Hyperstructures in Lie-Santilli admissibility and iso-theories

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Abstract

In the quiver of hyperstructures Professor R. M. Santilli, in early 90'es, tried to find algebraic structures in order to express his pioneer Lie-Santilli's Theory. Santilli's theory on 'isotopies' and 'genotopies', born in 1960's, desperately needs 'units e' on left or right, which are nowhere singular, symmetric, real-valued, positive-defined for n-dimensional matrices based on the so called isofields.These elements can be found in hyperstructure theory, especially in H_v-structure theory introduced in 1990. This connection appeared first in 1996 and actually several H_v-fields, the e-hyperfields, can be used as isofields or genofields so as, in such way they should cover additional properties and satisfy more restrictions. Several large classes of hyperstructures as the P-hyperfields, can be used in Lie-Santilli's theory when multivalued problems appeared, either in finite or in infinite case. We review some of these topics and we present the Lie-Santilli admissibility in Hyperstructures.

Key words: Lie-Santilli iso-theory, hyperstructures, hope, H_v-structures.
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1. Introduction

In T. Vougiouklis, *"The Santilli's theory 'invasion' in hyperstructures"* [24], there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics appeared by Santilli's pioneer research. We continue with new topics in this direction.

Last years hyperstructures have applications in mathematics and in other sciences as well. The applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For all the above applications the largest class of the hyperstructures, the *H_v-structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the *fundamental relations* which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on. *Hypernumbers or H_v-numbers* are called the elements of H_v-fields and they are important for the representation theory.

The hyperstructures were introduced by F. Marty in 1934 who defined the hypergroup as a set equipped with an associative and reproductive hyperoperation. M. Koskas in 1970 was introduced the fundamental relation β^*, which it turns to be the main tool in the study of hyperstructures. T. Vougiouklis in 1990 was introduced the H_v-structures, by defining the weak axioms. The class of H_v-structures is the largest class of hyperstructures.

Motivation for H_v-structures:

The quotient of a group with respect to an invariant subgroup is a group.
The quotient of a group with respect to any subgroup is a hypergroup.
The quotient of a group with respect to any partition is an H_v-group.

The Lie-Santilli theory on *isotopies* was born in 1970's to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive-defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit.

According to Santilli's iso-theory [14], [8] on a field $F=(F,+,\cdot)$, *a general isofield* $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new product

$$\hat{a}\hat{\times}\hat{b}:=\hat{a}\times\hat{T}\times\hat{b},\quad \text{with}\quad \hat{1}=\hat{T}^{-1},\quad \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of $\hat{\boldsymbol{F}}$,

$$\hat{1}\hat{\times}\hat{a} = \hat{a}\times\hat{1} = \hat{a}\ ,\ \forall \hat{a}\in \hat{\boldsymbol{F}}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1996 [15], and called *e-hyperfields*. They point out that in physics the most interesting hyperstructures are the one called e-hyperstructures which contain a unique left ant right scalar unit.

2. Basic definitions on hyperstructures

In what follows we present the related hyperstructure theory, enriched with some new results. However one can see the books and related papers for more definitions and results on hyperstructures and related topics: [2], [4], [17], [18], [19], [20], [23], [31], [33].

In a set H is called ***hyperoperation*** (abbreviated: ***hope***) or *multivalued operation*, any map from H×H to the power set of H. Therefore, in a hope

$$\cdot : H\times H\to \mathcal{P}(H): (x,y)\to x\cdot y\subset H$$

the result is subset of H, instead of element as we have in usually operations.

In a set H equipped with a hope $\cdot:H\times H\to P(H)-\{\varnothing\}$, we abbreviate by

WASS the *weak associativity*: $(xy)z\cap x(yz)\neq\varnothing,\ \forall x,y,z\in H$ and by

COW the *weak commutativity*: $xy\cap yx\neq\varnothing,\ \forall x,y\in H$.

The hyperstructure (H,·) is called *H_v-semigroup* if it is *WASS* and it is called ***H_v-group*** if it is reproductive H_v-semigroup, i.e. $xH=Hx=H,\ \forall x\in H$. The hyperstructure (R,+,·) is called *H_v-ring* if (+) and (·) are *WASS*, the reproduction axiom is valid for (+), and (·) is *weak distributive* to (+):

$$x(y+z)\cap(xy+xz) \neq \varnothing,\quad (x+y)z\cap(xz+yz) \neq \varnothing,\quad \forall x,y,z\in R.$$

An H_v-structure is ***very thin*** iff all hopes are operations except one, with all hyperproducts singletons except one, which is set of cardinality more than one.

The main tool to study all hyperstructures are the fundamental relations β*, γ* and ε*, which are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [17], [18].

A way to find fundamental classes is given by analogous to the following:

Theorem 2.1 Let (H,·) be H_v-group and *U* all finite products of elements of H. Define the relation β by setting xβy iff $\{x,y\}\subset u,\ u\in U$. Then β* is the transitive closure of β.

Let $(R,+,\cdot)$ be H_v-ring, U all finite polynomials of R. Define γ in R as follows: $x\gamma y$ iff $\{x,y\}\subset u$ where $u\in U$. Then γ^* is the transitive closure of γ.

An element is called *single* if its fundamental class is singleton.

The fundamental relations are used for general definitions. Thus, to define the H_v-field the γ^* is used [17], [18]: A H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field. In the sequence the *H_v-vector space* is defined.

Let $(F,+,\cdot)$ be H_v-field, $(\boldsymbol{V},+)$ a COW H_v-group and there exists an external hope

$$\cdot : F\times\boldsymbol{V}\rightarrow P(\boldsymbol{V}): (a,x)\rightarrow ax$$

such that, $\forall a,b\in F$ and $\forall x,y\in \boldsymbol{V}$, we have

$$a(x+y)\cap(ax+ay)\neq\varnothing, \quad (a+b)x\cap(ax+bx)\neq\varnothing, \quad (ab)x\cap a(bx)\neq\varnothing,$$

then V is called an ***H_v-vector space*** over F. In the case of an H_v-ring instead of H_v-field then the ***H_v-modulo*** is defined.

In the above cases the fundamental relation ε^* is the smallest equivalence rsuch that the quotient $\boldsymbol{V}/\varepsilon^*$ is a vector space over the fundamental field F/γ^*.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called *smaller* than $(*)$, and $(*)$ *greater* than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \text{ such that } xy\subset f(x*y),\ \forall x,y\in H.$$

Then we write $\cdot\leq*$ and we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

The Little Theorem. Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

The definition of H_v-field introduced a new class of hyperstructures:

The H_v-semigroup $(H,\cdot)$ is called ***h/v-group*** if the quotient H/β^* is a group.

In [20] the 'enlarged' hyperstructures were examined if an element, outside the underlying set, appears in one result. In enlargement or reduction, most useful in representations are H_v-structures with the same fundamental structure.

The Attach Construction. Let $(H,\cdot)$ be an H_V-semigroup and $v\notin H$. We extend $(\cdot)$ into $\underline{H}=H\cup\{v\}$ as follows: $x\cdot v=v\cdot x=v$, $\forall x\in H$, and $v\cdot v=H$.

Then $(\underline{H},\cdot)$ is an h/v-group where $(\underline{H},\cdot)/\beta^*\cong Z_2$ and v is single element.

We call the hyperstructure $(\underline{H},\cdot)$ *attach h/v-group* of $(H,\cdot)$.

Definitions 2.2 Let $(H,\cdot)$ be a hypergroupoid. We say that *remove* $h\in H$, if simply consider the restriction of $(\cdot)$ on $H-\{h\}$. We say that $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h, whenever it appears, by $\underline{h}$. We say that $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of

$x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

The *uniting elements* method was introduced by Corsini & Vougiouklis [3]. With this method one puts in the same class more elements. This leads, through hyperstructures, to structures satisfying additional properties. The ***uniting elements*** method is the following: Let G be algebraic structure and d be a property, which is not valid and it is described by a set of equations; then, consider the partition in G for which it is put in the same partition class, all pairs that causes the non-validity of d. The quotient G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)/\beta^*$ for which the property d is valid, is obtained.

An application is when more than one properties are desired then:

Theorem 2.3 [18] Let $(G,\cdot)$ be a groupoid, and $F=\{f_1,\ldots, f_m, f_{m+1},\ldots, f_{m+n}\}$ be a system of equations on G consisting of two subsystems $F_m=\{f_1,\ldots,f_m\}$ and $F_n=\{f_{m+1},\ldots, f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure using the systems F and F_m resp., and let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m=(G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*.$$

In a groupoid with a map on it, a hope is introduced [22]:

Definitions 2.4 Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be map. We define a hope (∂), called ***theta*** and we write ***∂-hope***, on G as follows

$$x\partial y= \{f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G.\ \ (\text{resp. } x\partial y= (f(x)\cdot y)\cup(x\cdot f(y),\ \forall x,y\in G)$$

If $(\cdot)$ is commutative then (∂) is commutative. If $(\cdot)$ is *COW*, then (∂) is *COW*.

Motivation for a ∂-hope is the map *derivative* where only the product of functions is used. Thus for two functions s(x), t(x), we have $s\partial t=\{s't, st'\}$ where $(')$ is the derivative.

A large class of hyperstructures based on classical ones are defined by [18]:

Definition 2.5 Let $(G,\cdot)$ be groupoid, then for every $P\subset G$, $P\neq\emptyset$, we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}:\ x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r:\ x\underline{P}_ry=(xy)P\cup x(yP),\quad \underline{P}_l:\ x\underline{P}_ly=(Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. The usual case is for $(G,\cdot)$ semigroup, then

$$x\underline{P}y=(xP)y\cup x(Py)=xPy$$

and $(G,\underline{P})$ is a semihypergroup.

3. Representations. H_v-Lie algebras.

Representations of H_v-groups, can be faced either by H_v-matrices or by generalized permutations [18], [20], [31].

H_v-matrix (or *h/v-matrix*) is called a matrix with entries elements of an H_v-ring or H_v-field (or h/v-field). The hyperproduct of H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type $m\times n$ and $n\times r$, respectively, is a set of $m\times r$ H_v-matrices, defined in a usual manner:

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{\mathbf{C}=(c_{ij}) \mid c_{ij}\in \oplus\mathbf{\Sigma} a_{ik}\cdot b_{kj}\},$$

where ($\oplus$) is the *n-ary circle hope* on the hypersum: the sum of products of elements is considered to be the union of the sets obtained with all possible parentheses. In the case of 2×2 H_v-matrices the 2-ary circle hope which coincides with the hypersum in the H_v-ring. Notice that the hyperproduct of H_v-matrices does not nessesarily satisfy *WASS*.

The representation problem by H_v-matrices is the following:

Definition 3.1 Let $(H,\cdot)$ be H_v-group, $(R,+,\cdot)$ be H_v-ring and $\mathbf{M}_R=\{(a_{ij}) \mid a_{ij}\in R\}$, then any

$$\mathbf{T}:H\to\mathbf{M}_R:\ h\to\mathbf{T}(h) \ \text{ with } \ \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2)\neq\varnothing,\ \ \forall h_1,h_2\in H,$$

is called ***H_v-matrix representation*** If $T(h_1h_2)\subset T(h_1)T(h_2)$, then T is an *inclusion representation,* if $T(h_1h_2)=T(h_1)T(h_2)$, then T is *a good representation*. If T is one to one and good then it is a *faithful representation*.

The main theorem of representations of H_v-structures is the following:

Theorem 3.2 A necessary condition in order to have an inclusion representation T of an H_v-group $(H,\cdot)$ by $n\times n$ H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following:
For all $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,...,n\}$ such that

$$T(\beta^*(a)) \subset \{ A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i,j\in\{1,...,n\} \}$$

Therefore, every inclusion representation $T:H\to M_R$: $a\mapsto T(a)=(a_{ij})$ induces an homomorphic representation T^* of H/β^* over R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$. Then T^* is called *fundamental induced representation* of T.

The ***helix hopes*** can be defined on any type of ordinary matrices [33], [34]:

Definition 3.3 Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$, with $1\le s\le m$, $1\le t\le n$. The *helix-projection* is a map $\underline{st}:\mathbf{M}_{m\times n}\to\mathbf{M}_{s\times t}:A\to A\underline{st}=(\underline{a}_{ij})$, where $A\underline{st}$ has entries

$$\underline{a}_{ij} = \{ a_{i+\kappa s,j+\lambda t} \mid 1\le i\le s,\ 1\le j\le t \ \text{ and } \ \kappa,\lambda\in N,\ i+\kappa s\le m,\ j+\lambda t\le n \}$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ be matrices and $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called *helix-sum*, by

$$\oplus : \mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})=(a_{ij}+b_{ij})\mid a_{ij}\in \underline{a}_{ij} \text{ and } b_{ij}\in \underline{b}_{ij})\}$.

Let $A=(a_{ij})\in \mathbf{M}_{m\times n}$, $B=(b_{ij})\in \mathbf{M}_{u\times v}$ and $s=\min(n,u)$. Define the *helix-product*, by

$$\otimes: \mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\boldsymbol{P}(\mathbf{M}_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj})\mid a_{ij}\in \underline{a}_{ij} \text{ and } b_{ij}\in \underline{b}_{ij})\}$.

The helix-sum is commutative, WASS, not associative. The helix-product is WASS, not associative and not distributive to the helix-addition.

Using several classes of H_v-structures one can face several representations. Some of those classes are as follows [18], [19], [7]:

Definition 3.4 Let $M=M_{m\times n}$, the set of m×n matrices on R and $P=\{P_i:i\in I\}\subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P}: M\times M\to P(M): (A,B)\ A\underline{P}B=\{AP^t_iB: i\in I\}\subseteq M$$

where P^t is the transpose of P. $\underline{P}$ is bilinear Rees' like operation where instead of one sandwich matrix a set is used. $\underline{P}$ is strong associative and inclusion distributive to sum:

$$A\underline{P}(B+C) \subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C\in M.$$

So $(M,+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.

Definition 3.5 Let $\mathbf{M}=\mathbf{M}_{m\times n}$ be module of m×n matrices on R and take the sets

$$\mathbf{S}=\{s_k:k\in K\}\subseteq R,\ \ \mathbf{Q}=\{Q_j:j\in J\}\subseteq\mathbf{M},\ \ \mathbf{P}=\{P_i:i\in I\}\subseteq\mathbf{M}.$$

Define three hopes as follows

$$\underline{S}: R\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (r,A)\to r\underline{S}A = \{(rs_k)A: k\in K\}\subseteq \boldsymbol{M}$$

$$\underline{Q}_+: \boldsymbol{M}\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{Q}_+B = \{A+Q_j+B: j\in J\}\subseteq \boldsymbol{M}$$

$$\underline{P}: \boldsymbol{M}\times\boldsymbol{M}\to\boldsymbol{P}(\boldsymbol{M}): (A,B)\to A\underline{P}B = \{AP^t_iB: i\in I\}\subseteq \boldsymbol{M}$$

Then $(\boldsymbol{M},\underline{S},\underline{Q}_+,\underline{P})$ is a hyperalgebra on $\boldsymbol{R}$ called *general matrix P-hyperalgebra*.

The general definition of an H_v-Lie algebra is the following [26], [31], [16]:

Definition 3.6 Let $(\boldsymbol{L},+)$ be H_v-vector space on $(F,+,\cdot)$, $\varphi: F\to F/\gamma^*$, canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Similarly, let ω_L be the core of the canonical map $\varphi': \boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Consider the bracket hope (commutator):

$$[\ ,\ ] : \boldsymbol{L}\times\boldsymbol{L}\to P(L): (x,y)\to [x,y]$$

then $\boldsymbol{L}$ is an ***H_v-Lie algebra*** over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e. $\forall x,x_1,x_2,y,y_1,y_2\in L$ and $\forall \lambda_1,\lambda_2\in F$

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing,\ [x,\lambda_1y_1+\lambda_2y]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing$$

(L2) $[x,x]\cap\omega_L\neq\varnothing,\ \forall x\in \boldsymbol{L}$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing,\ \forall x,y\in \boldsymbol{L}$

4. The Santilli's: e-hyperstructures, iso-hyper theory.

The e-hyperstructures where introduced in [15], [25] and where investigates in several aspects depending from applications [5], [6], [16], [31].

Definition 4.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, i.e. $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

Definition 4.2 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. In the case that the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we have a *strong e-hyperfield.*

Definition 4.3 *Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, we define in G, an extremely large number of hopes $(\square)$ as follows:

$$x\square y=\{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\text{ and } g_1, g_2,\ldots\in G-\{e\}$$

$g_1, g_2,\ldots$ are not necessarily the same for each pair (x,y). $(G,\square)$ is an H_v-group, it is an H_b-group which contains the $(G,\cdot)$. $(G,\square)$ is an e-hypergroup. Moreover, if for each x,y such that xy=e, so we have $x\square y=xy$, then $(G,\square)$ becomes a strong e-hypergroup.

The proof is immediate since for both cases we enlarge the results of the group by putting elements from the set G and applying the Little Theorem. Moreover it is easy to see that the unit e is unique scalar element and for each x in G, there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$. Finally if the last condition is valid then we have $1=x\cdot x^{1}=x^{-1}\cdot x$, so the hyperstructure $(G,\square)$ is a strong e-hypergroup.

Example 4.4 Consider the quaternion group $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$ with defining relations $i^2=j^2=-1$, $ij=-ji=k$. Denoting $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$ we may define a very large number $(*)$ hopes by enlarging only few products. For example, $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$. Then the hyperstructure $(Q,*)$ is a strong e-hypergroup.

Construction 4.5 [31], [32]. On the ring $(\boldsymbol{Z}_4,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal,

(b) COW (non-commutative),

(c) they have 0 and 1, scalars.

We have the isomorphic cases: $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. The fundamental classes are $[0]=\{0,2\}$, $[1]=\{1,3\}$ and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$.

Thus it is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element has a unique double inverse.

We can also define the analogous cases for the rings $(\boldsymbol{Z}_6,+,\cdot)$, $(\boldsymbol{Z}_9,+,\cdot)$, and $(\boldsymbol{Z}_{10},+,\cdot)$.

In order to transfer Santilli's iso-theory theory into the hyperstructure case we generalize only the new product $\hat{\times}$ by replacing it by a hope including the old one [15], [27], [29], [32] and [1], [5], [6], [13], [14], [21], [24]. We introduce two general constructions on this direction as follows:

Construction 4.6 *General enlargement*. On a field $\boldsymbol{F}=(\boldsymbol{F},+,\cdot)$ and on the isofield $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\hat{\times}\hat{b}=\hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}$, containing $\hat{T}$, for all $\hat{a}\hat{\times}\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{\boldsymbol{F}}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore the new iso-hope is

$$\hat{a}\hat{\times}\hat{b}=\hat{a}\times\hat{H}_{ab}\times\hat{b}=\hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}}$$

$\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH_v-field*. The elements of $\boldsymbol{F}$ are called *isoH_v-numbers* or *isonumbers*.

More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable one. Thus, this special case is if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\hat{\times}\hat{b}=\hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\ \ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}}$$

and the rest are ordinary results, then we have a *very thin isoH_v-field*.

The assumption $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall\hat{a}\in\hat{\boldsymbol{F}}$ one inverse.

A ***generalization of P-hopes***, used in Santilli's isotheory, is the following [5], [28], [31]: Let $(G,\cdot)$ be abelian group and P a subset of G with $\#P>1$. We define the hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x \cdot P \cdot y = \{x \cdot h \cdot y \mid h \in P\} & \text{if } x \neq e \text{ and } c \neq e \\ x \cdot y & \text{if } x = e \text{ or } y = e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G, \times_P)$ is abelian H_v-group.

Construction 4.7 *The P-hope*. Consider an isofield $\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where $a\in F$, and $\hat{1}$ is positive-defined outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-product

$$\hat{a}\hat{\times}\hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \ \forall \hat{a},\hat{b} \in \widehat{\boldsymbol{F}}$$

Take a set $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\widehat{\boldsymbol{F}}-\{\hat{0},\hat{1}\}$, define the ***isoP-$H_v$-field***, $\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times}_P)$, where the hope $\hat{\times}_P$ as follows:

$$\hat{a}\hat{\times}_P\,\hat{b} := \begin{cases} \hat{a}\times\hat{P}\hat{}\times\hat{b} = \{\hat{a}\times\hat{h}\hat{}\times\hat{b} \mid \hat{h}\hat{}\in\hat{P}\hat{}\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{T}\hat{}\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1} \end{cases}$$

The elements of $\widehat{\boldsymbol{F}}$ are called *isoP-H_v-numbers*.

Remark. If $\hat{P}=\{\hat{T},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in isoP-H_v-fields, are not necessarily unique.

Example 4.8 Non degenerate example on the above constructions:

In order to define a generalized P-hope on $\hat{\boldsymbol{Z}}_7=\hat{\boldsymbol{Z}}_7(\hat{\underline{a}},\hat{+},\hat{\times})$, where we take $\hat{P}=\{\hat{\underline{1}},\underline{6}\}$, the weak associative multiplicative hope is described by the table:

$\hat{\times}$	$\hat{\underline{0}}$	$\hat{\underline{1}}$	$\hat{\underline{2}}$	$\hat{\underline{3}}$	$\hat{\underline{4}}$	$\hat{\underline{5}}$	$\hat{\underline{6}}$
$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$	$\hat{\underline{0}}$
$\hat{\underline{1}}$	$\hat{\underline{0}}$	$\hat{\underline{1}}$	$\hat{\underline{2}}$	$\hat{\underline{3}}$	$\hat{\underline{4}}$	$\hat{\underline{5}}$	$\hat{\underline{6}}$
$\hat{\underline{2}}$	$\hat{\underline{0}}$	$\hat{\underline{2}}$	$\hat{\underline{4}},\hat{\underline{3}}$	$\hat{\underline{6}},\hat{\underline{1}}$	$\hat{\underline{1}},\underline{6}$	$\hat{\underline{3}},\hat{\underline{4}}$	$\hat{\underline{5}},\hat{\underline{2}}$
$\hat{\underline{3}}$	$\hat{\underline{0}}$	$\hat{\underline{3}}$	$\hat{\underline{6}},\underline{1}$	$\hat{\underline{2}},\underline{5}$	$\hat{\underline{5}},\underline{2}$	$\hat{\underline{1}},\hat{\underline{6}}$	$\hat{\underline{4}},\hat{\underline{3}}$
$\hat{\underline{4}}$	$\hat{\underline{0}}$	$\hat{\underline{4}}$	$\hat{\underline{1}},\underline{6}$	$\hat{\underline{5}},\hat{\underline{2}}$	$\hat{\underline{2}},\hat{\underline{5}}$	$\hat{\underline{6}},\underline{1}$	$\hat{\underline{3}},\hat{\underline{4}}$
$\underline{\mathbf{5}}$	$\hat{\underline{0}}$	$\hat{\underline{5}}$	$\hat{\underline{3}},\underline{4}$	$\hat{\underline{1}},\underline{6}$	$\hat{\underline{6}},\underline{1}$	$\hat{\underline{4}},\underline{3}$	$\hat{\underline{2}},\underline{5}$
$\underline{\mathbf{6}}$	$\hat{\underline{0}}$	$\hat{\underline{6}}$	$\hat{\underline{5}},\hat{\underline{2}}$	$\hat{\underline{4}},\underline{3}$	$\hat{\underline{3}},\underline{4}$	$\hat{\underline{2}},\hat{\underline{5}}$	$\hat{\underline{1}},\hat{\underline{6}}$

The hyperstructure $\hat{\boldsymbol{Z}}_7=\boldsymbol{Z}_7(\hat{\underline{a}},\hat{+},\hat{\times})$ is commutative and associative on the product hope. Moreover the β^* classes on the iso-product $\hat{\times}$ are $\{\hat{\underline{1}},\hat{\underline{6}}\}$, $\{\hat{\underline{5}},\hat{\underline{2}}\}$, $\{\hat{\underline{3}},\underline{4}\}$.

5. The Lie-Santilli's admissibility.

Another very important new field in hypermathematics comes straightforward from Santilli's Admissibility. We can transfer Santilli's theory in admissibility for representations in two ways: using either, the ordinary matrices and a hope on them, or using hypermatrices and ordinary operations on them [10], [11], [12], [14], [16] and [7], [9], [30], [31], [34].

Definition 5.1 Let $\boldsymbol{L}$ be H_v-vector space over the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\rightarrow\boldsymbol{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\boldsymbol{F}/\gamma^*$. Let ω_L be the core of the canonical map $\varphi':\boldsymbol{L}\rightarrow\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Take two subsets $\boldsymbol{R},\boldsymbol{S}\subseteq\boldsymbol{L}$ then a ***Lie-Santilli admissible hyperalgebra*** is obtained by taking the Lie bracket*,* which is a hope:

$$[\ ,\ ]_{\mathbf{RS}}:\boldsymbol{L}\times\boldsymbol{L}\rightarrow\boldsymbol{P}(\boldsymbol{L}):\ [x,y]_{\mathbf{RS}}=x\boldsymbol{R}y-y\boldsymbol{S}x=\{xry-ysx\mid r\in\boldsymbol{R},\ s\in\boldsymbol{S}\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) When only $\boldsymbol{S}$ is considered, then $[x,y]_{\mathbf{S}}=xy-y\boldsymbol{S}x=\{xy-ysx\mid s\in\boldsymbol{S}\}$

(b) When only $\boldsymbol{R}$ is considered, then $[x,y]_{\mathbf{R}}=x\boldsymbol{R}y-yx=\{xry-yx\mid r\in\boldsymbol{R}\}$

(c) When $\boldsymbol{R}=\{r_1,r_2\}$ and $\boldsymbol{S}=\{s_1,s_2\}$ then

$$[x,y]_{\mathbf{RS}}=x\boldsymbol{R}y-y\boldsymbol{S}x=\{xr_1y-ys_1x,\ xr_1y-ys_2x,\ xr_2y-ys_1x,\ xr_2y-ys_2x\}.$$

(d) We have one case which is 'like' P-hope for any subset $\boldsymbol{S}\subseteq\boldsymbol{L}$:

$$[x,y]_{\mathbf{S}}=\{xsy-ysx\mid s\in\boldsymbol{S}\}$$

On non square matrices we can define admissibility, as well:

Construction 5.2 Let $(\boldsymbol{L}=\boldsymbol{M}_{m\times n},+)$ be H_v-vector space of $m\times n$ hyper-matrices on the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\rightarrow\boldsymbol{F}/\gamma^*$, canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of the field $\boldsymbol{F}/\gamma^*$. Similarly, let ω_L be the core of $\varphi':\boldsymbol{L}\rightarrow\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Take any two subsets $\boldsymbol{R},\boldsymbol{S}\subseteq\boldsymbol{L}$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket*,* which is a hope:

$$[\ ,\ ]_{\mathbf{RS}}:\boldsymbol{L}\times\boldsymbol{L}\rightarrow\boldsymbol{P}(\boldsymbol{L}):\ [x,y]_{\mathbf{RS}}=x\boldsymbol{R}^{t}y-y\boldsymbol{S}^{t}x.$$

Notice that $[x,y]_{\mathbf{RS}}=x\boldsymbol{R}^{t}y-y\boldsymbol{S}^{t}x=\{xr^{t}y-ys^{t}x\mid r\in\boldsymbol{R}\text{ and }s\in\boldsymbol{S}\}$

Special cases, but not degenerate, is the 'small':

$\boldsymbol{R}=\{r_1,r_2\}$ and $\boldsymbol{S}=\{s_1,s_2\}$ then

$$[x,y]_{\mathbf{RS}}=x\boldsymbol{R}^{t}y-y\boldsymbol{S}^{t}x=\{xr_1^{t}y-ys_1^{t}x,xr_1^{t}y-ys_2^{t}x,xr_2^{t}y-ys_1^{t}x,xr_2^{t}y-ys_2^{t}x\}$$

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