above occurrences identify the primary objective of this paper, which consists in studying a new realization of conventional QM axioms which preserves conventional quantum laws, yet permits a quantitative treatment of the deformability of the charge distribution of nucleons.

### 2.2 Needs for a new realization of quantum axioms and physical laws

The need to reach an exact representation of total nuclear magnetic moments evidently constitutes the first motivation for a new realization of quantum axioms and physical laws. The same need can however be illustrated via a number of additional arguments.

QM was established for the characterization of action-at-a-distance interactions solely derivable from a potential and this confirms again its exact validity for the atomic structure, this time on dynamical grounds. The need for a new realization of quantum axioms and physical laws then emerges from a simple comparison of the atomic and nuclear structures.

Computer visualizations of the fundamental symmetry of QM, the Poincaré symmetry, show the characterization of a *Keplerian system*, namely, a system having the heaviest constituent in the center (*Keplerian nucleus*), thus confirming again its exact validity for the atomic structure.

By comparison nuclei do not have *Keplerian nuclei*, thus establishing

the need for a new, nuclear realization of the Poincaré symmetry, which is capable of removing the need for a Keplerian center under the condition of preserving the abstract Poincaré structure, as studied later on in this paper.

### THEORETICAL INSUFFICIENCIES OF QUANTUM MECHANICS IN NUCLEAR PHYSICS

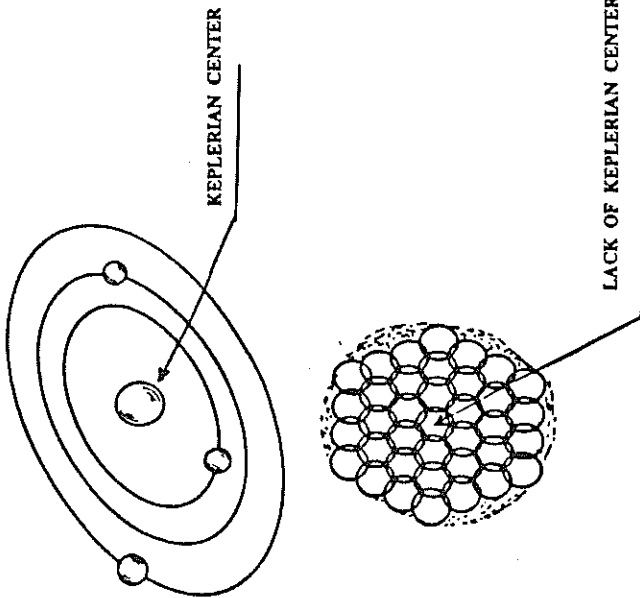


FIGURE 3. An illustration of the need for a new realization in nuclear physics of quantum axioms and physical laws, the *Keplerian character* of the atomic structure with consequential exact validity of the conventional realization of the Poincaré symmetry, and the *lack* of such Keplerian character for the nuclear structure evidently due to the lack of a Keplerian nucleus. *Prima facie*, such an occurrence would indicate the need of *breaking* the fun-

damental Poincaré symmetry, which would evidently imply a violation of quantum axioms and physical laws. An important result of this paper is to show that, thanks to new mathematical methods, this is not the case, because the *conventional* Poincaré symmetry admits a *new realization* capable of removing the Keplerian nucleus, thus preserving conventional quantum axioms and physical laws.

The most compelling needs for new realizations of conventional quantum axioms and physical laws originates from the *nuclear force*. The nucleons of a nuclear structure are not at large mutual distances as the atomic constituents, and are instead in an average state of mutual penetration of about  $10^{-3}$  parts of their charge distribution [4e]. But hadrons are some of the densest objects measured in a laboratory until now. These physical conditions indicates the presence in the nuclear force of interactions which are:

- 1) of *contact* type, thus having zero-range by assumption;
- 2) *nonlinear* in the wavefunctions and possibly their derivatives;
- 3) *nonlocal* in the sense of requiring an integral over the volume of overlapping;
- 4) *nonpotential-nonhamiltonian* in the sense of violating the conditions to be derivable from a potential or a Hamiltonian (the *conditions of variational selfadjointness* [6c]); and
- 5) of *nonunitary* type due to terms beyond the Hamiltonian structure.

By recalling the strictly action-at-a-distance, linear, local-differential, Hamiltonian and unitary character of QM, the preceding characteristics of the nuclear force due to mutual penetration of the hyperdense charge distribution of nucleons are clearly beyond the representational capabilities of QM.

The rather difficult task of this paper, which explains the considerable time for its achievement following the initiation of the studies in Ref. [4c], therefore consists in identifying a *new nonlinear, nonlocal and nonpotential-nonunitary realization of quantum mechanical axioms and physical laws, including its fundamental Poincaré symmetry*.

At a first inspection, such a task appears to be impossible, because nonlinear, nonlocal and nonunitary theories are known to be inequivalent to QM. A novelty of this paper is precisely that of showing that such an expectation

is in reality unfounded.

In fact, thanks to the use of a new mathematics specifically considered for the above objectives, we shall show that nonlinear, nonlocal and nonunitary theories can be *identically* reformulated in such a way to verify linearity, locality and unitarity on certain generalized spaces over generalized fields, thus being able to preserve quantum axioms and spacetime symmetries in their entirety. The preservation of quantum laws will then be consequential.

It should be noted that the Poincaré symmetry in its conventional quantum realization can only characterize linear, local and potential-unitary nuclear forces. A central objective of this work is therefore the identification of a novel realization of the Poincaré symmetry which can indeed provide an invariant description of nonlinear, nonlocal and nonunitary terms in the nuclear force.

### 2.3 Representation of extended and deformable charge distributions

The need for new realizations of quantum axioms and physical laws is effectively illustrated by the historical hypothesis on the deformability of the charge distribution of nucleons whose quantitative representation requires:

1) A representation of *extended, nonspherical and deformable shapes* of the charge distribution of nucleons, expectedly of spheroidal ellipsoidal character, hereon represented with the semiaxes  $n_1^2, n_2^2, n_3^2, n_k \neq 0, k = 1, 2, 3$ , which are functions of the intensity of external fields and any other needed local characteristic. For particles with spin along the third axis, the above quantities represent spheroidal ellipsoids which are *oblate* for  $n_1^2 = n_2^2 > n_3^2$  and *prolate* for  $n_1^2 = n_2^2 < n_3^2$ . The evident condition of preserving the original volume of nucleons then yields the normalization hereon assumed.

$$n_1^2 \times n_2^2 \times n_3^2 = 1, \quad n_1^2 = n_2^2 > \text{or} < n_3^2. \quad (2.1)$$

[It should be noted that in other cases the normalization  $n_1^2 + n_2^2 + n_3^2 = 3$  may be preferable].

2) A representation of the *density* of the medium in which motion occurs hereon represented with the functions  $n_4^2$  which, for the vacuum, is assumed to have the normalized value  $n_4^2 = 1$ , and we shall write

$$n_4^2 = 1, \quad < 1, \quad \text{or} > 1. \quad (2.2)$$

as we shall see in Sect. 3,  $n_4$  is in reality the local *index of refraction of light*, thus characterizing the local causal speed. We therefore recover the conventional case of speeds of light in media of low density (such as our atmosphere) smaller than that in vacuum ( $n_4 > 1$ ), but jointly admit speeds greater than that in vacuum ( $n_4 < 1$ ), as suggested by numerous recent measures (see Sect. 3.12).

3) A representation of the alteration called *mutation* [6b] of the intrinsic magnetic moment  $i_N$  of nucleons  $N = n, p$ , hereon expressed with the symbol  $i_N = \hat{\mu}_N(\mu_N, \mu, \dots)$ ,  $\mu = 1, 2, 3, 4$ , with  $\hat{\mu}_N > \mu_N$  for *oblate* deformations and  $\hat{\mu}_N < \mu_N$  for *prolate* ones, where the term "mutation" is preferred over "deformation" to indicate the fact that the underlying methods [6] (see the next sections) are structurally different than the known "quantum deformations" [4] of the current literature.

It is evident that the nonspherical and deformable characteristics (2.1) are beyond any representational capability of QM because the latter can only represent *perfectly spherical and perfectly rigid particles*, as necessary in order not to violate the fundamental rotational symmetry O(3). It should be stressed that the same occurrence persists in second-quantization and related form-factors which cannot represent the main characteristics of the historical hypothesis under study here. By comparison, any real treatment of the historical hypothesis requires *ab initio* the representation of *nonspherical and deformable particles*.

The above limitations of the conventional atomic realization of QM are well known to be inherent in the very structure of its fundamental carrier spaces, the Euclidean space  $E = E(r, \delta, \mathcal{R})$  with coordinates  $r = \{r^k\}$ ,  $k = 1, 2, 3$ , and metric  $\delta = \text{diag}(1, 1, 1)$  over the field of real numbers  $\mathcal{R} = \mathcal{R}(n, +, \times)$  and the Minkowski space  $M = M(x, \eta, \mathcal{R})$  with coordinates  $x = \{x^\mu\}$ ,  $\mu = 1, 2, 3, 4$ , and metric  $\eta = \text{diag}(+1, +1, +1, -1)$  over  $\mathcal{R}$ . In fact, the basic unit of  $E, I = \text{diag}(1, 1, 1)$  (which is the space component of the unit of  $M$ ) represents a *perfect and rigid sphere*. Moreover, the theory of deformations is well known to be incompatible with the above spaces and their symmetries.

Alternatively, it is easy to see that deviations from the exact  $720^\circ$  in

the  $4\pi$  interferometric measures (1.3) imply a deviation from the familiar *spinorial* component of Dirac's wavefunction,

$$\psi' = R(\theta_3) \times \psi = e^{i\gamma_3 \theta_3 / 2} \times \psi, \quad (2.3)$$

where the  $\gamma$ 's are the conventional gamma matrices. This is the very reason why the experiment is called the  $4\pi$  *spinorial symmetry test*.

In fact, mutations of the intrinsic magnetic moment of nucleons imply a departure from its characterization via Dirac's equation from which a departure from law (2.3) follows. At any rate, for an angle of spin flip different than  $720^\circ$ , spinorial law (2.3) cannot represent the physical reality.

The above occurrence illustrate the difficulties of our objective, which consists in achieving an exact-numerical representation of the *deviation* from  $720^\circ$  in interferometric measures [3] under the condition of the [exact validity of the SU(2)-spin symmetry, as an evident pre-requisite to preserve Pauli's exclusion principle.

## 2.4 Inapplicability of quantum deformations

As indicated earlier, this author initiated back in 1967 the study of quantum deformations with generalized brackets [4b]

$$(A, B) = pAB - qBA, \quad (2.4)$$

where  $p$  and  $q$  are non-null parameters, and subsequently submitted in 1980 the first quantum deformation of SU(2) on record [4c] in terms of the more general brackets

$$(A, B) = APB - BQA, \quad (2.5)$$

where  $P$  and  $Q$  are nonsingular operators evidently admitting the little  $p$  and  $q$  parameters as particular cases.

In 1989 Biedenham [4f] and Macfarlane [4h] studied the particular case for  $p = 1$  with brackets

$$(A, B) = AB - qBA, \quad (2.6)$$

subsequently called  $q$ -deformations, which were followed by a large number of papers (see, e.g., representative references [4i-4l] and literature quoted therein).

Ironically, by the time of the initiation of the large number of papers in quantum deformations, this author had already abandoned the field because of several reasons.

The first reason is the fact that, to be nontrivial, all quantum deformations must exit the class of equivalence of quantum mechanics. As such, they are not recommendable for a study of total nuclear magnetic moments because they imply it departs from quantum axioms and physical laws.

Additional reasons are rather serious physical shortcomings of all  $q$ -,  $k$ - and quantum deformations in general identified by our group (see Ref.s [5] and the first two sections of Ref. [6m]).

In fact, as a necessary condition to exit the class of equivalence of QM, all quantum deformations must have a nonunitary time evolution. In turn, when treated on conventional Hilbert spaces over conventional fields, all theories with nonunitary time evolutions possess the following physical shortcomings [5,6m]:

- 1) They do not possess invariant units of space and time, thus lacking unambiguous applications to measurements;
- 2) The initial Hermiticity of operators is not preserved in time, thus lacking physically acceptable observables;
- 3) The numerical predictions are not unique or invariant;
- 4) Causality and probability principles are generally violated;
- 5) The Pé symmetry and the special relativity are generally violated.

It is evident that the above shortcomings preclude any physically meaningful application of quantum deformation to the representation of total nuclear magnetic moments.

The interested reader may be intrigued by the fact that, following the construction of a yet broader new mathematics called *geromathematics*, quantum deformations with product  $(A, B) = APB - BQA$ , where  $P$  and  $Q$  are nonsingular operators admitting of the little  $p$  and  $q$  parameters as particular cases, can be identically reformulated on certain generalized spaces and fields which *resolve* the above physical shortcomings [4m].

Despite that, quantum deformations do not appear applicable to the representation of total nuclear magnetic moments. In fact, they were originally submitted [4a,4b] for the representation of *time rate of variations* of physical quantities, e.g.,



$$i dA/dt = (A, H) = pAH - qHA \neq I. \quad (2.7)$$

On the contrary, a physically consistent representation of total nuclear magnetic moments requires a generalized formalism which represents *total conservation laws*.

In fact, as discussed at the 1980 Clausthal meeting *Differential Geometric Methods in Theoretical Physics*, Santilli [4c] submitted deformations (2.5) for the possible characterization of time-rate of variations of the characteristics of *one hadron under "external strong" interactions*, under the condition of recovering total conservation laws. Because of certain technical reasons, such a setting is meaningful only for a minimum of *four* hadrons.

This provides additional deeper reasons for the inapplicability of deformations (2.5) for the total magnetic moment of the deuteron and, as such, they will be ignored hereon.

Perhaps, at some future time, when studying nuclei with four or more constituents, the use of deformations (2.5) in the invariant form [4m] may provide new insights on *individual constituents* under the validity of the formalism submitted below for the nucleus as a whole.

### 3 Nuclear realization of relativistic hadronic mechanics

#### 3.1 Conceptual foundations

In 1978 Santilli [6a-6c] suggested the construction of the *axiom-preserving isotopies of quantum mechanics*, namely a formulation in which the axioms are conventional, yet they are realized in a way more general than the familiar quantum version. To avoid confusion as well as to indicate the intended use, the name of *hadronic mechanics* was suggested in Ref. [6b] for the broader formulation. However, it should be indicated from the outset that *quantum and hadronic mechanics coincide at the realization-free level by conception and construction*. The terms "hadronic mechanics" therefore stand to indicate a *new realization* of conventional quantum axioms and physical laws, as we shall see below.

The physical motivation for the proposal was the achievement of a formulation permitting quantitative studies of the historical open legacy according to which strong interactions have a nonlinear, nonlocal and nonpotential component in addition to the conventional linear, local and potential ones. Recall that the range of strong interactions is of the same order of the charge radius of all hadrons (about 1 fm). A necessary condition to activate the strong interaction is therefore that hadrons enter into conditions of mutual penetration. But hadrons are some of the densest objects measured in laboratory until now. The mutual penetration of the hyperdense hadrons which is necessary to activate the strong interactions therefore implies the expectation of said nonlinear, nonlocal and nonpotential terms. Since the latter terms are conceptually, mathematically and physically outside any realistic treatment *via* quantum mechanics, the need for a new discipline specifically constructed for their treatment becomes evident.

Subsequently, hadronic mechanics was studied by numerous mathematicians, theoreticians and experimentalists (see Refs. [6,7] and contributions quoted therein), resulting in a discipline more recently known as *relativistic hadronic mechanics* (RHM). However, due to a number of previously unsolved problems illustrated later on during the course of our analysis, mathematical maturity of RHM was reached only recently in memoir [6], physical maturity in the axiomatic structure was only reached in the recent memoir [6m], and maturity in the formulation of the basic space-time and internal symmetries was reached in the recent memoir [7e].

This paper deals with the identification, apparently for the first time, of the specialization of the general formalisms of memoirs [6], [6m], [7e], specifically, for the nuclear structure under the conditions of: representing the historical hypothesis of Fig. 1 in an invariant form; preserving conventional quantum mechanical axioms and physical laws with particular reference to the special relativity; and avoiding the physical shortcomings of quantum deformations of Sect. 2.5. Our main objective of study could therefore be called *nuclear realization of relativistic hadronic mechanics*.

For certain reasons identified later on, RHM is also known as "isotopic completion" of *relativistic quantum mechanics* much along the historical argument by Einstein, Podolsky, and Rosen, as indicated beginning with the title of memoir [6m]. The view submitted in this work is therefore that the lack of achievement of an exact representation of total nuclear magnetic mo-

ments throughout this century (as well as of other nuclear data [1]), is a confirmation of the "lack of completeness" of conventional nuclear formulations.

RHM is constructed via maps of RQM called *isotopies* [6a,6m] from the Greek meaning of being "axiom-preserving" and referred to maps (also called *liftings*) of any given linear, local-differential and unitary theory into its most general possible nonlinear, nonlocal-integral and nonunitary extensions which are nevertheless capable of reconstructing linearity, locality and unitarity in certain generalized spaces called *isospaces*, and generalized fields called *isofields*.

It then follows that isotopic images of fields, spaces, algebras, etc., are isomorphic to the original structures by conception and construction, and they coincide at the abstract, realization free level, all this as preparatory grounds to preserve Einsteinian axioms of the special relativity. Nevertheless, as we shall see shortly, RQM and RHM are physically inequivalent because connected by nonunitary transforms.

Recall that the most dominant aspect of the predicted new terms in the nuclear force is that of not being representable with a Hamiltonian and, therefore, of being nonunitary (otherwise we trivially remain within the class of equivalence of RQM). As studied in detail in memoir [6m], the best way to construct the foundations of RHM is therefore subjecting the corresponding foundations of RQM to nonunitary transforms.

The fundamental quantities of RQM are: the basic unit of the underlying Minkowski space,  $I = \text{diag}\{1, 1, 1, 1\}$  representing in a dimensionless form the basic units of space, say, (1 cm, 1 cm, 1 cm, 1 cm) and of time, say, 1 sec. (with  $\hbar = 1$  assumed hereon); the basic associative product  $A \times B$  among generic quantities  $A, B$  (which is the same for all products of RQM, those of numbers, operators, etc., including the modular action  $H \times |\psi\rangle$  of operators  $H$  on Hilbert states  $|\psi\rangle$ ); and the fundamental relativistic canonical commutation rules  $[p_\mu, x^\nu] = p_\mu \times x^\nu - x^\nu \times p_\mu = -i\delta_\mu^\nu \times I$ .

Under nonunitary transforms, the above quantities become [6m]

$$U \times U^\dagger = \hat{I} = \hat{I}^\dagger \neq I, \quad (3.1a)$$

$$I \rightarrow \hat{I} = U \times I \times U^\dagger, \quad (3.1b)$$

$$A \times B \rightarrow \hat{A} \times \hat{B} = U \times A \times B \times U^\dagger = \hat{A} \times \hat{I} \times \hat{B}, \quad (3.1c)$$

$$U \times [p_\mu, x^\nu] \times U^\dagger = [\hat{p}_\mu, \hat{x}^\nu] = \hat{p}_\mu \times \hat{x}^\nu - \hat{x}^\nu \times \hat{p}_\mu = -i\delta_\mu^\nu \times \hat{I}, \quad (3.1d)$$

$$\hat{I} = (U \times U^\dagger)^{-1} = \hat{I}^{-1}, \quad \hat{K} = U \times K \times U^\dagger, \quad K = A, B, p, x. \quad (3.1e)$$

The above new images are then assumed as the fundamental quantities of RHM.

A most dominant aspect of the above nonunitary transforms is that they imply the joint lifting, of the unit  $I \rightarrow \hat{I} = \hat{I}^{-1}$  while the associative product is lifted in an amount which is the inverse of that of the unit,  $A \times B \rightarrow \hat{A} \times \hat{B} = \hat{A} \times \hat{I} \times \hat{B}$ , under which  $\hat{I} = \hat{I}^{-1}$  is the correct left and right unit of the new theory.

$$\hat{I} \times \hat{A} = \hat{I}^{-1} \times \hat{I} \times \hat{A} \equiv \hat{A} \times \hat{I} = \hat{A} \times \hat{I}^{-1} \equiv A, \quad \forall A, \quad (3.2)$$

in which case (only)  $\hat{I}$  is called the *isomit* and  $\hat{I}$  the *isotopic element* [6a,6b]. The emerging new operator envelope  $\xi$  is called *isassociative* [6a] because it verifies the associative law with respect to the isoproduct,  $\hat{A} \times (\hat{B} \times \hat{C}) = (\hat{A} \times \hat{B}) \times \hat{C}$ . Note that the new unit  $\hat{I}$  is Hermitean and will therefore be assumed hereon to be positive-definite. Under these conditions it is evident that the original envelope  $\xi$  and its isotopic image  $\hat{\xi}$  are isomorphic as desired,  $\xi \approx \hat{\xi}$ , and the map  $\xi \rightarrow \hat{\xi}$  is an *isotopy*. Yet they are physically nonequivalent because nonunitarily related.

As shown in memoir [6m], the remaining aspects of RHM are then constructed via a systematic use of the above nonunitary maps to all corresponding aspects of RQM, including numbers, spaces, algebras, geometries, functional analysis, etc., without any exception known to this author. In fact, the application of the nonunitary map to, say, the time evolution of RQM while preserving conventional Hilbert spaces and fields yields all the shortcomings of quantum deformations recalled in Sect. 2.5.

The motivations of this paper are the following. The isotopic completion of RQM presented in memoir [6m] is the most general possible one. As such, it can present extreme cases of interior particle systems, such as collapsing stars, in which case deviations from quantum mechanical laws may be significant, e.g., to represent the irreversibility of astrophysical structures. The first objective of this paper is to identify the subclass of the general formulation of memoir [6m] specifically constructed for nuclear structures, that is, restricted to verify all conventional quantum laws, such as Heisenberg's

uncertainties, Pauli's exclusion principle, etc. As we shall see, this is assured by the assumption of isounits which are not only positive-definite, but also diagonal and with generally small deviations from the unit of RQM  $I = \text{Diag}(1, 1, 1, 1)$ .

The second motivation of this paper is to present a direct experimental verification of the theory of memoir [6m], via the first exact-numerical representation on scientific records known to this author of the magnetic moment of the deuteron and of other few-body nuclei. Another motivation is to indicate the emerging, rather intriguing, novel possibilities in nuclear physics for study by the interested scholar.

The representation of systems with RQM is done via the knowledge of one operator only, the Hamiltonian  $H$ , under the tacit assumption of the simplest possible basic units  $I = \text{diag}(\{1, 1, 1, 1\})$ . The representation of systems via RHM requires the knowledge of two quantities, the conventional Hamiltonian  $H$  to represent conventional potential interactions, and a second quantity, the isounit  $\hat{I}$ , to represent all nonhamiltonian quantities.

We shall therefore assume hereon the realization of the isounit which is specifically restricted to be positive-definite and diagonal ( $\hbar = 1$ ),

$$\hat{I} = \text{diag}(\hat{I}_s, \hat{I}_t) = \text{diag}(\{n_1^2, n_2^2, n_3^2\}, n_4^2) \times \hat{\Gamma}(x, \dot{x}, \psi, \partial\psi, \dots) > 0, \quad (3.3a)$$

$$\hat{I}_s = \text{diag}\{n_1^2, n_2^2, n_3^2\} \times \hat{\Gamma}_s(x, \dot{x}, \psi, \partial\psi, \dots), \quad \hat{I}_t = n_4^2 \times \hat{\Gamma}_t(x, \dot{x}, \psi, \partial\psi, \dots), \quad (3.3b)$$

where  $\hat{I}_s$  and  $\hat{I}_t$  are called the *space and time isounits*, respectively, the  $n_k^2$ 's are quantities (2.1) representing the shape of the hadron considered,  $n_4^2$  is the quantity (2.2) representing the density of the medium in which motion occurs, and  $\hat{\Gamma}$  is a positive-definite  $4 \times 4$  matrix representing the contact, nonlinear, nonlocal, nonhamiltonian and nonunitary interactions (as identified in Sect. 2). The functional dependence of the isounit remains completely unrestricted in RHM and must be determined from the characteristics of the case at hand exactly as it is the case of the Hamiltonian in RQM.

It is important to know that identifications (3.3) are made following the historical teaching by Hamilton, Lagrange, Jacobi and other Founders of analytic dynamics [6m] according to which one quantity alone (we today call the Hamiltonian or the Lagrangian) simply cannot represent the entire

physical reality. For this reason they formulated their analytic equations with *external terms*.

Comprehensive classical studies not reported here for brevity (see the review and bibliography in [6j,6k]) have established that the use of isounits (3.3) is analytically equivalent to external terms and, in fact, they have the same number of independent elements. The reformulation of the external terms via the isounit has resulted to be necessary to preserve Einsteinian axioms of the special relativity beginning at the classical level because it permits the preservation of Lie's theory which would be otherwise lost in favor of the broader Lie-admissible theory under external terms [6a].

These classical studies have resulted in a new analytic mechanics, called *isohamiltonian mechanics* which is the unique and unambiguous classical image of the operator mechanics outlined in this section (see [6l,6m] for the latest studies and comprehensive bibliography).

A criticism is at times moved according to which RHM is "too broad" because the isounit can have infinitely possible values. This criticism is evidently equivalent to the statement that RQM is "too broad" because it admits infinitely possible Hamiltonians.

In reality, both the Hamiltonian and the isounits are selected via fully identified methods resulting in all applications considered until now in unique and unambiguous expressions. The Hamiltonian is selected via all conventional criteria of the *exterior problems*, such as mass, charge, potential, etc. The isounit is instead selected on the new grounds of the *interior problems*, thus requiring the description of *new characteristics* generally absent in the Hamiltonian, such as extended, nonspherical and deformable shapes of the particles, the density (or index of refraction) of the medium in which they move, contact-nonhamiltonian interactions, and other characteristics which are absent in the QM literature.

At any rate, any quantitative representation of the historical hypothesis of Fig. 1 requires the capability to represent all *infinitely possible different shapes of the same nucleon*, thus requiring for consistency infinitely possible isounits for each given Hamiltonian.

Also, the noninitiated reader should know since these introductory lines that, when an isolated interior system is considered from the outside, *internal nonpotential effects must be evidently averaged into constants because they are short range*, by therefore resulting in a mere rescaling of the shape and

density terms via the constant factor  $\hat{\Gamma}_0 = (\hat{\Gamma})$ . This occurrence renders preferable the scale invariant description of the characteristic  $n$ -quantities, which will be tacitly adopted hereon.

With the above conceptual foundations in mind, we now pass to the identification of the axiomatic structure of RHM specifically built for nuclear physics by preserving the notation of memoir [6m] according to which all quantities with a "hat" are computed in generalized spaces while those without are computed in conventional spaces.

### 3.2 Isofields

The first notion of RQM which must be isotopically lifted in order to achieve invariant units, Hermiticity and numerical predictions is that of the fields of ordinary real numbers  $\mathcal{R}(n, +, \times)$  and complex numbers  $\mathcal{C}(c, +, \times)$  with conventional sum  $a+b$ , additive unit 0, multiplication  $a \times b$  and multiplicative unit  $I$ ,  $a = n, c$ , resulting in the *isofields* [6f]  $\hat{\mathcal{R}} = \hat{\mathcal{R}}(\hat{n}, +, \hat{\times})$  and  $\hat{\mathcal{C}}(\hat{c}, +, \hat{\times})$  of *isoreal numbers*  $\hat{n} = U \times n \times U^\dagger = n \times \hat{I}$  and *isocomplex numbers*  $\hat{c} = U \times c \times U^\dagger = c \times \hat{I}$ ,  $n \in \mathcal{R}, c \in \mathcal{C}, \hat{I} \neq \mathcal{R}, \mathcal{C}$  equipped with the conventional sum  $\hat{+} \equiv +$  and related additive unit  $\hat{0} \equiv 0$ , as well as with the isoproduct and related isounit

$$\hat{a}\hat{\times}\hat{b} = U \times a \times b \times U^\dagger = \hat{a} \times \hat{I} \times \hat{b}, \quad \hat{I} = \hat{I}^{-1}, \quad (3.4)$$

$$\hat{I}\hat{\times}\hat{a} \equiv \hat{a}\hat{\times}\hat{I} \equiv \hat{a}, \quad \forall a = n, c.$$

The important property is that  $\hat{\mathcal{R}}$  and  $\hat{\mathcal{C}}$  preserve all axioms of a field [6j]. Thus, the liftings  $\mathcal{R} \rightarrow \hat{\mathcal{R}}$  and  $\mathcal{C} \rightarrow \hat{\mathcal{C}}$  are isotopies.

For consistency, all operations on numbers must then be isotopically lifted in a simple yet unique and significant way. We have in this way the following *isosquare, isosquare root, isoquotient, isonorm*, etc. (see [6f] for details).

$$\hat{a}^2 = \hat{a}\hat{\times}\hat{a} = a^2 \times \hat{I}, \quad \hat{a}^{\frac{1}{2}} = a^{\frac{1}{2}} \times \hat{I}^{\frac{1}{2}},$$

$$\hat{a}/\hat{b} = (a/b) \times \hat{I}, \quad |\hat{a}| = |a| \times \hat{I}, \quad a = n, c. \quad (3.5)$$

Thus the traditional statement " $2 \times 2 = 4$ " has meaning for RQM but has no meaning for RHM because one must identify first the selected unit and product for the operation " $2 \times 2$ " to have sense. This illustrates from the

outset the insidious inconsistencies in attempting to appraise the new RHM via the use of old mathematics.

In short, RQM is defined for numbers  $n$  whose basic unit is the quantity  $+1$  dating back to biblical times. RHM is instead defined for new numbers  $\hat{n} = n \times \hat{I}$  which admit arbitrary (positive-definite) units  $\hat{I}$ . As we shall see shortly, the introduction of the new isonumbers has deep and intriguing implications, including the possibility of defining new symmetries for conventional line elements and inner products.

The isonumbers were first submitted by Santilli [4c] at the meeting *Differential Geometric Methods in Theoretical Physics* held at the University of Clausthal, Germany in 1980; they were subsequently studied in various papers and books; received their first comprehensive treatment in Ref. [6f] (jointly with the broader genonumbers); and are today called *Santilli's isonumbers* [7].

### 3.3 Isohilbert spaces

The second notion of RQM which must be lifted for consistency is that of conventional Hilbert spaces  $\mathcal{H}$  with states  $|\psi\rangle, |\phi\rangle, \dots$ , inner product  $\langle\phi|\psi\rangle \in \mathcal{C}$  and normalization  $\langle\psi|\psi\rangle = 1$ , resulting in the *isohilbert space*  $\hat{\mathcal{H}}$  [6j] with the following *isostates, isoinner product and isonormalization*

$$\langle\hat{\phi}|\hat{\psi}\rangle = U \times \langle\phi|\psi\rangle \times (U \times U^\dagger)^{-1} \times U \times |\psi\rangle \times U^\dagger =$$

$$= \langle\hat{\phi}|\hat{\psi}\rangle \times \hat{I} \times |\hat{\psi}\rangle \times \hat{I} \in \hat{\mathcal{C}}, \quad (3.6a)$$

$$\langle\hat{\psi}|\hat{\psi}\rangle \times \hat{I} \times |\hat{\psi}\rangle = 1, \quad |\hat{\psi}\rangle = U \times |\psi\rangle, \quad \langle\hat{\phi}|\hat{\phi}\rangle = \langle\phi|\phi\rangle. \quad (3.6b)$$

Note that, again for consistency, the isoinner product must be an isocomplex number, i.e., must have the structure  $\hat{c} = c \times \hat{I}$ . The new composition is still inner (because  $\hat{I} > 0$ ) and, therefore,  $\hat{\mathcal{H}}$  is still Hilbert. Then,  $\hat{\mathcal{H}} \approx \mathcal{H}$  and the lifting  $\mathcal{H} \rightarrow \hat{\mathcal{H}}$  is again an isotopy.

The local isomorphism  $\mathcal{H} \approx \hat{\mathcal{H}}$  can also be seen from the following new *invariance law of the conventional Hilbert product* first introduced in [6m] here expressed for  $\hat{I}$  independent from the integration variable,

$$\langle \phi | \psi \rangle \times I \equiv \langle \phi | \times | \psi \rangle \times \hat{T} \times \hat{T}^{-1} \equiv \langle \phi | \times \hat{T} \times | \psi \rangle \times \hat{I} \equiv \langle \phi | \hat{\psi} \rangle. \quad (3.7)$$

Thus, RHM is based on conventional Hilbert spaces, only realized in a way more general than that of current use.

The isotopy  $\mathcal{H} \rightarrow \hat{\mathcal{H}}$  is also fundamental for the consistency of the theory. To see it, note that, under the mapping  $I \rightarrow \hat{I} = \hat{T}^{-1}$  and  $A \times B \rightarrow \hat{A} \times \hat{B} = \hat{A} \times \hat{T} \times \hat{B}$ , the action of an operator  $H$  on a state must be isotopic, i.e., of the type  $\hat{H} \hat{\times} \hat{\psi} = \hat{H} \times \hat{T} \times |\psi\rangle$  because this is the only one admitting the isomorphism  $\hat{I} \hat{\times} |\psi\rangle \equiv |\psi\rangle$ . Then the formulation of the above expression on a conventional Hilbert space with inner product  $\langle \hat{\phi} | \times | \hat{\psi} \rangle$  implies the general loss of Hermiticity. In fact, we would have the condition  $\{\langle \hat{\phi} | \hat{\times} \hat{H}^\dagger \rangle \times | \hat{\psi} \rangle = \langle \hat{\phi} | \times \{\hat{H} \hat{\times} | \hat{\psi} \rangle\}$ , i.e.,  $\hat{H}^\dagger = \hat{T}^{-1} \times H^\dagger \times \hat{T} \neq \hat{H}^\dagger$ . On the contrary, the use of the isohilbert space implies the conditions

$$\{\langle \hat{\phi} | \hat{\times} \hat{H}^\dagger \rangle \hat{\times} | \hat{\psi} \rangle = \langle \hat{\phi} | \hat{\times} \{\hat{H} \hat{\times} | \hat{\psi} \rangle\}, \quad \text{i.e.,} \quad \hat{H}^\dagger = \hat{H}^\dagger. \quad (3.8)$$

As a result, we have the properties: the conditions of Hermiticity and isohermicity coincide; quantities which are Hermitian-observable for RQM remain so for RHM; the eigenvalues of Hermitian operators of RHM are real; and other properties (see [6]) from brevity.

The only possible isoeigenvalues equations are then given by

$$\begin{aligned} \hat{H} \hat{\times} | \hat{\psi} \rangle &= \hat{H}(x, p) \times \hat{T}(x, p, \psi, \partial\psi, \dots) \times | \hat{\psi} \rangle = \hat{E} \hat{\times} | \hat{\psi} \rangle = \\ &= (E \times \hat{I}) \times \hat{T} \times | \hat{\psi} \rangle = E \times | \hat{\psi} \rangle. \end{aligned} \quad (3.9)$$

with corresponding isotopic expectation values

$$\langle \hat{H} \rangle = \frac{\langle \hat{\psi} | \hat{\times} \hat{H} \hat{\times} | \hat{\psi} \rangle}{\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle} = \frac{\langle \hat{\psi} | \times \hat{T} \times \hat{H} \times \hat{T} \times | \hat{\psi} \rangle}{\langle \hat{\psi} | \times \hat{T} \times | \hat{\psi} \rangle} = E, \quad (3.10)$$

which can be easily seen to coincide with the isoeigenvalues for the same operator. Note from Eq.s (3.9) that the "final numbers" of RHM to be confronted with experiments are conventional.

The fundamental axioms of RHM, are a simple isotopy of the axioms of RQM here omitted for brevity [6k,6m].

The above elements illustrate the main property that RHM coincides with RQM at the abstract realization-free level for which, from the positiveness of  $\hat{I}$ , we have  $\mathcal{R} \equiv \hat{\mathcal{R}}$ ,  $\mathcal{C} \equiv \hat{\mathcal{C}}$  and  $\mathcal{H} \equiv \hat{\mathcal{H}}$ . All other aspects of RHM are constructed following the same lines. Thus, RHM is not a new theory, but merely a new realization of the abstract axioms of RQM. These properties then establish the axiomatic consistency of RHM to such an extent that any criticism in its axiomatic structure is de facto a criticism on the axiomatic structure of RQM.

Despite the above abstract axiomatic identity, one should keep in mind that, as illustrated in Eq.s (3.1), RHM and RQM are physically inequivalent because the former is a nonunitary image of the latter. Moreover, isotopies imply the following mapping of eigenvalues

$$H \times | \psi \rangle = E_0 \times | \psi \rangle \rightarrow H \times T \times | \hat{\psi} \rangle = E \times | \hat{\psi} \rangle, \quad E \neq E_0, \quad (3.11)$$

according to which the same operator  $H$  has different eigenvalues in RQM and RHM, and this illustrates the nontriviality of the isotopies.

Isospaces (3.6) were first identified by Myung and Santilli [7p] in 1982; they were then used in all subsequent studies on hadronic mechanics; and are today called *Myung-Santilli iso-Hilbert spaces* [7].

### 3.4 Isoilinearity, isolocality, isounitariness

It is important to see that, despite their physical inequivalence, RHM preserves the conventional linearity, locality and unitarity of RQM. To begin, RHM is highly nonlinear in the wavefunctions (and their derivatives), as evident from isoeigenvalues expressions (3.9). Yet, the theory is isolinear, i.e., it verifies the linearity conditions in isospace, e.g., for all possible  $\hat{a} \in \hat{\mathcal{R}}$  or  $\hat{c}$  and  $|\hat{\phi}\rangle, |\hat{\psi}\rangle \in \hat{\mathcal{H}}$ , we have the identity

$$\hat{A} \hat{\times} (\hat{a} \hat{\times} |\hat{\psi}\rangle + \hat{b} \hat{\times} |\hat{\phi}\rangle) = \hat{a} \hat{\times} \hat{A} \hat{\times} |\hat{\psi}\rangle + \hat{b} \hat{\times} \hat{A} \hat{\times} |\hat{\phi}\rangle, \quad (3.12)$$

A similar situation occurs for locality. In fact, RHM is nonlocal-integral because interactions of that type are admitted in the  $\Gamma$ 's terms of the isounits, Eq.s (3.3). Nevertheless, RHM is isolocal, i.e., it verifies the condition of locality in isospace. In particular, RHM is everywhere local-differential except

at the isounit. On more technical grounds, RHM is equipped with a new topology called *Tsagas-Sourlas integro-differential topology* [7f].

By recalling that RQM is strictly local-differential, the above new topology has fundamental physical relevance inasmuch as it permits mathematically rigorous quantitative studies of the nonlocal-integral component of the nuclear force needed to represent the overlapping of the hyperdense charge distributions of nucleons in the nuclear structure [5f].

Next, RQM is said to be *unitary* in the sense that the only allowed transformations are of the unitary type,  $U \times U^\dagger = U^1 \times U = I$ . By comparison, RHM is *nonunitary* because its transformation theory is based on the requirement  $W \times W^\dagger = \hat{I} \neq I$ . Nevertheless, RHM reconstructs unitarity in isospace, a property called *isounitarity*. In fact, the above nonunitary transforms can be rewritten in the following identical isotopic form [6m]

$$W = \hat{W} \times \hat{T}^{1/2}, \quad W \times W^\dagger = \hat{I} \neq I, \quad (3.13a)$$

$$W \times W^\dagger \equiv \hat{W} \times \hat{W}^\dagger = \hat{W}^\dagger \times \hat{W} = \hat{I}, \quad (3.13b)$$

The necessity of the preceding reformulation is soon established by the fact that, even though derived via nonunitarity transforms, isotopic structures (3.1) and related properties are *not* invariant under additional nonunitary transforms. However, the needed form-invariance is readily achieved under the isounitary reformulation (3.13) for which [6m]

$$\hat{I} \rightarrow \hat{I} = \hat{W} \times \hat{I} \times \hat{W}^\dagger = \hat{W} \times \hat{T} \times \hat{T}^{-1} \times \hat{T} \times \hat{W}^\dagger \equiv \hat{I}, \quad (3.14a)$$

$$\hat{A} \times \hat{B} \rightarrow \hat{W} \times \hat{A} \times \hat{B} \times \hat{W}^\dagger = \hat{A} \times \hat{B},$$

$$\hat{K}' = \hat{W} \times \hat{K} \times \hat{W}^\dagger, \quad \hat{K} = \hat{A} \sigma \hat{B}, \quad (3.14b)$$

with a corresponding invariance of the condition of isohermiticity and all other properties. This illustrates again that the lack of application of the isotopies to any aspect of RQM implies insidious axiomatic inconsistencies.

Note that under isotransforms (3.14) the isounit and isotopic element remain *numerically invariant*. Note also that the transformation theory of RQM is restricted to transforms verifying the condition  $U \times U^\dagger = I$  for a fixed  $I$ . Similarly, the isotransforms of RHM are restricted to those verifying

the condition  $\hat{W} \times \hat{W}^\dagger = \hat{I}$ , this time, for fixed  $\hat{I}$  (because its change would imply the description of a *different system*).

The notions of isolinearity, isolocality and isocanoncity were first identified by Santilli in 1978 (see monographs [6c,6j,6k and literature quoted therein). The notion of isounitarity was first identified by Myung and Santilli [7p] in 1982.

### 3.5 Preservation of quantum laws

As indicated in Sect. 3.1, in this paper we study the simplest possible branch of RHM, that specifically formulated for applications to nuclear physics via a *Hermitian, positive-definite and diagonal* isounit (3.2). It is easy to see that the above branch does indeed preserve all conventional QM laws.

Recall that *generalizations of RQM which are conventionally nonlinear in the wavefunctions, i.e. of the type*  $H(x,p,\psi,\dots) \times \psi = E \times \psi$  [8] *imply the loss of the superposition principle, with consequential inapplicability to a consistent treatment of composite systems such as nuclei, besides having additional problematic aspects studied in details by Schuch [5e].*

RHM is also highly nonlinear in view of the eigenvalue structure (3.9), i.e.,  $H(x,p) \times \hat{T}(\psi,\dots) \times \hat{\psi} = E \times \hat{\psi}$ . However, the mathematical notion of isolinearity has the important physical implication that *RHM preserves the superposition principle in isospace*, as one can verify. This has the important implication that RHM can indeed be consistently applied to composite systems such as few-body nuclei.

Moreover, conventional nonlinear systems can be *identically* reformulated in the isotopic form,  $H(t,r,\psi) \times \psi \equiv H_0(t,r) \times \hat{T}(\psi,\dots)\psi = E \times \psi$ , by therefore recovering axiomatic consistency in isospace.

Next, it is important to see that *RHM preserves Heisenberg's uncertainty principle*. In fact, from isocommutators (3.1d) we have ( $\hbar = 1$ )

$$\Delta \hat{r}^i \Delta \hat{p}_j \geq \frac{1}{2} \langle \{\hat{r}^i, \hat{p}_j\} \rangle = \frac{1}{2} \delta^i_j. \quad (3.15)$$

This establishes that the deviations from Heisenberg's uncertainties predicted by quantum deformations (e.g., of the so-called squeezed states [4j]) can be removed via their reformulation in an invariant isotopic form (see [6k] for details).

Along similar lines, it is possible to prove that the notions of isolocality and isomilarity permit the preservation of causality under nonlocal-integral forces (see also [6k] for brevity). The preservation of the Fermi-Dirac statistics and related Pauli's exclusion principle will be indicated shortly. The proof of the preservation of other physical laws will be left to the interested reader.

The preservation of conventional laws can be seen from the fact that the fundamental quantity of representing deviations from RQM, the isounit, preserves all axiomatic properties of the conventional unit  $I$ , it is the basic invariant of the new theory and its isoepectation values recover the conventional value  $I$ ,

$$\hat{I}^n = \hat{I} \hat{\times} \hat{I} \hat{\times} \dots \hat{\times} \hat{I} \hat{\equiv} \hat{I}, \quad \hat{I}^{\frac{1}{2}} = \hat{I}, \quad \hat{I} / \hat{I} \hat{\equiv} \hat{I}, \quad \text{etc.} \quad (3.16a)$$

$$\hat{I}' = \hat{W} \hat{\times} \hat{I} \hat{\times} \hat{W}^\dagger \hat{\equiv} \hat{I}, \quad \text{id} \hat{I} / dt = \hat{I} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{I} = \hat{H} - \hat{H} \hat{\equiv} 0, \quad (3.16b)$$

$$\langle \hat{I} \rangle = \frac{\langle \hat{\psi} | \hat{\times} \hat{I} \hat{\times} | \hat{\psi} \rangle}{\langle \hat{\psi} | \hat{\times} | \hat{\psi} \rangle} = \frac{\langle \hat{\psi} | \hat{\times} \hat{T} \times \hat{T}^{-1} \times \hat{T} \times | \hat{\psi} \rangle}{\langle \hat{\psi} | \hat{\times} \hat{T} \times | \hat{\psi} \rangle} = I \quad (3.16c)$$

The above properties establishes the occurrence with far reaching implications according to which the validity of conventional QM laws for the nuclear structure, such as Heisenberg's uncertainty principle, Pauli's exclusion principle, etc., by no means, imply that the conventional formulation of RQM is the only applicable discipline because exactly the same laws are admitted by the structurally more general RHM.

It should be indicated for completeness that more general realizations of RHM exist [6k, 6l, 6m], e.g., those still of isotopic type with nondiagonal isounit  $\hat{I}$ , or the more general ones of genotopic type with nonhermitean basic unit and related transforms [6m]

$$I \rightarrow \hat{I} = U \times I \times U^\dagger \neq \hat{I}, \quad U \times U \neq I, \quad W \times W^\dagger \neq I, \quad (3.17)$$

which are particularly suited to represent irreversibility under open-nonconservative conditions, or those of hyperstructural type where  $\hat{I}$  is a set of

nonhermitean elements, which are particularly suited to represent irreversible biological systems [6k].

Original studies on the physical laws of hadronic mechanics implied departures from quantum laws. The invariant formulation of isotopic laws coinciding with conventional quantum laws was reached by Santilli in the recent memoir [6m].

### 3.6 Isotopic realization of "hidden variables" and "completion" of RQM

The reader should be aware that, as identified in memoir [6m], RHM provides an explicit and concrete realization of the theory of "hidden variables" [9a], which are actually realized via the operator  $\lambda = \hat{T}(x, \hat{x}, \hat{\psi}, \dots)$  and isoeigenvalues

$$\hat{H} \hat{\times}_\lambda | \hat{\psi} \rangle = \hat{H} \times \lambda(x, \hat{x}, \psi, \partial \psi, \dots) \times | \hat{\psi} \rangle = E_\lambda \times | \hat{\psi} \rangle. \quad (3.18)$$

In fact, the right modular actions " $H \times \psi$ " and " $H \hat{\times} \psi$ " lose any distinction at the abstract level and, in this sense, they are evidently "hidden" in the conventional realization.

As a result, RHM constitutes a form of "completion" of RQM, hereon called "isotopic completion", which results to be much along the celebrated argument by Einstein, Podolsky and Rosen [9b]. In particular, the completion is permitted by the fact that von Neumann's theorem [9c] and Bell's inequalities [9d] are inapplicable (and not "violated") for the isotopic completion due to its nonunitary structure.

More specifically, von Neumann theorem is inapplicable because the same Hamiltonian  $H$  has an infinite number of different sets of eigenvalues in RHM, one per each possible isotopic element ("hidden operator"  $\hat{T}$ ). Bell's inequalities are inapplicable, e.g., because RHM requires a nonunitary image of Pauli's matrices (see below for their outline). For detailed studies see ref. [6k], App. 4.C, where there is also a study of the classical limit.

A consequence of the above occurrences is that all applications of RHM outlined below, including the exact representation of nuclear magnetic moments of Sect. 4, are applications of the "isotopic completion" of RQM much along the celebrated  $E - P - R$  argument.

The isotopic realization of "hidden variables" and completion of quantum mechanics was first identified by Santilli (see Ref. [6k], App. 4.C and papers quoted therein). The most recent treatment is available in Ref. [9e].

### 3.7 Isominkowski spaces

The next notion of RQM which must be isotopically lifted for compatibility with basic structures (3.1) is that of the underlying carrier space, the Minkowskian space  $M(x, \eta, \mathcal{R})$  with space-time coordinates  $x = \{x^\mu\} = \{r, ct\}$ , where  $c_0$  is the speed of light in vacuum, metric  $\eta = \text{diag}(1, 1, 1, -1)$  and basic unit  $I = \text{diag}(\{1, 1, 1, 1\})$  on  $\mathcal{R}$ . The lifting yields the *isominkowski space*  $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$ , first proposed by Santilli [6e] in 1983, which is characterized by the lifting of: the coordinates  $x$  into the *isocoordinates*  $\hat{x} = U \times x \times U^\dagger = x \times I$ ; the basic unit of  $M$  into the isounit (here assumed to be diagonal from its Hermiticity),  $I \rightarrow \hat{I}$ , and the lifting of the metric  $\eta$  by the inverse of that of the unit,  $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$ . The basic *isointerval* is then given by or  $\hat{x}, \hat{y} \in \hat{M}$

$$\begin{aligned} (\hat{x} - \hat{y})^2 &= \{(\hat{x}^\mu - \hat{y}^\mu) \times \hat{N}_{\mu\nu}(x, \hat{x}, \hat{\psi}, \partial\psi, \dots) \times (\hat{x}^\nu - \hat{y}^\nu)\} \times \hat{I} = \\ &= (x - y)^2 = [(x^\mu - y^\mu) \times \hat{\eta}_{\mu\nu}(x, \hat{x}, \hat{\psi}, \partial\psi, \dots) \times (x^\nu - y^\nu)] \times \hat{I} = \\ &= \{(x_1 - y_1)^2 \hat{T}_1^2 + (x_2 - y_2)^2 \hat{T}_2^2 - (x_3 - y_3)^2 \hat{T}_3^2 - (x_4 - y_4)^2 \hat{T}_4^2\} \times \hat{I} \in \hat{\mathcal{R}}. \quad (3.19a) \end{aligned}$$

$$\begin{aligned} \hat{I} &= \text{diag}(\{\hat{T}_1^2, \hat{T}_2^2, \hat{T}_3^2, \hat{T}_4^2\}), \hat{I}_4 = \hat{T}^{-1} > 0, \\ \hat{T} &= \text{diag}(\{\hat{T}_1^2, \hat{T}_2^2, \hat{T}_3^2, \hat{T}_4^2\}), \hat{T}_4^2 > 0, \quad (3.19b) \end{aligned}$$

where  $\hat{N}$  is an *isomatrix*, i.e., a matrix whose elements are isoscalars  $\hat{N}_{\mu\nu} = \hat{\eta}_{\mu\nu} \times \hat{I} \in \hat{\mathcal{R}}$  (and, therefore, its operations and products are isotopic), while  $\hat{\eta}$  is an ordinary matrix, i.e., with elements  $\eta_{\mu\nu}$  given by ordinary scalars.

Note from the preceding structure that the use of isocoordinates  $\hat{x} = x \times \hat{I}$  is redundant in the isointerval. Nevertheless we shall keep using the scripture  $\hat{M} = \hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$  rather than  $\hat{M}(x, \hat{\eta}, \hat{\mathcal{R}})$  to recall that the coordinates are computed in isospace with respect to a generalized metric. Note also that the isospace  $\hat{M}$  and the underlying isofield  $\hat{\mathcal{R}}$  share the same generalized unit  $\hat{I}$ .

Note finally that the interval of  $\hat{M}$  is the most general possible one with signature  $(+, +, +, -)$  and a well behaved, yet arbitrary functional dependence on coordinates, wavefunctions, their derivatives of the needed order, as well as any additional quantity of the interior problem.

For this reason, as shown in details by Aringazin [10], the iso-Minkowski space  $\hat{M}(\hat{x}, \hat{\eta}, \hat{\mathcal{R}})$  is *directly universal*, i.e., admitting as particular case all possible signature-preserving generalizations of  $M$  (universality), directly in the coordinates of the observer (direct universality). In particular, the isominkowskian metric  $\hat{\eta}$  admits as particular cases the Riemannian, Finslerian, non-Desarguesian and all other possible metrics in  $(3+1)$ -dimension.

Despite the above arbitrariness, it has been proved that the original (abstract) Minkowskian axioms are preserved under the joint liftings  $I \rightarrow \hat{I} = \hat{T}^{-1}$  and  $\eta \rightarrow \hat{\eta} = \hat{T} \times \eta$ . Thus,  $\hat{M} \approx M$  and the lifting  $M \rightarrow \hat{M}$  is an isotopy.

The latter results can also be seen via the *new invariance law of the conventional Minkowskian interval*, first introduced in [6m] and (here expressed for a non-null scalar function  $\eta$ )

$$\begin{aligned} (x - y)^2 &= [(x^\mu - y^\mu) \times \eta_{\mu\nu} \times (x^\nu - y^\nu)] \times I = \\ &= [(x^\mu - y^\mu) \times (\eta^{-2} \times \eta_{\mu\nu}) \times (x^\nu - y^\nu)] \times (\eta^2 \times I) = \\ &= [(x^\mu - y^\mu) \times \hat{\eta}_{\mu\nu} \times (x^\nu - y^\nu)] \times \hat{I} = (x - y)^2, \quad (3.20) \end{aligned}$$

The new invariance identified by RHM is therefore  $[L = \text{length}] \times [I = \text{unit}] = \text{Inv}$ . As we shall see shortly, this is the mechanism which permits the preservation of spin and other conventional laws.

It is instructive to prove that the isoinvariant can also be obtained from the basic nonunitary transform (3.1a), i.e.  $\hat{x}^2 = U \times x^2 \times U^\dagger = [(x^\mu \times U^\dagger) \times (U^{-1} \times U^{-1}) \times \eta \times (U \times x)] \times (U \times U^\dagger) = \hat{x}^\dagger \times \hat{N} \times \hat{x}$  where " $\nu$ " stands for transposition. This construction also clarifies that the coordinate  $x$  on  $M$  is lifted into the form  $U \times x$  on  $\hat{M}$ .

Thus, RHM merely expresses "hidden" degrees of freedom of conventional quantum axioms. These degrees of freedom, expressed via the new invariance laws (3.7) and (3.20) have remained undetected through this century because they required the prior discovery of *new numbers*, those with arbitrary units [6f].



### 3.8 Isodifferential calculus

Despite the use of the isotopic theory, dynamical equations on  $\hat{M}$  are not invariant when expressed in terms of the conventional differential calculus. This has requested the construction of the *isodifferential calculus* [6] which is characterized by a simple, yet unique and effective isotopy of the conventional calculus based on the following *isodifferential, isoderivative* and related primary properties

$$\hat{d}\hat{x}^\mu = \hat{I}_\alpha^\mu \times d\hat{x}^\alpha, \quad \hat{\partial}/\hat{\partial}\hat{x}^\mu = \hat{I}_\mu^\alpha \times \partial/\partial\hat{x}^\alpha, \quad (3.21a)$$

$$\hat{\partial}\hat{x}^\mu/\hat{\partial}\hat{x}^\nu = \delta_\nu^\mu \times \hat{I}, \quad \hat{\partial}\hat{x}^\mu/\hat{\partial}\hat{x}_\nu = \hat{I}^{\mu\nu} \times \hat{I}, \quad \hat{\partial}\hat{x}_\mu/\hat{\partial}\hat{x}^\nu = \hat{I}_{\mu\nu} \times \hat{I}, \quad (3.21b)$$

where we have implied identities of the type

$$\hat{x} \hat{\times} |\hat{\psi}\rangle = x \times |\psi\rangle, \quad (\hat{\partial}/\hat{\partial}\hat{x}) \hat{\times} |\hat{\psi}\rangle \equiv \partial/\partial\hat{x} |\hat{\psi}\rangle, \quad \text{etc.} \quad (3.22)$$

The nontriviality of the isodifferential calculus is illustrated by the fact that it is commutative in isospace over isofields, e.g.,  $\hat{\partial}_\mu \hat{\partial}_\nu = \hat{\partial}_\nu \hat{\partial}_\mu$ , but is *not* commutative in its projection on the original space over conventional fields, e.g.,  $\hat{\partial}_\mu \hat{\partial}_\nu = \hat{I}_\alpha^\mu \times \partial_\alpha \times \hat{I}^\beta \times \partial_\beta \neq \hat{I}_\alpha^\nu \times \partial_\alpha \times \hat{I}^\beta \times \partial_\beta$ . This is evidently due to the x-dependence of the isotopic element  $\hat{I}$  which yields noncommutativity when referred to the unit I while the same dependence becomes "hidden" when referred to the isounit  $\hat{I} = \hat{I}^{-1}$ .

The above isocalculus has only recently permitted the achievement of an axiomatically consistent and form-invariant characterization of the isotopic linear momentum operator [6],[6m] which had escaped identification for over a decade and which can be written ( $\hbar = 1$ )

$$\hat{p}_\mu \hat{\times} |\hat{\psi}\rangle = p_\mu \times \hat{I}(x, \hat{x}, \psi, \partial\psi, \dots) \times |\hat{\psi}\rangle = -i \times \hat{\partial}_\mu |\hat{\psi}\rangle = -i \times \hat{I}_\mu^\alpha \times \partial_\alpha |\hat{\psi}\rangle, \quad (3.23)$$

which does indeed recover the fundamental isocommutation rules (3.1d).

Isomomentum (3.23) is of evident fundamental importance because it permits the explicit construction of the Hamiltonian, space-time and internal symmetries, specific applications, etc.

We are now in a position to indicate the reason why axiomatic consistency in the formulation of RHM has been reached only recently in memoir [6m]. In essence, the preceding formulations did use isofield, isospaces, and other isostructures, but they were formulated via the *conventional* differential equations. As such, they resulted to lack the invariance in the fundamental equations of the time evolution. The identification of the isodifferential calculus of memoir [6l] permitted in memoir [6m] the resolution of this last obstacle and the achievement of axiomatic consistency.

Note the main point here, that the differential calculus also depends on the assumed basic unit. This aspect is ignored in the rather vast literature in the field because of the tacit assumption of the trivial unit 1.

The integral calculus also admits a simple isotopy with basic definition  $\hat{I} = \int \times \hat{I}$  for which  $\int d\hat{x} = \hat{x}$ . For additional details, one may consult [6l].

The isodifferential calculus was first discovered by Santilli in the recent memoir [6l] and it is today essential for the invariant formulation of equations of motion in both classical and quantum descriptions.

### 3.9 Isofunctional analysis

It has been proved [6j] that the elaboration of data in RHM via ordinary and special functions and transforms is inconsistent because not invariant under the time evolution of the theory. This has required the isotopic lifting of functional analysis that we cannot possibly review here [loc. cit.].

We merely mention for subsequent use that in the spherical polar representation of the three-dimensional *iso-Euclidean space* with basic unit  $\hat{I} = \text{diag}(n_1^2, n_2^2, n_3^2)$ , the isounit assumes the value  $\hat{I}_\theta = n_1 \times n_2$  and  $\hat{I}_\phi = n_3$  while angles assume the isotopic value  $\hat{\theta} = \theta/n_1 \times n_2$ ,  $\hat{\phi} = \phi/n_3$ . This permits the construction of the *isotrigonometric functions* in the *isogauss plane*

$$\text{isocos } \hat{\theta} = n_1 \times \cos(\theta/n_1 \times n_2), \quad \text{isosin } \hat{\theta} = n_2 \times \sin(\theta/n_1 \times n_2), \quad (3.24)$$

with corresponding *isospherical coordinates* in three dimensions

$$x = n_1 \text{ isosin } \hat{\phi} \text{ isocos } \hat{\theta}, \quad y = n_2 \text{ isosin } \hat{\phi} \text{ isocos } \hat{\theta}, \quad z = n_3 \text{ isocos } \hat{\phi}. \quad (3.25)$$

The *isohyperbolic functions* and other structures are then constructed accordingly. Particular important for application is the *iso-Dirac deltas function*  $\delta(\hat{x})$  which, in general, has no longer a singularity at  $\hat{x}$ , thus having intriguing conceptual and technical implications in the possible removal of singularities *ab initio* (see [6j], for brevity).

The first isofunctions (e.g., the isoexponentiation) were identified by Santilli in the original proposal for the isotopies, Ref. [6a]. Additional important isofunctions were identified by Myung and Santilli [7p]. More recently, numerous authors have contributed to the construction of the isofunctional analysis, including J. V. Kadeisvili, M. Nishioka, A. K. Aringazin, D. A. Kirukhin, and others (see monograph [6j], Ch. 6, and references quoted therein).

### 3.10 Lie-Santilli isothery

The fundamental algebraic structure of RQM, Lie's theory, is linear, local-differential and canonical-unitary. As such, it is insufficient to characterize the desired nonlinear, nonlocal-integral and noncanonical-nonunitary component of the nuclear force due to mutual penetration of the hyperdense charge distributions of nucleons.

Because of the above insufficiencies, Santilli [6a,6b] submitted back in 1978 the *axiom-preserving isotopies of Lie's theory*, including the isotopies of universal enveloping associative algebras, Lie algebras, Lie groups, transformation and representation theories, etc., which were then studied by Santilli in a number of additional works (see monograph [6c] for a review as of 1983 and monograph [6k] for a review up to 1995). The emerging formulation is today called the *Lie-Santilli isothery* (see independent works [7] and references quoted therein).

Again, by conception and construction, the Lie-Santilli isothery is *not* a new theory, but merely a *new realization* of the abstract axioms of Lie's theory. Also, recall that all Lie algebras (over a field of characteristic zero) are known from Cartan's classification. Therefore, the isotopies of Lie's theory cannot possibly produce new algebras, and have been constructed instead to produce *novel realizations* of known Lie algebras.

The main lines of the Lie-Santilli isothery can be summarized as follows. Let  $\xi(L)$  be the universal enveloping associative algebra of an  $n$ -dimensional

Lie algebra  $L$  with: unit  $I$ ; (ordered set of) generators  $X = \{X_k\} = \{X_k\}$ ; associative product  $X_i \times X_j$ ; infinite-dimensional basis  $I, X_k, X_i \times X_j, i \leq j, X_i \times X_j \times X_k, i \leq j \leq k, \dots$  (Poincaré-Birkhoff-Witt theorem); and related exponentiation  $e^{\hat{x}w} = I + (i \times X \times w)/1! + (i \times X \times w) \times (i \times X \times w)/2! + \dots, w \in \mathcal{R}$ .

The *universal enveloping isoassociative algebra*  $\hat{\xi}(L)$ , first proposed in [6a], is the isotopic image of  $\xi(L)$  with isounit  $\hat{I}$ , the same generators  $\hat{X}_k = X_k$  only computed in isospace, isoassociative product  $\hat{X}_i \times \hat{X}_j$ , infinite dimensional isobasis  $\hat{I}, \hat{X}_k, \hat{X}_i \times \hat{X}_j, i \leq j, \hat{X}_i \times \hat{X}_j \times \hat{X}_k, i \leq j \leq k, \dots$  (isotopic Poincaré-Birkhoff-Witt theorem [6a,6d,7c]), and isoexponentiation

$$\hat{e}^{\hat{x}X \times w} = + (i \times X \times w)/1! + (i \times X \times w) \hat{X} (i \times X \times w)/2! + \dots = \{e^{\hat{x}X \times w}\} \times \hat{I}, \quad (3.26)$$

where  $w = \{w_k\} \in \mathcal{R}$ , and  $\hat{w} = x \times \hat{I}$  are the *isoparameters*. The script  $\hat{\xi}(L)$  rather than  $\xi(L)$  is used in the literature [6,7] to denote the *preservation of conventional generators under isotopies*. This is a physically important point because the generators  $X$  represent conventional physical quantities such as energy, linear momentum, angular momentum, etc. which apply for all possible interactions. The Lie-Santilli isothery preserves these generators unchanged by merely rewriting them in isospaces over isofields, and changes instead the *operations* on them.

In turn, the preservation of conventional generators assures the preservation by the Lie-Santilli isothery of *conventional total conservation laws*, as illustrated in more details below. This establishes *ab initio* that the fundamental physical notions of this memoir, the nonlinear, nonlocal and nonunitary terms in the nuclear force, are indeed fully compatible with *conventional total conservation laws*.

Note that, when  $\hat{I}$  is no longer positive-definite, in general  $\hat{I} \neq \xi(L)$ . This permits the study of a rather intriguing unification of *all simple, compact and noncompact Lie algebras of the same dimension into one unique isoalgebra* [6j]. Note the uniqueness of isoexponentiation (3.26) as compared to the lack of uniqueness of the exponentiation for  $q$ - and other deformations [5].

Let  $L$  be the Lie algebra homomorphic to the antisymmetric algebra  $[\xi(L)]^-$  of  $\xi(L)$  over a field  $F(a, +, \times)$  of real, complex or quaternionic numbers  $a$  with familiar Lie's second theorem  $[X_i, X_j] = X_i \times X_j - X_j \times X_i =$

$C_{ij}^k \times X_k$ . The Lie-Santilli isotope is the isospace  $\hat{L}$  with elements  $\hat{X}_k = X_k$  on  $\hat{\mathcal{H}}$  over  $\hat{F}$  with the isocommutation rules [6a-6c]

$$[\hat{X}_i, \hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i = \hat{C}_{ij}^k \hat{\times} \hat{X}_k, \quad (3.27)$$

whose brackets satisfy Lie's axioms in the isotopic form  $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ ,  $[\hat{A}, \hat{B}, \hat{C}] + [\hat{B}, \hat{C}, \hat{A}] + [\hat{C}, \hat{A}, \hat{B}] = 0$ , and the isodifferential rules  $[\hat{A} \hat{\times} \hat{B}, \hat{C}] = \hat{A} \hat{\times} [\hat{B}, \hat{C}] + [\hat{A}, \hat{C}] \hat{\times} \hat{B}$ .

Let  $G$  be the (connected) Lie transformation group characterized by the "exponentiation" of  $L$  into the elements  $U(w) = e^{i \times X \times w}$  with familiar laws  $U(w) \times U(w') = U(w + w')$ ,  $U(w) \times U(-w) = U(0) = I$ . Then the (connected) Lie-Santilli isotope transformation group  $\hat{G}$  is the "isoexponentiation" of  $\hat{L}$  according to Eqs (3.26) with isotopic laws

$$\begin{aligned} \hat{x}' &= \hat{U}(\hat{w}) \hat{\times} \hat{x} = \hat{e}^{i \times X \times w} \hat{\times} \hat{x} = \\ &= \{e^{i \times X \times \hat{r} \times w}\} \times \hat{I} \times \hat{T} \times \hat{x} = \{e^{i \times X \times \hat{r} \times w}\} \times \hat{x}, \end{aligned} \quad (3.28a)$$

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) \hat{\times} \hat{U}(-\hat{w}) = \hat{U}(0) = \hat{I}, \quad (3.28b)$$

verifying all conditions of  $G$  in isospace.

The nontriviality of the above isotopic theory over the conventional formulation is then established by the appearance of the isotopic element  $\hat{T}$  with an unrestricted functional dependence in the exponent of the group structure. This guarantees that the Lie-Santilli isotope has the most general possible nonlinear, nonlocal-integral and nonhamiltonian-nonunitary structure, although reformulated in an identical isolinear, isolocal and isounitary form.

A main difference between the Lie theory and the covering Lie-Santilli isotope is that the former admits only one formulation, while the latter admits two formulations, one in isospace over isofields, and the other given by its projection in the original space.

As a general rule, the Lie and Lie-Santilli theories coincide when formulated in their respective spaces, and this applies also for weights and the representation theory. However, the projection of the latter in the space of the former shows deviations called mutations which will be illustrated shortly.

We are now equipped to indicate the preservation of the Fermi-Dirac character of nucleons under the simplest possible isotope here considered, that is characterized by nonunitary transforms (3.1) with a diagonal isounit  $\hat{I}$ . The problem belongs to the study of the axiom-preserving isotopies  $S\hat{U}(2)$  of  $SU(2)$ -spin first studied in Ref. [6h] (which are different from the axiom-violating deformations  $SU_{p,q}(2)$  initiated in [4e]). The same isotopies are reformulated below via general rule (3.1), resulting in a new class of isorepresentations of  $S\hat{U}(2)$  of rather simple construction and effective applications.

Recall that the regular (two-dimensional) representation of  $SU(2)$  is characterized by the conventional Pauli matrices  $\sigma_k$  with familiar commutation rules  $[\sigma_i, \sigma_j] = 2i \times \epsilon_{ijk} \times \sigma_k$  and eigenvalues  $\sigma^2 \times |\psi\rangle = \sigma_k \times \sigma_k \times |\psi\rangle = 3 \times |\psi\rangle$ ,  $\sigma_3 \times |\psi\rangle = \pm 1 \times |\psi\rangle$  on  $\mathcal{H}$  over  $C$ .

RHM requires generalized Pauli's matrices first identified by Santilli in Ref. [6h] for the general case. For applications in nuclear physics we need the following new nonunitary image of Pauli's matrices here submitted for the first time

$$\hat{\sigma}_k = U \times \sigma_k \times U^\dagger, \quad U \times U^\dagger = \hat{I} \neq I, \quad (3.29a)$$

$$U = \begin{pmatrix} i \times m_1 & 0 \\ 0 & i \times m_2 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} -i \times m_1 & 0 \\ 0 & -i \times m_2 \end{pmatrix},$$

$$\hat{I} = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} m_1^{-2} & 0 \\ 0 & m_2^{-2} \end{pmatrix}, \quad (3.29b)$$

where the  $m$ 's are well behaved nowhere null functions, resulting in the regular iso-Pauli matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & m_1^2 \\ m_2^2 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \times m_1^2 \\ i \times m_2^2 & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}. \quad (3.30)$$

Another realization is given by nondiagonal nonunitary transforms

$$U = \begin{pmatrix} 0 & m_1 \\ m_2 & 0 \end{pmatrix}, \quad U^\dagger = \begin{pmatrix} 0 & m_2 \\ m_1 & 0 \end{pmatrix},$$

$$\hat{I} = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \quad \hat{T} = \begin{pmatrix} m_1^{-2} & 0 \\ 0 & m_2^{-2} \end{pmatrix}, \quad (3.31)$$