International Teleconference on

EINSTEIN'S ARGUMENT THAT "QUANTUM MECHANICS IS NOT A COMPLETE THEORY"

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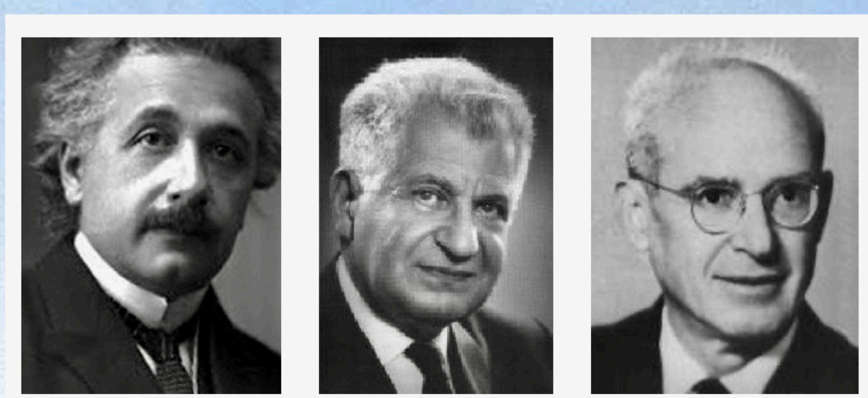
IN HONOR OF

Albert Einstein, Boris Podolsky and Nathan Rosen



IN RECOLLECTION OF THEIR 1935 HISTORICAL PAPER

A. Einstein, B. Podolsky, and N. Rosen
"Can quantum-mechanical description of physical reality be considered complete?"
Phys. Rev., vol.~47, p. 777 (1935)
http://www.eprdebates.org/docs/epr-argument.pdf

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Dedicated to the memory of

Prof. Zbigniew Oziewicz

Poland 1941 - Mexico 2020
Universidad Nacional Autónoma de México
Facultad de Estudios Superiores Cuautitlán

for his independence and depth of scientific thought.

Biographical Notes
http://eprdebates.org/docs/Zbigniew-Oziewicz-biography.pdf

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PREFACE

Almost from the moment it was first published in 1935, the famous - some might say infamous - article by Einstein, Podolsky and Rosen, claiming quantum mechanics to be an incomplete theory, has courted controversy. Following the initial furore, things calmed down with many accepting Bohr's refutation of the Einstein, Podolsky, Rosen argument. However, the topic has resurfaced periodically over the intervening years with no completely clear resolution emerging as far as some are concerned. In 2018 the entire issue resurfaced with the publication of some experimental results from a laboratory in Basle, results which served to support the view of that 1935 paper by Einstein, Podolsky and Rosen. This was followed by the international conference to be held in Florida in 2020 but which had to be turned into an online conference because of the covid-19 problem occurring in most, if not all, of the World. The conference ended up being a success at publicizing so many views, from so many people, which again supported the Einstein-Podolsky-Rosen line of thought. In particular, the conference brought to the fore the enormous, unheralded contribution to the debate by **Sir Ruggero Maria Santilli** (www.i-b-r.org/Ruggero-Maria-Santilli.htm - "Ruggero" hereon), with his first major contribution having come in 1998 after many years devoted to developing the new mathematics needed to cope adequately with the problems surrounding the Einstein, Podolsky, Rosen issue.

This book is intended to outline a collection of primary works from the mid 1960's to date by Ruggero and his collaborators on the verification and application of the Einstein-Podolsky-Rosen (EPR) argument that *Quantum mechanics* [and, therefore, quantum chemistry] *is not a complete theory,* in the expected recovering of classical determinism at least under limit conditions [1] (references indicated with square brackets refer hereon to those of Ruggero's Overview, while references indicated with upper numbers refer to additional works quoted in this preface).

This Preface is intended to provide a guide through a rather voluminous collection of works in various fields, as well as indicate important references following Ruggero's writing of the Overview.

During his Ph. D. studies at the University of Torino, Italy, in the mid 1960's, Ruggero discovered that the most advanced mathematics available at that time was insufficient for the representation of systems more complex than atomic structures, such as nuclear fusions, extended particles in deep mutual overlapping, combustion, biological structures, antimatter and other complex systems in the universe.

Extensive research carried out in 1964 at European mathematics libraries and repeated in 1978 at Cantabridgean mathematics libraries, convinced Ruggero that the mathematics needed for the effective representation of the indicated complex systems did not exist but had to be built and he had the courage to do it. In fact, Ruggero first recognized the need for new mathematics, constructed it himself and then proceeded to use it to examine complex systems not only in physics but in chemistry and biology as well.

In fact, throughout his research life, Ruggero first constructed the needed new mathematics and then passed to the treatment of complex systems in physics, chemistry or biology.

It then follows that no serious understanding of Ruggero's works is possible without a prior knowledge of the underlying new mathematics that, for the reader's convenience, are merely listed below with their primary references.

Quite notable is Ruggero's quote that "*There cannot exist a truly new physical theory without a new mathematics, and there cannot exist a new mathematics without new numbers.*"

I. NEW MATHEMATICS.

I-1. Lie-admissible mathematics.

As part of his Ph. D. curriculum, Ruggero studied the works by Giuseppe

Luigi Lagrange who lived in Torino and wrote a number of papers in Italian. In this way, Ruggero learned that Lagrange represented physical reality via his celebrated analytic equations containing potentials represented with Lagrangiansi L, plus *external terms* F representing non-conservative systems that, as such, are irreversible over time.

A comparison of Lagrange's original equations with the theories he was studying in Ph. D. courses soon revealed that Lagrange's external terms were not present in any of the available theories and that said external terms could not be represented with the available mathematics for various technical reasons.

A study of the works by Sir William Rowan Hamilton revealed their full parallelism with Lagrange's works, only formulated in phase space. In fact, the celebrated Hamilton's equations comprise a Hamiltonian H plus external terms F representing non-conservative and irreversible effects which were absent in the scientific literature of the mid 1960's.

Ruggero also learned that quantum mechanics cannot represent time irreversible systems because its main dynamical equations, Heisenberg's equation for an observable A, $idA/dt = [A, H] = AH - HA$ (where AH is the conventional associative product) can only represent the conservation of the energy, since $idH/dt = [H, H] = 0$.

To build the new mathematics needed for the representation of energy releasing, irreversible processes, Ruggero identified the main mathematical structure of quantum mechanics, which is given by Lie's theory with brackets $[A, B] = AB - BA$ and decided to generalize Lie's algebras into an algebra with brackets $(A, B) = A < B - B > A = ARB - BSA$ (where R, S are different, positive-definite operators) that turned out to be Lie-admissible according to the American mathematician, A. A. Albert [9]. The identification of the foundations of the new Lie-admissible mathematics immediately allowed Ruggero to generalize Heisenberg's equation into the form, today called *Heisenberg-Santilli Lie-admissible equations,* $idA/dt = (A, H) = A < H - H > A = ARH - HSA$ which represents the *time rate of variation* (rather than the conservation) of the energy $idH/dt = (H, H) = H(R - S)H \neq 0$ (see papers [6] [7] [8] of Ruggero's Ph. D. thesis although the unpublished version of the thesis better illustrates the originating thoughts).

The above studies in irreversibility attracted NASA attention and Ruggero moved to the U.S.A. in 1967 with his wife Carla and their newborn daughter Luisa for a one year appointment at the Center for Theoretical Physics of the University of Miami, Florida, with NASA support. He then accepted a faculty position at the Department of Physics of Boston University where he remained from 1968 to 1974 with partial support from the U. S. Air Force by teaching mathematics and physics at all levels and writing various papers with his associates and graduate students listed in his curriculum. From 1974 to 1977, Ruggero was a visiting scientist at the MIT Center for Theoretical Physics following an invitation from his Director Francis Low.

During his three year stay at the MIT-CTP, Ruggero wrote works[1−11] which are the *analytic foundation of verifications [210]-[214] of the EPR argument,* and comprise:

A. MIT-CTP preprints[1−5] on the necessary and sufficient conditions for the existence of a Lagrangian in field theory, technically known as the *conditions of variational selfadjointness* (SA), and their use for the representation of systems that are variationally non-selfadjoint (NSA) that are *analytic* in the sense of being derivable from a generalized action principle.

B. MIT-CTP memoirs[6,7,8] published at Annals of Physics which summarize the content of preprints[1−5].

C. MIT-CTP preprints[9,10] presenting the Newtonian particularization of the preceding field theoretical works, that Ruggero submitted in early 1977 to Springer-Verlag, Heidelberg, Germany, for publication as monographs.

D. MIT-CTP preprint[10] intended to be a third field theoretical volume of the Newtonian references[9,10], which preprint has remained unpublished, yet available as Annals of physics papers[6,7,8].

As we shall see in Section III, the above field theoretical works are important for the EPR completion of quantum electrodynamics into a form representing deviations of the theory from recent measures.

In September 1977, Ruggero joined Harvard University with a joint appointment as a visiting scientist at the Lyman Laboratory of Physics and at the Department of mathematics. In view of his works at MIT, Ruggero received on arrival an invitation from the U. S. Department of Energy for a grant intended to search for possible new clean nuclear energies. Soon thereafter, Ruggero also received the acceptance from Springer-Verlag for the publication of MIT preprints[9,10].

Encouraged by these openings, Ruggero plunged himself, firstly, in the construction of the Lie-admissible mathematics and secondly, in its application for the treatment of energy releasing irreversible systems.

The resulting main studies at Harvard University in the Lie-admissible mathematics, also called *genomathematics*[6,7,8], are given by: the 200 page memoir [19] of 1978 setting up the foundations of the new mathematics; Springer-Verlag monographs [21] [22] of 1978 proposing for the first time the completion of quantum mechanics into *hadronic mechanics* (see page 112) via the Lie-admissible generalization of Lie's theory and Heisenberg's equations and monographs [39] [40] on the Lie-admissble formulation of various aspects of 20th century applied mathematics.

In 1982, Ruggero accepted the position of President and Professor of Physics at the Institute for Basic Research (IBR) located at the Prescott House within the compound of Harvard University, which was moved to Florida in 1989 (www.i-b-r.org).

Among a rather large scientific production at the IBR reviewed in the

Overview, we here merely mention the systematic presentations of the various branches of hadronic mechanics in monographs [23]-[25] of 1996, and the five volumes [74] of 2006, all works having extensive references on independent contributions.

Various lectures on Lie-admissble mathematics are available on the website www.world-lecture-series.org. A tutoring lecture specifically intended to the verification of the EPR argument is available in Ref. [145].

Contributions in Lie-admissible mathematics that are important for the verification of the EPR argument are the following:

1. Paper [37] of 1979 established the $\widehat{\text{b}}$imodular structure of Lie-admissible mathematics, in the sense that the product $(A, B) = A < B - B < A = ARB - BSA$ can be reduced to two nodular actions, one to the right (representing motion forward in time) $H > |\psi >= HS|\psi >= E|\psi >$, and one to the left (representing motion backward in time) $< \psi| < H =< \psi|RH =< \psi|E', \quad E' \neq E$, whose in-equivalence assures the axiomatic representation of irreversibility. Said bimodular structure also assured the preservation of quantum mechanical axioms by the Lie- admissible branch of hadronic mechanics, since in a bimodular structure, quantum axioms are merely formulated per each selected time ordering.

2. In early 1980, it became known that physical applications of Lie-admissible methods are inconsistent when formulated over conventional numeric fields. This impasse was resolved with the discovery in the 1993 paper [33] of the genotopic numbers with multiplicative genounit to the right $I^> = 1/s$ and to the left $^<I = 1/R$.

3. In the mid 1990's it became also known that the representation of extended particles via Lie-admissible methods were inconsistent when elaborated via Newton-Leibnitz differential calculus due to its strict local character. In particular, hadronic mechanics was still missing in the mid 1990's a consistent generalized formulation of the angular momentum

due to the lack of a consistent generalization of the quantum mechanical linear momentum $p|\psi(r) >= -i\hbar\partial_r|\psi(r) >$. These additional impasses were resolved via the generalization-completion in Ref. [34] of 1996 of the conventional differential calculus into a form defined on *volumes,* rather than points, with *genomomentum* $p^> > |\psi^>(r^>) >= -i\partial^>_{r^>}|\psi^>(r^>) >= -iI^>\partial_{r^>}|\psi^>(r^>) >$ and to the left $< \psi^<(^<r)| <^< p = -i\partial^<_{<r} < \psi^<(^<r)| = -i^<I\partial_{<r}|\psi^<(^<r) >$ where volumes are represented by the *genounits* $I^> = 1/S$ and $^<I = 1/R$, respectively.

4. Paper [35] of 1989 identified a very simple method for the completion of 20th century mathematics into the Lie-admissible covering via the following two nonunitary transformations $UW^\dagger = I^>, \;\; WU^\dagger =^< I, \;\; UU^\dagger \neq I, \;\; WW^\dagger \neq I$ and the proof of the invariance over time of the genounits with consequential invariance of the shape and density of the represented particles.

5. Memoir [41] of 2006 proved the universality of the Heisenberg-Santilli genoequation f for the representation of all possible (regular) nonlinear, non-local, non-conservative, NSA systems via realizations of the type $R = 1, \; S = 1- = F/H$ with dynamical equation $idA/dt = (A, H) = A < H - H > A = AH - HA - AF$ where F is a suitably normalized operator form of Lagrange's and Hamilton's external terms.

In closing, we should indicate that there exists considerable literature on Lie-admissible algebras within the context of non-associative algebras in pure mathematics (see, e.g., Ref.[13] and Vol. I of Refs. [74]). However, these studies are formulated over conventional numeric fields, and even though mathematically correct, they cannot be used for the verification of the EPR argument because of a number of insufficiencies identified in Ruggero's Overview.

I-2. Lie-isotopic mathematics.

All objections against the EPR argument (see, e.g., Refs. [2]-[4]) are formulated for quantum mechanical, thus *conservative* systems for which Lie-admissible mathematics is inapplicable.

Additionally, in order to search for possible new clean nuclear energies under his DOE grant, Ruggero had to study nuclear structures, that when stable and isolated, verify all conventional total conservation laws, yet admit a more general formulation of internal strong interactions.

These requirements mandated the construction of a new mathematics for the representation of isolated, thus conservative systems with *extended* constituents in deep mutual penetration-entanglement, under non-linear, non-local and NSA interactions, yet such to verify conventional total conservation laws.

The needed new mathematics was identified in paper [19] of 1978 as the particular case of the Lie-admissible mathematics for $R = S = T > 0$ with basic brackets $[A, B]^* = A \star B - B \star A = ATB - BTA$, where T is called the isotopic element, which verify Lie's axioms, for which reason Ruggero called the new mathematics *Lie-isotopic* or $\widehat{\text{i}}$somathematics for short. Following the original proposal [19], isomathematics was studied in detail in monograph [22] of 1981 via the identification of its universal enveloping isoassociative algebra ξ with isoproduct $A \star B = ATB$, the initiation of the isofunctional analysis and the isotopic completion of the various branches of Lie's theory.

Isomathematics achieved maturity with the discovery of isonumbers [33], the isodifferential calculus [34] and the simple method for its construction in Ref. [35]. Systematic presentations of isomathematics were then provided in monographs [23]-[25] and [74].

Various lectures in isomathematics are available from www.world-lecture-series.org. A tutoring lecture in isomathematics specifically intended for the verification of the EPR argument is available from Ref. [143].

We should note the completion of the quantum entanglement of particles into the covering *EPR entanglement* (introduced by Ruggero in his

Overview and first released in paper[18]), which represents the non-linear, non-local and NSA interactions due to the overlapping of the wavepackets of particles via realizations of the isotopic element T, by jointly providing an explicit and concrete realizations of Bohm's *hidden variables* [17] (see Figure 2 and Section 7.3.2 of the Overview).

There is little doubt that Ruggero's EPR entanglement will have applications in all quantitative sciences. To see it, it is sufficient to note that the conception of a nucleus, or a molecule or a virus, as being composed of extended constituents under EPR entanglement, implies the inapplicability of all objections against the EPR argument [2]-[4], thus opening the door for a new physics as well as chemistry and biology.

We should also indicate that isomathematics has attracted considerable interest in mathematical circles and has seen a number of important contributions by pure mathematicians identified in the Overview, with complete listing in Vol. I of monographs[74].

I-3. Hypermathematics.

In line with his belief that mathematics will never admit final formulations, Ruggero has stated various times, that despite their vast representational capabilities, Lie-admissible and Lie-isotopic formulations cannot describe "all elements of reality" [1] because they are single-valued (in the sense that the multiplication of two quantities yields one single result, e.g., $2 \times 3 = 6$). This is an excessive limitation for the representation of complex systems, such as biological structures, which suggested Ruggero to turn Lie-admissible and Lie- isotopic mathematics into *multi-valued* (rather than multi-dimensional) forms via genotopic R, S and isotopic T elements representing an ordered set of values. As an example, the assumption that the isotopic element has three values $T = \{2, 3, 1/2\}$ implies that $2 \times 3 = \{12, 18, 1/3\}$ [27].

Even though sufficient for "simple" biological structures such as seashells, the above formulation of hypermathematics turned out to be insufficient for the representation of a living organism such as a cell.

Thanks to the participation of the Greek mathematician Thomas Vougiouklis, the above multi-valued formulations were generalized-completed into the most general and complex mathematics conceivable nowadays by the human mind, that of the *Lie-admissible and Lie-isotopic hyperstructures defined on hyperspaces over hyperfields* (see the Tutoring Lecture [146]).

I-4. Isodual mathematics.

Despite the advances indicated above, Ruggero still considered Lie-admissible and Lie-isotopic mathematics as being unable to represent *all* elements of reality [1].

During his graduate studies in the mid l960's, Ruggero wanted to study whether a far away galaxy is made up of matter or of antimatter, but was prohibited from doing such a study because the most advanced mathematics and physics available at that time identified no difference between matter and other complex systems in the universe.

Additionally, Ruggero has been a supporter of Dirac's view that *antimatter has negative energy* [11], as a pre-requisite for the representation of matter-antimatter annihilation.

Recall that, as discovered by Dirac himself, negative energies violate causality in the sense that the effect generally precedes the cause in the solution of quantum mechanical equations with negative energy).

For the intent of resolving Dirac's causality problem, while being at the Department of mathematics of Harvard University in the early 1980's, Ruggero decided to build the foundations of yet another new mathematics, this time based on the *negative unit* "-1" under the name of *isodual mathematics* where the word "isodual" stands to indicate an axiom-preserving duality of 20th century mathematics.

This lead to the construction of the isodual images of the conventional, Lie-isotopic and Lie-admissible mathematics [29].

In view of the construction of the above new mathematics and their application in physics, chemistry and biology, Ruggero was listed in 1990 by the Estonia Academy of Sciences among the most illustrious applied mathematicians of all times (see http://santilli-foundation.org/santilli-nobel-nominations.htm).

II. VERIFICATIONS OF THE EPR ARGUMENT.

Thanks to the new mathematics indicated in the preceding section, Ruggero and his associates have constructed the axiom-preserving completion of quantum mechanics into *hadronic mechanics* comprising the *Lie-isotopic, Lie-admissible, hyperstructural and isodual branches* [23]-[25] [74], with corresponding completions for *hadronic chemistry* [30] and *hadronic biology* [27] (see the outline in Section 6 of the Overview). Following these preparatory studies, as well as the teaching from historical completions of quantum mechanics by W. K. Heisenberg [16], Prince L. V. P. R. de Broglie [17], D. J. Bohm [18] and others reviewed in Section 5 of the Overview, Ruggero achieved the following verifications of the EPR argument (see also the presentation at the 2020 EPR conference by E.T. D. Boney om Gödel's incompleteness theorems [217] and by A.A. Nassikas on the minimum contradiction theory [218], as well as their recorded talks available from Ref. [0]):

II-1. Verification of the EPR argument for irreversible processes.

We are here referring to Ruggero's 1967 Ph. D. thesis [6]-[8] (see also memoir [41]) in which he proved the lack of completion of quantum mechanics for energy releasing, thus time irreversible processes and constructed the foundations of the Lie-admissible completion of quantum mechanics, also called genomechanics.

Besides scientific and industrial applications, the irreversible character of the Lie-admissible mechanics has stimulated studies to achieve a connection between mechanics and thermodynamics, of course, via the intermediate step of irreversible statistical mechanics, see P. Roman et al[14], A. A. Bhalekar[15], J. Fronteau et al [42], J. Dunning Davies [52] and other contributions.

II-2. Verification of the EPR argument for classical counterparts.

In a paper of 1964, J. S. Bell [3] proved a theorem essentially stating that a system of quantum mechanical particles with spin $1/2$ does not admit a classical counterpart.

Thanks to the prior development of the Lie-isotopic mathematics, Ruggero proved in his 1998 paper [210] that Bell's theorem is inapplicable (rather than being violated) for a system of extended particles with spin $1/2$ under deep EPR entanglement and said system does indeed admit a fully defined classical counterpart.

The proof was essentially based on the completion of Bell's theorem via the isoproducts $A \star B = A\hat{t}B,\ \hat{T} > 0$, which allows an explicit and concrete realization of Bohm's $hidden\ variables$ [17].

In the same paper [210], Ruggero illustrates the validity of hadronic mechanics via a numerically exact representation of nuclear magnetic moments that escaped a quantum mechanical representation for close to one century.

II-3. Verification of the EPR argument for classical determinism.

In a paper of 1981 [47], Ruggero introduced a generalization-completion of Heisenberg's uncertainties for strong interactions among extended hadrons in deep mutual entanglement (as occurring in a nuclear structure) when they are represented via hadronic mechanics.

In the 2019 paper [211], Ruggero proved the progressive recovering of Einstein's determinism in the interior of hadrons, nuclei and stars and its full recovering at the limit of gravitational collapse.

The proof is based on the realization of the isolinear momentum via the isodifferential calculus [34], $\hat{p} \star |\hat{\psi}(\hat{r}) >= -i\hat{\partial}_{\hat{r}}|\hat{\psi}(\hat{r}) >= -i\hat{I}\partial_{\hat{r}}|\hat{\psi}(\hat{r}) >$, $\hat{I} = 1/\hat{T} > 0$ (where "hat" denotes definition in hadronic mechanics).

Let Heisenberg's uncertainties be given by $\Delta p \Delta r = (1/2) < \psi(r)|[r,p]$-$|\psi(r) > \leq (1/2)\hbar$ under the normalization $< \psi||\psi >= \hbar$. The isotopies lead uniquely and unambiguously to the *isouncertainties* $\Delta\hat{p}\Delta\hat{r} = (1/2) < \hat{\psi}(\hat{r})| \star [\hat{r},\hat{p}] \star |\hat{\psi}(\hat{r}) > \leq (1/2)\hat{T}$ under the isonormalization $< \hat{\psi}| \star |\hat{\psi} >=< \hat{\psi}|\hat{T}|\hat{\psi} >= \hat{T}$. The results of paper [210] then follow from the fact that according to all fits of experimental data in hadron and nuclear physics, the isotopic element $\hat{T} = 1 - F/H$ has very small values and represents Schwartzchild's horizon at the limit of gravitational collapse.

II-4. Verification of the EPR argument for electron valence bonds.

In the final statement of their historic paper [1], Einstein, Podolsky and Rosen state that the *wavefunction of quantum mechanics* [and, therefore, of quantum chemistry] *cannot represent all elements of reality.*

In the 2001 monograph [30] on hadronic chemistry (see Chapter 4 on), Ruggero proved the above statement by showing that the wavefunction $\psi(r)$ of the Schrödinger equation of quantum chemistry cannot represent the *attraction* between the *identical* electrons in valence bonds since they experience at 10^{-13} cm the extremely big *repulsive* force of 230 $Newton$.

By subjecting the Schrödinger equation for electron valence bonds to a non-unitary transformation of the type $UU^\dagger = \hat{I} = exp\{-\bar{\psi}/\psi \int \psi^\dagger \psi d^3r\}$, and by using an appropriate selection of $\bar{\psi}$, the repulsive Coulomb potential is transformed into a strongly attractive Hulten potential, with corresponding completion of the wavefunction into a form representing the *element of reality* given by electron valence bonds.

The resulting bound state was called *isoelectronium* and allowed the numerically exact representation of the experimental data for the hydrogen [31] and water [32] molecules that have escaped quantum chemistry for half a century.

II-5. Verification of the EPR argument for antimatter.

Ruggero did not accept the conventional charge conjugation $\psi(t, r) \rightarrow \psi^c(t, r) = -\psi^\dagger(t, r)$ as being final because it provides no conjugation of matter into antimatter for neutral particles and prevents a representation of matter-antimatter annihilation (because antiparticles have the same positive energy of particles).

Hence, Ruggero developed the isodual mathematics for antimatter [29] outlined in the preceding section which is characterized by the isodual map (indicated with an upper letter "d") applied to the totality of quantities and their operations of quantum mechanics, resulting in the new *isodual charge conjugation* $\psi(t, r) \rightarrow \psi^d(t^d, r^d) = -\psi^\dagger(-t^\dagger, -r^\dagger)$, under which antimatter has negative energy as predicted by Dirac [11], evolves backward in time and all its characteristics are opposite those of matter, thus allowing a representation of matter-antimatter annihilation.

The violation of causality for particles with negative energies is resolved by the isodual mathematics because *particles with negative energy or negative time referred to negative units are as causal as positive energies or positive time referred to positive units.*

III. NEW APPLICATIONS.

Section 8 of Ruggero's Overview and the vast literature quoted therein, illustrate quite clearly that the verifications of the EPR argument, the new notion of EPR entanglement and their embodiment in the completion of quantum into hadronic mechanics, chemistry and biology, have important *new* implications (that is, applications not permitted by quantum mechanics) in all quantitative sciences.

We here merely note that what appears to be a central implication of the EPR argument, the existence of superluminal speeds under strong interactions (Section 8.4.4-VI), was first voiced by Ruggero in 1982[16].

The exact character of quantum electrodynamics (QED) has been recently disproved by accurate measurements[12,17] establishing a deviation of the measured muon g-factor from the QED prediction. Following the release of his Overview, Ruggero has provided a numerical representation of the anomalous muon g-factor via a branch of hadronic mechanics called *isoelectrodynamics* (IED)[18] whose analytic counterpart is given by the MIT-CTP papers[6,7,8].

We should finally mention that, following the release of the Overview, an independent review of the Einstein, Podolsky, Rosen argument has been published[19] and this is recommended for reading before embarking on a study of the technical issues as outlined in some detail in this book. In this article, inspired partly by the above-mentioned 2020 Florida Conference, much of the latest information, both experimental and theoretical, pertaining to this topic - a topic whose resolution is so important for the future of all science, not just physics - is provided. Also, reference is made to some earlier work supporting the Einstein-Podolsky-Rosen stance, which as far as many are concerned, has remained conveniently almost hidden ever since its ideas were advanced in the mid-1980's.

Further, the article contains some purely reflective thoughts on the position of probability theory in physics and other sciences, thoughts which may have relevance in other disciplines as well. The article also gives an independent view of some of the possible consequences of Ruggero's work - consequences which could affect each and every one of us, with the possibility of a new method for the quick and safe disposal of nuclear waste being probably the most important for many.

Even this, though, is merely one side product of his work which could lead to many more benefits for mankind.

In many ways, as with so many issues, the main problem encountered in discussions of the Einstein, Podolsky, Rosen issue has been an unwillingness to think 'outside the box', as the saying goes. However, the final resolution of this and other outstanding questions facing modern day science will surely rely on unorthodox thinking and in this respect, all should remember that final paragraph in the 1958 edition of Dirac's well-known text on quantum mechanics:

It would seem that we have followed as far as possible the path of logical development of the ideas of quantum mechanics as they are at present understood. The difficulties, being of a profound character, can be removed only by some drastic change in the foundations of the theory, probably a change as drastic as the passage from Bohr's orbit theory to the present quantum mechanics.

This powerful statement from such an eminent theoretical physicist surely deserves careful contemplation and certainly cannot be dismissed easily. It is the contention here that Ruggero has achieved, at least in part, that drastic change envisaged as necessary by Dirac. Not that Ruggero would claim this work constituted a complete theory, because as he has said on so many occasions, there cannot be a truly complete theory which successfully encompasses every possible situation. Physics, and indeed all of science, will always be continuously evolving subjects but it is felt the work discussed in this book represents a significant step forward in helping man gain a better understanding of the universe in which we all live.

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Figure 1: *Albert Einstein, Boris Podolsky and Nathan Rosen in the 1930s.*

OVERVIEW OF HISTORICAL AND RECENT VERIFICATIONS OF THE EPR ARGUMENT AND THEIR APPLICATIONS IN PHYSICS, CHEMISTRY AND BIOLOGY

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Abstract

In the 2020 Teleconference Ref. [0]: 1) We studied the 1935 objections against the **quantum entanglement** moved by Einstein, Podolsky and Rosen (EPR argument) [1]; 2) We pointed out that the interactions caused by wave-overlapping are of contact, zero-range, non-linear, non-local and non-Hamiltonian type; 3) We studied the new type of particle entanglement, here called **EPR entanglement**, consisting of particles in continuous and instantaneous communication via the overlapping of their wavepackets, thus without any need for superluminal speeds [1], whose non-Hamiltonian interactions are represented by the isotopic element $\hat{T}$ in the axiom-preserving product $A\hat{\times}B = A\hat{T}B$ of isomathematics and related hadronic mechanics (Tutoring Lectures [143] [144]); 4) We showed that the isotopic element $\hat{T}$ provides an explicit and concrete realization of Bohm's hidden variables; 5) We reviewed the recent verifications of the EPR argument by R. M. Santilli [210]-[214] showing the inapplicability of Bell's inequalities and other objections against the EPR argument for extended particles under non-potential interactions, with ensuing progressive recovering of classical images and Einstein's determinism in the structure of hadrons, nuclei and stars, and its full recovering at the limit of Schwartzchild's horizon **under the full preservation of quantum axioms merely subjected to a broader realization;** 6) We studied otherwise impossible advances in physics, chemistry and biology, including the exact representation of nuclear data, the achievement of an attractive force in valence electron bonds with ensuing exact representation of molecular data, and a new conception of life consisting of extended constituents under continuous E PR entanglement represented via hyperstructures [231]; 7) We showed the impossibility for the Copenhagen interpretation of quantum mechanics to solve our increasingly alarming environmental problems, such as recycling nuclear waste, achieving controlled nuclear fusions, and reaching the full combustion of fossil fuels; and pointed out their possible resolution under the EPR argument according to which "quantum mechanics is not a complete theory."

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1. FOREWORD.

As it is well known, Nazism reached in 1935 the peak of its military, political as well as, lesser known, scientific power, the latter being due to the conception and construction of quantum mechanics by German scientists, such as M. Planck, I. Schroedinger and W. Heisenberg and others.

The R. M. Santilli Foundation (http://www.santilli-foundation.org) and the Family of Israel Foundation (http://www.i-b-r.org/translational-medicine.htm) organized, conducted and recorded the International Teleconference from September 1 to 5, 2020 (see Ref. [0] and8 proceedings papers [210]-[233]) for the study of old and new verifications of the historical view by Albert Einstein, expressed jointly with his graduate students B. Podolsky and N. Rosen, that *"quantum mechanics is not a complete theory,"* with ensuing expectation that a suitable completion of quantum mechanics would recover classical determinism at least under limit conditions (EPR argument) [1].

This Overview is intended to provide an outline of studies conducted over one century in old and new verifications of the EPR argument and their applications in physics, chemistry and biology, with particular reference to the search for possible resolutions of our increasing alarming environmental problems via the new sciences permitted by the EPR argument, which resolutions appear to be impossible via quantum mechanics.

In Section 2, we present a brief outline of the EPR argument and its historical objections, including Bell's inequalities; in Section 3, we review known insufficiencies of quantum mechanics in various fields; in Section 4, we review the experimental verifications of the validity of Bell's inequality for point-like particles under linear, local and potential interactions as well its inapplicability (rather than violation) for extended particles under non-linear, non-local and non-potential interactions; in Section 5, we review the first historical completions of quantum mechanics essentially along the EPR argument by W. Heisenberg, L. de Broglie, D. Bohm et al.; in Section 5, we outline the completion of quantum mechanics,

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chemistry and biology into the various fields; in Section 7, we review five different verifications of the EPR Argument by R. M. Santilli [210]-[214]; in Section 8 we review applications and predictions in physics, chemistry and biology, as well as the implications of the EPR argument for high energy scattering experiments; and in Section 9, we present what appears to be the ultimate implications of the EPR argument.

To properly present and document a century of research in the field, we have made an effort to provide free pdf downloads of:

i) Reprints of Santilli's 1998 and 2019 verifications of the EPR argument [210]-[214];

ii) Papers [6]-[188] by numerous authors that resulted in time to be significant for the proofs the EPR argument, including:

ii-1) Refs. [32]-[36] that are important for the axiomatic structure and elaboration of Lie-isotopic theories;

ii-2) Refs. [49]-[52] that contain original important contributions by various authors for the construction of hadronic mechanics;

ii-4) Refs. [53]-[67] that provide a step-by-step Lie-isotopic completion of conventional space-time symmetries, their implications for physical laws and the proof of their isomorphism with conventional symmetries, which references play an important role in the construction of explicit and concrete realizations of Bohm's hidden variables with ensuing verification of the EPR argument.

iii) Monographs by various authors [154] [201]-[207] that review the process leading to verifications [210]-[214] of the EPR argument;

iv) References in the first representation of nuclear magnetic moments, spin, stability and other data with the consequential prediction and initial verification of new clean nuclear energies (Sections 8.1 and 8.2);

v) Internet debates for anonymous expressions of personal views in cosmology [76], neutrino [91], gravitation [180], applications of the EPR argument in physics [208], and applications of the EPR argument in chemistry [209];

vi) Tutorial lectures in isomathematics [143], verifications of the EPR argument [144], Lie-admissible formulations [145], and hyperstructures [146]; and

vii) Proceedings of international meetings in the field [188]-[200], including: five *Workshops on on Lie-Admissible Formulations* conducted at Harvard University from 1978 to 1982; twenty five *Workshops on Hadronic Mechanics* conducted from 1983 on at various locations in the U.S.A., Europe and China; and three *International Conferences on non-Potential interactions and their Lie-admissible treatments,* the first conducted in 1981 at the Université d'Orleans, France, the second conducted in 1995 at the Castle Prince Pignatelli, Molise, Italy, and the third conducted in 2011 at Katmandu University, Nepal. Out of all these meetings, the author could identify only twelve proceedings available with links for free pdf download which illustrate the number of scientists who contributed in the construction of hadronic mechanics, chemistry and biology.

It should be stressed that, by no means, the content of this Overview isexpected to be accepted by all participants of our 2020 Teleconference in the EPR Argument [0] because, particularly when dealing with fundamental open aspects, debates on qualified dissident views are essential for a serious scientific process.

2. THE EPR ARGUMENT.

A most mysterious experimental evidence in nature is the capability of particles to influence each other instantly at a distance. In view of the apparent influence in the 1930's by the Nazism, the scientific community of the time assumed that such an effect is predicted by quantum mechanics, for which reason the effect continues to be called to this day *quantum entanglement.*

By contrast, Albert Einstein noted that the Schrödinger equation of

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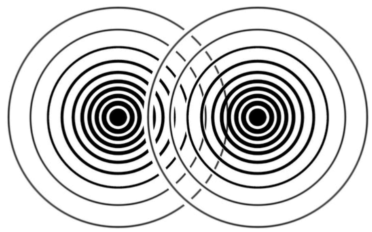

Figure 2: *A view of the new "EPR entanglement" introduced during the 2020 International Teleconference [0] (see Section 7.2.3 for a technical treatment), which consists of particles under continuous and instantaneous communication via the overlapping of their wavepackets, thus without any need for superluminal communications[1], with ensuing contact, zero-range, non-linear, non-local and non-potential interactions represented via isomathematics [54] and isomechanics [24].*

quantum mechanics (for $\hbar = 1$)

$$[-\frac{1}{2m}\Delta_r + V(r)]\psi(r) = E\psi(r), \tag{1}$$

can only represent point-like particles at a distance in vacuum and, therefore, cannot predict the entanglement of particles, in which case the sole possible representation of the entanglement is that via superluminal communications that would violate special relativity.

To avoid such a violation, Einstein Podolsky and Rosen argued that "quantum mechanics is not a complete theory" [1].

The EPR argument was quickly criticized by N. Bohr [2] a few months following its appearance although, under a sufficiently deep scrutiny, Bohr did not truly address the EPR argument, namely, the possible existence of "elements of realty" in the universe beyond those of the atomic structure which would require a suitable completion of quantum mechanics.

Despite such a visible insufficiency, Bohr's opposition to the EPR argument was supported quite widely by numerous scientists apparently because of the Nazism scientific power of the time.

In 1964, J. S. Bell [3] proved an inequality which essentially established that *quantum mechanical systems of particles with spin 1/2 do not admit classical counterparts.* Bell's inequality was considered by the mainstream scientific community, with due exceptions, to be the final dismissal of Einstein's dream of recovering classical determinism, and set the current widely accepted view that quantum mechanics is valid for whatever conditions exist in the universe (see review [4] and its comprehensive literature).

3. CONCEPTUAL FOUNDATIONS

3.1. Insufficiencies of quantum mechanics in particle physics.

The Copenhagen interpretation of quantum mechanics can only represent particles as being point-like at a distance in vacuum, because Schrödinger's equation (1) is characterized by wavefunctions $\psi(r)$, potentials $V(r)$, Laplacians Δ_r and other quantities that can only be defined at a finite set of isolated points r. Consequently, contrary to a rather popular belief, the Copenhagen interpretation of quantum mechanics cannot consistently predict or represent particle entanglements without superluminal communications, as correctly stated in the EPR argument [1].

The novel quantitative representation of particle entanglements studied at the 2020 teleconference [0] and called *EPR entanglement,* can be conceptually outlined as follows (see Figure 2 and Section 7.2.3 for a technical definition). Recall that the wavepacket of a particle fills up the entire universe with an intensity inversely proportional to the square of the distance. Hence, the entanglement of particles at a distance is characterized by the mutual penetration/overlapping of their wavepackets, under which conditions particles are in continuous instantaneous communication through the overlapping of their wavepackets without any need for superluminal speeds (Figure 2).

The technically challenging problem is that the interactions caused by the overlapping of the wavepackets of particles are: 1) Non-linear in the wavefunctions; 2) Non-local because defined in a volume, and 3) Not derivable from a potential because they are of contact, thus zero-range type, hereon referred to as *non-Hamiltonian interactions.*

Also recall that quantum mechanics is strictly Hamiltonian in the sense that interactions can only be represented via the Hamiltonian H. Consequently, the vast historical literature based on quantum mechanics (not listed in this Overview for brevity because easily identifiable with an internet search) represents particle entanglements with a Hamiltonian. By contrast, the 2020 Teleconference studied the alternative approach according to which the interactions occurring in particle entanglement should be represented with an operator *other* than the Hamiltonian, by therefore mandating a completion of quantum mechanics according to the EPR argument [1].

An important objective of the 2020 Teleconference has been that of reviewing half a century of research by R. M. Santilli as well as by various scholars in the completion of 20th century applied mathematics, from its sole validity at isolated points, into a covering mathematics providing a consistent representation of interactions 1), 2), 3) above (see the recorded Tutoring Lectures of Ref.[0]). Physical and chemical completions and applications of the new notion of particle entanglement were considered only thereafter.

Note that, from the basic axioms of the $SU(2)$-spin algebra and of Lie's theory at large, the validity of Bell's inequality [3] crucially depends for point-like particles under linear, local and potential interactions. Therefore, the study of systems of extended particles under non-linear, nonlocal and non-potential interactions automatically assures the *inapplica-*

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bility (rather than the violation of Bell's theorem), by therefore establishing rigorous grounds for the verification of the EPR argument.

A study of Goedel's incompleteness theorems presented at the 2020 Teleconference is available in Ref. [217] and in the recorded lecture by E. T. D. Boney [0].

The connection between causality and quantum mechanics was studied at the 2020 Teleconference by S. E. Johansen, see Ref. [224], and his recorded lecture [0].

The connecton between the EPR argument and the minimum contradiction theory was studied at the 2020 Teleconference by A. A. Nassikas, see contributed paper [228] and hiss recorded lecture [0].

3.2. Insufficiencies of quantum mechanics in nuclear physics.

On serious scientific grounds, a theory can be claimed to be *exactly valid* for given systems ("elements of reality" [1]) if and only if the theory provides an exact representation of *all* experimental data of the systems considered from first axiomatic principles without the adulterations appearing in the contemporary physics literature via manipulated form factors, venturing of particles and/or entities not directly testable, and the like.

Under the above serious scientific conditions, quantum mechanics can indeed be considered to be exactly valid for the structure of the hydrogen atom. However, with the understanding that the *approximate* validity of quantum mechanics in nuclear physics is out of question, the assumption of quantum mechanics as being exactly valid for the nuclear structure implies the exiting from the boundaries of serious science due to the well known failure by quantum mechanics to achieve an exact representation of nuclear experimental data, such as nuclear magnetic moments, nuclear spins, nuclear stability (recall that the neutron is unstable.....), and other data, despite the use of billions of dollars of public funds in about one century of research. This insufficiency begins with the

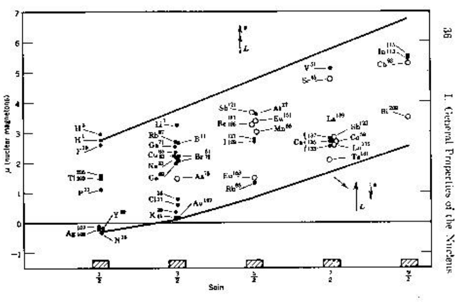

Figure 3: *A reproduction of the Schmidt limits providing a documentation of the deviations of the predictions of quantum mechanics from experimentally measured nuclear magnetic moments beginning with that of the smallest nucleus, the Deuteron. Similar insufficiencies exist for the representation of nuclear spins, nuclear stability and other data.*

lack of representation of experimental data on the smallest nucleus, the Deuteron, and becomes embarrassing for large nuclei such as the Zirconium (Figure 3) [6].

An understanding of the origin of the indicated insufficiency can be reached via the comparison of atomic and nuclear structures. In the hydrogen atom, the proton and the electron are at such a large mutual distance to allow their effective point-like approximation, with the ensuing exact validity of the theory. By contrast, experimental data on nuclear volumes and on the volume of individual nucleons, establishes that *nuclei are composed by a collection of extended and hyperdense nucleons in conditions of partial mutual penetration, thus EPR entanglement, of their charge distributions* (Figure 4) with ensuing non-Hamiltonian interactions (Section 3.1), with consequential need for a suitable completion of quantum mechanics.

3.3. Insufficiencies of quantum mechanics for irreversible

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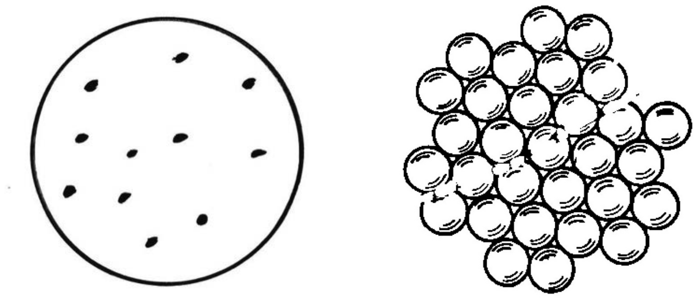

Figure 4: *An illustration on the left of the sole possible representation of nuclear structures by quantum mechanics due to its locality and linearity. An illustration on the right of the experimental reality establishing that nuclei are composed by extended and hyperdense nucleons in conditions of partial mutual overlapping with ensuing non-Hamiltonian interactions mandating a suitable completion of quantum mechanics [1].*

processes.
Beginning with his Ph. D. studies at the University of Torino, Italy, in the mid 1960's, Santilli dedicated his research life to the proof of the EPR argument [1] because quantum mechanics cannot achieve a consistent representation of energy releasing processes, such as combustion, nuclear fusions and others. This is due to the fact that all energy releasing processes are irreversible over time, while quantum mechanics can only represent systems whose time reversal image does not violate causality (Figure 5, left view) due to the invariance under anti-Hermiticity of Heisenberg's equation for Hermitean operators A

$$i\frac{dA}{dt} = [A, H] = AH - HA = -[A, H]^{\dagger}. \qquad (2)$$

The century old objection against the verification of the EPR argument via irreversible processes, which is still widely accepted nowadays, is

Figure 5: *A view on the left of the irreversibility of combustion and its lack of representation via quantum mechanics due to its reversibility. A view on the right of the clear irreversibility of high energy particle collisions, with ensuing need for an irreversible completion of quantum mechanics (Section 8.7).*

that irreversibility is 'illusory' (sic) because, when irreversible processes are reduced to their quantum mechanical elementary constituents, reversibility is fully regained. As part of his 1967 Ph. D. thesis [6]-[8], Santilli proved a number of theorems essentially stating that *a macroscopic irreversible system cannot be consistently decomposed into a finite number of quantum mechanical particles all in reversible conditions and, vice-versa, a finite number of quantum mechanical particles cannot recover a macroscopic irreversible system under the correspondence or other principles.*

An important discovery made during the 2020 Teleconference [0] is that *macroscopic irreversibility originates at the ultimate level of elementary particles, as established by the mere visual inspection of high energy scattering experiments at CERN, FERMILAB and other particle physics laboratories* (Figure 5, right view).

In view of the above insufficiency, Santilli proposed in 1967 (*loc. cit.*) the first known completion of quantum mechanics based on the embedding of Lie algebras of quantum mechanics with brackets $[A, B] = AB - BA$ into covering algebras with brackets (A, B) that are *Lie-admissible*

($Jordan$-$admissible$) according to the American mathematician A. A. Albert [8] when the attached anti-symmetric brackets $[A, B]^* = (A, B) - (B, A)$ verify the Lie algebra axioms (attached symmetric brackets $\{A, B\}^* = (A, B) + (B, A)$ verify the Jordan algebra axioms), with realization of the type

$$(A, B) = ARB - BSA = (ATB - BTA) + (AJB + BJA), \qquad (3)$$

where $R = T + J$ snd $S = -T + J$ are non-singular operators representing non-Hamiltonian interactions. The representation of irreversibility from first axiomatic principles is evidently assured by the violation of the invariance under anti-Hermiticity, $(A, B) \neq -(A, B)^\dagger$ which occurs whenever $R \neq S$. In particular, the Lie-isotopic operator T represents the non-Hamiltonian interactions of particle entanglement, while the Jordan-isotopic operator J represents the irreversibility of energy-releasing processes via a representation of the external terms in Lagrange's and Hamilton's equations that are completely absent in the Copenhagen interpretation of quantum mechanics (see Sections 6.2 and 7.6 for an overview).

3.4. Insufficiencies of quantum mechanics for antimatter.

Another majestic event in nature is given by the annihilation of particle-antiparticle pairs into light. The mechanism of this event cannot be represented via quantum mechanics because charge conjugation

$$\psi(r) \ \rightarrow \ \psi^c(r) = -\psi^\dagger(r), \qquad (4)$$

characterizes antiparticles in the same Hilbert space of the original particles, as a result of which antiparticles have the same positive energy of particles. Consequently, a particle-antiparticle pair both having positive energy cannot possibly annihilate into light, besides being in conflict with P. A. M. Dirac historical hypothesis that antiparticles have $negative\ energy$ [11].

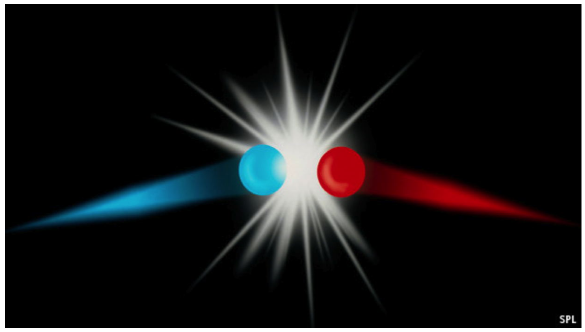

Figure 6: *Another topic studied at the 2020 Teleconference [0] has been the inability by quantum mechanics to represent the mechanism of particle-antiparticle annihilation into light because charge conjugation characterizes antiparticles with the same positive energy of particles against Dirac's view [11] that antiparticles should have negative energy.*

This occurrence clearly suggests the need for an additional completion of quantum mechanics, this time, for a representation of antiparticles in a way compatible with the experimental evidence on annihilation that can only be achieved when all characteristics of antiparticles are opposite those of particles (see Section 7.7 for an overview).

3.5. Insufficiencies of quantum mechanics in chemistry.

With the full admission of the historical advances achieved by quantum chemistry in the past century, the advancement of basic scientific knowledge requires the indication of the basic insufficiency of quantum chemistry given by the absence of a quantitative model of valence bonds due to the inability to represent the *attractive force* between *identical* valence electrons (Figure 7), with consequential lack of a quantitative model of molecular structures that carries evident environmental implications [12].

In fact, the Schrödinger equation of valence electron pairs is given by

$$[-\frac{1}{m}\Delta_r + \frac{e^2}{r}]\psi(r) = E\psi(r), \tag{5}$$

where m is the reduced mass, thus solely allowing a repulsion caused by the equal electron charge $-e$ which is represented by $+e^2/r$. Simple calculations show that the *repulsive force* between two valence electrons at $10^{-13}cm = 10^{-15}fm$ mutual distance is given by

$$F = k\frac{e^2}{r^2} = (8.99 \times 10^9)\frac{(1.60 \times x10^{-19})^2}{(10^{-15})^2} = 230\ N, \tag{6}$$

thus being so enormous for particle standards to prevent any realistic hope of being overcome by current models of valence bonds, thus confirming the need for a suitable completion of quantum chemistry. according to the EPR argument [1].

The lack of a sufficiently strong attractive force between identical valence electrons has rather serious implications in chemistry, such as:

a) The lack of an exact representation of molecular binding energies with deviations of the order of 2% that, rather than being ignorable, corresponds to about $950\ kcal/mole$.

b) The prediction that all substances are ferromagnetic in view of the lack of strongly bonded valence electron pairs.

c) The inability to achieve a quantitative representation of molecular electric and magnetic moments, with consequential inability to identify the attractive force between molecules in their liquid state, and other insufficiencies [*loc. cit.*].

4. EXPERIMENTAL FOUNDATIONS.

An important lecture delivered at the 2020 Teleconference has been that by Gerald Eigen [13] (see also Ref. [235]) who presented various experiments that essentially provide a *clear experimental verification of Bell's*

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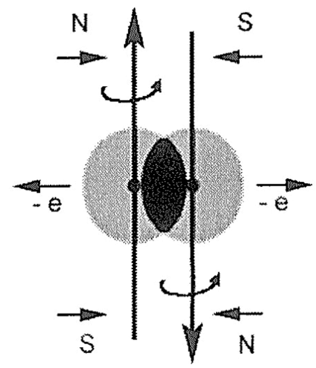

Figure 7: *A schematic view of the lack of a quantitative representation of molecular structures by quantum chemistry due to the large Coulomb repulsion between identical valence electrons according to Schrödinger equation (5) which is explicitly computed in equation (6).*

inequality for point-like particles in vacuum, thus including electromagnetic interactions.

Independently from the above tests, Matteo Fadel *et al* [14] conducted experiments similar to those of Ref. [13], but this time for the study of spin-correlations between space separated *atoms* of a Bose-Einstein condensate, by *measuring uncertainties below the bound of Heisenberg's uncertainty principle.*

Additionally, the ALICE experiments at CERN on *heavy ions* [15] have shown deviations from quantum mechanics bigger than those measured by experiments [14].

A view expressed at the 2020 teleconference is that tests [13]-[15] provide a conceptual and experimental verification of the EPR argument [1] because tests [13] confirm the acceptance by Einstein, Podolsky and Rosen of the validity of quantum mechanics for *point-like particles* in vacuum, while tests [14] [15] establish the existence of deviations from quantum mechanics for *extended particles.*

The main difference between tests [13] and [14] [15]] deal with the underlying interactions. In fact, for tests [13], the sole possible interactions are those at a distance, since the particles are dimensionless. By contrast, for the case of tests [14] [15], since the constituents are given by extended atoms and ions under the known compression caused by the Bose-Einstein condensate, the interactions are of the non-linear, non-local and non-potential type causing the *inapplicability* (rather than the violation) of Bell's inequality.

As we shall see, besides the above direct tests, there exist additional experimental evidence in support of the EPR argument in particle physics, nuclear physics, chemistry and other fields.

5. HISTORICAL EPR COMPLETIONS.

In general, the limitations of basic physical laws have been best identified by their originators. For example, following the discovery of the matrix realization of quantum mechanics on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$, with historical time evolution (2), Werner K. Heisenberg identified the limitations of his own theory caused by its linearity, and studied what can be well called the *first non-linear completion of quantum mechanics* [16].

Quantum mechanics is a strictly Hamiltonian theory and, therefore, can only represent non-linear interactions with the Hamiltonian, resulting in eigenvalues equations of the type

$$H(r, p, \psi)\psi(r) = E\psi(r). \tag{7}$$

As part of his decades of studies on completions according to the EPR argument, Santilli pointed out at the 2020 Teleconference [0] that, despite its clear historical value, Heisenberg's non-linear completion does not allow the characterization of the constituents $\psi_k(r),\ k = 1, 2, ..., N$ of a bound state with non-linear internal interactions because non-linear equations

(7) generally violate the superposition principle,

$$\psi(r) \neq \Sigma_{k=1,2,...,N}\psi_k(r). \quad (8)$$

The lack of a superposition principle then implies serious limitations on the representation of experimental data, e.g., on how the neutron, which is naturally unstable, becomes stable when bonded to a proton in a stable nucleus.

Therefore, in order to preserve the superposition principle under completion, Santilli suggested that non-linear interactions should be represented with a suitably selected operator *other than the Hamiltonian.* In any case, as indicated earlier, non-linear interactions are of contact, zero-range type thus carrying no potential energy. Their representation with the Hamiltonian would then cause consistency problems in the axiomatic structure of the completed theory.

Despite the indicated limitation, Heisenberg's non-linear completion of quantum mechanics played an important role in Santilli's studies. In fact, there is a mention in the Comments of Teleconference [0] that Santilli had various mail contacts with Heisenberg in the early 1970's precisely on the superposition principle when Santilli was in the faculty of the Department of Physics of Boston University.

As it is well known, Louis V. P. R. de Broglie was one of the primary contributors to the development of the wave structure of particles in quantum mechanics. Jointly, de Broglie was the first to identify the limitations of such a description caused by the locality of the wavefunction $\psi(r)$. The theory developed by de Broglie was semiclassical as a condition to by-pass Heisenberg's uncertainty principle and, as such, it can be considered to be one of the first attempts at recovering Einstein's determinism.

Unfortunately for science, de Broglie was dissuaded by mainstream scientists of the time to continue his studies on possible broadening of quantum mechanics. Nevertheless, de Broglie's ideas were resumed and further developed by David J. Bohm and the theory can be well called resulting the *de Broglie-Bohm non-local completion of quantum mechanics* [17].

This theory was discussed at the 2020 Teleconference [0] because of its important influence in the development of new completions of quantum mechanics. In particular, Santilli pointed out that, in his view, no consistent representation of the entanglement of particles can be achieved without a non-local theory since the overlapping of wavepackets occurs in a volume. Yet, to be effective, the representation of said non-locality should be done via an operator other than the Hamiltonian since said wave-overlapping carries no potential energy.

The third important historical completion of quantum mechanics studied at the 2020 Teleconference [0] is that by D. J. Bohm with his theory of *hidden variables* [18]. As it is well known, quantum mechanics is a probabilistic theory, with ensuing uncertainties in the position and momentum of particles. Bohm conjectured the possible existence of more fundamental physical laws hidden in the mathematics of quantum mechanics that would allow the theory to recover Einstein's determinism. Despite a large literature in the field, no concrete formulation of hidden variables, and related recovering of Einstein's determinism, was achieved up to the early 2000's . Nevertheless, Bohm's intuition on the lack of final character of the Copenhagen interpretation of quantum mechanics remained fundamental.

Santilli's view on Bohr's hidden variables presented at the 2020 Teleconference [0] is essentially the following:

1) Bell's inequality on the lack of existence of classical images should be considered valid for the infinite family of unitary equivalence of quantum mechanics. Consequently, the sole known possibility of bypassing Bell's inequality and achieving concrete realizations of hidden variables is the construction of a *non-unitary completion of quantum mechanics.*

2) Bell's inequality has been experimentally verified by G. Eigen and

others [13] for electromagnetic interactions of point-like particles in vacuum. Consequently, the most promising applications of non-unitary completions of quantum mechanics are those for strong nuclear interactions for which quantum mechanics is known to be incomplete (Section 2).

3) As it is the case for Heisenberg's non-linear theory, and de Broglie-Bohm on-local theory, Bohm hidden variables should be realized via an *operator*— other than the Hamiltonian which is hidden in the axioms of quantum mechanics.

A geometric interpretation of hidden variables was presented at the 2020 Teleconference by O. A. Olkhov, see contribution [239] and his recorded lecture [0].

6. HADRONIC MECHANICS, CHEMISTRY AND BIOLOGY.

6.1. Hadronic mechanics.

It is evident that the completion of quantum mechanics according to Einstein, Podolsky and Rosen [1] has implications for all quantitative sciences, including physics, chemistry and biology. In order to initiate an expectedly long process, in his Ph. D. thesis at the University of Torino, Italy, in the mid 1960's, Santilli proposed the completion of quantum mechanics for irreversible processes via Lie-admissible completion of the quantum mechanical Lie algebras [6]-[8] according to A. A. Albert [9], and continued the studies at Harvard University under DOE support in the late 1970's with further developments of the Lie-admissible formulations as well as their Lie-isotopic particularization [19]-[22].

Because of important contributions by numerous mathematicians, physicists and chemists, the above studies achieved in the late 1990's a completion/covering of quantum mechanics known as *hadronic mechanics* [23]-[28] for a consistent representation of extended particles under non-linear, non-local and non-Hamiltonian interactions due to wave-overlapping, thus EPR entanglement.

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An important feature of the studies herein considered is that *the Copenhagen interpretation of quantum mechanics resulted as being the simplest possible realization of quantum axioms, such as associativity, distributivity, linearity, locality, potentiality, etc., while hadronic mechanics is characterized by the progressive more general possible realizations of said axioms depending on the complexity of the considered "elements of reality"* [1], according to the following:

Classification of hadronic mechanics:

6-1-I) Quantum mechanics, with the familiar *Heisenberg's time evolution* (2) of an observable A in the infinitesimal form

$$i\frac{dA}{dt} = [A, H] = AH - HA = -[A, H]^\dagger, \;\; A = A^\dagger, \;\; H = H^\dagger, \tag{9}$$

and finite form

$$A(t) = UA(0)U^\dagger = e^{Hti}A(0)e^{-itH}, \;\; UU^\dagger = U^\dagger U = I, \tag{10}$$

characterized by the Lie's theory with brackets $[A, H]$ for the representation of stable, thus time-reversal invariant systems of point-like particles in vacuum via the Hamiltonian H.

6-1-II) Isomechanics (see Refs. [23] [24], reviews in Section 2 of Ref. [25] and Tutoring Lecture I [143][144]), with the *iso-Heisenberg time evolution* in its infinitesimal form (first introduced in Eqs. (18), page 153, Ref. [22])

$$i\frac{dA}{dt} = [A\hat{,}H] = A\hat{T}H - H\hat{T}A = -[A\hat{,}H]^\dagger, \hat{T} > 0, \tag{11}$$

and finite form

$$A(t) = WA(0)W^\dagger = e^{H\hat{T}ti}A(0)e^{-it\hat{T}H}, \;\; WW^\dagger \neq I, \tag{12}$$

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characterized by the axiom-preserving Lie-Santilli isotheory with brackets $[A\hat{,}B] = A\hat{T}B - B\hat{T}A$, for the representation of stable (thus time-reversal invariant) hadrons, nuclei and stars composed by extended, therefore deformable constituents in conditions of deep mutual entanglement with conventional interactions represented with the Hamiltonian H, and non-linear, non-local and non-potential interactions represented with the isotopic element $\hat{T}$.

6.1-III) Genomechanics (see Refs. [23] [24], reviews in Section 3 of Ref. [25] and Tutoring Lecture IV [145]), with the *geno-Heisenberg equation* in their infinitesimal form (first introduced in Eqs. (19), page 153, Ref. [22])

$$i\frac{dA}{dt} = (A, H) = A\hat{R}H - H\hat{S}A =$$
$$= (A\hat{T}H - H\hat{T}A) + (A\hat{J}H + H\hat{J}A) \neq -(A, B)^{\dagger},$$
$$\hat{R} = \hat{T} + \hat{J} \neq \hat{S} = -\hat{T} + \hat{J}, \ \ \hat{T} > 0, \ \ \hat{J} > 0, \tag{13}$$

and finite form

$$A(t) = WA(0)W^{\dagger} = e^{H\hat{R}ti}A(0)e^{-it\hat{S}H}, \ \ WW^{\dagger} \neq I, \tag{14}$$

characterized by the Lie-admissible/Jordan-admissible theory with product (3), for the representation of time-irreversible processes between extended constituents via: A)The conventional Hamiltonian H for the representation of conventional potential interactions; B) The Lie-isotopic operator $\hat{T}$ for the representation of non-Hamiltonian interactions due to mutual EPR entanglement as in Case II; and C) The Jordan-isotopic operator $\hat{J}$ for the representation of the external terms in the Lagrange's and Hamilton's equations that are completely missing in the Copenhagen interpretation of quantum axioms.

6.1-IV) Hypermechanics (see Ref. [27] [28] and Tutoring Lecture V [146]) characterized by the formulation of genomechanics in terms of

T. Vougiouklis H_v hyperstructures [233] for the representation of irreversible systems with a large number of extended particles in deep EPR entanglement.

6.1-V) Isodual quantum, iso-, geno- and hyper-mechanics [29] for the representation of antiparticles in condition of increasing complexity via an anti-Hermitean map (called *isoduality* and denoted with the upper index d) of the preceding time evolutions, such as

$$i^d \times^d \frac{d^d A^d}{d^d t^d} = [A^d, H^d]^d = A^d \times^d H^d - H^d \times^d A^d, \tag{15}$$

providing a causal representation of antimatter with *negative energy* according to P. A. M. Dirac [11] because necessary to represent particle-antiparticle annihilation (see Section 3.4 and Figure 6, contribution [227] to the 2020 Teleconference and the recorded lecture by A. S. Muktibodh [0]).

6.2. Hadronic chemistry.

Following the achievement of mathematical [23] and physical [24] maturity, as well as a number of experimental verifications [25] [26], hadronic mechanics was applied to chemistry, resulting in a completion-covering of hadronic chemistry for the representation of molecules composed by extended constituents in condition of EPR entanglement, called **hadronic chemistry** [30] which is also axiom-preserving, thus implying that *quantum and hadronic chemistry coincide at then abstract realization-free level.*

The most important function of hadronic chemistry is to show that the basic open problems in chemistry listed in Section 3.5 cannot be solved via quantum chemistry due to its linearity, locality and potentiality. By contrast, the achievement of an attractive force between identical valence electrons in molecular bonds has been possible [30] following the admission of the "element of reality" [1] given by the non-linearity, non-locality and non-potentiality of valence electron bonds (Figure 7). This new strong valence bond has permitted the achievement of exact representations of experimental data for the Hydrogen [31], water [32] and other molecules.

6.3. Hadronic biology.

Studies in the field were initiated by C. R. Illert [176] who proved that a conventional three-dimensional Euclidean space $E(r, \delta, I),\ r = (x, y, z),\ \delta = I = Dag.(1, 1, 1)$ can indeed represent all possible *shapes* of seashells, but the *representation of their growth in time requires a three-dimensional two-valued space* here denoted $\hat{E}(\hat{r}, \delta, I),\ \hat{r} = (\hat{r}_k),\ \hat{r}_k = (r_k^1, r_k^2), k = 1, 2, 3.$

In order to initiate the expectedly long process to represent the extreme complexity of the DNA code, and by recalling that all living organisms are irreversible over time, thus requiring genomathematics, Santilli introduced the representation of biological structures via three-dimensional multi-valued geno-Euclidean genospaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ over genofields with genounits having an arbitrary number N of ordered elements, $\hat{I} = Diag.(I_1, I_2, ..., I_N)$ [34], with classical operations, e.g., the genoproduct of two observables $A > B$ yields N ordered results [27].

T. Vougiouklis formulated the preceding results via the reformulation of genomathematics in terms of his H_v-hyperstructures on a hyperfield formulated in terms of hyperoperations called *hopes* (see Tutoring Lecture V [146] and Ref. [233]).

Vougiouklis' hyperformulation allowed the introduction at the 2020 Teleconference [0] of a new conception of living organisms presented in Ref. [231] as being composed by a very large number of molecules all in EPR entanglement, thus being in continuous and instantaneous communication, resulting in such a complex structure that can only be quantitatively represented with the most complex mathematics developed by the human mind.

It is significative that the 2020 Teleconference [0] ended up with the view: *There is hope that 'hope' can represent life.*

7. VERIFICATIONS OF THE EPR ARGUMENT.

7.1. Foreword.

A main topic of the 2020 Teleconference [0] has been the study of *isotopic verifications* [210]-[214] of the EPR argument [1], namely, verifications based on the preservation of the basic axioms of quantum mechanics and their most general possible realization which verifications are the outcome of the life-long research by R. M. Santilli with contributions by numerous scholars.

In this section, we review the indicated isotopic verifications in a language as simple as possible, including a rudimentary review of the needed basic formalism for minimal self-sufficiency of this Overview, as well as for the specialization of the formalism to the EPR problem.

A central concept of this section is that *extended* particles immersed within a hyperdense medium, as it is the case for a proton in the core of a star, experience a *pressure in all radial directions* proportional to the density of the medium (Figure 8) which evidently restricts Heisenberg's uncertainties in favor of Einstein's determinism [1]. Note that such a notion of pressure is completely absent in the 20th century physics due to the point-like approximation of particles.

With the understanding that the participants to the 2020 Teleconference are available for consultations, including this author, it should be indicated that a full understanding of the isotopic verifications reviewed in this section requires a technical knowledge of hadronic mechanics [24].

7.2. Recovering of classical images.

7.2.1. Rudiments of isomathematics. As recalled earlier, Bell [3] proved the lack of existence of classical images for systems of quantum mechanical point-like particles with spin $1/2$.

Since the systems studied by Bell are described by the time-reversible Heisenberg law (9)-(10) (Sections 3 and 6), Santilli studied in the 1998 paper [210] systems characterized by isomathematics (Tutoring Lecture

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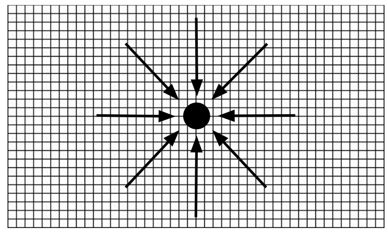

Figure 8: *This figure provides a conceptual rendering of the central notion used for the verifications of the EPR argument, namely, the inward 'pressure' experienced by 'extended' particles in 'all radial directions' when immersed in hyperdense media, such as a proton in the core of a star, which pressure evidently restricts uncertainties in favor of Einstein's determinism first identified in the 1981 paper [47] and then studied in detail in Refs.[210]-[214]. Note that the above notion of pressure did not exist in 20th century physics due to the approximation of particles as being point-like.*

I [143]) and related isomechanics [23] [24] with iso-Heisenberg's time evolution (11)-(12) for which the represented systems are equally time-reversible, yet they admit internal non-linear, non-local and non-potential interactions represented with the isotopic element $\hat{T}$.

Additionally, Bell [3] used the conventional $SU(2)$-spin symmetry for the derivation of his result. Consequently, Santilli constructed the Lie-isotopic completion of the various branches of Lie's theory, including the isotopies of universal enveloping associative algebras, Lie algebras, and Lie groups [22], resulting in a theory nowadays known as the *Lie-Santilli isotheory* [21]-[25] [50] [55] [201] (see also contribution [226] to the 2020 Teleconference and the recorded lecture by A. S. Muktibodh [0]).

The foundation of the Lie-Santilli isotheory is given by the *universal isoenveloping isoassociative isoalgebra* with *isoproduct* between arbitrary quantities A, B (first introduced in Eq.(5), page 71, Ref. [22])

$$A \hat{\times} B = A \hat{T} B, \hat{T} > 0, \tag{16}$$

where $\hat{T}$, called the *isotopic element,* is solely restricted by the condition of being positive-definite, but can otherwise possess an arbitrary (non-singular) dependence on all needed local variables of interior dynamical problems, such as a dependence on time t, coordinates r, momentum p, energy E, density μ, temperature τ, pressure γ, wavefunctions ψ, their derivatives $\partial_r\psi$, etc. $\hat{T} = \hat{T}(t, r, p, E, \mu, \tau, \gamma, \psi, \partial\psi, ...)$, which dependence is hereon omitted for brevity.

It is evident that product (16) verifies the axioms of associativity and distributivity, although in the following more general form

$$(A \hat{\times} B) \hat{\times} C = A \hat{\times} (B \hat{\times} C),$$

$$A \hat{\times} (B + C) = A \hat{\times} B + A \hat{\times} C, \tag{17}$$

for which reason it is called the isoproduct.

It should be indicated upfront that *the primary function of isoproduct (16) is that of providing novel explicit and concrete realizations of Bohm's hidden variables [18] via the the isotopic element $\hat{T}$ which is "hidden" in the associative and distributive laws as clearly indicated by Eqs. (17).*

Santilli realized in the early 1978 that, despite its simplicity, isoproduct (16) requires for consistency the isotopic completions of *all* aspects of 20th century applied mathematics with no known exception. As an illustration of the Lie-Santilli isoalgebra, we indicate the isotopic completion $\hat{L}$ of an N-dimensional Lie algebra L with Hermitean generators $X_k, k = 1, 2, ..., N$, characterized by the *isocommutator rules*

$$[\hat{X}_i \hat{,} \hat{X}_j] = \hat{X}_i \hat{\times} \hat{X}_j - \hat{X}_j \hat{\times} \hat{X}_i =$$

$$= \hat{C}_{ij}^k(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) \hat{\times} \hat{X}_k, \tag{18}$$

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whose verification of Lie algebra axioms is evident. Nowadays, Lie-Santilli isoalgebras are classified into *regular* when the structure quantities $\hat{C}$ are constant, and *irregular* when they are function of the local variables [55].

It should be noted, as illustrated below, that regular Lie-Santilli isoalgebras can be obtained via non-unitary transformations of the original Lie algebras, while irregular isoalgebras cannot, thus constituting new realizations of Lie's axioms.

The elaboration of isocommutation rules (18) with the conventional mathematics used for Lie's theory soon proved to lead to serious inconsistencies. This occurrence mandated the construction of isomathematics with the following main features:

1) The multiplicative unit of quantum mechanics $\hbar = 1$ is no longer invariant for isomechanics due to its non-unitary structure, Eq. (12). The consistent multiplicative unit under isoproduct (16) is given by the *isounit*

$$\hat{I} = 1/\hat{T}, > 0 \;\; \hat{I}\hat{\times}A = A\hat{\times}\hat{I} = A; \tag{19}$$

2) The isotopy of the unit $1 \rightarrow \hat{I}$ mandates, for consistency, the corresponding isotopic completion of numeric fields $F(n, \times, I)$ into *isofields* $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$, first introduced in Ref. [33] of 1993, with *isonumbers* $\hat{n} = n\hat{I}$ where $n \in F$ equipped with isoproduct (16) and isounit (19);

3) The elaboration of the Lie-Santilli isotheory needs to be done, also for consistency, via the *isofunctional isoanalysis* initiated in Refs. [47] [48] of 1992 (see Refs. [23] [24] for a general treatment) with expressions, for instance, for the *isoexponent*

$$\hat{e}^X = (e^{X\hat{T}})\hat{I} = \hat{I}(e^{\hat{T}X}); \tag{20}$$

4) Recall that non-unitary transformations on conventional spaces over conventional fields generally violate causality[24] [212]. To regain causal-

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ity, all non-unitary transformations can be reformulated into the *isounitary isotransforms*

$$U = \hat{U}\hat{T}^{1/2},\ \ UU^\dagger \neq I\ \rightarrow\ \hat{U}\hat{\times}\hat{U}^\dagger = \hat{U}^\dagger\hat{\times}\hat{U} = \hat{I}, \tag{21}$$

for which the non-unitary time evolution(12) is identically reformulated in the correct isounitary form

$$A(t) = \hat{W}\hat{\times}A(0)\hat{\times}\hat{W}^\dagger = \hat{e}^{Hti}\hat{\times}A(0)\hat{\times}\hat{e}^{-itH},$$

$$\hat{W}\hat{\times}\hat{W}^\dagger = \hat{W}^\dagger\hat{\times}\hat{W} = \hat{I}; \tag{22}$$

5) For consistency, scalar quantities must be elements of isofields, with ensuing expression for *isocoordinates* $\hat{r} = r\hat{I}$ and generic expression for *isofunction*

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}; \tag{23}$$

6) Also for consistency, Lie-Santilli isoalgebras must be formulated on *isospaces over isofields,* (first formulated in Ref. [34] with reference, as an illustration, for non-relativistic formulations to the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ with *isocoordinates* $\hat{r} = r\hat{I},\ r = (x, y, z)$, *isometric* $\hat{\delta} = (\hat{T}\delta)\hat{I}$ where $\delta = Diag.(1, 1, 1)$ and *isoinvariant* as an element of the isofield of isoreal isonumbers $\hat{\mathcal{R}}$

$$\hat{r}^{\hat{2}} = \hat{r}^i\hat{\times}\hat{\delta}_{i,j}\hat{\times}\hat{r}^j = (\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2})\hat{I}, \tag{24}$$

and the *iso-Minkowski isospace* for relativistic treatments$\hat{M}(\hat{x}, \hat{\eta}, \hat{I})$, with *space-time isocoordinates* $\hat{x} = x\hat{I} = (x, y, z, t)\hat{I}$, *space-time isometric* $\hat{\eta} = \hat{T}\eta,\ \eta = Diag.(1, 1, 1, -1)$ and *space-time isoinvariant*

$$\hat{x}^{\hat{2}} = \hat{x}^\mu\hat{\times}\hat{\eta}_{\mu\nu}\hat{\times}\hat{x}^\nu = (\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} - t^2\frac{c^2}{n_4^2})\hat{I}; \tag{25}$$

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7) Since the multiplicative isounit is generally dependent on local isocoordinates, $\hat{I} = \hat{I}(\hat{r}, ...)$, the elaboration of Lie-Santilli isoalgebras via the conventional Newton-Leibnitz calculus leads to axiomatic inconsistencies (see also Section 7.3), and must be lifted into the *isodifferential isocalculus,* first introduced by Santilli in Ref. [34] (see also monographs. [207], contributions [222] [223] to the 2020 Teleconference, and the recorded lectures by S. Georgiev [0]), with basic *isodifferential*

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}' \tag{26}$$

and *isoderivatives*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}; \tag{27}$$

8) Contrary to a popular belief that isomathematics is excessively complex for physicists, *all aspects of (regular) isomathematics can be very easily constructed via a non-unitary transform of the corresponding aspect of 20th century applied mathematics.* For instance, the trivial unit 1 can be transformed into the isounit $\hat{I}$, the conventional associative product AB can be transformed into the isoproduct $A\hat{T}B$, etc. [35]

$$1 \to U1U^\dagger = \hat{I} = 1/\hat{T} \neq I, \;\; \hat{T} = (UU^\dagger)^{-1},$$

$$AB \to U(AB)U^\dagger = (UAU^\dagger)(UU^\dagger)^{-1}(UBU^\dagger) = A'\hat{T}B',$$

$$E^X \to \hat{e}^X = (e^{X\hat{T}})\hat{I} = \hat{I}(e^{\hat{T}X}), etc.; \tag{28}$$

9) The crucial invariance of the numeric values of the isounit and isotopic element representing non-Hamiltonian interactions is verified by the appropriate time evolution, the isounitary form (21)-(22) [35]

$$\hat{I} \to \hat{U}\hat{\times}\hat{I}\hat{\times}\hat{U}^\dagger = \hat{I}' \equiv \hat{I},$$

$$A\hat{\times}B = A\hat{T}B \to \hat{U}\hat{\times}(A\hat{\times}B)\hat{\times}\hat{U}^\dagger = A'\hat{\times}'B' = A'\hat{T}'B', \;\; \hat{T}' \equiv \hat{T}. \tag{29}$$

Following the isotopic completion of Lie's theory in the period 1978-1982 [19]-[22], the step-by-step isotopies of conventional space-time mathematics, Santilli constructed in Refs. [60]-[70] the isosymmetries leaving invariant isospacetime (26).

These studies established that, rather then being violated for line element (25), the Lorentz symmetry remains valid because its isotopic image $\hat{SO}(3.1)$ is isomorphic to the original symmetry $SO(3.1)$ [60], and the same holds for all continuous as well as discrete space-time symmetries [23] [24].

We here merely recall for future comments: the *Lorentz-Santilli isotransformations* introduced in the 1983 paper [60], here indicated in the $(3,4)$-plane (see Ref. [24] for their full formulation)

$$x^{1'} = x^1, \ \ x^{2'} = x^2,$$
$$x^{3'} = \hat{\gamma}(x^3 - \hat{\beta}\frac{n_3}{n_4}x^4), \ \ x^{4'} = \hat{\gamma}(x^4 - \hat{\beta}\frac{n_4}{n_3}x^3), \tag{30}$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \ \ \hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}; \tag{31}$$

the second order iso-Casimir invariant of the *Lorentz-Poincaré-Santilli isosymmetry* [66]-[69]

$$\hat{C}_2 = (\hat{\eta}^{\mu\nu}P_\mu P_\nu)\hat{I}^c =$$
$$(n_1^2P_1^2 + n_2^2P_2^2 + n_3^2P_3^2 - n_4^2P_4^2)\hat{I}^c = m^2C^2, \tag{32}$$

where, from line element (25), C s the local speed of light left invariant by isosymmetry (30),

$$C = \frac{c}{n_4}, \tag{33}$$

and $\hat{I}^c$ is now the *contravariant isounit,* thus being equal to $\hat{T}$,

$$\hat{I}^c = \hat{T} = 1/\hat{T}^c = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}\frac{1}{n_4^2}); \tag{34}$$

and the *iso-Klein-Gordon isoequation* [24] derivable from iso-Casimir invariant (32) which, for the simplest possible assumptions $n_k = 1, \ k = 1, 2, 3$, $\hbar = 1$ and m_4 is a positive quantity given by

$$(-\frac{1}{c}\frac{\partial^2}{\partial t^2} + \frac{1}{n_4^2}\bigtriangledown)\hat{T}^c\hat{\psi}(\hat{x}) =$$

$$= (\frac{m^2}{n_4^2}\frac{c^2}{n_4^2})\hat{T}^c\hat{\psi}(\hat{x}) = (\bar{m}^2c^2)\hat{T}^c\hat{\psi}(\hat{x}), \tag{35}$$

where

$$\bar{m} = \frac{m}{n_4^2} \tag{36}$$

is the *isorenormalized energy* of an extended particle within a physical medium with $n_1^2 = n_2^2 = n_3^2 = 1$ holding for the particular case in which the particle is perfectly spherical and the medium is homogeneous and isotropic [24] (see the formal treatment in Section 8.4.4, Eq. (133) of Isoaxiom IV).

Finally, the isotopies $\hat{SU}(2)$ of the $SU(2)$-spin symmetry were constructed in Refs. [64] [65] (see the review in Section 3, Ref.[213]).

7.2.2. Recovery of classical images. Thanks to the above advances, Santilli introduced in paper [210] the following simplest possible realization of Bohm's hidden variables

$$\hat{U}U^\dagger = Diag(\lambda, 1/\lambda) \neq I, \quad Det\hat{I} = 1, \quad \lambda > 0, \tag{37}$$

applied to Pauli's matrices

$$\sigma_k \rightarrow \hat{\sigma}_k = U\sigma_kU^\dagger, \tag{38}$$

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with explicit form

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}, \tag{39}$$

and regular isoalgebra

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i \hat{T} \hat{\sigma}_j - \hat{\sigma}_j \hat{T} \hat{\sigma}_i = i2\epsilon_{ijk}\hat{\sigma}_k, \tag{40}$$

establishing the isomorphism $\hat{SU}(2) \approx SU(2)$, and *isoeigenvalue equations* (where all products, thus including squares, are isotopic)

$$\hat{\sigma}_3 \hat{\times} |\hat{b}>= \pm|\hat{b}>, \tag{41}$$

$$\hat{\sigma}^{\hat{2}} = (\hat{\sigma}_1 \hat{T} \hat{\sigma}_1 + \hat{\sigma}_2 \hat{T} \hat{\sigma}_2 + \hat{\sigma}_3 \hat{T} \hat{\sigma}_3)\hat{T}|\hat{b}>= 3|\hat{b}. \tag{42}$$

thus confirming the spin $1/2$ for *isoparticles,* namely, extended particles in deep EPR entanglement which are characterized by isounitary isoirreducible isorepresentation of the Lorentz-Poincaré-Santilli isosymmetry $\hat{P}(3.1)$ [66]-[69] (see Section 2 of paper [213] for a review).

Bell's proof of the lack of existence of classical images for two particles with spin $1/2$ is reduced a quantum mechanical (qm) quantity D^{qm} which, when computed via the use of Pauli's matrices, verifies the expression

$$D^{qm} \leq 2. \tag{43}$$

Santilli conducted in paper [210] a step-by-step *non-unitary/isounitary,* transformation of Bell's derivation along rules (28) for two isoparticles described by hadronic mechanics (hm) with Bohm's hidden variables λ_1, λ_2 in realization (37), resulting in the following property

$$D^{hm} = \frac{1}{2}(\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1})D^{qm}. \tag{44}$$

Since the factor $\frac{1}{2}(\lambda_1/\lambda_2 + \lambda_2/\lambda_1)$ can assume values arbitrarily larger than 2, Santilli proved in Ref. [210] that *a systems of isoparticles with spin* $1/2$ *in condition of mutual penetration, thus EPR entanglement and ensuing non-Hamiltonian interactions does admit classical images* with specific examples identified in paper [210].

The invariance of the results under the the time evolution of the theory was also proved according to rules (39).

7.2.3. Realization of the EPR entanglement. It should be noted that realization (28) of hidden variables is indeed the most elementary possible because the notion of EPR entanglement (Figure 2) is characterized by the isotopic element $\hat{T}$ with realization for the hadronic bound state of two isoparticles (such as the Deuteron) of the type

$$\hat{T} = \Pi_{k=1,2} Diag.(1/n_{1k}^2, 1/n_{2k}^2, 1/n_{3k}^2, 1/n_{4k}^2) \times$$
$$\times exp[-\Gamma(\psi, \hat{\psi}, ...)] > 0, \tag{45}$$

where $n_{k\alpha}^2, , \alpha = 1, 2, 3, k = 1, 2$, where: $n_{\alpha k}^2 = 1$ represents the deformable semi-axes of the extended k-particle normalized to the values for the sphere; n_{4k}^2 represents the density of the k-particle normalized to the value $n_{4k}^2 = 1$ for the vacuum; and the exponent is a positive-definite function representing non-linear, non-local and non-potential interactions with realizations of the type

$$exp[-\Gamma(\psi, \hat{\psi}, ...)] = exp[-|\psi/\hat{\psi}| \int \hat{\psi}_1 \hat{\psi}_2^\dagger d^3r,] \tag{46}$$

where ψ is a quantum mechanical wave function, and $\hat{\psi}$ is the completed wavefunction under isotopy.

In conclusion, Ref. [210] has shown that the historical intuition of *hidden variable* by David Bohm [18] is truly fundamental because it characterizes the completion of quantum into hadronic mechanics resulting in a basically new notion of particle entanglement with momentous applications indicated in the next sections.

The cosmological implication of the EPR argument based in the classification of elementary particles via Dinkins diagram was presented by the 2020 Teleconference [0] by E. Trell, See contributed paper [232].

7.3. Recovering of Einstein's determinism.

7.3.1. Rudiments of isomechanics. The proof of the recovering of Einstein's determinism achieved in the 2019 paper [211] requires the use of isomathematics outlined in the preceding section plus the use of isomechanics [24] whose basic elements can be briefly outlined for the non-initiated reader as follows:

1) The basic unit of isomechanics is given by the completion of Planck's unit $\hbar = 1$ into the isounit $\hat{I}$ which is the inverse of the iso' topic element (45)

$$\hat{I} = 1/\hat{T} = \Pi_{k=1,2} Diag.(n_{1k}^2, n_{2k}^2, n_{3k}^2, n_{4k}^2) > 0, \tag{47}$$

where exponent (46) is incorporated into the n-characteristic quantities. Isounit (47) represents the *volume* and *density* of isoparticles as well as the impossibility for *extended* protons in the core of a star to solely have discrete energy exchanges due to the extreme local densities and pressures, in favor of integro-differential energy exchanges;

2) The Lie-Santilli isotheory has be formulated on a *Hilbert-Myung-Santilli isospace* [45] $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [33];

3) Isostates $\hat{\psi}(\hat{r}) \in \hat{\mathcal{H}}$ should have *isonormalization*

$$< \hat{\psi}(\hat{r})|\hat{\times}|\hat{\psi}(\hat{r}) >=< \hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) >=$$

$$= \int_{-\infty}^{+\infty} \hat{\psi}^{\dagger}(\hat{r})\hat{T}\hat{\psi}(\hat{r})\hat{d}\hat{r} = \hat{T}, \tag{48}$$

were one should note the need for isointegrals and isodifferentials (23);

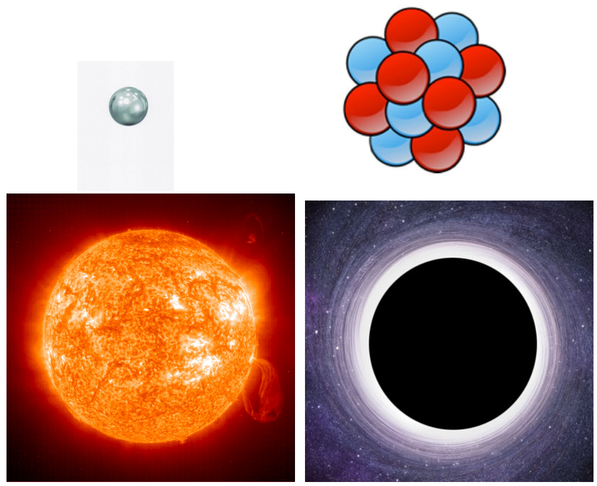

Figure 9: *Experimental evidence establishes that the wavepacket of particles has approximately the same size* 10^{-13} *cm as that of all hadrons (top left). Consequently, hadrons are composed by wavepackets in deep EPR entanglement (Sections 3.1 and 7.2.3) called 'hadronic medium.' A corresponding structure holds for nuclei, stars and black-holes due to partial or deep mutual penetration of hadron constituents. The realization via the isotopic element of hadronic mechanics of the pressure exercised by the hadronic medium on individual constituents (Figure 8) implies the progressive weakening of Heisenberg's uncertainties first identified in Ref. [47] of 1981, with progressive recovering of Einstein's determinism [1] in the structure of hadrons, nuclei and stars and its full recovering at the limit of gravitational collapse studied in detail in Refs. [210]-[214]. Note that Heisenberg's uncertainties remain valid for the center of mass of hadrons, but not so for black-holes, thus illustrating the indicated full recovering of Einstein's determinism.*

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4) The *isoexpectation values* of an observable A are given by

$$\hat{<}A\hat{>} = <\hat{\psi}(\hat{r})|\hat{\times}A\hat{\times}|\hat{\psi}(\hat{r})> = <\hat{\psi}(\hat{r})|\hat{T}A\hat{T}|\hat{\psi}(\hat{r})>; \tag{49}$$

5) The realization of the linear momentum on isospaces over isofield via the conventional differential calculus leads to major inconsistencies. The correct formulation of the *isolinear isomomentum* is that via isoderivative (27) and it is given by

$$\hat{p}\hat{\times}\hat{\psi}(\hat{r}) = -\hat{i}\hat{\times}\hat{\partial}_{\hat{r}}\hat{\psi}(\hat{r}) = -i\hat{I}\partial_{\hat{r}}\hat{\psi}(\hat{r}), \tag{50}$$

thus providing a forceful illustration of the need for the novel isodifferential calculus [34];

6) The *isocanonical isocommutation rules* are consequently given by

$$[r^i \hat{,} p_j] = \hat{i}\hat{\times}\hat{\delta}^i_j = i\hat{I}\delta^i_j, \quad [r^i \hat{,} r^j] = [p_i \hat{,} p_j] = 0, \tag{51}$$

by illustrating that isomechanics is *isolinear*, that is, linear on isospace over isofields because $[p_i \hat{,} p_j] = 0$, although the theory is highly non-linear when *projected* into our conventional spaces over conventional fields because, in that case, $[p_i, p_j] \neq 0$;

7) The *iso-Schrödinger isoequation* is then given by [20]-[24]

$$\hat{H}\hat{\times}\hat{\psi}(\hat{r}) = [-\frac{1}{2m}\hat{\triangle}_{\hat{r}} + \hat{V}(\hat{r})]\hat{\times}\hat{\psi}(\hat{r}) = \hat{E}\hat{\times}\hat{\psi}(\hat{r}) = E\hat{\psi}(\hat{r}), \tag{52}$$

and now represents extended, deformable and hyperdense particles at all levels, including isocoordinates $\hat{r}$, isostates $\hat{\psi}(\hat{r})$, isotopic element $\hat{T}$, isopotentials $\hat{V}(\hat{r})$ iso-Laplacian $\hat{\triangle}_{\hat{r}}$, etc. In particular, isostates verify the isosuperposition principle on isospaces over isofields

$$\hat{\psi}(\hat{r}) = \Sigma_{k=1,2}\hat{\psi}_k(\hat{r}), \tag{53}$$

because the theory is isolinear as indicated above. Therefore, hadronic mechanics resolves the inability by Heisenberg's non-linear completion of quantum mechanics [16] to represent the constituents of a bound state of particles with non-linear internal forces discussed in Section 5.

7.3.2. Confirmation of the EPR argument. It is evident that the recovering of classical images in paper [210] established the foundation for the recovering of Einstein's determinism. These studies were initiated by Santilli in the 1981 paper [47], continued in various papers due to the need to achieve maturity in the formulation of isomathematics (see monographs [23] [24]), and concluded in the 2019 paper [211] via the following simple isotopy of the conventional derivation of Heisenberg's uncertainties

$$\Delta r \Delta p = \frac{1}{2}| < \hat{\psi}(\hat{r})|\hat{\times}\,[\hat{r},\hat{p}]\,\hat{\times}|\hat{\psi}(\hat{r}) > \approx \frac{1}{2}\hat{T} \ll 1, \tag{54}$$

where the property $T \ll 1$ is established by all fits of experimental data to date [25] [26] (see Section 7.6 for the analytic aspects).

By recalling that the value of the isotopic element is inversely proportional to the increase of the density [211], isodeterministic principle (47) establishes *the progressive validity of Einstein's determinism in the interior of hadrons, nuclei and stars, and its full achievement in the interior of gravitational collapse* (Figure 9).

The latter result is due to the fact that the isotopic element admits a realization in terms of the space component of Schwartzchild's metric [211]

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \tag{55}$$

where M is the gravitational mass of the body considered with ensuing isodeterministic isoprinciple

$$\Delta\hat{r}\Delta\hat{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r \to 0} = 0. \tag{56}$$

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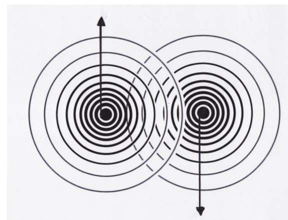

Figure 10: *A conceptual rendering of the 'attractive' force between 'identical' electrons in valence pairs first identified in Chapter 4 of the 2001 Ref. [30] against their very big Coulomb repulsion, Eq. (6), thanks to the representation of the extended character of the electron wavepackets via isomathematics and isochemistry with ensuing total EPR entanglement of the electrons. Note that this achievement proves the last statement of the EPR argument to the effect that quantum wavefunctions cannot represent 'all elements of reality' [1].*

Note that the center-of-mass of hadrons and nuclei verifies quantum uncertainties while that for stars and gravitational collapse verifies classical determinism.

Note also that isodeterministic principle (54) holds for the infinite class of isounitary time evolutions of the theory, Eqs. (29).

7.4. Achievement of attractive force between valence electrons.

Einstein, Podolsky and Rosen ended their historical paper [1] with the statement: *"While we have thus shown that the wavefunction does not provide a complete description of the physical [and we add chemical] reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible."*

In Section 3.5, we indicated that a primary insufficiency of quantum chemistry is the lack of an *attractive* force between the *identical* electrons in valence coupling, Eq. (5), and that their *repulsive* force is so big, Eq. (6), to prevent any realistic hope to achieve an attractive force via quantum chemistry due to its linear, local, Hamiltonian and unitary structure.

In monograph [30] of 2001, particularly Chapter 4, Santilli achieved the needed attractive force between identical valence electrons via a *non-unitary/isounitary completion of the wavefunction of quantum mechanics* which is a direct verification of the final EPR statement quoted above (see Figure 10 and Section 5 of paper [214]).

The above result can be outlined as follows. Assume that the electrons are perfectly spherical for which $n_k^2 = 1,\ k = 1, 2, 3$, and that their density is ignorable for which $n_4^1 \approx 1$. Consider then a non-unitary transformation with the following simple realization of Eqs. (45)(46)

$$\hat{I} = UU^\dagger = e^{\frac{V_{hm}}{V_{qm}}} \approx 1 + \frac{V_{hm}}{V_{qm}}$$

$$\hat{T} = (UU^\dagger)^{-1} = e^{-\frac{V_{hm}}{V_{qm}}} \approx 1 - \frac{V_{hm}}{V_{qm}} \tag{57}$$

where V_{hm} is a quantum mechanical (qm) potential, for instance, the *repulsive* Coulomb potential, and V_{hm} is a strongly *attractive* potential of hadronic mechanical (hm), such as the Hulten potential,

$$V_{qm} = +\frac{e^2}{r}$$

$$V_{hm} = -W\frac{e^{-br}}{1 - e^{-br}}, \tag{58}$$

where W is a normalization constant.

It is then easy to see that *a non-unitary transformation can turn the quantum mechanical repulsion between valence electrons into the attraction needed to represent the "element of reality" given by valence bonds*

$$UV_{qm}U^\dagger \approx V_{qm}(1 + \frac{V_{hm}}{V_{qm})} = V_{qm} + V_{hm} =$$

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$$= +\frac{e^2}{r} - W\frac{e{-}br}{1-e^{-br}} \approx K\frac{e^{-br}}{1-e^{-br}} \quad (59)$$

where the last step is due to the fact that the Hulten potential behaves at short distances like the Coulomb potential [20]. However, the former is much stronger than the latter, by therefore allowing the absorption of the Coulomb potential into the Hulten potential irrespective of whether the Coulomb potential is attractive or repulsive. In this way, the final equation solely shows the Hulten potential with the new, positive, renormalization constant K (see page 833 of Ref. [20], Chapter 4 of Ref. [30], and paper [56]).

To achieve the corresponding iso-Schrödinger equation, recall rules (28) for the nonunitary transformation of the linear into the isolinear momentum

$$Up\psi(r)U^\dagger = UpU^\dagger(UU^\dagger)^{-1}U\psi(r)U^\dagger = \hat{p}\hat{\times}\hat{\psi}(\hat{r}) =$$

$$= -\hat{i}\hat{\times}\frac{\hat{\partial}}{\hat{\partial}\hat{r}}\hat{\psi}(\hat{r}) = -i\hat{I}\partial_{\hat{r}}\hat{\psi}(\hat{r}). \quad (60)$$

Under the above properties, the application of non- unitary transformation (57) to the conventional Schrödinger equation (5) for the valence electron pair with repulsive force then yields the iso-Schrödinger equation for their attraction as in the physical reality, which we write in its projection into the conventional Hilbert space over the field of complex numbers (see Section 2 of Ref. [214] for its detailed derivation)

$$(-\frac{1}{\bar{m}}\hat{\Delta}_r - Ke^{\frac{-br}{1-e^{-br}}})\hat{\psi}(r) = E\hat{\psi}(r), \quad (61)$$

where v$\bar{m}$ is the reduced form of isorenormalized masses, Eq. (56).

The above results confirm the final statement by Einstein, Podolsky and Rosen quoted above because the $wavefunction$ $\psi(r)$ of quantum chemistry cannot represent the physical reality of the attraction between the identical valence electrons in molecular bonds in favor of its completion into the isowavefunction $\hat{\psi}(\hat{r})$ of hadronic chemistry. A more recent derivation of the attraction between valence electrons is available in Section 2.8 of paper [214].

It should be indicated that *valence electron bonds appear to be one of the most significant realizations of the EPR entanglement (Figures 2 and Section 7.2.3) apparently occurring at the limit of null mutual distance* with a considerable broadening of the implications of entanglement in physics, chemistry and biology. In fact, by merely admitting the evidence, we learn that the entanglement of wavepackets allows particle to influence each other at a distance, we learn from isotopic element (37)-(39) that entanglement plays a crucial role in nuclear forces (see also Section 8), and we now learn that, at the limit of null mutual distance, entanglement may alter the very characteristics of particles, a feature called *mutation,* which is evidently needed to turn a natural repulsion into an attraction.

It may be of some interest to note that the indicated mutation of valence electrons can be quantitatively represented by the transition from *particles* characterized by a unitary irreducible representation of the spinorial covering of the Poincaré symmetry $\mathcal{P}(.)$ into *isoparticles* characterized by isounitary isoirreducible isorepresentations of the Lorentz-Poincaré-Santilli-isosymmetry $\hat{\mathcal{P}}(.)$ [66] - [69] (see reviews Ref. [213]).

In fact, the transition from $\mathcal{P}(.)$ to $\hat{\mathcal{P}}(.)$ implies, in general, a mutation of all *intrinsic* characteristics of entangled particles also called *isorenormalization* because caused by the isotopic element representing non-Hamiltonian interactions. When the entanglement is limited, as occurring in a nuclear structure, the spin $1/2$ of the constituents is maintained, but when the entanglement is total, as occurring in valence bonds, all characteristics of the particles are expected to be mutated, including the charge.

7.5. Removal of quantum divergencies.

Recall that the divergencies of quantum mechanics originate on the sin-

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gularity of Dirac's delta function

$$\delta(r) = \frac{1}{2\pi} \int_{-} \infty^{+} \infty e^{ikr} dk, \tag{62}$$

at the origin $r = 0$.

It is easy to see that there exist an isotopic element for which the singularity at the origin $r = 0$ is removed for the *Dirac-Myung-Santilli isodelta isofunction* [45] (see also Refs. [47] - [51]), as illustrated by the following simplest possible formulation

$$\hat{\delta}(r) = \frac{1}{2\pi} \int_{-} \infty^{+} \infty \hat{e}^{ikr} \hat{d}k = \frac{1}{2\pi} \int_{-} \infty^{+} \infty e^{i\hat{T}kr} dk. \tag{63}$$

Similarly, recall that the isotopic product $A \star B = A\hat{T}B$ must be applied to the *totality* of the products, thus including all products appearing in perturbative series. But the isotopic element $\hat{T}$ has very small values in all known applications. Consequently, perturbative series that are generally divergent in quantum mechanics and chemistry are turned under isotopy into strongly convergent series. This is illustrated by fact that, given a divergent perturbative series

$$A(t) = A(0) + (AH - HA)/1! + ... \to \infty, \tag{64}$$

there always exist a value of the isotopic element $\hat{T}$ causing the strong convergence of the isotopic series

$$A(t) = A(0) + (A\hat{T}H - H\hat{T}A)/1! + ... = K \ll \infty, \ \ \hat{T} \ \ll \ 1. \tag{65}$$

Consequently, *the validity of Einstein's[determinism per Section 7.3 implies the removal of quantum mechanical divergencies* (see Corollary 3.7.1, page 128, Ref. [213]).

The actual verification of the above important property has been provided by D. D. Shillady and R. M. Santilli in papers [31] [32] with the

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proof that *the perturbative series of hadronic chemistry converge at least one thousand times faster than the corresponding quantum chemical series.*

In Santilli's view, the above rapid convergence of calculations for the EPR entanglement is an important advantage over the conventional quantum entanglement, particularly for applications such as that for computers based on the entanglement of electrons.

Note that the new computers are called in this Overview *EPR computers,* instead of 'quantum computers', due to the impossibility by quantum mechanics to provide a representation of the entanglement of electrons discussed throughout this work.

7.6. Representation of irreversible processes.

As indicated in Section 3.3 and Figure 5, the insufficiency of quantum mechanics that stimulated Santilli's lifelong research on the completion of quantum mechanics according to the EPR argument [1] is the inability to represent time-irreversible events, including energy-releasing processes ,due to the invariance under anti-Hermiticity of the Lie brackets for Hermitean operators , $[A, B] = -[A, B]^{\dagger}$.

The above occurrence stimulated the construction of the Lie-admissble branch of hadronic mechanics [6]-[9], [19]-[25], [38]-[39], with time evolution (13)-(14) (see Section 2 of Ref. [212] for a review).

Discussions during the 2020 Teleconference [0] established that genotopic elements $\hat{R}$ and $\hat{S}$ represent the external terms $F > 0$ in Lagrange's and Hamilton's equations

$$\frac{d}{dt}\frac{\partial L(t,r,v)}{\partial v} - \frac{\partial L(t,r,v)}{\partial r} = F(t,r,v),$$
$$\frac{dr}{dt} = \frac{\partial H(t,r,p)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H(t,r,p)}{\partial r} + F(t,r,p). \tag{66}$$

In fact, in his Ph. D. thesis of the mid l960s [6]-[8], Santilli accepted the EPR argument [1] on the lack of completeness of quantum mechanics because quantum mechanics cannot represent the external terms F of

Lagrange's and Hamilton's analytic equations which is the analytic representation of the irreversibility of nature.

Santilli noted that the brackets (A, F, H) of the classical time evolution with the external terms

$$\frac{dA}{dt} = (A, H, F) = \frac{\partial A}{\partial r}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial r}\frac{\partial A}{\partial p} + \frac{\partial A}{\partial r}F, \tag{67}$$

violate the right distributivity and scalar laws, thus prohibiting the construction of a completion of quantum mechanics characterized by an $algebra$ as understood in mathematics.

Therefore, Santilli turned brackets (67) into the identical but bilinear form verifying the axioms of an algebra

$$\frac{dA}{dt} = (A, H) = \frac{\partial A}{\partial r}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial r}S\frac{\partial A}{\partial p},$$

$$S = 1 - \frac{F}{\partial H \partial r}. \tag{68}$$

The algebra characterized by the new brackets (A, H) turned out to be Lie-admissible according to Albert [9].

In subsequent studies, Santilli constructed the operator image of brackets (57) with the following realization of the genotopic elements $\hat{R}$ and $\hat{S}$

$$i\frac{dA}{dt} = (A, H) = A < H - H > A =$$

$$= A\hat{R}H - H\hat{S}A = AH - HA - AF,$$

$$R = 1, \;\; \hat{S} = 1 - F/H, \tag{69}$$

that was proved in paper [41] to be $directly\ universal$ for the representation of all possible (non-singular) non-conservative, thus irreversible systems ("universality") directly in the coordinates of the experimentalist ("direct universality") without any need for the transformation theory.

Note that the isotopic element $\hat{T}$ is a particular case of the genotopic element $\hat{S}$ and that, in numeric values, $F < H$. Consequently, *Eqs. (69) establish the analytic origin of the important property that the isotopic element is smaller than* 1, $\hat{T} = 1 - F/H < 1$ *which property is important for the regaining Einstein's determinism.*

Additionally, *Eqs. (69) provide a quantitative identification of the insufficiency of quantum mechanics for nuclear fusions, evidently given by the absence in the quantum mechanical time evolution of the extra term* AF *representing irreversibility.*

It is important to clarify that, despite the lifting of Lie into Lie-admissible algebras, quantum mechanical axioms are indeed preserved. This important property can be best seen in the axiomatic representation of irreversible systems via Lie-admissible formulations whose main aspects are the following.

A tacit assumption of quantum mechanics is that the product of a quantity A times B to the right $A \rightarrow B$ is equal to the product of B times A to the left $A \leftarrow B = A \rightarrow B$ (which property is different than commutativity $AB = BA$), and the same property holds for isomechanics.

In Lie-admissible formulations, motion forward (backward) in time is represented by the axiom of restricting all products to be ordered to the right $A > B$ (to the left $A < B$). The representation of irreversibility is then assured when the numeric values of the two products are different, $A > B \neq A < B$. This results in the following ordered products and units called *genoproducts and genounits to the right and to the left*

$$A > B = A\hat{R}B, \;\; \hat{I}^{>} = 1/\hat{R},$$
$$A < B = A\hat{S}B, \;\; \hat{I}^{<} = 1/\hat{S},$$
$$\hat{R} > 0, \;\; \hat{S} > 0, \;\; \hat{R} \neq \hat{S}. \tag{70}$$

The above basic assumptions imply the duplication of isomathematics and related isomechanics, one for product to the right and one for

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product to the left whose review is omitted for brevity [24]. The combination of forward and backward mathematics and mechanics are known as *genomathematics and genomechanics.*

An important point is that *the abstract axioms of quantum mechanics remain fully valid for each direction of time, including the preservation of the abstract axioms of numeric fields [33], functional analysis [23] differential calculus [34], etc, merely realized in a time-ordered isotopic form.*

It is generally believed that genomathematics and genomechanics are too complex for physicists. This is unfortunate because, as shown in Ref. [35], the construction of genomathematics is truly elementary. The completion of a quantum mechanical model of nuclear fusion into a form inclusive of irreversibility, is given by subjecting all quantum mechanical quantities and operations to the following *two* non-unitary transformations U and W, including Planck's unit $\hbar = 1$, products, etc.

$$1 \to \hat{I}^{>} = U1W^{\dagger} = 1/\hat{S},$$
$$1 \to \hat{I}^{<} = W1U^{\dagger} = 1/\hat{R},$$
$$AB \to U(AB)W^{\dagger} = (UAW^{\dagger})(UW^{\dagger})^{-1}(UBW^{\dagger}) = A\hat{S}B = A > B,$$
$$AB \to W(AB)U^{\dagger} = (WAU^{\dagger})(WU^{\dagger})^{-1}(WBA^{\dagger}) = A\hat{R}B = A < B. \quad (71)$$

As illustrated in the next section, the treatment of energy releasing processes via the Lie-admissible branch of hadronic mechanics has been done via the simple, dual, non-unitary transformation (71) of quantum mechanical models.

7.7. EPR completion for matter-antimatter annihilation.

As indicated in Section 3.4 and Figure 6, the additional insufficiency of quantum mechanics studied at the 2020 teleconference [0] has been the apparent impossibility of representing the mechanism of particle-antiparticle annihilation via the conventional charge conjugation, Eq. (4).

This insufficiency lead Santilli to propose in papers [62] [63] of 1985 a fifth EPR completion of quantum mechanics, that characterizes the *isodual map* (denoted with an upper symbol d) which is also an anti-Hermitean map like charge conjugation, but it is applied to the *totality* of the quantities Q representing particles and their operations with no known exception, such as

$$Q(t, r, E, \psi, ...) \rightarrow Q^d(t^d, r^d, E^d, \psi^d, ...) =$$
$$= -Q^\dagger(-t^\dagger, -r^\dagger, -E^\dagger, -\psi^\dagger, ...). \quad (72)$$

The above map implies that antiparticles have the *negative energy* $E^d = -E < 0$ predicted by Dirac [11]which is necessary to represent particle-antiparticle annihilation [29].

The novel mathematics characterized by isodual map (72) is known as *isodual mathematics* [29] [57], and it is based on the *isodual unit*

$$1^d = -1, \quad (73)$$

isodual numbers [33]

$$n^d = n1^d = -n, \quad (74)$$

isodual product

$$A \times^d B = AT^d B = -AB, \;\; 1^d = 1/T^d = -1, \quad (75)$$

isodual spacetime

$$x^{d2d} = x^{d\mu} \times^d \eta^d_{\mu\nu} \times^d x^{d\nu} = -x^2, \quad (76)$$

and isodual image of 20th century applied mathematics [34] [32].

On axiomatic grounds, isodual mathematics resolves the violation of causality by particles with negative energy identified by Dirac [11] because *negative energies referred to negative units are as causal as positive* *energies referred to positive units,* and the same holds for time and other physical quantities.

On experimental grounds, the isodual map is equivalent to charge conjugation at the particle level. Consequently, the isodual theory of antimatter represents all available experimental data in particle physics [32].

A basic novelty of the isodual theory of antimatter is that isoduality (72) predicts that antimatter emits a new light called *isodual light* [75] whose characteristics are *all* opposite to those of matter emitted light, thus implying new annihilations of the type

$$e^- + e^+ \ \rightarrow \ \gamma + \gamma^d. \tag{77}$$

which are independently requested by the symmetry of Dirac's equation and of the l. h. s. of the above decay known as *isoselfduality,* namely, the invariance under the isodual map (72) [32].

In particular, *light emitted by antimatter is predicted not to be visible by any Galileo-type refractive telescopes with convex lenses available on Earth or in terrestrial orbits,* because it requires for its detection a basically new type of telescopes currently under development. The cosmological implications of the isodual theory of antimatter are discussed in debate [76] and Ref. [163].

8. APPLICATIONS OF EPR COMPLETIONS

8.1. Foreword.

Following various requests, in this section we provide an outline of the novel applications permitted by the recent verifications [210]-[214] of the EPR argument [1] in: nuclear physics (Section 8.2), chemistry (Section 8.3), relativities (Section 8.4), and high energy scattering experiments (Section 8.5).

Since the outline presented in this section is mainly conceptual with a bare minimum of quantitative treatments, the study of the quoted literature is essential for a serious understanding of the treated topics. In addition, the literature in the field is quite vast by therefore forcing us for brevity to quote only the originating papers. Comprehensive bibliographies can be obtained from the quoted reviews.

Needless to say, the fundamental methods of this section are the *iso-, geno- and hyper-mathematics,* with particular reference to the *iso-, geno- and hyper-completions of Lie's theory,* and corresponding *iso-, geno- and hyper-mechanics and chemistry* [21]-[30]. To assist the non-initiated reader, Tutoring Lectures in the above methods have been provided in Refs. [143]-[146].

In inspecting this section, interested readers should keep in mind that the basic axioms of 20th century theories are preserved in their entirety, and merely subjected to the broadest possible realization.

Consequently, this section is intended to illustrate the remarkable broadening of the prediction and representational capabilities of the basic axioms of 20th century theories, as well as to point out that technically unsubstantiated criticisms of the applications reviewed in this section are, in reality, criticisms on 20th century theories.

Another important aspect needed for the understanding of the applications herein reviewed is that 20th century theories have one single formulation, the conventional one. By contrast, their isotopic completions have *two* formulations, that on isospaces over isofield and their *projection* on conventional spaces over conventional fields.

Recall that 20th century Hamiltonian interactions between *point-like* particles cause alterations of the characteristics of particles called *renormalization.* It is known since the original 1978 proposal to construct hadronic mechanics that contact non-Hamiltonian interactions between *extended* particles causes different alterations of the characteristics of particles called *isorenormalizations* (see Section 5 of Ref. [20], and Refs [23] [24]). The latter renormalizations are triggered by the isotopic element $\hat{T}$. Consequently, the conventional notion of *particle* is no longer applicable under EPR completion in favor of the covering notion of *isoparticles* pointed out

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in the preceding section [66]-[70].

As an illustration, the assumption that the particles in a valence bond are ordinary electrons leads to major inconsistencies that remain undetected by the non-initiated reader. Similarly, the assumption that the particles in the synthesis of the neutron from the hydrogen in the core of stars are conventional protons and electrons leads to inconsistencies so serious to prevent a consistent synthesis in disagreement with the experimental evidence on the existence of the neutron synthesis in nature.

8.2. Applications of EPR completions in nuclear physics.

8.2.1. Representation of nuclear magnetic moments. As it is well known, the magnetic moment of a rotating charged sphere increases (decreases) when said sphere is deformed into an oblate (prolate) spheroid while keeping constant its angular momentum. Enrico Fermi, Victor Weisskopf and other founders of nuclear physics suggested in the 1940's that the anomalous values of nuclear magnetic moments (Section 3.2 and Figure 3) may be due to a *deformation* of the charge distribution of protons and neutron caused by strong nuclear interactions.

Despite its manifestly plausible character and authoritative origination, the hypothesis of the deformability of nucleons has not yet been accepted by the mainstream physics community to date (with due exceptions) because it would require a modification of quantum mechanics essentially along the EPR argument [1]. In any case, point-like particles cannot be deformed.

The Lie-isotopic completion of quantum mechanics with two-body isounit (45), related deformation of charge distributions (Figure 14), and exact representation if the Deuteron magnetic moment first achieved in Ref. [40] of 1994, have been proposed to honor the legacy by Fermi, Weisskopf and others without altering the basic axioms of quantum mechan-

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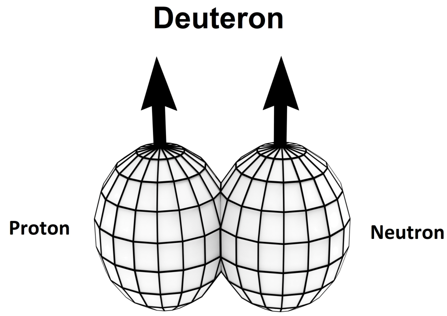

Figure 11: *A view of the deformation of the proton and the neutron in the Deuteron structure that permitted the first exact representation of the deuteron magnetic moment in Ref. [40] of 1994. Note that the triplet coupling used to represent the Deuteron spin* 1 *is unstable according to quantum mechanics. The resolution of this paradoxical occurrence is outlined in Section 8.2.5 [26] [213].*

ics.

However, in so doing the constituents of nuclei are no longer point-like protons and neutrons per 20th century definition, but extended, thus deformable *isoprotons* $\hat{p}^+$, *isoneutrons* $\hat{n}^0$, *or isonucleons,* as defined by the LPS isosymmetry. In fact, the 1998 proof of the EPR argument [210] applies the underlying isomathematics and isomechanics for the *exact* representation of the nuclear magnetic moment of the Deuteron $D = (\hat{p}^+, \hat{n}^0)_{hm}$, as well as of heavier nuclei, precisely along the legacy by Fermi, Weisskopf and others indicated above.

It should be noted that the deformability of nucleons was preliminarily confirmed by the 1981 neutron interferometric experiments [43] and the adjoining theoretical study on the Lie-admissible completion of the rotational symmetry [44].

Additionally, paper [210] used isomathematics and isomechanics for

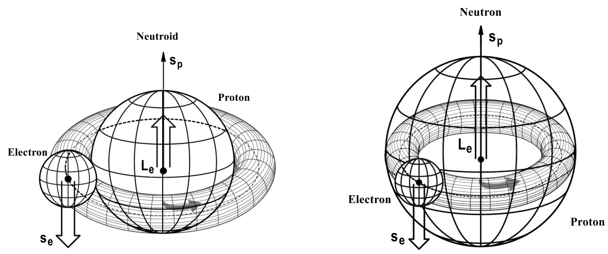

Figure 12: *At the mutual distance of* 10^{-13} *cn, the proton and the electron experience the extremely big Coulomb attraction of* 230 *Newtons, Eq. (6), although quantum mechanics prohibits their bound state. In this figure we depict the two bound states predicted by hadronic mechanics [68] [84] [214]: the neutroid in the left for a singlet coupling with total spin* 0 *and* 7 *s mean-life, and the neutron in the right with spin* 1/2 *and* 900 *s mean-life under the total compression of the electron inside the proton in the core of stars (Section 8.2.2).*

the reconstruction of the *exact* $\hat{SU}(2)$-isospin symmetry when believed to be broken by electromagnetic interactions. This result was achieved via the representation of the proton-neutron mass difference with Bohm's hidden variable λ.

The systematic reconstruction of all space-times continuous and discrete symmetries at the isotopic level when believed to be broken is presented in Refs. [23] [24].

Note that, according to the basic axioms of quantum mechanics, the triplet coupling of the proton and the neutron of Figure 11 generally used to represent the spin 1 of the deuteron, is highly unstable, The only stable quantum mechanical bound state is the singlet, but in this case the total spin of the deuteron would be 0 contrary to experimental evidence. The resolution of this paradoxical situation via hadronic mechanics has been achieved in Refs. [26] [213] and will be outlined in Section 8.2.5.

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8.2.2. Representation of the neutron synthesis. Nuclei are assumed to be bound states of protons and neutrons under strong interactions. However, isolated neutrons are naturally unstable. Therefore, a deeper representation of nuclear structures requires the understanding of the mechanism in which neutrons become stable when they are members of a nuclear structure. In turn, this understanding can best be achieved in the study of the neutron synthesis from a proton and an electron in the core of stars.

The problem is that the neutron synthesis is impossible for quantum mechanics despite the enormous Coulomb attraction between the proton and the electron at short distances per Eq. (6), because the mass of the neutron is $0.784\ MeV$ bigger than the sum of the masses of the proton and of the electron, in which case the Schrödinger equation would require a *positive potential energy* producing a *mass excess* which is anathema for quantum mechanics.

The 1978 Harvard University paper [20] assumed the above insufficiency of quantum mechanics as a clear evidence on the lack of completion of quantum mechanics according to the EPR argument [1] and proposed the construction of the axiom-preserving hadronic mechanics precisely for the representation of the neutron synthesis from the hydrogen.

In view of the list of insufficiencies of the conjecture that the hypothetical and directly undetectable quarks are the physical constituents of hadrons [38], paper [20] presented in Section 5 a structure model of the octet of mesons as hadronic bound states of actual physical particles produced free in their spontaneous decays with the lowest mode. Note that the model achieved compatibility with the $SU(3)$ model of the time by merely restricting it to the sole quantum mechanical *classification* of mesons, and the use of hadronic mechanics for the *structure* of individual mesons due to the known inapplicability of quantum mechanics in the interior of hadrons, much along the dichotomy classification/structure set in history for atoms and other structures.

Following the development of the Lie-Santilli isotheory [22] and its application to the isotopies of the $SU(2)$-spin symmetry [62] [63], hadronic mechanics was used in Ref. [84] for the first known non-relativistic representation of all characteristics of the neutron as a hadronic bound state of an isoelectron $\hat{e}^-$ and an isoproton $\hat{p}^+$

$$n = (\hat{e}^-, \hat{p}^+)_{hm}, \tag{78}$$

resulting in the prediction of two hadronic bound states: the first is called $neutroid$ and occurs for the singlet coupling isoelectron-isoproton with spin 0 and $7\ s$ mean-life (left of Figure 12); the second is given by the neutron with spin $1/2$ and $900\ s$ mean-life, which occurs under the total compression of the isoelectron inside the isoproton (right of Figure 12).

The representation of the rest energy, mean life and charge radius of the neutron were represented by a non-unitary transformation of the conventional Schrödinger equation (1) with isotopic element of type (57) and ensuing iso-Schrödinger equation of type (61) in which the fine spectrum of the Hulten potential is constrained to have one value only, the neutron.

The representation of the spin $1/2$ required the irregular isotopies $\hat{SO}(3)$ and $\hat{SU}(2)$ to represent the $hadronic\ angular\ momentum$ of the isoelectron when compressed inside the hyperdense isoproton, thus being forced to rotate with its spin, resulting in an $orbital$ eigenvalue $1/2$ which is impossible for the $unitary\ SU(2)$ symmetry of quantum mechanics, but readily possible for the $non\ \text{-}unitary\ \hat{SU}(2)$ isosymmetry.

The representation of the anomalous magnetic moment of the neutron became possible thanks to the contribution from the rotation of the isoelectron inside the isoproton which is completely absent in quantum mechanics (see the review and upgrade in paper [214]).

The relativistic representation of all characteristics of the neutron in its synthesis from the hydrogen was achieved in the 1993 paper [68] written at the Joint Institute for Nuclear Research, Dubna, Russia (see also the 1995 paper [69] published in China).

By ignoring in first approximation the negative binding energy due to the Coulomb interactions between the isoelectron and the isoproton (because the Coulomb attraction is absorbed by the Hulten potential), iso-Schrödinger equation (62) admits physically meaningful solutions for the Hulten potential only when the isorenormalized rest energy of the isoelectron, Eq. (36), acquires the value

$$\bar{m}_{\hat{e}} = \frac{m_e}{n_4^2} = 0.511/n_4^2 = 1.295\ MeV, \tag{79}$$

from which we obtain the density value

$$n_4^2 = 0.394, \tag{80}$$

which is fully within the values of similar densities, such as that of the proton-antiproton fireball obtained from the fit of the experimental data of the Bose-Einstein correlation, which are given by the value $n_4^2 = 0.428$ of Ref. [85] , Eq.(10.27a), page 127 and the value $n_4^2 = 0.364$ of Ref. [86], Table I, page 441.

As recalled earlier, the radial equation of iso-Kein-Gordon equation (35) or the iso-Dirac equation (163), do not admit physically meaningful solution in the event the rest energy of the neutron is $smaller$ than the sum of the rest energies of the electron and of then proton, thus suggesting isorenormalization (79) that includes the missing $0.782\ MeV$.

A primary function of the iso-Minkowskian geometry [70] is the characterization of the isorenormalization of the rest energy of the electron caused by non-Hamiltonian interactions in the interior of the neutron in such a way to avoid the indicated "mass-energy defect" and allow isorelativistic equations to have physical solutions.

Needless to say, the above relativistic effects mandate the study of the laws applicable in the hyperdense medium inside the neutron, which study is rudimentarily done in Section 8.4.4, see in particular Isoaxiom IV, Eqs. 133 (see alsoVolume IV or Ref. [87] for a 2008 review and Ref. [214] for a recent update).

In Section 8.2.5, we shall outline the significance of the neutron synthesis for the representation of nuclear data, such as the stability of nuclei as bound state of protons and neutrons despite the natural instability of the neutron when isolated.

8.2.3. Etherino or neutrino? Due to the respect toward the memory of W. Pauli and E. Fermi, it took years to indicate that in both, the non-relativistic [84] and relativistic [68] derivations, there is no energy available for the neutrino due to the fact that the neutrino is written in the *right-hand-side* of the synthesis,

$$p^+ + e^- \ \rightarrow \ n + \nu, \tag{81}$$

while $0.784\ MeV$ are missing for the synthesis of the neutron itself.

Additionally, the isotopic $\hat{SU}(2)$-spin isosymmetry established that there is no spin available for the neutrino because the total spin of the isoelectron inside the neutron must be 0. This is due to the fact that the orbital motion of the isoelectron inside the isoproton must be equal to the isoproton spin to avoid huge resisting forces and be opposite to the isoelectron sped for a stable singlet coupling (right of Figure 12).

These stability conditions confirm the original conception of the neutron synthesis by Rutherford in 1920 [88],

$$\hat{p}^+ \hat{e}^- \ \rightarrow \ n. \tag{82}$$

In summary, despite a widespread oblivion in the best Ph. D. schools of physics, the neutron synthesis is indeed the most fundamental event in nature with yet unsolved basic problems. For instance, it is generally believed that the missing energy of $0.784\ MeV$ is provided by the

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latent energy in the interior of stars. However, at the initiation of the production of light, our Sun synthesizes 10^{38} neutrons $per\ seconds.$ In the event the missing energy is provided by the latent energy, stars would $lose$ 10^{38} MeV per seconds, thus cooling down and not being able to produce light.

It is also believed that the missing energy in the neutron synthesis is provided by the $p^+ - e^-$ relative kinetic energy. However, the scattering cross section of protons and electrons at about $1MeV$ is so small, 10^{-20} $Barns$, to prohibit any synthesis.

Independently from the above aspects, in supernova explosions we have the emission of many times the entire energy released by the Sun in its entire life of 10 billion years. Calculations have shown that the emission of such enormous amount of energy in one single explosion cannot be consistently represented via the sole admission of quantum mechanical processes.

In view of the above, paper [89] of 2007 suggested that the missing energy in the neutron synthesis is provided by the *ether* as a universal substratum with extremely high energy density (to propagate light at $300K\ km/s$) via a longitudinal, spin 0 impulse called *etherino,* (denoted with the letter "a" from the Latin *aether*) and placed in the *left-hand-side* of the synthesis,

$$p^+ + a + e^- \ \rightarrow \ n. \tag{83}$$

It appears that the experimental data of $indirect\ events$ predicted by the neutrino hypothesis can be numerically interpreted via the etherino hypothesis, of course, jointly with corresponding reinterpretations in the standard model for its sole use for the classification of particles.

The historical criticism against the ether that it would violate special relativity has been long dismissed because we would never be able to identify a reference frame at rest with the ether. The additional historical criticism against the ether as a physical medium given by the *aethereal wind* that should be experienced by moving masses, has also been debunked long time ago (see the 1956 paper [90]) because, from the known law $E = h\nu$, the electron is characterized by oscillations with the frequency of

$$Hz_e = 1.25 \times 10^{20} Hz. \tag{84}$$

The idea that the above oscillations are those of a tiny mass inside the electron has no scientific credibility. Hence, the sole known plausible hypothesis is that *the electron is characterized by oscillations of one point of the ether with frequency* (84). When the electron moves, said oscillations are transferred to different points of the ether with no possible "aethereal wind," and the same holds for all particles and, therefore, for matter. Inertia is the resistance by the ether against changes in the transfer speed of the oscillations to new points.

Note that, far from being pure philosophical considerations, the problem of the structure of the electron is crucial to achieve an understanding of the mechanism according to which two identical valence electrons bond themselves in molecular structures against their huge Coulomb repulsion (see Section 8.3.1).

Note also that the *transverse* character of electromagnetic waves (oscillations perpendicular to the direction of motion) mandates that the ether should have a "rigid-type" medium, while, by contrast, a "fluid-type" structure of the ether would be irreconcilable with the transverse characteristics of light.

To illustrate the implications of the problem here considered, the above view on the ether would imply that *matter is completely empty and space is completely full* to such an extent that, in case we could stop time, the entire universe would disappear.

Also, the etherino hypothesis provides a concrete example of the historical hypothesis of the *continuous creation of matter in the universe,* occurring not only for the neutron synthesis in the core of stars, but also for supernova explosions and other possible events.

Recall that, according to available measurements, the Sun loses via radiation the rather sizable amount of

$$\Delta M_{sun}^{loss} = -4.26 \times 10^{9}\ Kg\ per\ second. \quad (85)$$

In the event the etherino hypothesis is correct, the above loss would be mostly counterbalanced by the synthesis in the Sun of 10^{38} neutrons per second implying an *increase* of its mass by about [163]

$$\Delta M_{sun}^{gain} = +2.87 \times 10^{8}\ kg\ per\ second, \quad (86)$$

resulting in the rather small loss by the Sun of about $3f73\ Kg/s$, which is well within current errors, plus secondary contributions, such as accretion of particles during travel in intergalactic space.

Rather than rejecting the above view due to its novelty, an important scientific question is whether current knowledge in planetary trajectory do set an upper value on the possible loss of mass by the Sun per second (additional aspects are treated in Ref. [163] and in the EPR Debate [91] and Ref. [136]).

8.2.4. Industrial synthesis of neutrons from the hydrogen. Tests on the laboratory synthesis of the neutron from the hydrogen were initiated in the late 1940's with support by Albert Einstein, but the related papers were never accepted for publication by scientific journals of the time. During the last of these initial tests, don Carlo Borghi detected nuclear transmutations apparently caused by particles he called *neutroids* emitted by the apparatus without the direct detection of synthesized neutrons (see the historical account [92]).

Following, and only following, the theoretical understanding of the neutron synthesis via hadronic mechanics (Section 8.2.2.), systematic tests on the laboratory synthesis of the neutron were initiated in the early 2000's. The first actual detection of the emission of neutrons synthesized from a hydrogen gas were presented in paper [93] of 2007, with the repeated detection of synthesized neutrons as well as the confirmation of

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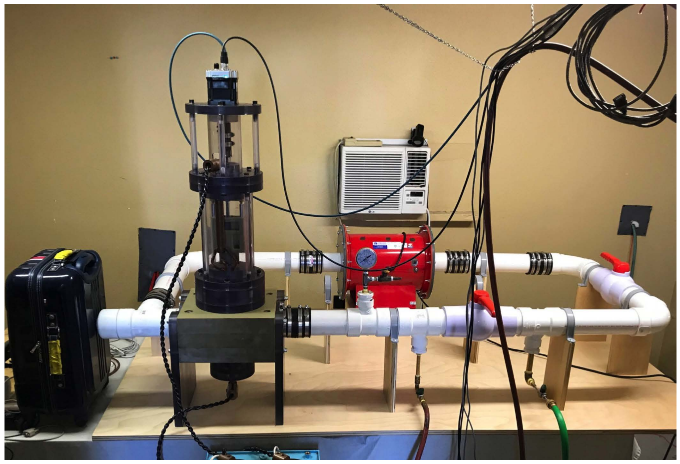

Figure 13: *A view of the Directional Neutron Source (DNS) described in Section 8.2.4 which synthesizes on demand neutrons from the hydrogen in the desired direction, flux and energy.*

the synthesis of don Borghi's neutroids via their stimulated nuclear transmutations (see paper [214] for a recent review).

The neutron synthesis was achieved via the use of a special electric arc submerged within a hydrogen gas which: ionizes the gas, synthesizes the neutroid and then synthesizes the neutron by "compressing" the isoelectron inside the isoproton much according Rutherford's original conception [88].

These tests led to the formation of the U. S. publicly traded company *Thunder Energies Corporation* (now Hadronic Technologies Corporation) for the manufacturing and sale of equipment called *Directional Neutron Source* (DNS) which produces on demand a neutron beam synthesized from the hydrogen gas in the desired direction, flux and energy (see Figure 13, the latest systematic tests [94] and the lectures by S. Beghella Bartoli in the 2020 Teleconference [0]).

Subject to appropriate funding, the DNS is intended to: 1) Provide the clear detection of nuclear weapons or materials smuggled in luggage; 2) Detect precious metals in mining operation; 3) Test weldings of thick plates in naval constructions; and other applications.

8.2.5. Representation of Deuteron data. As indicated in Section 3.2, a second important verification of the EPR argument in nuclear physics is the inability by quantum mechanics to represent the Deuteron magnetic moment, spin, stability and other data *for isolated Deuterons in their ground state* (because representations in excited orbits exist but they do not represent the physical reality).

The representation of all data of the neutron in its synthesis from an isoelectron and an isoproton allowed a new structure model of the Deuteron as a hadronic bound state of one isoelectron and two isoprotons

$$D = (\hat{p}^+_\downarrow, \hat{e}^-_\uparrow, \hat{p}^+_\downarrow)_{hm}, \tag{87}$$

first presented in Ref. [26], Section 2.5, page 181 on, continued in various works (see the references of paper [212]), and completed in Ref. [214].

These studies allowed the representation of all experimental data of the Deuteron *in its true ground state* as follows:

1) The representation of the total spin $J = 1$ of the deuteron by apparently confirming that *the spin* 1 *suggests the Deuteron to be a three-body bound state* according to the structure (87). Note that the sole spin representable via quantum mechanics for a *stable* bound state of a proton and a neutron *in the ground state* is the singlet

$$D = (p^+_\uparrow, n^0_\downarrow)_{qm}, \tag{88}$$

for which $J = 0$;

2) The representation of the anomalous magnetic moment of the Deuteron thanks to the anomalous magnetic of the neutron (Section 8.2.2.).

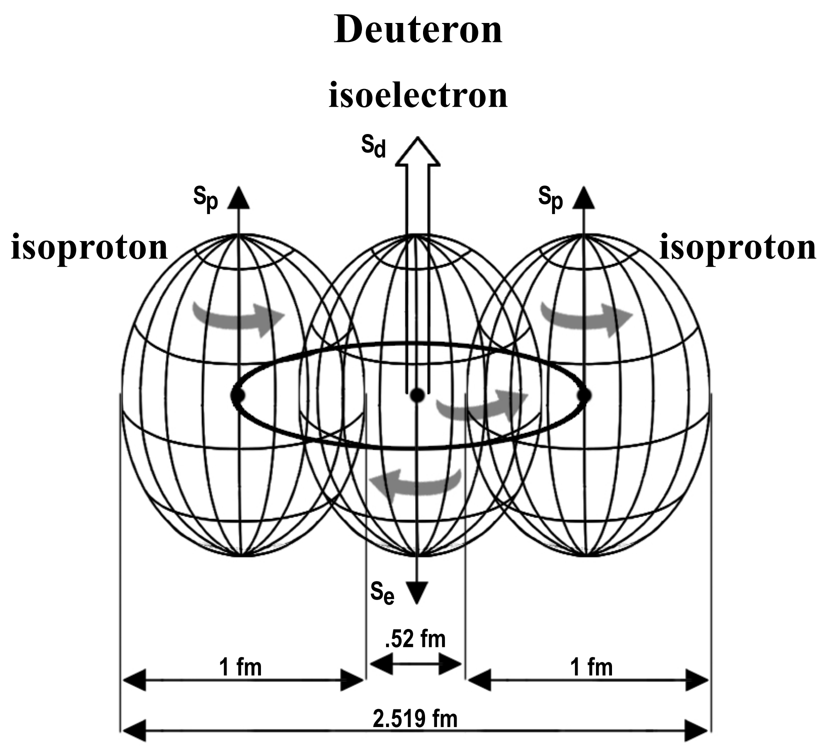

Figure 14: *An illustration of the structure of the Deuteron following the neutron synthesis according to hadronic mechanics (Figure 12), composed by two isoprotons with parallel spin* 1/2 *and one interconnecting isoelectron in singlet coupling with the isoprotons and orbital momentum inside the isoprotons constrained with total angular momentum* 0 *(Section 8.2.2). These features imply a very stable, gear-type, three-body configuration, Eq. (87), which has allowed the first known representation of all Deuteron data in its ground state, including magnetic moment, spin, stability, and other features [26] [214].*

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3) The representation of the stability of the Deuteron despite the natural instability of the neutron because isoprotons and isoelectrons are permanently stable particles.

Section 7 of paper [214] presents an upgrade of the Deuteron model of Ref. [26] with a more accurate representation of the magnetic moment and other nuclear data, thanks to the first known *explicit representation of strong nuclear interactions with isotopic element (38)(39).*

Paper [95] of 2016 used the above results to achieve a representation of the magnetic moment and spin of all stable nuclei.

8.2.6. Reduction of matter to isoprotons and isoelectrons. The preceding results, with particular reference to the reduction of the neutron to a hadronic bound state of an isoproton and an isoelectron, as well as the ensuing representation of nuclear data in their ground state, imply the reduction of all matter in the universe to isoprotons and isoelectrons, including protons and electrons as a particular case [26]. Note that neutron stars are equally reduced to isoprotons and isoelectrons.

8.2.7. The synthesis of the pseudo-proton. Following the synthesis of the neutron via the compression of an electron within the hyperdense proton, by following knowledge established fro the neutron synthesis (Figure 12), the Directional Neutron Source (DNS) of Section 8.2.4, progressively synthesizes, in a statistical lesser amount, *two negatively charged strongly interacting particles.* The first is given by a hadronic singlet coupling of an electron and a neutron called *protoid* and denoted $\bar{p}_1^-$ [96] with spin 0, mass essentially that of the neutron and predicted mean-life of about $5\ s$ (left of Figure 12) whose existence is predicted from the fact that non-Hamiltonian interactions caused by deep EPR entanglement according to structure equation (61) are insensitive to charges. The second particle is given by the compression of the electron, this time, inside the hyperdense neutron, called *pseudo-proton* $\bar{p}_2^-$ [96] (Figure 15) with

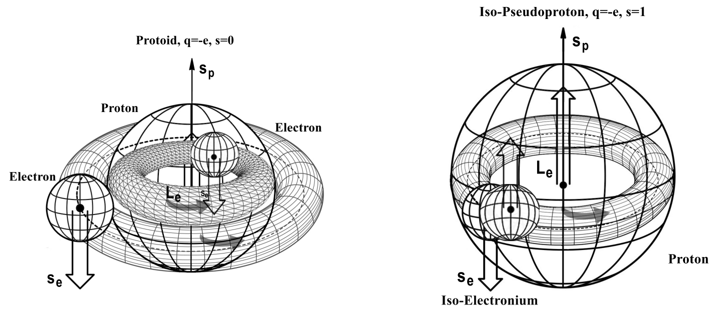

Figure 15: *An illustration of two negatively charged strongly interacting particles predicted by the EPR completion of quantum into hadronic mechanics with mean-lives of the order of seconds and mass close to that of the neutron: the protoid (on the left) with spin* 0*, which is a hadronic singlet state of an electron and a neutron, and the pseudo-proton (on the right) which occurs when the electron is totally compressed inside the neutron with spin* 1 *(Section 8.7).*

spin 1, mass essentially that of the neutron and mean-life of the order of 7 s (see also [214] for a recent review).

In regard to the spin of the pseudo-proton, we recall that electrons couple in singlet. Therefore, when the external electron is compressed inside the neutron, it couples in singlet with the pre-existing internal electron, resulting in a singlet electron pair with total spin 0 which is *constrained* to rotate by the hyperdense medium inside the neutron with the neutron spin $1/2$, resulting in the total spin 1. Different models are faced with the extreme resistance that would be experienced by an extended wavepacket to rotate inside and against the motion of the hyperdense neutron medium.

The emerging new technology, called *pseudo-proton irradiation* is made possible by the fact that, even though the pseudo-proton is evidently unstable, its mean-life is nevertheless of the order of a few seconds (a fraction of the $900\ s$ mean life of the neutron), thus being suitable for applications.

On industrial grounds, by noting that *pseudo-protons are strongly attracted by nuclei,* pseudoproton irradiation is significant to study new clean nuclear energies, recycling of nuclear waste, and other intriguing applications studied in Section 8.2.10.

On medical grounds, pseudo-proton irradiation is under consideration for cancer treatment via a localized low energy beam in replacement of proton irradiation due to its excessive invasive character caused by the high energies needed to overcome proton-nucleus repulsion.

It should be noted that the existence of the pseudo-proton, let alone its long mean life for particle standards, is impossible without the EPR completion of quantum into hadronic mechanics.

8.2.8. The synthesis of the pseudo-Deuteron. The Directional Neutron Source of Section 8.2.5 synthesizes neutrons and pseudo-protons (as well as their intermediate states) when filled up with a commercial grade hydrogen gas.

When filled up with Deuterium gas, the same DNS is predicted to synthesize a *negatively charged nucleus* called the *pseudo-Deuteron* [96] [214] with the structure

$$\hat{D} = [\hat{p}^+_\uparrow, (\hat{e}^-_\downarrow, \hat{e}^-_\uparrow), \hat{n}^0_\uparrow]_{hm}, \tag{89}$$

namely, via the compression of an electron pair $(\hat{e}^-_\downarrow, \hat{e}^-_\uparrow)$ inside the Deuteron $D = (p^+_\uparrow, n^0_\uparrow)$ resulting in a new nucleus with spin 1 that (in nuclear symbols A = atomic number, Z = charge and J= spin) is denoted $\hat{D}(-1, 2, 1)$ with evident decay $\hat{D} \rightarrow D + 2\beta$.

To understand the above synthesis, let us recall that the valence electrons of the deuterium molecule are bonded into the valence pair $(\hat{e}^-_\downarrow, \hat{e}^-_\uparrow)$ called *isoelectronium* [30] (Section 8.3.1) with charge $2\ e$, total spin 0, total angular momentum 0 and total magnetic moment 0.

Under sufficient power, a submerged DC arc separates Deuterium

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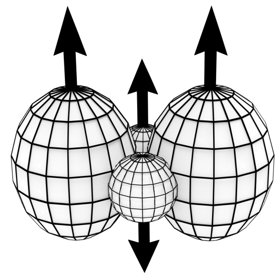

Figure 16: *An illustration of the negatively charged Deuteron, called pseudo-Deuteron denoted* $\hat{D}(-1,2,1)$*, according to structure (89), which is given by the compression within a Deuterium gas of a valence electron pair (known as isoelectronium) inside a natural Deuteron via a sufficiently powerful DC electric arc.*

molecules by forming a plasma mostly composed by protons and valence pairs due to their strong bond (Section 8.1). Next we should recall the *very big Coulomb attraction between valence electron pairs and natural Deuteron* according to Eq. (6) which prepares synthesis (89) by forming an intermediate state called *Deuteroid* [96]. The same DC arc then compresses the valence pair inside the Deuteron according to the configuration of Eq. (89) and Figure 16.

This process creates a negatively charged nucleus which is evidently unstable, but its main-life (computed via hadronic mechanics) is of the order of *seconds,* thus allowing industrial applications.

Recall that the structure $D = (p^+_\uparrow, n^0_\uparrow)$ is *unstable* due to the triplet coupling of the proton and the neutron. Therefore, the compression of the electron pair $(\hat{e}^-_\downarrow, \hat{e}^-_\uparrow)$ inside the Deuteron *stabilizes* the structure due to the physical separation of the proton and the neutron.

Needless to say, the plasma created by the submerged DC electric arc also contains isolated electrons that are also very strongly attracted by Deuterons, and can form other hadronic bound states with smaller statistical probabilities and mean-lives [96].

Again, the pseudo-Deuteron is unstable, but again its mean life is estimated to be of the order of seconds, thus being suitable for applications.

8.2.9. New clean nuclear energies. In this section, we provide a conceptual outline of new nuclear energies without harmful radiations called *hadronic energies* originally proposed in paper [97] of 1994 , which are not possible for quantum mechanics, but they are indeed possible for the completion of quantum into hadronic mechanics [23]-[25] according to the Einstein-Podolsky-Rosen argument [1] following the exact representation of nuclear spin, magnetic moments, stability and other data presented in the preceding sections.

The *existence* of the new energies in parts per million (ppm) has been experimentally verified by verious independent laboratories thanks to funds provided by Magnegas Corporation (now Taronis Corporation), but no funds could be located for their development up to the level of *industrial* production of new nuclear energies, which development is therefore left to interested scientists and governmental officers.

Among the variety of hadronic energies predicted in Ref. [97] (thanks to the Lie-admissible completion of quantum mechanics for irreversible processes) we outline the following three clean energies (see Refs. [25] and [98] for reviews):

8.2.9-I: Nuclear energies via stimulated neutron decays. Incontrovertible experimental evidence establishes that the neutron is synthesized from the proton and the electron and decays into the proton and the electron. Hadronic mechanics has permitted the representation of *all* characteristics of the neutron at both non-relativistic and relativistic levels (Section 8.2.3) when assumed to be a hadronic bound state of an isoproton $\hat{p}^+$ with conventional rest energy of $938.272\ MeV$ and an isoelectron $\hat{e}^-$ with isorenormalized rest energy of $1.295\ MeV$, Eq.(80) due to its immersion within the hyperdense medium inside the proton with density (79).

In view of the above data, Ref. [97] submitted the hypothesis that *when the member of a suitably selected, light, natural, and stable nucleus, the neutron can be stimulated to decay* via a resonating frequency which is an integer multiple or submultiple of $\hat{\gamma} = 1.295\ MeV$ corresponding to the resonating frequency (see Eq. (3.6), page 326, Ref. [87])

$$\nu_{reson} = 3.1289 \times n \times 10^{20}\ Hz,\ n = 1, 2, 3, ... \tag{90}$$

or

$$\nu_{reson} = 3.129/n \times 10^{20}\ Hz. \tag{91}$$

By recalling that $1.295\ MeV = 0.0014\ amu$, and by ignoring integer multiples or submultiples for a first analysis, the indicated hypothesis can be written

$$\hat{\gamma}(0, 0, 1, 0.0014\ amu) + N(Z, A, J, amu) \rightarrow$$

$$\rightarrow \hat{N}(Z + 1, A, J + 1, amu) + \hat{e}^-(-1, 0, 0, 0.0014\ amu). \tag{92}$$

Note that the energy supplied by the resonating photon is reemitted by the isoelectron which has the rapid spontaneous decay in vacuum

$$\hat{e}^- \rightarrow e^- + \nu\ (or\ a), \tag{93}$$

where the etherino alternative a is outlined in Section 8.2.4.

Among various possible hadronic energies, Ref.[97] suggested in page 340 the test of the stimulated decay of a neutron of $30 - Zn - 70$ according to the reaction (see Ref.[99] for nuclear data)

$$\hat{\gamma} + Zn(30, 70, 0, 69.9253) \rightarrow$$

$$\rightarrow \hat{Ga}(31, 70, 1, 69.9253) + \hat{\beta}^-, \tag{94}$$

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where $\hat{Ga}$ is a highly unstable pseudo- nuclide due to the missing energy of $0.0007\ amu$ to have the tabulated mass for $31 - Ga - 70$ of $69.9260\ amu$, with spontaneous decay

$$\hat{Ga}(31, 70, 1, 69.9253) \rightarrow Ge(32, 70, 0, 69.9242) + \beta^- + \nu, \qquad (95)$$

where $32-Ge-70$ is a stable light natural element. Note that the indicated transmutation triggered by the stimulated neutron decay provides two different new clean hadronic energies without the emission of harmful radiation or release of radioactive waste: 1) Heat produced in the amount of $0.0011\ amu = 1.024\ MeV$ per reaction, and 2) Electricity generated by the production of two electrons per reaction which can be easily trapped via a thin metal shield to form a $nuclear\ battery$ (see Sections 4-I and 4-II of Ref. [97] for details.).

The stimulated double beta decay was tested in Ref. [100] (see also the subsequent presentation [121]) with preliminary positive results reported in Ref. [87].

The tests can nowadays be done by irradiating a plate of the selected isotope (e.g., $30 - Zn - 70$) with resonating frequency available from radioactive isotopes for a few days, and then have comparative mass spectroscopic analyses of the untreated and the treated plate to ascertain the possible presence in the treated plate of the predicted new isotope $32 - Ge - 70$ in ppm.

The direct costs for the repetition of the tests are approximately given by: 1) $1,200 for a sample of the needed pure $30 - Zn - 70$ isotope with certified content; 2) $2,200 for the purchase of the needed radioactive isotope; and 3) $800 for mass spectroscopic analyses, for the total direct cost of $4,200$.

It should be noted that there exist a considerable amount of scientific literature on double beta decays (which is easily identifiable via an internet search), although none of them, to our knowledge, considers the $stimulated$ double beta decay proposed in Ref. [97] of 1994 and tested in

Ref. [100] of 1996. The study of the triggering of known double beta decays with resonating photons to search for new clean energies is left to interested colleagues.

8.2.9-II: Intermediate Controlled Nuclear Fusions. It is generally believed in mainstream physics circles that *nuclear fusions cannot occur without the emission of neutrons.* It is important for our environment to disregard such a view because generally based on the dismissal of the insufficiencies of quantum mechanics in nuclear physics (Section 3.2), and expressed in oblivion of Einstein's view on the limitation of quantum mechanics. This section is devoted to physicists interested in testing nuclear fusions without the emission of neutrons, as predicted by the completion of quantum into hadronic mechanics according to the EPE argument.

Additionally, this section is intended for physicists interested in verifying or dismissing in refereed scientific publications that the sole scientific, i.e., quantitative interpretation of thunder is that via nuclear fusions of light, natural and stable elements triggered by lighting without the emission of neutrons.

As it is well known, the *existence* of nuclear fusions (also called syntheses) at low energies has been established by the so-called *cold fusions,* but the energy available turned out to be insufficient to power all engineering means needed for industrial production of clean energy. The existence of nuclear fusions has also been established by the so-called *hot fusions,* but their energies and temperatures are so big to create uncontrollable instabilities at the time of the initiation if the fusion process.

Despite the investment of billions of dollars of public funds over half a century, cold and hot fusions have been unable to achieve the much needed, industrially viable, clean nuclear energies.

The apparent primary reason is that all research in the field was based on quantum mechanics in oblivion of the EPR argument [1] as well as

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of the known impossibility for quantum mechanics to provide an exact representation of nuclear data, as well as the impossibility to provide a correct representation of energy releasing processes due to their time irreversibility (Section 3.3).

In view of the above occurrence, systematic research was initiated via funds provided by Magnegas Corporation in the search of new clean nuclear energies based on the completion of quantum mechanics into the irreversible, Lie-admissible branch of hadronic mechanics. This research can be briefly outlined as follows.

Following, and only following the achievement of sufficient mathematical [23] and physical [24] maturity on he Lie-admissible branch of hadronic mechanics (see also lecture [101]), a study was conducted on the formulation of the *new* physical laws of nuclear fusions according to hadronic mechanics, which laws were presented in the 2008 paper [102] to characterize the new *Intermediate Controlled Nuclear Fusions* (ICNF), namely, nuclear fusions with energy intermediate between the cold and hot fusions.

The main differences between quantum fusions and ICNF are: 1) Nuclei are massive points for quantum fusions, while nuclei are extended for ICNF; 2) Quantum fusions solely admit action-at-a-distance potential interactions, while ICNF are primarily based on contact non-potential interactions; and 3) Quantum fusions are solely based on physical processes, while ICNF are based on physical as well as certain crucial chemical processes indicated in Section 8.3, thus requiring both hadronic mechanics and chemistry (see the review in Section 3.25 of Ref. [25]).

A hadronic reactor for the test of ICNF was built in 2009 comprising:

1) A metal vessel housing in its interior a pair of Carbon electrodes whose gap is electronically controlled from the outside;

2) A selected light, natural and stable gas called *hadronic fuel* which is stored at pressure in the metal vessel;

3) A DC arc between a pair of electrodes submerged within the hadronic

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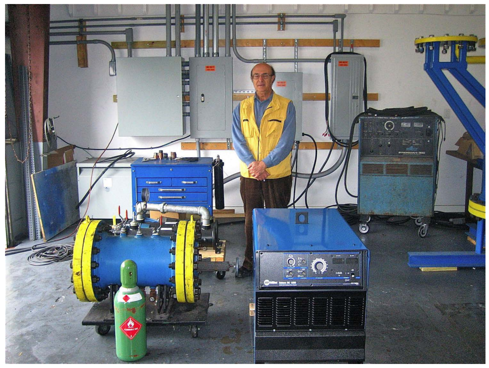

Figure 17: *A view of the hadronic reactor used for the Intermediate Controlled Nuclear Fusion (ICNF) of the Nitrogen from Deuterium and Carbon without the emission of neutronic or other harmful radiations first achieved in Ref. [103] of 2010.*

fuel powered by a commercially available $50\ KW$ DC welder produced by Miller Electric corporation which delivers up to $1K\ A$ in the arc between the submerged electrodes with about $7\ mm$ gap;

4) Means for the control of the internal nuclear fusions including the control of the DC power, gas pressure, electrode gap, and other engineering means;

5) The hadronic reactor is then completed with a number of peripheral equipment, such as: a variety of radiation detectors with alarms preset at minimal reading for safety; vacuum and pressure pumps; vacuum and pressure gages; internal and external temperature gages; electric panel for the monitoring and control of the pressure, temperature, radiation; and other equipment (see Figure 19).

Samples of the gaseous hadronic fuel are taken before and after the activation of the reactor whose operation is limited to a few minutes for evident safety due to a generally rapid increase of the temperature. ICNF occur when laboratory analyses signed by the laboratory director estab-

Figure 18: *A view of the team of scientists who confirmed the ICNF of the Deuterium and Carbon into the Nitrogen without the emission of harmful radiations and without the release of radioactive waste [100]-[112].*

lish the presence of new light, natural and stable elements at least in ppm, under the condition that said new elements do not exist in the original hadronic fuel.

Following extensive tests, Ref. [103] of 2010 announced the ICNF in ppm of the Nitrogen from the Carbon of the electrodes and a commercial grade Deuterium gas as hadronic fuel (Figure 17) according to the ICNF

$$D(1, 2, 1^+, 2.0141) + C(6, 12, 0^+, 12.0000) \rightarrow \\ \rightarrow N(7, 14, 1+, 14.0030) + \Delta E_{heat}, \quad (96)$$

confirmed by chemical analyses [104]-[107] signed by the laboratory director, where the released heat is given by

$$\Delta E_{heat} = 10.2231\ MeV = 1.57 \times 10^{-15} BTU \quad (97)$$

per fusion.

It should be stressed that the indicated Nitrogen synthesis can only occur **without any production of neutron or other harmful radiation or release of radioactive waste** wince it deals with the synthesis

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Figure 19: *A view of the third hadronic reactor for the ICNF of the Carbon and Oxygen into the silicon successfully tested in 2011 [109].*

of two light, natural and stable elements into a third light, natural and stable element with a smaller mass, as well as due to the limited power of the DC source ($50\ KW$) which is clearly insufficient to crack nuclei for the production of the popularly expected neutrons. A recording of the sound of the hadronic reactor during operation is available from Ref. [108].

The above results were all confirmed by systematic tests and verifications [109] - [112] conducted by an independent team of scientists (Figure 18).

Among a number of additional hadronic reactors build for the test of ICNF, we mention the third hadronic reactor (Figure 16) built in 2011 for a Chinese client for the ICNF of Carbon and Oxygen into the Silicon,

$$C(6, 12, 0^+, 12.0000) + O(8, 16, 0^+, 17.9991) \rightarrow$$
$$\rightarrow Si(14, 28, 0^+, 27.9769) + \Delta E, \tag{98}$$

were

$$\delta E = 2.0222\ amu = 1,882.6682\ MeV = 2.8604 \times 10^{-13} BTU/reaction, \tag{99}$$

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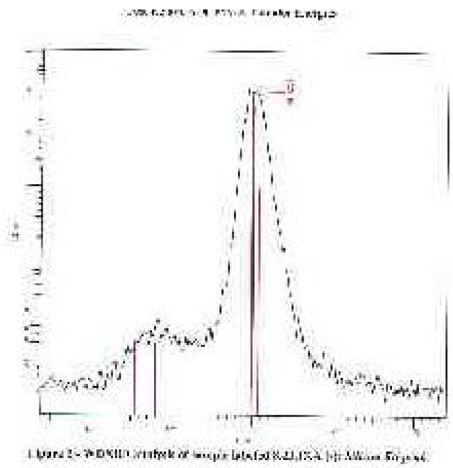

Figure 20: *A view of one of the analyses [116]-[118] signed by laboratory directors showing the pick of the Silicon synthesized from Carbon and Oxygen.*

the ICNF

$$O(8,16,0^+,17.9991) + O(8.16,0^+,17.9991) \rightarrow$$
$$\rightarrow S(16,32,0^+,31.9720) + \Delta E, \qquad (100)$$

where

$$\Delta E = 4.0262\ amu = 3,7483,3922\ MeV =$$
$$= 5.6950 \times 10^{-13} BTU\ per\ synthesis, \qquad (101)$$

and other ICNF that were announced in video [113] and paper [114] of 2011 (see also lecture [115]) following systematic analyses [116] - [120] also signed by the laboratory director (see the sample analysis of Figure 20).

It should be stressed that the above tests were primarily done to establish the *existence* and *control* of ICNF in parts per millions under the expectation that such an evidence was sufficient for funding the construction of industrial hadronic reactors, by keeping in mind that the efficiency of the reactors was limited by the low operating pressure, the low power of the DC generator and other reasons.

The rather voluminous amount of information collected on ICNF indicates that the achievement of a an *industrial* hadronic reactor capable of producing truly controlled and radiation free nuclear energy is indeed within current technological reach via the use of: a specially built, $100\ KW$ DC power unit of the particular type used for the synthesis of the neutron (Section 8.2.3); the use of bigger pressures; and other engineering means. To see it, recall that the Nitrogen synthesis of Ref. [103] produces $1.57 \times 10^{-15} BTU\ per\ fusion.$ The plasma surrounding a '1 cm DC arc contains about 10^{30} atoms. The extrapolation of available data indicates the realistic possibility of producing of 10^{21} fusions *per hour* with the ensuing delivery of $10^6\ BTU/h$ which are fully sufficient to generate the needed electric power plus the delivery of clean energy of nuclear origin. Alternatively, as shown in video [113], the third hadronic reactor for the synthesis of the Silicon was already producing electricity when operated at $1,000\ psi$ pressure under $50\ KW$ power. The production of free, clean, nuclear energy when operated at $5,000\ psi$ pressure with $100\ KW$ DC power is rather plausible.

A study of ICNF presented at the Teleconference is available in Ref. [219] and in the lecture recorded by I. B. Das Sarma [0].

It is regrettable for the environment that billions of dollars of public funds have been and continue to be invested in conventional nuclear fusions despite known insufficiencies of quantum mechanics, while it has not been possible to secure until now comparatively small funds for the test of the indicated ICNF.

8.2.9-III. HyperFusions of natural nuclei and pseudo-nuclei. The biggest problem that has prevented the achievement to date

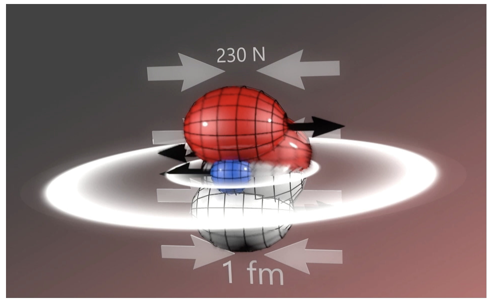

Figure 21: *A first serious environmental problem facing mankind is the lack of achievement of 'new' clean nuclear energies (Section 8.2.9). In this picture, we illustrate the new HyperFusion without harmful radiation between natural Deuterons and synthesized negatively charged pseudo-Deuterons solely permitted by the EPR completion of quantum into hadronic mechanics.*

of controlled nuclear fusions is the *Coulomb barrier,* namely, the *repulsive Coulomb force* between natural nuclei caused by their positive charge which, from Eq. (6), has the extremely big value for nuclear standards of hundreds of Newtons. In this section, we report the research done to date on a basically new type of nuclear fusion, called *HyperFusion* occurring between natural, positively charged nuclei and synthesized negatively charged nuclei, by therefore turning the repulsive Coulomb force into an attraction [96] [214].

The principle of HyperFusion is elementary. Recall from Section 8.2.2 that at 10^{-13} cm the electron and the proton experience the extremely big *attractive* Coulomb force of 230 Newtons. Nevertheless, quantum mechanics admits no bound state between the electron and the proton at short distance, by therefore confirming Einstein's view on the lack of completeness of quantum mechanics beyond scientific doubts [1]. The completion of quantum into hadronic mechanics then allowed the identification of all engineering needs for the synthesis of the electron and the roton into an unstable particle with 900 seconds mean life, the neutron.

The principle of the new HyperFusion is essentially the same. At $10^13\ cm$ mutual distance, electrons are attracted by nuclei with a Coulomb force of hundreds of Newtons. The technology established for the Rutherford compression of the electron inside the proton is also applicable to the compression of electrons inside nuclei, resulting in the synthesis of negatively-charged nuclei, called *pseudo-nuclei* [214] that are evidently unstable, but possess nevertheless mean lives of the order of seconds, thus being usable for industrial applications. HyperFusion is then completed by established technologies for the controlled separation of pseudo-nuclei from their originating plasma, and controllable engineering means for their natural attraction by natural nuclei, activation at contact of strongly attractive nuclear forces, and then inevitable HyperFusion.

As indicated in Section 8.2.4, when filled up with hydrogen, the Directional Neutron Source (DNS) produces neutroids, neutrons, and pseudo-protons. When the same DNS is filled up with a commercially available Deuterium gas, it produces negatively charged pseudo- Deuterons via the synthesis [96] [214]

$$D(1,2,1) + (e^-_{\downarrow}, e^-_{\uparrow}) \rightarrow \hat{D}(-1,2,1). \tag{102}$$

The existence of the pseudo-Deuteron, and its separation from the DNS plasma via standard technologies, allow the test of the HyperFusion between natural Deuterons and pseudo-Deuterons according to the reaction [96] [214]

$$D(1,2,1^+_{\uparrow}, 2.0141\ amu) + \hat{D}(-1,2,1^-_{\downarrow}, 2.0141\ amu) \rightarrow$$

$$\rightarrow He(2,4,0,4.0026\ amu) + 2\beta^- + \Delta E, \tag{103}$$

where [*loc. cit.*]

$$\Delta E = 0.0222amu = 20.67\ MeV. \tag{104}$$

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Note that the above fusion not only eliminates the *repulsive* Coulomb force between natural nuclei, but turn the repulsion into an *attraction* due to the opposite charges, as a result of which Deuterons and pseudo-Deuterons are naturally attracted to the mutual distance of $10^{-13}\ cm$ needed to activate nuclear forces.

Recall that a controlled quantum mechanical fusion of two Deuterons into the Helium has been essentially impossible due to the need for a *controlled* singlet coupling of the two Deuterons which is needed for the conservation of the angular momentum (see the hadronic laws for nuclear fusoins in Ref. [102]). By comparison, the coupling of a Deuteron and a pseudo-Deuteron is naturally singlet as expressed in reaction (103) due to their opposite charges and magnetic moments, by therefore providing a second facilitation for a controlled nuclear fusion over conventional attempts.

Note finally that the fusion $D + \hat{D} \rightarrow He + 2\beta^0$ can be initiated and stopped on demand, and can be controlled via the control of: the pressure of the Deuterium gas; the energy of the DC arc; its voltage; the flow of the Deuterium gas through the arc; and other engineering means.

8.2.10. The problem of recycling nuclear waste. As it is well known, automotive production is under reorganization in the U.S.A. and in other developed countries to replace fossil fuels operated cars with electrically operated cars. This reorganization implies a consequential, not sufficiently spoken, exponential increase of electricity produced from nuclear power plants and other sources. As a consequence, the problems of recycling highly radioactive nuclear waste and the achievement of clean combustion of fossil fuels are becoming some of the most serious environmental problems facing mankind today.

In this section we shall indicate the implications of the EPR argument for the recycling of nuclear waste, while the achievement of full combustion of fossil fuels is treated in the next section.

Due to various oppositions, it has not been possible to store radioactive nuclear waste in depositories such as that in the Yucca Mountain in the U.S.A. Consequently, nuclear waste is stored nowadays by the nuclear power plants themselves in their own facilities. Such a storage has already reached safety limits with an expected unreassuring surpassing of safety limit under the ongoing automotive reorganization from fossil fuels to electric cars.

Also, all attempts at an effective recycling of nuclear waste via quantum mechanical technologies have failed to date. Therefore, a most unreassuring aspect of the current condition is the lack of a visible search for *new* forms of waste recycling, despite the availability of large corporate and governmental funds, and the continuation of the century old oblivion of the EPR argument [1].

By recalling that any transportation of radioactive nuclear waste is prohibited by popular opposition, the indicated unreassuring condition mandates at least the *search* by responsible societies of the only viable solution, that of developing new technologies for the recycling of nuclear waste by the nuclear power plants themselves in their own facilities. Among all possible solutions, the most desirable recycling is that via the *stimulated decay of radioactive waste* in such a manner of reducing mean lives from thousands of years down to days or minutes.

It is known that such a recycling is impossible for quantum mechanics because the alteration of the mean-life of radioactive nuclei would imply a violation of the Poincaré symmetry which is at the foundation of special relativity.

In this section, we outline the following three recycling of nuclear waste proposed in Ref. [97], Section 4-III-A, page 342 on, which are made possible by the completion of quantum into hadronic mechanics:

8.2.10-I. Recycling of radioactive hospital waste. Tests conducted in 2011 at the U. S. publicly traded company Magnegas Cor-

Figure 22: *A second serious environmental problem facing mankind is the recycling of radioactive nuclear waste now stored by nuclear power plants in their facilities. It is unfortunate for mankind that the century old opposition to the the EPR argument by mainstream physics generally discredits the search for stimulated recycling of nuclear waste because not permitted by quantum mechanics.*

poration have indicated that a $300\ KW$ PlasmaArcFlow Hadronic Reactor (Figure 22) can apparently recycle lightly radioactive liquid waste from hospitals by triggering their decays via the deformation of their nuclei when exposed to intense electric and magnetic fields into such a prolate shape to cause the disintegration of naturally unstable nuclei due to internal Coulomb repulsion between aggregated protons at the extremities.

Note that such a deformation is impossible for quantum mechanics due to the representation of nuclei as massive points. By contrast, said deformations are fully possible for hadronic mechanics due to the representation of nuclei as extended and, therefore, deformable according to Eqs, (45)-(46) for which the stimulated decay of nuclear waste is reduced to the intensity of the electric and magnetic fields.

Jointly, the indicated $300\ KW$ Plasma-Arc-Flow hadronic reactor sterilizes the liquid waste due to its exposure to the high temperature of the arc.

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Figure 23: *In this figure, we show a* 300 *kw PlasmaArcFlow Recycler produced and sold by Magnegas Corporation (now Taronis Corporation) which is recommended for the recycling of radioactive hospital waste, since the extreme electric and magnetic fields of the submerged electric arc predict their stimulate decay.*

8.2.10-II. Recycling nuclear waste via stimulated neutron decays. A primary reason for suggesting the rather inexpensive test of the stimulated neutron decay in Section 8.2.9-I is that, in the event successful, it would allow an effective recycling of radioactive nuclear waste via the stimulated decay of some of its peripheral neutrons. This possibility is illustrated in Ref. [97], page 340, with the predicted stimulated decay of $42-Mo-100$, which is an unstable nuclide with mean -life of 10^{19} years. Yet, a stimulated neutron decay would imply the transmutation of $42-Mo-100$ into $43-Tc-100$ according to the reaction

$$\hat{\gamma}(0,0,1,1.295\ MeV) + Mo(42,100,0,99.9074\ amu) \rightarrow$$

$$\rightarrow Tc(43,100,1,99.9076 amu) + \hat{e}^-(-1,0,0,1.295\ MeV), \tag{105}$$

by turning the mean life of 10^{19} *years* of $42-Mo-100$ into the mean-life of18.5 *seconds* for $43-Tc-100$, with natural decay

$$Tc(43,100,1,99.9076\ amu) \rightarrow$$

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$$\rightarrow Ru(44,100,0,99.9042\ amu) + e^- + \nu\ (or\ a), \quad (106)$$

where one should note that the final nuclide is stable and there is no emission of neutronic or other damaging radiation.

From data [99], we know that the indicated transmutation produces heat for $3.475\ MeV$ per reaction plus electricity generated by easily trappable electrons.

For brevity, interested readers may inspect Ref. [97], page 342 for a possible industrial recycling of nuclear waste via a beam of resonating photons causing two or more stimulated neutron decay per nucleus.

8.2.10-III. Recycling nuclear waste via pseudo-proton irradiation. The 2020 Teleconference [0] included discussions on the possible recycling of radioactive nuclear waste via their irradiation with pseudo-protons (Section 8.2.7). In this case, unlike stimulated decay (105)-(105), we have the predicted reaction via the use of two pseudo-protoids with spin 0

$$2\bar{p}^-(-1,1,01.0073\ amu) + Mo(42,100,0,99.9074\ amu) \rightarrow$$

$$\rightarrow Zr(40,102,0,101.9229\ amu)n + \Delta E, \quad (107)$$

in which case the $10^{19}\ years$ main-life of $40-Zr-102$ is reduced to the of $2.9\ seconds$ mean-life of $40-Zr-102$.

Since the corporate handling of nuclear waste is prohibited by law, systematic research on the suggested stimulated recycling of radioactive nuclear waste by nuclear plants themselves is left to governmental laboratories of countries interested in developing new technologies permitted Einstein's legacy.

In closing this section, it should be indicated that mainstream physicists generally attempt to discredit the search for the new clean nuclear energies on grounds that they are not predicted by quantum mechanics, without any repetition of the numerous inexpensive tests done to date, by ignoring the gross insufficiencies of quantum mechanics in nuclear physics (Section 3.2), and in oblivion of Einstein's view that "quantum mechanics is not a complete theory" (Figure 21).

Due to its importance for society, the problem of the recycling of nuclear waste was addressed in the opening lecture of the 2020 Teleconference by J. Dunning-Davies (see the recorded lecture [0]) and Ref. [220].

8.3. Applications of EPR completions in chemistry.

8.3.1. Representation of valence electron bonds. There is no doubt that, beginning with the 1916 pioneering contribution by G. N. Lewis, followed by numerous advances, the 20th century valence bind theory has allowed historical discoveries in chemistry.

Yet, it is well known that science at large, and chemistry in particular, will never admit final theories. Advances generally depend on the identification of open problems followed by due scientific process in the proposed solutions.

One of the best kept secrets of the best Ph. D. schools in chemistry is that there is no possibility for quantum mechanics and, therefore, for quantum chemistry, to represent the *attraction* between the *identical* electrons in valence bonds because the Schrödinger equation for an electron pair, Eq. (1), can only predict the extremely big *repulsion,* of 230 Newtons at the 10^{-13} cm mutual distance of a valence electron bond (Eq. (6) and Section 3.5).

A truly *attractive* force between identical valence electron pair was achieved in 2001 (see Chapter 4 of Ref. [30]) inspired by the last statement of the 1935 paper by Einstein, Podolsky and Rosen [1] according to which the *wavefunction* of quantum mechanics and chemistry cannot describe all "elements of reality."

As reviewed in Section 7.4, the much needed attractive force was achieved thanks to the completion of quantum into hadronic chemistry,

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Figure 24: *The third serious environmental problem facing mankind is the lack of full combustion of fossil fuels. It is unfortunate for our environment that mainstream chemistry generally discredits its solution via the EPR completion of quantum chemistry in oblivion of the inability by quantum chemistry to represent the attraction between valence electron pairs and Einstein's legacy that quantum wavefunctions cannot represent all "elements of reality."*

resulting in a Hulten-type new bond, of type (61) called *strong isovalence bond,* which is so strongly attractive to generate a quasi-particle called *IsoElectronium* (IE)

$$IE = (\hat{e}^-_\downarrow, \hat{e}^-_\uparrow)_{hm}, \tag{108}$$

with double elementary charge, null spin and null magnetic moment.

In this section, we hope to indicate the unreassuring implications for our environment of molecular models without a clear attraction between valence electron pairs (Figure 19).

8.3.2. Representation of molecular data. Historical advances have been made during the 20th century in the collection and representation of molecular data. Yet, a belief in the achievement of final knowledge would imply the existing from the boundaries of science since so much remains to be understood.

Quantum mechanics allowed the representation of the experimental data of the Hydrogen atom from first axiomatic principles with an incredible accuracy. By comparison, when two Hydrogen atoms are bonded together in the Hydrogen molecule, quantum chemistry misses 2% of the molecular binding energy from first axiomatic principles without adulterations, which percentage is far from being ignorable because it corresponds to about $950\ BTU$.

The inability by quantum chemistry to achieve an exact representation of the binding energy of the Hydrogen and other molecules is clear evidence of the appearance in valence electron bonds of short range nonlinear, non-local and non-potential, thus non-Hamiltonian interactions typical of the EPR entanglement (Section 7.2.3) that are absent at the large mutual distances of the atomic structure.

In fact, the EPR completion of quantum into hadronic chemistry, and the achievement of a strong valence bond, have allowed the following advances:

1) The representation of the binding energies of the Hydrogen [31] and water [32] molecules which is *exact to the desired decimal value.*

2) Perturbative calculations used in the above results that have a convergence at least one thousand times faster than the convergence of the corresponding quantum chemical calculations, thanks to the very low value of the isotopic element $\hat{T}$ inserted in all associative products $A\hat{\times}B = A\hat{T}B$ (Section 7.5).

3) The representation of the diamagnetic character of the Hydrogen molecule which is not possible for 20th century weak valence bonds.

4) The explanation of the reason why the Hydrogen molecule can only accommodate *two* Hydrogen atoms in its stable configuration, resulting in a *restricted three-body model of the hydrogen molecule,* with the consequential, first known admission of analytic solutions that have evident importance for the environment, e.g., to achieve full combustion of fossil fuels.

5) Consequential advances in other molecules, such as the *iso-Helium*

model [122] and other advances [123].

8.3.3. The new chemical species of magnecules. Nowadays, we release in our atmosphere about *35 billion tons of contaminants per year* mostly composed by Carbon Monoxide $CO = C - O$ (where $-$ represent valence bond) and HydroCarbons HC. As it is well known, this environmental problem is caused by the incomplete combustion of fossil fuels (Figure 24). By noting that CO and HC are themselves combustible, the indicated environmental problem is caused by the lack of complete combustion of fossil fuels. In turn, this is due to a combustion insufficient for the dissociation of the valence bonds of fuels such as gasoline or diesel. Note that the dissociation of fuel molecules into their atomic constituents is a necessary pre-requisite for the chemical synthesis of CO and HC.

In view of the above unreassuring data, a considerable scientific and industrial effort was done in early 2000 to conceive, test and produce *fuels with the new chemical species of magnecules* (see Chapter 8 of Ref. [30] and the U. S. patent [123]). The new species of magnecules (to distinguish them from molecules) has essentially the same *atomic* components of fossil fuels and it is stable in pressure thanks at ambient temperatures, yet *the binding energy of magnecules is much smaller than that of molecules* as a necessary condition to achieve full combustion, namely, a combustion without detectable CO and MHC n the exhaust.

The new species of magnecules is essentially composed by clusters of atoms (such as H, C, etc.), dimers (such as $H - O$, $C - H$, etc.), and ordinary molecules (such as $H - H$, $C - O$, etc.) bonded together by the attractive force called *magnecular bond* between opposing magnetic polarities of toroidal configurations of the orbits of peripheral electrons, which magnecular bond is stable under ambient temperature.

As an illustration, by denoting the magnecular bond with the symbol

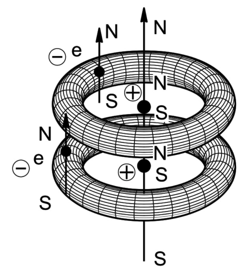

Figure 25: *An illustration of the elementary magnecule such as $H \times H$, $O \times O$, $C \times O$, etc., where one can see: the toroidal polarization of atomic orbitals; the attraction between opposing magnetic polarities; and the exposure of nuclei out of their electron cloud with the proper spin coupling, which features are essential for the ICNF of Section 9.2.9-II.*

$''\times''$, samples of *elementary magnecules* are given by (see Figure 25)

$$H \times H, \;\; O \times O, \;\; C \times O, etc. \tag{109}$$

By noting that electron orbitals can be controlled under very big electric and magnetic fields [30], new fuels with magnecular structure are produced via specially designed hadronic reactors converting fossil fuels into their gaseous magnecular form via a submerged electric arc. Water is added to fossil fuels to increase the content of Hydrogen and Oxygen of the magnecular species by therefore improving its environmental qualities [124].

Stock cars produced to run on compressed natural gas, but operated on compressed magnegas release an exhaust containing no appreciable Carbon Monoxide CO or HydroCarbon HC; and Oxygen O_2 up to 14% (see Figure 23 and the documentation presented at the 2016 International Summit on the Environment, Hainan Island, China [126]).

In view of the importance of the above features for the environment, the publicly traded company *Magnegas Corporation* was organized for

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Figure 26: *A view in the left of a Ford Cavalier produced to run on compressed natural gas and run on compressed magnegas with no appreciable CO and HC in the exhaust, and a view in the right of a Ferrari 308 GTS 1981 converted to run on compressed magnegas and shown at the Moroso International Race Track, Florida, to be competitive with same Ferraris running on gasoline.*

the production and sale world wide of the gasification of fossil fuels into the environmentally friendly *magnegas* (chemical symbol MG, Figure 26) [125].

It should be indicated that *the author does not recommend the large scale use of Hydrogen as an alternative automotive fuel to gasoline* because: 1) Cars running on compressed Hydrogen are expected to produce neutrons (Section 8.2.3); 2) The large scale use of Hydrogen would cause a prohibitive Oxygen depletion; 3) Hydrogen seeps through walls and immediately raises to the Ozone Layer with ensuing rapid chemical reaction $H_2 + O_3 \rightarrow O_2 + H_2O$; and for other environmental problems. [30].

8.3.4. Experimental verifications of the new species of magnecules. The chemical composition of gases is nowadays analyzed via Gas Chromatographers Mass Spectrometers (GC-MS), InfraRed Detectors (IRD) and other equipment all designed, specifically, to detect *molecules,* that is, atoms bonded together by valence bonds with ensuing

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large binding energy. In said instruments, molecules are first exposed to an ionization beam whose energy is below the molecular dissociation energy; the molecules are then processed by a reactant in the column; and then eluded with a speed inversely proportional to their masses for identification.

By noting that magnecules have a binding energy which is about 10% that of molecules, GS-MS are basically unable to detect magnecules because their ionization beams destroy the very species to be detected, since they terminate all magnecular bonds and reduce the species to conventional molecules.

In view of the above, magnecules can only be indirectly detected via GC-MS/IRD and the findings confirmed via direct measurements of the characteristics of the magnecular gas, such as flame temperature, BTU content, etc. [126].

The instruments used so far for the analysis of fuels with magnecular structure are GC-MS equipped with IRD (GC-MS/IRD) so as to test first the gas un the GC-MS used with special provisions, such as the lowest possible ionization energy, the lowest possible column temperature, and the longest possible elusion time so as to minimize the destruction of the species to be detected. The clusters identified in the GC-MS are then tested via the IRD *without transferring the gas to a separate IRD because of the impossibility to combine with certainty results from different tests.* Magnecules are detected when the clusters identified in the GC-MS have no IR signature because lack of IR signature implies the lack if existence of internal molecular bonds.

The GC-MS/IRD tests that were originally done by the author in 1998 are reported in Chapters 8 and 9 of Ref.[30], Figures 8.7 on). More recent independent experimental verifications via GC- MS/IRD are available in Refs. [127] [128]. Comparison of the original tests [30] with then more recent tests [127] [128] shows that the detection of magnecular clusters is decreased in the latter tests due to the increase of the ionization energy of the GC-MS which was confirmed by comparison of data in the relative manuals.

The best evidence establishing the existence of a *new* chemical species is given by direct measurements of the characteristics of magnegas since said characteristics cannot be explained with quantum chemistry.

The first of these anomalous characteristics is given by extensive tests conducted by scientists of the *Institute for Ultra Fast Spectroscopy and Laser of the City College of New York,* [129] [130] which established that *magnegas synthesized from fossil fuels has a flame temperature in air of* $10,597\ F = 3,647\ C$ which is 56% *bigger* than the hydrogen flame temperature in air $2,045\ C$. The increased flame temperature explains the absence of CO and HC in magnegas exhaust due to their combustion. Additionally, the increased flame temperature makes magnegas a fuel particularly suited for steel mills, refineries and other applications [124]-[126].

An intriguing feature of magnegas is its energy content because *magnegas cuts a 12" thick steel slab faster than acetylene* (C_2H_2) *which possesses* $1,498\ BTU/cf$ [132]. This anomaly property is established by the fact that a conventional chemical analysis of magnegas done at maximal ionization energy and column temperature reveals that magnegas is composed by

$$MG:\ 54\%\, H_2,\ \ 31\%\, CO,\ \ 15\%\, HC, \tag{110}$$

with corresponding energy content of $325,\ 323\ , 1,500\ BTU/cf$. Hence, according to quantum chemistry, magnegas synthesized from petroleum should contain a maximum of

$$54\%\, 325 + 31\%\, 323 + 15\%\, 1,500\ BTU/cf = 431\ BTU/cf, \tag{111}$$

namely, an amount basically insufficient to cut any metal, let alone 12" thick steel plates, thus confirming that *magnegas cannot be quantitatively described via quantum chemistry.*

The description of the energy content of magnegas according to hadronic chemistry is essentially the following. Let us denote the molecular bond with the symbol "$-$" and the magnecular bond with "$\times$." It is generally believed that the combustion of Hydrogen and Oxygen according to the reaction

$$H - H + (1/2)O - O \rightarrow H - O - H + 57\,BTU/cf. \tag{112}$$

However, the two Hydrogen atoms are separated in the $H - O - H$ molecule. Therefore, in order to verify the principle of conservation of the energy, the combustion of Hydrogen and Oxygen must produce $104\,kcal/mole$ for the $H - H$ dissociation, plus $45\,kcal/mole$ for the $O - O$ dissociation, plus $57\,kcal/mole$ of produced heat, for a total of $206\,kcal/mole$. In the event the Hydrogen and Oxygen species have a magnecular structure $H \times H$ and $O \times O$, the binding energy is only 10% of the molecular value. In this case, the combustion of the *magnecular* species $H \times H$ and $O \times O$ produces the total energy of $97 + 40 + 57\,kcal/mole = 194\,kcal/mole$

$$H \times H + (1/2)O \times O \rightarrow H - O - H + 194\,kcal/mole, \tag{113}$$

which is 3.4-times the energy produced by the molecular species, Eq. (113), resulting in the value of $1,465\,BTU/cf$ which is comparable to the energy content of acetylane $1,498\,BTU/cf$, by providing in this way the only known quantitative explanation of the reason magnegas cuts faster than acetylane under essentially the same atomic structure.

In conclusion, *the energy content of molecular gases is constant while, by comparison, the energy content of magnecular gases is variable because it depends on the original liquid feedstock, the power of the hadronic reactor, and other engineering data.*

8.3.5. Magne-Hydrogen and Magne-Oxygen. The anomalous temperature and energy content of magnegas is due to the anomalous character of its primary components known as *Magne-Hydrogen* (symbol MH), *Magne-Oxygen* (MO) and *Magne-Carbon-Monoxide* (MCO) [133].

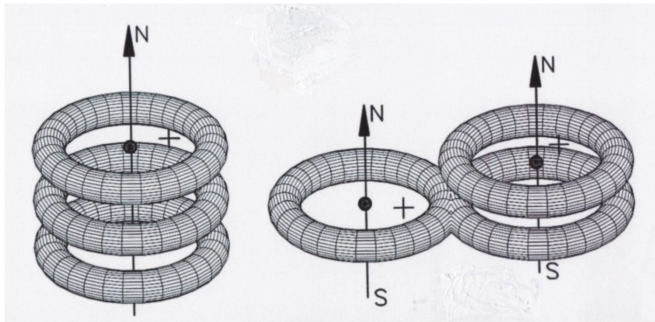

Figure 27: *An illustration of the magnecule $3H = H \times H \times H$ that should be compared with the molecule $H_3 = H - H - H$. The main difference is that the latter is unstable, while the former is stable at ambient temperature according to repeated detections of 3 amu in the same MH gas. Such a difference is due to the independence of the magnecular bond from the number of bonded atoms compared to the lack of existence of a valence electron triplet.*

The indicated new species have been separated from magnegas via Pressure Swing Adsorption (PSA) equipment, also called molecular seeving, and their anomalous specific weight has been confirmed by independent measurements [134]-[137]. By recalling that MH constitutes more than 50% of MG, the anomalous energy content of MH is established by that of MG. Tests on the increase of the efficiency of fuel cells via the use of MH and Oxygen are reported in Ref. [124]. The case of a 3-atomic magnecule or molecule is indicated in Figure 27.

An important experimental evidence in the above tests is that the specific weight of the magnecular species MH, MO and MCO increases with the number of times the separated gas is passed again through the PSA equipment. This is a clear indication of the effect known as *magnecular accretion,* namely, the increase of the mass of magnecule with the increase of pressure and other treatments.

In turn, the above effect indicates that *the Avogadro number is not expected to be a constant for magnecular gases* [30], which feature is suggested for test by interested chemists jointly with other innovative measurements.

An understanding of the above intriguing feature is the following. Recall from Section 3.5 that another insufficiency of quantum chemistry is the impossibility of achieving an *attractive* force between the water molecules if the liquid state due to their diamagnetic and dielectric characters. It appears that such an attractive force is of magnecular type since the Hydrogen atoms H in the water molecule $H-O-H$ has a toroid polarization of its orbit for the proper bonding to the corresponding valence electron of the Oxygen, thus allowing a magnecular attraction of the type [138]

$$H-O-H_{\downarrow}\times H^{\uparrow}-O-H^{\uparrow}\times H_{\downarrow}-O-.... \tag{114}$$

According to this view, the magnecules of the new species MH, MO and MCO are *quasi-liquid* with consequential magnecular accretion and the predicted lack of constancy of the Avogadro number.

Chemical analyses have repeatedly established that MH is composed by atomic masses from $1\ amu$ to hundreds of amu, thus suggesting a structure of the type

$$\begin{aligned} MH:\ \ &\{H,\ H-H,\ H\times H,\ H\times H,\\ &H-H\times H,\ H-H\times H,....\}, \end{aligned} \tag{115}$$

with a corresponding structure for MO and MCO.

In view of its special features, including the increased specific weight, increased energy content and predicted increase of the liquefaction temperature, MH is expected to be particularly significant for the aerospace industry as well as for gasoline refineries, fertilize production and other fields. MO is expected to be signifiant for medical applications, e.g., for use in ventilators for persons affected by the Corona Virus because its

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magnecular structure is expected to decompose at contact with lungs, with ensuing release of sterilizing UV radiation as it is the case for the decomposition of Ozone. MCO has been conceived and tested for the primary aim of developing the HyperCombustion indicated (Section 8.3.7).

8.3.6. The HHO combustible gas. One of the most intriguing fuels with magnecular structure is the *HHO gas* [139] which is produced in the gasification of water via a specially designed water electrolyzer developed by the U. S. company *Hydrogen Technologies Applications, Inc.*

Some of the unique features of the HHO gas, which are manifestly outside any serious representational capability by quantum chemistry, are:

1) The HHO gas instantly cuts bricks, tungsten and other hard material at content in air or under water;

2) The combustion of the HHO gas occurs without any need for atmospheric Oxygen, since HHO is composed by a stochiometric ratio of Hydrogen and Oxygen; and

3) The combustion exhaust of the HHO gas are composed by water vapor without any contaminant.

By using a cautious scientific language, Ref. [139] (see also Ref. [124]) presents a series of *measurements* on the HHO gas conducted with various instruments and their tentative representation with the new chemical species of magnecules.

It should be indicated that all hadronic reactors indicated in this section have a *negative* energy balance in the sense that they require energy to produce a gaseous, environmentally friendly fuel, unless, jointly with the gasification of the liquid feedstock, the hadronic reactos synthesize new elements, such as the synthesis of $Si-28$ from $C-12$ and $O-16$, in which case the energy balance is positive even for nuclear synthesis in ppm.

8.3.7. Hyper-Combustion. At the dawn of the third millennium,

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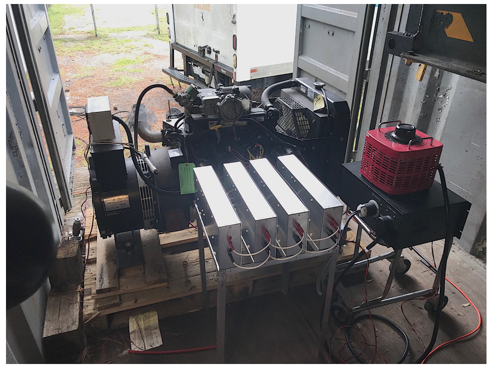

Figure 28: *A view of the equipment developed by Hadronic Technologies Corporation to test Hyper-Combustion in a four cylinder electric generator showing (from bottom right) Control Module, Variac and four ICNF Activation Banks (Section 3.3.7).*

the combustion of fossil fuels is essentially the same as it was some fifty thousands years ago, because in our civilian, industrial and military engines we essentially strike a spark and lit the fuel.

This section is intended for scientists interested in the search for a basically new form of combustion as a necessary condition for the future achievement of a sustainable life of our planet.

Recall from Section 8.3.3 that magnegas does achieve *full combustion,* namely a combustion without appreciable combustible contaminants in the exhaust.

Based on such a result, a main drive of the physical and chemical studies reported in this section has been the achievement of full combustion, this time, for fossil fuels as commercially available, including gasoline, diesel, methane, acetylane, et al.

Recall that our combustion of about 36 *billion* barrels of crude oil per year releases in the environment about 10 billion barrels per year of contaminants such as CO and HC. However, it is important to note that CO and HC are themselves combustible. This establishing that *the temperature of combustion in our civilian, industrial and military engines is insufficient to achieve full combustion.*

The studies reported in this section have led to the formulation of a new form of combustion known as *Hyper-Combustion* [140] under development by the privately held *Hadronic Technologies Corporation* which can be defined as follows: *the hyper-combustion of Carbon with atmospheric Oxygen comprises the conventional chemical combustion plus a controllable number of engineering means causing the synthesis of Silicon and other light, natural and stable elements without the emission of harmful radiation and without the release of radioactive waste.*

The engineering realization of the hyper-combustion is based on the use of a specially designed DC power unit delivering an arc that: 1) Ionizes the fuel; 2) Creates magnecules $C \times O$ between toroidally polarized stable isotopes of Carbon and Oxygen; and 3) Triggers Intermediate Controlled Nuclear Fusions (ICNF, Section 8.2.9) in parts per million (ppm) of the indicated magnecule Carbon and Oxygen into the Silicon.

Recall from Nuclear Data [99] that Carbon has the following two stable isotopes with relative abundance $6-C-12,\ 98.898\%,\ 6-C-13,\ 1.11\%$ and that the Oxygen has the following three stable isotopes with relative abundance $8-O-16,\ 99.76\%,\ 8-O-17,\ 0.037\%,\ 8-O-18,\ 0.200\%$. Consequently, when dealing with nuclear fusions in ppm, the significant isotopes are $6-C-12$ and $8-O-16$ with related nuclear fusion into the $14-Si-28$ studied in Section 9.2.9-II, Eqs. (98)-(99). This fusion has been experimentally verified with the Third Hadronic Reactor (Figures 16 and 17, video [113], sound [108], and verifications [114]-[120]).

Note that the aim of hyper-combustion *is not* that of replacing fossil fuels with controlled nuclear fusions, but that of using Carbon and Oxygen as fuels for nuclear fusions mainly intended to maximize the energy output of crude oil and achieve full combustion without harmful radiations or radioactive waste.

To minimize possible misrepresentations, it should be reported that Hadronic Technologies Corporation has manufactured an equipment (Figure 28) testing Hyper-Combustion in a four cylinders electric generator with tungsten tip spark (patent pending) which comprises: 1) A control module to set maximal power and arc settings per selected engine; 2) A variac to operate with minimal and maximal power; and 3) Four ICNF Activation Banks, one per cylinder. The equipment produced the desired increase of the combustion temperature at which CO, HC and other combustible contaminants burn by producing a first increase of power output. The same equipment has produced a second increase of energy output due to the activation of ICNF in ppm of Carbon and Oxygen into Silicon within the thermodynamical limit of the electric generator.

In closing this section, it should be reported that mainstream chemists generally dismiss the *anomalous characteristics* of magnegas as being due to *anomalous measurements,* without repeating the measurements and without considering the fact that said measurements were done by the highly authoritative Institute for Ultra Fast Spectroscopy and Laser of the City College of New York [130] [131].

Additionally, the same chemists generally dismiss the existence of the new species of magnecules on grounds that the identified *anomalous clusters* can be reduced by the GC-MS to their *molecular* constituents, by ignoring that: 1) Stable clusters systematically detected by GC-MS must have an internal atomic or molecular bond to exist; 2) The ionization energy used in the decomposition of the clusters to their molecular constituents destroys the very species to be detected; and 3) The dismissal is done in oblivion of Einstein's view on the limitations of quantum mechanics and chemistry, as well a without the proof in refereed journals that quantum chemistry provides a quantitative representation of magnegas anomalous features.

Objections against magnegas and magnecules voiced in social media, rather than via publications in refereed journals, has significantly undermined and delayed the solution to our environmental problems. In fact, as it is well known, social media plays in the hands of those who profit by shorting the stock of publicly traded companies, by therefore causing significant losses to investors eager to support the development of new environmental technologies for a sustainable future of our planet.

8.4. Applications of EPR completions to special relativity.

8.4.1. Foreword.

In this section, we show that the methods developed for the proof of the EPR argument [1] imply *a geometric unification of Einstein's special and general relativities for the exterior problem of point-like particles in vacuum, as well as their extension for interior dynamical problems of extended particles within physical media .*

8.4.2. Applications of the EPR completion to Galileo's relativity. As it is well known, the Galileo symmetry $G(3.1)$ and relativity are exactly valid for the description of non-relativistic conservative systems of point-like particles moving in vacuum (*non-relativistic exterior dynamical systems*), thus without any resistive force or contact interaction.

Since point-like particles are an approximation of the physical reality, the verifications of the EPR argument reported in Section 7 mandate the completion of Galileo symmetry and relativity for *non-relativistic interior dynamical systems* comprising extended particles in deep EPR entanglement, with ensuing resistive, as well as non-linear, non-local and non-Hamiltonian interactions (Figure 29).

The above need suggested the construction of the Lie-isotopic completion of the Galileo symmetry $\hat{G}(3.1)$ and relativity, called *Iso-Galilean isorelativity,* for the axiom-preserving representation of extended masses moving within physical media, thus experiencing resistive as well as con-

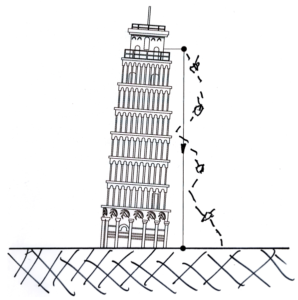

Figure 29: *A schematic view of the experiments done by Galileo Galilei at the end of the 16th century to measure the acceleration due to gravity by dropping from the top of the Pisa tower balls with different masses assumed to be point-like, thus ignoring the resistance due to our atmosphere (vertical line). The axiom - preserving iso-Galilean isorelativity aims at the dynamics of extended bodies, thus including atmospheric resistance (wiggly line).*

tact interactions. Note that the axiom- preserving condition restricts the system to have a conserved total energy, as it is the case for the non-relativistic description of an isolated nucleus with contact internal forces.

Since conservative systems are an evident particular case of nonconservative/irreversible systems, the studies here considered required the construction of the broader Lie-admissible completion of the Galileo symmetry $\hat{G}^{>}(3.1)$ and relativity for the description of extended particles in nonconservative conditions, as it is the case for the non-relativistic representation of nuclear fusions.

Since non-relativistic studies are an evident pre-requisite for relativistic counterparts, we regret not to be able to review them to avoid a prohibitive length of this Overview. Nevertheless, we indicate for interested colleagues that non-relativistic studies were initiated in paper [19] of 1978 with the Lie-admissible completion of Galileo symmetry and relativity for non-conservative and Galileo- non-invariant systems.

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The first direct study of Galileo's isosymmetry was done in Section 5.3, pages 225 on, of the 1983 monograph [22] formulated over conventional fields. These isotopies were then systematically studied and upgraded in the 1991 volumes [71] [72]. The formulation of Galilean isosymmetries with the full use of isomathematics was done in the 1995 monographs [23] [24] [73].

The above studies attracted the attention of Abdus Salam, founder and President of the *International Center for Theoretical Physics* (ICTP), Trieste, Italy, who invited Santilli in 1991 to deliver at his Center a series of lectures in the isotopies of the Galileo symmetry and relativity, said invitation being apparently the last one by Salam prior to his death. During his visit at the ICTP, Santilli wrote papers [147]-[153]. The notes from Santilli's lectures at the ICTP were collected by some of the attendees and published in volume [154] of 1992.

8.4.3. Special Relativity (SR). Mainly for the sake of notation, we recall that special relativity (SR) is defined on a Minkowski space $M(x, \eta, I)$ over the field of real numbers $\mathcal{R}$ with local coordinates $x = (x,^{\rho}),\ \rho = 1, 2, 3, 4,\ x^4 = ct$, metric $\eta = Diag.(1, 1, 1, -1)$, unit $I = Diag.(1, 1, 1, 1)$, and spacetime interval

$$(x - y.)^2 = (x - y)^{\mu}\eta_{\mu\ni}(x - y)^{\nu} =$$
$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (t_1 - t_2)^2c^2, \quad (116)$$

which is left invariant by the Lorentz symmetry $SO(3.1)$, the Lorentz-Poincaré symmetry $P(3.1)$ and its spinorial covering $\mathcal{P}(3.1)$.

The above methods uniquely and unambiguously characterize the following basic axioms for a *time-reversal invariant relativistic system of point-like particles and electromagnetic waves propagating in vacuum:*

AXIOM I: The speed of light c is the maximal causal speed for point-like particles propagating in vacuum

$$V_{max} = c \quad (117)$$

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AXIOM II: The addition of speeds follows the relativistic law

$$V_{tot} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \tag{118}$$

AXIOM III: The dilation of time, the contraction of lengths, the variation of mass and the mass-energy equivalence follow the relativistic laws:

$$t' = \gamma\, t, \tag{119}$$

$$\ell' = \gamma^{-1}\, \ell, \tag{120}$$

$$m' = \gamma\, m, \tag{121}$$

$$E = mc^2 \tag{122}$$

where

$$\beta = \frac{v}{c}, \ \ \gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{123}$$

AXIOM IV: The frequency shift due to relative speed follows the law (for null aberration)

$$\omega' = \gamma\, [1 - \beta\, cos(\alpha)]\, \omega. \tag{124}$$

The above axioms are hereon assumed to be exactly valid for the assumed time-reversal invariant systems.of point-like particles and electromagnetic waves in vacuum.

8.4.4. Special isorelativity (SIR). As a consequence of the widely assumed reduction of the entire universe to point-like particles, it is generally believed that special relativity and the constancy of the speed of light c, are valid for whatever conditions exist in the universe.

In the preceding sections we have shown that such a conception can be considered as being *approximately valid,* although it implies a number of insufficiencies, such as the inability to achieve exact representations of

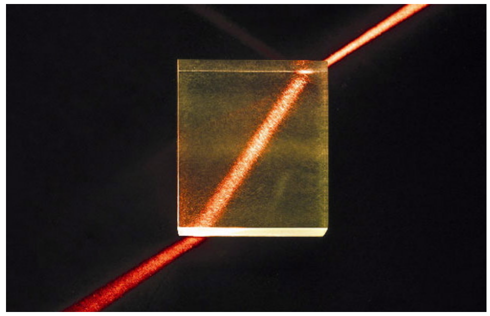

Figure 30: *The light beam passing through a glass of water depicted in this figure is generally reduced to photons for the intent of maintaining special relativity in interior conditions. However, such a reduction is afflicted by the widely propagated (yet ignored) Inconsistencies 1) to 7) of Section 8.4.1 whose resolution mandates the completion of SR for interior dynamical conditions.*

nuclear and molecular data due to the extended character of particles at a mutual distances of the order of 10^{-13} cm (relativistic interior dynamical problems).

When the reduction of the universe to point–like constituents is assumed as being *exactly valid,* it leads to serious inconsistencies that remain generally ignored by the mainstream physics community, although they need to be brought to the attention of the scholars in the field. For instance, the belief that electromagnetic waves propagating in water can be reduced to photons propagating in vacuum and scattering among the water molecules is afflicted by the following [24]:

INCONSISTENCY I: The reduction of light to photons cannot represent the angle of refraction of light in water, evidently because photons will scatter in all directions at the impact with the water surface.

INCONSISTENCY II: The reduction of light to photons cannot represent the decrease of the speed of light in water by $100,000$ km/s, because calculations have shown that the scattering of photons among water molecules can at best represent a reduction of the speed of light of about $7,000\ km/s$.

INCONSISTENCY III: The reduction of light to photons cannot represent the propagation of light in water as a beam, again, because a beam of photons will scatter among the water molecules and disperse in water.

INCONSISTENCY IV: The reduction light to photons has no physical sense for infrared light or radio waves with $1\ m$ wavelength that experience the same phenomenology as that for electromagnetic waves at large.

INCONSISTENCY V: The reduction of light to photons scattering among water molecules, thus propagating in vacuum at speed c, cannot represent the Cherenkov light because said light can only occur when electrons travel *faster* than the local speed of light.

INCONSISTENCY VI: The reduction of light to photons cannot resolve the inapplicability of the relativistic sum of speeds in water, Eq. (118), since the sum of two light speeds in water does not yield the light speed in water.

INCONSISTENCY VII: The reduction of light to photons cannot be tested experimentally due to the lack of an inertial frame in water.

Following the construction of the axiom-preserving isotopies of the various branches of Lie's theory in the 1983 monograph [22], Santilli constructed the **axiom-preserving isotopies of special relativity for extended particles with interval** (25) in Ref. [60] for the classical part and Ref. [61] for the operator counterpart, with the first known construction of the the Lorentz-Isotopic symmetry $\hat{SO}(3.1)$, today known as the *Lorentz-Santilli isosymmetry,* with isotransformations (30).

Subsequently, Santilli conducted systematic studies [62]-[74] of the isotopies of all conventional spacetime symmetries resulting in the Lorentz-Poincaré- Santilli isosymmetry $\hat{P}(3.1)$ [67] and related isospinorial covering $\hat{\mathcal{P}}(.)$ [68] [69] which are formulated on an iso-Minkowskian isospace first introduced in Ref. [60] (see Ref. [70] for detailed studies) $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ with isospacetime isocoordinates $\hat{x} = x\hat{I}, \ \hat{y} = y\hat{I}$, isometric correctly written as having elements in the isofield $\hat{\mathcal{R}}$

$$\hat{\Gamma} = \{\hat{\eta}_{\mu\nu}\}\hat{I} = (\hat{T}\eta)\hat{I}, \tag{125}$$

and isounit ($\hat{I} = 1/\hat{T} > 0$ given in Eq. (34), with invariant (25),which we now rewrite in the form

$$(\hat{x} - \hat{y})^{\hat{2}} = (\hat{x} - \hat{y})^{\mu} \hat{\times} \hat{\Gamma}_{\mu\nu} \hat{\times} (\hat{x} - \hat{y})^{\nu} =$$

$$= [(x - y)\hat{I}]\hat{T}(\hat{\eta}\hat{I})\hat{T}[(x - y)\hat{I}] = [(x - y)^{\mu}\hat{\eta}_{\mu\nu}(x - y)^{\nu}]\hat{I} =$$

$$[\frac{(x_1 - y_1)^2}{n_1^2} + \frac{(x_2 - y_2)^2}{n_2^2} + \frac{(x_3 - y_3)^2}{n_3^2} - (t_1 - t_2)^2 \frac{c^2}{n_4^2}]\hat{I}, \tag{126}$$

where: the multiplication of the interval by the isounit $\hat{I}$ is necessary for its value to be an element of the isofield; and the characteristic quantities $n_\rho, \ \rho = 1, 2, 3, 4$ are solely restricted by the condition of being positive-definite $n_\rho > 0$, but possess otherwise an unrestricted functional dependence on all needed local variables (which shall be hereon tacitly assumed) such as local coordinates x, velocity v, momentum p, energy E, density ρ, temperature τ, pressure γ, frequency ω, wavefunctions ψ, their derivatives $\partial_r\psi$, etc. $n_\rho = n_\rho(t, r, v, p, E, \rho, \tau, \gamma, \omega, \psi, \partial\psi, ...)$.

A feature important for the understanding of this section is that all (non-singular) infinitely possible realizations of the iso-Minkowski isospace $\hat{M}$ on an isoreal isofield $\hat{\mathcal{R}}$ are locally isomorphic to the conventional space M over the reals $\mathcal{R}$. This property was first proved in the 1983 paper [60] (see Refs. [23] [72] for a detailed treatment) and can be seen from the fact that the Minkowski metric η is completed into the isometric $\hat{T}\eta$ while, jointly, the basic unit I is completed by the *inverse* amount $\hat{I} = 1/\hat{T}$, thus preserving the original metric η.

Alternatively, one can see from Eq. (125) that the numeric value of the *Minkowskian* metric η is preserved under isotopies since $(\hat{T}\eta)\hat{I} \equiv \eta$.

Note the duality of the formulation, namely, the iso-Minkowski isospace can be first written on isospace over isofields (see the first line of interval (126)), in which case SR applies identically, and then *projected* on the conventional Minkowski space over a conventional field (see the third line of interval (126))where novelties appear.

Recall that the Minkowskian geometry represents a *homogeneous and isotropic* 3*-dimensional space* while, by comparison, the iso-Minkowskian isogeometry represents an *inhomogeneous space,* due to the local variation of the density, as well as an *anisotropic space,* due to the change of geometry with the change of the direction.

Consequently, axioms (117)-(124) of SR do not need the identification of the selected direction, while such an identification is necessary for the SIR due to the indicated variation of physical characteristics with the variation of the space direction.

Under the above clarifications, special isorelativity can be defined as the *axiom-preserving formulation of special relativity on iso- Minkowski isospaces over isoreal isofields.* Its universal LPS isosymmetry characterizes uniquely and unambiguously the following isoaxioms first formulated systematically in Refs, [72] of 1991 on conventional fields and completed in Ref.[24] of 1995 over isofields, here expressed for the selected k-direction, e.g., that of the third space component,

ISOAXIOM I: The speed of light within (transparent) physical media is given by the locally varying speed:

$$C = \frac{c}{n_4}. \tag{127}$$

ISOAXIOM II: The maximal causal speed within physical media is given by:

$$V_{max,K} = c\frac{n_k}{n_4}. \tag{128}$$

ISOAXIOM III: The addition of speeds within physical media follows the isotopic law:

$$V_{tot} = \frac{\frac{V1.k}{n_k} + \frac{c_{2.k}}{nn_k}}{1 + \frac{v_1 v_2}{c^2}\frac{n_4^2}{n_k^2}}. \tag{129}$$

ISOAXIOM IV: The isodilation of time, the isocontraction of lengths, the isovariation of mass and the mass-energy isoequivalence (isorenormalization) within physical media follow the isotopic laws:

$$t'_k = \hat{\gamma}_k\, t, \tag{130}$$

$$\ell'_k = \hat{\gamma}_k^{-1}\, \ell, \tag{131}$$

$$m'_k = \hat{\gamma}_k\, m, \tag{132}$$

$$\hat{E}_k = m\, V_{max,k}^2 = \hat{m}_k c^2, \;\; \hat{m}_k = m\frac{n_k^2}{n_4^2}, \tag{133}$$

where, from Eqs. (*31)*

$$\hat{\beta}_k = \frac{v_3/n_k}{c_o/n_4}, \;\; \hat{\gamma}_k = \frac{1}{\sqrt{1-\hat{\beta}_k^2}}. \tag{134}$$

ISOAXIOM V: The frequency shift within physical media follows the isotopic law (for null aberration)

$$\omega' = \hat{\gamma}\, [1 - \hat{\beta}\; cos(\hat{\alpha})]\, \omega. \tag{135}$$

Note that *the maximal causal speed in SIR is no longer given by the speed of light,* and it is given instead by value (128), because physical media are generally opaque to light, thus requiring the broader geometric notion derivable from the expression in $(k-4)$-space

$$\frac{dx_k^2}{n_k^2} - dt^2 \frac{c^2}{n_4^2} = 0. \tag{136}$$

Among a variety of verifications of Isoaxioms I to V, we indicate here the following representative examples:

8.4.4-I. Verification of SIR within liquid media. It is an instructive exercise for the interested colleague to verify that Isoaxioms I to V resolve all Inconsistencies I to7 (Figure 30) in the use of conventional axioms.

This includes the verification via isoaxiom (118) that the sum of two local speeds of light $C = c/n_4$ yields the local speed of light $C = c/n_4$. Moreover, water can be assumed to be homogeneous and isotropic for which $n_\mu = 1,\ \mu - 1, 2, 3, 4$ for which interval (126) becomes

$$(\hat{x} - \hat{y})^{\hat{2}} = (x-y)^2 \hat{I} =$$
$$= [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 - (t_1 - t_2)^2 c^2] \hat{I}, \tag{137}$$

in which case the maximal causal speed *in water* is the speed of light c *in vacuum,* by therefore providing the first known quantitative representation that *electrons can indeed travel in water faster than the local speed of light.*

Additionally, the iso-Minkowskian geometry [70] provides a geometric representation of the difference between the actual size d_{act} of an object in water and that perceived by an external observer d_{ext}. For this purpose, note that in water we have the value $\hat{I} = n_4^2 = 9/4 > 1$. The use of the basic law for isotopy [23] [24] $d_{act}^2 \hat{I} = d_{ext}^2 I$,, then yields the value

$$d_{ext} = n_4 d_{act} = \frac{3}{2} d_{act} \tag{138}$$

Figure 31: *A view of the apparent increase of size of objects submerged in water as seen by an outside observer. This increase is quantitatively represented by Special IsoRelativity (SIR) via Eq. (138) of Section 8.4.4-I with intriguing geometric implications.*

which is essentially verified by visual inspection as one can see from Figure 31.

Intriguing similar properties occur for other external and internal characteristics. For instance we have a similar distinction between the observer time called *external time* t and related unit $I_t = 1$ (which is the time of the external observer), and the *intrinsic time* $\hat{t}$ with related isounit $\hat{I}_t$ (which is the time for an internal observer). Said two times are interconnected by the isotopic law $tI_t = \hat{t}\hat{I}_t$.

The above notion of *isotime* appears to be significant for biological structures [165] because we generally assume that the time felt by a seashell $\hat{i}$ is identical to our time t, while in reality the external and internal times may be different.

8.4.4-II. Verification of SIR within gaseous media. Recall that the Doppler law in vacuum, Eq. (124), can be written in first order

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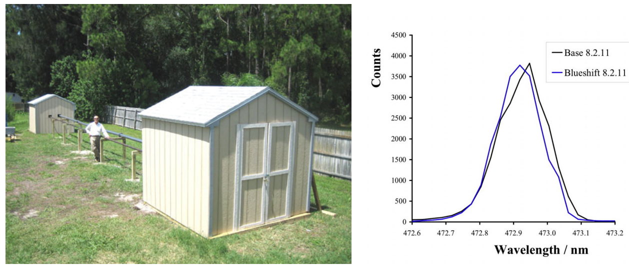

Figure 32: *A view on the left of the* $60'$*-long Isoshift Testing Station that established the energy loss (gain), thus frequency decrease (increase) without any relative motion of a blue laser light passing through air at* 1000 *psi and at* -10^0 *C* *(*$+150^0$ *C) called isoredshift (isoblueshift), which was first predicted in the 1991 Refs. [71] [72], and first detected in the 2010 paper [155] via the scan shown in the right view.*

approximation

$$z = \frac{\omega'}{\omega} - 1 \approx \pm \frac{v}{c}, \tag{139}$$

where the minus (plus) sign occurs for the source moving away (toward) the observer. The corresponding expansion for the iso-Doppler law (135) yields the expression

$$\hat{z} = \frac{\omega'}{\omega} - 1 \approx \pm \frac{v_k}{c} \frac{n_4}{n_k}, \tag{140}$$

where the factor n_4/n_k can be expanded in terms of $(c/v_k)d$, d being the travel of light in the medium

$$\frac{n_4}{n_k} \approx 1 \pm S(\rho, \tau,) \frac{c}{v_k} d, \tag{141}$$

where S s a function of the local density ρ, temperature τ and possibly other local variables.

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The iso-Doppler law (135) then assumes the form in first order approximation, first derived in the 1991 Refs. [71] [72],

$$\hat{z} = \frac{\omega'}{\omega} - 1 \approx \pm\frac{v}{c} \pm S(\rho, \tau, ...)d, \tag{142}$$

which represents two independent frequency shifts, the conventional Doppler's shift $\pm v/c$ due to relative motion between observer and source, and the isoshift $\pm S(\rho, \tau, ...)d$ due to loss (absorption) of energy $E = h\nu$ to (from) a gas. at temperatures lower (bigger) than $0\ C$, called *isoredshift (isoblueshift),* in which the prefix "iso" denotes the sole possible derivation via isomathematics.

Following a decade of failed attempts to locate a physics laboratory interested in proving or disproving the prediction of the isoshift [72], Santilli conducted systematic tests in collaboration with the technicians of *Magnegas Corporation* at its former laboratories located at 150 Rainville Rd, Tarpon Springs, Florida. An *isoshift Test Station* (left view of Figure 32) was built consisting of a front (rear) air-conditioned cabins containing a blue laser light (wavelength analyzers), the two cabins being interconnected by a $60\ ft$ long tube containing air at $1,000\ psi$. The air temperature was varied from $-30\ C$ to $+200\ C$ via commercially available cooling or heating means..

Systematic tests reported in the 2010 paper [155], established that a blue laser light loses (gains) energy, thus decreasing (increasing) its frequency, when passing through air contained in the indicated tube at $1,000$ psi and $-10\ C$ ($+150\ C$).

Since none of the measured frequency shifts occurred with any relative motion between the source and the observer, tests [155] established the existence of the isoshift as predicted in the 1991 monograph [72].

Systematic additional measurements were conducted in the USA and Europe via the best available wavelength analyzers by following Sunlight with a telescope from the Zenith to the horizon. These tests established

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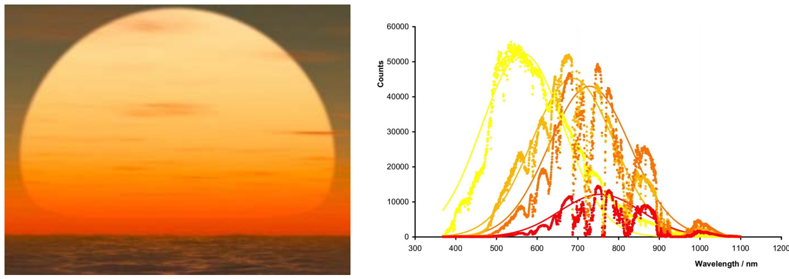

Figure 33: *In order to maintain the validity of special relativity within our inhomogeneous and anisotropic atmosphere, it is generally believed that the redness of Sunlight at Sunset is due to the absorption of blue light resulting in the residual red light, against evidence known since Newton's time that blue light is the most penetrant and red light is quickly absorbed by media. Systematic measurements [159] done in the U.S.A. and in Europe established that the redness of the Sun at Sunset (left view) is due to loss of energy to our atmosphere with ensuing isoredshift of blue light into the red red light (right view).*

that the redness of the Sun at the horizon (left view of Figure 33) is due to a loss of energy to our atmosphere such to cause the isoredshift of the blue light into the red light (right view of Figure 33) [157] (see also lecture f). It should be noted that the scattering of photons among the molecules of our atmosphere is basically insufficient to represent the large redshift from the blue all the way to the red.

8.4.4-III. Verification of SIR in astrophysics. Measurements [159] provided an experimental verification on Earth of Zwicky's hypothesis of *Tired Light* [160], according to which galactic light loses energy to the very cold intergalactic medium (mostly composed by Hydrogen at absolute zero degree temperature) in a way essentially proportional to the covered distance d in a static universe without motion of galaxies at such a velocity v/c to provide a contribution to the redshift.

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In particular, SIR provides a direct representation of Hubble's law with the identification of the S quantity of Eq. (142) with the Hubble constant H_0 without any contribution from the Doppler shift

$$\hat{z} = \frac{\omega'}{\omega} - 1 = -\frac{H_0}{c}d. \tag{143}$$

The above isolaw provides an excellent representation of astrophysical data on cosmological redshift as shown in Ref.[161], including a proportionality from the distance, the representation of the very large redshift of light from galaxies at the end of the known universe without any need for superluminal speeds of entire galaxies or the hyperbolic and unverifiable assumption that space itself is expanding.

The numerical representation of internal galactic redshift anomalies occurs via the use of the isoredshift for star light propagating through the cold peripheral intergalactic medium, and the isoblueshift for star light passing trough hot intergalactic medium near a black hole [162].

The acceleration of the cosmological redshift with the distance d can be beautifully represented via a gravitational contribution to the redshift of galactic light passing near stars or galaxies in their long travel to Earth [163].

The above results set the foundations of the new *Tired Light Cosmology representing a static universe in total EPR entanglement* (Figure 2) according to the view preferred by Einstein, Hubble, Hoyle, Zwicky, Fermi, and others who died without accepting the expansion of the universe.

Note that for about one century the cosmological redshift z has been interpreted to be a direct "measurement" of the speed v of galaxies in an expanding universe according to the SR law (139), while in reality the identification $z = v/c$ is an *assumption* due to the existence of the Tired Light and other interpretations of the cosmological redshift.

SIR has established that the identification $z = v/c$ is an experimentally unverifiable *assumption* since the cosmological redshift z can be equally interpreted via Zwicky's Tired Light without any expansion of the universe, and via other models such as that of the $Tired\ Time$ [85].

Note that the assumption $z = v/c$ implies

$$\frac{v}{c} = \frac{H_o}{c}d, \tag{144}$$

from which

$$v = H_o d, \tag{145}$$

which establishes the $radial$ character of the conjectured expansion of the universe necessarily implying Earth at its center [163].

Following one century of oblivion of Einstein's rejection of the expansion of the universe as well as the oblivion of Einstein's view that "quantum mechanics is not a complete theory, " it is hoped that cosmologists will compare the new Tired Light Cosmology with EPR entangled universe [155]-[162] with the forgotten conceptual, geometrical and physical insufficiencies or sheer inconsistencies of the unverifiable conjecture of the expansion of the universe [163] [164] (see also debate [76] for details).

8.4.4-IV. Verification of SIR with the mean-life of unstable hadrons. Ref.[166] of 1964 suggested that non-linear and non-local effects in the interior of hadrons caused by their high density can manifest themselves in the outside via deviations of the behavior with energy of their mean lives τ from time evolution law (119), i.e.,

$$\tau' = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{146}$$

Numerous generalizations of said law were then proposed, but Ref. [167] showed that, in view of the universality of the Lorentz-Santilli isosymmetry $\hat{SO}(3.1)$ for all possible symmetric space-tines (116), all said generalized time evolution laws are particular cases of the SIR isolaw (130),

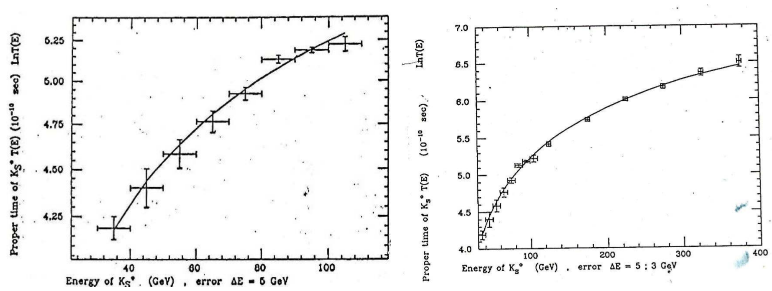

Figure 34: *In this figure, we show the exact fit by special isorelativity experimental data [168] showing deviations from special relativity law (119) in the behavior of the mean-life of Kaons from* 1 *to* 100 GeV *(left fit) as well as experimental data [168] joined with those of Ref. [169] claiming the confirmation of special relativity law (119) between* 100 *to* 400 GeV *(right fit) [170]-[172].*

i.e.

$$\tau' = \frac{\tau}{\sqrt{1 - \frac{v_k^2}{c^2}\frac{n_4^2}{n_k^2}}}. \tag{147}$$

since they can be all obtained via different expansions of the term n_4^2/n_k^2 in different variables and with different truncations. Consequently, Ref. [167] provided a significant clarification for experiments since they can be all restricted to test isolaw (130).

Experiments on the indicated prediction of deviation from law (146) were conducted in 1983 [168] for the behavior of the unstable kaons with speed and established deviations from law (119), consequently in favor of isolaw (130) from 1 to 100 GeV. A counter-experiment was done in 1987 [169] claiming the confirmation of law (119) between the *different* range of 100 to 400 GeV.

Refs. [170] [171] showed that the data from both experiments [168] and [169] can be exactly fit with the iso-Minkowskian geometry of relativistic hadronic mechanics [60] [70] (Figure 34). The 1992 Ref. [172]

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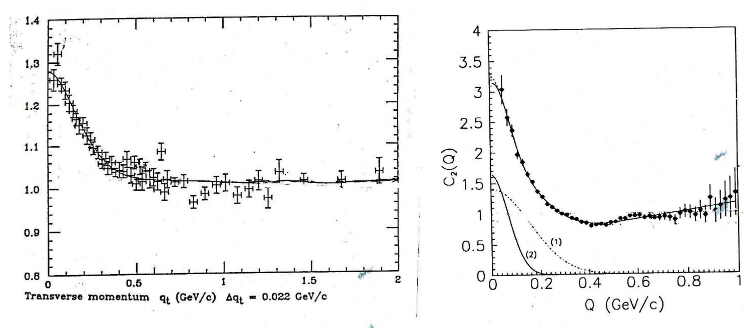

Figure 35: *The fit of experimental data for the two-point correlation function of proton-antiproton annihilation in the Bose-Einstein correlation requires 'four' arbitrary parameters of unknown origin [173]. Refs. [85] [86] (see also review [174]) have shown the inapplicability of special relativity for the proton-antiproton annihilation in favor of the exact fit via special isorelativity at high energy (left plot) and low energy (right plot) from which the four characteristic quantities n_μ of the iso-Minkowskian geometry represent the very elongated fireball of proton-antiproton annihilations, Eqs. (150)-(151).*

confirmed the findings of Refs. [170] [171], by indicating flaws in the theoretical elaboration of the form factors of counter-experiments [169]. A detailed presentation is available in Ref. [74], Vol. IV, Section 9.

It is unfortunate for our scientific knowledge that. according to the official position of the editors of the journals of major physical societies, SR is claimed to be exactly valid in the interior of hadrons following the 1987 counter-experiment [169] despite the fact that the deviations from SR laws remained established between 100 to 400 GeV and in complete oblivion of refs. [170] [171], while requests made to various particle physics laboratories over decades to run the experiment again rom 1 to 400 GeV have been generally discredited.

8.4.4-V. Verification of SIR in the Bose-Einstein cor-

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relation. As it is well known (see, e.g., Ref. [173]), the fit of the experimental data of the *two*-point correlation function of the Bose-Einstein correlation for proton-antiproton annihilation requires *four* different arbitrary parameters of unknown origin or meaning called the *chaoticity parameters.* It has been shown in Refs. [85] [86] that this occurrence is clear evidence on the lack of exact character of SR for proton-antiproton annihilation for a number of reasons, including the fact that the vacuum expectation value of the two-point correlation operator $C_{2\times 2}$ is given by the known quantum mechanical expression

$$< C_{2\times 2} >=< \psi|C_{2\times 2}|\psi >, \tag{148}$$

which, being two-dimensional and real-valued, can at best allow *two* parameter following due manipulation of the basic axioms.

By contrast, the use of relativistic hadronic mechanics [24] implies the following isoexpectation value

$$\hat{<} C_{2\times 2} \hat{>} =< \hat{\psi}|\hat{\times} C_{2\times 2} \hat{\times}|\hat{\psi} >=< \hat{\psi}|\hat{T}_{2\times 2} C_{2\times 2} \hat{T}_{2\times 2}|\hat{\psi} > . \tag{149}$$

It is then easy to see that the positive-definite 2×2-dimensional isotopic element $\hat{T}_{2\times 2} > 0$ does indeed allow in its non-diagonal realization the introduction of four characteristic quantities $n_\mu^2, \ \mu = 1, 2, 3, 4$ from first axiomatic principles without any adulteration, whose value is fit from the experimental data resulting in the values (Ref. [85], Eqs. (10.27a), page 127, Ref. [86], Table I, page 441, and review [174])

$$n_1^2 = n_2^2 = 10.666, \ \ n_3^2 = 0.355,$$

$$n_1 = n_2 = 3.265, \ \ n_3 = 0.595 \tag{150}$$

$$n_4^2 = 0.429, \ \ n_4 = 0.654. \tag{151}$$

One can see from values (150) that the fit of the experimental data (Figure 35) characterizes the known prolongated toroid of the proton-antiproton fireball, while value (151) characterizes its density (for details, see Ref. [75]. Volume IV, Section 10, page 802-809 and review [174]).

8.4.4-VI. Verification of SIR with superluminal speeds.
8.4.4.VIA. Solution of the historical Lorentz problem.

The most important implication of the axiom-preserving isotopies of the Lorentz symmetry achieved in the 1983 paper [60] is the *solution of the historical Lorentz problem,* namely, the invariance of the locally varying speeds of light $C = c/n$ that Lorentz could not solve due to the linearity cf Lie's theory, by being therefore forced to restrict his invariance to the particular case of a constant c.

In the author's view, the most important implication of the EPR argument [1] applied to isosymmetry [60] can be expressed with the following property from Ref. [175] [176]: :

LEMMA 8.4.1. The axioms of Einstein's special relativity provide a representation invariant over time of speeds of light through transparent media that can be arbitrarily bigger or smaller than the speed of light in vacuum,

$$C = \frac{c}{n_4} \lesseqgtr c. \tag{152}$$

PROOF:
Isosymmetries $\hat{SO}(3.1)$, $\hat{P}(3.1)$ and $\hat{\mathcal{P}}$ of isospacetime(126) [60]-[70] (see Refs. [24] and [213] for a review) require no restrictions on the value and functional dependence of the characteristic quantities of the medium, except for the condition of being positive definite, $n_\mu > 0$, $\mu = 1, 2, 3, 4$. Consequently, said isosymmetries provide a characterization of arbitrary local speeds of light, Eq. (152), in a way fully compatible with the axioms of SR because said isosymmetries are locally isomorphic to the corresponding conventional spacetime symmetries $SO(3.1)$, $P(3.1)$, $\mathcal{P}$. The proof of the invariance over time of arbitrary speeds (152) can be done via a simple isotopy of the invariance over time of the speed of light c under the conventional symmetries $SO(3.1)$, $P(3.1)$, $\mathcal{P}$.
Q.E.D.

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Alternatively, SR can be fully realized on the Minkowski-Santilli isospace [70] $\hat{M}(\hat{x}, \hat{\Gamma}, \hat{I})$ over the isoreal isofields $\hat{\mathcal{R}}$ with isounit $\hat{I} = 1/\hat{T} > 0$, in which case the line element given by the top line of Eq. (126).

The realization of the isocoordinates

$$\hat{x}^\mu = \frac{x^\mu}{n_\mu} \hat{I}, \tag{153}$$

then yields isosymmetries $\hat{SO}(3.1)$, $\hat{P}(3.1)$, $\hat{\mathcal{P}}$ with arbitrary speeds (152).

The propagation of light within transparent liquids with densities $n_4 > 1$ and light speeds *smaller* than that in vacuum, $C = c/n_4 < c$, has been known for centuries (Section 8.4.4-I), e.g., for the case of water with density characterized by $n_4 = 1.5$ for which we have (Figures 30, 31)

$$C = \frac{c}{1.5} = 0.666c = 200K\ km/s. \tag{154}$$

Note that the characteristic quantity n_4 provides a *geometric characterization* of the density of the medium whose actual value is given by $D = 1/n_4$.

Note also Inconsistencies 1) to 7) in the event mainstream physicists attempt to maintain in water the speed of light in vacuum via the usual reduction of electromagnetic waves to photons.

Recall that the Cherenkov light we see in the pools of nuclear power plants establishes the propagation of electrons in water at speed *bigger* than the local speed of light.

SIR has generalized the above Cherenkov effect to hyperdense physical media with geometrized densities $n_4 < 1$ resulting in arbitrary speeds $C = c/n_4 > 1$. This result is implicit in all experimental fits of particle physics experiments to date via the SIR (e.g., Figures 34, 35). The result is also implicit in all structure models of hadrons, nuclei and stars based on the EPR completion of quantum into hadronic mechanics [36] [74].

As an illustration, recall from Section 8.2.2 that the *sole* known possibility to represent *all* characteristics of the neutron in its synthesis from

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the hydrogen is that the energy of the isoelectron is isorenormalized according to Eq. (36) from $m_e = 0.511\ MeV$ to $m_{\hat{e}} = 0.511 + 0.784 = 1.295\ MeV$, by therefore yielding the expected geometrized density of the proton

$$n_4 = \frac{0.511}{1.295} = 0.394 < 1, \tag{155}$$

whose order of magnitude is confirmed by the geometrized density of the proton-antiproton fireball in the Bose-Einstein correlation (Ref. [85], Eqs. (10.27a), page 127).

This implies that *the isoelectron is rotating inside the hyperdense proton (Figure 12) with the superluminal tangential speed*

$$C = \frac{c}{0.394} = 2.538\ c. \tag{156}$$

Superliuminal speeds are generally obtained within the π^0 meson [20] and hold for all remaining hadrons since they all have masses bigger than that of the π^0, while having essentially the same size of the π^0. Consequently, SIR suggests that *particles travel at a superluminal speed that begins with the interior of the π^0, and increases with the increase of the mass of the hadrons in a way parallel to the progressive regaining of Einstein's determinism with the increase of the density of hadrons* [210]-[214].

8.4.4.VIB. Geometric locomotion. It may be of some interest to mention the *mathematical* prediction by SIR of spaceships traveling at speeds much bigger than the speed of light in vacuum which speeds are evidently necessary for any interstellar travel. The prediction was submitted in Section 4.3.3, page 281 on (see also Figure 4.5) of the 2006 monograph [29] under the name of *geometric locomotion,* also known as *isolocomotion* due to the need of its treatment via isomathematics. Isolocomotion is an intrinsic feature of the iso-Minkowskian geometry [70] and its Lorentz-Santilli isosymmetry [60].

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Recall that the main notion of any geometry dating back to Euclid is the distance d between two points, which we write $d1$, where 1 is the unit of measurement (as well as the unit of the basic numeric field), e.g. $1\ m$. Recall the preservation of numeric values under axiom-preserving isotopies [24], which we write $d\ 1 \equiv \hat{d}\hat{1}$. Isolocomotion is based on means capable of altering the local geometry in such a way to $increase$ the value of the isounit $\hat{1}$ in the desired direction, with consequential $decrease$ of the distance $\hat{d}$ in the selected direction. Isolocomotion then occurs without any possible geometric singularity, such as instantaneous accelerations, sharp changes of direction, or arbitrary speeds because the interior observer in the iso-Minkowski isospace is at rest since locomotion is achieved via the change of the geometry in its environment. By contrast, an observer in our external Minkowski space may see the distance d being covered at a multiple the speed of light c.

Alternatively, the aim of isolocomotion is to achieve arbitrary speeds without violating SR. This is $mathematically$ achieved by turning a spaceship into rather complex $interior$ conditions via the mutation of the surrounding geometry, with ensuing local applicability of SIR and related arbitrary speeds.

8.4.4.VIC. Interstellar travel. Recall that no interstellar travel is nowadays possible because of: 1) The need for superluminal speeds, 2) The impossibility of carrying along the necessary fuel, 3) The inability to change directions to avoid collisions under very high speeds.

The possible achievement of superluminal speeds has been discussed in the preceding section. Studies on a future resolution of the fuel problem have been initiated in Refs. [29] [175]. The main result is that quantitative studies for future interstellar travels can indeed be conducted provided that space is conceived as a universal substratum (ether) with an extremely big energy density for the characterization and propagation of particles and electromagnetic waves (Section 8.2.3).

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In different words, the sole known possibility for interstellar travel is that the needed fuel has to be extracted from the ether.

The complexity of the problem soon emerges when considering the need for *two* co-existing ethers, one with positive energy for the characterization of matter, and one with negative energy for the characterization of antimatter [29]. This structure is of such a complexity that, in the author's view, can be best represented via the hyperstructural branch of hadronic mechanics [24].

Despite its complexity, the above conception of a universal substratum implies that immense values of positive and negative energy are available everywhere in the universe. Future attempts at the realization of isolocomotion are then reduced to engineering means for the directional transfer of *negative* energy from the ether to the spaceship. Under the indicated conditions, isolocomotion occurs via matter-antimatter repulsion.

The third problem (how to avoid collisions with astrophysical objects) is perhaps the most complex of all problems connected with interstellar travel since its sole solution os a change of the very structure of the spaceship in such a way to pass through planets without any damage, as shown in reports by the U. S. Navy of UFO entering in the sea at high speeds without causing any wave. Conceivably, the alteration of the geometry of the ether can *mathematically* be such to produce the needed "de-materialization."

In short, the ether appears to have a truly fundamental role for interstellar travel due to various reasons treated in Ref. [175], as a result of which the ether is considered by an increasing number of scientists as the most important frontier of the third millennium.

In closing, we point out that the entire content of this section, including superluminal speeds, is fully compatible with the axioms of Einstein's special relativity, only realized in their most general possible form [175] [176].

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An additional study of the implications of the EPR argument for superliuminal speeds was presented at the 2020 Teleconference [0] by Y. F. Chang from China and it is available in Ref. [218].

8.5. Application of EPR completions to exterior gravitational problems.

8.5.1. Universal invariance of Riemannian intervals.

As it is well known, Lie's theory c an only provide the invariance of *linear* theories and this explains the reason for the inability to construct Lie symmetries of Riemannian intervals due to their non-linearity.

A primary motivation for the construction of the *non-linear* completion of Lie's theory into the covering Lie-Santilli isotheory [21]-[25] [50] [55] [201] [226] has been the construction of the universal isosymmetry of Riemannian intervals.

The above objective was studied for the particular case of isointerval (126) in which the isometric $\hat{\eta}_{\mu\nu}$ coincides with a Riemannian metric $g(x)$, and resulted in the following basic isosymmetries (see the review in Ref. [213]): 1) The iso-Lorentz symmetry, nowadays called the *Lorentz-Santilli (LS) isosymmetry* $\hat{SO}(3.1)$ first achieved at the classical level in the 1983 paper [60] with operator counterpart in Ref. [61];

2) The *iso-Poincaré isosymmetry,* nowadays called the *Lorentz-Poincaré-Santilli (LPS) isosymmetry* $\hat{P}(3.1)$, first achieved in the 1993 paper [67] written at Moscow State University; and

3) The spinorial covering of the LPS isosymmetry $\hat{\mathcal{P}}(3.1)$, first achieved at the 1994 paper [68] written at the JINR, Dubna, Russia (see also the 1995 paper [69] published in China).

The understanding of the remaining parts of this section requires a technical knowledge that the isotopic symmetries formulated on iso-Minkowskian isospaces over isofields are locally isomorphic to the corresponding conventional symmetries formulated on a Minkowski space over a numeric field, i.e., $\hat{SO}(3.1) \approx SO(3.1), \quad \hat{P}(3.1) \approx P(3.1), \;\; \hat{\mathcal{P}}(3.1) \approx$

$\mathcal{P}(3.1)$.

8.5.2. Isogeometric unification of special and general relativity.

As indicated earlier, the *iso-Minkowskian geometry* [60] [70] includes as particular cases all possible geometries with symmetric space-time intervals (126), thus including the Minkowskian, Riemannian, Fynslerian and other geometries.

Consequently, the iso-Minkowski geometry is particularly suited for the *isogeometric unification of Einstein's special and general relativities under the universal LPS isosymmetry,* namely, a unification based on isomathematics while maintaining identically Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}R_{\mu\nu}R = 0. \tag{157}$$

Consider an arbitrary non-singular Riemannian metric $g(x)$ where x are the conventional space-time coordinates. Its *identical* reformulation via isomathematics requires the decomposition of $g(x)$ into the product of the Minkowski metric η multiplied by a 4×4 positive-definite *gravitational isotopic element* $\hat{T}_{gr}(x)$ [177] [178],

$$g(x) = T_{gr}(x) \times \eta = \hat{\eta}_{gr}. \tag{158}$$

The resulting iso-Minkowskian isospace $\hat{M}(\hat{x}_{gr}, \hat{\eta}_{gr}, \hat{I}_{gr})$ is then formulated over an isoreal isofield $\hat{\mathcal{R}}$ with *gravitational isounit* given by the *inverse* of the gravitational isotopic element,

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x), \tag{159}$$

and local isocoordinates

$$\hat{x}_{gr} = x\hat{I}_{gr}. \tag{160}$$

Exterior general isorelativity (EGIR) [177]- [182] can be defined as *the formulation of general relativity on isospace $\hat{M}(\hat{x}_{gr}, \hat{\eta}_{gt}, \hat{I}_{gr})$ over an isoreal* *isofield $\hat{\Re}$ under the condition that its projection on a Riemannian space over a conventional field recovers general relativity uniquely and identically.* Consequently, by construction, isogravitation is a mere reformulation of general relativity via the use of isomathematics. The terms *exterior isogravitation* are used to recall that general relativity describes the exterior gravitational field in vacuum, in preparation of the complementary interior gravitational problem studied in the next section.

Note that, in view of the dependence of the isometric $\hat{\eta}(x)$ on spacetime coordinates, the iso-Minkowski isogeometry is formulated with the entire mathematical machinery of the Riemannian geometry, although expressed in terms of the isodifferential isocalculus [70].

Note also that isogravitation provides the reformulation of general relativity with the axioms of *special* relativity in their isotopic formulation, Isoaxioms I to V.

The isogeometric unification of special and general relativities is then assured by the fact that isogravitation is an identical reformulation of general relativity while admitting special relativity at the simple limit

$$Lim \hat{I}_{gr} \, I. \tag{161}$$

8.5.3. Resolution of century-old controversies in gravitation? As it is well known to experts in gravitation, although rarely admitted, general relativity has been plagued by a host of controversies that have not been resolved in about one century of studies, such as the apparent incompatibility of general relativity with special relativity, quantum mechanics, grand unifications and other 20th century theories.

In the author's view, protracted *physical* controversies are generally due to the fact that the used *mathematics* is insufficient for the solution of the problem considered. For the case of general relativity, it has been shown that the origin of the controversies can be reduced to the incompatibility of the Riemannian geometry with the axiomatic structure of

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20th century theories, since the latter are all defined on flat spaces, while the formed is define on a curved space.

Isogravitation (GIR), that is, the reformulation of general relativity via the iso-Minkowskian geometry, appears to offer realistic possibilities of resolving the indicated century-old controversies, as illustrated by the following comments:

1) Isogravitation is isoflat, namely, it is flat on isospace over isofield. This important feature can be seen from the fact that the isogravitational isometric $\hat{\eta}_{gr}$ is given by the Minkowski metric η multiplied by the isotopic element $\hat{T}_{gr}$ according to Eq. (158), while jointly the basic unit of the Minkowski geometry $I = Diag.(1,1,1,1)$ is completed by the $inverse$ amount, Eq. (150), resulting in no curvature. In the author's view, the isoflatness of GIR is important for the achievement of compatibility between gravitation and 20th century theories for the indicated reason that the latter are formulated in flat spaces.

2) Invariance under isosymmetries that are locally isomorphic to the corresponding conventional symmetries (Section 8.5.1). Recall that another reason for the incompatibility of GR with 20th century theories is the invariance of the former under the Lorentz and Poincaré symmetries compared to the lack of invariance for GR. It is then evident that the reformulation of gravitation in a form admitting symmetries locally isomorphic to the conventional symmetries provides serious support for the compatibility of isogravitation and 20th century theories.

3) Unique and unambiguous limit of isogravitation into special relativity [177]. Recall that the limit of general into special relativity has remained controversial for a century due to numerous reasons [181], such as the impossibility of recovering from the Riemannian geometry the $Poincaré\ symmetry$ of special relativity, let alone its generators (conserved quantities). These controversies appear to be resolved by isogravitation due to its formulation via the axioms of $special$ relativity. Additionally, SR is recovered from GIR in full via simple limit (161).

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4) Unique and unambiguous operator formulation of gravitation [177]. An additional reason for controversies is the lack of a clear and unambiguous quantum mechanical formulation of GR. This is due to the fact that GR is a *non-canonical* theory whose consistent operator image is then given by *non-unitary* theories. But the latter theories formulated in conventional fields violate causality as indicated various times in this Overview, resulting in the lack of a unique, physically consistent operator image of GR. This additional, century-old controversy is resolved by the fact that the operator image of GIR is uniquely and unambiguously given by relativistic hadronic mechanics [36] when characterized by the gravitational isounit $\hat{I}_{gr}$, E q. (159). Additionally, relativistic hadronic mechanics is *isounitary,* that is, unitary on isospaces over isofields, thus recovering causality.

5) Unique and unambiguous grand unification [178]. Recall the impossibility to achieve a consistent grand unification of gravitation with other interactions beginning with Einstein's own failed attempts. In addition to a number of problematic aspects [181], this impossibility is primarily due to the *curvature* of the Riemannian geometry because, when combined with electromagnetic and/or weak interactions, curvature causes the collapse of their axiomatic structure, beginning with the loss of spacetime, gauge and other symmetries. This additional century-old controversy appears to be resolved by isogravitation because of its isoflatness, as studied in detail in monograph [29]. Note that the resulting grand unification required, for consistency, the addition of the gravity of antimatter via the isodual cf completion of charge conjugation (Section 7.7).

6) Historical objections against the curvature of space. As it is well known, when looking at the Sun at Sunset, the Sun is already below the horizon due to the *refraction* of Sun light in our atmosphere without any possible curvature of space. The historical, well known (but rarely mentioned) objection against the curvature of space is that the bending of light passing near our Sun is due to its refraction of light within the Sun chromospheres. Additional controversies occur from the fact that space is assumed to be empty. It is therefore counter-intuitive for a number of physicists to accept the idea that the curvature of an empty space can control the trajectory of large planets such as Jupiter. Additional controversies on curvature can be found in the debate [192].

7) Lack of time-invariant numerical predictions. Recall that a majestic feature of special relativity is the preservation over time of numerical prediction due to the *invariance* of the theory under the Lorentz-Poincaré symmetry. The canonical character of SR then assures the *uniqueness of the space-time metric for all experimental verifications.* As it is well known to historians (but also rarely spoken), the lack of Lie symmetries in the Riemannian geometry mandated the replacement in GR of the notion of invariance with that of *covariance.* However, such a replacement triggered a number of controversies, the first controversy being due to the the lack of uniqueness of the Riemannian metric for all experimental verifications in exterior conditions, with ensuing lack of final experimental results. Secondly, the use of covariance instead of invariance implies that numerical predictions of GR are not preserved over time, with the ensuing additional reason for the lack of final character of experimental verifications [161]. Additional controversies have occurred for experimental verifications of GR due to the apparent *ad hoc* selection of the PPN approximation for the selected experiment. It appears that GIR offers realistic possibility ofd resolving the indicated controversies on experimental verifications in case seeded in a receptive scientific environment (for additional controversies, see next section and Ref. [182]).

8.6. Application of EPR completions to interior gravitational problems.

8.6.1. Exterior and interior gravitational problems. In the first part of the 20th century, dynamical problems were called *exterior problems* when dealing with point-particles in vacuum and *interior*

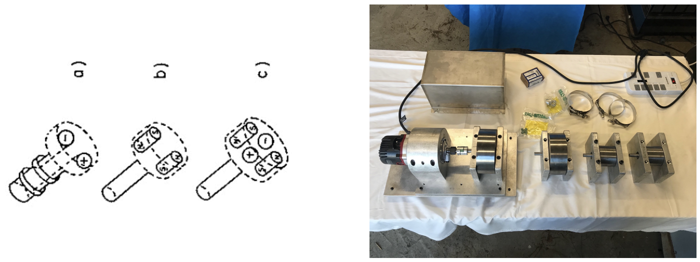

Figure 36: *On the left of this figure, we reproduce the proposal in page 145 of the 1974 MIT paper [185] to test the Poincaré hypothesis that the exterior gravitational field of a mass is entirely due to the electric and magnetic fields of the charged constituents in high dynamical conditions. In the right of this figure, we show the Gravity Generator Equipment (GGE) manufactured by Hadronic Technologies Corporation to test the Poincaré hypothesis in the expectation of its conduction by a physics laboratory with a sufficiently sensitive gravity meter.*

problems when dealing with particles in the interior of physical media. For instance, Schwartzschild wrote *two* important papers: the first paper [183] in the *exterior gravitational problem* that became justly famous, and the second paper [184] on the *interior gravitational problem* that continues to remain vastly ignored because not compatible with Einstein general relativity.

This view is essentially based on the fact that the reduction of matter to *point-like* constituents according to quantum mechanics, essentially eliminates any major distinction between exterior and interior problems.

The admission of the physical reality that particles in general, and hadrons in particular, are *extended,* and the ensuing verifications [210]-[214] of the EPR argument [1], reinstate the structural difference between exterior and interior gravitational problems adopted by Schwartzschild [183] [184[, due to the very complex interactions occurring for particles

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in interior conditions, ultimately reducible to deep EPR entanglements (Figure 2 and Section 7.2.3), which internal interactions are completely absent for the same particles in exterior conditions.

8.6.2. Interior General Isorelativity (IGIR). As noted in Section 8.5, general isorelativity (GIR) is characterized by an isometric solely dependent on space-time isocoordinates $\hat{\eta}(\hat{x})$, in which case the theory can solely represent the exterior gravitational field.

In general, the isometric has an unrestricted dependence on all needed local variables [70]

$$\hat{\eta} = \hat{\eta}(x, v, p, E, \mu, \tau, \rho, ...) = \hat{T}_{gr}(x, v, p, E, \mu, \tau, \rho, ...)\eta. \quad (162)$$

Consequently, isogravitation may indeed allow the study of interior gravitational problems, resulting in a theory called *interior general isorelativity* (IGIR).

One of the best illustrations of IGIR is given by the *iso-Dirac isoequation,* also known as *Dirac-Santilli isoequation,* on an iso-Minkowskian isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{I})$ first introduced in Ref. [70] of 1995

$$(-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_\mu\partial_\nu + mC)|\hat{\psi}(\hat{x}) >= 0. \quad (163)$$

In this case, the *Dirac-Santilli isogamma isomatrices* $\hat{\gamma}\hat{I}$ are given by

$$\hat{\gamma}_k = \tfrac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \quad \hat{\gamma}_4 = \tfrac{i}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \quad (164)$$

where $\hat{\sigma}_k$ are the *regular Pauli-Santilli isomatrices* used for the EPR verification [211] (see realization(39) via Bohm's hidden variables and Section 3.3 of Ref. [213] for the general case), with *anti-isocommutation rules*

$$\{\hat{\gamma}_\mu \hat{,} \hat{\gamma}_\nu\} = \hat{\gamma}_\mu\hat{T}\hat{\gamma}_\nu + \hat{\gamma}_\nu\hat{T}\hat{\gamma}_\mu == 2\hat{\eta}_{\mu\nu}. \quad (165)$$

The simpler case of GIR holds when the isometric coincides with a Riemannian metric, $\hat{\eta}_{\mu\nu} = g(x)_{\mu\nu}$, including the Schwartzschild metric, by therefore describing an isoelectron-isopositron pair under an external gravitational field.

The isogeometric unification of this section is illustrated by the fact that Eq.(163) allowed the representation of *all* characteristics of the neutron in its synthesis from the hydrogen with the sole change of the metric into one used for SIR, such as that of the EPR entanglement of Section 7.2.3 [70].

8.6.3. Black or brown holes? The study of IGIR has been rudimentarily initiated in Ref. [180] with intriguing outcome, such as the apparent *reformulation of black holes into brown holes* in representation of the lack of existence of singularities in nature.

Alternatively, the reformulation of black into brown holes appears to indicate the existence of a limit in the compressibility of protons at the divergence of their entangled number

8.6.4. Studies in the origin of gravitation. Recall that the gravitational field of a mass originates in the interior of the mass. Hence, a historical open problem is that of the *origin* rather than the sole description of a gravitational field.

It may be of some interest to know that IGIR can be defined as *a representation of the origin of the gravitational field with particular reference to the study of matter-matter gravitational attraction and matter-antimatter gravitational repulsion.*

As an example, a typical problem of the IGIR is the study of Poincaré's hypothesis that the exterior gravitation field of a mass is entirely generated by the electric and magnetic fields of its charged constituents.

The Poincaré hypothesis has been studied in detail in the 1974 MIT paper [185] via advanced and retarded formulations of quantum field theory. This study confirmed that the electromagnetic field of all charged components of a mass, including atomic and nuclear constituents, can indeed be the complete source of the exterior gravitational field of a mass

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even when the total charge is zero.

This study implies the existence of a gravitational source $\hat{F}^{elm}_{\mu\nu}$ representing the origin of the gravitational field, with ensuing completion of field equations (157) [180] into the form

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}_{\mu\nu}\hat{R} = k\hat{F}^{elm}_{\mu\nu}, \qquad (166)$$

which does verify the forgotten Freud identity of the Riemannian geometry [181].

Note that Eq. (163) no longer represent gravitation via curvature and this explains the reason for which the study of the Poincaré hypothesis in the origin of the gravitational field has been vastly ignorewd by mainstream physics for one full century (see debate [91] for details) despite Einstein's doubts on the r.h.s. of field equations (162) which he called *"a house made of wood"* while called the l.f.s. *"a house made of marble."*

For non-initiated readers, we should recall that Eq. (163) is fully admitted by conventional GR although, in this case, $\hat{F}^{elm}_{\mu\nu}$ represents the field of the *total charge* of the mass considered, with ensuing extremely small contribution to the gravitational field that, as such, remains represented by curvature.

By contrast, in field equation (163) according to the 1974 paper [185], the term $\hat{F}^{elm}_{44}$ represents the *total mass of the body assumed to be neutral,* the extension to the a non-null total charge being trivial. In this case, the gravitational field cannot be represented via curvature, because it is correctly represented by the iso-Minkowskian geometry that, as indicated earlier, is flat.

The basic open problems of the IGIR can then be tentatively formulated as the study of: the gravitational attraction of two masses represented with two equations of type (163)a; and the study of the gravitational repulsion between one mass represented with Eqs. (163) and its antimatter image represented via the isodual iso-Minkowskian geometry

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and isodual image of Eqs.m(163) [29] [70].

Additional studies in the general relativistic formulation of the electromagnetic field and its implications for quantum gravity are available from contributions [225] and [234] of the 2020 Teleconference [0].

8.6.5. Experimental test of the origin of gravitation. An experiment to prove or disprove the Poincaré hypothesis was suggested in page 145 of the 1974 MIT paper [185] (Figure 36). The U. S. *Hadronic Technologies Corporation* has constructed the equipment according proposed in Ref. [185] called *gravity generation equipment* (GGE), essentially consisting of a series of discs with *null* total charge of 1" thickness and diameter varying from 3" to 5".The discs are composed by various material, such as aluminum or Iron, which discs are pout in rotation up to $100,000\ rpm$ by a specially designed sequence of spindles (Figure 35).

The proposed experiment essentially consists in: 1) placing the indicated GGE next to a highly sensitive gravity detector; 2) measuring the local gravitational field when the GGE is disconnected; and 3) measuring the local gravitational field when the GGE is progressively activated up to the maximal $100,000\ rpm$. The detection of any gravitational field when the GGE discs are rotating would establish the first known *creation in laboratory of a gravitational field.* Note that there is no need for the GGE discs to be charged (see Ref.[185] for details).

In the author's view, Einstein's geometric conception of gravitation as being entirely due to curvature is one of the most beautiful and important discoveries by the human mind. However, such a conception has merely initiated, rather than ended, the studies in gravitation in view of century-old controversies yet to be resolved, and so much remains to be discoveresd theoretically and experimentally.

8.7. Application of EPR completions to high energy scattering experiments

The non-relativistic and relativistic elaboration of high energy scattering

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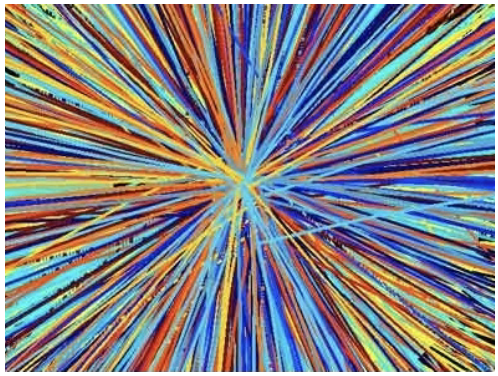

Figure 37: *It is generally accepted in the physics community that quantum mechanics is inapplicable in the interior of a black hole. Consequently, quantum mechanics should not be expected to be applicable in the interior of the scattering region of current high energy particle experiments in accordance with the EPR argument [1].*

experiments with the EPR completion of the quantum into hadronic mechanics is known as:

i) *isoscattering theory,* when possessing a Lie-isotopic algebraic structure for the representation of *elastic,* thus time-reversal invariant scattering experiments on an iso-Hilbert isospace $\hat{\mathcal{H}}$ over the isocomplex isofield $\hat{\mathcal{C}}$;

ii) *genoscattering theory,* when possessing a Lie-admissible algebraic structure for the representation of *inelastic,* thus time irreversible scattering experiments on an geno-Hilbert genospace $\hat{\mathcal{H}}^{>}$ over the genocomplex genofield $\hat{\mathcal{C}}^{>}$; and

iii) *hyperscattering theory,* when elaborated via hyperstructures for the representation of time irreversible multi-particle scattering experiments requiring a three-dimensional multi-valued representation on a hyper-Hilbert hyperspace over the hypercomplex hyperfield [24] [233].

Studies in the field were initiated by R. Mignani in 1984 [186] and continued by other authors (see the 1995 review in Chapter 12 of Ref. [24]). Subsequent detailed studies have been conducted by A. K. Aringazin *et al.* [54], A. O. E. Animalu *et al.* [187] [215].

Numerous additional works exist in the formulation of *non-unitary scattering theories,* but they are formulated on a conventional Hilbert space $\mathcal{H}$ over the conventional field of complex numbers $\mathcal{C}$, thus generally violating causality [23] [212]. Nevertheless the latter scattering theories are significant because they can be easily reformulated on the appropriate iso-, geno- or hyper-space over a corresponding iso-, geno- or hyperfield, by becoming iso-, geno- or hyper-unitary theories and regaining in this way causality.

By recalling the preservation of quantum mechanical axioms by hadronic mechanics (Section 8.4.4-I), and by remembering the dual formulation of hadronic mechanics on isospaces over isofields and their projection on conventional spaces over conventional fields, to the author's best knowledge the application of the EPR completions to high energy scattering experiments essentially implies the following:

a) The simplest possible Copenhagen realization of quantum mechanical axioms for point-particles in vacuum is no longer exactly valid for the hyperdense interior of high energy scattering regions due to their approaching the density of black holes, in favor of the broadest possible realization of quantum mechanical axions according to the iso-, geno- and hyper-structural branches of relativistic hadronic mechanics (Figure 37).

b) All theoretical as well as numerical results of high energy scattering experiments released by particle physics laboratories to date remain valid when formulated on an iso-, geno-, or hyper-Minkowskian isospace over an iso-, geno-, or hyper-complex fields.

c) The projection of the preceding formulations on our physical Minkowski space-time over the field of real numbers requires isorenormalization (133) of Isoaxiom IV of the energy and other isorenormalizations of the characteristics of intermediate particles which have been crucial for the exact representation of the neutron synthesis (Section 8.2.2), nuclear data (Section 8.2.5), molecular data (Section 7.2), and other data.

9. CONCLUDING REMARKS

The author has stated various times in his works that *the basic axioms of quantum mechanics and special relativity are majestic* in view of their mathematical consistency, predictive power, and preservation over time of numerical results for the conditions of their original conception (point particles in vacuum).

A central feature of the research outlined in this Overview is that the basic axioms of quantum mechanics are preserved identically in the verifications and applications [210]-[214] of the EPR argument [1], and they are merely realized in their most general possible form.

A similar situation occurs for special relativity because, when equally realized in their most general possible form, its axioms appear to allow: a geometric unification of Einstein's special and general relativities; their extension to interior dynamical problems; the apparent resolution of century-old controversies in gravitation; the study of the *origin* (rather than the sole description) of the gravitational field; the study of the mechanism of matter-matter gravitational attraction and matter-antimatter gravitational repulsion; and other intriguing open problems.

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Isorepresentations of the Lie-Isotopic $SU(2)$ Algebra with Applications to Nuclear Physics and to Local Realism

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Abstract. In this note, we study the nonlinear-nonlocal-noncanonical, axiom-preserving isotopies/Q-operator deformations $S\hat{U}_Q(2)$ of the $SU(2)$ spin-isospin symmetry. We prove the local isomorphism $S\hat{U}_Q(2) \approx SU(2)$, construct and classify the isorepresentations of $S\hat{U}_Q(2)$, identify the emerging generalizations of Pauli matrices, and show their lack of unitary equivalence to the conventional representations. The theory is applied for the reconstruction of the exact $SU(2)$-isospin symmetry in nuclear physics with equal p and n masses in isospaces. We also prove that Bell's inequality and the von Neumann theorem are inapplicable under isotopies, thus permitting the isotopic completion/Q-operator deformation of quantum mechanics studied in this note which is considerably along the celebrated argument by Einstein, Podolsky and Rosen.

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1. Statement of the Problem

According to current knowledge (see, e.g., [1, 4]), the $SU(2)$ spin or isospin symmetry can solely characterize the familiar eigenvalues $j(j+1)$ and m, $j = 0, \frac{1}{2}, 1, \ldots, m = j, j-1, \ldots, -j$.

In this note, we show that the isotopic generalization of $SU(2)$, herein denoted $S\hat{U}_Q(2)$, while being locally isomorphic to $SU(2)$, can characterize more general eigenvalues of the type

$$\hat{J^2} \Rightarrow f(\Delta)j[f(\Delta)j+1], \qquad J_3 \Rightarrow f(\Delta)m, \tag{1.1}$$

where j and m have conventional values and $f(\Delta)$ is a real valued, positive-definite function of the determinant of the background metric $\Delta = \operatorname{Det} g = \operatorname{Det} Q\delta$ such that $f(1) = 1$.

For the two-dimensional case, the condition $\det g = 1$ for $g = \operatorname{diag}(g_{11}, g_{22})$ is realized by $g_{11} = g_{22}^{-1} = \lambda$. This implies the preservation of the conventional value $\frac{1}{2}$ of the spin, but the appearance of a nontrivial generalization of Pauli's matrices, herein called *iso-Pauli matrices*, with an explicit realization of the 'hidden variable' λ in the structure of the spin $\frac{1}{2}$ itself.

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As a first application, we construct the isotopies of the conventional isospin (see, e.g., [2, 6]), and show that the iso-Pauli matrices permit the reconstruction of an *exact* $S\hat{U}(2)$-isospin symmetry *under electromagnetic and weak interactions* because protons and neutrons acquire equal masses in the underlying isospace.

As a second application, we show that Bell's inequality and the von Neumann theorem (see, e.g., review [7]) are inapplicable under isotopies, thus permitting the isotopic completion of quantum mechanics studied in this note, which is considerably much along the lines of the celebrated Einstein–Podolski–Rosen (EPR) argument.

It should be noted that, at the International Workshop on Symmetry Methods in Physics held at the JINR in July 1993, Lopez [9] showed that the so-called *q-deformations* (see, e.g., [17, 45]) can be put in an axiomatic form precisely via the isotopic Q-operator deformations studied in this note.

One isotopy of Pauli matrices was first presented by this author at the Third International Wigner Symposium (held at Oxford University in September 1993, [13]). In this note, we present, apparently for the first time, a systematic study and classification of the fundamental (adjoint) isorepresentation of the Lie-isotopic $S\hat{U}_Q(2)$ algebra, their applications to the reconstruction of the exact isospin symmetry as well as to the limitation of Bell's inequality and von Neumann's theorem. Additional applications to nuclear magnetic moments, particle physics and other fields will be presented elsewhere.

2. Isotopies of $SU(2)$ Symmetry

The understanding of this note requires a knowledge of the nonlinear-nonlocal-noncanonical, axiom-preserving isotopies of the theory of numbers [11] and of Lie's theory as reviewed in the article [8] in this issue and studied in detail in the monographs [16, 18].

The fundamental notion is the *isotopy of the unit* of the theory considered [4, 6–14], in this case, the unit $I = \mathrm{diag}(1, 1)$ of $SU(2)$, into a two-dimensional matrix $\hat{I}$ whose elements have the most general possible dependence on complex coordinates z, $\bar{z}$ of the underlying carrier space of $SU(2)$, their derivative with respect to time of arbitrary order, the wave functions ψ, $\psi^\dagger$ and their derivatives also of arbitrary order, and any needed additional quantity, subject to the condition of preserving the original axioms of I (smoothness, boundedness, nonsingularity, Hermiticity and positive-definiteness, as a necessary condition for isotopy),

$$I = \mathrm{diag}(1, 1) > 0 \Rightarrow \hat{I} = \hat{I}(t, z, \bar{z}, \dot{z}, \bar{z}\psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \ldots) > 0. \qquad (2.1)$$

The isotopy of the unit then demands, for consistency, a corresponding, compatible lifting of all associative products AB among generic QM quantities A, B, into the isoproduct

$$AB \Rightarrow A * B := AQB, \quad Q \text{ fixed}, \qquad (2.2)$$

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where the isotopic character of the lifting is established by the preservation of associativity by the isoproduct, $A * (B * C) = (A * B) * C$.

The assumption $\hat{I} = Q^{-1}$ then implies that $\hat{I}$ is the correct left and right unit of the theory, $\hat{I} * A = A * \hat{I} \equiv A$, in which case Q is called the *isotopic element*, and $\hat{I}$ is called the *isounit*. Note the invariant appearance of q-deformations in their Q-operator form at the very foundation of the theory, provided that they are reformulated with respect to the new unit $\hat{I} = q^{-1}$ ([10, 11]).

The isotopies of the unit $I \Rightarrow \hat{I}$ and of the product $AB \Rightarrow A * B$ then imply the necessary lifting of *all* mathematical structures of quantum mechanics (QM) into those of a covering discipline called *hadronic mechanics* (HM) [16]. Here we mention the lifting of the field of complex numbers $C(c, +, \times)$, with elements c, ordinary sum $+$ and multiplication $c \times c' = cc'$, into the infinitely possible isotopes $\hat{C}_Q(\hat{c}, +, *)$, with *isocomplex numbers* $\hat{c} = 'C\hat{I}$, conventional sum $+$ and isomultiplication $\hat{c}_1 * \hat{c}_2 = \hat{c}_1 Q \hat{c}_2 = (c_1 c_2)\hat{I}$. Note for future use that, for an arbitrary quantity A, $\hat{c} * A = c\hat{I}QA \equiv cA$.

The isotopies of the unit, multiplication and fields then demand, for mathematical consistency, corresponding compatible isotopies of the basic carrier space, the two-dimensional complex Euclidean space $E(z, \bar{z}, \delta, C)$ with familiar metric $\delta = \mathrm{diag}(1, 1)$ into the complex two-dimensional *iso-Euclidean spaces* introduced in [15, 16]

$$\hat{E}_Q(z, \bar{z}, \hat{\delta}, \hat{C}):\ z = (z_1, z_2), \quad \hat{\delta} = Q\delta \equiv g = \mathrm{diag}(g_{11}, g_{22}) = g\dagger > 0, \tag{2.3a}$$

$$z \dagger_i g_{ij}(t, z, \bar{z}, \ldots) z_j = \bar{z}_1 g_{11} z_1 + \bar{z}_2 g_{22} z_2, \tag{2.3b}$$

where the assumed diagonalization of Q is always possible (although not necessary) from its positive-definiteness.

The isotopic character (as well as novelty) of the generalization is established by the fact that, under the *joint* lifting of the metric $\delta \Rightarrow \hat{\delta} = Q\delta = g$ and of the field $C \Rightarrow \hat{C}_Q$, $\hat{I} = Q^{-1}$, all infinitely possible isospaces $\hat{E}_Q(z, \bar{z}, \hat{\delta}, \hat{C})$ are locally isomorphic to the original space $E(z, \bar{z}, \delta, C)$ under the sole condition of positive-definiteness of the isounit $\hat{I}$ [15]. In turn, this evidently sets the foundation for the local isomorphism of the corresponding symmetries.

Note that separation (2.3) is the most general possible nonlinear, nonlocal and noncanonical generalization of the original separation $z \dagger z$ under the sole condition of remaining positive-definite, i.e., of preserving the topology $\mathrm{sig}\,\delta = \mathrm{sig}\,\hat{\delta} = (+, +)$. The symmetries of invariant (2.3) are then expected to be nonlinear, nonlocal and noncanonical, as desired.

The preceding isotopies imply, for consistency, the isotopies of Hilbert spaces $\mathcal{H}$: $\langle\psi|\phi\rangle \in C$ into the so-called *iso-Hilbert space* $\hat{\mathcal{H}}_Q$ with *isoproduct* and *isonormalization*

$$\hat{\mathcal{H}}_Q:\ \langle\hat{\psi}\hat{|}\hat{\phi}\rangle = \langle\hat{\psi}|Q|\hat{\phi}\rangle\hat{I} \in \hat{C}_Q; \quad \langle\hat{\psi}\hat{|}\hat{\psi}\rangle = \hat{I}. \tag{2.4}$$

Then, operators which are Hermitian (observable) for QM remain Hermitian (observable) for HM [16].

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The liftings of the Hilbert space require corresponding isotopies of all operations in $\mathcal{H}$ [13, 14]. We here mention isounitarity $\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{I}$; isoeigenvalue equations $H * |\hat{\psi}\rangle = HQ|\hat{\psi}\rangle = \hat{E} * |\hat{\psi}\rangle \equiv E|\hat{\psi}\rangle$; isoexpectation values $\hat{\langle A \rangle} = \langle\hat{\psi}|QAQ|\hat{\psi}\rangle / \langle\hat{\psi}|Q|\hat{\psi}\rangle$; etc.

The lifting of the unit, base field and carrier space then require, for mathematical consistency, the lifting of the entire structure of Lie's theory first submitted in [10]. We are here referring to the isotopies of enveloping associative algebras ξ, Lie algebras L, Lie groups G, representation theory, etc., today called Lie–Santilli theory. Here we mention the *isoassociative enveloping operator algebras* $\hat{\xi}_Q$ with isoproduct (2.2), $A * B \equiv AQB$; the *Lie-isotopic algebras* $\hat{L}_Q$ with isoproduct

$$[A, B]_{\hat{\xi}_Q} = [A, \hat{,} B] = A * B - B * A \equiv AQB - BQA; \tag{2.5}$$

the (connected) *Lie-isotopic groups* $\hat{G}_Q$ of *isolinear isounitary transforms* on $\hat{E}_Q(z, \bar{z}, \hat{\delta}, \hat{C})$

$$z' = \hat{U}(w) * z = \hat{U}(w)Qz = \hat{U}(w)Q(z, \bar{z}, \dot{z}, \dot{\bar{z}}, \hat{\psi}, \hat{\psi}\dagger, \ldots)z, \tag{2.6a}$$

$$\hat{U}(w) = \mathrm{e}^{iX*\hat{w}}_{|\hat{\xi}_Q} = \hat{I} + (iXw)/1! + (iXw) * (iXw)/2I + \cdots$$
$$= \{\mathrm{e}^{iXQw}\}\hat{I}, \tag{2.6b}$$

$$\hat{U}(w) * \hat{U}(w') = \hat{U}(w') * \hat{U}(w) = \hat{U}(w + w'),$$
$$\hat{U}(w) * \hat{U}(-w) = \hat{U}(0) = \hat{I}, \tag{2.6c}$$

where the reformulation in terms of the conventional exponentiation has been done for simplicity of calculations.

The *isounitary* $\hat{U}_Q(2)$ *symmetry* is the most general possible, nonlinear, nonlocal and noncanonical, simple, Lie-isotopic invariance group of separation (2.3b) with realization in terms of isounitary operators on $\hat{\mathcal{H}}_Q$

$$\hat{U} * \hat{U}^{\dagger} = \hat{U}^{\dagger} * \hat{U} = \hat{I} = Q^{-1}, \tag{2.7}$$

verifying isotopic laws (2.6). $\hat{U}(2)$ can be decomposed into the *connected, special isounitary symmetry* $S\hat{U}_Q(2)$ for

$$\det(\hat{U}Q) = +1, \tag{2.8}$$

plus a discrete part which is similar to that for $\hat{O}(3)$ [10] and is here ignored for brevity.

The connected $S\hat{U}_Q(2)$ components admit the realization in terms of new generators $\hat{J}_k$ and the same parameters $\theta_k \in R(n, +, \times)$ of $SU(2)$ although re-expressed in the isofield $\hat{R}(\hat{n}, +, *)$

$$\hat{U} = \prod_k \mathrm{e}^{i\hat{J}_k * \hat{\theta}_k}_{\hat{\xi}} = \left\{\prod_k \mathrm{e}^{i\hat{J}_k Q \theta_k}\right\}\hat{I}, \tag{2.9}$$

under the conditions (necessary for isounitarity)

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$$\mathrm{tr}(\hat{J}_k Q) \equiv 0, \quad k = 1,2,3. \tag{2.10}$$

The *isorepresentations of the isotopic algebras* $S\hat{U}_Q(2)$ can be studied by imposing that the isocommutation rules have the same structure constants of $SU(2)$, i.e., for the rules

$$[\hat{J}_i, \hat{\ } \hat{J}_j] = \hat{J}_i Q \hat{J}_j - \hat{J}_j Q \hat{J}_i = i\varepsilon_{ijk}\hat{J}_k. \tag{2.11}$$

with iso-Casimir

$$\hat{J}^{\hat{2}} = \sum_k \hat{J}_k * \hat{J}_k. \tag{2.12}$$

The maximal isocommuting set is then given by $\hat{J}^{\hat{2}}$ and $\hat{J}_3$ as in the conventional case. These assumptions ensure the local isomorphism $S\hat{U}(2) \approx SU(2)$ by construction.

Let $|\hat{b}^d_k\rangle$ be the d-dimensional isobasis of $S\hat{U}_Q(2)$ with iso-orthogonality conditions

$$\langle \hat{b}^d_i | * |\hat{b}^d_j\rangle := \langle \hat{b}^d_i | Q | \hat{b}^d_j\rangle = \delta_{ij}, \quad i,j = 1,2,\ldots,n. \tag{2.13}$$

By putting as in the conventional case $\hat{J}_\pm = \hat{J}_1 \pm \hat{J}_2$, and by repeating the same procedure as the familiar one [1], we have

$$\hat{J}_3 * |\hat{b}^d_k\rangle = b^d_k |\hat{b}^d_k\rangle, \qquad \hat{J}^{\hat{2}} * |\hat{b}^d_k\rangle = b^d_1(b^d_1 - 1)|\hat{b}^d_1\rangle, \tag{2.14a}$$

$$d = 1,2,\ldots, \quad k = 1,2,\ldots,d,$$

$$b^d_1 \equiv -b^d_d, \qquad b^d_1(b^d_1 - 1) \equiv b^d_d(b^d_d + 1). \tag{2.14b}$$

A consequence is that *the dimensions of the isorepresentations of* $S\hat{U}_Q(2)$ *remain the conventional ones*, i.e., they can be characterized by the familiar expression $n = 2j + 1$, $j = 0, \frac{1}{2}, 1, \ldots$ as expected from the isomorphism $S\hat{U}_Q(2) \approx SU(2)$.

However, *the explicit form of the matrix representations are different than the conventional ones*, as expressed by the rules

$$(\hat{J}_1)_{ij} = \tfrac{1}{2} i \langle \hat{b}^d_i | * (\hat{J}_- - \hat{J}_+) * |\hat{b}^d_j\rangle, \tag{2.15a}$$

$$(\hat{J}_2)_{ij} = \tfrac{1}{2} i \langle \hat{b}^d_i | * (\hat{J}_- + \hat{J}_+) * |\hat{b}^d_j\rangle, \tag{2.15b}$$

$$(\hat{J}_3)_{ij} = \langle \hat{b}^d_i | * \hat{J}_3 * |\hat{b}^d_j\rangle, \tag{2.15c}$$

under condition (2.10).

The isorepresentations of the desired dimension can then be constructed accordingly. In the next section we shall compute the two-dimensional isorepresentations, while those of higher dimensions will be studied in a subsequent paper.

A new image of the conventional $SU(2)$ symmetry is characterized by our isotopic methods via the antiautomorphic map $I = \mathrm{diag}(1,1) \Rightarrow I^d = -I$ called

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isoduality ([12, 16]), which provides a novel and intriguing characterization of antiparticles. The corresponding *isodual isosymmetry* $S\hat{U}^d_Q(2)$ will be studied in a separate work.

In summary, our isotopic methods permit the identification of four physically relevant isotopies and isodualities of $SU(2)$ which, for the case of isospin, are given by the broken conventional $SU(2)$ for the usual treatment of $p-n$; the exact isotopic $S\hat{U}_Q(2)$ for the characterization of $p-n$ (see next section); the broken isodual $SU^d(2)$ symmetry for the characterization of the antiparticles $\bar{p}-\bar{n}$ in isodual spaces; and the exact, isodual, isotopic $S\hat{U}^d_Q(2)$ for the characterization of antiparticles $\bar{p}-\bar{n}$ in isodual isospace.

The reader may be interested in knowing that, when the positive- (or negative-) definiteness of the isotopic element Q is relaxed, the isotopes $S\hat{U}(2)$ unifies all three-dimensional simple Lie groups of Cartan classification over a complex field (of characteristic zero). In fact, we have the compact isotopes $S\hat{U}_Q(2) \approx SU(2)$ for $g_{11} > 0$, $g_{22} > 0$, and the noncompact isotopes $S\hat{U}_Q(2) \approx SU(1,1)$ for $g_{11} > 0$ and $g_{22} < 0$ (see [8] for the corresponding unification of orthogonal groups over the reals). In this note we consider only positive-definite isotopic elements Q.

3. Isotopics of Pauli Matrices

Recall that the conventional Pauli matrices σ_k (see, e.g., [2, 6]) verify the rules $\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k$, i, j, $k = 1,2,3$. In this section we identify and classify the generalizations of these familiar matrices implied by the isoalgebra $S\hat{U}_Q(2)$.

To have a guiding principle, we recall that ([8, 15]), in general, *Lie-isotopic algebras are the image of Lie algebras under nonunitary transformations*. In fact, under a transformation $UU\dagger = \hat{I} \neq I$, a Lie commutator among generic matrices A, B, acquires the Lie-isotopic form

$$U(AB - BA)U\dagger = A'QB' - B'QA',$$
$$A' = UAU\dagger, \qquad B' = UBU\dagger, \qquad Q = (UU\dagger)^{-1} = Q\dagger. \tag{3.1}$$

We therefore expect a first class of fundamental (adjoint) isorepresentations, here called *regular adjoint isorepresentations*, which are characterized by the maps $J_k = \frac{1}{2}\sigma_k \to \hat{J}_k = UJ_kU\dagger$, $UU\dagger\hat{I} \neq I$ with isotopic contributions that are factorizable in the spectra, $\pm\frac{1}{2} \to +\frac{1}{2}f(\Delta)$, $3/4 \to (3/4)f^2(\Delta)$, where $\Delta = \det Q$ and $f(\Delta)$ is a smooth nowhere-null function such that $f(1) = 1$.

An example is readily constructed via Equations (2.15) resulting in the following generalization of Pauli's matrices here called *regular iso-Pauli matrices*

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -ig_{11} \\ +ig_{22} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}, \tag{3.2}$$

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where $\Delta = \det Q = g_{11}g_{22} > 0$. The above isorepresentation verifies the isotopic rules $\hat{\sigma}_i Q \hat{\sigma}_j = i\Delta^{\frac{1}{2}}\varepsilon_{ijk}\hat{\sigma}_k$ and, consequently, the following isocommutator rules and generalized isoeigenvalues for $f(\Delta) = \Delta^{\frac{1}{2}}$

$$[\hat{\sigma}_1, \hat{\sigma}_j] = \hat{\sigma}_i Q \sigma \partial_j - \hat{\sigma}_j Q \hat{\sigma}_i = 2i\Delta^{\frac{1}{2}}_{ijk}\hat{\sigma}_k, \tag{3.3a}$$

$$\hat{\sigma}_3 * |\hat{b}^2_i\rangle = \pm\Delta^{\frac{1}{2}}|\hat{b}^2_i\rangle, \tag{3.3b}$$

$$\hat{\sigma}^{\hat{2}} * |\hat{b}^2_i\rangle = 3\Delta|\hat{b}^2_i\rangle, \qquad i = 1, 2, \tag{3.3c}$$

which confirm the 'regular' character of the generalization here considered (that is, the factorizability of the isotopic contribution in the spectrum of eigenvalues). The isonormalized isobasis is then given by a trivial extension of the conventional basis $|\hat{b}\rangle = Q^{-\frac{1}{2}}|b\rangle$.

Recall that Pauli's matrices are essentially unique in the sense that their transformations under unitary equivalence do not yield significant changes in their structure, as well known ([1, 4]). The situation is different for the iso-Pauli matrices, because isorepresentations are based on various degrees of freedom which are absent in the conventional $SU(2)$ theory, such as: (1) infinitely possible isotopic elements Q; (2) formulation of the isoalgebra in terms of structure functions [7, 9]; (3) use of an isotopic element for the iso-Hilbert space different than that of the isoalgebra [13, 14]; and others.

In fact, we can identified a second class of isorepresentations, here called *irregular adjoint isorepresentations*, in which the isotopic contributions is no longer factorizable in the entirety of the spectra of eigenvalues. A first example is given by the following *irregular iso-Pauli matrices*

$$\hat{\sigma}'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \qquad \hat{\sigma}'_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$
$$\hat{\sigma}'_3 = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta\hat{I}\sigma_3, \tag{3.4}$$

which verify the isocommutation rules

$$[\hat{\sigma}'_1, \hat{\sigma}'_2] = 2i\hat{\sigma}'_3, \quad [\hat{\sigma}'_2, \hat{\sigma}'_3] = 2i\Delta\hat{\sigma}'_1, \quad [\hat{\sigma}'_1, \hat{\sigma}'_1] = 2i\Delta\hat{\sigma}'_2, \tag{3.5}$$

without evidently altering the local isomorphism $S\hat{U}_Q(2) \approx SU(2)$. The new isoeigenvalue equations are given by

$$\hat{\sigma}'_3 * |\hat{b}^2_i\rangle = \pm\Delta|\hat{b}^2_i\rangle, \quad \hat{\sigma}'^{\hat{2}} * |\hat{b}^2_i\rangle = \Delta(\Delta+2)|\hat{b}^2_i\rangle, \tag{3.6}$$

which confirm the 'irregular' character consideration (that is, the lack of factorizability of the isotopic contributions in the entirety of the spectrum of eigenvalues). Isorepresentation (3.4) also provide an illustration of Equations (1.1) with the nontrivial lifting of the spin $s = \frac{1}{2} \to \hat{s} = \frac{1}{2}\Delta$.

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The 'degrees of freedom' of isorepresentations are then illustrated via the following second example of irregular Pauli matrices

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & g_{22}^{-\frac{1}{2}} \\ g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \qquad \hat{\sigma}''_2 = \begin{pmatrix} 0 & -ig_{22}^{-\frac{1}{2}} \\ ig_{11}^{-\frac{1}{2}} & 0 \end{pmatrix},$$
$$\hat{\sigma}''_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \tag{3.7}$$

with isocommutation rules and isoeigenvalues for $\hat{J}_k = \frac{1}{2}\hat{\sigma}''_k$

$$[\hat{J}'_1, \hat{,} \hat{J}''_2] = i\Delta \hat{J}''_3, \qquad [\hat{J}''_2, \hat{,} \hat{J}_3] = i\hat{J}''_1, \qquad [\hat{J}''_3, \hat{,} \hat{J}''_1] = i\hat{J}''_2, \tag{3.8a}$$
$$\hat{J}''_3 * |\hat{b}^2_i\rangle = \pm\frac{1}{2}|\hat{b}^2_i\rangle, \qquad \hat{J}''^{\hat{2}} * |\hat{b}^2_i\rangle = \frac{1}{2}\left(\frac{1}{2} + \Delta\right)|\hat{b}^2_i\rangle, \tag{3.8b}$$

where, as one can see, the eigenvalue of the third component is conventional, but that of the magnitude is generalized with a nonfactorizable isotopic contribution.

Intriguingly, the isorepresentations generally occurring in physical applications are the irregular ones ([15, 16]) because the generators represent physical quantities and, as such, are not changed under isotopies [7–9]. Their embedding in an isotopic algebra then generally implies the appearance of the structure functions and irregular isorepresentations.

By no mean do the above two classes exhaust all possible, physically significant isorepresentations. We therefore introduce a third class of isorepresentations without any claim of completeness (in fact, we do not study here for brevity the isorepresentations with different isotopic elements for the isoenvelopes and iso-Hilbert space which characterize yet more general isorepresentations).

We here define as *standard adjoint isorepresentations* those occurring when the spectra of eigenvalues are conventional, but the representations are nontrivially generalized, i.e., remain nonunitarily equivalent to the conventional representations. In fact, regular iso-Pauli matrices (3.2) admit the conventional eigenvalues 1/2 and 3/4 for $\Delta = 1$. This condition can be verified by putting $g_{11} = g_{22}^{-1} = \lambda$. We discover in this way the existence of the *standard iso-Pauli matrices* first presented in [13]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-1} \\ g_{11}^{-1} & 0 \end{pmatrix}, \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-1} \\ ig_{11}^{-1} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \tag{3.9}$$

which admit all conventional structure constants and eigenvalues for $\hat{J}_k = \frac{1}{2}\hat{\sigma}_k$,

$$[\hat{J}_i, \hat{,} \hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k, \qquad \hat{J}_3 * |\hat{b}\rangle = \pm\tfrac{1}{2}|\hat{b}\rangle, \qquad \hat{J}^{\hat{2}} * |\hat{b}\rangle = \tfrac{3}{4}|\hat{b}\rangle \tag{3.10}$$

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yet exhibit the 'hidden functions' g_{kk} in their structure.

Needless to say, isorepresentation (3.9) remains standard under the physically significant condition

$$\det Q = g_{11} g_{22} = 1,$$

which is realized for

$$g_{11} = g_{22} = \lambda \neq 0,$$

where is a sufficiently smooth, real-valued and nowhere-null function of the local variables. In this case, isorepresentation (3.9) assumes the form used in physical applications (see the next sections)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{3.11}$$

Similarly, irregular isorepresentations also become standard under condition (3.9) and realization (3.10). We therefore have the following additional standard iso-Pauli matrices

$$\hat{\sigma}'_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}'_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2,$$
$$\hat{\sigma}'_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{3.12a}$$

$$\hat{\sigma}''_1 = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ \lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}''_2 = \begin{pmatrix} 0 & -i\lambda^{\frac{1}{2}} \\ i\lambda^{-\frac{1}{2}} & 0 \end{pmatrix},$$
$$\hat{\sigma}''_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}. \tag{3.12b}$$

Iso-Pauli matrices with generalized eigenvalues are useful for interior structural problems, i.e., the description of a neutron in the core of a neutron star or, along the same lines, for a hadron constituent. As such, the applications of the general case of the $\hat{SU}_Q(2)$ isosymmetry is studied elsewhere [16].

When studying conventional particles, e.g., those of nuclear physics, the physically relevant subclass of $\hat{SU}_Q(2)$ is the special one with conventional eigenvalues, which is studied in the next sections. The image $\hat{\sigma}^d_k$ under isoduality, called *isodual Pauli matrices*, will be studied elsewhere.

4. Application to Isospin

As is well known (see, e.g., [2, 6]), the conventional $SU(2)$-isospin symmetry is broken by electromagnetic and weak interactions. One of the first applications

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of our isotopic/Q-operator-deformation theory is to show that the $SU(2)$-isospin symmetry can be reconstructed as exact at the isotopic level, namely, there exists a realization of the underlying isospace $\hat{E}_Q(z,\bar{z},\hat{\delta},\hat{C})$ in which protons and neutrons have the same mass, although the conventional values of mass are recovered under isoexpectation values.

The main idea is that the $SU(2)$-isospin symmetry is broken when realized via the simplest conceivable Lie product $AB - BA$. However, when the same symmetry is realized via a lesser trivial product, such as our Lie-isotopic product $AQB - BQA$ [7], it can be proved to be exact even under electromagnetic and weak interactions. In this case, the elements of the Q-matrix are constants and acquires the meaning of average of these interactions.

The reader should be aware that this is an isolated occurrence, because it represents a rather general capabilities of the Lie-isotopic theory. In fact, it is referred to as the *isotopic reconstruction of exact spacetime and internal symmetries when conventionally broken.* For example, the rotational symmetry has been reconstructed as exact for all infinitely possible ellipsoidical deformations of the sphere; the Lorentz symmetry has been reconstructed as exact at the isotopic level for all possible signature preserving deformations $\hat{\eta} = Q\eta$ of the Minkowski metric, etc. [15].

The reconstruction of the exact $S\hat{U}_Q(2)$-isospin symmetry is so simple to appear trivial. Consider a 2-component isostate

$$\hat{\psi}(x) = \begin{pmatrix} \hat{\psi}_p(x) \\ \hat{\psi}_n(x) \end{pmatrix}, \tag{4.1}$$

where $\hat{\psi}_p(x)$ and $\hat{\psi}_n(x)$ are solutions of the isodirac equation [19] which transforms isocovariantly under a standard isorepresentation of $\hat{\mathcal{P}}_Q(3.1) \times S\hat{U}_{Q'}(2)$. In this note we study only the $S\hat{U}_Q(2)$ part without any iso-Minkowskian coordinates, thus restricting our attention to the isonormalized isostates

$$|\hat{\psi}_p\rangle = \begin{pmatrix} \lambda^{-\frac{1}{2}} \\ 0 \end{pmatrix}, \qquad |\hat{\psi}_n\rangle = \begin{pmatrix} 0 \\ \lambda^{\frac{1}{2}} \end{pmatrix},$$
$$\langle\hat{\psi}_k|Q|\hat{\psi}_k\rangle = 1, \; k = p, n, \tag{4.2}$$

where $Q = \mathrm{diag}(\lambda, \lambda^{-1})$, $\hat{I} = Q^{-1} = \mathrm{diag}(\lambda^{-1}, \lambda)$.

We then introduce the $S\hat{U}_Q(2)$-isospin with isorepresentation (3.11) admitting conventional eigenvalues $\pm 1/2$ and $3/4$, defined over the isospace $\hat{E}_Q(z,\bar{z},\hat{\delta},\hat{C})$, $\hat{\delta} = Q\delta$.

We now select such an isospace to admit the same masses for the proton and the neutron. This is readily permitted by the 'hidden variable' λ when selected in such a way that

$$m_p\lambda^{-1} = m_n\lambda, \quad \text{i.e.,} \quad \lambda^2 = m_p/m_n = 0.99862. \tag{4.3}$$

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The mass operator can then be defined by

$$\begin{aligned}\widehat{M} &= \left\{\frac{1}{2}\lambda(m_p + m_n)\widehat{I} + \frac{1}{2}\lambda^{-1}(m_p - m_n)\widehat{\sigma}_3\right\}\widehat{I} \\ &= \begin{pmatrix} m_p\lambda^{-1} & 0 \\ 0 & m_n\lambda \end{pmatrix}, \end{aligned} \tag{4.4}$$

and manifestly represents equal masses $\widehat{m} = m_p\lambda^{-1} = m_n\lambda$ in isospace.

The recovering of conventional masses in our physical space is readily achieved via the isoeigenvalue expression on an arbitrary isostate

$$\widehat{M} * |\widehat{\psi}\rangle = M\widehat{I}Q|\psi\rangle = M|\widehat{\psi}\rangle = \begin{pmatrix} m_p & 0 \\ 0 & m_n \end{pmatrix}|\widehat{\psi}\rangle \tag{4.5}$$

or, equivalently, via the isoexpectation values

$$\langle\widehat{\psi}_p|Q\widehat{M}Q|\widehat{\psi}_p\rangle = m_p, \qquad \langle\widehat{\psi}_n|Q\widehat{M}Q|\widehat{\psi}_n\rangle = m_n. \tag{4.6}$$

Similarly, the charge operator can be defined by

$$q = \frac{1}{2}e(\widehat{I} + \widehat{\sigma}_3) = \begin{pmatrix} e\lambda^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.7}$$

Thus, the $S\widehat{U}_Q(2)$ charges on isospace are $q_p = e\lambda^{-1}$ and $q_n = 0$. However, the charges in our physical space are the conventional ones

$$\langle\widehat{\psi}_p|QqQ|\psi_p\rangle = e, \qquad \langle\widehat{\psi}_n|QqQ|\widehat{\psi}_n\rangle = 0. \tag{4.8}$$

The *isodual* $S\widehat{U}^Q_Q(2)$*-isospin* characterizing the antiparticle $\bar{p}$ and $\bar{n}$ will be studied elsewhere. The entire theory of isospin and its application can then be lifted in an isotopic form which remains exact under all interactions. This is not a mere mathematical curiosity, because it carries a corresponding isotopy of the nuclear force, e.g., via $S\widehat{U}_Q(2)$-isotopic exchange mechanism, essentially representing the old legacy of a (generally small) nonlocal component in the nuclear structure. These dynamical implications are studied elsewhere.

5. Applications to Local Realism

The $S\widehat{U}_Q(2)$ theory studied above is based on a structural generalization of QM of nonlinear-nonlocal-non-Hamiltonian, although axiom-preserving type. However, in the so-called literature of *local realism* (see, e.g., [7]) there exist certain arguments, most notably Bell's inequality and von Neumann's theorem, *prohibiting* a generalization of quantum mechanics.

This note would therefore not be completed without an inspection of these issue and the proof that both Bell's inequality and von Neumann's theorem are *inapplicable* (and not 'violated') under isotopies. This then sets the foundations

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for the isotopic completion of QM studied in this note. The study also serves as an application of the $\widehat{SU}_Q(2)$ symmetry to spin.

The lack of applicability of Bell's inequality and von Neumann's theorem under regular and irregular isotopies is transparent from the alteration of the spectra of eigenvalues and, as such, deserves no additional comment.

In the following we show that the above inapplicability persists not only for standard isorepresentations (3.9) but also for the particular case of $\det Q = 1$, isorepresentations (3.11).

Consider two *standard isoparticles* with spin $\frac{1}{2}$, i.e., particles characterized by standard iso-Pauli matrices (3.9). Even though their spin is the same, there is no necessary reason to restrict their isotopic degrees of freedom λ to be the same outside isospin treatments (e.g., because their density may be different). We can therefore assume

$$\text{Particle 1: } Q = \mathrm{diag}(\lambda, \lambda^{-1}), \quad \Delta = \det Q = 1, \ \text{spin} \tfrac{1}{2}, \tag{5.1a}$$

$$\text{Particle 2: } Q' = \mathrm{diag}(\lambda', \lambda'^{-1}), \quad \Delta' = \det Q' = 1, \ \text{spin}' \tfrac{1}{2} \tag{5.1b}$$

Next, consider the composite system of the two isoparticles 1 and 2 which is characterized by the isounit

$$\hat{I}_{\mathrm{tot}} = \hat{I}_1 \times \hat{I}_2 = Q_{\mathrm{Tot}}^{-1} = (Q \times Q')^{-1}. \tag{5.2}$$

To properly recompute the isotopies of Bell's inequality (see, e.g., [13] for the conventional case), it is necessary to identify the isonormalized basis $|\widehat{S}_{1-2}\rangle$, that is, the basis of the total spin of the particles 1 and 2 normalized to $\hat{I}_{\mathrm{tot}}$,

$$\langle \widehat{S}_{1-2} \widehat{|} \widehat{S}_{1-2}\rangle = \langle \widehat{S}_{1-2}| Q_{\mathrm{tot}} |\widehat{S}_{1-2}\rangle \hat{I}_{\mathrm{tot}} = \hat{I}_{\mathrm{tot}}. \tag{5.3}$$

A simple isotopy of the conventional case (see, e.g., [3], Sect. 17.9) then leads to the *isobasis for the singlet state*

$$|\widehat{S}_{1-2}\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} \lambda^{-\frac{1}{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda'^{\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda'^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda'^{-\frac{1}{2}} \\ 0 \end{pmatrix} \right\}. \tag{5.4}$$

It is a tedious but instructive exercise for the interested reader to verify isonormalization condition (5.3) by constructing the adjoint of basis (5.4), by sandwiching the quantity $T_{\mathrm{tot}} = Q \times Q'$, by contracting only quantities of the same particle, and then multiplying the scalar results of the two different particles.

Next, recall that the conventional scalar product $\boldsymbol{\sigma} \cdot \mathbf{a}$, where a is a three-vector, has no mathematical or physical meaning in isospace $\widehat{E}_Q(z, \bar{z}, \hat{\delta}, \mathbb{R})$ and must be replaced by the isoscalar product for isorepresentation (3.11)

$$\widehat{\sigma} Q \mathbf{a} = \begin{pmatrix} a_z & (a_x - i a_y) \\ (a_x + i a_y) & -a_z \end{pmatrix}. \tag{5.5}$$

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The tedious but straightforward repetition of the conventional procedure [7] under isotopy then leads to the expression

$$\langle\hat{S}_{1-2}|(Q\times Q')\{(\hat{\sigma}*\mathbf{a})\times(\hat{\sigma}'*\mathbf{b})\}(Q\times Q)|\hat{S}_{1-2}\rangle$$
$$= -a_x b_x - a_y b_y - \tfrac{1}{2}(\lambda\lambda'^{-1}+\lambda^{-1}\lambda')a_z b_z. \quad (5.6)$$

Consider now unit vectors a, b, a', b' along the z-axis. Then the Bell's inequality under the conventional $SU(2)$ symmetry [7]

$$D_{\rm Bell} = {\rm Max}|P(a,b)-P(a,b')|+|P(a',b)+P(a',b')| \leqslant 2, \quad (5.7a)$$
$$P(a,b) = \langle S_{1-2}|(\sigma_1\cdot a)\times(\sigma_2\cdot b)|S_{1-2}\rangle = -a\cdot b \quad (5.7b)$$

admits the following isotopic image under the covering $S\hat{U}_Q(2)$ symmetry

$$D_{\rm Bell} \leqslant D^{\rm HM}_{\rm Max} = \tfrac{1}{2}(\lambda\lambda'^{-1}+\lambda^{-1}\lambda')D_{\rm Bell}. \quad (5.8)$$

But, the factor $\frac{1}{2}(\lambda\lambda'^{-1}+\lambda^{-1}\lambda')$ can be easily proved to admit values bigger than one. This establishes the statement of Section 1, to the effect that Bell's inequality is not universally valid, but holds, specifically, for the conventional, linear, local and canonical realization of the $SU(2)$ symmetry. The proof for arbitrary orientations of the unit vectors follows the conventional one [3] and it is here omitted for brevity.

Similarly, von Neumann theorem [7] is inapplicable under isotopies because based on the uniqueness of the spectrum of eigenvalues of Hermitian operators. In fact, isotopic theories establish that the *same* Hermitian operator H admits *an infinite variety of different spectra of eigenvalues*, trivially, because of the infinitely possible isotopic elements Q, $H*|\hat{\psi}\rangle = HQ|\hat{\psi}\rangle = E_Q|\hat{\psi}\rangle$ [13].

Similar obstacles to the completion of QM into a covering theory are removed under isotopies as shown elsewhere [16]. We here merely mention the reason why HM is indeed a completion of QM much along the EPR argument [3]. Recall that

$$D^{\rm Class.}_{\rm Max} = {\rm Max}|a\cdot b - a\cdot b'| + |a'\cdot b + a'\cdot b'| = 2\sqrt{2} > 2. \quad (5.9)$$

and that $D_{\rm Bell} < D^{\rm Class.}_{\rm Max}$, thus preventing the completion of quantum mechanics.

However, under isotopic liftings, one can assume a classical iso-Euclidean space $\hat{E}(r,\hat{\delta},\hat{\mathbb{R}})$ (representing motion of extended objects within physical media [11]) with isotopic scalar product

$$a*b = a^t Q b = a_x g_{11} b_x + a_y g_{22} b_y + a_z g_{33} b_z. \quad (5.10)$$

Then, there *always exists a realization of $\hat{E}(r,\hat{\delta},\mathbb{R})$ under which we have the identity of the maximal operator and classical values*, $\hat{D}^{\rm HM}_{\rm Max} \equiv \hat{D}^{\rm Classical}_{\rm Max}$, as it is the case for the orientation of the unnit vectors as above, and values

$$g_{11} = g_{22} = 1, \qquad g_{33} = \frac{1}{2}(\lambda\lambda'^{-1}+\lambda^{-1}\lambda') = \sqrt{2}. \quad (5.11)$$

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A number of additional, intriguing completions of QM are provided by HM along the EPR argument, such as the recovering of classical determinism for a particle in the interior of a gravitational singularity and others [16].

In closing, it is hoped that systematic studies on the isorepresentations of Lie-isotopic algebras, such as the isotopic $\hat{O}(3)$, $\hat{O}(3.1)$, $S\hat{L}(2.\hat{\mathbb{C}})$, $\hat{P}(3.1)$, $S\hat{U}(3)$, etc., are conducted by interested colleagues because of their capabilities of novel applications, that is, results beyond the capacity of the conventional Lie theory.

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Studies on the classical determinism predicted by A. Einstein, B. Podolsky and N. Rosen

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Abstract

In this paper, we continue the study initiated in preceding works of the argument by A. Einstein, B. Podolsky and N. Rosen according to which quantum mechanics could be "completed" into a broader theory recovering classical determinism. By using the previously achieved isotopic lifting of applied mathematics into isomathematics and that of of quantum mechanics into the isotopic branch of hadronic mechanics, we show that extended particles appear to progressively approach classical determinism in the interior of hadrons, nuclei and stars, and appear to recover classical determinism at the limit conditions in the interior of gravitational collapse.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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1. Introduction

1.1. The EPR argument

As it is well known, Albert Einstein did not consider quantum mechanical uncertainties to be final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein accepted quantum mechanics for atomic structures and other systems, but believed that quantum mechanics is an "incomplete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

Soon after the appearance of paper [1], N. N. Bohr published paper [2] expressing a negative judgment on the possibility of "completing" quantum mechanics along the lines of the EPR argument.

Bohr's paper was followed by a variety of papers essentially supporting Bohr's rejection of the EPR argument, among which we recall *Bell's inequality* [3] establishing that the $SU(2)$ spin algebra does not admit limit values with an identical classical counterpart.

We should also recall *von Neumann theorem* [4] achieving a rejection of the EPR argument via the uniqueness of the eigenvalues of quantum mechanical Hermitean operators under unitary transforms.

The field became known as *local realism* and was centered on the rejection of the EPR argument via additional claims that hidden variables [5] are not admitted by quantum axioms (see the review [6]).

1.2. The 1998 apparent proof of the EPR argument

In 1998, the author published paper [7] presenting an apparent proof of the EPR argument based on the following main steps that we here outline to render this paper minimally self-sufficient:

Step 1: The proof that Bell's inequality, von Neumann's theorem and other similar objections against the EPR argument [6] are indeed correct, but under the generally tacit assumptions of point-like particles moving in vacuum under sole potential/Hamiltonian interactions (*exterior dynamical systems*) when the systems are treated via quantum mechanics and its underlying 20th century mathematics, including Lie's theory and the

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Newton-Leibnitz differential calculus;

Step 2: The proof that the above treatments are not applicable for extended, therefore deformable and hyperdense particles under conditions of mutual penetration or entanglement occurring in the structure of hadrons, nuclei, stars, and gravitational collapse such as for black holes, with novel non-linear, non-local, and non-potential/non-Hamiltonian interactions (*interior dynamical systems*);

Step 3: The treatment of interior systems via the axiom-preserving lifting of 20th century applied mathematics known as *isomathematics,* whose study was initiated by the author in the late 1970's when he was at Harvard University under DOE support, Refs. [8] to [12] and then continued by various mathematicians. Isomathematics is based on:

3-A) The axiom-preserving isotopy of the conventional associative product between generic quantities a, b (numbers, functions, operators, etc.) first introduced in Eq. (5), p. 71 of Ref. [11]

$$ab \ \rightarrow a \star b = a\hat{T}b, \tag{1}$$

where $\hat{T}$ is a positive-definite quantity called the *isotopic element* providing a representation of the dimension, deformability and density of particles and physical media in which they are immersed via realizations of the type

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2})e^{-\Gamma}, \tag{2}$$

where: n_4^2 represents the density; $n_k^2, \ k = 1, 2, 3$ represents the deformable share of particles; $n_\mu^2, \mu = 1, 2, 3, 4$, and Γ are solely restricted to be positive-definite but otherwise admit a functional dependence on any needed local variables, such as time t, coordinates r, momenta p, energy E, density d, temperature τ, pressure π, wavefunctions ψ, their derivatives $\partial\psi$, etc.

$$n_\mu = n_\mu(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \ \mu = 1, 2, 3, 4, \tag{3}$$

$$\Gamma = \Gamma(t, r, p, E, d, \tau, \pi, \psi, \partial\psi, ...) \gg 0. \tag{4}$$

$$e^{- \ \Gamma(t,r,p,E,d,\tau,\pi,\psi,\partial\psi,....)} \ll 1. \tag{5}$$

3-B) The formulation of isoassociative algebras on an *isofield* $\hat{F}(\hat{n}, \star, \hat{I})$ first introduced in Ref. [13] (see also independent work [14]), with *isounit*

$$\hat{I} = 1\hat{T}, \tag{6}$$

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and *isoreal, isocomplex and isoquaternionic isonumbers* $\hat{n} = n\hat{I}$ under isoproduct (1), with ensuing isooperations such as the isosquare

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n}. \tag{7}$$

Isofields also imply the lifting of functions into *isofunctions* [11] [20]

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}, \tag{8}$$

among which we quote the *isoexponentiation*

$$\hat{e}^X = (e^{X\hat{T}})\hat{I} = \hat{I}\ (e^{\hat{T}X}), \tag{9}$$

where X is a Hermitean operator.

3-C) The ensuing axiom-preserving lifting of Lie's theory into a non-linear, non-local and non-Hamiltonian form first introduced in Ref. [11] (see also the recent paper [15] and independent work [16]), which theory is today known as the *Lie-Santilli isotheory,* with isobrackets at the foundation of Ref. [7]

$$[X \hat{,} Y] = X \star Y - Y \star X = X\hat{T}Y - Y\hat{T}X. \tag{10}$$

3-D) The isotopic lifting of the Newton-Leibnitz differential calculus, from its historical definition at isolated points, into a form defined on volumes, first introduced in Ref. [17] (see Refs. [18] for vast independent works) with *isodifferential*

$$\begin{aligned} \hat{d}\hat{r} = \hat{T}(r, ...)d\hat{r} = \\ = \hat{T}(r, ...)d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \end{aligned} \tag{11}$$

and corresponding *isoderivatives*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{12}$$

Step 4: The axiom-preserving lifting of quantum mechanics into the *isotopic branch of hadronic mechanics,* or *isomechanics* for short, whose study was initiated in Refs. [8] to [12] (see the 1995 monographs [19] [20] [21] with 2008 upgrade [22] and independent studies [23][24]).

Isomechanics is formulated on a *Hilbert-Myung-Santilli (HMS) isospace* [25] $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$, and it is based on the *iso-Heisenberg isoequations* for the time evolution of a Hermitean operator $\hat{Q}$ in the infinitesimal form

$$\hat{i} \star \frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [\hat{Q},\hat{H}] = \hat{Q} \star \hat{H} - \hat{H} \star \hat{Q} = \\ = \hat{Q}\hat{T}\hat{H} - \hat{H}\hat{T}\hat{Q}, \tag{13}$$

and the finite form

$$\hat{Q}(\hat{t}) = \hat{U}(\hat{t})^{\dagger} \star \hat{Q}(0) \star \hat{U}(\hat{t}) = \\ = \hat{e}^{\hat{H} \star \hat{t} \star \hat{i}} \star \hat{Q}(0) \star \hat{e}^{-\hat{i} \star \hat{t} \star \hat{H}} = \\ = e^{\hat{H}\hat{T}ti} Q(0) e^{-it\hat{T}\hat{H}}, \tag{14}$$

with the following rules for the basic *isounitary isotransforms*

$$\hat{U}(\hat{t})^{\dagger} \star \hat{U}(\hat{t}) = \hat{U}(\hat{t}) \star \hat{U}(\hat{t})^{\dagger} = \hat{I}, \tag{15}$$

where $\hat{t} = t\hat{I}_t$ is the *isotime* which is assumed hereon to coincide with conventional time, $\hat{I}_t = 1$. Dynamical equations (13) to (15) were first presented in Eq. (4.16.49), page 752 of Ref. [9] over conventional fields and reformulated via the full use of isomathematics in Ref. [17]).

Isomechanics is also based on the *iso-Schrödinger isorepresentation* characterized by the fundamental representation of the *isomomentum* permitted by the isodifferential isocalculus, Eq. (12),

$$\hat{p}\hat{|}\psi(\hat{t},\hat{r}) >= -\hat{i} \star \hat{\partial}_{\hat{t},\hat{r}}\hat{|}\psi(\hat{t},\hat{r}) >= \\ = -i\hat{I}\partial_{\hat{r}}\hat{|}\psi(\hat{t},\hat{r}) >, \tag{16}$$

from which one can derive the *iso-Schrödinger isoequation,* [12] [17] [20]

$$\hat{i} \star \hat{\partial}_{\hat{t}}|\hat{\psi}(\hat{t},\hat{r}) >= \hat{H} \star |\hat{\psi}(\hat{t},\hat{r}) >= \\ = \hat{H}(r,p)\hat{T}(t,r,p,E,d,\tau,\pi,\psi,\partial\psi,....)|\hat{\psi}(\hat{t},\hat{r} >) = \\ = \hat{E} \star |\hat{\psi}(\hat{t},\hat{r}) >= E|\hat{\psi}(\hat{t},\hat{r}) > \tag{17}$$

and the *isocanonical isocommutation rules,*

$$[\hat{r}_i,\hat{p}_j]|\hat{\psi} >= \hat{i} \star \hat{\delta}_{i.j} \star |\hat{\psi} >= i\delta_{ij}|\hat{\psi} > \tag{18}$$

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$$[\hat{r}_i \hat{,} \hat{r}_j]|\hat{\psi}>= [\hat{p}_i \hat{,} \hat{p}_j]|\hat{\psi}>= 0. \qquad (19)$$

Note that the characterization of extended particles at mutual distances smaller than their size requires the knowledge of *two quantities,* the conventional Hamiltonian H for the representation of potential interactions, and the isotopic element $\hat{T}$ for the representation of dimension, shape, density as well as of non-linear, non-local and non-potential interactions.

Step 5: The proof in Ref. [7] that the isotopic $\hat{SU}(2)$-spin symmetry for extended particles immersed within a dense hadronic medium admits an explicit and concrete realization of *hidden variables* [5], e.g., of the type

$$\hat{T} = Diag.(\lambda, 1/\lambda), \quad Det\hat{T} = 1. \qquad (20)$$

In particular, the isotopic $\hat{SU}(2)$-spin isosymmetry admits limit conditions with identical classical counterpart, Eq. (5.4) page 189 Ref. [7].

One aspect of isomathematics and isomechanics which is crucial for this paper is that *in all applications to date, the isotopic element* $\hat{T}$ *has values much smaller than* 1, Eqs. (4) (5), as it has been the case for: the synthesis of the neutron from the hydrogen in the core of stars; the representation of nuclear magnetic moments and spin; new clean energies; and other applications [21].

It should be also noted that thanks to the new interactions represented by $\hat{T}$, isomathematics and isomechanics have permitted the first known identification of the *attractive force between identical valence electron pairs* in molecular structures [26]. A significant confirmation of values $|\hat{T}| \ll 1$ is provided by the fact that exact representations of binding energies for the hydrogen and water molecules have been achieved with *isoseries based on isoproduct (1) that are at least one thousand times faster than conventional quantum chemical series* [27] [28].

We should finally indicate that the numerical invariance of the isotopic element $\hat{T}$ and therefore, of the isounit $\hat{I} = 1/\hat{T}$, under isounitary time evolutions (14) (15) was proved in Ref. [29]. Detailed reviews and upgrades of isomathematics, isomechanics, and their applications to interior problems which are specifically written for the EPR argument should soon be available in Refs. [30] [31].

1.3. Aim of the paper

In this work, we shall attempt to complete the proof of the EPR argument of Ref. [7] by showing that extended particles in interior dynamical conditions appear to progressively recover classical determinism in interior dynamical conditions with the increase of the density and other characteristics, as indicated at the end of Ref. [7].

It should be stressed that a technical understanding of this work requires technical knowledge of hadronic mechanics, e.g., from Refs. [19] [20] [21] or from the forthcoming reviews and upgrades [30] [31].

We should indicate that the words "completion of quantum mechanics" is used in Einstein's sense for the intent of honoring his memory. For instance, the conventional associative product ab of Eq. (1), which is at the foundation of quantum mechanics, admits a "completion" into the equally associative, yet more general isoproduct $a\hat{T}b$. Under no conditions Einstein's word "completed theory" should be confused with a 'final theory,' that is a theory admitting no additional Einstein's "completions." In fact, the time-reversal invariant, Lie-isotopic isomathematics and isomechanics studied in this work admit the "completion" into the covering, irreversible *Lie-admissible genomathematics and genomechanics* (in which $\hat{T}$ is no longer Hermitean) which, in turn, admit a covering via the most general mathematics and mechanics conceived by the human mind, the multi-valued *hypermathematics and hypermechanics* [32] [33], with additional "coverings" remaining possible in due time [19] [20] [21] .

The reader should be finally aware that the isotopic element $\hat{T}$ and isounit $\hat{I} = 1/\hat{T}$ are inverted in some of the early quoted literature not dealing with determinism without affecting their consistency. An important aim of this paper has been that of achieving the final selection of isotopic element and isounit which is compatible with studies on determinism.

2. Recovering of determinism in interior conditions?

2.1. Heisenberg uncertainty principle

Consider an electron in empty space represented with the 3-dimensional Euclidean space $E(r, \delta, I)$, where r represents coordinates, $\delta = Diag.(1, 1, 1)$ represents the Euclidean metric and $I = Dian(1, 1, 1,)$ is the space unit.

Let the operator representation of said electron be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\Psi(r)$ and familiar normalization

$$< \Psi(r)| \, |\Psi(r) >= \int_{-\infty}^{+\infty} \Psi(r)^{\dagger}\Psi(r)dr = 1. \qquad (21)$$

As it is well known, the primary objections against the EPR argument

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[2] [3] [4] were based on the *uncertainty principle* formulated by Werner Heisenberg in 1927, according to which *the position r and the momentum p of said electron cannot both be measured exactly at the same time.*

By introducing the *standard deviations* Δr and Δp, the uncertainty principle is generally written in the form (see, e.g., [5])

$$\Delta r \Delta p \;\geq\; \frac{1}{2}\hbar, \tag{22}$$

easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p \;\geq\; |\frac{1}{2i} < \Psi|\, [r,p]\, |\Psi > | = \frac{1}{2}\hbar. \tag{23}$$

The standard deviations have the known form [34] (with $\hbar = 1$)

$$\begin{aligned} \Delta r &= \sqrt{< \Psi(r)|[\, r - (< \Psi(r)|\, r\, |\Psi(r) >)]^2 |\Psi(r) >}, \\ \Delta p &= \sqrt{< \Psi(p)|\, [p - (< \Psi(p)|\, p\, |\Psi(p) >)]^2 |\Psi(p) >}, \end{aligned} \tag{24}$$

where $\Psi(r)$ and $\Psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

2.2. Particle in interior conditions

We consider now the electron, this time, in the core of a star classically represented with the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ [17] with basic isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{25}$$

and isotopic element of type (2) under conditions (3) to (5).

Besides being immersed in the core of a star, the electron has no Hamiltonian interactions. Consequently, we can represent the electron in the *HMS isospace* $\hat{\mathcal{H}}$ [25] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [13], and introduce the time independent *isoplanewave* [20]

$$\begin{aligned} &\hat{\Psi}(\hat{r}) = \hat{\psi}(\hat{r})\hat{I} = \\ &= \hat{N} \star (\hat{e}^{\hat{i}\star\hat{k}\star\hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{26}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar,* $\hat{k} = k\hat{I}$ is the *isowavenumber*, and the isoexponentiation is given by Eq. (9).

The corresponding representation in isomomentum isospace is given by

$$\hat{\Psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i} \star \hat{n} \star \hat{p}}, \tag{27}$$

where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

2.3. Isodeterministic isoprinciple

The *isopropability isofunction* is given by [20]

$$\hat{\mathcal{P}} = \hat{<} | \star | \hat{>} = < \hat{\Psi}(\hat{r}) | \, T \, | \hat{\Psi}(\hat{r}) > I =$$

$$= [\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^\dagger \star \hat{\Psi}(\hat{r}) \star \hat{d}\hat{r}] \hat{I} = \tag{28}$$

$$= [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{\psi}(\hat{r}) \hat{d}\hat{r}] \hat{I},$$

where one should keep in mind that the isodifferential $\hat{d}\hat{r}$ is now given by Eqs. (11).

The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [20]

$$\hat{<} | \star \hat{Q} \star | \hat{>} = < \hat{\Psi}(\hat{r}) | \star \hat{Q} \star | \hat{\Psi}(\hat{r}) > \hat{I} =$$

$$= [\int_{-\infty}^{+\infty} \hat{\Psi}(\hat{r})^\dagger \star \hat{Q} \star \hat{\Psi}(\hat{r}) \hat{d}\hat{r}] \hat{I} = \tag{29}$$

$$= [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{Q} \hat{\psi}(\hat{r}) \hat{d}\hat{r}] \hat{I},$$

with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

We now introduce, apparently for the first time in this paper, the *isotopic operator*

$$\hat{\mathcal{T}} = \hat{T}\hat{I} = I, \tag{30}$$

that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the transition of the isoproduct from its scalar form (1) into the isoscalar form

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n} = \hat{n} \star \hat{\mathcal{T}} \star \hat{n} = n^2 \hat{I}. \tag{31}$$

Since the identity I can be inserted anywhere in the expectation values of quantum mechanics without altering the results, realization (33) illustrates the central feature of the isotopies, namely, the property that the abstract axioms of quantum mechanics admit a "hidden" realization broader

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than that of the Copenhagen School whose degrees of freedom have been used in Ref.[7] for the proof of the EPR argument [1].

We now introduce the isoexpectation isovalue of the isotopic operator

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =< \hat{\Psi}(\hat{r})| \star \hat{\mathcal{T}} \star |\hat{\Psi}(\hat{r}) > \hat{I} = \\ = [\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r}] \hat{I}, \tag{32}$$

and assume the isonormalization

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} = \\ = \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = \hat{T}. \tag{33}$$

We then introduce, in this paper apparently for the first time, the *isostandard isodeviation* for isocoordinates $\Delta\hat{r} = \Delta r \hat{I}$ and isomomenta $\Delta\hat{p} = \Delta p \hat{I}$, where Δr and Δp are the standard deviations in our space.

By using isocanonical isocommutation rules (18), we obtain the expression

$$\Delta\hat{r} \star \Delta\hat{p} = \Delta r \Delta p \hat{I} \approx \quad \frac{1}{2}| < \hat{\Psi}(\hat{r})| \star [\hat{r}\hat{,}\hat{p}] \star \hat{\Psi}(\hat{r}) > | = \\ = \frac{1}{2}| < \hat{\Psi}(\hat{r})| \hat{T} [\hat{r}\hat{,}\hat{p}] \hat{T} |\hat{\Psi}(\hat{r}) > . \tag{34}$$

By eliminating the common isounit $\hat{I}$, we then have the desired *isodeterministic isoprinciple* here proposed apparently for the first time

$$\Delta r \Delta p \approx \frac{1}{2}| < \hat{\Psi}(\hat{r})| \star [\hat{r}\hat{,}\hat{p}] \star |\hat{\Psi}(\hat{r}) >= \\ = \frac{1}{2}| < \hat{\Psi}(\hat{r})| \hat{T} [\hat{r}\hat{,}\hat{p}] \hat{T} |\hat{\Psi}(\hat{r}) >= \\ \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = T \ll 1 \tag{35}$$

where the property $\Delta r \Delta p \ll 1$ follows from the fact that *the isotopic element $\hat{T}$ has always a value smaller than* 1 (Section 1.2).

It is now necessary to verify isoprinciple (35) by proving that the isostandard isodeviations tend to null values when $\hat{T} \to 0$.

For this purpose, we introduce the following simple isotopy of Eqs. (24) (where we ignore the common multiplication by the isounit)

$$\Delta r = \sqrt{< \hat{\Psi}(\hat{r})|[\, \hat{r} - < \hat{\Psi}(\hat{r})| \star \hat{r} \star |\hat{\Psi}(\hat{r}) >]^{\hat{2}} \hat{|} \Psi(\hat{r}) >}, \\ \Delta p = \sqrt{< \hat{\Psi}(\hat{p})|\, [\hat{p} - < \hat{\Psi}(\hat{p})| \star \hat{p} \star |\hat{\Psi}(\hat{p}) >]^{\hat{2}} |\hat{\Psi}(\hat{p}) >}, \tag{36}$$

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where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity.

It is then easy to see that the isosquare (7) implies the covering forms of the isostandard isodeviations

$$\Delta r = \sqrt{\hat{T} < \hat{\Psi}(\hat{r})|[\,\hat{r} - < \hat{\Psi}(\hat{r})| \star \hat{r} \star |\hat{\Psi}(\hat{r}) >]^2|\hat{\Psi}(\hat{r}) >},$$
$$\Delta p = \sqrt{\hat{T} < \hat{\Psi}(\hat{p})|\,[\hat{p} - < \hat{\Psi}(\hat{p})| \star \hat{p} \star |\hat{\Psi}(\hat{p}) >]^2|\hat{\Psi}(\hat{p}) >}, \quad (37)$$

that indeed approach null value under the limit conditions

$$Lim_{\hat{T}=0}\Delta r = 0,$$
$$Lim_{\hat{T}=0}\Delta p = 0, \quad (38)$$

thus confirming isodeterministic isoprinciple (35).

2.4. Particles under pressure

To illustrate the above expressions, we consider an electron in the center of a star, thus being under extreme pressures π from the surrounding hadronic medium in all radial directions, while ignoring particle reactions in first approximation or under a sufficiently short period of time.

These conditions are here rudimentarily represented by assuming that the $\Gamma > 0$ function of the the isotopic element (2) is a constant linearly dependent on the pressure π, resulting in a realization of the isotopic element of the type

$$\hat{T} = e^{-w\pi} \ll 1, \;\; \hat{I} = e^{+w\pi} \gg 1, \quad (39)$$

where w is a positive constant.

The isodeterministic isoprinciple for the considered particle is then given by

$$\Delta r \Delta p \approx \frac{1}{2} e^{-w\pi} \;\ll\; 1, \quad (40)$$

and tends to null values for diverging pressures.

The above example illustrates the consistency of isorenormalization (33) because, a constant isotopic element implies the consistent expression

$$\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} =$$
$$T < \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) > \hat{I} = \quad (41)$$
$$< \hat{\psi}(\hat{r})|\,|\hat{\psi}(\hat{r}) >,$$

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while, by contrast, the following alternative isonormalization

$$\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \tag{42}$$

would imply the expression

$$< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) > \hat{I} = \hat{I}, \tag{43}$$

which is manifestly inconsistent since $< \hat{\psi}(\hat{r})||\hat{\psi}(\hat{r}) >$ is an ordinary number while $\hat{I}$ is a matrix with integro-differential elements.

Note that we have considered a free particle immersed in a hadronic medium, rather than a bound state of extended particles in condition of mutual penetration. Consequently, in our view, isotopic element (2) represents a *subsidiary constraint* caused by the pressure of the hadronic medium encompassing the particle considered, by therefore restricting the values of the isostandard isodeviations for isocoordinates and isomomenta.

Illustrations of the isodeterministic isoprinciple in specific structure models of hadrons and related aspects have been studied in Ref. [21] and their interpretation in terms of the isodeterministic isoprinciple will be studied in future works.

2.5. Gravitational example

To provide a gravitational illustration, recall that isotopic element (2) contains as particular cases all possible symmetric metrics in $(3+1)$-dimensions, thus including the Riemannian metric [20].

We then consider the 3-dimensional sub-case of isotopic element (2) and factorize the space component of the Schwartzchild metric $g_s(r)$ according to isotopic rule introduced in Refs. [35] [36]

$$g_s(r) = \hat{T}(r)\delta, \tag{44}$$

where δ is the Euclidean metric.

We reach in this way the following realization of the isotopic element

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \tag{45}$$

where M is the gravitational mass of the body considered, with ensuing isodeterministic isoprinciple

$$\Delta\hat{r}\Delta\hat{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r\rightarrow 0} = 0, \tag{46}$$

which confirms the statement in page 190 of Ref. [7], on the possible recovering of full classical determinism in the interior of gravitational collapse (see Ref. [37], Chapter 6 in particular, for a penetrating critical analysis of black holes).

It should perhaps be indicated that Refs. [35] [36] introduced the factorization of a full Riemannian metric $g(x),\ x = (r, t)$ in $(3+1)$-dimensions

$$g(x) = \hat{T}_{gr}(x)\eta, \tag{47}$$

where $\hat{T}_{gr}$ is the *gravitational isotopic element*, and η is the Minkowski metric $\eta = Diag.(1, 1, 1, -1)$.

Refs. [35] [36] then reformulated the Riemannian geometry via the transition from a formulation over the field of real numbers $\mathcal{R}$ to that over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ where the *gravitational isounit* is evidently given by

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x). \tag{48}$$

The above reformulation turns the Riemannian geometry into a new geometry called *iso-Minkowskian isogeometry,* which is locally isomorphic to the *Minkowskian* geometry, while maintaining the mathematical machinery of the Riemannian geometry (covariant derivative, connection, geodesics, etc.) us fully maintained, although reformulated in terms of the isodifferential isocalculus [38].

The apparent advantages of the *identical* iso-Minkowskian reformulation of Riemannian metrics and Einstein's field equations (see, e.g., Eqs. (2.9), page 390 of Ref. [38]) are:

1) The achievement of a consistent operator form gravity in terms of *relativistic hadronic mechanics* [39] whose axioms are those of quantum mechanics, only subjected to a broader realization;

2) The achievement of a universal *symmetry* of *all* non-singular Riemannian metrics, which symmetry is locally isomorphic to the Lorentz-Poincaré symmetry, today known as the *Lorentz-Poincaré-Santilli (LPS) isosymmetry* [40], and it is notoriously impossible on a conventional Riemannian space over the reals;

3) The achievement of clear compatibility of Einstein's field equation with 20th century sciences, such as a clear compatibility of general relativity with special relativity via the simple limit $\hat{I}_{gr} = I$ implying the transition from the universal LPS isosymmetry to the Poincaré symmetry of special relativity with ensuing recovering of conservation and other special relativity laws [41] [42]; the achievement of axiomatic compatibility of gravitation with electroweak interactions thanks to the replacement of curvature into the new notion of isoflatness with the ensuing, currently

impossible, foundations for a grand unification [43]; and other intriguing advances.

3. Concluding remarks

t In this paper, we have continued the study of the EPR argument [1] conducted in Ref. [7] and preceding works, with particular reference to the study of the uncertainties for extended particles immersed within hyperdense medias with ensuing linear and non-linear, local and non-local and Hamiltonian as well as non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics characterized by the isotopic element $\hat{T}$ of Eq. (1) which represents the non-linear, non-local and non-Hamiltonian interactions of the particles with the medium [19] [20] [21].

The main result of this paper is that the standard deviations of coordinates and momenta for particles within hyperdense media are characterized by the isotopic element that, being always very small, $\hat{T} \ll 1$, reduces the uncertainties in a way inversely proportional to a non-linear increase of the density, pressure, temperature, and other characteristics of the medium, while admitting the value $\hat{T} = 0$ under extreme/limit conditions with ensuing recovering of full determinism as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

We can, therefore, tentatively summarize the content of this paper with the following:

ISODETERMINISTIC ISOPRINCIPLE: The product of isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as the individual isodeviations, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

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Studies on A. Einstein , B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," I: Basic methods

Ruggero Maria Santilli*

Abstract

In 1935, A. Einstein expressed his view, jointly with B. Podolsky and N. Rosen, that "quantum mechanics is not a complete theory" (*EPR argument*). Following decades of preparatory studies, R. M. Santilli published in 1998 a paper showing that the objections against the EPR argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. In a more recent paper, Santilli has shown that quantum uncertainties of extended particles appear to progressively tend to zero when in the interior of hadrons, nuclei and stars, and appear to be identically null at the limit of gravitational collapse, essentially along the EPR argument. In this first paper, we review, upgrade and specialize the basic mathematical, physical and chemical methods for the study of the EPR argument. In two subsequent papers, we review the above results and provide specific illustrations and applications.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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1. INTRODUCTION.

1.1. The EPR argument.

As it is well known, quantum mechanics does not admit classical precision in the measurement of the position or the mutual distance of particles (Figure 1) in view of Heisenberg's uncertainty principle and other physical laws.

Albert Einstein did not accept this uncertainty as being final for all possible conditions existing in the universe and made his famous quote *"God does not play dice with the universe."*

More specifically, Einstein accepted quantum mechanics for atomic structures and other systems of point-like particles in vacuum (conditions known as *exterior dynamical problems*), but believed that quantum mechanics is an "incomplete theory," in the sense that it could admit a "completion" into

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Figure 1: *In this figure, we present a conceptual rendering of the sole representation of particles permitted by the differential calculus underlying quantum mechanics, namely, the representation as isolated points in empty space which particles, being dimensionless, can only be at a distance, with ensuing EPR argument on the need for superluminal interactions to explain quantum entanglement [1].*

such a form to recover classical determinism at least under limit conditions.

Einstein communicated his view to the post doctoral associates, B. Podolsky and N. Rosen at the Institute for Advanced Study, Princeton, NJ, and all three together published in the 1935, May 15th issue of the Physical Review, the paper entitled *"Can Quantum Mechanical Description of Physical reality be Considered Complete?"* which paper became known as the *EPR argument* [1].

Soon after the appearance of paper [1], N. Bohr published paper [2] expressing a negative judgment on the possibility of "completing" quantum mechanics along the EPR argument.

Bohr's paper was followed by a variety of papers essentially supporting Bohr's rejection of the EPR argument, among which we recall *Bell's inequality* [3] establishing that the $SU(2)$ spin algebra does not admit limit values with an identical classical counterpart.

We should also recall *von Neumann theorem* [4] achieving a rejection of the EPR argument via the uniqueness of the eigenvalues of quantum mechanical Hermitean operators under unitary transforms.

The field became known as *local realism* and was centered on the rejection of the EPR argument on claims of *lack of existence of hidden variables* λ *[5] in quantum mechanics* (see the review [6] with a comprehensive literature).

Nowadays, the EPR argument is generally ignored in view of the widespread belief that quantum mechanics is universally valid for whatever conditions may exist in the universe without any scrutiny of the limitations and/or insufficiencies of quantum mechanics in various fields reviewed in this section.

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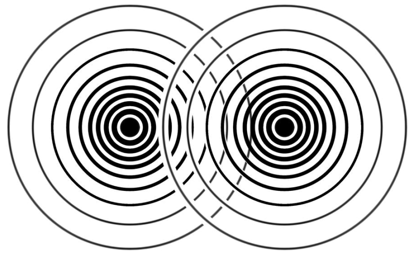

Figure 2: *A conceptual rendering of the main assumption of the apparent proofs [7] [8] of the EPR argument [1], consisting in the representation of particles as extended, deformable and hyperdense in conditions of mutual overlapping with ensuing continuous contact at a distance eliminating the need for superluminal interactions to explains quantum entanglement. Despite its simplicity, the quantitative treatment of said view required decades of studies due to the need for a "completion" of the mathematics underlying quantum mechanics. Intriguingly, the "completion" here considered turned out to be of isotopic/axiom-preserving type, thus being fully admitted by quantum mechanical axioms, merely subjected to a realization broader than that of the Copenhagen school.*

1.2. Apparent proofs of the EPR argument.

In Vol. 50, pages 177-190, 1998, of *Acta Applicandae Mathematica,* R. M. Santilli published paper [7] entitled "Isorepresentation of the Lie-isotopic $SU(2)$ Algebra with Application to Nuclear Physics and Local Realism," which paper appears to confirm Einstein's view on the existence of a "completion" of quantum mechanics into the *isotopic branch of hadronic mechanics,* or *isomechanics* for short and a "completion" of quantum chemistry into a form known as *isochemistry.* These "completions" are based on a broadening of applied mathematics known as *isomathematics* and admit progressive conditions of particles in the interior of hadrons, nuclei, stars and black holes that appear to recover classical determinism.

The proof presented in paper [7] was done via the following three main steps:

1.2.1. The proof that Bell's inequality, von Neumann's theorems and other similar objections of the EPR argument [6] are indeed correct, but under the generally tacit assumptions:

A) The point-like approximation of particles moving in vacuum (Figure 1);

B) The sole admission of Hamiltonian interactions [18];

C) The treatment of assumptions A and B via 20th century applied mathematics, including Lie's theory and the Newton-Leibnitz differential

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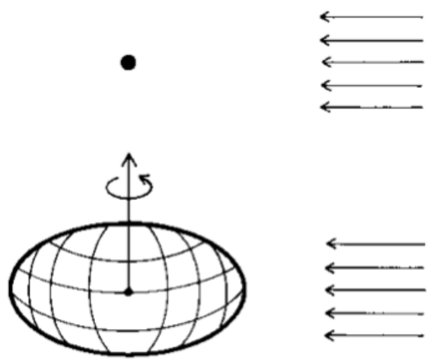

Figure 3: *An illustration of the central objective for the proof of the EPR argument consisting in the transition from the quantum mechanical representation of the Newtonian notion of massive points moving in vacuum under linear, local and potential interactions (top view), to the time invariant representation of extended particles moving within physical media under linear and non-linear, local and non-local and potential as well as non-potential interactions (bottom view).*

calculus;

1.2.2. The proof that the above treatments are not applicable to Einstein's vision on the existence of a "completion" of quantum mechanics based on the following assumptions:

A′) The representation of extended, therefore deformable and hyperdense particles under conditions of mutual penetration/entanglement known (Figure 2) as occurring in the structure of hadrons, nuclei, stars and black holes (systems known as *interior dynamical problems*);

B′) The emergence under condition A′ of Hamiltonian as well as contact non-Hamiltonian interactions of non-linear, non-local and non-potential character;

C′) The treatment of assumptions A′ and B′ via isomathematics that, as we shall see in Section 2, is based on:

i) The axiom-preserving isotopy $ab = ab \to a{\star}b = a\hat{T}b$ of the associative product ab between generic quantities a, b (numbers, functions, operators, etc.), where $\hat{T}$ is a positive-definite quantity called the *isotopic element* representing the dimension, deformability and density of particles;

ii) The ensuing axiom-preserving "completion" of Lie's theory with isotopic product $[x\hat{;}y] = x \star y - y \star x$ between Hermitean operators x, y;

iii) The reconstruction of the 20th century applied mathematics into a form compatible with isoproduct $a \star b$, including most importantly the isotopic lifting of the Newton-Leibnitz differential calculus from its centuries

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Figure 4: *A first illustration of the lack of "completion" of quantum mechanics beyond scientific doubt is the time reversal invariance of the theory with equal probability for events forward and backward in time. Such a time reversibility is acceptable for atomic structures, particles in accelerators, crystals and other reversible systems, but it does not allow a consistent physical or chemical representation of energy-releasing process, such as the coal burning depicted in this figure. In fact, the time reversal image of coal burning implies that smoke must reconstruct coal with evident violation of causality.*

old definition at isolated points to its definition in the volumes of particles represented by $\hat{T}$.

1.2.3. The proof that the Lie-isotopic $\hat{SU}(2)$ algebra with isoproduct $[x\hat{,}y]$ admits limit conditions with an identical classical counterpart.

More recently, R. M. Santilli completed the above proof in paper [8] by showing that, under the above indicated conditions, the standard deviations for coordinates Δr and momenta Δp appear to progressively tend to zero for extended particles within hadrons, nuclei and stars, and appear to be identically null for extended particles within the limit conditions in the interior of gravitational collapse, essentially along Einstein's vision.

It should be noted that the above proofs of the EPR argument are centered in the *preservation* of the basic axioms of quantum mechanics, only submitted to their broadest possible *realization*.

It should be also noted that, under said broadest possible realization, quantum axioms do admit an *explicit and concrete realization of hidden variables* embedded in the structure of the Lie-isotopic product $[x\hat{,}y] = x\hat{T}y - y\hat{T}x$, for instance, via realization $\hat{T} = Diag.(\lambda, 1/\lambda), \quad Det\hat{T} = 1$ [7].

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1.3. Insufficiencies of quantum mechanics for irreversible processes.

One of the insufficiencies of quantum mechanics well known since the 1930's is the inability to represent physical or chemical energy releasing processes, such as nuclear fusions, fuel combustion and other processes.

This is due to the fact that energy releasing processes are *irreversible over time* (namely, the time reversal image of the processes violates causality), while quantum axioms were conceived for and remain solely applicable to *systems reversible over time* (namely, the time reversal image of the systems verifies causality), such as atomic structures, particles in accelerators, crystals and other systems (Figure 4).

It is hoped that the ongoing alarming deterioration of our environment, with the consequential need for *new* clean energies, illustrate the need for a "completion" of quantum mechanics for irreversible processes outlined for completeness in Section 2.

It should be indicated that, except for the short presentation in Section 2, this and the following paper are solely devoted to the apparent proofs of the EPR argument for reversible interior systems, while the study for the broader irreversible interior systems is done elsewhere (see Refs. [9] to [83]).

This is due to the fact that the objections against the EPR argument were formulated for reversible exterior systems. Consequently, proofs [7] and [8] studied in these papers were formulated for reversible interior systems.

1.4. Insufficiencies of quantum mechanics in particle physics.

Quantum mechanics is justly considered to be *exactly valid* for the structure of the hydrogen atom because it achieved a numerically exact representation of *all* experimental data for the system considered.

It is hoped serious scholars will admit that quantum mechanics cannot be considered as being exactly valid for particle physics because of the known inability to achieve an exact representation of *all* experimental data of any given family of particles, despite the admission of a number of hypothetical neutrinos and other *ad hoc* conjectures.

Recall that, with the exception of electrons, protons and the hypothetical neutrinos, *all particles produced by contemporary accelerators are structurally irreversible because unstable, thus being outside the capability of a full representation via quantum mechanics (Section 1.3), e.g., because they require a covering Lie-admissible (rather the Lie) treatment.* [22] [23].

Additionally, quantum mechanics is completely inapplicable (rather than violated) for the most fundamental synthesis in nature, that of the

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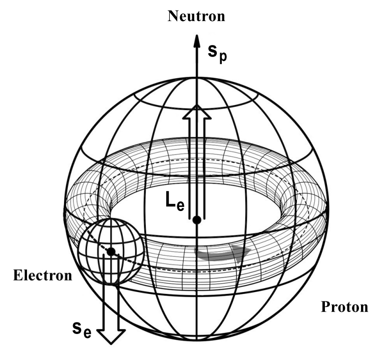

Figure 5: *A conceptual rendering of the synthesis of the neutron as a "compressed hydrogen atom" in the core of stars according to H. Rutherford [84], which synthesis cannot be represented by quantum mechanics due to a mass excess and other reasons (see Section 1.4). By contrast, all characteristics of the neutron in said synthesis have been represented at the non-relativistic and relativistic levels by the "completion" of 20th century applied mathematics and physics studied in this paper (Refs. [85] [95]).*

neutron from a proton and an electron as occurring in the core of stars [84], with ensuing inapplicability to the synthesis of other particles, such as that of the π^0 meson from the positronium [17].

This is due to the fact that quantum mechanical axioms have been conceived for the synthesis of particles in which the mass of the final state is *smaller* than the sum of the masses of the original constituents, resulting in the well known *mass defect* caused by *negative potentials.*

By contrast, the mass of the neutron $E_n = 939.565\ MeV$ is $0.782\ MeV$ *bigger* than the sum of the masses of the proton $E_p = 938.272\ MeV$ and of the electron $E_e = 0.511\ MeV$, resulting in a *mass excess* requiring a *positive potential* for which Schrödinger, Dirac and other quantum mechanical equations admit no physically meaningful solutions, with similar cases occurring for the synthesis of other particles [17].

The inability by quantum mechanics to represent the fundamental synthesis of the neutron in a star is ultimately due to the *point-like characterization of particles* since it is mathematically and physically impossible to fuse together two point-like particles (the proton and the electron) into a third point-like particle (the neutron).

In turn, this insufficiency identified the need for the representation of hadrons as extended, deformable and hyperdense, which representation is at the foundation of the EPR proof [7].

Another insufficiency of quantum mechanics is its sole capability of representing *linear interactions* (i.e., interactions linear in the wave function). By contrast, the sole known possibility of achieving a bound state with excess mass is that via the admission of non-linear, generally non-Hamiltonian interactions for extended particles in conditions of mutual penetration/entanglement, as it is the case for the electron when totally compressed inside the proton [16] [17].

In turn, non-linear interactions are crucial for the "completion of the wavefunction" advocated by Einstein, Podolsky and Rosen [1] as we shall see in Paper III of this series.

It is nowadays known that the above insufficiencies originate from the theory at the foundation of quantum mechanics, Lie's theory, because said theory solely admits Hamiltonian linear interactions [19].

It is hoped that the above insufficiencies illustrate the significance of the "completion" of Lie's theory used for the proof of the EPR argument [7].

In fact, only following the achievement of a "completion" of 20th century mathematical and physical methods for extended, deformable and hyperdense particles in interior dynamical conditions, Santiili achieved a numerically exact representation of *all*— characteristic of the neutron in its synthesis from a proton and an electron at the non-relativistic (Refs. [85] to [87]), relativistic (Refs. [88] to [89]) and experimental (Refs. [90] to [95]) levels (see Sections 2, 3 and Paper II of this series).

1.5. Insufficiencies of quantum mechanics in nuclear physics.

There is no doubt that quantum mechanics has permitted historical achievements in nuclear physics.

However, quantum mechanics is only *approximately valid* in nuclear physics because of the inability to achieve over the last century a representation of the characteristics of the simplest nucleus, the deuteron, with embarrassing deviations of the prediction of the theory from experimental data for heavier nuclei, such as the Zirconium [59].

In Santilli's view, the primary reason for the indicated insufficiency is that the *mathematics* underlying quantum mechanics, with particular reference to the *Newton-Leibnitz differential calculus,* imply the conception of nuclei as ideal spheres with isolated points in its interior (Figure 6) while in the physical reality, nuclei are composed by extended and hyperdense protons and neutrons in conditions of partial mutual penetration established by the comparison of nuclear volumes with the constituent volumes [59] (Figure 7).

The inability to represent nuclei as they are in the physical reality im-

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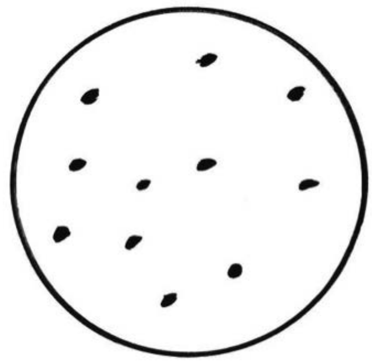

Figure 6: *The mathematics underlying quantum mechanics is local-differential, thus solely admitting a point-like approximation of particles. This figure illustrates the consequential conception of nuclei as ideal spheres with isolated points in their interior with ensuing insufficiencies beyond scientific doubt.*

plies the inability to achieve an exact representation of nuclear magnetic moments. In fact, the quantum mechanical representation of the anomalous magnetic moment of the deuteron still misses $1\ \%$ despite all possible relativistic or quark-based corrections.

Additionally, quantum mechanics misses much bigger percentages of nuclear magnetic moments for heavier nuclei (as illustrated in Figure 8).

In turn, the inability to represent protons and neutrons as extended charge distributions implies the inability to represent deformations under strong nuclear forces, with related deformation of their angular momenta.

In fact, J. M. Blatt and V. F. Weisskopf state on page 31 of their treatise in nuclear physics [96]: *It is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon* (Figure 9).

The representation of nucleons as extended, thus deformable and hyperdense charge distributions via the "completion of 20th century mathematics (Section 2) and physics (Section 3) has permitted the exact representation of the anomalous magnetic moment of the deuteron [97], as well as of heavier nuclei [98].

It should be noted that the representation of nuclear magnetic moments is presented in Ref. [7] as an illustration of the implications of the proof of the EPR arguments for extended particles in interior conditions.

A second insufficiency of quantum mechanics in nuclear physics is the lack of a consistent representation of nuclear spins despite efforts also conducted for about one century.

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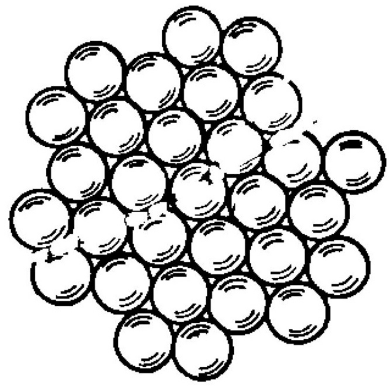

Figure 7: *A conceptual rendering of nuclei as they occur in the physical reality, i.e., a collection of extended and hyperdense protons and neutrons in conditions of partial mutual penetration, with ensuing non-linear, non-local and non-potential interactions beyond any dream of quantitative treatment via quantum mechanics.*

Recall that the proton and the neutron have both spin $1/2$ and that the only stable bound state predicted by quantum mechanics between two particles with spin $1/2$ is the singlet with spin 0.

Therefore, quantum mechanics predicts that the deuteron in its ground state must have spin 0, while experimental data establish that the deuteron has spin 1.

In an attempt of salvaging quantum mechanics, the spin of the deuteron is generally represented via a combination of *excited orbital states* which, even though significant, does not represent the spin 1 of the deuteron *in its ground state*.

The achievement of the synthesis of the neutron (Refs. [85] to [95]) has permitted a resolution of the above impasse because *the deuteron emerges as being a three-body state composed by two protons and one exchange electron, with ensuing spin* 1 *in the ground state* [59].

Subsequent studies by A. A. Bhalekar and R. M. Santilli [100] based on the "completion" of 20th century mathematical and physical methods have achieved a representation of the spin of stable nuclei in their bound state.

It should be finally noted that the biggest insufficiency of quantum mechanics in nuclear physics is given by the inability to achieve a consistent representation of nuclear forces in one century of efforts.

This is due to the fact that the sole forces permitted by quantum mechanics are of *potential,* thus of action- at-a-distance type, which is solely

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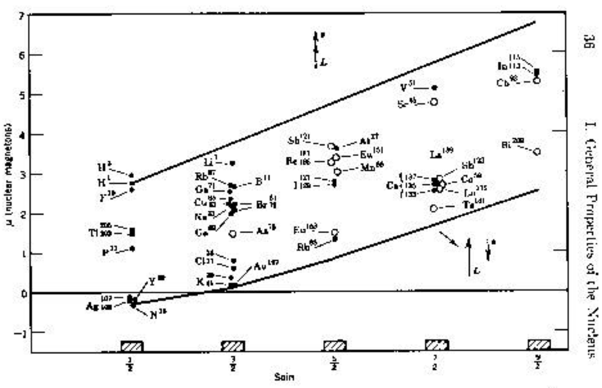

Figure 8: *A view of the experimental data of nuclear magnetic moments that cannot be exactly represented by quantum mechanics. Similar insufficiencies exist for nuclear spins.*

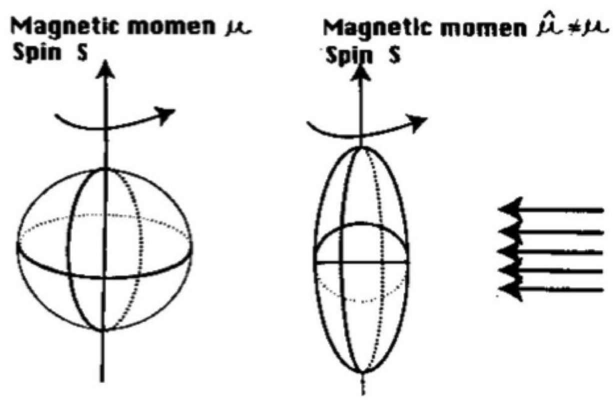

Figure 9: *A conceptual rendering of the deformability of protons and neutrons under strong nuclear interactions predicted by J. M. Blatt and V. F. Weisskopf [96] as being the origin of the inability by quantum mechanics to represent nuclear magnetic moments.*

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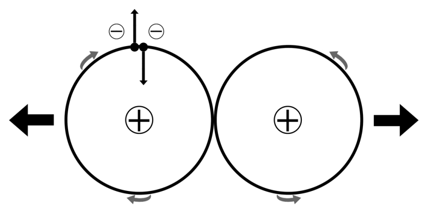

Figure 10: *This picture depicts the hydrogen molecule at absolute zero degree temperature to illustrate the fact that, despite historical achievements, quantum mechanics and chemistry have been unable to identify the force attracting identical electrons in valence bonds, because the sole admitted forces are of potential-Coulomb type, thus implying a repulsion in valence bonds with ensuing lack of a "complete" representation of molecular structures [60].*

possible, mathematical and physical conception for nuclei as ideal spheres with point-like particles in their interior (Figure 6).

One of the most important applications of the new methods studied in this work is that of representing nuclear forces as being non-linear, non-local and non-potential forces due to the mutual penetration/entanglement of the charge distribution of the hyperdense nucleons. The latter forces have emerged as being strongly attractive thus allowing the first known initiation of the understanding of the charge independence of nuclear forces [59].

1.6. Insufficiencies of quantum mechanics in chemistry.

Without doubt, quantum mechanics and chemistry have permitted chemical discoveries of historical proportions. Hence, the historical and scientific value of quantum chemistry is out of question.

Yet, it is the fate of all theories to admit, with the advancement of scientific knowledge, suitable coverings and this is the fate of quantum chemistry as well.

In fact, on strict scientific grounds quantum chemistry is only *approximately valid* in chemistry, thus admitting a suitable "completion," because of the inability in one century of efforts to achieve an exact representation of molecular experimental data from first axiomatic principles without *ad hoc* form factors and other adaptations.

Recall that quantum mechanics has achieved a numerically exact representation of all experimental data of the hydrogen atom.

By contrast, when two hydrogen atoms are bonded into the hydrogen molecule H_2, quantum mechanics and chemistry still miss the represen-

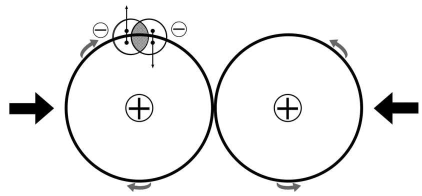

Figure 11: *This picture depicts the hydrogen molecule at absolute zero degree temperature with the strongly attractive force between identical valence electrons permitted by the representation of valence pairs as being composed by electrons with extended wavepackets in conditions of mutual penetration/entanglement with ensuing non-linear, non-local and non-potential interactions that result to be so strongly attractive [17] to overcome repulsive Coulomb forces [60]. The mutual distance d of said valence pair approaches Einstein's classical determinism and achieves it fully when it is in the interior of black holes [7].*

tation of $1\ \%$ of the H_2 binding energy, which is not insignificant since it corresponds to about $940\ kcal/mole$.

An in depth study of the impasse has shown that the above insufficiency is due to *the inability by quantum mechanics and chemistry to represent the attractive force between identical valence electrons in molecular bonds* (Figure 10) because according to basic axioms, *two identical valence electrons must repel each other according to quantum mechanics and chemistry due to their equal charge* [60].

As it had been the case for other problems, the primary difficulty to achieve an attractive force between identical valence electrons was of *mathematical* rather than of physical or chemical character, because quantum mechanics and chemistry solely admit potential forces that, in this case, can only be of repulsive Coulomb type.

By contrast, the sole possibility of resolving the impasse was the representation of electrons as extended wavepackets, that when in conditions of mutual penetration at $1\ fm$ mutual distance, admit contact, non-linear, non-local and non-potential interactions of Hulten type.

These new interactions result to be so strong to "absorb" repulsive Coulomb forces resulting in the needed attraction [60] (Figure 11).

The new valence force was first identified in Table 5 of the 1978 paper [17] as responsible for the birth of strong interactions in the synthesis of the π^0 meson from an electron and a positron.

In 1995, A. O. E. Animalu and R. M. Santilli published paper [101] establishing that the Hulten force of paper [17] is so strong to account for the

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bond of the two identical electrons in the Cooper pair of superconductivity.

A generalization of superconductivity based on the new methods was then developed and it is today known as *Animalu isosuperconductivity* [102].

In 2000, R. M. Santilli and D. D. Shillady showed that valence electron pairs with a strongly attractive force, called *isoelectronia,* permit numerically exact representations from first axiomatic principles of experimental data on the hydrogen [103] and water [104] molecule, which representation had escaped quantum chemistry for about one century (see also review [105]).

The achievement of a new model of molecular structures based on the isoelectronium valence bond has permitted novel advances in larger molecules whose study has been initiated by A. A. Bhalekar and R. M. Santilli [106] with intriguing implications, e.g., possible improvements in the combustion of fossil fuels based on a more accurate representation of their molecular structure [53].

Additionally, Santilli and Shillady showed that *perturbative series of the resulting "completion" of quantum chemistry converge at least one thousand times faster than the corresponding series of quantum chemistry* (see Section 4.13).

1.7. Implications of the EPR argument.

It is hoped that the preceding sections have indicated the truly vast implications for all quantitative sciences of Einstein's view that "quantum mechanics is not a complete theory" [1], thus warranting due scientific process.

In this paper we outline the main aspects of the new mathematical and physical methods underlying proof [7], with the understanding that a technical knowledge can be solely achieved via a study of the original literature.

The reader should be aware that the literature accumulated in half a century of research in the field by numerous scientists is rather vast. Consequently, in this paper we can only quote the most important original contributions and provide comprehensive references for interested readers.

2. LIE-ADMISSIBLE "COMPLETION" OF 20TH CENTURY APPLIED MATHEMATICS.

2.1. Foreword.

R. M. Santilli never accepted quantum mechanics as a "complete" theory

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beginning with his graduate studies in physics at the University of Torino, Italy, in the mid 1960's because quantum axioms are invariant under time reversal, due to the invariance under anti-Hermiticity of the Lie product between Hermitean operators

$$[x, y] = -[x, y]^\dagger, \tag{1}$$

and other physical laws.

By recalling the fundamental character of Lie's theory, it follows that the *mathematics* (more than physical laws) underlying quantum mechanics does not allow a consistent representation of nuclear fusions and other physical or chemical energy releasing processes, due to their known irreversibility over time (Figure 4).

2.2. The historical teachings by Lagrange and Hamilton.

In view of the above lack of "completeness" of quantum mechanics, R. M. Santilli initiated his Ph. D. studies with the reading of the original works by J. L. Lagrange and studying his true analytic equations, those with external terms [9]

$$\frac{d}{dt}\frac{\partial L(r,v)}{\partial v} - \frac{\partial L(r,v)}{\partial r} = F_{ak}(t,r,v), \tag{2}$$

as well as the true Hamilton's equations, those with external terms [10]

$$\begin{aligned} \tfrac{dr}{dt} &= \tfrac{\partial H(r,p)}{\partial p}, \\ \tfrac{dp}{dt} &= -\tfrac{\partial H(r,p)}{\partial r} + F(t,r,p), \end{aligned} \tag{3}$$

where the Lagrangian L and Hamiltonian H were used to represent conservative and potential, thus notoriously reversible forces, while the irreversibility of nature was represented with their *external forces* F.

2.3. The "No Reduction Theorem."

The external terms have been truncated in the 20th century sciences on claims that irreversible systems can be decomposed into their elementary particle constituents at which level the validity of quantum mechanics is fully recovered.

However, Santilli proved the following theorem as part of his Ph. D. thesis (see Refs. [56] [25]).

THEOREM 2.3.1 (No reduction Theorem): A macroscopic time irreversible system cannot be consistently decomposed into a finite number of quantum mechanical particles and vice versa, a finite collection of quantum mechanical particles *cannot reproduce a macroscopic irreversible system under the correspondence or other principle.*

Consequently, a serious study of irreversible systems requires a return to the true Lagrange's and Hamilton's equations, those with external terms.

2.4. The inevitability of the Lie-admissible "completion" of 20th century science.

As it is well known, the true Lagrange and Hamilton equations cannot be assumed as a "completion" of 20th century sciences because they are not derivable from a potential.

Additionally, the brackets of the time evolution of an observable Q represented via Hamilton's equations with external terms

$$\begin{aligned} \tfrac{dA}{dt} &= (Q, H, F) = \\ &= \tfrac{\partial Q}{\partial r}\tfrac{\partial H}{\partial p} - \tfrac{\partial H}{\partial r}\tfrac{\partial Q}{\partial p} + \tfrac{\partial Q}{\partial r}F, \end{aligned} \tag{4}$$

characterizes the *triple system* (Q, H, F) that, in view of the external terms, violate the right scalar and associative axioms to characterize an algebra as currently understood in mathematics.

In the absence of a consistent algebra in the brackets of the time evolution, it was not possible to achieve a "completion" of quantum mechanics via a covering for irreversible systems.

Hence, Santilli was forced to seek the needed "completion" on algebraic grounds.

Following a year of research in the European mathematics libraries, Santilli did his Ph. D. thesis in 1965 on the "completion" of Lie algebras into A. A. Albert's *Lie-admissible and Jordan-admissible algebras* [11] with product [12]

$$(a, b) = pab - qba, \tag{5}$$

later on known as *(p, q)-deformations [13]*, where $p,\ q, p \pm q$ are non-null scalars, and time irreversibility is assured for $p \neq q$ for which irreversiblity is ensured for $p \neq q$ by the property $(x, y) \neq -(x, y)^{\dagger}$.

To achieve a first approximation of Hamilton's equations with external terms, Santilli introduced the following *parametric Lie-admissible generalization of Hamilton's equation* [14] [15]

$$\frac{dr}{dt} = p\frac{\partial H(r,p)}{\partial p}, \quad \frac{dp}{dt} = -q\frac{\partial H(r,p)}{\partial r}, \tag{6}$$

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with corresponding *parametric Lie-admissible generalization of Heisenberg equation* for the time evolution of a Hermitean operator Q (for $\hbar = 1$)

$$i\frac{dQ}{dt} = (Q, H) = pQH - qHQ. \tag{7}$$

As one can see, all dynamical equations are manifestly irreversible over time, as desired.

2.5. Lie-admissible genomathematics and genomechanics.

In September 1977, Santilli joined the Department of Mathematics of Harvard University under DOE support during which stay he introduced the most general known realization of irreversible Lie-admissible algebras (see Refs. [16] to [23]) based on the generalization and differentiation of the ordinary product ab of arbitrary quantities (numbers, functions, operators, etc.) into the *ordered genomodular product to the right*

$$a > b = a\hat{R}b, \tag{8}$$

and that to the left

$$a < b = a\hat{S}b, \tag{9}$$

where $\hat{R}, S$ and $R \pm S$ are positive-definite operators with an unrestricted functional dependence on wavefunctions $\psi(t, r)$ and any other needed variables.

The operators R and S were called *genotopic element to the right and to the left,* respectively, where the prefix "geno" was suggested by Carla Santilli in the Greek sense of "inducing a new structure" [16].

The new genomodular products permitted the construction of new mathematics known as *genomathematics to the right and to the left,* [19] with corresponding "completion" of quantum mechanics into an irreversible covering known as *genotopic branch of hadronic mechanics* or *genomechanics for short* [19] in which irreversible energy releasing processes are represented with ordered genomodular product to the right, as it is the case for the *geno-Schrödinger equation* or *Schrödinger-Santilli genoequation* [20]

$$H > \psi = H(r, p)R(\psi, ...)\psi = E\psi, \tag{10}$$

Irreversibility is then assured whenever the genotopic elements R and S are not invariant under time reversal.

The time evolution of a Hermitean operator Q is given by the *Lie-admissible generalization of Heisenberg equations,* also known as *Heisenberg-Santilli genoequations* (see, e.g., Ref. [107]) first introduced in Eqs. (4.15.34), page 746 of Ref. [17] (see the 2006 general treatment [24] and the 2016 update [25]) which can be written in the infinitesimal form

$$i\frac{dQ}{dt} = (Q.H) = Q < H - H > Q = \\ = QSH - HRQ, \tag{11}$$

and the finite form

$$Q(t) = e^{HRti} Q(0) e^{-itSH}. \tag{12}$$

The representation of Lagrange's and Hamilton's external terms is provided by the difference $R - S$ at both classical and operator levels [24].

2.6. Universality of Lie-admissible formulations.

The following simple realization of the genotopic elements

$$S = 1, \quad R = 1 - \frac{1}{H} K(\psi, \partial\psi, ...), \\ i\frac{dQ}{dt} = (Q.H) = [Q, H] + QK, \tag{13}$$

where K is a positive-definite operator representing non-Hamiltonian interactions illustrates that the Lie-admissible generalization (11) of Heisenberg's equations constitute an operator image of Hamilton's equations with external terms (3) [56].

The double infinity of possible realizations of the genotopic elements R and S then allows Lie-admissible equations (11) to be "directly universal" for the representation of all possible (regular) non-linear, non-local and non-Hamiltonian interactions in the sense of representing all of them ("universality") directly in the frame of the experimentalism without the use of the transformation theory ("direct universality") (for details, see Ref. [24]).

It should be finally indicated that the original 1978 proposal [16] established the universality of Lie-admissible algebras because the product $(A, B) = A < B - B > A$ admit as particular case the product of all possible "algebras" as commonly understood in mathematics, including Associative, Lie, Jordan, Lie-isotopic, Jordan-isotopic, alternative, super-associative, super-Lie, super-Jordan, nilpotent, flexible and other possible algebras.

2.7. Prediction of new clean nuclear fusions.

Scientific and industrial applications to search for new clean energies were initiated in the late 1990's only following the achievement of maturity in the mathematical and physical methods needed for the representation of irreversible processes.

The first application was the conception of the new *Intermediate Controlled Nuclear Fusion* (ICNF) of light, natural and stable elements into light, natural and stable elements with smaller mass which occur without the emission of harmful (e.g., neutron) radiations and without the release of radioactive waste (see Ref. [26] to [32] for originating papers and Refs. [33] to [40] for independent studies). Laboratory analyses [41] to [52] confirmed the *existence* oof ICNF at the particle levels, with the understanding that the related industrial production of new clean energy require additional extensive research.

2.8. Prediction of a new clean combustion of fossil fuels.

To illustrate the implications of the lack of "completion" of quantum mechanics for energy releasing processes, we should note that the current combustion of fossil fuels is essentially that at the dawn of our civilization, because we essentially strike a spark and ignite the fuel with known alarming environmental deterioration of our planet.

The achievement of the Lie-admissible representation of energy releasing processes has permitted the first known conception and initiation of tests for a new principle of combustion called *HyperCombustion* which is based on the conventional combustion of carbon and oxygen dating back to the dawn of our civilization, plus the novel synthesis of a limited number of nuclei C-12 and O-16 into Si-28 to achieve full combustion of fossil fuels as well as a significant increase of energy output [53].

2.9. Literature on hadronic mathematics and mechanics.

Due to the prediction of new clean nuclear energies, the connection between irreversible mechanics and thermodynamics and other features, the literature on the foundations of hadronic mechanics is rather vast.

Ref. [54] provides a summary of the formalism of hadronic mechanics. Refs. [55] to [61] provide general presentations of hadronic mechanics. Vol. I of Refs. [61] contains a comprehensive literature up to 2008 with an upgrade to 2016 in Ref. [25].

Additional references are available in the reprint volumes [62] [63] and in the proceedings of five *Workshops on Lie-Admissible Algebras,*, twenty five *Workshops on Hadronic Mechanics,* and three international conferences on the *Lie-admissible treatment of Irreversible Processes* whose references are available from Ref. [61]. Representative independent papers are available from Refs. [64] to [74] and independent monographs are available from Refs. [75] to [83].

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3. LIE-ISOTOPIC "COMPLETION" OF 20TH CENTURY APPLED MATHEMATICS.

3.1. Foreword.

As it is well known, the objections against the EPR argument (Section 1.1) were formulated for *isolated reversible systems of point particles in vacuum with linear, local and Hamiltonian interactions* called *reversible exterior dynamical problems*.

Hence, the proof of the EPR argument had to study *isolated reversible systems of extended, therefore deformable and hyperdense particles in conditions of mutual penetration/entanglement with ensuing linear and non-linear, local and non-local and Hamiltonian as well as non-Hamiltonian internal interactions.* The latter systems are called *reversible interior dynamical problems*, and they occur in the structure of hadrons, nuclei, stars and black holes.

Despite their reversible character, the latter systems could not be studied with 20th century applied mathematics, including Lie's theory, due to its strictly Hamiltonian character. Reversible interior dynamical systems could not be studied with Lie-admissible formulations due to their irreversible character. Hence, the needed new mathematics had to be built.

In this section, we review the foundations of the new mathematics for the consistent representation of reversible interior dynamical systems which was essentially constructed as a reversible particular case of universal Lie-admissible formulations.

3.2. Isoproduct.

In order to achieve a representation of the latter systems, Santilli introduced in the original proposal [16] of 1978 the axiom-preserving particular case of genomathematics called *isomathematics,* which is characterized by the genoproduct to the right being equal to that to the left, $R = S = \hat{T}$, resulting in the time-reversal invariant *isoproduct,* (first introduced in classical realization in Eqs. (3.7.10), page 352 of Ref. [16], introduced in operator form in Eq. (4.15.46), page 751, Ref. [17] and then studied in details in Ref. [19], Eq. (2), page 71 on)

$$a \star b = a \,\hat{T}\, b, \tag{14}$$

where $\hat{T}$, called the *isotopic element,* is a function, matrix or operator solely restricted to be positive-definite, but possesses otherwise an unrestricted functional dependence on all needed local variables, such as: spacetime coordinates $x = (t, r)$; linear momentum p; energy E; frequency ν; density of the medium α; temperature τ; pressure π; wavefunctions ψ; their

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derivatives $\partial\psi$; and any other needed variable

$$\hat{T} = \hat{T}(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \tag{15}$$

where the prefix "iso" was also suggested by Carla Santilli in its Greek meaning of preserving the axions.

The *physical* significant of isoproduct (14) is illustrated by nothing that it allows "ab initio" a *direct representation of extended, thus deformable and hyperdense particles and their non-Hamiltonian interactions* illustrated in Figure 3 (Section 1.4). This important task is achieved via simple realizations of the isotopic element of the type (needed for the neutron synthesis from the hydrogen studied in Section 2.4) [55] [56]

$$\hat{T} = Diag.(\frac{1}{n_1^2}, \frac{1}{n_2^2}, \frac{1}{n_3^2}, \frac{1}{n_4^2})e^{-\Gamma}, \tag{16}$$

with subsidiary conditions

$$\begin{gathered} n_\mu = n_\mu(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \mu = 1, 2, 3, 4, \\ \Gamma(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) \geq 0. \end{gathered} \tag{17}$$

The isoproduct also allows a direct representation of nuclei as a collection of extended nucleons in conditions of mutual penetration/entanglement as presented in Figure 7 and Section 1.5 with broader realizations of the type (needed to represent nuclear magnetic moments and spins, or for the achievement of an attractive force between identical valence electron bonds in molecular structures) [57]

$$\begin{gathered} \hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2})e^{-\Gamma}, \\ k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4. \end{gathered} \tag{18}$$

In the above realizations of the isotopic element, n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_1^2 = n_2^2, = n_3^2 = 1$, for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

The *mathematical* significance of basic assumption (14) is that it requires, for consistency, a compatible "completion" of *all* aspects of 20th century applied mathematics without any known exception.

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This program was initiated in the 1978 proposal [16], continued in the 1981 monograph [19] and completed in numerous works by various mathematicians (see the 1995 monograph [55] for a comprehensive presentation).

Regrettably we cannot provide a technical review of isomathematics to prevent excessive length. Nevertheless, a rudimentary outline of the main aspects of isomathematics appears to be recommendable for an understanding of proof [7] of the EPR argument.

3.3. Isonumbers.

As it is well known, physical theories are formulated over a field $F(n, \times, 1)$ of real, complex or quaternionic numbers n with product $nm = n \times m$ and multiplicative unit 1. Said field remains invariant under the unitary time evolution of quantum mechanics, thus allowing the prediction of the same numerical values under the same conditions at different times.

But the time evolutions of hadronic mechanics, such as Eqs. (11), are *non-unitary* when formulated on conventional space over a conventional field (not so for isomathematics as shown below).

This implies the loss over time of the multiplicative unit 1, and consequently, of the entire numeric field, with ensuing lack of consistent experimental verifications.

To resolve this impasse, Santilli had no other choice than that of reinspecting the historical classification of numbers, by discovering in this way that *the abstract axioms of a numeric field do not necessarily restrict the multiplicative unit to be the number* 1*, and allow for unit an arbitrary positive-definite quantity* $\hat{I}$ *provided that the multiplication is redefined for* $\hat{I}$ *to verify the unit axiom* [107].

This lead to "completion" of numeric fields $F(n, \times, 1$ into *isofields* $\hat{F}(\hat{n}, \star, \hat{I}$ of *isoreal, isocomplex, and isoquaternionic isonumbers* $\hat{n} = n\hat{I}$ with *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{19}$$

isoproduct (14), $\hat{n} \star \hat{m} = (nm)\hat{I}$, and isounit isoaxiom

$$\hat{I} \star \hat{n} = \hat{n} \star \hat{I} = \hat{n}, \;\; \forall \hat{n} \; \in \; \hat{F}. \tag{20}$$

Isofields are completed by compatible redefinitions of all numeric operations, such as *isoquotient, isosquare, isosquareroot,* etc. [107].

It should be indicated that *isofields verify all axioms of a field. Hence, isonumbers are fully acceptable for experimental verifications* [57].

Isofields are classified into those of the *first kind* (*second kind*) depending on whether the isounit $\hat{I}$ is (is not) an element of the original field.

In this paper, we shall solely consider isofields of the second kind since the representation of extended particles and their non-Hamiltonian interactions is achieved via the isounit or equivalently, the isotopic element.

Paper [107] has stimulated various studies in number theory, among which we mention the study by C-Xu. Jiang [79], A. K. Aringazin [108], C. Corda [109] and others.

3.4. Isofunctions.

As it is well known, a necessary condition for a variable to be measurable is that it is an element of the base field, and the same holds for functions of said variable.

The implementation of the same rule under isotopic "completion" stimulated the construction of the *isofunctional isoanalysis* initiated by: J. V. Kadeisvili [110][111]; A. K. Aringazin, D. A. Kirukhin and R. M. Santilli [112]; Raul M. Falcon Ganfornina and Juan Nunez Valdes [80]; and others.

We here limit ourselves to indicate: the *isotime* $\hat{t} = t\hat{I}_t$, *isospace isocoordinates* $\hat{r} = r\hat{I}_r$, and the *isofunctions of isovariable*,

$$\hat{f}(\hat{r}) = [f(r\hat{I})]\hat{I}, \tag{21}$$

such as the *isoexponentiation*

$$\hat{e}^{\hat{X}} = [e^{\hat{X}\hat{T}}]\hat{I} = \hat{I}[e^{\hat{T}\hat{X}}]. \tag{22}$$

Similar expressions hold for virtually all conventional functions used in applications [55].

3.5. Isospaces.

The initial construction of isomathematics [19] was formulated via conventional vector or metric spaces over conventional fields.

The consistent need to formulate spaces over isofields triggered the isotopic "completion" of metric spaces into *isospaces* whose study was initiated by the mathematician Gr. Tsagas and his school [113]. In turn, these studies triggered the construction of the *isotopology* by R. M. Falcon Ganfornina and J. Nunez Valdes [114], yielding the first known topology for the characterization of extended particles, known as the *Tsagas, Sourlas, Santilli, Ganfornina and Nunez (TSSGN) isotopology.*

Let $E(r, \delta, I)$ be the conventional Euclidean space with space coordinates $r = (x, y, z)$, metric $\delta = Diag.(1, 1, 1)$, unit $I = Diag(1, 1, 1)$ and invariant

$$r^2 = (x^2 + y^2 + z^2)1, \tag{23}$$

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where one should note the trivial multiplication by 1 for compatibility with the isotopic image studied below.

The representation isospace of the non-relativistic proof of the EPR argument is given by the infinite family of *iso-Euclidean isospaces,* $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I})_r$, first formulated in the 1978 Ref. [16] and treated in detail in Ref. [19] , finalized in 1996 Ref. [115] and extensively treated in monograph [55].

For the simple realization of the isotopic element

$$\hat{T} = Diag.(1/n_1^2, 1/n_2^2, 1/n_3^2), \tag{24}$$

iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\Delta}, \hat{I}_r)$ are characterized by: the *isocoordinates* $\hat{r} = r\hat{I}$, the *iso-Euclidean isometric* $\hat{\Delta} = (\hat{T}\delta\hat{I}$, and the *isospace isounit* $\hat{I}_r = 1/\hat{T} > 0$ resulting in the *iso-Euclidean isoinvariant*

$$\begin{aligned} \hat{r}^{\hat{2}} = (\hat{r}^j \star \hat{\Delta}_{jm} \star \hat{r}^m) = (r^j \hat{\delta}_{jm} r^m)\hat{I}_r = \\ = (\tfrac{r_1^2}{n_1^2} + \tfrac{r_2^2}{n_2^2} + \tfrac{r_3^2}{n_3^2})\hat{I}_r, \end{aligned} \tag{25}$$

where we should recall that, for consistency, all scalar quantities have to be elements of an isofield $\hat{F}$.

The above conditions require that: squares must be isosquares $\hat{r}^{\hat{2}} = \hat{r} \star \hat{r} = \hat{r}^2\hat{I}_r$; coordinates have to be isocoordinates $\hat{r} = r\hat{I}_r$; to be isomatrices, isometrics must have the structure $\hat{\Delta} = \hat{\delta}\hat{I}_r$; and the elaboration requires the use of the *isotrigonometric isofunctions* as well as of the *isospherical isocoordinates* (see Ref. [55] for details).

Let $M(x, \eta, I)$ be the conventional Minkowski space with spacetime coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = Diag.(1, 1, 1, -1)$, unit $I = Diag(1, 1, 1, 1)$ and invariant

$$\begin{aligned} x^2 = (x^\mu \eta_{\mu\nu} x^\nu)I = \\ = (x_1^2 + x_2^2 + x_3^3 - c^2t^2)I. \end{aligned} \tag{26}$$

The isospace for relativistic treatments of extended particles is given by the infinite family of *iso-Minkowski spaces* $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I}_x)$ also known as *Minkowski-Santilli isospaces,* (see e.g., Ref. [107]) first introduced in Ref. [116] and then treated in details in Ref. [56].

Iso-Minkowskian isospaces are characterized by the *isospace-time isocoordinates* $\hat{x} = x\hat{I}$; isounit $\hat{I} = 1/\hat{T}$, and *isometric* $\hat{\Omega} = \hat{\eta}\hat{I}_x = (\hat{T}\eta)\hat{I}_x$ formulated on the isoreal isonumbers $\hat{\mathcal{R}}$.

For the simple realization of the isotopic element

$$\hat{T} = Diag(1/n_1^2, 1/n_2^2, 1/n_3^2, 1/n_4^2), \tag{27}$$

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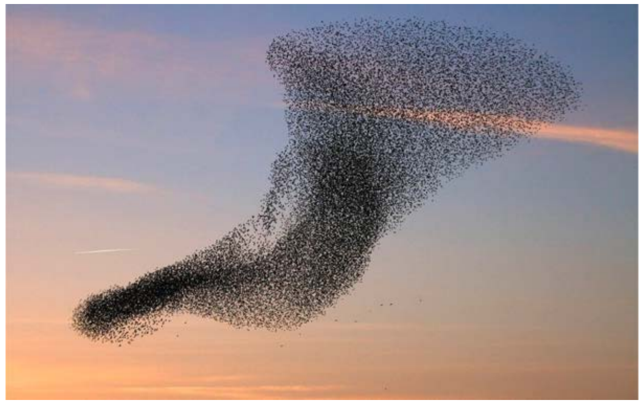

Figure 12: *This picture illustrates the isodifferential calculus [115] via birds flying in close formation without wing interferences, which can be best understood by assuming that birds conceive themselves as a volume encompassing their wings, rather than a mass concentrated in their center of gravity as it would be requested by the Newton-Leibnitz differential calculus.*

we have the infinite family of realizations of the *isospace-time isoinvariant*

$$\hat{x}^{\hat{2}} = \hat{x}^{\mu} \star \hat{\Omega}_{\mu\nu} \star \hat{x}^{\nu} = (x^{\mu}\hat{\eta}_{\mu\nu}x^{\nu})\hat{I} = \\ = (\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2\frac{c^2}{n_4^2})\hat{I}, \tag{28}$$

where the final multiplication by the isounit is again necessary for the invariant to be an isoscalar.

It should be noted that, in addition to the use of the isospherical isocoordinates, data elaborations in the iso-Minkowskian isospace requires the use of *isohyperbolic isofunctions* (see Ref. [56], Chapters 5 and 6).

Note also that invariant (30) is the most general possible symmetric (non-singular) invariant in $(3+1)$-dimensions, thus including as particular cases all possible Minkowskian, Riemannian, Fynslerian and all other geometries.

3.6. Isodifferential isocalculus.

Despite the above advances, numerical predictions of isomathematics lacked the crucial property of invariance over time.

In addition, isomechanics and hadronic mechanics at large, were incomplete due to the inability to formulate the isotopies and genotopies of the linear and angular momenta (see next section).

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In order to resolve this impasses, Santilli had no other choice than that of reinspecting the Newton-Leibnitz differential calculus, by discovering in this way that, contrary to rather popular belief for four centuries, *the differential calculus depends on the multiplicative unit of the base field* because, when the unit depends on the variable of differentiation, said calculus has to be "completed" into the infinite family of *isodifferentials,* e.g., of the isocoordinates $\hat{r}$, first formulated in memoir [115] submitted in 1995 and published in 1996 and then treated in details in Refs. [55] [56]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r,...)] = dr + r\hat{T}d\hat{I}(r,...), \tag{29}$$

with corresponding *isoderivative* [114]

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{30}$$

Following decades of searches, the discovery of the isodifferential isocalculus finally permitted the achievement of the invariance over time of numerical predictions, the formulation of isolinear and genolinear momenta and signaled the achievement of maturity for applications and experimental verifications (see Refs. [56] [57] for details).

All novel applications of isomathematics in physics, chemistry and other fields, including the proof of the EPR argument [7], originate from the extra term $r\hat{T}d\hat{I}(r,...)$ in isodifferential (29), which is absent in the mathematics for point particles.

The covering character of the isodifferential isocalculus over the conventional calculus is illustrated by the fact that whenever the isounit is independent from the differentiation variable or it is a constant, the conventional calculus is recovered uniquely and identically.

It should be indicated that the biggest difficulty in the use of the isodifferential isocalculus is of *conceptual,* rather than of mathematical character, because it requires the transition from the visualization of the calculus at individual points to volumes (or surfaces) represented by the isotopic element $\hat{T}$ (Figure 12).

By looking in retrospect, it appears nowadays evident that a "completion" of quantum mechanics for the representation of extended particles is fundamentally inconsistent when formulated via the conventional differential calculus, because of its sole possible characterization of particles as being point-like.

Consequently, the generalization of the differential calculus into a form defined on volumes represented by $\hat{T}$, rather than defined on coordinate points r, is essential for a consistent representation of extended particles.

Nineteen years following the discovery of the isodifferential isocalculus for extended particles, comprehensive studies in the field have been conducted by the mathematician S. Georgiev in the series of six monographs [83] which consider the broadest possible formulation of the isodifferential calculus, including the case of different isotopic elements for different isovariables.

3.7. Lie-Santilli isotheory.

In Santilli's view, the physically most important part of isomathematics is given by the isotopic "completion" of the various branches of Lie's theory, today known as the *Lie-Santilli isotheory,* which was first formulated in papers [16] [17] of 1978, systematically studied in monograph [19], finalized in Refs. [55] [56] of 1995 following the discovery of the isodifferential calculus [115] and recently studied in paper [117].

In this section, we follow the presentation of Ref. [19] of 1981 upgraded into a formulation on isospaces over isofields and elaborated via the isodifferential isocalculus.

Let L be a n-dimensional Lie algebra with Hermitean generators $X_k,\ k = 1, 2, ...n$ defined on a conventional space over a conventional numeric field. Then, the infinite family of isotopies $\hat{L}$ of L are characterized by the following main theorems:

THEOREM 3.7.1 [19]: (Poincaré-Birkhoff-Witt-Santilli isotheorem): The isocosets of the isounit and of the isostandard isomonomials

$$\hat{I},\ \ \hat{X}_k,\ \ \hat{X}_i \star \hat{X}_j,\ i \leq j,\ \ \hat{X}_i \star \hat{X}_j \star \hat{X}_k,\ i \leq j \leq k,\ \ldots, \tag{31}$$

form an infinite dimensional isobasis of the universal enveloping isoassociative isoalgebra $\hat{E}(\hat{L})$ (also called isoenvelope for short) of a Lie-Santilli isoalgebra $\hat{L}$.

The first illustration of the above theorem is given by isoexponential isofunction (22) whose correct derivation requires infinite basis (31).

The appearance of the non-linear, non-local and non-potential isotopic element $\hat{T}$ in the *exponent* illustrates the non-trivial character of the Lie-Santilli isotheory.

THEOREM 3.7.2 [19]: (Lie-Santilli isoalgebras) The antisymmetric isoalgebras $\hat{L}$ attached to the isoenveloping algebras $\hat{E}(\hat{L})$ verify the isocommutation rules

$$\begin{aligned}[\hat{X}_i \hat{,} \hat{X}_j] &= \hat{X}_i \star \hat{X}_j - \hat{X}_j \star \hat{X}_i = \\ &= \hat{C}^k_{ij}(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) \star \hat{X}_k,\end{aligned} \tag{32}$$

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where the quantities $\hat{C}$ are called the structure quantities.

The above isocommutation rules show the axiom-preserving character of the isotopies in view of their evident verification of the Lie axioms, although via a broader realization.

The Lie-Santilli isotheory is called *regular* or *irregular* depending on whether the structure quantities $\hat{C}_{i,j}^k$ are isoscalars or isofunctions, respectively.

This classification is important because as shown in the next section, the regular Lie-Santilli isotheory can be constructed via a well defined transform of corresponding Lie theory, while irregular Lie-Santilli isotheories are truly new theories verifying Lie's axioms without a known map form the conventional formulation.

THEOREM 3.7.3 [19]: (Lie-Santilli isogroups) The isoexponentiated form $\hat{G}$ of isocommutation rules (32) defined on an isospace $\hat{S}$ with local isocoordinates $\hat{x}$ over an isofield $\hat{F}$ with isounit $\hat{I} = 1/\hat{T} > 0$ is a group mapping each element $\hat{x} \in \hat{S}$ into a new element $\hat{x}' \in \hat{S}$ via the isotransformations

$$\hat{x}' = \hat{g}(\hat{w}) \star \hat{x}, \ \ \hat{x}, \hat{x}' \in \hat{S}, \ \ \hat{w} \in \hat{F}, \tag{33}$$

verifying the following isomodular isoaction to the right:

1) The isomap of $\hat{g} \star \hat{S}$ into $\hat{S}$ is isodifferentiable $\forall \hat{g} \in \hat{G}$;

2) $\hat{I}$ is the left and right isounit of $\hat{G}$,

$$\hat{I} \star \hat{g} = \hat{g} \star \hat{I} \equiv \hat{g}, \ \ \forall \hat{g} \in \hat{G}; \tag{34}$$

3) The isomodular isoaction is isoassociative,

$$\hat{g}_1 \star (\hat{g}_2 \star \hat{x}) = (\hat{g}_1 \star \hat{g}_2) \star \hat{x}, \ \ \forall \hat{g}_1, \hat{g}_2 \in \hat{G}; \tag{35}$$

4) In correspondence with every element $\hat{g}(\hat{w}) \in \hat{G}$ with $\hat{w} \in \hat{F}$ there exists the inverse element $\hat{g}(-\hat{w})$ such that

$$\hat{g}(\hat{0}) = \hat{g}(\hat{w}) \star \hat{g}(-\hat{w}) = \hat{I}; \tag{36}$$

5) The following composition laws are verified

$$\hat{g}(\hat{w}) \star \hat{g}(\hat{w}') = \hat{g}(\hat{w}') \star \hat{g}(\hat{w}) = \hat{g}(\hat{w} + \hat{w}'), \forall \hat{g} \in \hat{G}, \ \hat{w} \in \hat{F}; \tag{37}$$

with corresponding isomodular action to the left, and general expression

$$\hat{g}(\hat{w}) = \prod_k \hat{e}^{\hat{i} \star \hat{w}_k \star \hat{X}_k} \star \hat{g}(0) \star \prod_k \hat{e}^{-\hat{i} \star \hat{w}_k \star \hat{X}_k}. \tag{38}$$

Nowadays, *isomathematics* is referred to as the infinite family of isotopies of 20th century applied mathematics, with particular reference to the isotopies of Lie's theory when formulated on isospaces over isofields and elaborated via the isodifferential isocalculus. Isomathematics is then classified into *regular and irregular isomathematics* depending on whether the structure quantities $\hat{C}_{ij}$ are isoscalars or isofunctions of local isovariables, respectively.

It should be indicated that, the proof [7] of the EPR argument uses both regular and irregular Lie-Santilli isoalgebras.

Since Lie's theory is at the foundation of the axiomatic structure and applications of quantum mechanics, the covering Lie-Santilli isotheory predictably stimulated a number of independent contributions, such as the studies by the mathematicians: D. S. Sourlas and Gr. T. Tsagas [76], J. V. Kadeisvili [118], T. Vougiouklis [119] and papers quoted therein.

3.8. Simple construction of regular isomathematics.

A simple method for physicists has been identified in Ref. [120] of 1997 for the construction of regular isomathematics. The method consists of: 1) Selecting the desired representation of extended particles with non-Hamiltonian interactions via isotopic elements $\hat{T}$ of type (16) or (18); 2) Identifying a *non-unitary* transformation representing the selected isounit $\hat{I} = 1\hat{T}$

$$UU^\dagger = \hat{I}; \tag{39}$$

3) Subjecting the *totality* of conventional applied mathematics to the above nonunitary transform with no known exception, resulting in expressions of the type

$$I \to \hat{I} = UIU^\dagger = 1/\hat{T}, \tag{40}$$

$$n \to \hat{n} = UnU^\dagger = nUU^\dagger = n\hat{I},\ n \in F, \tag{41}$$

$$f(r) \to \hat{f}(\hat{r}) = Uf(r)U^\dagger, \tag{42}$$

$$e^A \to Ue^AU^\dagger = \hat{I}e^{\hat{T}\hat{A}} = (e^{\hat{A}\hat{T}})\hat{I}, \tag{43}$$

$$\begin{gathered} AB \to U(AB)U^\dagger = \\ = (UAU^\dagger)(UU^\dagger)^{-1}(UBU^\dagger) = \hat{A} \star \hat{B}. \end{gathered} \tag{44}$$

It should be indicated that the above transformations imply the possibility of constructing the infinite family of Lie-Santilli isoalgebras via nonunitary transforms of the considered Lie algebra. This is possible in view of the transformation of commutation rules into their covering isocommutator forms,

$$[A, B] \rightarrow U[A, B]U^{\dagger} = [\hat{A},\hat{B}]. \tag{45}$$

This property is evidently important for the construction of the isorepresentation of regular isoalgebras used for physical and chemical applications.

Note that *serious inconsistencies occur, at times without their detection by non-experts, in the event only one single quantity or operation of 20th century applied mathematics is not subjected to the above non-unitary map.*

We should finally indicate that *the proof of the EPR argument used in Ref. [7] is of 'non-unitary' character,* thus implying that the physical conditions for said proof are outside the class of equivalence of quantum mechanics.

3.9. Invariance of regular isomathematics.

An additional contribution of paper [120] is the proof that *the dimension, shape and density of extended particles and their non-Hamiltonian interactions are represented by isomathematics in a form invariant over time.*

Firstly, Ref. [120] showed that, following the construction of regular isomathematics via non-unitary transformations (Section 2.2.8), isomathematics is *not* invariant under additional non-unitary transforms, e.g., because of the lack of invariance of the basic isounit

$$\hat{I} \rightarrow \hat{I}' = W\hat{I}W^{\dagger} \neq \hat{I}, \quad WW^{\dagger} \neq I, \tag{46}$$

with consequential physical inconsistencies since any structural change of the isounit implies the transition to a different physical or chemical system.

However, non-unitary transforms can always be identically rewritten as *isounitary isotransforms* according to the rule [56]

$$WW^{\dagger} = \hat{I}, \quad W = \hat{W}\hat{T}^{1/2}, \tag{47}$$

$$WW^{\dagger} = \hat{W} \star \hat{W}^{\dagger} = \hat{W}^{\dagger} \star \hat{W} = \hat{I}, \tag{48}$$

under which reformulation we have the following *invariance of the isounit of the isotopic element and of the isoproduct of regular isomathematics* [120]

$$\hat{I} \rightarrow \hat{I}' = \hat{W} \star \hat{I} \star \hat{W}^{\dagger} \equiv \hat{I}, \tag{49}$$

$$\hat{A} \star \hat{B} \rightarrow \hat{W} \star (\hat{A} \star \hat{B}) \star \hat{W}^{\dagger} =$$

$$= \hat{A}' \star \hat{B}' = \hat{A}'\hat{T}\hat{B}', \tag{50}$$

$$\hat{A}' = \hat{W} \star A \star \hat{W}^{\dagger}). \; \hat{B}' = \hat{W} \star \hat{B} \star \hat{W}^{\dagger}, \; \hat{T} = (W^{\dagger} \star \hat{W})^{-1}.$$

The invariance of the entire isomathematics follows. Note that the invariance is ensured by the *invariant numeric values of the isounit and therefore, of the isotopic element under isounitary isotransforms,*

$$\hat{I} \rightarrow \hat{I}' \equiv \hat{I}, \tag{51}$$

$$\begin{aligned} A \star B = A\hat{T}B \rightarrow A' \star' B' = A'\hat{T}'B' \\ = \hat{A}' \star \hat{B}' = \hat{A}'\hat{T}\hat{B}' \end{aligned} \tag{52}$$

$$\hat{T} \rightarrow \hat{T}' \equiv \hat{T}. \tag{53}$$

By noting that, as shown in the next section, the time evolution of the isotopic "completion" of quantum mechanics is an isounitary isotransform, paper [120] established that *isomechanics is an axiom-preserving "completion" of quantum mechanics capable of representing extended particles under Hamiltonian as well as non-Hamiltonian interactions in a form invariant over time.*

In closing, we should recall other generalizations of 20th century mathematics and their applications, such as the so-called *deformations.* These generalizations are mathematically correct, but physically inconsistent because they violate causality laws (for brevity, see the *Theorems of Inconsistency of Non- Unitary Theories* in Vol. I of Refs. [61]). These inconsistencies arise from a structural generalization of Lie's algebras and other quantum laws when formulated on conventional spaces over conventional fields, thus preventing their reformulation as isounitary theories.

Note that the invariance of isomathematics reviewed in this section implies the verification of causality on isospaces over isofields in the same way as quantum mechanics verify causality laws.

4. LIE-ISOTOPIC "COMPLETION" OF QUANTUM MECHANICS.

4.1. Foreword.

In this section, we review the foundation of the *isotopic branch of hadronic mechanics,* also known as *isomechanics*, which is used in the proof of the EPR argument [7].

An important difference between preceding works and the presentation in this section is that the previous works generally present isomechanics in its *projection* on conventional spaces over conventional fields.

By contrast, in this section we put the emphasis in the full formulation of isomechanics, that on isospaces over isofields because important to illustrate that, contrary to opposing view (Section 1.1), proof [7] of the EPR argument is fully compatible with quantum axioms, only subjected to a broader realization.

Isomechanics was first introduced via isoproducts defined on conventional spaces over conventional fields in the 1978 papers [16] [17] and in the 1981 monograph [19]; it first achieved mathematical maturity in the 1996 memoir [115] thanks to the discovery of the isodifferential isocalculus; isomechanics was finally presented in a systematic way in monographs [55] [56] [57] with a 2016 upgrade in Section 2 of memoir [25].

Independent studies are available in Refs. [75] to [83]. A comprehensive list of references up to 2008 is available in Vol. I of Refs. [61] while a 2016 upgrade is available in Ref. [25]. We regret the inability of reviewing all important contributions on isomechanics to prevent an excessive length and are forced to outline only the most salient structural contributions.

4.2. Iso-Newton isoequations.

As it is well known, the fundamental equations of mechanics are the historical *Newton's equations*, representing systems of point-particles with Hamiltonian (that is, *variationally selfadjoint,* SA [18]) and non-Hamiltonian (*variationally non selfadjoint,* NSA [18]) forces [18] defined on a conventional Euclidean space, $E(r, \delta, I)$ over the field of real numbers $\mathcal{R}$

$$m\frac{dv_{ak}}{dt} - F^{SA}_{ak}(r, v) - F^{NSA}_{ak}(t, r, v) = 0,$$
$$k = 1, 2, 3, \quad a = 1, 2, ..., N, \quad N \geq 2. \tag{54}$$

It is generally believed that Newton's equations with non-conservative forces can solely represent open, irreversible systems. In Ref. [19] Section 6.3, Page 236, Santilli introduced *closed non-self-adjoint systems,* which are given by systems (54) violating the integrability conditions for their representation via Lagrange's or Hamilton's equations, yet verifying all ten conservation laws of Galileo relativity under the conditions

$$\sum\nolimits_{ak} F^{NSA}_{ak} = 0,$$
$$\sum\nolimits_{ak} \mathbf{r} \cdot \mathbf{F}^{NSA} = 0, \tag{55}$$
$$\sum\nolimits_{ak} \mathbf{r} \mathbf{F}^{NSA} = 0.$$

which conditions are evidently applicable only for $N \geq 2$, since the case of one particle $N = 1$ is trivial.

The fundamental equations of isomechanics are given by the isotopic "completion" of Eqs. (54) known as *iso-Newton isoequations*, which were first introduced in Ref. [115] immediately following the discovery of the isodifferential isocalculus, and they are also known as *Newton-Santilli isoequations,* (see, e.g., Refs. [107] [75] [81]) defined on iso-Euclidean isospaces $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ (Section 2.2.5) over isoreal isonumbers $\hat{\mathcal{R}}$ (Section 2.2.3)

$$\hat{m}_a \star \frac{\hat{d}\hat{v}_{ak}}{\hat{d}\hat{t}} - F^{SA}_{ak}(\hat{r}, \hat{v}) = 0. \tag{56}$$

A first important feature of Eqs. (56) is that of providing the first known consistent representation of the actual shapes and dimensions of the particles considered via the isodifferential calculus, with realization of the isotopic element of type (16).

A second important feature of Eqs. (56) is that of representing all potential-(SA) forces via conventional Newtonian forces $F^{SA}(r, v)$ while representing all non-potential-(NSA) forces via the isodifferential calculus.

This feature is treated in details in Ref. [115] and can be summarized as follows.

Note that the basic isounit of Eqs. (56) is the *isovelocity isounit*, $\hat{I} = \hat{I}_v$. Assume for simplicity that the isotime is equal to the conventional time,

$$\hat{t} = t \ \ \hat{I}_t = 1. \tag{57}$$

Consequently, from isoderivative (30), we have

$$\frac{\hat{d}\hat{v}}{\hat{d}\hat{t}} = I_t d\hat{v}/d\hat{t} = d\hat{v}/dt. \tag{58}$$

Consider then the projection of Eqs. (56) in the Euclidean space $E(v, \delta, I)$ and use the various rules of isomathematics (Section 2.2). Then Eqs. (56) can be written in the projected form (where all multiplications are conventional),

$$\begin{gathered}(m\hat{I})\hat{T}\tfrac{\hat{d}(v\hat{I})}{dt} - F^{SA}\hat{I} = \\ = \hat{I}[m\tfrac{dv}{dt} - F^{SA}] + mv\hat{T}\tfrac{d\hat{I}}{dt} = 0.\end{gathered} \tag{59}$$

By dividing the above equation with $\hat{I} > o$, one obtains Newton's equations with the following realizations of the NSA forces

$$F^{NSA}(t, r, v) = mv\hat{T}\frac{d\hat{I}}{dt}. \tag{60}$$

In conclusion, *the iso-Newton isoequations (56) embed all NSA forces into the isoderivatives by therefore allowing the first known consistent operator image of non-Hamiltonian forces studied in subsequent sections.*

4.3. Iso-Lagrangian and iso-Hamiltonian isomechanics.

Iso-Newton isoequations (56) admit a representation in terms of the *iso-Lagrange isoequations* first formulated in memoir [115] via the isodifferential calculus, thus being defined on an iso-Euclidean isospace over the isoreal isofield

$$\frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}\hat{L}(\hat{r},\hat{v})}{\hat{\partial}\hat{v}_{ak}} - \frac{\hat{\partial}\hat{L}(\hat{r},\hat{v})}{\hat{\partial}\hat{r}_{ak}} = 0, \tag{61}$$

where $\hat{L} = L\hat{I}$ is an *iso-Lagrangian*, namely, a conventional Lagrangian formulated on isospaces over isofields thus being multiplied by the isounit to be an isoscalar.

Eqs. (56) also admit the isocanonically isoequivalent isorepresentation in terms of the *iso-Hamilton equations* first formulated in memoir [115] on an isophase isospaces over an isoreal isofield and also known as *Hamilton-Santilli isoequations* (see, e.g., Ref. [107])

$$\begin{aligned} \frac{\hat{d}\hat{r}_{ak}}{\hat{d}\hat{t}} &= \frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{p}_{ak}}, \\ \frac{\hat{d}\hat{p}_{ak}}{\hat{d}\hat{t}} &= -\frac{\hat{\partial}\hat{H}(\hat{r},\hat{p})}{\hat{\partial}\hat{r}_{ak}}, \end{aligned} \tag{62}$$

where $\hat{H} = H\hat{I}$ is the iso-Hamiltonian, that is, a conventional Hamiltonian formulated on an isophase isospace over an isofield.

Note that iso-Hamiltonian isomechanics admits the following time evolution for a quantity $\hat{Q}$

$$\frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}} = [\hat{Q}\hat{,}\hat{H}] = \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}_{ak}}\frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_{ak}} - \frac{\hat{\partial}\hat{H}}{\hat{\partial}\hat{p}_{ak}}\frac{\hat{\partial}\hat{Q}}{\hat{\partial}\hat{r}_{ak}}, \tag{63}$$

where the brackets $[\hat{Q}\hat{,}\hat{H}]$ constitute a classical realization of Lie-Santilli isoalgebras.

4.4. Isovariational isoprinciple.

Another important feature of Eqs. (56) is that of permitting the first known representation of variationally non- selfadjoint/non-Hamiltonian systems via an *isovariational isoprinciple* [115],

$$\hat{\delta}\hat{A} = \hat{\delta}\hat{\int}(\hat{p}_{ak}\star\hat{d}\hat{r}_{ak} - \hat{H}\star\hat{d}\hat{t}) = 0, \tag{64}$$

which representation is notoriously impossible for NSA Newton's equations, with the consequential lack of achievement of consistent operator forms of nonconservative forces.

In view of the universality of Eqs. (56), the above isovariational isoprinciple is *directly universal,* that is, capable of representing all infinitely possible, regular, time reversal invariant Newtonian systems (54) ("universality") directly in the coordinates of the experimenter ("direct universality").

4.5. Iso-Hamilton-Jacobi isoequations.

Much along conventional analytic procedures, it is easy to prove that isovariational isoprinciple (64) implies the following *iso-Hamilton-Jacobi isoequations* also called *Hamilton-Jacobi-Santilli isoequations* [115] [25] that are at the foundation of the *isoquantization* reviewed in the next section

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{t}} + \hat{H} = 0, \tag{65}$$

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}_{ak}} - \hat{p}_{ak} = 0, \tag{66}$$

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{p}_{ak}} = 0. \tag{67}$$

We should recall from Section 2.2.5 that isodynamical isoequations of classical isomechanics require two different isofields, the first being the *isotime isofield* with isounits $\hat{I}_t$ and the second being the *isovelocity isofield* with isounits $\hat{I}_v$.

However, the direct universality is already achieved with the sole use of the isovelocity isounit. Hence, the isotime isounit can be assumed to be 1 without any loss of direct universality.

For non-relativistic formulations, we shall use isotime in the isodynamical equations for completeness, with the tacit understanding that, unless otherwise specified, isotime will be assumed to be equal to the conventional time.

As it will soon be evident, the Hamilton-Jacobi-Santilli isoequations (65)-(67) are truly fundamental for the construction of operator isomechanics, as well as for the proof of the EPR argument because said equations have permitted:

1) The achievement from Eqs. (65) of a unique and unambiguous map of classical into operator isomechanics;

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2) The achievement from Eqs. (66) of the first known operator form of the isolinear isomomentum;

3) The achievement, from Eqs. (67) of an operator isomechanics whose *isowave Isofunctions* solely depend on local isocoordinates $\hat{\psi}(\hat{r})$. This feature is necessary for a consistent isotopic "completion" of quantum mechanics since conventional wave functions $\psi(t, r)$ do not depend on linear momenta p. Hence, any "completion" of quantum mechanics whose wavefunctions also depend on linear momenta would not be an axiom-preserving map.

It should be noted that, by comparison, the use of the Birkhoffian mechanics would imply broader Hamilton-Jacobi-Santilli isoequations (see page 205 of Ref. [19]) with 'wave functions' depending also on isomomenta, $\hat{\psi}(\hat{t}, \hat{r}, \hat{p})$, resulting in an operator mechanics beyond our current knowledge for quantitative treatments.

4.6. Naive isoquantization.

As it is well known, the conventional "naive quantization" of Hamiltonian mechanics into quantum mechanics is based on the following map generated by the conventional Hamilton-Jacobi equations

$$\begin{aligned} A = \int (p_k dx^k - H dt) \rightarrow \\ \rightarrow -i\hbar log\psi(t, r), \end{aligned} \tag{68}$$

that identifies Planck's constant $\hbar = 1$ as the fundamental unit of the theory.

The isotopic lifting of the naive quantization, called *naive isoquantization* (first identified by A. E. O. Animalu and R. M. Santilli in Ref. [121]), characterizes the following map of (classical) iso-Hamiltonian mechanics into (operator) isomechanics via Hamilton-Jacobi-Santilli isoequation (65) (where the sum over the indices ak is omitted for simplicity)

$$\begin{aligned} \hat{A} = \hat{\int} (\hat{p} \star \hat{d}\hat{r} - \hat{H} \star \hat{d}\hat{t}) \rightarrow \\ \rightarrow -i\hat{I} Log \hat{\psi}, \end{aligned} \tag{69}$$

with the following fundamental identification of the *isolinear isomomentum* from Eq. (66)

$$\hat{p} \star \psi = -\hat{i} \star \hat{\partial}_r \hat{\psi}, \tag{70}$$

and the equally fundamental, independence of the isowavefunction from the isolinear isomomentum from Eq. (67)

$$\hat{\partial}_{\hat{p}} \hat{\psi} = 0, \tag{71}$$

where we use the notion of *isolog* $\hat{log}\hat{\psi} = \hat{I} \log \psi$ (see [55]).

The above naive isoquantization identifies the central assumption of isomechanics, namely, *the map of Planck's constant* $\hbar$ *into an integro-differential operator, the isounit* $\hat{I}$,

$$\hbar \rightarrow \hat{I}(t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) > 0. \quad (72)$$

that, from Section 2.2.8, can be achieved via a non-unitary transform of Plank's constant selected in such a way to represent the desired systems, e.g., as in model (16),

$$\hbar = 1 \quad \rightarrow \hat{\hbar} = U\hbar U^{\dagger} = UU^{\dagger} = \hat{I}. \quad (73)$$

The above transform is then restricted by the subsidiary condition that isomechanics must recover quantum mechanics at mutual distances of particles bigger than their size d

$$Lim_{r>d/2}\hat{I} = 1. \quad (74)$$

Therefore, the studies herein reported assume that isomechanics is solely valid within the volume occupied by hadrons, nuclei or stars, while quantum mechanics is assumed to be exactly valid everywhere else (Figure 13).

Note that the structure of the isotopic element (16) permits a smooth transition from isomechanics to quantum mechanics.

The above condition means that, for the case of the structure of a hadron, isomechanics is solely valid within a sphere with diameter $d \approx 1fm = 10^{-15}\ cn$. For the case of the structure of the deuteron, isomechanics is solely valid within a volume with diameter $d \approx 2.50\ fm$; and the same applies for nuclei, stars and black holes.

The main objective of the "completion" of Planck's constant $\hbar$ into the integro-differential isounit $\hat{I}$ is to represent the expected, generalized, energy exchanges of particles in interior dynamical conditions (as expected for an electron in the core of a star), which exchanges cannot be the same as those occurring when particles moves in vacuum due to the surrounding pressures and other factors (see paper II for details).

4.7. Iso-Hilbert isospaces.

Another basic notion of isomechanics is its formulation on the *iso-Hilbert isospaces* $\hat{\mathcal{H}}$, also called the *Hilbert-Myung-Santilli isospace* (HMS isospace) because first introduced by H.C. Myung and R.M. Santilli in Ref. [122] of 1982 over a conventional field of complex numbers $\mathcal{C}$ and then formulated on an isocomplex isofield $\hat{\mathcal{C}}$ in Ref. [115].

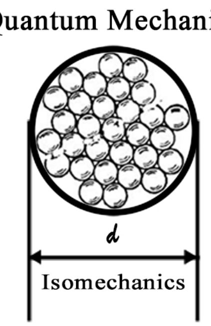

Figure 13: *An illustration of the central assumption according to which isomechanics is solely valid within the volume occupied by hadrons, nuclei, stars or black holes, while quantum mechanics is valid everywhere else thanks to the rapid convergence of isotopic elements, such as Eq. (16), to the unit value* 1.

HMS isospaces are characterized by (see Ref. [55] for details): *isostates* $\hat{\psi}$, with *isonormalization*

$$< \hat{\psi}| \star |\hat{\psi} >= \hat{I}, \tag{75}$$

isoexpectation isovalues of an iso-Hermitean operator $\hat{A}$,

$$\hat{<}\hat{A}\hat{>} =< \hat{\psi}| \star \hat{A} \star |\hat{\psi} >, \tag{76}$$

and basic isoidentity

$$\hat{<}\hat{I}\hat{>} = \hat{I}, \tag{77}$$

where the "hat" denotes definition on isospace over isofields.

It should be recalled from Ref. [122] that the condition of iso-Hermiticity coincides with that of Hermiticity. Therefore, *all quantities that are observable in quantum mechanics remain observable in isomechanics* (see monograph [56] for details).

4.8. Iso-Schrödinger isorepresentation.

Recall that the Schrödinger representation is crucially dependent on the realization of the linear momentum in term of the differential calculus,

$$p\psi(t, r) = -i\hbar\partial_r\psi(t, r). \tag{78}$$

Consequently, the "completion" of quantum mechanics mandated the search for the "completion" of the differential calculus [115] to achieve a consistent formulation of the linear momentum such as that of Eq. (70), that we rewrite in the more detailed form on the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ over the isofield of isocomplex isonumbers $\imath c\hat{al}C$ (with $\hbar = 1$)

$$\hat{p}_k \star |\hat{\psi}(\hat{t}, \hat{r}) >= -\hat{i} \star \hat{\partial}_{\hat{r}^k} |\hat{\psi}(\hat{t}, \hat{r}) >= \\ = -i\hat{I}\partial_{\hat{r}^k} |\hat{\psi}(\hat{t}, \hat{r}) >, \tag{79}$$

where: $\hat{r} = r\hat{I}$ are the isocoordinates on an iso-Euclidean isospace over an isofield, from which one can derive the *iso-Schrödinger isoequation,* also called *Schrödinger-Santilli isoequation* [17] [115]

$$\hat{i} \star \hat{\partial}_{\hat{t}} |\hat{\psi}(\hat{t}, \hat{r}) >= \hat{H} \star |\hat{\psi}(\hat{t}, \hat{r}) >= \\ = \hat{H}(r,p)\hat{T}(t, r, p, \psi, \partial\psi, ...)|\hat{\psi}(\hat{t}, \hat{r} >) = \\ = \hat{E} \star |\hat{\psi}(\hat{t}, \hat{r}) >= E|\hat{\psi}(\hat{t}, \hat{r}) >, \tag{80}$$

where $\hat{E} = E\hat{I}$ is an isoeigenvalue defined on the isoreal isofield $\hat{\mathcal{R}}$, and E is an ordinary eigenvalue defined on the field of real numbers.

The iso-Schrödinger isorepresentation is completed by the *isocanonical isocommutation rules,* solely definable thanks to the isodifferential realization (79) of the isolinear isomomentum [115]

$$[\hat{r}_i \hat{,} \hat{p}_j]|\hat{\psi} >= \hat{i} \star \hat{\delta}_{i.j}|\hat{\psi} >= i\delta_{i.j}|\hat{\psi} >, [\hat{r}_i \hat{,} \hat{r}_j]|\hat{\psi} >= [\hat{p}_i \hat{,} \hat{p}_j]|\hat{\psi} >= 0. \tag{81}$$

Note that the characterization of extended particles at short mutual distances requires the knowledge of *two isoobservables,* the conventional Hamiltonian H for the representation of SA interactions and the isotopic element $\hat{T}$ for the representation of dimension, shape, density and NSA interactions.

On more technical grounds, Eqs. (80) are referred to as *regular iso-Schrödinger equations* to emphasize, in the sense of Theorem 3.7.2, the fact that they can be derived from the conventional Schrödinger equation via non-unitary transformations. The broader *irregular iso-Schrödinger equation* which cannot be derived via non-unitary transformations due to the addition of strong interactions, are studied in Paper II, Section 4.3., Eqs. (89).

4.9. Iso-Heisenberg isorepresentation.

Non-relativistic isomechanics is additionally based on the *iso-Heisenberg isoequations,* also called *Heisenberg-Santilli isoequations* (first formulated in

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Eq. (4.15.49), page 752 of the 1978 paper [17] over conventional fields and formulated via the full use of isomathematics in the 1996 memoir [115]), here written for the iso infinitesimal isotime evolution of an iso-Hermitean operator $\hat{Q}$

$$\begin{gathered}\hat{i}\star\frac{\hat{d}\hat{Q}}{\hat{d}\hat{t}}=[\hat{Q},\hat{H}]=\hat{Q}\star\hat{H}-\hat{H}\star\hat{Q}=\\=\hat{Q}\hat{T}(\psi,...)\hat{H}(r,p)-\hat{H}(r,p)\hat{T}(\psi,...)\hat{Q},\end{gathered}\tag{82}$$

and then in their isoexponentiated form

$$\begin{gathered}\hat{Q}(\hat{t})=\hat{e}^{\hat{H}\star\hat{t}\star\hat{i}}\star\hat{Q}(0)\star\hat{e}^{-\hat{i}\star\hat{t}\star\hat{H}}=\\=e^{\hat{H}\hat{T}ti}Q(0)e^{-it\hat{T}\hat{H}},\end{gathered}\tag{83}$$

where we have used isoexponentiation (22).

Note the characterization of isoinfinitesimal isoequations (82) via the Lie-Santilli isoalgebras (Theorem 3.7.2.) and the characterization of their finite form (83) via the Lie-Santilli isogroups (Theorem 3.7.3).

Note also that the Lie-isotopic equations (82) (83) are a particular case of the broader Lie-admissible equations (11) (12), respectively.

4.10. Iso-Klein-Gordon isoequation.

The relativistic isoequations of hadronic mechanics are characterized by the iso-Casimir isoinvariants of the basic symmetry of the iso-Minkowski isospace-time, the *Lorentz-Poincaré-Santilli isosymmetry* studied in paper II.

At this stage of our analysis, we merely consider the following isotopic "completion" of the second order invariant of the *Lorentz-Poincaré symmetry* formulated on iso-Minkowskian isospace $\hat{M}(\hat{x},\hat{\Omega},\hat{I}$ over the isoreals $\hat{\mathcal{R}}$ (Section 2.5)

$$\hat{p}^{\hat{2}}=\hat{p}_\mu\star\hat{p}^\mu=(\hat{M}\star\hat{C})^{\hat{2}}=(mC)^2\hat{I},\tag{84}$$

where

$$\hat{M}=m\hat{I},\tag{85}$$

is the *isomass,* and

$$\hat{C}=C\hat{I}=\frac{c}{n_4}\hat{I},\tag{86}$$

is the *light isospeed* from isoinvariant (26).

By using the isolinear isomomentum (79) isoinvariant (84) characterizes the second order isoequation of isomechanics known as *iso-Klein-Gordon* *isoequation* [123]

$$\hat{p}_\mu \star \hat{p}^\mu|\hat{\psi}(\hat{x}) >= \hat{\Omega}^{\mu\nu} \star \hat{p}_\mu \star \hat{p}_\nu|\hat{\psi}(\hat{x}) >=$$
$$= \hat{\eta}^{\mu\nu}(-i\hat{I}\partial_\mu)\hat{T}(-i\hat{I}\partial_\nu)|\hat{\psi}(\hat{x}) >= \tag{87}$$
$$= -\hat{I}\hat{\eta}^{\mu\nu}\partial_\mu\partial_\nu|\hat{\psi}(\hat{x}) >= \hat{I}(mC)^2|\hat{\psi}(\hat{x}) >,$$

also called *Klein-Gordon-Santilli isoequation,* first introduced with the isodifferential isocalculus in Chapter 9 of Refs. [56] and in papers [123] [124], (see also review [25]).

4.11. Iso-Dirac isoequation.

The first-order relativistic isoequation of hadronic mechanics is given by the isolinearization [124] of isoinvariant (84) and it is called the *iso-Dirac isoequation,* or *Dirac-Santilli isoequation* (see Refs. [115] [56] [123] [124])

$$[\hat{\Omega}^{\mu\nu} \star \hat{\Gamma}_\mu \star \hat{\partial}_\nu + \hat{M} \star \hat{C}]|\hat{\psi}(\hat{x}) >=$$
$$= (-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_\mu\partial_\nu + mC)|\hat{\psi}(\hat{x}) >= 0, \tag{88}$$

where the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\hat{\gamma}_k = \frac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix},$$
$$\hat{\gamma}_4 = \frac{i}{n_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{89}$$

where $\hat{\sigma}_k$ are the *regular iso-Pauli isomatrices* studied in Section 3.3 of paper II, with *anti-isocommutation rules*

$$\{\hat{\gamma}_\mu\hat{,}\hat{\gamma}_\nu\} = \hat{\gamma}_\mu\hat{T}\hat{\gamma}_\nu + \hat{\gamma}_\nu\hat{T}\hat{\gamma}_\mu =$$
$$= 2\hat{\eta}_{\mu\nu}. \tag{90}$$

One should note that the anti isoanticommutators of the Dirac-Santilli isogamma isomatrices yield the isometric $\hat{\eta}_{\mu\nu}$ of the iso-Minkowski isospace-time (Section 2.5).

Recall that the iso-Minkowski isospace-time includes as particular cases all possible, non-singular, symmetric space-times, thus including the Riemannian space-time reformulated with isomathematics. In fact, the iso-Minkowskian isometric $\eta^{\mu\nu}$ admits as a particular case the Schwartzchild metric via the following simple realization of the isotopic element

$$\hat{T}_{kk} = 1/(1 - 2M/r),$$
$$\hat{T}_{44} = 1 - 2M/r. \tag{91}$$

Additionally, the iso-Minkowskian isometric admits the following combination of the Schwartzchild metric for exterior gravitational problems and that for interior problems (see, Eqs. (9.5.18) page 448, Ref. [56]

$$\hat{T}_{kk} = 1/(1 - 2M/r)n_k^2,$$
$$\hat{T}_{44} = (1 - 2M/r)/n_4^2. \tag{92}$$

Consequently, the Dirac-Santilli isoequation with realization (91) of the isotopic element permits the study of an electron in an *exterior gravitational field,* while realization (92) permits the study of *electrons in interior gravitational fields.*

Relativistic isoequations are far from being mere academic curiosities because, as we shall see in paper II, they have provided the first and only known relativistic representation of *all* characteristics of the neutron in its synthesis from a proton and an electron [124], *none* of which characteristics are representable via quantum mechanics. Additional advances permitted by relativistic isomechanics will be indicated in paper II.

More technically, Eq. (88) is referred to as *regular iso-Dirac equations* to emphasize, in the sense of Theorem 3.7.2, the fact that they can be derived from the conventional Dirac equation via non-unitary transformations. The broader *irregular iso-Dirac equation* which cannot be derived via non-unitary transformations due to the addition of strong interactions, are studied in Paper II, Section 4.4., Eqs. (97).

4.12. Representation of non-linear interactions.

An important insufficiency of quantum mechanics is the inability to characterize individual constituents under non-linear internal forces in view of the inapplicability of the superposition principle.

In fact, the sole possible, quantum mechanical representation of non-linear interactions is that via the Hamiltonian

$$H(r, p, \psi, ...)|\psi(t, r) >= E|\psi(t, r) >, \tag{93}$$

under which the total state $|\psi(t, r) >$ does not admit a consistent decomposition into the individual states.

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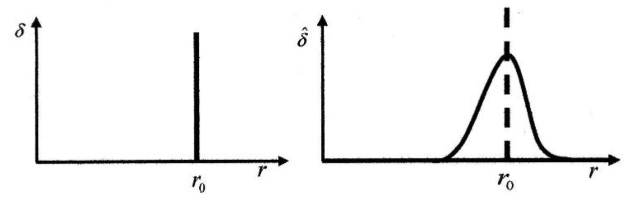

Figure 14: *An illustration on the left of the divergence of the Dirac delta distribution at its origin caused by the point-like approximation of particles, with ensuing divergencies in quantum mechanics. An illustration on the right of the removal of said divergencies by the Dirac-Myung-Santilli isodelta isofunction thanks to the representation of particles as extended, with ensuing lack of divergencies in isomechanics.*

It is easy to see that this insufficiency is resolved by isomechanics thanks to the embedding of all non-linear forces in the basic invariant of the theory, the isounit (or isotopic element).

In fact, the Schrödinger-Santilli isoequation (80) can be explicitly written

$$\hat{H} \star |\hat{\psi} >= \hat{H}(\hat{r}, \hat{p})\hat{T}(\hat{\psi}, ...)|\hat{\psi} >= E|\hat{\psi} >, \tag{94}$$

and its total isostate verifies the factorization

$$\hat{\psi} = \Pi_k \hat{\psi}_k, \ \ k = 1, 2, ..., N, \tag{95}$$

called *isosuperposition isoprinciple* [56].

It is evident that factorization (94) allows the characterization of individual constituents under non-linear internal interactions, thus permitting new structural models of hadrons, nuclei, stars and black holes.

4.13. Isostrong isoconvergence.

In all applications to date, the basic isotopic element (16) resulted to have a numeric value smaller than one

$$||\hat{T}|| \ \ll \ 1. \tag{96}$$

This feature has the important consequence that *perturbative and other series that are slowly convergent or divergent in quantum mechanics become strongly convergent under their isotopic "completion"* [56].

To illustrate this important feature, consider a divergent quantum mechanical series, such as the canonical series

$$A(w) = A(0) + (AH - HA)/1! + ... =\rightarrow \\ \rightarrow \infty, \ \ w > 1. \tag{97}$$

But the value of the isotopic element is much smaller than the parameter w. Therefore, the isotopic "completion" of the above series

$$A(w) = A(0) + (A\hat{T}H - H\hat{T}A)/1! + ... \rightarrow \\ \rightarrow N < \infty, \tag{98}$$

is strongly convergent.

Specific examples of convergence of isoperturbative isoseries much more rapid than corresponding quantum mechanical series have been provided in Refs. [103] [104] (see Section 1.6 for additional comments).

4.14. Removal of quantum divergencies.

As it is well known, the divergencies of quantum mechanics originate from the singularity existing at the origin of the *Dirac delta distribution* (Figure 14) which divergence originates from the point-like approximation of particles.

Another important feature of isomechanics is that of avoiding these singularities as illustrated by the isotopic image of Dirac's delta "distribution", known as *Dirac-Myung-Santilli isodelta isofunction* first introduced in Ref. [122] (see also Nishioka's studies [125] to [128])

$$\hat{\delta}(\hat{r}) = \hat{\int} \hat{e}^{\hat{k} \star \hat{r}} \star \hat{d}\hat{k} = \int e^{\hat{k}\hat{T}\hat{r}} d\hat{k}, \tag{99}$$

where we have used isoexponentials and isointegrals [56].

As illustrated in Figure 14, the appearance of the isotopic element in the *exponent* of the integrant changes a sharp singularity at the origin $r = 0$ into a bell-shaped function.

In summary, *the singularities of quantum mechanics are ultimately due to the point-like abstraction of particles or equivalently, to the formulation of the differential calculus at isolated points. Whenever particles are represented with their actual extended size, and the differential calculus is extended to formulations over volumes, quantum singularities no longer hold.*

4.15. Isoscattering isotheory.

Another important application of isomechanics is the isotopic "completion" of the conventional, potential, scattering theory into the covering *isoscattering isotheory* studied by R. Mignani [129] to [131], A. K. Aringazin and D. A. Kirukhin [132], A. O. E. Animalu and R. M. Santilli [133] and others (see Chapter 12 of Ref. [56]).

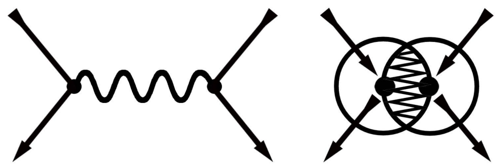

Figure 15: *The left view illustrates the scattering of point-particles, with ensuing realization of interactions via point-particle exchanges, that have been experimentally established for electromagnetic interactions. The right view illustrates the scattering of extended hadrons at high energy implying the presence of contact, non-linear, non-local and non-potential interactions in the scattering region. These conditions imply the impossibility of conventional particle exchanges evidently because of the extremely big density of the scattering regions that may approach the density of black holes, thus requiring broader scattering theories.*

Regrettably, we cannot review these studies to avoid a prohibitive length. We limit ourselves to recall that the potential scattering theory was originally conceived by Feynman and others for the electromagnetic interactions of *point-like* particles in vacuum.

Its dominant notion is the characterization of interactions via the *exchange of point-like particles.*

The historical experimental verifications of the potential scattering theory under the indicated conditions triggered its use for the scattering of *extended* hadrons all the way to their recent very high energies.

The studies herein reported on the covering isoscattering isotheory have indicated that the consistent representation of hadrons as extended, therefore deformable and hyperdense, implies necessary revisions of hadron physics beginning with a "completion" of its mathematical foundations, then passing to the "completion" of quantum laws. These advances then require, for consistency, the "completion" of the potential scattering theory into a covering theory in which the hyperdense character of the scattering region implies the presence of non-linear, non-local and non-potential effects preventing any consistent representation of interactions via the sole exchanges of extended hadrons in favor of covering vistas (Figure 15).

It should be stressed that the isoscattering isotheory has a Lie-Santilli algebraic structure, thus being solely applicable to time reversal invariant collisions generally given by elastic scattering. The study of inelastic scattering of extended hadrons at high energy requires the broader *genoscattering genotheory* with the covering Lie-admissible algebraic structure, which broader theory cannot be considered here for brevity (see Refs. [56] [133]).

To avoid major misconceptions, it should be indicated that *the isoscattering isotheory cannot change the numeric values of scattering angles, cross sections and other experimentally measured quantities.* However, the isoscattering isotheory does require serious revisions of the *theoretical interpretation* of measured quantities.

In closing, we should recall that rather vast studies have been conducted on *non-unitary scattering theories* (see, e.g., Ref. [134] and papers quoted therein).

As it is well known, these studies had to be abandoned because nonunitary theories violate causality laws. By contrast, *the isoscattering isotheory is isounitary on the iso-Hilbert isospace over an isofield,* thus restoring causality (Section 2.9).

This occurrence illustrates again the importance of isomathematics for the verification of the EPR argument and related applications.

4.16. Geno- and Iso-chemistry.

The Lie-admissible mathematical and physical methods of Section 2 have allowed the "completion" of quantum chemistry into *Lie-admissible hadronic chemistry,* also known as *genochemistry* [60], which is the first known chemical formulation specifically built for the consistent treatment of chemical reactions at large and energy releasing chemical processes in particular.

As it is well known but generally ignored, chemical reactions are generally *irreversible over time,* while quantum chemistry is strictly reversible. Hence, the EPR argument on the "lack of completion of quantum mechanics" does indeed apply to quantum chemistry.

In addition to a "completion" for chemical reactions, quantum chemistry needs an additional "completion," this time, for the achievement of an *attractive* force between the *identical* electrons of valence couplings in molecular structures (Section 1.6).

Since isolated molecules existing in nature are stable, thus being *reversible— over time,* and so are their valence electron bonds, there was the need of building the *Lie-isotopic particularization of the Lie-admissible hadronic chemistry* which became known as *isochemistry* because based on the isomathematics of Section 3.

Isochemistry did indeed achieve the first known attractive force between identical electrons in valence couplings [60] in a form permitting the exact representation of experimental data on the hydrogen [103] and the water [104] molecules.

As reviewed in details in Paper II, these studies essentially established that *the contact, non-potential interactions occurring in deep mutual penetration/entanglement of the wavepackets of particles (Figure 2) are strongly attrac-*

tive, thus being responsible for the neutron and other hadron syntheses (Section 1.4), as well as for the attractive force between identical electrons in valence pairs (Section 1.6), thus illustrating the truly fundamental character of short range, contact, non-potential interactions in the ultimate structure of nature.

5. CONCLUDING REMARKS.

In 1935, A. Einstein, B. Podolsky and N. Rosen presented the view that *quantum mechanics is not a complete theory* [1].

Following decades of preparatory works, R. M. Santilli published in 1998 paper [7] that:

1) Confirmed the validity of the objections against the EPR argument by Bohr [2], Bell [3], von Neumann [4] and others for point-particles in vacuum under linear, local and potential/Hamiltoian interactions *exterior dynamical problems;*

2) Proved the inapplicability (rather than the violation) of said objections for extended, thus deformable particles within hyperdense physical media with ensuing linear and non-linear, local and non-local and potential as well as non-potential/non-Hamiltonian interactions expected in the structure of hadrons, nuclei, stars and black holes (*interior dynamical systems*);

3) Proved the existence of *hidden variables* [5] for interior dynamical systems when represented via the isotopic branch of hadronic mathematics and mechanics;

4) Achieved a consequential exact representation of nuclear magnetic moments; and

5) Showed the apparent existence of identical classical counterparts for extended particles in interior dynamical conditions.

More recently, Santilli completed the above proof in paper [8] by showing that the standard deviations for coordinates Δr and momenta Δp appear to progressively tend to zero for extended particles within hadrons, nuclei and stars, and appear to be identically null for extended particles at the limit conditions in the interior of gravitational collapse, essentially along Einstein's vision.

In this paper, we have reviewed and upgraded the mathematical, physical and chemical methods used for proofs [7] [8], with particular reference to the following aspects:

1) Review and upgrade the Lie-admissible and Lie-isotopic "completions" of 20th century applied mathematics for the representation of time irreversible and reversible interior systems, respectively,, which "completions" were initiated by Santilli in 1978 [16] when at Harvard University

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with DOE support under the names of *geno- and iso-mathematics,* respectively, and waere subsequently studied by various authors [55];

2) Review and upgrade the "completions" of quantum mechanics and chemistry into the Lie-admissible and Lie-isotopic branches of hadronic mechanics and chemistry that were also initiated by Santilli in 1978 [17] under the names of *genotopic and isotopic branches of hadronic mechanics and chemistry,* which "completions" were then studied by various authors [56];

3) Review and upgrade the main aspect of the studies herein considered, namely, the "completions" of the Newton-Leibnitz differential calculus into forms applicable to irreversible and reversible interior systems of extended particles, which "completions" were initiated by Santilli in the 1996 paper [115] under the name of *geno- and iso-differential isocalculus* and studied in details by S. Georgiev [83] and other mathematicians [55].

In a second paper of this series, we review and upgrade *geno- and iso-symmetries* for interior systems and then review the apparent proofs [7] [8] of the EPR argument.

In the third paper of this series, we study specific cases of interior dynamical systems in particle physics, nuclear physics and chemistry that progressively approach classical determinism, and present applications to new clean energies that are prohibited by quantum mechanics, yet fully admitted by its "completion."

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ous other mathematicians.

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Studies on A. Einstein, B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," II: Apparent confirmation of the EPR argument

Ruggero Maria Santilli*

Abstract

In 1935, A. Einstein expressed his historical view, jointly with B. Podolsky and N. Rosen, that quantum mechanics could be "completed" into a form recovering classical determinism at least under limit conditions (*EPR argument*). In the preceding Paper I, we have outlined the novel methods underlying the "completion" of quantum mechanics into hadronic mechanics for the representation of extended, thus deformable particles within physical media. In this Paper II, we study the *isosymmetries* for interior dynamical systems; we confirm the 1998 apparent proof that interior dynamical systems admit a classical counterpart; we confirm the 2019 apparent proof that Einstein's determinism is progressively approached for extended particles in the interior of hadrons, nuclei and stars, while being fully verified in the interior of gravitational collapse; and we show for the first time that the recovering of Einstein's determinism in interior systems implies the apparent removal of quantum mechanical divergencies. In the subsequent Paper III, we present a number of illustrative examples and novel applications in mathematics, physics and chemistry.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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1. INTRODUCTION

1.1. The EPR argument.

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe."

More particularly, Einstein believed that "quantum mechanics is not a complete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions.

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

In view of the rather widespread belief that quantum mechanics is a final theory valid for all conceivable conditions existing in the universe, objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3] [4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature). The field became known as *local realism* and included the dismissal of the EPR argument based on claims that quantum axioms do not admit *hidden variables* λ [7] [8].

1.2. Outline of Paper I.

This paper, and the preceding Ref. [9] (hereinafter referred to as Paper I), are dedicated to the review and upgrade of decades of studies by mathematicians, physicists, and chemists (see Refs. [10] to [71] and papers quoted therein) on the apparent proof of the EPR argument via the "completion," also called *isotopic lifting,* of quantum mechanics into the axiom-preserving *hadronic mechanics* (see the 1995 monographs [29] [30] [31] and literature quoted therein).

More specifically, in Section I-1.1, we have outlined the EPR argument [1] jointly with representative objections [2] to [6].

In Section I-1.2, we have outlined the apparent proof by R. M. Santilli [10] (See also the detailed study in monograph [30], particularly Chapter 4 and Appendix 4C, page 166) that interior dynamical systems represented

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Figure 1: *In this figure, we present a conceptual rendering of the tacit assumption underlying the objections against the EPR argument [2] - [6], namely, the representation of particles as being point-like because mandated by the Newton-Leibnitz differential calculus underlying quantum mechanics, namely, the representation of particles as isolated points in empty space. A first consequence is that, being dimensionless, particles can only be at a distance, with ensuing Einstein's argument on the need for superluminal interactions to explain quantum entanglement [1]. A second consequence is that, being at a distance, the sole possible interactions are of linear, local and potential type, under which assumptions the objections against the EPR argument are indeed valid.*

with hadronic mathematics and mechanics admit classical counterparts.

In the same Section I-1.2, we have outlined the apparent second proof by Santilli [11] that classical determinism is progressively approached in the interior of hadrons, nuclei, stars and gravitational collapse as predicted by Einstein.

In support of the plausibility of the EPR argument, in the subsequent Sections I-1.3 to I-1.7, we have outlined insufficiencies of quantum mechanics for time-irreversible processes, particle physics, nuclear physics, chemistry, and other fields. We have also provided various references indicating the apparent resolution of said insufficiencies by hadronic mechanics.

In Section I-2, we have outlined the *Lie-admissible covering of Lie's theory* [12] [13], with ensuing time-irreversible *Lie-admissible brach of hadronic mechanics,* also known as *genomechanics,* [12] [14] allowing studies on the compatibility of mechanics with thermodynamics, said compatibility being notoriously impossible for quantum mechanics.

Quantum mechanics and the objections against the EPR argument are formulated for time-reversal invariant systems of exterior dynamical systems. Therefore, in preparation for the proof of the EPR argument studied in Section 3, we have outlined and upgraded in Section I-3 the time-reversal invariant *Lie-isotopic subclass of Lie-admissible mathematics,* also known as *isomathematics,* [15] [18] which is used for the representation of time-reversible invariant interior dynamical systems.

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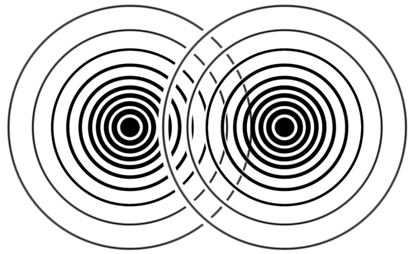

Figure 2: *A conceptual rendering of the main assumption of the apparent proofs [10] [11] of the EPR argument [1], is the representation of particles as extended, deformable and hyperdense in conditions of mutual overlapping/entanglement with ensuing continuous contact at a distance which eliminates the need for superluminal interactions to explain quantum entanglement. A first implication is the need, for consistency, of generalizing Newton-Leibnitz differential calculus from its historical form solely definable at isolated points, to a covering form definable in volumes [21]. Another implication is the emergence of contact, non-linear, non-local and non-potential interactions that, being not representable by Hamiltonians are Lagrangians, require a structural lifting of the Lie algebra of quantum mechanics under which the objections against the EPR argument are inapplicable (Section 3). Intriguingly, the "completions" here considered turned out to be of isotopic/axiom-preserving type, thus being fully admitted by quantum mechanical axioms, merely subjected to a realization broader than that of the Copenhagen school. The apparent proofs of the EPR argument [10] [11] become an unavoidable consequence of the indicated "completions" (Section 3).*

In the same Section I-3, we have devoted particular attention to the "completion" of conventional Hilbert spaces [19], numeric fields [20] and Newton-Leibnitz differential calculus [21] into forms defined on *volumes,* rather than points.

In the same Section I-1.3, we have provided particular attention to the main methods for the proofs of the EPR argument, namely, the *axiom-preserving, isotopic lifting of Lie's theory* [26], today known as the *Lie-Santilli isotheory* [38].

Finally, in Section I-4, we have outlined and upgraded the time-reversal invariant *isotopic branch of hadronic mechanics,* also known as *isomechanics* [30] which provides the dynamical foundations of the proofs of the EPR argument [10] [11].

1.3. Basic assumptions.

The most dominant aspects underlying the studies here considered are:

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1) The validity of quantum mechanics for point-like particles in vacuum with ensuing linear, local and action-at-a-distance/potential interactions (*exterior dynamical problems*) occurring in atomic structures, particles in accelerators, crystals and numerous other systems in nature (Figure 1);

2) The "completion" of quantum mechanics into hadronic mechanics for the representation of extended, therefore deformable and hyperdense particles within physical media with ensuing, additional, non-linear, non-local and contact/non-potential interactions (*interior dynamical problems*), occurring in the structure of hadrons, nuclei and stars, with limit conditions occurring in the interior of gravitational collapse where the inapplicability (rather than the violation) of quantum mechanics is already accepted by the majority of serious scholars (Figure 2, 3).

The central assumption of these studies is the axiom-preserving lifting of the conventional associative product $ab = a \times b$ between *all* possible quantum mechanical quantities (numbers, functions, matrices, etc.) into the *isoproduct* [14] [26] (Section 3)

$$a \star b = a\,\hat{T}\,b, \tag{1}$$

where $\hat{T}$, called the *isotopic element,* is restricted to be positive-definite, $\hat{T} > 0$, but possesses otherwise an unrestricted functional dependence on all needed local variables.

Refs. [14] [26] constructed an axiom-preserving isotopy of the various branches of Lie's theory, resulting in a theory today known as the *Lie-Santilli isotheory* [38] (Section I-3.7) with isotopic lifting of lie algebras of the type [10]

$$[X_i \hat{,} X_j] = X_i \star X_j - X_j \star X_i = C_{ij}^k X_k. \;\; i,j = 1,2,...,N. \tag{2}$$

Following laborious efforts for the achievement of mathematical maturity, Ref. [10] applied the Lie-Santilli isotheory to the isotopy $\hat{SU}(2)$ of the $SU(2)$ spin with three-dimensional isoalgebras of type (2) and introduced the realization of hidden variables [7] [8] of the type

$$\hat{T} = Diag.(1/\lambda, \lambda), \;\; Det\hat{T} = 1. \tag{3}$$

Ref. [10], therefore establishing that, contrary to objections [2] to [6], *the abstract axioms of quantum mechanics do indeed admit explicit and concrete realizations of hidden variables.*

The proof in Ref. [10] that interior systems admit identical classical counterparts was consequential (Section 3).

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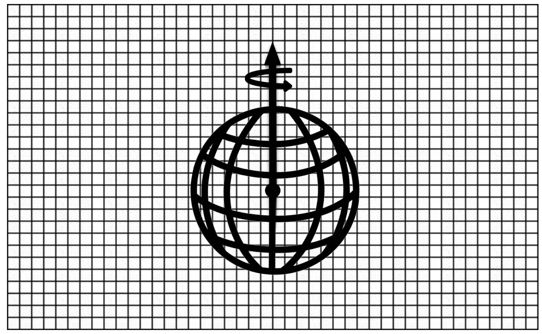

Figure 3: *A conceptual rendering of the central notion used for the study of the EPR argument, namely, a mathematically consistent representation invariant over time of extended, deformable and hyperdense particles in interior conditions, such as protons in the interior of a star, thus being under the most general known (non-singular) non-linear, non- local and non-potential interactions fully representable via the isotopic element of isoproduct (1).*

Isoproduct (1) also allows a direct and immediate representation of extended particles in conditions of mutual penetration with realizations of the type (Figure 3) [33]

$$\hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2}) e^{-\Gamma}, \quad k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4, \tag{4}$$

where n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_k^2 = 1, `k = 1, 2, 3$ for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

The smaller than 1 absolute value of the isotopic element $\hat{T}$ occurring in all known applications [26]-[36]

$$|\hat{T}| \leq 1, \tag{5}$$

permitted Ref. [**?**] to show that *the standard deviations Δr and Δp appear to progressively tend to zero with the increase of the density of the medium, and appear to achieve full classical determinism in the interior of gravitational collapse, as originally conceived by Einstein.*

The initial construction of the isotopies of 20th century applied mathematics with isoproduct (1) defined over conventional numeric fields $F(n, \times,$-$1)$ [26] turned out to be inconsistent because the underlying time evolution is *non-unitary,* thus causing the lack of invariance over time of the traditional basic unit 1, with ensuing inapplicability over time of the entire field $F(n, \times, 1)$.

The above occurrence mandated the construction of *isofields* $\hat{F}(\hat{n}, \star, \hat{I})$ [20] [41](Section I-3.3) with basic *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{6}$$

and *isonumbers* $\hat{n} = n\hat{I}$ equipped with isoproduct (1).

Ref. [20] essentially established that *the abstract axioms of a numeric field do not require that the multiplicative unit of the field be the trivial number* 1*, since said unit can be an arbitrary quantity with an unrestricted functional dependence on local variables, provided that said multiplicative unit is positive definite and the field is lifted into a compatible form.*

Despite all the above efforts, the ensuing isomathematics was still inapplicable to the proof of the EPR argument because it lacked the crucial *invariance over time,* namely, the prediction of the same interior dynamical systems under the same conditions but at different times.

The above occurrence forced the construction of the covering of the Newton-Leibnitz differential calculus into the covering *isodifferential isocalculus* [21] [44] (Section I-3.6) with basic *isodifferential* (Figure 2) [**?**]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \tag{7}$$

and corresponding *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}. \tag{8}$$

In essence, Ref. [21] established the inapplicability of the conventional differential calculus whenever the axioms of numeric fields admit multiplicative units with a dependence on the differentiation variable, with ensuing inapplicability of quantum mechanics, as well as of the objections against the EPR argument, for interior dynamical systems.

The "completion" of the differential calculus into an isotopic form compatible with basic isoproduct (1) finally allowed the achievement of invariance over time (Section I-3.9), thus signaling the achievement of maturity for the apparent proof of the EPR argument reviewed.

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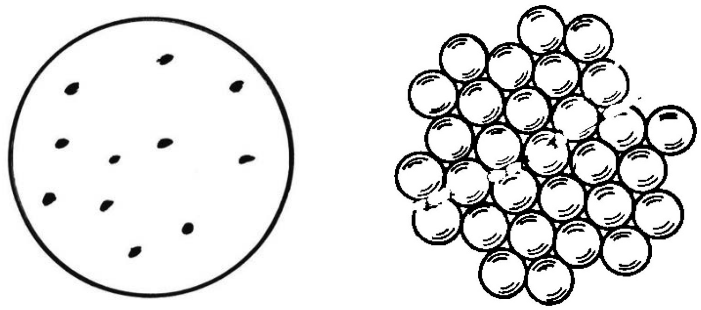

Figure 4: *In the l.h.s. of this picture, we present a conceptual rendering of the structure of nuclei as ideal spheres with isolated point-like particles in their interior. This view is an inevitable consequence of the elaboration of quantum mechanics via the conventional differential calculus, resulting in rather serious insufficiencies in nuclear physics outlined in Section I-1.5. In the r.h.s. of this picture, we present a conceptual rendering of the representation of nuclei as occurring in the physical reality, namely, as a collection of extended, therefore deformable charge distributions in condition of partial mutual penetration according to Eq. (4) of isomathematics and related isomechanics, thus permitting the resolution of at least some of the insufficiencies of quantum mechanics in nuclear physics reviewed in Section I-1.5.*

In Section 2 of this paper, we complete the methodological needs by outlining and upgrading the time-reversal invariant coverings of conventional spacetime symmetries, known as *isosymmetries,* for systems of extended particles in interior conditions; in Section 3, we review and upgrade the Lie-isotopic $SU(2)$-spin symmetry and related proofs [10] [11] of the EPR argument.

A few comments on terminologies appear to be recommendable.

The word "completion" is used in these studies to honor the memory of Albert Einstein and should not be intended to indicate "final" theories. In fact, isomathematics and isomechanics admit coverings of Lie-admissible character [12] (Section I-2) that, in turn, admits coverings of hyperstructural character [43], with additional coverings remaining possible in due time.

The terms "non-Hamiltonian interactions" are intended to indicate interactions that are not representable with a Hamiltonian, and are technically identified as interactions violating the integrability conditions for the existence of a Hamiltonian, namely, the *conditions of variational self-adjointness* [25].

When dealing with stable and isolated interior dynamical systems, the

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terms "non-conservative forces" are strictly referred to *internal* non-Hamiltonian exchanges verifying conditions (1-55) for the verification of the ten conventional total conservation laws for the total energy, momentum, angular momentum and the uniform motion of the center of mass.

The terms "physical media" refer to media composed by matter in its various states, and are often referred to as *hadronic media,* in the sense that the media are *not* composed by empty space, thus requiring the use of hadronic mathematics and mechanics for their quantitative treatment.

The terms "extended particles" refer to: the wavepacket of elementary particles such as the electron assumed to be of about $1\ fm = 10^{-15}\ cm$; extended charge distributions for protons and neutrons when members of a nuclear structure, also assumed to have a diameter of about $1\ fm$; and stable nuclei when considering the structure of stars. Due to its crucial significance for the structure of interior systems, a technical definition of the notion of "extended particles" will be given in Section 3 via the notion of *isoparticle* as isorepresentations of space-time isosymmetries.

2. ISOSYMMETRIES

2.1. Foreword.

In this section, we study the axiom-preserving "completion" (or isotopic lifting) of conventional space-time symmetries, known as *Lie-isotopic symmetries,* or *isosymmetries* for short, which provide the invariance of stable and isolated (thus time reversible) interior dynamical systems of extended particles at mutual distances smaller than their size as occurring, e.g., in nuclear structures (Figure 3).

Lie-isotopic symmetries were first introduced by Santilli in the 1978 Harvard University paper [13] as a particular case of the broader *Lie-admissible symmetries* for irreversible, non-conservative systems [14]. Isosymmetries were then studied in various subsequent works quoted in this section.

The understanding of this section requires a knowledge of the *Lie-Santilli isotheory* (Section I-3.7), which was first formulated in monographs [25] [26] over the field of real numbers. Isosymmetries were then formulated in monographs [29] [30] with the full use of isomathematics, including the use isofields [20] [41] and the isodifferential calculus [21] [44] (see Refs. [38] [46] [47] [48] for works on the Lie-Santilli isotheory, and Ref. [45] for a general review with applications and experimental verifications).

The assumption at the foundation of isosymmetries is *the preservation of the abstract axioms of 20th century space-time symmetries, and the mere construction of their broadest possible realization permitted by isomathematics.*

Consequently, criticisms of isosymmetries and their novel implications

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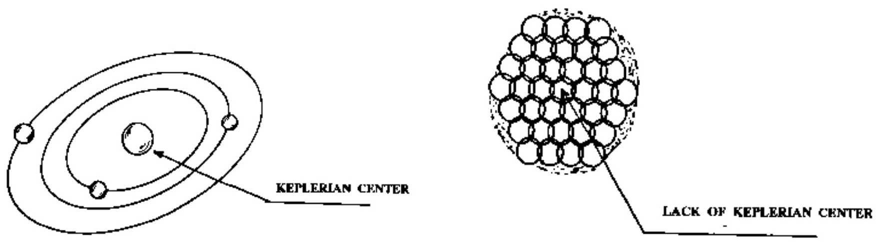

Figure 5: *The l.h.s. of this figure illustrates Keplerian systems for which space-time symmetries have been constructed, namely, exterior dynamical systems of point-like masses orbiting in vacuum around a heavier point-like mass known as the Keplerian center. The r.h.s. of this figure illustrates interior systems for which isosymmetries have been built, namely, systems of extended particles in conditions of mutual penetration without any Keplerian center.*

are *de facto* criticisms on 20th century space-time symmetries and their implications

2.2. Inapplicability of Lie symmetries for interior systems.

A rather widespread view of 20th century physics is the lack of any difference between exterior and interior dynamical systems on grounds that the latter can be reduced to their elementary constituents, by therefore recovering exterior conditions.

The above view was disproved by R. M. Santilli in his Ph. D. thesis (see the review in Ref. [30]) on numerous grounds, the first being the notorious incompatibility of quantum mechanics with thermodynamics whose resolution motivated the Lie-admissible generalization of Lie algebras and related physical theories [29] [30].

The absence of structural differences between exterior and interior systems was dismissed more directly with the following [26] [28]:

NO REDUCTION THEOREM 2.2.1: A classical dynamical system with non-conservative interior forces cannot be consistently reduced to a finite number of isolated particles all in conservative conditions and, vice- versa, the latter system cannot reproduce the former under the correspondence or other principles.

The first direct consequence of the above No Reduction Theorem is the "inapplicability" (rather than the "violation") for interior dynamical systems of conventional space-time symmetries that have been proved to be so effective for exterior dynamical systems.

Said inapplicability was also proved [*loc. cit.*] from the fact that the

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Galileo and the Lorentz-Poincaré symmetries can only provide a non-relativistic and relativistic characterization, respectively, of *Keplerian systems*, namely, systems of point-like masses orbiting in vacuum around a heavier mass called the *Keplerian nucleus* [26].

However, interior dynamical systems do not admit a Keplerian structure because *nuclei have no nuclei* [33] and the same happens for hadrons, stars and gravitational collapse (Figure 5).

It is then possible to prove, e.g., via the imprimitivity theorem, that the lack of existence of a Keplerian structure implies the lack of exact validity of conventional space-time symmetries [26] [30].

On more technical grounds, Lie's theory is known to be solely applicable to exterior systems of point-like particles in vacuum with ensuing sole possible, linear, local and Hamiltonian interactions.

Experimental evidence on interior dynamical systems, e.g., on nuclear volumes compared to the volumes of individual nucleons, establishes that nuclei are composed of extended charge distributions in conditions of partial mutual penetration/entanglement with the ensuing existence of additional, non-linear, non-local and non-Hamiltonian interactions under which Lie's theory is inapplicable.

Hence, the transition of particles from exterior to interior conditions implies the inapplicability of the $SU(2)$-spin symmetry with consequential inapplicability of Bell's inequality [3] and other objections against the EPR argument [6] in favor of suitable covering vistas [10] [11].

In any case, the $SU(2)$ symmetry, while unquestionable effective for exterior dynamical systems, has been unable to provide a consistent representation of the spin of particles and nuclei, thus warranting the search for a suitable "completion."

2.3. The fundamental theorem on isosymmetries.

The construction of isosymmetries requires the full use of isomathematics with particular reference to the Lie-Santilli isotheory formulated on isospaces over isofield and elaborated via the isodifferential calculus (Section I-2.7).

Said construction can be done with the following theorem (for brevity, see the proof in Section 1.2 , Vol. I of Refs. [36]):

THEOREM 2.3.1: Let G be an N-dimensional Lie symmetry of the line element of a k-dimensional metric or pseudo-metric space $S(x, m, I)$ over a numeric field F

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with coordinates x, metric m over a numeric field F with conventional unit I,

$$G: \quad x' = \Lambda(w)x, \quad y' = \Lambda(w)y, \quad x, y \in S,$$
$$(x' - y')^\dagger \Lambda^\dagger m \Lambda (x' - y') \equiv (x - y)^\dagger m (x - y), \tag{9}$$
$$\Lambda^\dagger(w) m \Lambda(w) \equiv m. \;\; w \in F.$$

Then, all infinitely possible (non-singular) Lie-Santilli isotopies $\hat{G}$ of G on isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$ with isocoordinates

$$\hat{x} = xI, \tag{10}$$

isometric

$$\hat{M} = \hat{m}\hat{I} = (\hat{T}_i^k m_{kj})\hat{I}, \tag{11}$$

and isounit

$$\hat{I} = 1/\hat{T} > 0, \tag{12}$$

over an isofield $\hat{F}$ with isounit v$\hat{I}$ leave invariant the isoline element of the isospace $\hat{S}(\hat{x}, \hat{M}, \hat{I})$:

$$\hat{G}: \quad \hat{x}' = \hat{\Lambda}(\hat{w}) \star \hat{x}, \quad \hat{y}' = \hat{\Lambda}(\hat{w}) \star \hat{y}, \quad \hat{x}, \hat{y} \in \hat{S},$$
$$(\hat{x}' - \hat{y}')^\dagger \star \hat{\Lambda}^\dagger \star \hat{M} \star \hat{\Lambda} \star (\hat{x}' - \hat{y}') \equiv (x - y)^\dagger \hat{m} (x - y), \tag{13}$$
$$\hat{\Lambda}^\dagger(\hat{w}) \star \hat{M} \star \hat{\Lambda}(\hat{w}) \equiv \hat{M}.$$

All infinitely possible so constructed isosymmetries $\hat{G}$ are locally isomorphic to the original symmetry G.

The reader should note that, while a given Lie symmetry G is unique as well known, there can be an infinite number of covering isosymmetries $\hat{G}$ with generally different explicit forms o the isotransformations due to the infinite number of possible isotopic elements representing the infinitely different internal interactions of extended particles within physical media.

Note also that all possible isotopic images of a given Lie symmetry can be explicitly and uniquely constructed via the sole knowledge of the original Lie symmetry and of the isotopic element $\hat{T} > 0$, or of the isounit $\hat{I} = 1/\hat{T}$, which property shall be hereon tacitly assumed.

2.4. Isospaces and isogeometries.

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As it is well known, the fundamental representation space of relativistic space-time symmetries is the conventional *Minkowski space* $M(x, \eta, I)$ formulated on the field of real numbers $\mathcal{R}$ with coordinates $x = (x^1, x^2, x^3, x^4 = ct)$, metric $\eta = Diag.(1, 1, 1, -1)$, unit $I = Diag(1, 1, 1, 1)$ and invariant

$$x^2 = (x^\mu \eta_{\mu\nu} x^\nu) I = \\ = (x_1^2 + x_2^2 + x_3^3 - c^2 t^2) I, \tag{14}$$

where the trivial multiplication by the conventional unit $I = Diag.(1, 1, 1, 1)$ is done for compatibility with isomathematics.

The fundamental isospaces of space-time isosymmetries are given by the infinite family of *iso-Minkowski isospaces,* also called *Minkowski-Santilli isospaces,* $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ formulated on the isofield of isoreal isonumbers $\hat{\mathcal{R}}$. (Section I-3.9), which isospaces were first introduced by R. M. Santilli in Ref. [23] of 1983 and then treated in details in works [29] [30].

Iso-Minkowskian isospaces are characterized by *space-time isocoordinates* $\hat{x} = x\hat{I}$; *isounit* $\hat{I} = 1/\hat{T}$, *isometric*

$$\hat{\Gamma} = (\hat{T}_\mu^\rho \eta_{\rho\nu}) \hat{I}, \tag{15}$$

(where one should note the necessary structure of an isomatrix [29]), positive-definite *isotopic element* (4) representing a system of extended particles in interior dynamical conditions with a restricted functional dependence on local quantities such as coordinates x, momenta p, energy E, frequency ν, density α, temperature τ, pressure pi, wavefunction ψ, etc., under the conditions

$$n_\mu = n_\mu(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) > 0, \ \mu = 1, 2, 3, 4, \tag{16}$$

$$\Gamma(x, p, E, \nu, \alpha, \tau, \pi, \psi, \partial\psi, ...) \geq 0, \tag{17}$$

$$\hat{T} = e^{-\Gamma} \ \ll \ 1. \tag{18}$$

Iso-Minkowskian isospaces are characterized by the infinite family of isoinvariants (I-28) with isotopic element (4) that, for the case of one single extended particle can be written

$$\hat{x}^{\hat{2}} = \hat{x}^\mu \star \hat{\Omega}_{\mu\nu} \star \hat{x}^\nu = (x^\mu \hat{\eta}_{\mu\nu} x^\nu) \hat{I} = \\ = (\tfrac{x_1^2}{n_1^2} + \tfrac{x_2^2}{n_2^2} + \tfrac{x_3^2}{n_3^2} - t^2 \tfrac{c^2}{n_4^2}) \hat{I}, \tag{19}$$

where the exponential $exp{-\Gamma}$ has been absorbed in the characteristic quantities n_μ, and the final multiplication by the isounit is necessary for the

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isoinvariant to be an isoscalar, namely, an element of the isoreal isofield [20] (Section I-3-5).

The following aspects treated in Paper I are important for the understanding of the apparent proof of the EPR argument:

1. The characteristic quantities n_1^2, n_2^2, n_3^2, admit the first interpretation as representing the deformable semi-axes of elementary or composite particles normalized to the values for the sphere $n_1^2 = n_2^2, = n_3^2 = 1,$.

2. The characteristic quantity n_4^2 admit the first interpretation as representing the *density* of the hadronic medium normalized to the value $n_4 = 1$ for the vacuum.

3. The function $\Gamma \geq 0$ provides an invariant representation (Section I-3-9) of all non-linear, non-local and non-Hamiltonian interactions.

4. Property (18) is verified for all applications of isosymmetries to date [10] to [68].

5. The correct elaboration of iso-Minkowskian isospaces requires the use of the *isospherical and isohyperbolic isocoordinates* (see Refs. [29] [30]).

6. Isoinvariant (19) provides a unified representation of both exterior and interior gravitational problems. In fact, K. Schwartzchild wrote in 1916 *two* important papers, the first paper [49] on the *exterior gravitational problem* which became world famous for its initiation of gravitational singularities, and the second paper [50] in the *interior gravitational problem* which has been vastly ignored, except rare studies (such as that in Section 23.2, page 609, Ref. [52]). Such an oblivion is essentially due to the fact that Schwartzchild's second paper is not aligned with the widespread tendency of reducing masses to point-like constituents, in which case all differences between exterior and interior gravitational problems disappear to the detriment of the depth of the gravitational analysis. Readers should keep in mind the full parallelism between exterior and interior dynamical problems for *particles and gravitation.*

7. The *exterior gravitational interpretation* of isoinvariant (19) is given by the following identical representation of Schwartzchild's exterior metric [50]

$$\hat{T}_{kk} = \frac{1}{1 - \frac{2M}{r}}, \quad \hat{T}_{44} = 1 - \frac{2M}{r}. \tag{20}$$

The corresponding *interior gravitational representation* is given by the fol-

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lowing isotopy of Schwartzchild exterior metric

$$\hat{T}_{kk} = \frac{1}{(1 - \frac{2M}{r})n_k^2}, \quad \hat{T}_{44} = (1 - \frac{2M}{r})/n_4^2. \tag{21}$$

In view of the arbitrariness of the functional dependence of the characteristic quantities n_μ, it is easy to prove that Schwartzchild's interior metric [50] is a particular case of the much broader class of interior gravitational models (21).

8. The geometry of the iso-Minkowskian isospaces, first presented by Santilli in Ref. [24] under the name of *iso-Minkowskian isogeometry,* contains the machinery of the Riemannian geometry (due to the dependence of the isometric $\hat{\eta}$ on the local coordinates x), although such a machinery is formulated for consistency over isofields [20] and elaborated via the isodifferential isocalculus [21] (Section I-3.5). Hence, *the isominkowskian isogeometry can unify exterior and interior problems for both particles and gravitation.*

9. Recall that iso-Minkowskian isospaces are locally isomorphic to the conventional Minkowski space (Refs. [23] [24] and Theorem 2.3.1). Therefore, *the iso-Minkowskian isogeometry has a null curvature.* This is due to the fact that, under isotopic lifting, the conventional Minkowski metric $\eta = Diag.(1, 1, 1, -1)$ is lifted into a coordinate-dependent isometric $\hat{T}(x)\eta = \hat{\eta}(x)$ which is *identical* to any given Riemannian metric

$$\eta \rightarrow \hat{\eta}(x) = \hat{T}(x)\eta = g(x). \tag{22}$$

Jointly, the original unit of the Minkowski space $\hat{I} = Diag.(1, 1, 1, 1)$ is lifted by the *inverse* amount

$$I > 0 \rightarrow \hat{I}(x) = 1/\hat{T}(x) > 0, \tag{23}$$

resulting in no actual curvature. The above features have suggested the introduction of the new notion of *isoflat isospace,* referred to an isospace that has null curvature when formulated on isofields, while recovering conventional curvature when formulated on conventional fields. Readers should be aware that the achievement of the universal symmetry of (non–singular) Riemannian line elements studied in the next sections are due precisely to the isoflatness of the iso-Minkowski isospace since no such symmetry is possible for a convemtional Riemannian space, as well known.

Recall that the fundamental representation space of symmetries in 3-space dimensions is the conventional Euclidean space $E(r, \times, I$ with coor-

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dinates $r = (x^1, x^2, x^3)$, metric $\delta = Diag.(1, 1, 1)$ and unit $I = Diag.(1, 1, 1)$ on the conventional field of real numbers.

Similarly, the fundamental representation space of isosymmetries in 3-dimensions is the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$, also called *Euclid-Santilli isospace* (Refs. [14] [26] [29] and Section I-3.5) which is the space component of the iso-Minkowskian isospace. As such, the iso-Euclidean isospace is hereon tacitly assumed to be known.

2.5. Lorentz-Poincaré-Santilli isosymmetries.

2.5.1. Main references. Following, and only following the construction of the isotopies of Lie's theory, Santilli conducted systematic studies on the isotopies of the various aspects of the Lorentz-Poincaré symmetry for the achievement of the universal invariance of spacetime isoinvariant (19), including:

1) The classical isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [53];

2) The operator isotopies $\hat{SO}(3.1)$ of the Lorentz symmetry $SO(3.1)$ [54];

3) The isotopies $\hat{SO}(3)$ of the rotational symmetry $SO(3)$ [55] [56] [57];

4) The isotopies $\hat{SU}(2)$ of the $SU(2)$ spin symmetry [10] [58];

5) The isotopies $\hat{P}(3.1)$ of the Poincaré symmetry $P(3.1)$ [59] [60], which included the universal symmetry of (non-singular) Riemannian line elements;

6) The isotopies $\hat{\mathcal{P}}(3.1)$ of the spinorial covering $\mathcal{P}(3.1)$ of the Poincaré symmetry [61] [62];

7) The isotopies $\hat{M}(3.1)$ of the Minkowskian geometry $M(3.1)$ [24].

A general presentation is available in the 1995 monographs [29] [30] with the full use of isomathematics, including isofields and isodifferential calculus, with upgrades in the 2008 monographs [36].

The resulting infinite family of isosymmetries $\hat{SO}(3.1)$ are known as the *Lorentz-Santilli (LS) isosymmetries* while the broader isosymmetries $\hat{P}(3.1)$ and $\hat{\mathcal{P}}(3.1)$ are known as *Lorentz-Poincaré-Santilli isosymmetries* (see Refs. [37] [42] [45] and papers quoted therein).

Experimental verifications of LPS isosymmetries for interior dynamical systems are available in monographs [31] and in Section 3 of the more recent review [63].

In inspecting the subsequent sections, the reader should be aware of the "direct universality" of the LPS isosymmetries for the considered infinite family of interior dynamical systems [64], including the treatment of exterior and interior, particle and gravitational problems (Section 4).

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2.5.2. Basic definitions. As it is well known, the conventional Lorentz-Poincaré (LP) symmetry is the symmetry of line element (14) which we rewrite in the form

$$(x-y)^2 = (x^\mu - y^\mu)\eta_{\mu\nu}(x^\nu - y^\nu)I =$$

$$= \left[(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 - (t_1-t_2)^2c^2)\right] I, \tag{24}$$

$$\eta = Diag.(1,1,1,-c^2),, \quad I = Diag.(1,1,1,1),$$

where the exponential component $exp{-}\Gamma$ is again embedded for simplicity in the characteristic quantities $n\mu^2$.

The LPS isosymmetry is the universal symmetry of the isoline element (19) in the iso-Minkowski isospace $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ over the isoreal isonumbers $\hat{\mathcal{R}}$ rewritten in the form

$$(\hat{x}-\hat{y})^{\hat{2}} = \left[(\hat{x}^\mu - \hat{y}^\mu) \star \hat{\Omega}_{\mu\nu} \star (\hat{x}^\nu - \hat{y}^\nu)\right] =$$

$$= \left[x^\mu - y^\mu)\hat{\eta}_{\mu\nu}(x^\nu - y^\nu)\right] \hat{I} =$$

$$= \left[\frac{(x_1-y_1)^2}{n_1^2} + \frac{(x_2-y_2)^2}{n_2^2} + \frac{(x_3-y_3)^2}{n_3^2} - (t_1-t_2)^2\frac{c^2}{n_4^2}\right] \hat{I}, \tag{25}$$

$$\hat{\eta} = \hat{T}\eta, \quad \hat{T} = Diag((\tfrac{1}{n_1^2}, \tfrac{1}{n_2^2}, \tfrac{1}{n_3^2}, \tfrac{1}{n_4^2}),$$

$$n_\mu = n_\mu(x, v, a, E, d, \omega, \tau, \psi, \partial\psi, ...) > O, \quad \hat{I} = 1/\hat{T} > 0.$$

2.5.3. Isotransformations. By following Theorem 2.3.1, the isotransformations of the LPS isosymmetries can be written

$$\hat{x}' = \hat{\Lambda}(\hat{w}) \star \hat{x}, \tag{26}$$

where $\hat{\Lambda}(\hat{w}) = \Lambda(\hat{w})\hat{I}$, resulting in generally *non-linear* isotransformations, including isotranslations of the type

$$\hat{x}' = \hat{x} + \hat{A}(\hat{x}, ...), \tag{27}$$

verifying the following property

$$\hat{\Lambda}^\dagger \star \hat{\eta} \star \hat{\Lambda} = \Lambda\hat{\eta}\Lambda^\dagger. \tag{28}$$

Under the *condition of isomodularity*

$$\hat{D}et\ (\hat{\Lambda}) = +\hat{I}, \tag{29}$$

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we have the *isoconnected LS isosymmetries* $\hat{SO}^0(3.1)$ and the *isoconnected LPS isosymmetries* $\hat{P}^0(3.1)$.

Consider the conventional generators of the Poincaré symmetry

$$(J_k) = (J_{\mu\nu}),\ P_\mu, k = 1, 2, 3, 4, 5, 6,\ \mu, \nu = 1, 2, 3, 4. \tag{30}$$

By keeping in mind isoexponentiation (I-16), the isotransformations of $\hat{SO}^0(3.1)$ can be written [60]

$$\begin{aligned} \hat{x}' &= (\hat{e}^{iJ_k w_k}) \star \hat{x} \star (\hat{e}^{-iJ_k w_k}) = \\ &= \left[(e^{iJ_k \hat{T} w_k}) x (e^{-i w_k \hat{T} J_k})\right] \hat{I}, \end{aligned} \tag{31}$$

and the isotranslations $\hat{A}(3.1)$ can be written

$$\begin{aligned} \hat{x}' &= (\hat{e}^{iP_\mu a_\mu}) \star \hat{x} \star (\hat{e}^{-iP_\mu a_\mu}) = \\ &= \left[(e^{iP_\mu \hat{T} a_\mu}) x (e^{-i a_\mu \hat{T} P_\mu})\right] \hat{I}. \end{aligned} \tag{32}$$

It is evident that the above isotransformations do constitute Lie-Santilli isogroups according to Theorem I-2.7.3.

2.5.4. Isocommutation rules. As recalled earlier, the total quantities of an isolated, stable, interior system must be conserved for consistency.

In order to represent this evidence, the Lie-Santilli isotheory was constructed [26] in such a way to preserve conventional generators, because they represent total conservation laws, and isotopically lift their product.

By expanding the preceding finite isotransforms in terms of the isounit, the *LPS isoalgebra* $\hat{so}^0(3.1)$ is characterized by the conventional generators of the LP algebra and the isocommutation rules [30] [60] (here written in their projection on conventional spaces over conventional fields)

$$\begin{aligned} &[J_{\mu\nu} \hat{,} J_{\alpha\beta}] = \\ &= \imath(\hat{\eta}_{\nu\alpha} J_{\beta\mu} - \hat{\eta}_{\mu\alpha} J_{\beta\nu} - \hat{\eta}_{\nu\beta} J_{\alpha\mu} + \hat{\eta}_{\mu\beta} J_{\alpha\nu}), \end{aligned} \tag{33}$$

$$[J_{\mu\nu} \hat{,} P_\alpha] = i(\hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu)$$

$$[P_\mu \hat{,} P_\nu] = 0, \tag{34}$$

$$\hat{\eta}_{\mu\nu} = \hat{T}\eta = (\hat{T}^\rho_\mu \eta_{\rho\nu}). \tag{35}$$

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where one should note the appearance of the *structure functions* $\hat{\eta}(x, p,$-$E, \nu, \alpha, \tau, \psi, ...)$, rather than the traditional structure constants (Theorem I-2.7.2).

The presence of structure functions $\hat{\eta}$ in isocommutation rules (33)-(35), Theorem I-3.7.2 and the analysis of Section I-3.8 imply the following important property (Section I-3.8):

LEMMA 2.5.1: LPS isosymmetries cannot be derived via non-unitary transformations of the conventional LP symmetry.

Despite the above non-equivalence, the property $\hat{T} > 0$, the topological structure $(+1, +1, +1, -1)$ of the isometric $\hat{\eta} = \hat{T}\eta$ and Theorem 2.3.1 imply that:

LEMMA 2.5.2. *All LPS isosymmetries are locally isomorphic to the conventional LP symmetry.*

Recall from Section I-1 that an important limitation of quantum mechanics for the study of the EPR argument is the inability to achieve a consistent and effective treatment of non-linear interactions that are expected in the structure of hadrons, nuclei and stars. In Section I-4.12, we have shown that the isotopic "completion" of quantum mechanics into hadronic mechanics does indeed allow a consistent and effective treatment of non-linear interactions via their embedding in the isotopic element $\hat{T}$.

Due to the unrestricted functional dependence of the isotopic element $\hat{T}$ and, therefore, of the isometric $\hat{\eta} = \hat{T}\eta$, it is easy to see that the LPS isosymmetries are indeed non-linear as a necessary condition to provide the invariance of non-linear dynamical equations.

Note that *isolinear isomomenta $\hat{P}_\mu$ isocommute on isospaces over isofields, but they do not commute on conventional spaces over conventional fields,* Eqs. (35), thus confirming that *the LPS isosymmetry is isolinear, that is, linear on isospaces over isofields but generally non-linear in their projection on conventional spaces over conventional fields.*

This important property can be illustrated by recalling the isolinear isomomentum (I-79) on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $\hat{\psi} >$ over the isocomplex isonumbers $\hat{\mathcal{C}}$

$$\hat{P}_\mu \star |\hat{\psi} >= -i\hat{I}\partial_\mu|\hat{\psi} > . \tag{36}$$

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Isocommutators (35) on $\hat{\mathcal{H}}$ over $\hat{\mathcal{C}}$ can then be explicitly written

$$[\hat{P}_\mu,\hat{P}_\nu] \star |\hat{\psi}>= (\hat{P}_\mu \star \hat{P}_\nu - \hat{P}_\nu \star \hat{P}_\mu) \star |\hat{\psi}>=$$
$$= (-i\hat{I}\partial_\mu)\hat{T}(-i\hat{I}\partial_\nu) - (-i\hat{I}\partial_\nu)\hat{T}(-i\hat{I}\partial_\mu)\hat{T}|\hat{\psi}>= \tag{37}$$
$$= (i\hat{I}\partial_\mu\partial_\nu - i\hat{I}\partial_\nu\partial_\mu)|\hat{\psi}>= 0.$$

By contrast, the projection of the same isocommutators (35) on a conventional Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ no longer commutes,

$$[\hat{P}_\mu, \hat{P}_\nu]|\hat{\psi}>= (\hat{P}_\mu\hat{P}_\nu - \hat{P}_\nu\hat{P}_\mu)|\hat{\psi}>=$$
$$= (-i\hat{I}\partial_\mu)(-i\hat{I}\partial_\nu) - (-i\hat{I}\partial_\nu)(-i\hat{I}\partial_\mu)|\hat{\psi}>\neq 0. \tag{38}$$

because, in general, $\partial_\mu\hat{I} \neq \partial_\nu\hat{I}$, and this proves the isolinear character of the isomomentum.

Besides a direct relevance for the structure of hadrons, nuclei and stars, the above isolinearity has important implications, such as a new consistent operator form of gravitation, a new grand unification and other advances [35].

The presence of the structure functions in the isocommutation rules, the capability to provide the invariance under non-linear interactions and other features and applications outlined in Section 4 illustrate the non-triviality of the Lie-Santilli isotheory.

2.5.5. Iso-Casimir Isoinvariants. The simple direct use of isocommutation rules (33)-(35) establishes that the *iso-Casimir-isoinvariants* of $\hat{p}^0 0(3.1)$ are given by [60]

$$\hat{C}_1 = \hat{I}((t, r, p, E, \mu, \tau, \psi, \partial\psi, ...) > 0,$$
$$\hat{C}_2 = \hat{P}^{\hat{2}} = \hat{P}_\mu \star \hat{P}^\mu = (\hat{\eta}_{\mu\nu}P^\mu P^\nu)\hat{I} =$$
$$= (\textstyle\sum_{k=1,2,3} \frac{1}{n_k^2}P_k^2 - \frac{c^2}{n_4^2}p_4^2)\hat{I}, \tag{39}$$
$$\hat{C}_3 = \hat{W}^{\hat{2}} = \hat{W}_\mu \star \hat{W}^\mu, \;\; \hat{W} = W\hat{I},$$
$$\hat{W}_\mu = \hat{\epsilon}_{\mu\alpha\beta\rho} \star J^{\alpha\beta} \star P^\rho,$$

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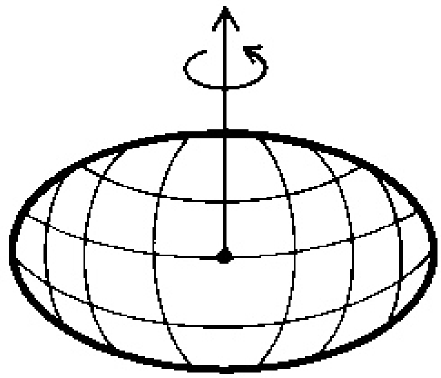

Figure 6: *It was generally believed in the 20th century physics that the rotational symmetry is broken for ellipsoids. Santilli isorotational isosymmetry has restored the exact character of the rotational symmetry for all possible (topology preserving) deformations of the sphere [30].*

and they are at the foundation of classical and operator *relativistic isomechanics* (Section I-4) with deep implications for structure models of interior dynamical systems [31].

2.5.6. Isorotations. By using isotransforms (32), the explicit form of the isorotations $\hat{SO}(3)$, first derived in Refs. [55] [56], can be written in the isoplane $(\hat{x},^1, \hat{x}^2)$ of iso-Euclidean isospaces $\hat{E}(\hat{x}, \hat{\Delta}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, here formulated for simplicity in their projection on the conventional Euclidean space (see Ref. [30] for the general case)

$$x^{1'} = x^1 \cos[\theta(n_1 n_2)^{-1}] - x^2 \tfrac{n_1^2}{n_2^2} \sin[\theta(n_1 n_2)^{-1}],$$
$$x^{2'} = x^1 \tfrac{n_2^2}{n_1^2} \sin[\theta(n_1 n_2)^{-1}] + x^2 \cos[\theta(n_1 n_2)^{-1}]. \tag{40}$$

It was generally believed in the 20th century that the $SO(3)$ symmetry is broken for ellipsoid deformations of the sphere. By contrast, as shown by isotransforms (40) the $\hat{SO}(3)$ isosymmetry achieves the invariance of ellipsoids (Figure 6). But $SO(3)$ and $\hat{SO}(3)$ are locally isomorphic (Theorem 2.3.1). We therefore have the following property [55] [56]:

LEMMA 2.5.3: The Lie-Santilli $\hat{SO}(3)$ isosymmetry restores the exact character of the rotational symmetry for all ellipsoid deformations of the sphere.

This property is due to the fact that the mutation of the semiaxes of the sphere occur jointly with the *inverse*, mutation of the related units, thus maintaining the perfect spherical shape in isospaces over isofields

$$Radius \quad 1_k \to 1/n_k^2, \;\; Unit \quad 1_k \to n_k^2. \tag{41}$$

Note the crucial role of isonumbers for the reconstruction of the exact rotational symmetry because said reconstruction occurs thanks to the isoinvariant by the isounit.

2.5.7. Lorentz-Santilli isotransforms. The infinite family of isoconnected Lorentz-Santilli (LS) isotransforms $\hat{SO}^0(3.1)$ on iso-Minkowskian isospaces $\hat{M}(\hat{x}, \hat{\Omega}, \hat{I})$ over the isoreals $\hat{\mathcal{R}}$, first derived by in Ref. [53] of 1983, can be written in the $(\hat{x}^3, \hat{x}^4)$-isoplane in their projection in the conventional l Minkowski space $M(x, \eta, I)$, as follows (see Ref. [30] for the general case):

$$\begin{aligned} x^{1'} &= x^1, \;\; x^{2'} = x^2, \\ x^{3'} &= \hat{\gamma}(x^3 - \hat{\beta}\tfrac{n_3}{n_4}x^4), \\ x^{4'} &= \hat{\gamma}(x^4 - \hat{\beta}\tfrac{n_4}{n_3}x^3), \end{aligned} \tag{42}$$

where

$$\hat{\beta} = \frac{v_3/n_3}{c/n_4}, \;\; \hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}. \tag{43}$$

A significant aspect of Ref. [53] is the solution of the *historical Lorentz problem,* namely, the invariance of locally varying speeds of light within physical media

$$C = \frac{c}{n_4}. \tag{44}$$

In fact, Lorentz first attempted the invariance of the speed of light $C = c/n_4$, but had to restrict his study to the invariance of the constant speed of light in vacuum c, due to insurmountable technical difficulties. Santilli has shown that Lorentz's difficulties were due to the use of Lie's theory, because, under the use of the covering Lie-Santilli isotheory, the invariance of $C = c/n_4$ was achieved in two pages of the 1983 letter [53].

A second significant aspect of Ref. [53] is the achievement of the first invariant formulation of extended, thus deformable and hyperdense particles, as stated beginning with the title of the quoted paper.

It was generally believed in the 20th century that the Lorentz symmetry $SO^0(3.1)$ is broken for locally varying speed of light within physical media represented with the wiggly circle of Figure 7. Ref. [53] proved that the isosymmetry $\hat{SO}^0(3.1)$ achieves the invariance of $C = c/n_4$. But $SO^0(3.1)$

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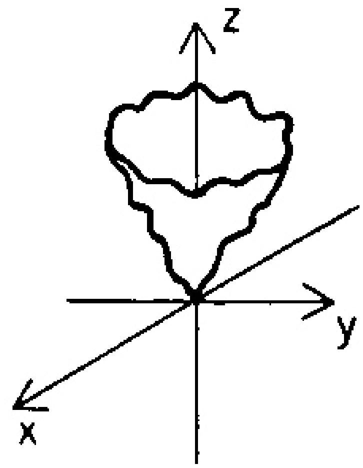

Figure 7: *It was generally believed in the 20th century physics that the Lorentz symmetry is broken for locally varying speeds of light within physical media (here represented with a wiggly light cone). The Lorentz-Santilli isosymmetry has restored the exact validity of the Lorentz symmetry for interior dynamical problems [53] [30].*

and $\hat{SO}^0(3.1)$ are locally isomorphic, thus restoring the exact character of the abstract axioms of the Lorentz for all possible values $C = c/n_4$. We therefore have the following important property [30]:

LEMMA 2.3.5: The Lie-Santilli $\hat{SO}^0(3.1)$ isosymmetry restores the exact validity of Lorentz's axioms for locally varying speeds of light.

This property is due to the reconstruction of the exact light cone on the iso-Minkowskian isospace over isofields with maximal causal value c, called the *light isocone,*

$$\hat{x}^{\hat{2}} = \hat{x}_3^2 + \hat{x}_4^2 = 0, \tag{45}$$

while its projection on the conventional Minkowski space over conventional fields represents a locally varying speed

$$\hat{x}^2 = (\frac{x_3^2}{n_3^2} - t^2\frac{c^2}{n^4})\hat{I} = 0. \tag{46}$$

This property is due to the fact that the mutation of the $\hat{x}^3$ and $\hat{x}^4$ isocoordinates occurs jointly with the *inverse* mutation of the corresponding isounits, by therefore preserving the original perfect light cone with c as

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the maximal causal speed (see the 1966 monograph [30] for details)

$$x^3 \rightarrow \tfrac{x^3}{n_3}, \quad I_3 = 1 \rightarrow \hat{I}_3 = n_3$$
$$x^4 = tc \rightarrow \tfrac{x^4}{n_4} = t\tfrac{c}{n_4}, \quad I_4 = 1 \rightarrow \hat{I}_4 = n_4. \tag{47}$$

Another significant aspect of Ref. [53] is the achievement of the first known invariance of non-linear, non-local and non-Hamiltonian interactions thanks to their embedding in the characteristic n-quantities of the isoinvariant (25).

2.5.8. Isotranslations. In view of their non-linearity, isotranslations in four parameters a_μ can be written in their projection in the conventional Minkowski space [30]

$$x'^\mu = x^\mu + A^\mu(a, x, \ldots), \tag{48}$$

and can be written via a power series expansion of the general expression

$$A^\mu = a^\mu(n_\mu^{-2} + a^\alpha[n_\mu^{-2} \hat{,} P_\alpha]/1! + \ldots), \tag{49}$$

The understanding of the isotopic completion of 20th century spacetime symmetries requires the knowledge that, when properly written on iso-Minkowskian isospace over isofields, isotranslations recover their conventional form . [30].

2.5.9. Isodilatations. Santilli introduced in Ref. [60] a novel one-dimensional isoinvariance denoted $\hat{D}$ which is given by the dilatation of the isometric caused by its multiplication by as parameter w, while the isounit is jointly subjected to the *inverse* dilatation

$$\hat{\Omega} = \hat{\eta}\hat{I} \rightarrow \hat{w} \star \hat{\Omega} = w\hat{\eta}\hat{I}'$$
$$\hat{I} \rightarrow \hat{I}' = \tfrac{1}{w}\hat{I}, \tag{50}$$

under which isoinvariant (25) remain manifestly unchanged.

In essence, the new symmetry originates from the fact that, for mathematical consistency, isoinvariants must be elements of t isofields, thus having structure (25), namely, isoinvariants must be given by a conventional invariant multiplied by the isounit.

Ref. [60] showed that, by writing conventional invariants with the multiplication, in this case, by the trivial unit 1, the new dilatation symmetry persists for conventional space-time symmetries,

$$\eta \rightarrow \eta' = w\eta, \quad 1 \rightarrow 1' = \frac{1}{w}1. \tag{51}$$

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The above properties imply the following:

LEMMA 2.5.5: The conventional Lorentz-Poincaré symmetry is eleven-dimensional with structure

$$P^0(3.1) = so^0(3.1) \times A(3.1) \times D, \tag{52}$$

and, consequently, the Lorentz-Poincaré-Santilli isosymmetry is also eleven-dimensional with the structure

$$\hat{P}^0(3.1) = \hat{so}^0(3.1) \star \hat{A}(3.1) \star \hat{D}. \tag{53}$$

The above seemingly trivial property has permitted Santilli the study of a new grand unification of electroweak and gravitational interactions based on the embedding of gravitation in the isotopic degree of freedom of the theory [35].

2.5.10. Isoinversions. The isotopic "completion" of conventional inversions has been studied in details in Refs. [30] and consists of the *isotime isoinversions*

$$\hat{\tau}\hat{t} = (\tau\hat{t})\hat{I} \tag{54}$$

plus the *isospace isoinversions*

$$\hat{\pi}\hat{r} = (\pi\hat{r})\hat{I} \tag{55}$$

where τ and π are conventional time and space inversions, respectively.

Despite their simplicity, Santilli has shown in Ref. [30] that *not only continuous, but also discrete space-time symmetries can be reconstructed as being exact on isospaces over isofields when assumed to be broken on conventional spaces over conventional fields.*

2.5.11. Isospinorial LPS isosymmetry. Recall that the spinorial covering $\mathcal{P}^0(3.1)$ of the connected component of the LP symmetry $P^0(3.1)$ is constructed via the use of the Dirac gamma matrices. In fact, the conventional generators are realized via suitable combination of Dirac gamma matrices.

By following the same historical pattern, Santilli proposed in the 1995 communication [61] of the *Joint Institute for Nuclear Research,* Dubna, Russia (see also the subsequent paper [62]) the following eleven-dimensional isotopic "completion" $\hat{\mathcal{P}}^0(3.1)$ of $\mathcal{P}^0(3.1)$

$$\hat{\mathcal{P}}(3.1) = \hat{SL}(2.\hat{C}) \star \hat{\mathcal{A}}(3.1) \star \hat{\mathcal{D}}, \tag{56}$$

with realization of the generators in terms of the Dirac-Santilli isogamma isomatrices $\hat{\Gamma}_\mu = \hat{\gamma}_\mu \hat{I}$, Eqs. (I-89),

$$\hat{SL}(2.\hat{C}): \quad \hat{R}_k = \tfrac{1}{2}\epsilon_{kij}\hat{\Gamma}_i \star \hat{\Gamma}_j, \quad \hat{S}_k = \tfrac{1}{2}\hat{\Gamma}_k \star \hat{\Gamma}_4,$$

$$\hat{\mathcal{A}}(3.1): \quad \hat{P}_\mu, \tag{57}$$

$$k = 1, 2, 3, 4, 5, 6, \quad \mu = 1, 2, 3, 4.$$

The verification by the above isogenerators of isocommutation rules (33)-(35) is an instructive exercise for the interested reader. The proof that the Dirac-Santilli isoequations (I-88) transform isocovariantly under $\hat{\mathcal{P}}^0(3.1)$ is equally instructive.

2.5.12. Galilean isosymmetries.

As it is well known, the Galileo symmetry $G(3.1)$ characterizes the non-relativistic motion of point particles in vacuum, with consequential absence of resistive or non-potential forces (see the vertical line of Figure 8).

The isotopies of the Galileo symmetry are intended to characterize the non-relativistic motion of extended particles within physical media, by therefore experiencing resistive non-potential forces (see the wiggly line of Figure 8).

The resulting infinite family of isosymmteries $\hat{G}(3.1)$ are here called *Galilean isosymmetries* to stress the preservation of the basic axioms of the Galileo symmetry and the mere construction of the broadest possible realizations permitted by isomathematics.

The Lie-isotopic lifting of the Galileo symmetry were introduced by Santilli in the 1978 paper [12] as a particular case of the covering *Lie-Admissible symmetries,* also called *genosymmetries,* which are intended fo characterize the *time rate of variation of physical quantities.*

The first direct study of Galilean isosymmetries was done in Section 5.3, pages 225 on, of the 1981 monograph [26] formulated over conventional fields. These isotopies were then systematically studies and upgraded in the two 1991 volumes [27] [28]. The formulation of Galilean isosymmetries with the full use of isomathemaics was done in the 1995 monographs [29] [30] with a final study presented in Ref. [32].

The above studies attracted the attention of Abdus Salam, founder and president of the *International Center for Theoretical Physics* (ICTP), Trieste, Italy, who invited Santilli in 1991 to deliver at his Center a series of lectures

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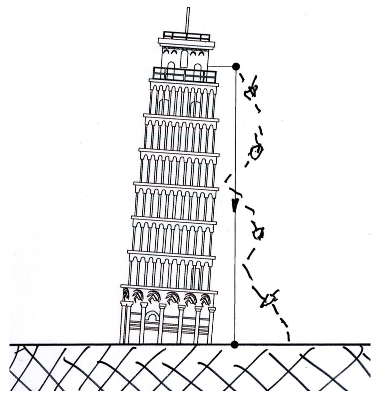

Figure 8: *This figure presents a conceptual rendering of the free fall of point-masses in vacuum studied by Galileo (represented with a straight line), and the free fall of extended masses experiencing resistive forces from our atmosphere studied by Santilli (represented with a wiggling line) [26] [30] [37]. It is symptomatic to note that the achievement of the symmetry for extended masses required the construction of a covering of the mathematics used for the point masses with particular reference to the generalization of Newton-Leibnitz differential calculus, from its historical formulation for isolated point, to a covering formulation for volumes [21].*

in the isotopies of the Galileo symmetry and relativity, said invitation being apparently the last by Salam prior to his death.

During his visit at the ICTP, Santilli wrote papers [65] through [71]. The notes from Santilli's lectures were collected by A. K. Aringazin, A. Jannussis, F. Lopez, M. Nishioka and B. Vel-janosky and published in volume [37] of 1992.

This work is primarily intended for relativistic isosymmetries. Additionally, all primary applications require relativistic treatments. Therefore, we regret to be unable to review Galilean isosymmetries to prevent a prohibitive length.

Nevertheless, the reader should be aware that an introductory knowledge of the Galilean isosymmetries is suggested, e.g., from the reading of the ICTP papers [65] to [71].

3. APPARENT PROOFS OF THE EPR ARGUMENT

3.1. Foreword.

As it is well known, the conventional Pauli matrices $\sigma_k,\ k = 1, 2, 3$, are the fundamental (also called adjoint), irreducible unitary representation of the $SU(2)$-spin symmetry and play a crucial role for the objections against the

EPR argument [2] - [6] .

In this section, we review the isotopic "completion'" of Pauli's matrices into isomatrices

$$\hat{\Sigma}_k = \hat{\sigma}_k \hat{I} \tag{58}$$

which constitute the isofundamental, isoirreducible, isounitary isorepresentation of the Lie-Santilli $\hat{SU}(2)$ isosymmetry and play a crucial role in the apparent proof of the EPR argument for extended particles within physical media studied later on in this section.

By recalling that the $SU(2)$ symmetry characterizes the spin of point-particles in vacuum, the "completed $\hat{SU}(2)$ isosymmetry is intended to characterize the spin of extended particles within hyperdense media called *hadronic spin,* such as the spin of an electron in the core of a star.

The isotopic "completion" of Pauli's matrices was introduced by Santilli in 1993 while visiting the *Joint Institute for Nuclear Research,* Dubna, Russia [58]. Said "completion" was presented systematically in Refs. [29] [30], used for the apparent proofs of the EPR argument [10] [11], and they are nowadays known as the *Pauli-Santilli isomatrices* [45].

In particular, the preceding studies have shown that, unlike the case for the $SU(2)$ symmetry, the isotopic $\hat{SU}(2)$ isosymmetry admits an explicit and concrete realization of hidden variables λ [3] [4] via realizations of the isotopic element of type $\hat{T} = Diag.(1.\lambda, \lambda)$ Eq. (3).

In this section, we shall review the construction of $\hat{SU}(2)$ isosymmetry and of Pauli-Santilli isomatrices of regular and irregular type with hidden variables. We shall then use the methods acquired in this and in the preceding paper [9], for the proof that *interior dynamical systems represented via isomathematics and isomechanics appear to admit identical classical counterparts* [10] (Section 3.7), and to *progressively approach the classical EPR determinism [1] in the structure of hadrons, nuclei and stars, while achieving the EPR determinism in the interior of gravitational collapse* [11] (Section 3.8).

A first understanding of this section requires a knowledge of the Lie-Santilli isotheory (Section I-2.7) [26] [30] [38] [46] [47] [49]. A technical understanding of this section requires a technical knowledge of hadronic mechanics [29]- [31].

3.2. Pauli matrices.

As it is well known (see, e.g., Ref. [72]), the carrier space of the two-dimensional group of special unitary transformations $SU(2)$ is the two-dimensional complex Euclidean space $E(z, \delta, I)$ with coordinates $z = (z_1, z_2)$, metric $\delta = Diag.(1, 1)$ and unit $I = Diag/(1, 1)$.

The two-dimensional, fundamental (also called adjoint), irreducible,

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unitary representation of the special unitary Lie algebra $su(2)$ of the $SU(2)$-spin symmetry is given by the celebrated *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{59}$$

with commutations rules

$$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = i2\epsilon_{ijk}\sigma_k, \tag{60}$$

and eigenvalues on a Hilbert space $calH$ over the field of complex numbers $\mathcal{C}$ with basis $|b>$

$$\begin{gathered} \sigma^2|b>= (\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3)|b>= 3|b>, \\ \sigma_3|b>= \pm 1|b> . \end{gathered} \tag{61}$$

Among the various properties of Pauli's matrices, we should recall their uniqueness in the sense that their expression is invariant under the class of equivalence admitted by quantum mechanics, that under unitary transformation.

We should also recall that Pauli's matrices are also fundamental for the structure of Dirac's equation, Eq. (I-9) since they appear in the very definition of Dirac's gamma matrices, Eqs. (I-89).

3.3. Regular Pauli-Santilli isomatrices.

By following Ref. [58], the carrier isospace of the two-dimensional Lie-Santilli isogroup of isospecial isounitary isotransformations $\hat{SU}(2)$ is the isocomplex iso-Euclidean isospace $\hat{E}(\hat{z}, \hat{\Delta}, \hat{I})$ with isocoordinates

$$\hat{z} = z\hat{I} = (z_1, z_2) = (z_1, z_2)\hat{I}; \tag{62}$$

isounit and isotopic element

$$\hat{I} = \begin{pmatrix} n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix} = 1/\hat{T} > o, \tag{63}$$

$$\hat{T} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2}; \end{pmatrix} \tag{64}$$

isometric

$$\hat{\Delta} = \hat{\delta}\hat{I} = (\hat{T}_i^k \delta_{ki})\hat{I} = \begin{pmatrix} n_1^{-2} & 0 \\ 0 & n_2^{-2} \end{pmatrix} \hat{I}; \tag{65}$$

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positive-definite characteristic quantities n_k with unrestricted functional dependence on the variables for interior dynamical problems

$$n_k = n_k(z, \bar{z}, E, \mu, \alpha, \tau, \psi, \partial\psi, ...) > 0, \quad k = 1, 2; \tag{66}$$

and basic isoinvariant

$$\hat{z}_i \star \hat{\Delta}_{ij} \star \hat{\bar{z}}_j = (z_i \hat{\delta}_{ij} z_j)\hat{I} = \\ = (\tfrac{z_1 \bar{z}_1}{n_1^2} + \tfrac{z_2 \bar{z}_2}{n_2^2})\hat{I}. \tag{67}$$

By also following Refs. [29] [58], the isogroup of regular, isospecial, isounitary, isotransformations $\hat{SU}(2)$ leaving invariant isoline element (67), is characterized by the isotransforms

$$\hat{z}' = \hat{U}(\hat{\theta}) \star z = \hat{U}(\hat{\theta})\hat{T}\hat{z}, \tag{68}$$

verifying the following conditions [30]:

1. Isounitarity

$$\hat{U}(\hat{\theta}) \star \hat{U}^\dagger(\hat{\theta}) = \hat{U}^\dagger(\hat{\theta}) \star \hat{U}(\hat{\theta}) = \hat{I}; \tag{69}$$

2. Isogroup isoaxioms

$$\hat{U}(\hat{\theta}_1) \star \hat{U}(\hat{\theta}_2) = \hat{U}(\hat{\theta}_1 + \hat{\theta}_2), \\ \hat{U}(\hat{\theta}) \star \hat{U}(-\hat{\theta}) = \hat{U}(0) = \hat{I}, \quad k = 1, 2, 3; \tag{70}$$

and

3. Isospecial isounitarity

$$IsoDet\hat{U}(\hat{\theta}) = \hat{I}, \ Det(U\hat{T}) = 1. \tag{71}$$

The latter condition essentially restricts the isogroup $\hat{SU}(2)$ to its isoconnected component $\hat{SU}^0(2)$, which is hereon tacitly assumed.

The above conditions imply the local isomorphism

$$\hat{SU}(2) \approx SU(2), \tag{72}$$

and the following explicit realization in terms of isoexponential (I-22)

$$\hat{U}(\hat{\theta}) = \Pi_k U_k(\theta_k)\hat{I} = \Pi_k \hat{e}^{\hat{i} \star \hat{J}_k \star \hat{\theta}_k} = \Pi_k (e^{iJ_k \hat{T} \theta_k})\hat{I}, \\ U_k(\theta_k) = e^{iJ_k \hat{T} \theta_k} \quad , k = 1, 2, 3, \tag{73}$$

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where $\hat{J}_k$ represents the isogenerators of the Lie-Santilli isoalgebra $\hat{su}(2)$ verifying the conditions

$$IsoTr\hat{J}_k = 0, \ \ Tr(\hat{J}_k\hat{T}) = 0, \tag{74}$$

and the isocommutation rules

$$\begin{aligned}[\hat{J}_i\hat{,}\hat{J}_j] = \hat{J}_i \star \hat{J}_j - \hat{J}_j \star \hat{J}_i = \\ = \hat{J}_i\hat{T}\hat{J}_j - \hat{J}_j\hat{T}\hat{J}_i = \epsilon_{ijk}\hat{J}_k.\end{aligned} \tag{75}$$

Note that, in accordance with Theorem I-2.7.2, the isorepresentations here considered are called *regular* because they can be constructed via non-unitary transformations of the conventional $su(2)$ algebra, resulting in the preservation of the conventional structure *constants* ϵ_{ijk}.

However, as we shall see in the next section, the isotopies of the $su(2)$ algebra imply realizations called *irregular* that cannot be constructed via non-unitary representations of $su(2)$ [58], in which case the structure constants ϵ_{ijk} are replaced by *structure functions* with an arbitrary (non-singular) functional dependence on local variables,

$$\hat{C}_{ijk} = C_{ijk}(z, \bar{z}, E, \nu, \alpha, \tau, \psi, \partial\psi, ...)\hat{I}. \tag{76}$$

As one can verify, $\hat{su}(2)$ admits the following iso-Casimir isoinvariant

$$\begin{aligned}\hat{J}^{\hat{2}} = \Sigma_k\hat{J}_k \star \hat{J}_k = \\ = \hat{J}_1\hat{T}J_1 + \hat{J}_2\hat{T}J_2 + \hat{J}_3\hat{T}J_3.\end{aligned} \tag{77}$$

The maximal set of isocommuting isooperators is then given by $\hat{J}_3$ and $\hat{J}^{\hat{2}}$.

By again following Ref. [58], in order to compute the explicit form of the isorepresentations of $\hat{su}(2)$, we introduce the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$[19] over the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ [20] with d-dimensional isobasis $|\hat{b}^d_k>$ verifying isonormalization (I-75),

$$\begin{aligned}< \hat{b}^d_k| \star |\hat{b}^d_k >=< \hat{b}^d_k|\hat{T}|\hat{b}^d_k >= \hat{I}, \\ d = 1, 2, 3, ...N, , \quad k = 1, 2, , 3.\end{aligned} \tag{78}$$

From the local isomorphism $\hat{su}(2) \approx su(2)$ we know that the isoeigenvalue equations have the structure

$$\begin{aligned}\hat{J}_k \star |\hat{b}^d_k >= b^d_k|\hat{b}^d_k >, \\ \hat{J}^{\hat{2}} \star |\hat{b}^d_k >= \Sigma_k b^d_k(b^d_k + W)|\hat{b}^d_k > \ , \\ W = Det\hat{T} = 1/n_1^2 n_2^2,\end{aligned} \tag{79}$$

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where $W = 1$ for regular isorepresentation, otherwise W is an arbitrary function of local quantities to be identified via subsidiary constraints from the medium in which extended particles are immersed.

The explicit form of the isorepresentations of $\hat{su}(2)$ is then given by the simple isotopy of the conventional case [72]

$$\hat{J}_{\pm} = \hat{J}_1 \pm \hat{J}_2,$$
$$(\hat{J}_1)_{ij} = i\tfrac{1}{2} < \hat{b}^d_k | \star (\hat{J}_- - \hat{J}_+) \star |\hat{b}^d_k >,$$
$$(\hat{J}_2)_{ij} = i\tfrac{1}{2} < \hat{b}^d_k | \star (\hat{J}_- + \hat{J}_+) \star |\hat{b}^d_k >, \tag{80}$$
$$(J_3)_{ij} =< \hat{b}^d_k | \star \hat{J}_3 \star |\hat{b}^d_k > .$$

By continuing to follow Ref. [58], we now restrict our attention to the two-dimensional isofundamental (isoadjoint) isorepresentation of $\hat{su}(2)$ occurring for $d = 2$, in which case we assume

$$\hat{J}_k = \frac{1}{2}\hat{\sigma}_k, \quad k = 1, 2, 3, \tag{81}$$

and select the basic isounitary isotransform according to Sections I-2.8 and I-2.9

$$UU^{\dagger} = f(W) > 0, \ \ W = Det.\hat{I} = n_1^2 n_2^2, \tag{82}$$

where $f(W)$ is a smooth function such that $f(1) = 1$.

By using the above procedure, we have the following *regular Pauli-Santilli isomatrices* first introduced by Santilli in Ref. [58], Eqs. (3.2) (where the isometric elements are denoted $g_{kk} = n_k^{-2}, \ k - 1, 2,$

$$\hat{\Sigma}_k = \hat{\sigma}_k \hat{I},$$
$$\hat{\sigma}_1 = (n_1 n_2) \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \ \ \hat{\sigma}_2 = (n_1 n_2) \begin{pmatrix} 0 & -in_1^{-2} \\ in_2^{-2} & 0 \end{pmatrix}, \tag{83}$$
$$\hat{\sigma}_3 = (n_1 n_2) \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2}, \end{pmatrix}.$$

with isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = i2\epsilon_{ijk}\hat{\sigma}_k, \tag{84}$$

in which the 'regular' character of the isomatrices is established by the presence of the conventional (constant) structure constants.

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We then have the isoeigenvalues isoequations

$$\hat{\sigma}_3 \star |\hat{b}^2_m >= \hat{\sigma}_3 \hat{T} |\hat{b}^2_m >= \pm \tfrac{1}{n_1 n_2} |\hat{b}^2_m >$$

$$\hat{\sigma}^{\hat{2}} = (\sigma_1 \hat{T} \hat{\sigma}_1 + \sigma_2 \hat{T} \hat{\sigma}_2^{+} \sigma_3 \hat{T} \hat{\sigma}_3) \hat{T} |\hat{b}^2_m >= \tag{85}$$

$$= 3 \tfrac{1}{n_1^2 n_2^2} |\hat{b}^2_m >,$$

showing that the regular Pauli-Santilli isomatrices preserve the conventional structure constants ϵ_{ijk} of Pauli matrices, but exhibit structure (84) with generalized isoeigenvalues containing two characteristic quantities n_1^2, n_2^2.

It is evident that, under isounimodularity condition (71),

$$Det \hat{T} = 1, \;\; n_1 = 1/n_2, \tag{86}$$

isomatrices (83) reduce to

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1^{-2} \\ i n_2^{-2} & 0 \end{pmatrix},$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{87}$$

by verifying conventional commutation rules (84) and conventional eigenvalues

$$\hat{\sigma}_3 \star |\hat{b}^2_m >= \pm |\hat{b}^2_m >$$

$$\hat{\sigma}^{\hat{2}} \star |\hat{b}^2_m >= 3 |\hat{b}^2_m > . \tag{88}$$

In order to search for additional realizations of regular Pauli-Santilli isomatrices, we now assume the following non-unitary transform

$$U = \begin{pmatrix} n_1 & 0 \\ 0 & n_2 \end{pmatrix} = U^\dagger, \tag{89}$$

under which we have the following second realization of regular Pauli-Santilli isomatrices

$$\hat{\sigma}_k = U \sigma_k U^\dagger,$$

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1 n_2 \\ i n_1 n_2 & 0 \end{pmatrix}, \tag{90}$$

$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}.$$

It is an instructive exercise for the interested reader to verify that the above isomatrices verify the isocommutation rules (84) and conventional isoeigenvalue (99), namely, *the second realization of the Pauli-Santilli isomatrices, Eqs. (83), also admit conventional structure constants and eigenvalues despite the degrees of freedom permitted by the two characteristic quantities* $n_1^2,\ n_2^2$.

We now assume the following non-diagonal realization of the non-unitary transform

$$U = \begin{pmatrix} 0 & n_1 \\ n_2 & 0 \end{pmatrix}, U^\dagger = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \qquad (91)$$

$$UU^\dagger = \hat{I} = 1/\hat{T} > 0,$$

which characterizes the following third realization of the regular Pauli-Santilli isomatrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_1 n_2 \\ n_1 n_2 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -i n_1 n_2 \\ i n_1 n_2 & 0 \end{pmatrix}, \qquad (92)$$

$$\hat{\sigma}_3 = \begin{pmatrix} -n_1^2 & 0 \\ 0 & n_2^2 \end{pmatrix}.$$

It is easy to see that the above third realization of the regular Pauli-Santilli isomatrices also verify conventional commutation rules (84) and eigenvalues (88).

Note that, while Pauli's matrices are invariant under unitary transforms, there exist a number of Pauli-Santilli isomatrices each of which is invariant under isounitary isotransforms (Section I-3.9).

3.4. Irregular Pauli-Santilli isomatrices.

3.4.1. Historical notes. One of the most fundamental, yet unresolved processes in nature is the synthesis of the neutron from the hydrogen in the core of stars, which is a pre-requisite for the production of light and, therefore, for the existence of life itself.

In this section, we would like to outline the main historical aspects on the synthesis of the neutron and identify the open problems because truly fundamental for the construction of the new mathematics needed for their solution.

Recall that stars initiate their life as an aggregate of hydrogen and grow via the accretion of hydrogen existing in interstellar spaces.

In 1910, H. Rutherford [73] conjectured that, when the pressure and temperature at the core of the star reaches certain values, the hydrogen

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atom is "compressed" into a neutral particle n which is called the *neutron* according to the reaction (see Section I-4)

$$e^- + p^+ \ \rightarrow \ n. \tag{93}$$

Rutherford hypothesis was experimentally confirmed by J. Chadwick in 1932 [74], and the neutron became part of scientific history.

Following the experimental verification that the neutron has the same spin $1/2$ of the electron and of the proton, in an attempt at maintaining the conservation of the angular momentum, E. Fermi [75] suggested that the synthesis of the neutron occurs with the emission of a hypothetical, massless and chargeless particle ν with spin $1/2$ which he called the *neutrino* (meaning "little neutron" in Italian), according to the reaction widely accepted by the scientific communioty for about one century

$$e^- + p^+ \ \rightarrow \ n + \nu. \tag{94}$$

After joining Harvard University in September 1977 under DOE support, R. M. Santilli [15]-[17] noted that, despite the salvaging of space-time symmetries and related conservation laws, reaction (93) is not compatible with quantum mechanical laws because the rest energy of the neutron E_n is $0.782\ MeV$ *bigger* than the sum of the rest energies of the proton E_p and of the electron E_e,

$$\begin{gathered} E_p = 938.272\ MeV, \ \ E_e = 0.511\ MeV, \ \ E_n = 939.565\ MeV, \\ E_n - (E_p + E_e) = 0.782\ MeV > 0. \end{gathered} \tag{95}$$

Therefore, Santilli presented a number of arguments according to which the synthesis of the neutron is clear evidence of Einstein's view on the lack of "completion" of quantum mechanics (see Einstein's name in the title of the 1981 paper [17] released from the Department of Mathematics of Harvard University). Subsequently, Santilli achieved in Ref. [76] (see the independent review [**?**] the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen representation, with a relativistic representation subsequently achieved in Ref. [62] (see the independent review in ref. [45]).

The technically most difficult problem of the above representation was the identification of the spin-orbit coupling for the electron when totally immersed inside the proton which was first solved non-relativistically in Ref. [76] and relativistically in Ref. [62] (see the review in Ref. [30], Chapter 6 in particular).

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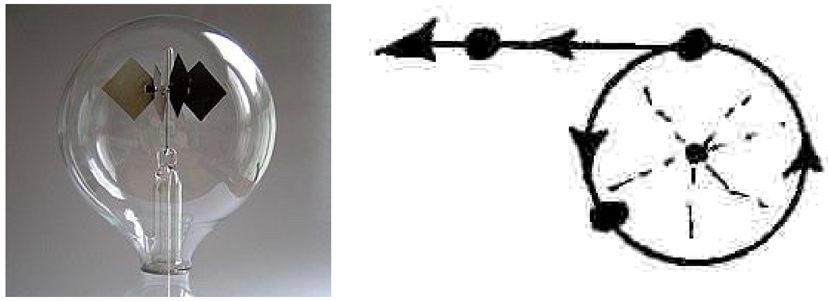

Figure 9: *To illustrate some of isosymmetries for interior systems, in the left we show the conversion of linear momentum into angular momentum for the case of photons causing the rotation of a small propeller in a vacuum chamber. In the right, we show the opposite conversion of angular momentum into linear momentum, as it is the case for the sling shot. The left view illustrates that the neutrino hypothesis is not necessary for the synthesis of the neutron when particles are represented as being extended. The right view illustrates that the emission of an antineutrino is not necessary in the neutron decay because the internal angular momentum of the electron can be converted into its external linear momentum without any violation of physical laws.*

Recall that quantum mechanics has an excellent consistency for bound states with *negative potentials* causing a *mass defect.* Santilli's first argument is that a representation of experimental data (95) via quantum mechanics is impossible because it would require a *positive potential* capable of producing a *mass excess,* which features imply the loss of physical consistency of Schrödinger equation *for bound states* (and not for free particles with positive kinetic energy) because the indicial equation of Schrödinger equation admits no consistent solutions for positive potential energies, (see Section I-4).

The inability of Dirac's and other quantum mechanical equations for the representation of experimental data (95) then followed.

Various conjectures, aimed at maintaining for the neutron structure the theory so effective for the hydrogen atom, were proved not to be consistent. For instance, the hypothesis that the missing energy of $0.782\ MeV$ is provided by the star via a relative energy between the electron and the proton had to be dismissed because the cross section $e - p$ at $0.782\ MeV$ is essentially null, thus preventing any fusion between the electron and the proton.

Similarly, the hypothesis that the missing energy is provided by the

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antineutrino $\bar{\nu}$ via reactions of the type

$$e^- + \bar{\nu} + p^+ \ \rightarrow \ n, \tag{96}$$

had to be equally abandoned because the cross section between neutrinos or antineutrinos and individual particles is identically null.

As it is well known, the neutrino hypothesis is necessary for the quantum mechanical treatment of synthesis (94), namely, for the point-like characterization of the proton, the electron and the neutron. Ref. [17] indicated that the neutrino hypothesis cannot any longer be consistently applied for the neutron synthesis whenever particles are represented as being extended.

Independently from that, there exist no known conventional possibility of identifying the *energy* needed for the creation of the neutrino since synthesis (94) already misses $0.782\ MeV$ for the synthesis of the neutron.

Another argument of Ref. [17] is that the conservation of the angular momentum is necessary for the synthesis of bound states with a Keplerian center under the validity of conventional space-time symmetries, such as for the synthesis of the hydrogen atom form an electron and a proton.

However, said conservation is no longer necessary for bound states at short distances without a Keplerian center, since in that case we have the validity of space-time isosymmetries for which the angular momentum can be transformed into linear momentum and vice-versa without any violation of physical laws (see Section 2.2 and Figure 9).

But *the neutron has no Keplerian center,* with the consequential lack of applicability of the Lorentz-Poincaré symmetry, and the ensuing lack of necessary conservation of the angular momentum in favor of alternative hypotheses.

In view of the above (and other) insufficiencies of the neutrino hypothesis, Santilli suggested in Ref. [78] the introduction in the *l.h.s.* (rather than the r.h.s.) of the synthesis of the mass-less, charge-less and spin-less particle called the *etherino* and denoted with the symbol a (from the Latin *aether*)

$$p^+ + a + e^- \ \rightarrow \ n, \tag{97}$$

whose scope is to represent the delivery of the missing $0.782\ MeV$ to the neutron.

An intriguing aspect is that *the etherino hypothesis can be shown to be compatible with the experimental data of the so-called "neutrino experiments," of course, under the condition of abandoning point-like abstractions of hadrons and representing them as they are in the physical reality, i.e., extended, deformable and hyperdense.*

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By recalling that there is no known consistent way of accounting for the missing $0.782\ MeV$ as originating from the star, Ref. [**?**] submitted the hypothesis that the missing energy may originate from the ether as a universal medium of extremely high energy density for the characterization and propagation of particles and electromagnetic waves.

However, it should be stressed that *the etherino is not intended to be a particle, but to be an "impulse" representing the mechanism of supplying the missing* $0.782\ MeV$ *for the neutron, the origin of the missing energy from the ether being only one among other possibilities.*

In view of the above insufficiencies of quantum mechanics for the synthesis of the neutron, Santilli initiated in Refs. [15] - [17] the search for a "completion" of quantum mechanics into hadronic mechanics with particular reference to the "completion"of Lie's theory at large, and the $SU(2)$-spin symmetry in particular, for the characterization of the spin of the electron when "compressed" inside the hyperdense proton.

It should also be recalled that, during the same period, Santilli conducted a post Ph. D. Seminar Course at the Lyman Laboratory of Physics of Harvard University with a technical treatment of the insufficiency of quantum mechanics for the neutron synthesis via the *conditions of variational self-adjointness* for the existence of a Lagrangian or a Hamiltonian. This Seminar Course was eventually published by Springer-Verlag in monographs [25] [26] whose primary aim is the first known presentation of the axiom-preserving Lie-Santilli isotheory and the axiom-inducing Lie-Santilli genotheory.

In fact, possible representations of experimental data (95) for the neutron synthesis violate the conditions of variational self-adjointness, thus mandating the search for a covering theory.

The subsequent 1995 papers [55] [56] [58] achieved the regular isotopies $\hat{SU}(2)$ of the spin symmetry (reviewed in the preceding section).

However, *regular isotopies of the* $SU(2)$ *spin symmetry are insufficient for the neutron synthesis because it requires alterations (called mutations) of conventional eigenvalues that can be solely represented via irregular isorepresentations.*

The irregular isorepresentations of the $SU(2)$-spin symmetry were identified, apparently for the first time, by Santilli in the 1990 paper [78] and used to achieve the non-relativistic representation of *all* characteristics of the neutron in its synthesis from the hydrogen.

In the 1995 paper [62], Santilli presented a relativistic study of $\hat{SU}(2)$ as an isosubalgebra of the irregular isospinorial covering of the Lorentz-Poincaré symmetry (Section 2.5.11) and used the results to achieve a relativistic representation of the neutron synthesis.

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The indicated irregular representations of the $SU(2)$-spin symmetry were then instrumental for the apparent confirmations of the EPR argument in Refs.[10] [11] reviewed in this section.

3.4.2. Non-relativistic formulation. The first irregular isotopies of Pauli's matrices, today known as *irregular Pauli-Santilli isomatrices* [45], have been introduced in Eqs. (2.32) of Ref. [78] via the use of the isorepresentations of $\hat{SU}(2)$ worked out in the preceding papers [55] [56], and are given by

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & -n_1 \\ n_2 & 0 \end{pmatrix}, \ \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_1 \\ in_2 & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = \tfrac{1}{n)1n_2} \begin{pmatrix} n_1^2 & 0 \\ 0 & -n_2^2 \end{pmatrix}, \tag{98}$$

with irregular isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = 2i \frac{1}{n_1 n_2} \epsilon_{ijk} \hat{\sigma}_k, \ \ i, j, k, = 1, 2, 3 \tag{99}$$

and isoeigenvalues

$$\hat{\sigma}_3 \star |\hat{u} >= \pm \tfrac{1}{n_1 n_2} |\hat{u} >, \\ \hat{\sigma}^{\hat{2}} \star |\hat{u} >= \tfrac{1}{n_1 n_2} (\tfrac{1}{n_1 n_2} + 2) |\hat{u} > . \tag{100}$$

It is easy to see that, when the hyperdense medium surrounding the immersed particle is homogeneous and isotropic, the characteristics quantities can be normalized to the values $n_1 = n_2 = n_3 = 1$, in which case isoeigenvalues (100) are conventional. We therefore have the following

LEMMA 3.1: Irregular isorepresentations of the Lie-Santilli isosymmetry $\hat{su}(2)$ represent the inhomogeneity and anisotropy of media in which extended particles are immersed.

Among a number of additional irregular Pauli-Santilli isomatrices with isotopic element $\hat{T}$ in Eq. (64) we quote Eqs. (3.2) of Ref. [10]

$$\hat{\sigma}_1 = n_1 n_2 \begin{pmatrix} 0 & n_1^{-2} \\ n_2^{-2} & 0 \end{pmatrix}, \ \ \hat{\sigma}_2 = n_1 n_2 \begin{pmatrix} 0 & -in_1^{-2} \\ in_2^{-2} & 0 \end{pmatrix}, \\ \hat{\sigma}_3 = n_1 n_2 \begin{pmatrix} n_2^{-2} & 0 \\ 0 & -n_1^{-2} \end{pmatrix}, \tag{101}$$

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with irregular isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = 2i \frac{1}{n_1 n_2} \hat{\sigma}_k, \tag{102}$$

and isoeigenvalues

$$\hat{\sigma}_3 \star |\hat{u}> = \pm \tfrac{1}{n_1 n_2} |\hat{u}>,$$
$$\hat{\sigma}^{\hat{2}} \star |\hat{u}> = 3 \tfrac{1}{n_1^2 n_2^2} |\hat{u}>, \tag{103}$$

The above isorepresentation appears to be significant when the medium causes a proportional alteration/mutation of both the third component as well as the total value of the spin of a particle having the value $1/2$ in vacuum.

Another example of irregular Pauli-Santilli isomatrices is given by Eqs. (3.7) of Ref. [10]

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & n_2 \\ n_1 & 0 \end{pmatrix}, \ \hat{\sigma}_2 = \begin{pmatrix} 0 & -in_2 \\ in_1 & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = \begin{pmatrix} n_1^2 & \\ 0 & -n_2^2 \end{pmatrix}, \tag{104}$$

with irregular isocommutation rules

$$[\hat{\sigma}_1 \hat{,} \hat{\sigma}_2] = 2i \tfrac{1}{n_1^2 n_2^2} \hat{\sigma}_3, \quad [\hat{\sigma}_2 \hat{,} \hat{\sigma}_3] = 2i \hat{\sigma}_1,$$
$$[\hat{\sigma}_3 \hat{,} \hat{\sigma}_1] = 2i \hat{\sigma}_2, \tag{105}$$

and mutated isoeigenvalues

$$\hat{\sigma}_3 \star |\hat{u}> = \pm |\hat{u}>,$$
$$\hat{\sigma}^{\hat{2}} \star |\hat{u}> = \tfrac{2}{n_1^2 n_2^2} |\hat{u}> . \tag{106}$$

The above isorepresentation may be useful when the anisotropy and inhomogeneity of the medium maintain the spin value $1/2$ along the third axis, yet they are such to deform the remaining components.

Additional example of irregular Pauli-Santilli isomatrices are available from Refs. [10] and [58], and can be readily constructed by interested readers.

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3.4.3. Relativistic formulation. Consider the iso-Minkowskian isospace $\hat{M}(\hat{x}, \bar{\Omega}, \hat{I})$ with isometric

$$\hat{\Omega} = \hat{\eta}\hat{I}, \quad \bar{\eta} = \hat{T}\eta,$$

$$\hat{T} = Diag.(\tfrac{1}{m_1^2}, \tfrac{1}{m_2^2}, \tfrac{1}{m_3^2}, \tfrac{1}{m_4^2}), \tag{107}$$

$$m_\mu = m_\mu(r, p, E, \nu, \alpha, \tau, \pi, \psi, ...) > 0. \;\; \mu = 1, 2, 3, 4,$$

where the new characteristic quantities m_μ have been introduced to avoid confusion with the previously used symbols n_μ.

The relativistic formulation of the irregular isorepresentation of $\hat{SU}(2)$ were derived, apparently for the first time, in Eqs. (6.4c)-(6.4d) of Ref. [62], and can be written

$$J_k = \frac{1}{2}\epsilon_{kij}\hat{\gamma}_i \star \gamma_j, \tag{108}$$

where $\hat{\gamma}$ are the regular Dirac-Santilli isomatrices (I-89), i.e.,

$$\hat{\gamma}_k = \tfrac{1}{m_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix},$$

$$\hat{\gamma}_4 = \tfrac{i}{m_4}\begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{109}$$

and $\hat{\sigma}_k$ are the *regular Pauli-Santilli isomatrices.*

The irregular character of isorepresentation (108) is established by the presence of structure functions in the isocommutation rules, Eqs, (6.4c) of Ref. [62],

$$[J_i \hat{,} J_j] = \epsilon_{ijk}\frac{1}{m_k^2}J_k, \tag{110}$$

and in the irregular isoeigenvalues

$$J_3 \star |\hat{\psi}> = \pm\tfrac{1}{2}\tfrac{1}{m_1 m_2}|\hat{\psi}>,$$

$$J^{\hat{2}} \star |\hat{\psi}> = \tfrac{1}{4}(\tfrac{1}{m_1 m_2} + \tfrac{1}{m_2 m_3} + \tfrac{1}{m_3 m_1})|\hat{\psi}>, \tag{111}$$

that, as shown in Ref. [62], permit a relativistic representation of the spin of the neutron in its synthesis from the hydrogen.

Again one should note that, when the medium is homogeneous and isotropic, isoeigenvalues (101) are conventional.

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Note that the assumption of mutated spin for an extended particle within a hyperdense medium implies the *inapplicability (rather than the violation) of the Fermi-Dirac statistics, Pauli's exclusion principle and other quantum mechanical laws* with the understanding that said mutations are internal, thus solely testable under *external* strong interactions, as indicated beginning with the *title* of Harvard's 1978 paper [15].

3.5. Isotopies of hadronic spin and angular momentum.

3.5.1. Historical notes. An electron orbiting in vacuum around the proton in the hydrogen atom experiences no resistive forces, thus verifying known symmetries and conservation laws.

When the same electron has been "compressed" inside the proton according to Rutherford [73], Santilli [78] argued that the sole possible angular moment is that permitted by *constraints* exercised on the electron by the internal medium.

Since the electron is about $2,000$ times lighter than the proton, *the most stable configuration is that for which the electron is "constrained" to orbit with a value of the angular momentum equal to the proton spin,* since any different configuration would imply big resistive forces (Figure 9).

Needless to say, fractional angular momenta are anathema for the quantum mechanical description of point-particles in vacuum.

However, the angular momentum of extended particles immersed within hyperdense hadronic media can acquire values other than integers, depending on the local physical conditions of the medium surrounding the particle, such as pressure, density, anisotropy, inhomogeneity, etc.

The first known quantitative study of *constrained angular momenta* of extended particles within hyperdense hadronic media was done at the non-relativistic level by Santilli in Ref. [78] of 1990 following the preceding isotopies of the rotational symmetry, Refs. [55] [56]. The study was then extended to the relativistic level in Ref. [62] of 1990.

These studies are crucial for quantitative representations of the synthesis of hadrons providing apparent verifications of the EPR argument, and can be summarized as follows.

3.5.2. Non-relativistic representation. Recall the central assumption of isosymmetries according to which conventional generators are preserved (because representing conventional total conservation laws), and only their product is lifted into the isotopic form (1) (to represent the extended character of the particles and their non-Hamiltonian interactions).

Hence, the definition of the *isoangular isomomentum,* also called *hadronic angular momentum,* on an iso-Euclidean isospace is the same as that of

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quantum mechanics

$$L_k = \epsilon_{ijk}\hat{r}_i \star \hat{p}_j = \epsilon_{ijk} r_i p_j, \tag{112}$$

although it is defined on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ with isostates $|\hat{\psi}>$ on an isocomplex isofield $\hat{\mathcal{C}}$, with *isolinear isomomentum* Eqs, (I-79), and isocommutation rules are then given by Eqs. (I-81).

It is then easy to verify the following isocommutation rules for the hadronic angular isomomentum, Eqs. (2.22b) [78]

$$[L_i \hat{,} L_j] = i\hat{I}\epsilon_{ijk}L_k, \tag{113}$$

where, as one can see, the characteristics of the medium, represented by the isounit $\hat{I}$, enter directly in the isocommutation rules.

The use of the *isosperical isoharmonic isofunctions* (see page 240 of Ref. [30] for details)

$$\begin{aligned} \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = UY(\theta, \phi)U^\dagger = \hat{T}^{-1}Y_{\ell m}(\theta, \phi), \\ UU^\dagger = \hat{I} = 1/\hat{T} \neq I, \end{aligned} \tag{114}$$

where $Y_{\ell m}(\theta, \phi$ are the conventional spherical harmonic functions, yields the following isoeigenvalues, Eqs. (2.25), Ref. [78],

$$\begin{aligned} L_3 \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I}m\hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ L^{\hat{2}} \star \hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}) = \hat{I}\ell(\hat{I}\ell + 1)\hat{Y}_{\ell m}(\hat{\theta}, \hat{\phi}), \\ m = \ell, \ell - 1, ..., -\ell, \;\; m = 1, 2, 3, ... \end{aligned} \tag{115}$$

where one can see again the mutation of the eigenvalues caused by the surrounding medium.

Applications to particle physics then require specific realizations of the isounit $\hat{I}$, such as the simple assumption of expressions (4) used in Ref. [78]

$$\rho = |\hat{I}| =\approx |e^\gamma|, \tag{116}$$

where ρ is a function of all possible or otherwise needed local variables of the medium.

3.5.3. Isotopies of non-relativistic spin-orbit coupling. As one can see, isoeigenvalues (115) do not allow a representation of the constrained hadronic angular momentum of the electron when compressed inside the proton (Figure 9).

In view of this insufficiency, Santilli conducted in Ref. [76] (see Also Ref. [30], Chapter 6) a study of the eigenvalues of the combined spin and angular momentum of the electron in the indicated interior conditions.

We consider then the *total hadronic momentum*

$$J_{tot} = L_\ell \hat{\otimes} J_s, \tag{117}$$

with corresponding basis $|\hat{Y} \hat{\otimes} \hat{u}>$ and isoexpectation values, Eqs. (2.34), Ref. [78],

$$\begin{gathered} J_{3,tot}|\hat{Y}\hat{\otimes}\hat{u}> = (\rho_m(\ell) \pm \tfrac{m(s)}{n_1 n_2})|\hat{Y}\hat{\otimes}\hat{u}> \\ J^{\hat{2}}_{tot} \star |\hat{Y}\hat{\otimes}\hat{u}> = (\rho\ell \pm \tfrac{s}{n_1 n_2})(\rho\ell \pm \tfrac{s}{n_1 n_2} + 1)|\hat{Y}\hat{\otimes}\hat{u}> \\ \ell = 0, 1, 2, 3, ... \;\; s = 0, \tfrac{1}{2}1, \tfrac{3}{2}, ..., \\ m(\ell) = \ell, \ell - 1, ..., -\ell, \;\; m(s) = s, s-1, ..., -s. \end{gathered} \tag{118}$$

Following a laborious journey initiated in 1977, isoeigenvalues (118) finally permitted Santilli to achieve the desired solution for $\ell = 1$ and $s = \frac{1}{2}$, Eq. (2.36), Ref. [78],

$$\rho = \frac{1}{2} \frac{1}{n_1 n_2}, \tag{119}$$

for which *the total hadronic angular momentum of the electron in the synthesis of the neutron is identically null,* $J_{tot} = 0$, *and the spin of the neutron coincides with that of the proton.*

More detailed studies pertaining to electric and magnetic dipoles excluded the alternative $J = 1$ of eigenvalues (118), as well as total hadronic angular momenta of the electron other than zero.

The preceding studies permitted a quantitative non-relativistic representation of the spin of the neutron in its synthesis from the hydrogen atom. A representation of the remaining characteristics of the neutron (mass, radius, charge, dipole moments, etc.) is reviewed in Section 4.5.

3.5.4. Isotopies of relativistic spin-orbit couplings. The hadronic spin $\hat{S} = S\hat{I}$ is a realization of the $\hat{SU}(2)$ isosubalgebra of $\hat{\mathcal{P}}(3.1)$ with generators (57), while the hadronic angular momentum $\hat{L} = L\hat{I}$ is a realization of the isorotational $\hat{SO}(3)$ isosubalgebra. Their relativistic formulation on iso-Minkowskian isospace (107) has been first derived in Eqs. (6.4a) (6.4b), Ref. [62] and are given by

$$\begin{gathered} S_k = 2\epsilon_{kij}\hat{\gamma}_i \star \hat{\gamma}_j, \\ L_k = \epsilon_{kij} r_i \star p_j, \end{gathered} \tag{120}$$

where $\hat{\gamma}_k$ are the Dirac-Santilli isomatrices.

We then have the irregularisocommutation rules

$$[S_i \hat{,} S_j] = \epsilon_{kij} m_k^2 \hat{S}_k,$$
$$[L_i \hat{,} L_j] = \epsilon_{ijk} m_k^2 L_k, \tag{121}$$

and isoeigenvalues, Eqs, (6.4d) Ref. [62]

$$\hat{S}_3 \star |\hat{\psi}>= \pm \tfrac{1}{m_1 m_2} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}} \star |\hat{\psi}>= (m_1^{-2} m_2^{-2} + m_2^{-2} m_3^{-2} + m_3^{-2} m_1^{-2}) |\hat{\psi}>,$$
$$\hat{L}_3 \star |\hat{\psi}>= \pm m_1 m_2 |\hat{\psi}>, \tag{122}$$
$$\hat{L}^{\hat{2}} \star |\hat{\psi}>= (m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2) |\hat{\psi}> .$$

The most salient difference between relativistic isoeigenvalues (122) and their non-relativistic counterparts (155) is that *the former admit fractional hadronic angular momenta while the latter do not.*

In fact, for the following values admitted by a homogeneous and isotropic medium [62]

$$m_1 = m_2 = m_3 = \frac{1}{\sqrt{2}}, \tag{123}$$

isoeigenvalues (122) become

$$\hat{S}_3 \star |\hat{\psi}>= \pm \tfrac{1}{2} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}} \star |\hat{\psi}>= \tfrac{3}{4} |\hat{\psi}>,$$
$$\hat{L}_3 \star |\hat{\psi}>= \pm \tfrac{1}{2} |\hat{\psi}>, \tag{124}$$
$$\hat{L}^{\hat{2}} \star |\hat{\psi}>= \tfrac{3}{4} |\hat{\psi}> .$$

Consequently, *isoeigenvalues (122) permit a quantitative representation of the hadronic angular momentum of the electron as being constrained to be equal to the proton spin* [61] [62] (Figure zzzz).

In this case too, the total hadronic angular momentum of the electron is null because the only stable hadronic spin-orbit coupling is in singlet, and the spin of the electron can be assumed in first good approximation not to be mutated since the electron is about $2,000$-times lighter than the proton.

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Hadronic spins, hadronic angular momenta and hadronic spin-orbit couplings were studied in detail Chapter 6 of Ref. [30] resulting in Lemma 6.12.1 here reproduced without proof:

LEMMA 3.2: When immersed within hadrons or nuclei with spin $1/2$*, an elementary particle having spin* $1/2$ *in vacuum can only have a null total hadronic angular momentum.*

As we shall see in Section 4.6, the above configuration of the synthesis of the neutron from the hydrogen is an apparent verification of the EPR argument.

3.6. Realization of hidden variables.

As recalled in Section 1.1, the conventional quantum mechanical realization of the Lie symmetry $SU(2)$ does not allow a consistent representation of hidden variables λ [3] [4].

It is easy to see that, despite the local isomorphism $\hat{SU}(2) \approx SU(2)$, the Lie-Santilli isosymmetry $\hat{SU}(2)$ does indeed allow explicit and concrete realizations of hidden variables thanks to the degree of freedom permitted by the isotopic element (1) in the structure of the Lie-Santilli isoproduct (2) with realizations of the isotopic element of type (3).

In this section, we review the explicit and concrete realization of *regular hidden variables,* namely, realizations that can be derived via non-unitary transforms of the Lie algebra $su(2)$, and then review *irregular hidden variables,* namely, realizations that do not admit such a simple derivation.

Regular and irregular realizations of hidden variables have been first identified by Santilli in Ref. [58] of 1993, and then used for the proof of the EPR argument [10] reviewed in Section 3.7.

Realizations of regular hidden variables are easily provided by Pauli-Santilli isomatrices (83) with the identifications

$$n_1^2 = \lambda_1, \;\; n_2^2 = \lambda_2, \tag{125}$$

yielding the desired realization, Eqs. (3.9), Ref. [58],

$$\hat{\sigma}_1 = (\lambda_1\lambda_2)\begin{pmatrix} 0 & \lambda_1^{-1} \\ \lambda_2^{-1} & 0 \end{pmatrix}, \;\; \hat{\sigma}_2 = (\lambda_1\lambda_2)\begin{pmatrix} 0 & -i\lambda_1^{-1} \\ i\lambda_2^{-1} & 0 \end{pmatrix},$$
$$\hat{\sigma}_3 = (\lambda_1\lambda_2)\begin{pmatrix} \lambda_2^{-1} & 0 \\ 0 & -\lambda_1^{-1} \end{pmatrix} \tag{126}$$

verifying isocommutation rules

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = i\epsilon_{ijk}\hat{\sigma}_k, \tag{127}$$

and isoeigenvalue isoequations

$$\hat{\sigma}_3 \star |\hat{b}>= \pm(\lambda_1\lambda_2)|\hat{b}>$$
$$\hat{\sigma}^{\hat{2}} = 3(\lambda_1\lambda_2)^2|\hat{b}> . \tag{128}$$

We consider now the particular case of Eq. (3), i.e.,

$$Det.\hat{T} = 1, \;\; n_1^2 = 1/n_2^2 = \lambda, \tag{129}$$

derivable via the basic non-unitary transformation

$$\hat{T} = (UU^\dagger)^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}. \tag{130}$$

In this case, isomatrices (83) become (Eqs. (3.9) of [58])

$$\hat{\sigma}_1(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \;\; \hat{\sigma}_2(\lambda) = \begin{pmatrix} 0 & -i\lambda^{-1} \\ i\lambda & 0 \end{pmatrix},$$
$$\hat{\sigma}_3(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^{-1} \end{pmatrix}. \tag{131}$$

It is an instructive exercise for the interested reader to verify that the above realization of the regular Pauli-Santilli isomatrices verifies isocommutation rules with the same stricture constants of the $SU(2)$ algebra

$$[\hat{\sigma}_i(\lambda)\hat{,}\hat{\sigma}_j(\lambda)] = i2\epsilon_{ijk}\hat{\sigma}_k(\lambda), \tag{132}$$

and admit conventional eigenvalues

$$\hat{\sigma}_3(\lambda) \star |\hat{b}>= \pm|\hat{b}>$$
$$\hat{\sigma}(\lambda)^{\hat{2}} = 3|\hat{b}> . \tag{133}$$

Consequently, we have the following property [58]:

LEMMA 3.3. Regular Pauli-Santilli isomatrices provide an explicit and concrete realization of regular hidden variables directly in the spin $1/2$ algebra.

Note that, besides being positive-definite, hidden variables have an unrestricted functional dependence on all needed local variables, Eqs. (66).

An example of irregular hidden variables is provided by the $\hat{SU}(2)$ component of the spinorial covering of the Lorentz-Poincaré-Santilli isosymmetry.

To illustrate this realization, introduce *three* additional hidden variables for the characterization of isospace (107)

$$m_\mu = \lambda_\mu, \quad \mu = 1, 2, 3, 4. \tag{134}$$

Realization (108) then implies the following irregular Dirac-Santilli isomatrices

$$\hat{\gamma}_k(\lambda) = \frac{1}{\gamma_k} \begin{pmatrix} 0 & \hat{\sigma}_k(\lambda) \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \qquad \hat{\gamma}_4 = \frac{i}{m_4} \begin{pmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{pmatrix}, \tag{135}$$

where $\hat{\sigma}_k$ are the *regular or irregular Pauli-Santilli isomatrices,* with isocommutation rules

$$[S_i(\lambda)\hat{,}S_j(\lambda)] = \epsilon_{ijk} \frac{1}{\lambda_k} S_k, \tag{136}$$

and isoeigenvalues

$$S_3 \star |\hat{\psi}>= \pm\frac{1}{2}\frac{1}{\sqrt{\lambda_1\lambda_2}}|\hat{\psi}>, \qquad S^{\hat{2}} \star |\hat{\psi}>= \frac{1}{4}\left(\frac{1}{\sqrt{\lambda_1\lambda_2}} + \frac{1}{\sqrt{\lambda_2\lambda_3}} + \frac{1}{\sqrt{\lambda_3\lambda_1}}\right)|\hat{\psi}> . \tag{137}$$

Consequently, we have the following property [62]

LEMMA 3.4: The axioms of Dirac's equation admit up to five generally different, regular or irregular hidden variables.

Additional realizations of irregular hidden variables can be found in Eqs. (3.11) of Ref. [58] or can be easily derived from the preceding realization of the Pauli-Santilli isomatrices.

3.7. Apparent admission of classical counterparts.

As it is well known, Bell's inequality [3] [4], von Neumann's theorem [5], and the theory of local realism at large (see review [6] with a comprehensive literature) are generally assumed to be evidence of the impossibility of "completing" quantum mechanics into a broader theory, with ensuing rejection of the EPR argument [1].

Following decades of preparatory works reviewed in Paper I [9] and in the preceding sections of this paper, Santilli proved in Ref. [10] of 1998 (see also the detailed study in Ref. [30], particularly Chapter 4 and Appendix 4C, page 166) that:

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1) Bell's inequality, von Neumann's theorem and related studies are indeed valid, but under the *tacit* assumption of representing particles as being point-like, with ensuing sole admission of linear, local and potential interactions (exterior dynamical problems).

2) Bell's inequality, von Neumann's theorem and related studies are *inapplicable* (rather than being violated) for extended particles within physical media, due to the presence of additional non-liner, non-local and non-potential interactions (interior dynamical systems).

3) The latter systems represented with the axiom-preserving "completion" of 20th century applied mathematics into isomathematics and the ensuing "completion" of quantum mechanics into hadronic mechanics [29]-[31] verify Statement 2 and admit well defined classical counterparts.

To review the preceding advances, consider two quantum mechanical particles with spin $1/2$ denoted 1 and 2 which verify the $SU(2)$ spin symmetry.

Assume that, as a result of some interaction, the two particles have antiparallel spins represented in the Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$. The total state in $ca\hat{l}H$ is then given by

$$|S_{1-2}>= \frac{1}{\sqrt{2}}(|S_{1\uparrow}> \times |S_{2\downarrow}> - |S_{1\downarrow}> \times |S_{2\uparrow}>), \tag{138}$$

with conventional l normalization

$$< S_{1-2}|S_{1-2}>= 1, \tag{139}$$

where $\times$ is the conventional associative product.

Let a_1, b_1 and a_2, b_2 be unit vectors along the z-axis of a conventional Euclidean space $E(r, \delta, I)$ for particle 1 and 2, respectively. Introduce the quantum mechanical probability

$$P(a_1, b_1) =< S_{1-2}|(\sigma_1 \otimes a_1) \times (\sigma_2 \otimes b_1)|S_{1-2}>= -a_1 \otimes b_1, \tag{140}$$

where $\otimes$ is the conventional scalar product.

Then, Bell's inequality can be written [4] (see Ref. [6] for numerous equivalent formulations)

$$D^{QM}_{Bell} = Max|P(a_1, b_1) - P(a_1, b_2) + P(a_2, b_1) + P(a_2, b_2)| \leq 2, \tag{141}$$

and implies the following property:

LEMMA 3.5: Particles in vacuum verifying the Lie symmetry $SU(2)$ admit no classical counterparts.

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PROOF: The classical counterpart of Bell's inequality is given by

$$D_{Max}^{Classical} = Max|a_1 \otimes b_1 - a_1 \otimes b_2| + |a_2 \otimes b_1 + a_2 \otimes b_2| = 2\sqrt{2}. \qquad (142)$$

But the quantum mechanical value of D_{Bell}^{QM} is *always* smaller than its classical counterpart $D_{Max}^{Classical}$,

$$D_{Bell}^{QM} < D_{Max}^{Classical}, \qquad (143)$$

by therefore establishing the impossibility for an $SU(2)$-invariant system to admit identical classical images. Q. E. D.

Santilli [10] has shown that inequality (141) is inapplicable for the same particles when they are in interior dynamical conditions, e.g., when they are in the core of a star, or at the limit, when they are in the interior of a gravitational collapse.

Considers two extended particles also denoted 1 and 2. Suppose that said particles verify the regular $\hat{SU}(2)$ isosymmetry with spin $1/2$ (Section 3.3), thus implying the elaboration via isomathematics (Section I-3) and the verification of the isotopic branch of hadronic mechanics (Section I-4).

Suppose that the two extended particles with spin $1/2$ are characterized by the following isotopic elements:

$$\begin{aligned} &Particle\ 1:\ \hat{T}_1 = Diag(\lambda_1, 1/\lambda_1), \\ &Particle\ 2:\ \hat{T}_2 = Diag(\lambda_2, 1/\lambda_2), \end{aligned} \qquad (144)$$

with realization (83) of the Pauli-Santilli isomatrices.

Suppose that, due to preceding interactions, the two extended particles are in single overlapping/entanglement thus having opposite spins.

Let $\hat{I}_1$ and $\hat{I}_2$ be the isounits for particles 1 and 2, respectively. The systems of the assumed two isoparticles is then characterized by the total isounit

$$\hat{I}_{tot} = \hat{I}_1 \times \hat{I}_2 = \frac{1}{\hat{T}_{tot}} = \frac{1}{\hat{T}_1 \times \hat{T}_2}. \qquad (145)$$

In this case, the total isostate on the Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ [19] over the isofield of isocomplex isonumbers $ca\hat{l}C$ [20] is given by

$$|\hat{S}_{1-2}> = \frac{1}{\sqrt{2}}(|\hat{S}_{1\uparrow}> \hat{\times}|\hat{S}_{2\downarrow}> - |\hat{S}_{1\downarrow}> \times|\hat{S}_{2\uparrow}>). \qquad (146)$$

The lack of validity of inequality (141) for irregular isorepresentations of $\hat{SU}(2)$ is evident (e.g., because of the anomalous spin isoeigenvalues) and, as such, it is ignored.

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A significant aspect of Ref. [10] is the proof of the inapplicability of inequality (141), not only for regular isorepresentation of $\hat{SU}(2)$, but also when such isorepresentations are isounimodular, Eqs. (144).

Let a_1, b_1, a_2, b_2 be unit vectors along the z-axis of an iso-Euclidean isospace. Introduce the isoprobability (Eq. (32.39), page 99, Ref. [30])

$$\hat{P}(a,b) =< \hat{S}_{1-2}| \star (\hat{\Sigma}_1 \hat{\otimes}_1 a) \times (\hat{\Sigma}_2 \hat{\otimes}_2 b)|\hat{S}_{1-2} > \hat{I}_{tot} = \\ =< \hat{S}_{1-2}| \star (\hat{\sigma}_1 \otimes a) \times (\hat{\sigma}_2 \otimes b)|\hat{S}_{1-2} > \hat{I}_{tot}, \tag{147}$$

with isonormalization (here referred to individual diagonal elements of isotopic elements and isounits)

$$< \hat{S}_{1-2}| \star |\hat{S}_{1-2} >=< \hat{S}_{1-2}|\hat{T}_{tot}|\hat{S}_{1-2} >= \hat{I}_{tot} \tag{148}$$

where: $\star$ is the total isoproduct; $\hat{\otimes}_k,\ k = 1,2,$ is the isoscalar isoproduct; and we have used simplifications of the type

$$\hat{\Sigma}_1 \hat{\otimes}_1 a = (\hat{\sigma}_1 \hat{I}_1)(\hat{T}_1 \otimes) a = \hat{\sigma}_1 \otimes a. \tag{149}$$

An isotopy of the conventional case yields the following isobasis, Eq. (6.5) of Ref. [10],

$$|S_{1-2} >= \frac{1}{2}\left\{ \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \begin{pmatrix} o \\ \lambda_2^{1/2} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda_2^{1/2} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1/2} \\ 0 \end{pmatrix} \right\}. \tag{150}$$

The appropriate use of products and isoproducts then yield expression (5.6) Ref. [10], i.e.,

$$< \hat{S}_{1-2}|\hat{T}_{tot}(\hat{\sigma}_1 \otimes_1 a) \times (\hat{\sigma}_2 \otimes_2 b)\hat{T}_{tot}|\hat{S}_{1-2} >= \\ = -a_x b_x - a_y b_y - \tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1}\lambda_2) a_z b_z. \tag{151}$$

The continuation of the isotopy of the conventional case, yields the main result, Eq. (5.8) of Ref. [10], which provides the following *isotopic "completion" of Bell's inequality,*

$$\hat{D}^{HM}_{Max} = D^{HM}_{Max} \hat{I}_{tot} = \\ Max|\hat{P}(a_1,b_1) - \hat{P}(a_1,b_2) + \hat{P}(a_2,b_1) + \hat{P}(a_2,b_2)| = \\ = [\tfrac{1}{2}(\lambda_1 \lambda_2^{-1} + \lambda_1^{-1}\lambda_2) D^{QM}_{Bell}\, \hat{I}_{tot}, \tag{152}$$

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with consequential:

LEMMA 3.6. Extended particles within physical media that are invariant under the Lie-Santilli isosymmetry $\hat{SU}(2)$ admit identical classical counterparts.

PROOF: Isoinequality (141) establishes the lack of universal validity of Bell's inequality (128) because the factor $\frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2)$ can have values *bigger* than one, thus implying

$$D^{HM}_{Max} \geq D^{QM}_{Bll}. \tag{153}$$

Consider then a classical iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ representing motion of classical extended particles 1 and 2 within physical media [30] with isometric elements

$$\hat{\delta}_{11} = 1, \;\; \hat{\delta}_{22} = 1, \;\; \hat{\delta}_{33} = \frac{1}{2}(\lambda_1\lambda_2^{-1} + \lambda_1^{-1}\lambda_2) = 2, \tag{154}$$

in which case

$$D^{HM}_{Max} \equiv htD^{Classical}_{Max}, \tag{155}$$

by therefore establishing that systems of extended particles within physical media verifying the $\hat{SU}(2)$ isosymmetry admits an identical classical counterpart along the EPR argument. Q.E.D.

It is an instructive exercise for the interested reader to prove that the above lemma also holds for different isorenormalizations, e.g., Eqs. (171) of next section, with the understanding that different isorenormalizations imply different isobasis and different hidden variable terms in Eqs. (151).

Note the crucial role of hidden variables for the proof of Lemma 3.6. It is an instructive exercise for interested readers to prove that Lemma 3.6 holds for any other regular, isounimodular isorepresentation of the isotopic $\hat{SU}(2)$ symmetry in terms of hidden variables presented in Section 3.3.

The proof of the lack of applicability of von Neumann's theorem [5] for extended particles in interior conditions is elementary. Recall that von Neumann's theorem is based on the uniqueness of the eigenvalues E of a Hermitean operator $H, \; H|\psi >= E|\psi >$ under unitary transformation on $\mathcal{H}$,

$$UH|\psi > U^\dagger = UE|\psi > U^\dagger = EU|\psi > U^\dagger, \;\; UU^\dagger = U^\dagger U = I, \tag{156}$$

under the tacit assumption of point particles in vacuum.

By contrast, when the same particles is in interior conditions, it is subjected to an infinite number of different physical different interactions with

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the medium represented by the isotopic element $\hat{T}$ with ensuing isoeigenvalue equation (Section I-4), [9],

$$abel1H \star |\hat{\psi}_{\hat{T}}r >= HT|\hat{\psi}_{\hat{T}} >= E_{\hat{T}}|\hat{p}si_{\hat{T}} >, \tag{157}$$

thus establishing that a given quantum mechanical operator H representing the energy of an extended particle in interior conditions has an infinite number of *generally different* isoeigenvalues $E_{\hat{T}}$ depending on the infinite number of different interior conditions.

Note that, for each given $\hat{T}$ the isoeigenvalue $E_{\hat{T}}$ is invariant under isounitary isotransformations (Section I-3-9).

3.8. Apparent admission of classical determinism.

Consider a point-like particle in empty space represented in the 3-dimensional Euclidean space $E(r, \delta, I)$, where r represents coordinates, $\delta = Diag.$-$(1, 1, 1)$ represents the Euclidean metric and $I = Diag.(1, 1, 1,)$ represents the space unit.

Let the operator representation of said point-like particle be done in a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$ with states $\psi(r)$ and familiar normalization

$$< \psi(r)| \, |\psi(r) >= \int_{-\infty}^{+\infty} \psi(r)^\dagger \psi(r) dr = 1. \tag{158}$$

As it is well known, the primary objections against the EPR argument [2]- [6] were based on Heisenberg uncertainty principle according to which *the position r and the momentum p of said particle cannot both be measured exactly at the same time.*

By introducing the *standard deviations* Δr and Δp, the uncertainty principle is generally written in the form

$$\Delta r \Delta p \ \geq \ \frac{1}{2}\hbar, \tag{159}$$

which is easily derivable via the vacuum expectation value of the canonical commutation rule

$$\Delta r \Delta p \ \geq \ |\frac{1}{2i} < \psi| \, [r, p] \, |\psi > | = \frac{1}{2}\hbar. \tag{160}$$

Standard deviations have the known form (see, e.g., Ref. [79]) with $\hbar = 1$

$$\begin{aligned} \Delta r &= \sqrt{< \psi(r)|[\, r - (< \psi(r)| \, r \, |\psi(r) >)]^2 |\psi(r) >}, \\ \Delta p &= \sqrt{< \psi(p)| \, [p - (< \psi(p)| \, p \, |\psi(p) >)]^2 |\psi(p) >}, \end{aligned} \tag{161}$$

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where $\psi(r)$ and $\psi(p)$ are the wavefunctions in coordinate and momentum spaces, respectively.

We consider now an extended particle, this time, in interior conditions, e.g., in the core of a star, classically represented by the *iso-Euclidean isospace* $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ with isounit $\hat{I} = 1/\hat{T} > 0$, isocoordinates $\hat{r} = r\hat{I}$, isometric

$$\hat{\delta} = \hat{T}\delta, \tag{162}$$

and isotopic element (4)) under conditions (5).

For simplicity, we assume that the extended particle has no Hamiltonian interactions due to the dominance of the latter interactions over the former.

Consequently, we can represent the extended particle in the isospace $\hat{\mathcal{H}}$ over the isofield $\hat{\mathcal{C}}$ and introduce the time independent *isoplanewave* [18]

$$\begin{aligned} \hat{\psi}(\hat{r}) = \tilde{\psi}(\hat{r})\hat{I} = \\ = \hat{N} \star (\hat{e}^{\hat{i} \star \hat{k} \star \hat{r}})\hat{I} = N(e^{ik\hat{T}\hat{r}})\hat{I}, \end{aligned} \tag{163}$$

where $\hat{N} = N\hat{I}$ is an *isonormalization isoscalar,* $\hat{k} = k\hat{I}$ is the *isowavenumber,* and the isoexponentiation is given by Eq. (I-22) [26].

The corresponding representation in isomomentum isospace is given by

$$\tilde{\psi}(\hat{p}) = \hat{M} \star \hat{e}^{\hat{i} \star \hat{n} \star \hat{p}}, \tag{164}$$

where $\hat{M} = M\hat{I}$ is an isonormalization isoscalar and $\hat{n} = n\hat{I}$ is the isowavenumber in isomomentum isospace.

The *isopropability isofunction* is then given by (Ref. [30] page 99)

$$\hat{\mathcal{P}} = \hat{<} | \star | \hat{>} =< \hat{\psi}(\hat{r})| \, T \, |\hat{\psi}(\hat{r}) > I, \tag{165}$$

that, written in terms of isointegrals (Ref. [29] page 354), becomes

$$\begin{aligned} \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^\dagger \star \hat{\psi}(\hat{r}) \star \hat{d}\hat{r} = \\ = \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^\dagger \tilde{\psi}(\hat{r})(dr + r\hat{T}d\hat{I}), \end{aligned} \tag{166}$$

where one should keep in mind that the isodifferential $\hat{d}\hat{r}$ given by Eqs. (I-29).

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The *isoexpectation isovalues* of a Hermitean operator $\hat{Q}$ are then given by [30]

$$\hat{<}| \star \hat{Q} \star |\hat{>} =< \hat{\psi}(\hat{r})| \star \hat{Q} \star |\hat{\psi}(\hat{r}) >=$$
$$= \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \star \hat{Q} \star \hat{\psi}(\hat{r}) \hat{d}\hat{r} = \quad (167)$$
$$= \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^{\dagger} \hat{Q} \tilde{\psi}(\hat{r}) \hat{d}\hat{r},$$

with corresponding expressions for the isoexpectation isovalues in isomomentum isospace.

Santilli then introduced apparently for the first time in Ref. [11] the *isotopic operator*

$$\hat{\mathcal{T}} = \hat{T}\hat{I} = I, \quad (168)$$

that, despite its seemingly irrelevant value, is indeed the correct operator formulation of the isotopic element for the "completion" of the isoproduct from its scalar form (1) to the isoscalar form

$$\hat{n}^{\hat{2}} = \hat{n} \star \hat{n} = \hat{n} \star \hat{\mathcal{T}} \star \hat{n} = n^2 I. \quad (169)$$

In Sections 3.6, 3.7, we have shown that the Lie-Santilli isosymmetry $S\hat{U}(2)$ admits an explicit and concrete realization of hidden variables that allowed the construction of identical classical counterparts for interior dynamical systems.

Ref. [11] introduced the isoexpectation isovalue of the isotopic operator

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =< \hat{\psi}(\hat{r})| \star \hat{\mathcal{T}} \star |\hat{\psi}(\hat{r}) > \hat{I} =$$
$$= \int_{-\infty}^{+\infty} \tilde{\psi}(\hat{r})^{\dagger} \hat{T} \tilde{\psi}(\hat{r}) \hat{d}\hat{r}, \quad (170)$$

and assumed the isonormalization (again, intended for diagonal matrix elements)

$$\hat{<}| \star \hat{\mathcal{T}} \star |\hat{>} =$$
$$= \int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) \hat{d}\hat{r} = \hat{T}. \quad (171)$$

Consider then the *isostandard isodeviation* for isocoordinates $\Delta\hat{r} = \Delta r\hat{I}$ and isomomenta $\Delta\hat{p} = \Delta p\hat{I}$, where Δr and Δp are the standard deviations in our space.

By using isocanonical isocommutation rules (I-81), we obtain the expression

$$\Delta\hat{r} \star \Delta\hat{p} = \Delta r \Delta p \hat{I} \;\approx\; \tfrac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r},\hat{p}] \star \hat{\psi}(\hat{r}) > |\hat{I} =$$
$$= \tfrac{1}{2}| < \hat{\psi}(\hat{r})|\hat{T}\, [\hat{r},\hat{p}]\, \hat{T}|\hat{\psi}(\hat{r}) > \hat{I}, \quad (172)$$

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One should note the replacement of the symbol $\geq$ in Eq. (160) with the symbol $\approx$ in Eq. (172). This is due to the fact that the historical arguments applying for a point-like particle in vacuum no longer apply for an interior system because the pressure exercised by the medium on the particle (Figure 4) reduce the lower limit of Eq. (160) to the approximate value of Eq. (172).

Under the above assumptions, by eliminating the common isounit $\hat{I}$, Ref. [11] achieved the desired result here called *isodeterministic isoprinciple*

$$\Delta r \Delta p \approx \frac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r},\hat{p}] \star |\hat{\psi}(\hat{r}) >=$$
$$= \frac{1}{2}| < \hat{\psi}(\hat{r})|\hat{T}\,[\hat{r},\hat{p}]\,\hat{T}|\hat{\psi}(\hat{r}) >= \tag{173}$$
$$\frac{1}{2}\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger}\hat{T}\hat{\psi}(\hat{r})d\hat{r} = \frac{1}{2}T \ll 1$$

where the property $\Delta r \Delta p \ll 1$ follows from the fact that the isotopic element $\hat{T}$ has values smaller than 1 in the fitting of all experimental data dealing with hadronic media such as hadrons, nuclei and stars, and null value for gravitational collapse [31].

In the event Eq. (35), page 14 of Ref. [11] should be compatible with Eq. (173) above, it is sufficient to turn into a comma the sign $=$ in the right of the central expression of Eq. (35), or absorb the factor $1/2$ of Eq. (173) into the isorenormalization.

In this way, thanks to a laborious scientific journey initiated at Harvard University in late 1977, and thanks to contributions by numerous mathematicians, theoreticians and experimentalists, Santilli reached the following verification of the EPR argument [11]:

LEMMA 3.7 (ISODETERMINISTIC PRINCIPLE): The isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as their product, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei, and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

PROOF: Define the isostandard isodeviations via the following isotopy of quantum mechanical expressions (161) (where we ignore the common multiplication by the isounit)

$$\Delta r = \sqrt{< \hat{\psi}(\hat{r})|[\,\hat{r} - < \hat{\psi}(\hat{r})| \star \hat{r} \star |\hat{\psi}(\hat{r}) >]^{\hat{2}}|\hat{\psi}(\hat{r}) >},$$
$$\Delta p = \sqrt{< \hat{\psi}(\hat{p})|\,[\hat{p} - < \hat{\psi}(\hat{p})| \star \hat{p} \star |\hat{\psi}(\hat{p}) >]^{\hat{2}}|\hat{\psi}(\hat{p}) >}, \tag{174}$$

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where the differentiation between the isotopic elements for isocoordinates and isomomenta is ignored for simplicity. But the isotopic element represents the interactions of the particle with the physical medium and tends toward null values for gravitational collapse, Eqs. (I-91) (I-92). Therefore, isosquare in expression (171) implies the expressions

$$\Delta r = \sqrt{\hat{T} < \hat{\psi}(\hat{r})|[\ \hat{r} - < \hat{\psi}(\hat{r})| \star \hat{r} \star |\hat{\psi}(\hat{r}) >]^2 |\hat{\psi}(\hat{r}) >},$$
$$\Delta p = \sqrt{\hat{T} < \hat{\psi}(\hat{p})|\ [\hat{p} - < \hat{\psi}(\hat{p})| \star \hat{p} \star |\hat{\psi}(\hat{p}) >]^2 |\hat{\psi}(\hat{p}) >}, \tag{175}$$

that approach indeed null value under the indicated limit conditions of gravitational collapse

$$Lim_{\hat{T}=0} \Delta r = 0,$$
$$Lim_{\hat{T}=0} \Delta p = 0, \tag{176}$$

Q.E.D.

3.9. Apparent removal of quantum divergencies.

Recall from Section I-4.13 that, under condition (I-96), corresponding to condition (173), there is a rapid convergence of isoseries (I-97), as well as the removal of the singularity of Dirac's delta distribution, Eq. (I-98) (Figure I-14).

The above properties can be now formalized according to the following:

COROLLARY 3.7.1. Einstein's determinism according to Lemma 3.7 implies the removal of quantum mechanical divergencies.

PROOF. Lemma 3.7 is based on values of the isotopic element $\hat{T}$ being smaller than 1, which values imply in turn the rapid convergence of perturbative series without divergencies (Section I-4-13).
Q.E.D.

The removal of quantum divergencies, that have been cause of controversies for about one century, illustrates the far reaching implications of Einstein's determinism for interior dynamical systems.

3. CONCLUDING REMARKS.

Following the study of basic methods in Paper I, in this paper we have provided an apparent confirmation of proofs [10] [11] of the EPR argument [1] for extended, thus deformable particles within hyperdense media with

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ensuing linear and non-linear, local and non-local and potential as well as non-potential/non-Hamiltonian interactions.

This study has been conducted via the use of isomathematics and isomechanics admitting a conventional Hamiltonian H or Lagrangian L for the invariant representation of linear, local and potential interactions, plus the isotopic element $\hat{T}$ of isoproducts $A \star B = A\hat{T}B$, Eq. (1), for the invariant representation of non-linear, non-local and non-Hamiltonian interactions.

Following the outline and upgrade of isosymmetries for time-reversal invariant interior systems (Section 3), we have apparently confirmed the proof of Ref. [10] according to which extended particles in interior dynamical conditions admit identical classical counterpart.

We have then apparently confirmed the proof of Ref. [11] according to which extended particles progressively approach classical determinism when in the interior of hadrons, nuclei and stars, and achieve full determinism at the limit of gravitational collapse, essentially as predicted by A. Einstein, B. Podolsky and N. Rosen [1].

To illustrate the far reaching implications of what appears to be Einstein's most important legacy, we have shown for the first time that the recovering of Einstein's determinism for interior conditions appears to imply the removal of quantum divergencies due to the rapid convergence of the isoseries of hadronic mechanics, the removal of the singularity in Dirac's delta distribution and other features.

A number of illustrations and novel applications in mathematics, physics and chemistry are presented in the forthcoming Paper III.

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study of the most important vision by Albert Einstein.

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Studies on A. Einstein, B. Podolsky and N. Rosen argument that "quantum mechanics is not a complete theory," III: Illustrative examples and applications

Ruggero Maria Santilli*

Abstract

In the preceding Papers I and II of this series, we have presented a review and upgrade of novel mathematical, physical and chemical methods, and shown their use for a confirmation of the apparent proof of the EPR argument that extended particles within physical media admit classical counterparts, while Einstein's determinism appears to be progressively verified with the increase of the density of the medium. In this third paper, we have additionally shown, apparently for the first time, the validity of the EPR final statement to the effect that *the wavefunction [of quantum mechanics] does not provide a complete description of the physical reality.*" In fact, we have studied the axiom-preserving "completion" of the quantum mechanical wavefunction due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature.

Keywords: EPR argument, isomathematics, isomechanics.
2010 AMS subject classifications: 05C15, 05C60. [1]

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[1]Received 26th of April, 2020, accepted 19th of June, 2020, published 30th of June, 2020, doi 10.23755/rm.v38i0.517, ISSN 1592-7415; e-ISSN 2282-8214. ©Ruggero Maria Santilli

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1. INTRODUCTION.

1.1. The EPR argument.

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe."

Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the *EPR argument.*

Objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3] [4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature).

The field became known as *local realism* and included the dismissal of the EPR argument based on claims that quantum axioms do not admit *hidden variables* λ [7] [8].

1.2. Outline of preceding works.

Following various preparatory works, in Ref. [9] of 1998, R. M. Santilli:

1) Assumed the validity of quantum mechanics, with consequential validity of the objections against the EPR argument [2] - [6], for point-like particles in empty space under linear, local and potential interactions (*exterior dynamical problems*);

2) Proved the inapplicability (and not their violation) of said objections for the broader class of extended, deformable and hyperdense particles within physical media under the most general known linear and non-linear, local and non-local and potential as well as non-potential interactions (*interior dynamical problems*); and

3) Provided the apparent proof that *interior dynamical systems admit classical counterparts in full accordance with the EPR argument* via the representation of interior systems with of *isomathematics* also called *isotopoic branch of hadronic mathematics,* and *isomechanics,* also called *isotopic branch of hadronic mechanics*.

In the 2019 paper [10], Santilli provided the apparent proof that *Einstein's determinism is progressively approached in the interior of hadrons, nuclei and stars and it is fully achieved in the interior of gravitational collapse.*

In Ref. [11], herein referred to as Paper I, we provided a review of isomathematics and isomechanics.

In the subsequent Ref. [12], hereinafter referred as Paper II, we provided a review of isosymmetries, with particular reference to the isotopies of the spin and rotational symmetries, and provided an apparent confirmation of the proofs [9] and [10] of the EPR argument.

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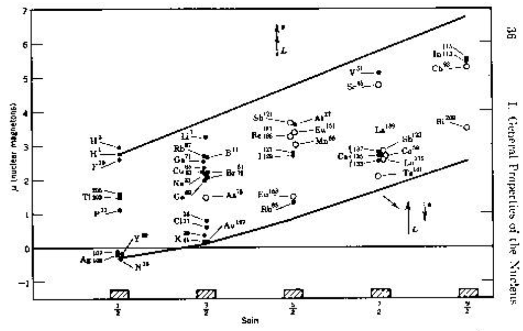

Figure 1: *In this figure, we present the so-called Schmidt limits essentially representing the deviations of experimental data on nuclear magnetic moments from the predictions of quantum mechanics. In Santilli's view [11], this occurrence is clear evidence on the need for a "completion" of quantum mechanics along Einstein's legacy [1], beginning with a "completion" of applied mathematics into a form suitable to represent extended and deformable nucleons according to the founders of nuclear physics, such as E. Fermi [13] and V. F. Weisskopf who states in page 31 of his treatise in nuclear physics with J. M. Blatt [14]: " it is possible that the intrinsic magnetism of a nucleon is different when it is in close proximity to another nucleon. " For penetrating appraisals of the insufficiencies of 20th century science, one should also see K. R. Popper [15]. J. Dunning-Davies [16], J. Horgan [17] and others. In this paper, we outline and update the achievement by hadronic mathematics and mechanics of exact representations of nuclear experimental data along the indicated historical legacies, and point out important implications for much needed new, clean energies and fuels (Sections 2.6.5, 2.7.3, 2.8.3).*

In this third and final paper of this series: we present apparent experimental verifications of the EPR argument in physics and chemistry; we outline expected industrial applications; and identify the most salient implications of the EPR argument.

1.3. Basic notions.

The basic assumptions underlying the apparent proofs of the EPR argument are the following::

1) The axiom-preserving lifting of the conventional associative product $ab = a \times b$ between *all* possible quantum mechanical quantities (numbers, functions, matrices, etc.) into the *isoproduct,* first introduced in the 1978 Harvard university paper [18] (see also Refs. [19] to [23])

$$a \star b = a\,\hat{T}\,b, \tag{1}$$

where $\hat{T}$, called the *isotopic element,* is restricted to be positive-definite,

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$\hat{T} > 0$, but possesses otherwise an unrestricted functional dependence on all needed local variables, including wavefunctions and their derivatives.

20th century applied mathematics and quantum mechanics are reformulated in an axiom-preserving, thus isotopic form, via isoproduct (1) (Paper I and Section II-2).

3) The axiom-preserving isotopy of the various branches of Lie's theory, first achieved by Santilli in Refs. [18] [23] and are today known as the *Lie-Santilli isotheory* [24] (Section I-3.7) with Lie-Santilli algebras of the type [9]

$$[X_i \hat{,} X_j] = X_i \star X_j - X_j \star X_i = C_{ij}^k X_k. \; i, j = 1, 2, ..., N. \tag{2}$$

and ensuing systematic isotopy of space-time and internal symmetries (Section II-2).

3) Immediate explicit and concrete realizations of "hidden variables" [7] [8] of the type

$$\hat{T} = Diag.(1/\lambda, \lambda), \;\; Det\hat{T} = 1. \tag{3}$$

Ref. [9], therefore establishing that, contrary to objections [2] to [6], *the abstract axioms of quantum mechanics do indeed admit explicit and concrete realizations of hidden variables.*

4) Representation of extended particles in conditions of mutual penetration via realizations of the isotopic element $\hat{T}$ of isoproduct (1) of the type [25]

$$\hat{T} = \Pi_{k=1,...,N} Diag.(\tfrac{1}{n_{1k}^2}, \tfrac{1}{n_{2k}^2}, \tfrac{1}{n_{3k}^2}, \tfrac{1}{n_{4k}^2}) e^{-\Gamma},$$
$$k = 1, 2, ..., N, \quad \mu = 1, 2, 3, 4, \tag{4}$$

where n_1^2, n_2^2, n_3^2, (called *characteristic quantities*) represent the deformable semi-axes of the particle normalized to the values $n_k^2 = 1, `k = 1, 2, 3$ for the sphere; n_4^2 represents the *density* of the particle considered normalized to the value $n_4 = 1$ for the vacuum; and Γ represents non-linear, non-local and non-Hamiltonian interactions caused by mutual penetrations/entanglement of particles.

In particular, the isotopic element $\hat{T}$ has resulted to have numeric values *smaller* than 1 in all known applications [23]-

$$|\, \hat{T} \,| \; \leq \; 1, \tag{5}$$

which property permitted Ref. [10] to show that *the standard deviations Δr and Δp progressively tend to zero with the increase of the density of the medium of interior problems.*

5) Isosymmetries do not preserve over time the basic unit 1 of conventional numeric fields $F(n, \times, 1)$ with consequential lack of experimental

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verifications. This occurrence mandated Santilli to formulate isomathematics on *isofields* $\hat{F}(\hat{n}, \star, \hat{I})$ [26] [27](Section I-3.3) with *isonumbers* $\hat{n} = n\hat{I}$ equipped with isoproduct (1) and basic *isounit*

$$\hat{I} = 1/\hat{T} > 0, \tag{6}$$

which remains numerically invariant under isosymmetries as necessary for consistency (Sections I-3.8, I-3.9 and II-2).

5) Recall that the Newton-Leibnitz differential calculus provides the ultimate characterization of the point-like character of particles admitted by quantum mechanics due to the known feature that said calculus can be solely defined as a finite number of isolated points.

This occurrence mandated Santilli to construct the covering of the Newton-Leibnitz differential calculus into the *isodifferential isocalculus* (Section I-3.6) with basic *isodifferential* [28] [29]

$$\hat{d}\hat{r} = \hat{T}d[r\hat{I}(r, ...)] = dr + r\hat{T}d\hat{I}(r, ...), \tag{7}$$

and corresponding *isoderivative*

$$\frac{\hat{\partial}\hat{f}(\hat{r})}{\hat{\partial}\hat{r}} = \hat{I}\frac{\partial\hat{f}(\hat{r})}{\partial\hat{r}}, \tag{8}$$

which are defined on *volumes* represented by the isotopic element $\hat{T}$, rather than points.

Recall that Bell's inequality [3] D^{qm}_{Bell} does not admit a classical counterpart D^{class} because always smaller than the classical counterpart,

$$D^{qm}_{Bell} < D^{class}. \tag{9}$$

In Ref. [9], Eqs. (5.8), page 189, proved that, thanks to the existence in hadronic mechanics of hidden variables of type (3), the corresponding inequality in hadronic mechanics (hm), D^{hm} is bigger than Bell's inequality, and always admits a classical counterpart

$$D^{hm} \equiv D^{class}, \tag{10}$$

resulting in the following:

LEMMA II-3.6 (EXISTENCE OF CLASSICAL COUNTERPART). Extended particles within physical media that are invariant under the Lie-Santilli isosymmetry $\hat{SU}(2)$ admit identical classical counterparts.

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The main result of Ref. [10], Eqs. (35), page 14,

$$\Delta r \Delta p \approx \frac{1}{2}| < \hat{\psi}(\hat{r})| \star [\hat{r} \hat{,} \hat{p}] \star |\hat{\psi}(\hat{r}) >=$$

$$= \frac{1}{2}| < \hat{\psi}(\hat{r})| \hat{T} \, [\hat{r} \hat{,} \hat{p}] \, \hat{T} |\hat{\psi}(\hat{r}) >= \tag{11}$$

$$\int_{-\infty}^{+\infty} \hat{\psi}(\hat{r})^{\dagger} \hat{T} \hat{\psi}(\hat{r}) d\hat{r} = T \ \ll \ 1,$$

verifying Einstein's determinism, was expressed with the following:

LEMMA II-3.7 (EINSTEIN DETERMINISM): The isostandard isodeviations for isocoordinates $\Delta\hat{r}$ and isomomenta $\Delta\hat{p}$, as well as their product, progressively approach classical determinism for extended particles in the interior of hadrons, nuclei and stars, and achieve classical determinism at the extreme densities in the interior of gravitational collapse.

By recalling that isoperturbative series of hadronic mechanics has based on isoproduct (1) (Section I-4),

$$A(0) = \hat{I} + (A\hat{T}H - H\hat{T}A)/1! + ... \tag{12}$$

and that $\hat{T} \ll 1$ from Lemma II-3.7, the following consequential property was introduced, apparently for the first time in Paper II:

COROLLARY 3.7.1 (LACK OF DIVERGENCIES) Einstein's determinism according to Lemma II-3.7 implies the lack of quantum mechanical divergencies in hadronic mechanics.

It should be indicated that the technical understanding of the verifications and applications presented in this paper require a knowledge of isomathematics and isomechanics.

2. VERIFICATIONS AND APPLICATIONS.

2.1. Foreword.

Being dimensionless, *Newtonian massive points cannot experience resistive or contact force of any type.* By recalling that Newton's equation have been the foundations of physics for the past four centuries, 20th century mainstream particle physics has been developed without the notion of resistive force, with ensuing lack of treatment of the *pressure* exercised by a medium on a particle in its interior, contrary to clear evidence that a proton in the core of a star is exposed to extremely big pressures (Figure 1).

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In this section, we shall illustrate the fact that, once admitted, the pressure exercised on extended particles characterizes in their interior characterizes standard deviations $\Delta\hat{r}$ and $\Delta\hat{p}$ that, being *constrained* by said pressure, verify the isodeterministic principle of Lemma II-3.7 as well as the rapid convergence of isoperturbative series according to Corollary II-3.7.1, by progressively approaching classical determinism with the increase of the pressure, up to the apparent achievement of classical determinism for the interior of gravitational collapse as predicted by Einstein, Podolsky and Rosen [1].

A physically important notion emerging from the examples provided below is that *the EPR argument appears to be verified by strong interactions* because, as indicated by Santilli in the 1978 paper [19], contact non-Hamiltonian interactions responsible for the synthesis of the neutron and other hadrons are short range, strongly attractive, charge independent and non-Hamiltonian (technically identified as *variationally non-selfadjoint interactions* [22]), thus providing a conceivable, first known, explicit and concrete representation of strong interactions.

The models outlined in this section were first proposed by Santilli in Ref. [19] in their time irreversible form, as requested for decaying bound states, thus being elaborated with Lie-admissible genomathematics, in which case, the need for a "completion" of quantum mechanics is beyond scientific doubt (Section I-1.3).

However, the objections against the EPR argument [2] - [6] have been formulated for conventional quantum axioms, thus implying the sole consideration of time-reversal invariant states. In this section, we illustrate the need for a "completion" of quantum mechanics also for time-reversal invariant systems of extended particles in interior conditions.

Therefore, unstable strongly interacting particles (hadrons) are hereon studied for such a small period of time to allow their time-reversal invariant approximation.

2.2. Particles under pressure.

One of the simplest illustrations of Lemma II-3.7 is given by a particle in the center of a star, thus being under extreme pressures π from the surrounding hadronic medium in all radial directions (Figure 1).

By ignoring particle reactions in first approximation, the conditions here considered can be rudimentarily represented for very short periods of time by assuming that the function $\Gamma > 0$ in the exponent of the isotopic element (4) is linearly dependent on the pressure π, resulting in a

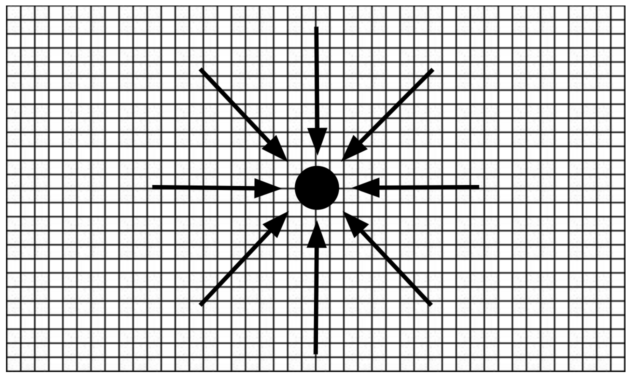

Figure 2: *In this figure, we present a conceptual rendering of the central notion used for the verification of the EPR argument, namely, the pressure experienced by extended particles immersed in hyperdense media, such as a proton in the core of a star, which pressure evidently restricts uncertainties in favor of Einstein's determinism (Lemma II-3.7). Note that the notion of pressure does not exist in 20th century physics due to the approximation of particles as being point-like.*

realization of the isotopic element of the simple type

$$\hat{T} = e^{-w\pi} \ll 1, \ \ \hat{I} = e^{+w\pi} \gg 1, \tag{13}$$

where w is a positive constant.

Isodeterministic principle (11) for the considered particle is then given by

$$\Delta r \Delta p \approx \frac{1}{2} e^{-w\pi} \ \ll \ 1, \tag{14}$$

and tends to null values for diverging pressures.

The above example illustrates the consistency of isorenormalization (II-171) because a constant isotopic element verifies the isonormalization

$$\begin{aligned} &\hat{<}\hat{\psi}(\hat{r})|\hat{T}|\hat{\psi}(\hat{r}) > \hat{I} = \\ &= T < \hat{\psi}(\hat{r})| \, |\hat{\psi}(\hat{r}) > \hat{I} = \\ &=< \hat{\psi}(\hat{r})| \, |\hat{\psi}(\hat{r}) >, \end{aligned} \tag{15}$$

but not necessarily other isorenormalizations.

Note that we have considered an individual extended particle immersed in a hadronic medium, rather than the bound state of extended particles in condition of mutual penetration that are studied in the next sections.

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Consequently, isotopic element (14) represents a *subsidiary constraint on standard deviations* caused, as indicated, by the pressure of the surrounding hadronic medium on the particle considered.

It is easy to see that, since $\Gamma(\pi) > 0$, more complex functional dependence on the pressure π continue to verify Lemma II-3.7 as well as Corollary II-3.7.1.

2.3. Non-relativistic hadronic bound states.

As recalled in Paper I, the dynamical equations of quantum mechanics, such as the Schrödinger and the Dirac equation, are characterized by the conventional differential calculus which can solely be defined at a finite set of isolated points on a given representation space. Consequently, quantum mechanical bound states are solely possible for *point-like constituents* under linear, local and potential interactions (technically identified as *variationally self-adjoint interactions* [22]). This it is the case for the familiar Schrödinger equation for the bound state of two point-like particles of mass m with Coulomb potential $V(r)$ in a Euclidean space $E(r, \delta, I)$ formulated on a Hilbert space $\mathcal{H}$ over the field of complex numbers $\mathcal{C}$

$$i\frac{\partial}{\partial t}\psi(t,r) = H\psi(t,r) = \left[\frac{\hbar^2}{m}\Sigma_k p_k p_k - V(r)\right]\psi(t,r) =$$
$$= \left[\frac{1}{m}\Sigma_k(-i\hbar\partial_k)(-i\hbar\partial_k) - V(r)\right]\psi(t,r) = \tag{16}$$
$$= \left[-\frac{\hbar^2}{m}\Delta_r - V(r)\right]\psi(t,r) = E\psi(t,r).$$

By contrast, *hadronic bound states* are bound states of *extended particles* at mutual distances smaller or equal to the *hadronic horizon* (Figure I-13)

$$R = \frac{1}{b} \approx 10^{-13}\, cm. \tag{17}$$

In such a region, bound states verify hadronic mechanics, can be represented with isomathematics (Section I-3) and isomechanics (Section I-4) for the case of time-reversal invariant bound states (such as the deuteron), or genomathematics and genomechanics for time irreversible bound states (such as all hadrons produced in particle physics laboratories).

By using the methods outlined in Paper I and in the preceding sections, when assumed to be stable in first approximation (thus being time reversal invariant), hadronic bound states are characterized by the following main features:

1) The bound states occur between *isoparticles,* namely, isoirreducible, isounitary isorepresentations of the isospinorial covering of the Galileo-Santilli isosymmetry $\hat{\mathcal{G}}$ for non-relativistic treatments (Sections II-2.5.1 and II-3.9) or of the Lorentz-Poincaré-Santilli isosymmetry $\hat{\mathcal{P}}$ for relativistic treatments (Sections II-2.5.11 and II-3.9) represented by hadronic mechanics including the verification of Einstein's determinism (Lemma II-3.7) and the absence of divergencies (Corollary II-3.7.1).

2) The representation of the extended character of the isoparticles is done with isoproduct (1) and isotopic element (4), resulting in iso-Schrödinger equations of type (I-80), while the deep mutual penetration of the wavepackets and/or charge distributions of isoparticles generates novel non-linear, non-local and non-potential interactions represented by the exponent of the the isotopic element (4) and other means. Note that the latter interactions are short range, strongly attractive, charge independent, and non-Hamiltonian according to all studies conducted to date in the field, thus allowing an initial yet explicit and concrete realization of strong interactions [19].

3) By recalling that isosymmetries $\hat{\mathcal{G}}$ and $\hat{\mathcal{P}}$ are all *irregular* realizations of the Lie-Santilli isotheory (Sections I-2.7 and II-2.5.4), a necessary condition for the invariance of hadronic dynamical equations under isosymmetries is that contact interactions *cannot* be derived via non-unitary transforms of quantum mechanical potentials, thus being *basically new interactions.* The physically equivalent property is that, as it is well known, *strong interaction cannot be derived via non-unitary or other known transformations of electromagnetic interactions,* thus confirming the necessary use of the irregular Lie-Santilli isotheory and hadronic dynamical equations.

The notion of hadronic bound states was proposed, apparently for the first time, by Santilli in the 1978 Ref. [19] and was extensively studied hereafter in various works by various authors (see the 2001 monograph [30], the 2011 review [31], and papers quoted therein).

It is important to review the derivation of the basic non-relativistic and relativistic isoequations for hadronic bound states to show their apparent verification of the isodeterministic principle of Lemma 3.7.

The fundamental *non-relativistic, irregular isoequation of a time-reversal invariant hadronic bound state* of two isoparticles of mass m at mutual distances of the order of the hadronic horizon $R = 10^{-13}\ cm$ in an iso-Euclidean isospace $\hat{E}(\hat{r}, \hat{\delta}, \hat{I})$ formulated on a Hilbert-Myung-Santilli isospace $\hat{\mathcal{H}}$ over

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the isofield of isocomplex isonumbers $\hat{\mathcal{C}}$ can be written

$$i\frac{\hat{\partial}}{\hat{\partial}\hat{t}}\hat{\psi}(\hat{t},\hat{r}) = \left[(\hat{1}\hat{/}\hat{m}) \star \Sigma_k \hat{p}_k \star \hat{p}_k \pm \hat{V}(\hat{r}) - \hat{S}(\hat{\psi})\right] \star \hat{\psi}(\hat{t},\hat{r}) =$$

$$= \left[(1/m)\Sigma_k \hat{p}_k \hat{T} \hat{p}_k \hat{T} \pm V(\hat{r}) - S(\hat{\psi})\right] \hat{\psi}(\hat{t},\hat{r}) = \tag{18}$$

$$\hat{E} \star \hat{\psi}(\hat{t},\hat{r}) = E\hat{\psi}(\hat{t},\hat{r}),$$

where $\hat{V}(\hat{r}) = V(\hat{r})\hat{I}$; $\hat{S}(\hat{\psi}) = S(\hat{\psi})\hat{I}$ represents the novel short range, strongly attractive force; the value $-\hat{V}(\hat{r})$ occurs for bound states with opposite charge (as it is the case for the synthesis of hadrons reviewed below); the value $+\hat{V}(\hat{r})$ occurs for isoparticles with the same charge (as occurring for valence electron bonds reviewed below); and one should note that isoeigenvalues can always be reduced to conventional eigenvalues, thus allowing experimental verifications.

Due to the large representational capabilities of isoequations (182), we use the following simplifying assumptions:

1) The isotime is equal to the conventional time, $\hat{t} = t\hat{I}_t = t,\ \hat{I}_t = 1$;

2) Being extremely small, the orbits of the isoparticles are assumed to be nearly constant circles, thus implying that the n_k characteristic quantities of then isotopic element (4) can be normalized to the sphere, $n_k = 1,\ k = 1, 2, 3$;

3) The isotopic element is assumed to be given by the exponential term of Eq. (4) with realizations of the non-linear, non-local and non-potential interactions of the type (Eq. (4.7), page 170 Ref. [30])

$$\hat{T} = e^{-\Gamma} = e^{-N\psi/\hat{\psi}} \approx 1 - N\psi/\hat{\psi}, \tag{19}$$

where ψ behaves like the solution of quantum equation (180),

$$\psi(r) \approx W_1 e^{-br}, \tag{20}$$

and $\hat{\psi}$ behaves like the solution of the hadronic equation expected to be of the type

$$\hat{\psi} \approx W_2(1 - e^{\frac{(1--br)}{r}}, \tag{21}$$

where W_1 and W_2 are positive normalization constants.

We therefore have the following explicit form of the isotopic element

$$\hat{T} = e^{-W_2\frac{e^{-br}}{(1-e^{-br})/r}} \approx 1 - W\frac{e^{-br}}{(1-e^{-br})/r}, \tag{22}$$

exhibiting the *Hulten potential*

$$V_{Hult} = W_2 \frac{e^{-br}}{1 - e^{-br}}, \tag{23}$$

directly in the exponent of the isotopic element, where W is normalization constants.

It should be recalled that Santilli suggested the use of the Hulten potential in the 1978 paper [19], Eq. (5.1.6), page 833, as an *initial yet explicit and concrete representation of strong interactions.*

Under the above assumptions, isotopic element (186) verifies the central condition for the validity of the isodeterministic principle as well as the rapid convergence of isoperturbative series inside the hadronic horizon (Lemma II-3.7 and Corollary II-3.7.1), in a way fully compatible with the validity of conventional uncertainties as well as divergence of perturbative series outside said horizon

$$\begin{aligned} |\hat{T}| &\ll 1, \\ Lim_{r \gg R} \hat{T} &= 1. \end{aligned} \tag{24}$$

The use of the isolinear isomomentum (I-79), the projection of isodynamical equation (II-182) into our Euclidean space can be written in the form first derived in Eq. (5.1.9), page 833, Ref [19]

$$\begin{aligned} i\tfrac{\partial}{\partial t}\hat{\psi}(t,r) &= \left[\tfrac{1}{m}\Sigma_k \hat{p}_k \star \hat{p}_k \star \pm V\tfrac{e^2}{r} - W\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = \\ &= \left[\tfrac{1}{m}\Sigma_k(-i\hat{I}\partial_k)(-i\hat{I}\partial_k) \pm W_1\tfrac{e^2}{r} - W_2\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = \\ &= \left[-\tfrac{1}{\bar{m}}\Delta_r \pm V\tfrac{e^2}{r} - W\tfrac{e^{-br}}{1-e^{-br}}\right]\hat{\psi}(t,r) = E\hat{\psi}(t,r), \end{aligned} \tag{25}$$

where: W_1' and W_2 are renormalization constants; E is the binding energy of the hadronic bound state; the total energy E_{tot} is given by

$$E_{tot} = E_1 + E_2 + -E, \tag{26}$$

where $E_k, k = 1,2$ are the total energies of particles 1 and 2, respectively, and

$$\bar{m} = \frac{m}{\hat{I}^2}. \tag{27}$$

The nonrelativistic expression of the mean-life of the hadronic bound state can be derived via an isotopy of the quantum mechanical form, yielding the expression (Ref. [19], Eq. (5.1.13), page 835)

$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_k}{\hbar},\ k = 1, 2. \tag{28}$$

The angular component of Eqs. (85) has been studied in detail in Ref. [33] via the isospherical isoharmonics.

The radial component of the non-relativistic, irregular, hadronic isoequation for the characterization of the total energy E_{tot}, mean life τ and charge radius R of a time-reversal invariant hadronic bound state can be written (Ref. [19], Eqs. (5.1.40, page 835)

$$\begin{gathered}\left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2\tfrac{e^{-br}}{1-e^{-br}})\right] = 0,\\ E_{tot} = E_1 + E_2 - E,\ \ \bar{m} = \tfrac{m}{I^2}\\ \tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_k}{\hbar},\ k = 1, 2,\\ R = b^{-1},\end{gathered} \tag{29}$$

where the last two equations are subsidiary constraints on the first two.

The analytic solution of the above equations has been studied in detail in Section 5, Ref. [19], including boundary conditions requested by the subsidiary constraints, and we cannot review it here for brevity. We merely limit ourselves to the indication that said analytic solution was reduced to the solution of the following two algebraic equations on the parameters k_1 and k_2 (Ref. [19], Eq. (5.1.32), page 840)

$$\begin{gathered}k_1[k_1 - (k_2 - 1)^3] = \tfrac{1}{2c}E_{tot}R,\\ \tfrac{(k_2-1)^3}{k_1} = \tfrac{9\times 10^6}{3\pi c}\tfrac{R}{\tau}.\end{gathered} \tag{30}$$

It is now important to evaluate the numeric value of the binding energy E in Eq. (198). For this purpose, we recall that the Hulten potential behaves like the Coulomb potential at short distances

$$V_{hp} \approx K\frac{1}{r}, \tag{31}$$

where K is a positive constant. Consequently, *the Hulten potential can absorb the Coulomb potential resulting in a short range strongly attractive force irrespective of whether the Coulomb force is attractive or repulsive.*

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The radial equation can then be reduced in first approximation to the expression (Eq. (5.1.14a), page 836, Ref. [19])

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E + W_2\frac{e^{-br}}{1-e^{-br}})\right] = 0, \quad (32)$$

and its energy spectrum results to be the typical *finite* spectrum of the Hulten potential

$$|E_{hp}| = \frac{1}{4R^2\bar{m}}(k_2\frac{1}{n} - n)^2, \quad n = 1, 2, 3, ... \quad (33)$$

An important feature of hadronic bound states is that, since we can ignore Coulomb interactions and solely assume *contact* interactions represented with the exponent of the isotopic element (4), *the value of the binding energy E is expected to be small or null,*

$$|E| = \frac{1}{4R^2\bar{m}}(k_2\frac{1}{n} - n)^2 = 0. \quad (34)$$

This is due to the fact that *contact interactions do not carry potential energy* (this is classically the case for a balloon moved by winds in our atmosphere).

Therefore, *the Hulten potential for consistent hadronic bound states is expected to admit one single energy level, the ground state* since all possible excited states imply radial distances bigger than the hadronic horizon R, with consequential recovering of quantum mechanics.

In particular, property (34) is solely possible for the following values of the k-[arameters

$$k_1 \ > \ 0, \ \ k_2 \ \geq \ 1. \quad (35)$$

The absence of a spectrum of energies was called in Section 5 of Ref. [19] the *hadronic suppression of quantum mechanical energy spectra* in order to differentiate the *quantum mechanical classification of hadrons into families* (which is characterized by energy spectra) from the *structure of individual hadrons of a given classification family* (which is expected to require different constituents for different particles due to the general difference of spontaneous decays with the lowest mode).

2.4. Relativistic hadronic bound states.

The relativistic counterpart of Eqs. (25) was identified, apparently for the first time, in Refs. [35] [36] and was formulated in the isoproduct of a real-valued iso-Minkowski isospace for orbital motions and a complex valued iso-Euclidean isospace for the hadronic spin

$$\hat{S}_{tot} = \hat{M}(\hat{x}, \hat{\eta}, \hat{I}_{orb}) \star \hat{R}(\hat{z}\hat{\delta}, \hat{I}_{spin}), \quad (36)$$

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resulting in the following *irregular extension of the Dirac-Santilli isoequation (I-88)* ,

$$[\hat{\Omega}^{\mu\nu} \star \hat{\Gamma}_{\mu} \star \hat{\partial}_{\nu} + \hat{M} \star \hat{C} - \hat{V}_{hp}]\hat{\psi}(\hat{x}) = \\ = (-i\hat{I}\hat{\eta}^{\mu\nu}\hat{\gamma}_{\mu}\partial_{\nu} + mC - V_{hp})\hat{\psi}(\hat{x}) = 0, \tag{37}$$

where $\hat{S} = S\hat{I}_{orb}$ represents strong interactions, and the *Dirac-Santilli isogamma isomatrices* $\hat{\Gamma} = \hat{\gamma}\hat{I}$ are given by

$$\hat{\gamma}_k = \frac{1}{n_k}\begin{pmatrix} 0 & \hat{\sigma}_k \\ -\hat{\sigma}_k & 0 \end{pmatrix}, \\ \hat{\gamma}_4 = \frac{i}{n_4}\begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}; \tag{38}$$

where $\hat{\sigma}_k$ are the *irregular Pauli-Santilli isomatrices* studied in Section II-3.4 with the following *anti-isocommutation rules*

$$\{\hat{\gamma}_{\mu}\hat{,}\hat{\gamma}_{\nu}\} = \hat{\gamma}_{\mu}\hat{T}\hat{\gamma}_{\nu} + \hat{\gamma}_{\nu}\hat{T}\hat{\gamma}_{\mu} = \\ = 2\hat{\eta}_{\mu\nu}. \tag{39}$$

where $\hat{\eta}$ is the isometric of the orbital iso-Minkowskian isospace.

2.5. Einstein's determinism in the structure of mesons.

2.5.1. Insufficiencies of quark conjectures. While the classification of hadrons into families has received a rather large consensus since its initiation by M. Gell-Mann in the 1960's [75], the conjecture that the hypothetical quarks are the actual physical constituents of hadrons has been controversial since its inception. These problematic aspects were reviewed in detail in the 1979 paper [20] (written at the Department of Mathematics of Harvard University under DOE support), and can be summarized as follows:

1) The quantum mechanical classification of point-like particles permitted by the $SU(3)$ model and, more recently, by the standard model, has indeed achieved a satisfactory *classification* of hadrons into families.

2) Quarks are purely mathematical representations of a purely mathematical unitary symmetry defined on a purely mathematical complex-valued internal space and, as such, quarks cannot be the actual physical constituents of hadrons for numerous insufficiencies or sheer inconsistencies, such as:

2A) By recalling that quarks have to be point-like as a necessary condition to maintain the validity of quantum mechanics in the interior of hadrons, the ensuing conception of the hyperdense hadrons as ideal spheres with point-particles in their interior is not realistic;

2B) Quarks cannot be physical particles in our spacetime, and the same holds for their masses, because they cannot be defined as unitary irreducible representations of the Lorentz-Poincaré symmetry (Section 3.9);

2C) Quarks cannot be rigorously confined inside hadrons (i.e,. confined with a rigorously proved, identically null probability of tunnel effect) due to the uncertainty principle;

2D) Quarks have not been directly detected under collisions at the extremely high energies achieved at CERN and at other particle physics laboratories;

2E) The wavepackets of all particles are of the same order of magnitude of the size of all hadrons. Hence, the hyperdense character of hadrons is due to the total mutual penetration of the wavepackets of their constituents, resulting in non-linear, non-local and non-Hamiltonian internal interactions under which the SU(3) and other symmetries cannot be consistently defined (Sections 3, 4).

3) History has thought that the study of atoms (as well as of other natural systems) required *two* different yet compatible models, one for the *classification* of atoms into family, and a different model for the *structure* of each atom of a given classification family. Particularly significant is the fact that the classification of atoms could be achieved via the use of *preexisting mathematics,* while the structure of atoms required *new mathematics,* such as the Hilbert spaces, that are unnecessary for the classification of atoms.

In order to resolve the insufficiencies of the conjecture that quarks are physical particles, Santilli [20] suggested to follow the teaching of the history of science, and study hadrons via *two* different models, the standard model for the classification of hadrons and a different, yet compatible, model for the structure of individual hadrons of a given classification multiplet.

In particular, the classification of hadrons can be effectively done via quantum mechanics because individual hadrons can be well approximated as being point-like particles in vacuum. By contrast, the structure of hadrons requires a necessary "completion" of quantum mechanics into a covering theory suggested beginning with the title of Ref. [20] in view of the unavoidable, internal, non-linear, non-local, and non-potential interactions.

2.5.2. Hadronic structure model of mesons. A primary aim of papers [18] [19] [20] and monographs [22] [23] of 1978-1979 was the "completion" of quantum mechanics (qm) into the covering hadronic mechanics (hm) for the specific intent of achieving a representation of *all* characteristics of mesons as hadronic bound states of actual, massive, physical particles

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Table I. The Technical Origin of Some of the Controversies in Hadron Physics[a]

$\{\pi^0 : 1^-, 0^-, +, 135, 0.8 \times 10^{-16}, \sim 1\ F\}$, $\{\pi^\pm : 1^-, 0^-, ..., 139, 2.6 \times 10^{-8}, \sim 1\ F\}$
$\{K^\pm : \frac{1}{2}, 0^-, ..., 439, 1.2 \times 10^{-8}, \sim 1\ F\}$, $\{K^0_S : \frac{1}{2}, 0^-, ..., 498, 0.9 \times 10^{-10}, \sim 1\ F\}$
$\{K^0_L : \frac{1}{2}, 0^-, ..., 498, 5.1 \times 10^{-8}, \sim 1\ F\}$, $\{\eta : 0^+, 0^-, +, 549, \Gamma = 0.8\ \text{keV}, \sim 1\ F\}$

$\pi^0 \to \gamma\gamma$, 98.8%	$K^\pm \to e\pi^0\nu\gamma$, 3.7×10^{-4}
$\pi^0 \to \gamma e^+e^-$, 1.15%	$K^\pm \to \pi e^+e^-$, 2.6×10^{-7}
$\pi^0 \to \gamma\gamma\gamma$, $<5 \times 10^{-6}$	$K^\pm \to \pi^\mp e^\pm e^\pm$, $<1.5 \times 10^{-8}$
$\pi^0 \to e^+e^-e^+e^-$, 3.3×10^{-5}	$K^\pm \to \pi\mu^+\mu^-$, $<2.4 \times 10^{-6}$
$\pi^0 \to \gamma\gamma\gamma\gamma$, $<6 \times 10^{-5}$	$K^\pm \to \pi\gamma\gamma$, $<3.5 \times 10^{-5}$
$\pi^0 \to e^+e^-$, $<2 \times 10^{-6}$	$K^\pm \to \pi\gamma\gamma\gamma$, $<3.0 \times 10^{-4}$
$\pi^\pm \to \mu\nu$, 100%	$K^\pm \to \pi\nu\nu$, $<0.6 \times 10^{-6}$
$\pi^\pm \to e\nu$, 1.2×10^{-4}	$K^\pm \to \pi\gamma$, $<4 \times 10^{-6}$
$\pi^\pm \to \mu\nu\gamma$, 1.2×10^{-4}	$K^\pm \to e\pi^\mp\mu^\pm$, $<2.8 \times 10^{-8}$
$\pi^\pm \to \pi^0 e\nu$, 1.02×10^{-8}	$K^\pm \to e\pi^\pm\mu^\mp$, $<1.4 \times 10^{-8}$
$\pi^\pm \to e\nu\gamma$, 3×10^{-8}	$K^\pm \to \mu\nu\nu\nu$, $<6 \times 10^{-6}$
$\pi^\pm \to e^+e^-\nu$, $<3.4 \times 10^{-8}$	$K^0_S \to \pi^+\pi^-$, 68.7%
$K^\pm \to \mu\nu$, 63.6%	$K^0_S \to \pi^0\pi^0$, 31.3%
$K^\pm \to \pi\pi^0$, 21.05%	$K^0_S \to \mu^+\mu^-$, $<3.2 \times 10^{-7}$
$K^\pm \to \pi\pi^-\pi^+$, 5.6%	$K^0_S \to e^+e^-$, $<3.4 \times 10^{-8}$
$K^\pm \to \pi\pi^0\pi^0$, 1.7%	$K^0_S \to \pi^+\pi^-\gamma$, 2.0×10^{-3}
$K^\pm \to \mu\pi^0\nu$, 3.2%	$K^0_S \to \gamma\gamma$, $<0.4 \times 10^{-3}$
$K^\pm \to e\pi^0\nu$, 4.8%	$K^0_L \to \pi^0\pi^0\pi^0$, 21.4%
$K^\pm \to \mu\nu\gamma$, 5.8×10^{-3}	$K^0_L \to \pi^+\pi^-\pi^0$, 12.2%
$K^\pm \to e\pi^0\pi^0\nu$, 1.8×10^{-5}	$K^0_L \to \pi\mu\nu$, 27.1%
$K^\pm \to \pi\pi^\pm e^\mp\nu$, 3.7×10^{-5}	$K^0_L \to \pi e\nu$, 39.0%
$K^\pm \to \pi\pi^\pm e^\pm\nu$, $<5 \times 10^{-7}$	$K^0_L \to \pi e\nu\gamma$, 1.3%
$K^\pm \to \pi\pi^\mp\mu^\pm\nu$, 0.9×10^{-5}	$K^0_L \to \pi^+\pi^-$, 0.2%
$K^\pm \to \pi\pi^\pm\mu^\pm\nu$, $<3.0 \times 10^{-6}$	$K^0_L \to \pi^0\pi^0$, 0.09%
$K^\pm \to e\nu$, 1.5×10^{-5}	$K^0_L \to \pi^+\pi^-\gamma$, 6.0×10^{-5}
$K^\pm \to e\nu\gamma$, 1.6×10^{-5}	$\eta \to \gamma\gamma$, 38.0%
$K^\pm \to \pi\pi^0\gamma$, 2.7×10^{-4}	$\eta \to \pi^0\gamma\gamma$, 3.1%
$K^\pm \to \pi\pi^+\pi^-\gamma$, 1×10^{-4}	$\eta \to 3\pi^0$, 29.9%
$K^\pm \to \mu\pi^0\nu\gamma$, $<6 \times 10^{-5}$	$\eta \to \pi^+\pi^-\pi^0$, 23.6%
$K^0_L \to \pi^0\gamma\gamma$, $<2.4 \times 10^{-4}$	$\eta \to \pi^+\pi^-\gamma$, 4.9%
$K^0_L \to \gamma\gamma$, 4.9×10^{-4}	$\eta \to e^+e^-\gamma$, 0.5%
$K^0_L \to e\mu$, $<2.0 \times 10^{-9}$	$\eta \to \pi^0e^+e^-$, <0.04%
$K^0_L \to \mu^+\mu^-$, 1.0×10^{-8}	$\eta \to \pi^+\pi^-$, <0.15%
$K^0_L \to \mu^+\mu^-\gamma$, $<7.8 \times 10^{-6}$	$\eta \to \pi^+\pi^-e^+e^-$, 0.1%
$K^0_L \to \mu^+\mu^-\pi^0$, $<5.7 \times 10^{-5}$	$\eta \to \pi^+\pi^-\pi^0\gamma$, $<6 \times 10^{-4}$
$K^0_L \to e^+e^-$, $<2.0 \times 10^{-9}$	$\eta \to \pi^+\pi^-\gamma\gamma$, <0.2%
$K^0_L \to e^+e^-$, $<2.8 \times 10^{-6}$	$\eta \to \mu^+\mu^-$, 2.2×10^{-5}
$K^0_L \to \pi^0\pi^\pm e^\mp\nu$, $<2.2 \times 10^{-5}$	

[a] In this table we have listed most of the experimental data on the octet of light mesons from the particle data[180] with only one addition, the inclusion of the (approximate) charge radius of the particles. The data are divided into two groups: those for total

Figure 3: *A reproduction of Table 1, page 429, Ref. [20] used to identify the physical constituents of mesons in the massive particles produced free in the spontaneous decays, generally those with the lowest mode. The "completion" of quantum mechanics into hadronic mechanics then becomes recommendable for any quantitative treatment of the indicated structure model.*

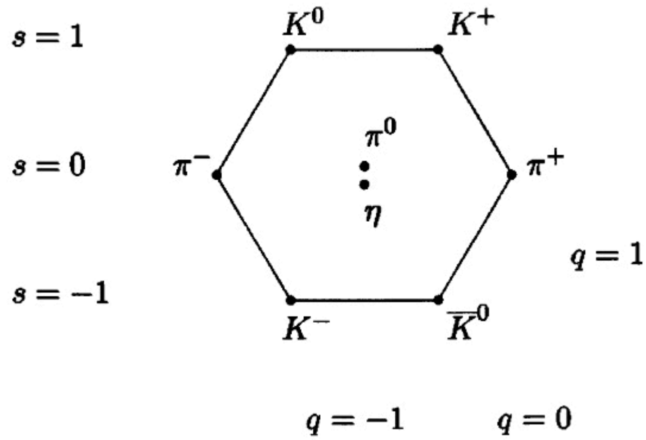

Figure 4: *An illustration of the compatibility between the hadronic structure model of mesons, Eqs. (40)-(45), with physical constituents and the conventional $SU(3)$ classification. This compatibility is achieved via the identification of the isounits or isotopic elements per each meson, Eq. (46), and their use as the hyperunit of the $\tilde{SU}(3)$ hypersymmetry in view of its local isomorphism to the conventional symmetry.*

produced free in the spontaneous decays , generally those with the lowest mode illustrated in Table 1, page 429, Ref. [20] (reproduced in Figure 2).

The above view suggested the following new structure models for the octet of mesons:

$$\pi^0 = (\tilde{e}^-, \tilde{e}^+)_{hm}, \tag{40}$$
$$\pi^\pm = (\tilde{\pi}^0, \tilde{e}^\pm)_{hm}, \tag{41}$$
$$K^0 = (\tilde{\pi}^0, \tilde{\pi}^0)_{hm}, \tag{42}$$
$$K^\pm = (\tilde{\pi}^0, \tilde{\pi}^\pm)_{hm}, \tag{43}$$
$$K_S = (\tilde{\pi}^-, \tilde{\pi}^+)_{hm}, \tag{44}$$
$$K_L = (\tilde{K}^-, \tilde{\pi}^+)_{hm}. \tag{45}$$

In the above models, the role of positrons as physical constituents of mesons provides the first known numerical representation of their very short meanlives, while all other characteristics are numerically represented via the hadronic bound states of Sections 2.3 and 2.4.

Structure models (40) - (45) are incompatible with quantum mechanics because the rest energy of all particles are *bigger* than the sum of the rest energies of the constituents (thus requiring *positive binding energies,*

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with ensuing *mass excesses* that are anathema for quantum mechanics as discussed in Section I.3), and for other reasons.

Necessary conditions to prevent misrepresentations is that models (40) - (45) are treated with hadronic mechanics and their constituents are *isoparticles and anti-isoparticles* (Section II-3.9) hereon denoted with an upper tilde.

Therefore, the elementary constituents of the π^0 in Eq.(40) are *one isoelectron* $\tilde{e}^-$ *and one isopositron* $\tilde{e}^+$ (called *eletons* in Ref. [19]); the constituents of the $\pi^\pm$, Eq. (51) are *one iso-meson* $\tilde{\pi}^0$ and *one isoelectron or isopositron;* etc.

Following their emission in the spontaneous via isotunnel effects (i.e., tunnel effects in the iso-Hilbert isospace), the isoconstituents assume conventional quantum mechanical characteristics plus possible secondary effects with the emissions of massless particles.

Note that *all models (40)-(45) either are or can be reduced to two-body hadronic bound states,* thus admitting analytic solutions in their representation via Eqs. (29). Note also that models (40)-(45) have a kind of bootstrap structures, since a given meson appears in the structure of heavier mesons.

Note additionally that *models (40)-(45) imply the increase of the number of "elementary" constituents with the increase of the rest energy.* In fact, the π^0 has only two elementary constituents while K_L has eight elementary constituents.

Note finally that the role of isopositrons as actual physical constituents of mesons provides the *only* known mechanism via particle-antiparticle annihilation for the *quantitative representation* of the very small mean-lives of mesons, by keeping in mind that the rest energy of electron-positron bound states is positive in our world and negative in the antimatter world (Section 2.5.3).

Since different structure models are characterized by numerically different isounits, the compatibility of the above hadronic structure model of mesons with their classification is readily achieved at the higher level of the *hyperstructural branch of hadronic mechanics* [33] [37], with the following total multi-valued *hyperunit*

$$\hat{I}_{tot} \ = \ \{\hat{I}_{\pi^0}, \hat{I}_{\pi^\pm}, \hat{I}_{K^0}, \hat{I}_{K^\pm}, \hat{I}_{K_S}, \hat{I}_{K_L}\}. \tag{46}$$

2.5.3. Positive energy of particle-antiparticle bound states. The *positive* total energy of particle-antiparticle hadronic bound states has been studied in detail in monograph [38] (see, e.g., Section 2.3.14, page 131), and in various additional works, including all historical references.

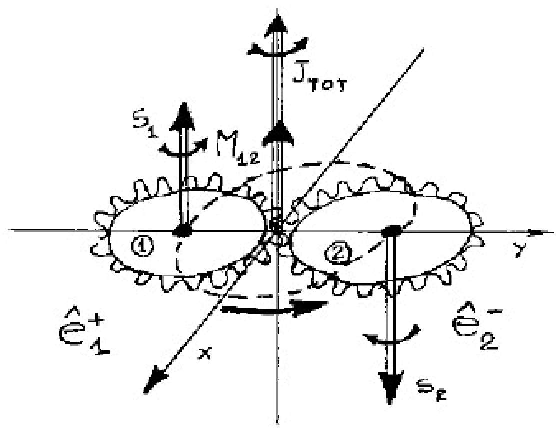

Figure 5: *A reproduction of "gear model" used in Section 5, page 852, Ref. [19] to illustrate the strongly attractive character of contact non-Hamiltonian forces when the constituents are in singlet coupling and their strongly repulsive character when in triplet couplings.*

We cannot possibly review here the underlying *isodual theory of antimatter,* but an indication of the following main notions appears recommendable to avoid basic misconceptions.

Recall that *negative energies violate causality,* for which reason P. A. M. Dirac was forced to work out his "hole theory."

Santilli resolved the problem of causality by referring all physical quantities, thus including energies, to *isodual units defined on isodual fields with a 'negative' unit.*

Hence, the terms "negative energy of antiparticles" have no meaning in the field here considered, because the correct statement is "negative energies of antiparticles measured to negative units of energy."

Within the above setting, Ref. [38], Eqs. (2.3.77), page 131, shows that *the total energy of a particle-antiparticle bound state is positive when measured in our world (with positive units) and negative when measured in the antimatter word (with negative units).*

The above property is truly fundamental for a consistent quantitative representation of all characteristics of unstable particles.

In fact, the presence of antiparticles in the structure of mesons and (unstable) baryons, first proposed by Santilli in 1978 [19], appears to be the most plausible origin of the extreme instability of mesons and (unstable) baryons with mean lives such as $\tau = 10^{-16}\ s$ or less, with no equally effec-

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tive or plausible alternative known to this day.

2.5.4. Hadronic structure model of the π^0 meson. The characteristics of the π^0 meson are:

1. Rest energy $E = 134.96 MeV$,
2) Mean-life $\tau = 0.828 \times 10^{-16}\ s$,
3) Charge radius $R = 10^{-13}\ cm$,
4) Null charge and spin.
5) Null electric and magnetic moments,
6) Negative parity; and
7) Primary decay

$$\pi^0 \rightarrow \gamma + \gamma, \ \ 98.85\ \%. \tag{47}$$

The above characteristics are *all* numerically represented by hadronic bound state (29)) as a "compressed" form of the positronium (Pos), resulting in hadronic bound state (40) of one isoelectron and one isopositron (Ref. [19], Section 5, page 828 on)

$$Pos = (e^-, e^+)_{qm} \ \ \rightarrow \ \ \pi^0 = (e^-, e^+)_{hm}, \tag{48}$$

with hadronic structure equations

$$\begin{gathered} \left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2 \tfrac{e^{-br}}{1-e^{-br}})\right] = 0, \\ E_{tot} = E_{\tilde{e}^-} + E_{\tilde{e}^+} - E = 135\ MeV \ \ \bar{m} = \tfrac{m}{I^2} \\ \tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2 \tfrac{\alpha^2 E_{\tilde{e}}}{\hbar} = 10^{-16}s \\ R = b^{-1} = 10^{-13}\ cm = 1\ fm, \end{gathered} \tag{49}$$

and numeric solution

$$k_1 = 0.34, \ \ k_2 = 1 + 4.27 \times 10^{-2}, \tag{50}$$

verifying conditions (35) as expected.

In appraising representation (48), the reader should keep in mind the assumption of Section 2.3 of '*constrained, nearly constant and circular orbits of the constituents*' under which the isounit can be considered independent from local coordinates.

The above results confirm all expectations indicated in preceding sections, namely,

1) Hadronic spectrum (33) admits one and only one energy level, the π^0, since all excited states are those of the positronium;

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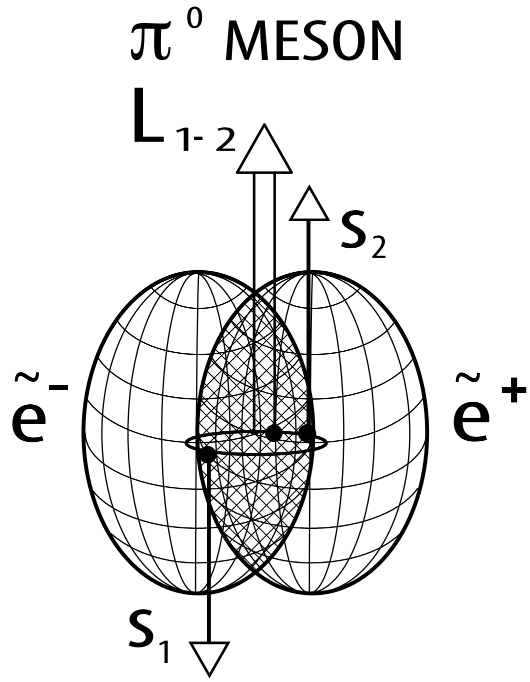

Figure 6: *A conceptual rendering of the structure model of the* π^0 *meson as a hadronic bound state of one isoelectron* $\tilde{e}^-$ *and one isopositron* $\tilde{e}^+$ *in singlet coupling with unmutated spins* $s_1 = -s_2 = 1/2$ *and orbital hadronic momentum in the ground state* $L_{1-2} = 0$. *The dashed area represents the contact, non-linear, non-local and non-potential interactions responsible for the bound state that, being not representable with a Hamiltonian* H, *are representable via the isotopic element* $\hat{T}$ *in the iso-Schrödinger equation* $H \star \hat{\psi} = H\hat{T}\hat{\psi} = E\hat{\psi}$.

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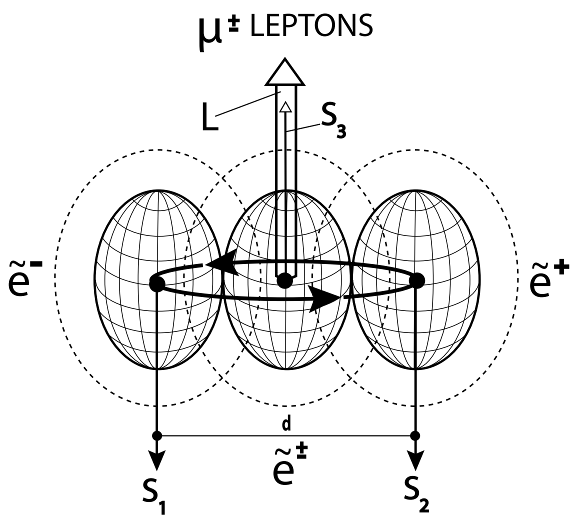

Figure 7: *To achieve compatibility with the standard model, it is generally believed that muons are elementary particles in contrast with the experimental evidence that they are naturally unstable with various decay modes (51) that strongly suggest a composite structure. In this figure, we present a conceptual rendering of the structure of the* $\mu^{\pm}$ *leptons predicted by hadronic mechanics as a three-body of elementary isoconstituents identified in the muon decay with the lowest mode. The weak interaction character of the particles is derived from a weak hadronic bound state essentially given by the overlapping of the classical radius of the constituents indicated by dashed ellipses. By recalling that all three constituents have a point-like charge, the dimension* d *of the charge radius of the two peripheral constituents is purely nominal.*

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2) The presence of the antiparticle $\tilde{e}^+$ in the π^0 structure (40) explains its very short mean-life as well as its main decay (47);

3) Isodeterministic conditions (11) are verified by model (40), as a result of which standard deviations Δr and Δp have individual values smaller than one (Lemma II-3.7) and the isoperturbation series have no divergences (Corollary II-3.7.1).

2.5.5. Hadronic structure model of the remaining mesons. Recall the main characteristics of the muons $\mu^\pm$ are:

1) Rest energy $105, 658\ MeV$;
2) Charge radius $R = 10^{-13}\ cm$;
3) Mean-life $\tau = 2.19703 \times 10^{-6}\ s$;
4) Spin $1/2$ and elementary charge;
5) Spontaneous decay

$$\begin{aligned} \mu^\pm &\to e^- + \nu + \bar{\nu}, \\ \mu^\pm &\to e^- + \gamma, \\ \mu^\pm &\to e^- + 2\gamma, \\ \mu^\pm &\to e^- + e^\pm + e^+. \end{aligned} \tag{51}$$

For the intent of achieving compatibility with the standard model, the $\mu^\pm$ leptons are considered to be "elementary particles." Santilli [19] cannot accept such a view because in contrast with the experimental evidence that *the muons are naturally unstable particles with decays (51).*

Therefore, Ref. [19], Section 5, proposed the structure model of the muons with the physical constituents identified by the decay mode (51) with the smallest mode $< 10 \times 10^{-12}$ (Figure 5)

$$\mu^\pm = (e^-, e^\pm, e^+)_{hm}. \tag{52}$$

The birth of weak interactions was suggested as being due to a weak form of wave-overlapping essentially that of the classical size of the constituents

$$R_{e^\pm} = 2.28 \times 10^{-13}\ cm, \tag{53}$$

which said weak contact interactions can be quantitatively represented by hadronic mechanics via an isotopic element $\hat{T}$ with values close to 1.

A main aspect is that there is the birth of strong interactions in model (52) because we do not have a wave-overlapping inside the hadronic horizon, as it is the case for the π^0.

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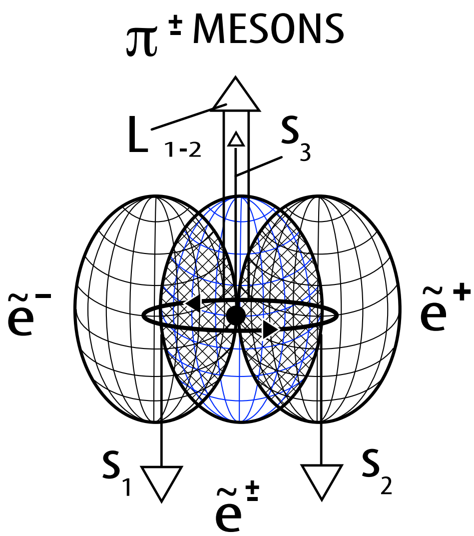

Figure 8: *A conceptual rendering of the structure of the $\pi^{\pm}$ mesons as a three-body hadronic bound states (57) of elementary isoparticles which can be interpreted as a "compressed $\mu^{\pm}$ lepton" (Figure 6). Note that individual pairs of isoconstituents are coupled in singlet as necessary for consistency. Note also that the isoconstituents have a point-like charge structure. Therefore, the charge diameter of the structure is given by the distance between the centers of the two peripheral isoconstituents.*

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It is easy to see that model (52) can represent the muon spin $1/2$ and decay (51) with the lowest mode as a tunnel effect of the constituents.

The remaining characteristics can be represented via a direct analytic solution of model (52) as a restricted three-body problem.

Ref. [19] suggested the approximation of the model into a two-body structure with weakly bounded constituents and the structure equations

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2 \frac{e^{-br}}{1-e^{-br}})\right] = 0,$$
$$E_{tot} = E_{100\ Mev} + E_{e^\pm} - E, = 105\ MeV \quad \bar{m} = \frac{m}{I^2}$$
$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_e}{\hbar} = 10^{-6}\ s \qquad (54)$$
$$R = b^{-1} = 10^{-13\ cm},$$

admitting the following values for the the k-parameters

$$k_1 = 0.93, \quad k_2 = 1 + 8.47 \times 10^{-2}, \qquad (55)$$

that also verify conditions (35).

The main characteristics of the $\pi^\pm$ mesons are:

1) Rest energy $139.570\ MeV$;

2) Charge radius $R = 10^{-15}\ cm$;

3) Mean-life $\tau = 2.603 \times 10^{-8}\ s$;

4) Spin $J = 0$ and elementary charge;

5) Decay with lowest mode $< 3.2 \times 10^{-9}$)

$$\pi^\pm \rightarrow e^- + e^\pm + e^+ + \nu\pi^\pm \qquad (56)$$

.

The above characteristics are all represented by the hadronic structure model of the $\pi^\pm$ mesons first proposed in Section 5, Ref. [19]

$$\pi^\pm\ = (\tilde{e}^-, \tilde{e}^\pm, \tilde{e}^+)_{hm} \approx (\tilde{\pi}^0, \tilde{e}^\pm)_{hm}, \qquad (57)$$

for which, unlike the case of the muons, the constituents are elementary isoparticles in condition of deep mutual penetration inside the hadronic horizon (Figure 7).

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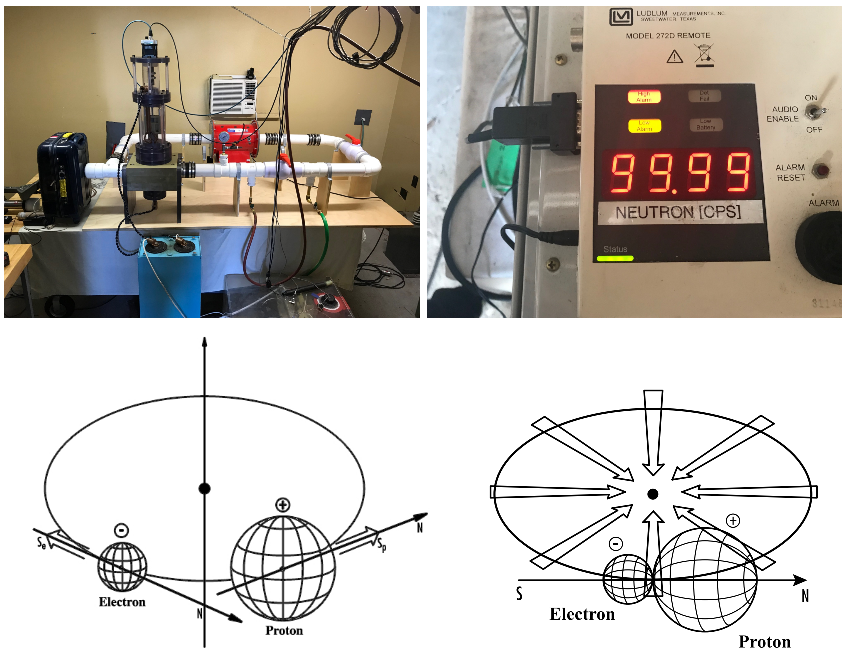

Figure 9: *In this figure we show: 1) At the top left a picture of the Directional Neutron Source (DNS) developed by Santilli at Thunder Energies Corporation, now Hadronic Technologies Corporation [39] which synthesizes on demand neutral and negatively charged hadrons from a hydrogen gas with preferred directionality, energy and flux; 2) At the top right a typical case of neutron CPS; 3) At the bottom left a conceptual rendering of the ionization of the hydrogen gas by the electric arc during its activation and the proper axial alignment of the proton and electron with opposite charge and magnetic polarities; and 4) At the bottom right a conceptual rendering of the "compression" of the electron inside the proton by the electric arc during its disconnection.*

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The rest energy, mean-life and charge radius of the $\pi^{\pm}$ are readily represented by model (57) with the hadronic structure equations

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2 \frac{e^{-br}}{1-e^{-br}})\right] = 0,$$

$$E_{tot} = E_{\tilde{\pi}^{\pm}} + E_{\tilde{e}^{\pm}} - E = 139\ MeV, \ \ \bar{m} = \frac{m}{I^2} \tag{58}$$

$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_{\tilde{e}^{\pm}}}{\hbar} = 2.603 \times 10^{-8}\ s,$$

$$R = b^{-1} = 10^{-13}\ cm,$$

admitting solutions for the k-parameters

$$k_1 = 0.34, \ \ k_2 = 1 + 3.67 \times 10^{-3}, \tag{59}$$

that again verify conditions (35).

The main recent advance since the above 1978 proposal [19] is the achievement of a consistent representation of the total angular momentum $J = 0$ of the $\pi^{\pm}$ meson thanks to the irregular $SU(2)$ isosymmetry (Section 3.5). In essence, when the three isoparticles are all compressed inside the hadronic horizon, the orbital motion of the two peripheral isoconstituents L_{1-2} is constrained, for stability, to be equal to the spin $S_{\tilde{e}^{\pm}}$ of the central isoparticle, thus having value $L_{1-2} = S_{tildee^{\pm}} = 1/2$ with total angular momentum (Sections II-3.4 and II-3.5, Section II-3.5.3 in particular).

$$J_{tot} = s_1 + s_2 + s_3 + L_{1-2} = -1/2 + 1/2 - 1/2 - 1/2 = 0. \tag{60}$$

Note that model (57) can be interpreted as a form of "compressed muons" (52) (see Figures 6 and 7). The orbital motion of the peripheral electrons is unrestricted in model (52), yielding a total angular momentum $1/2$. By contrast, the peripheral isoelectrons and isopositron in model (57) are *constrained to orbit inside the central isoelectron or isopositron*, thus being forced to have the orbital value equal to the spin of the isoparticle. In fact, orbital values different than $1/2$ would imply extreme resistive forces with ensuing excessive instabilities.

In Santilli's view, an important aspect of model (57) is its apparent verification of the isodeterminism of Lemma II-3.7 as well as of the lack of divergencies in the isoperturbation seres (Corollary II-3.7.1) due to the small value of the isotopic element assured by values (59).

Interested readers can verify that values (35) remain verified under the use of more accurate experimental values for the characteristics of muons and mesons.

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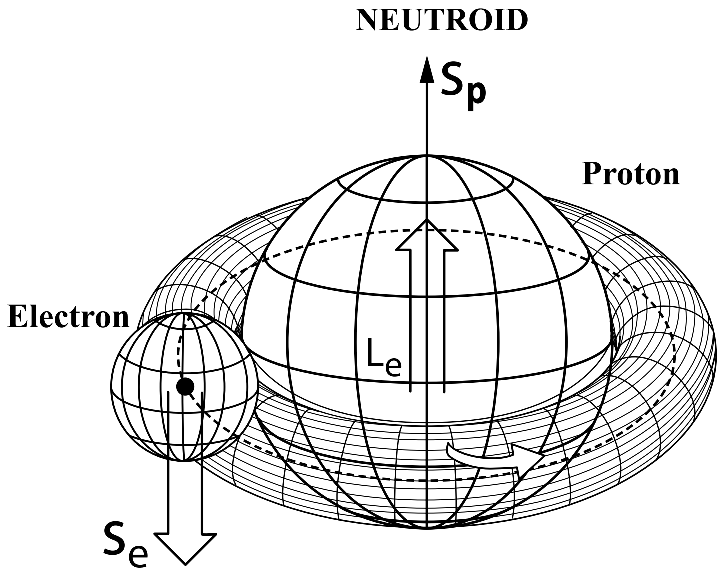

Figure 10: *A conceptual rendering of the structure of a particle called the neutroid $\tilde{n} = (e^-, p^+)_{hm}$, Eq. (61), with spin $J = 1$, charge radius $R = 10^{-13}$ cm, mass essentially that of the proton and mean-life of the order of 5 s which has been undetected in all experiments to date [I-90]-[I-95] by neutron detectors, yet has caused nuclear transmutations typical of neutron irradiation. The bound state is created by the extremely strong, $r - p$ Coulomb attraction which is of the order of 10^{24} N plus weak contact interactions implying ignorable mutations of the electron and the proton. The $e - p$ singlet coupling and the orbital motion in the ground state imply the spin $J = 1$ explaining the lack of detection of neutroids by neutron detectors. Note that the neutroid is impossible for quantum mechanics.*

It is an instructive exercise for the interested reader to work out the hadronic structure models of the remaining mesons, Eqs. (42)-(45) showing the increase of the k_1 value and the decrease of the k_2 value with the increase of the rest energy (see Section 6.2 of Ref. [31] for an independent review).

2.6. Einstein's determinism in the structure of baryons.

2.6.1. Structure of the neutroid. Stars initiate their lives as being composed of hydrogen; they grow in time via the accretion of interstellar-intergalactic hydrogen; and eventually reach such pressures and temperatures in their core to synthesize the neutron as a ”compressed” hydrogen atom according to H. Rutherford [40].

A main aspect of the studies herein considered is that, in Santilli’s view [20], the synthesis of the neutron from the hydrogen in the core of stars is one of the best illustrations of the validity of the EPR argument [1] because said synthesis cannot be represented with quantum mechanics for various technical reasons reviewed in Section I-1, in denial of the experimental evidence that, *having opposite charges, the proton and the electron at mutual distances of* 10^{-13} cm *are attracted to each other with a Coulomb force of about* 230 N *(See Eq, (107) for its calculation). Such an attractive force is so big for particle standards to dismiss as inapplicable any theory unable of producing a bound state under such an anormous attractive force.*

Consequently, Santilli dedicated decades of mathematical, theoretical, experimental and industrial research on the neutron synthesis due to its truly fundamental character for all quantitative sciences (see Refs. (I-85] to [I-95] and independent reviews [31] [41]).

Besides a number of preceding attempts without neutron detection, the first indirect experimental detection of the synthesis of the neutron from the hydrogen was initiated by in the 1960’s by Don Carlo Borghi [76], confirmed in subsequent tests [I-90] - [I-95], and routinely verifiable with the Directional Neutron Source developed by Hadronic Technologies Corporation [39] (Figure 8), indicate the existence of an unstable, neutral, intermediate state called *neutroid* (denoted with the symbol $\tilde{n}$) which is unidentified by neutron detectors, while causing nuclear transmutations typically triggered by neutron irradiation.

Hadronic mechanics suggests the following representation of the structure of the neutroid (Figure 9)

$$\tilde{n} = (\tilde{e}^-_\downarrow, p^+_\uparrow)_{hm}, \tag{61}$$

consisting of an isoelectron $\tilde{e}^-$ with spin $s_1 = 1/2$ in singlet contact coupling with a standard proton p^+ with spin $s_2 = 1/2$ and orbital motion $L_{1-2} = 0$ in the ground state.

These assumptions suggest the following predicted features of the neutroid:

1) Rest energy estimated to be about $940\ MeV$,
2) Mean-life estimated to be of at least $\tau = 5\ s$,
3) Charge radius estimated to be of about $R \approx 10^{-13}\ cm$,
4) Spin predicted to be $J = 0$, and
5) Spontaneous decay

$$\tilde{n} \rightarrow e^- + p^{\cdot} \tag{62}$$

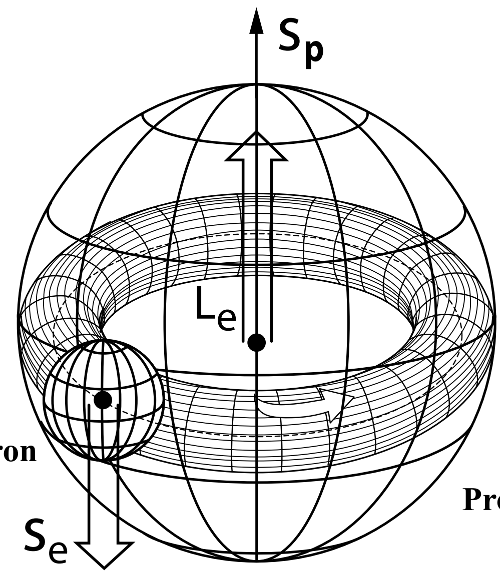

Figure 11: *A conceptual rendering of the structure of the neutron as "compressed" hydrogen atoms in the core of stars according to H. Rutherford [40]. When the proton is represented as an extended particle, there is the emergence of a constrained orbital motion of the electron inside the proton verifying the isodeterminism of Lemma 3.7, which allows the first known exact representation of all the characteristics of the neutron at the nonrelativistic [42] and relativistic [36] levels without need for the neutrino hypothesis.*

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The above features can be represented with hadronic bound state (29) specialized to the indicated values

$$\begin{gathered}\left[\tfrac{1}{r^2}(\tfrac{d}{dr}r^2\tfrac{d}{dr}) + \bar{m}(E \pm W_1\tfrac{e^2}{r} + W_2\tfrac{e^{-br}}{1-e^{-br}})\right] = 0,\\ E_{tot} = E_{\tilde{e}} + E_p - E = 940\ MeV\\ \tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\tfrac{\alpha^2 E_1}{\hbar} = 5\ s\\ R = b^{-1} = 10^{-13}\ cm,\end{gathered} \tag{63}$$

with values of the k-parameters

$$k_1 \approx 1, \ \ k_2 \approx 1 + 10^{-6}, \tag{64}$$

verifying the crucial condition (35).

Once absorbed by a stable nucleus $N(A, Z, J)$, the neutroid is transformed by strong interactions into a neutron plus secondary emissions, resulting in a generally untabulated unstable nucleus $\tilde{N}(A+1, Z, J+1/2)$ that decays into stable nuclei plus radiations.

The spin 0 of the neutroid explains the impossibility of its detection via conventional neutron detectors.

The decay of the nucleus $\tilde{N}(A+1, Z, J+1/2)$ explains the triggering by undetectable particles of conventional nuclear transmutations normally triggered by neutron irradiations.

Hadronic bound state (61) is clearly impossible for quantum mechanics, but readily possible for hadronic mechanics via structure model (29) due to the combination of strongly attractive Coulomb and contact forces at mutual distances of the order of $10^{-13}\ cm$ with isotopic elements of the simple type (19) verifying isodeterministic Lemma II-3.7. and the rapid convergence of isoperturbation series of Corollary II-3.7.1.

2.6.2. Nonrelativistic representation of the neutron synthesis. We consider now the synthesis of the neutron on demand from a hydrogen gas in the needed directionality, (low) energy and CPS.

Such a synthesis has been first achieved by the Directional Neutron Source (DNS) developed by Santilli at Thunder Energies Corporation, now Hadronic Technologies Corporation [39].

Recall that, when seen in an oscilloscope set for milliseconds, electric arcs between carbon electrodes, are continuously connected and disconnected to seek the shortest distance for the discharge.

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With reference to Figure 9, the electric arc ionizes the hydrogen gas during its connection and creates an axial alignment along a magnetic line of electrons and protons with opposite charge and magnetic polarities.

By contrast, during their disconnection electric arcs “compress” the electron inside the proton by synthesizing in this way the neutron in a way crucially dependent on the shape of the arc, its power and other factors.

With reference to Figure 11, when compressed inside the hyperdense proton, the much lighter electron is “constrained” to have an orbital motion equal to the proton spin so as to avoid extreme resistive forces with ensuing high instabilities.

Note that the neutroid (figure 10) appears to be an unavoidable intermediate step prior to the full synthesis of the neutron (Figure 11).

The well known main characteristics of the neutron are the following:

1) Rest energy $939.565\ MeV$;

2) Charge radius $R = 1.73 \times 10^{-13}\ cm$;

3) Mean-life $\tau = 881\ s(about\ 15\ m)$;

4) Spin $1/2$;

5) Charge ;0

6) Anomalous magnetic moment $\mu_n = -1.9\frac{e}{2m_p c}$ and null electric dipole moment;

7) Decay

$$n \ \rightarrow \ p^+ + e^- + \bar{\nu}. \tag{65}$$

The nonrelativistic representation of *all*— characteristics of the neutron in its synthesis from the hydrogen was first achieved by Santilli in the 1990 paper [43] via hadronic bound state

$$n \ \ = \ \ (\tilde{e}^-_{\downarrow}, \tilde{p}^+_{\uparrow})_{hm}, \tag{66}$$

where one should note that, unlike the case of the neutroid (61), both the electron and the proton are mutated into isoparticles.

Note also that model (66) represents a *compressed neutroid* in a way much similar to the structure of the $\pi^\pm$ mesons as compressed muons , $\mu^\pm$ (Section 2.5.4).

The representation of the rest energy, charge radius, mean-life, charge, parity and tunnel effect decay of the neutron have been first achieved in

Ref. [43] Eqs. (2.19), page 521, via hadronic two-body bound state

$$\left[\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d}{dr}\right)+\bar{m}\left(E\pm W_1\frac{e^2}{r}+W_2\frac{e^{-br}}{1-e^{-br}}\right)\right]=0,$$

$$E_{tot}=E_{\tilde{e}}+E_{\tilde{p}}-E=939\ MeV$$

$$\tau^{-1}=2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2E_{\tilde{e}}}{\hbar}=881\ s\ (about 15m) \quad (67)$$

$$R=b^{-1}=10^{-13}cm,$$

with values of the k-parameters

$$k_1=2.6,\ \ k_2=1+0.81\times10^{-8}, \quad (68)$$

verifying conditions (35) for the validity of isodeterminism inside the neutron and the validity of conventional uncertainties in its outside.

The representation of the spin $1/2$ of the neutron was also achieved for the first time in Ref. [43], Eqs. (2.22)-(2.37), thanks to the appearance of a constrained *orbital* motion of the isoelectron when totally "compressed" inside the proton (which orbital motion is completely non-existent for quantum mechanics).

The representation of the spin $1/2$ is additionally permitted by the *isotopies of spin-orbit couplings* (see Chapter 6, page 209 on, Ref. [33], for a detailed treatment), which the hadronic angular momentum of the isoelectron is constrained to be equal to the spin of the isoproton, as a necessary condition to avoid big instabilities according to Eqs. (iI-118) under constraint (II-119). Hence, $J^n_{tot}=s_p+s_e+L_e=1/2$, namely, *the spin of the neutron coincides with that of the proton.*

Recall the following experimental values of magnetic moments in nuclear units $\mu_N=\frac{e}{2m_pc}$

$$\mu_n^{exp}=-1.91\ \mu_N$$

$$\mu_p^{exp}=2.7\ \mu_N \quad (69)$$

$$\mu_e^{exp}=1\ \mu_B=5\times10^{-4}\ \mu_N,$$

where one can see that the magnetic moment of the neutron is "anomalous" because outside the prediction of quantum mechanics both in its direction and numeric value.

The first known exact representation of the anomalous magnetic moment of the neutron was achieved in Ref. [43], Eqs. (2.39)-(2.41), page 526, via the novel contribution of the internal orbital motion of the isoelectron.

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By noting from values (233) that the magnetic moment of the electron is very small for nuclear standard, thus being ignorable in first approximation, the main assumption of Ref. [43] is that *the magnetic moment of the neutron of model (66) is given by the magnetic moment of the proton plus the orbital magnetic moment ot the isoelectron inside the proton.*

Note that said contribution is completely absent in quantum mechanics due to its point-like approximation of particles as being point-like when at mutual distances *smaller* than their charge diameter.

Note that the electron is negatively charged and, consequently, the contribution to the magnetic moment of the neutron from its rotation inside the proton in the direction of the proton spin is *negative,* thus providing the first known representation of the *anomalous direction* of the neutron magnetic moment.

Calculations done in Ref. [43] for the magnetic moment of the isoelectron orbiting inside the proton yield the value $\mu_{\tilde{e}}^{orb} \approx -4.6\ \mu_N$, by therefore reaching the first known representation of the *anomalous value* of the neutron magnetic moment (Eqs. (2.40), page 526, Ref. [43], see also Ref. [31], Section 6.3.D and paper [41], Section 4, page 41)

$$\begin{aligned} \mu_n = \mu_{\tilde{p}} + \mu_{\tilde{e}}^{orb} + \mu_{\tilde{e}}^{intr} \approx \\ \approx \mu_{\tilde{p}} + \mu_{\tilde{e}}^{orb} = \\ = 2.7\ \mu_N - 4.6\ \mu_N = -1.91\ \mu_N = \mu_n^{exp}. \end{aligned} \tag{70}$$

Note that *the negative sign of the neutron magnetic moment can be considered as direct evidence of the presence of an electron in singlet internal coupling in the proton,* since no other known particle besides the electron can provide the internal, rather large contribution $\mu_{\tilde{e}}^{orb} = -4.6\ \mu_N$.

In Santilli's view, *the inability by quantum mechanics to represent both the anomalous direction and value of the neutron magnetic moment constitutes an additional evidence beyond scientific doubt supporting Einstein's vision on the lack of "completion" of quantum mechanics.*

Furthermore, *the impossibility for quantum mechanics to characterize the motion of the electron compressed inside the proton, constitutes evidence that the most important "completion" of quantum mechanics needed to represent experimental values is the characterization of particles with their actual shape and density.*

The parity and null value of the electric dipole moment were represented via a isotopy of conventional lines.

For historical comments on the birth of the neutrino hypothesis in quantum mechanics [13] and its possible replacement with the etherino hypoth-

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esis [42] in hadronic mechanics, one may consult Section II-3.4.1 on Irregular Pauli-Santilli isomatrices.

The reader should keep in mind that the neutron is naturally unstable (when isolated). Consequently, the synthesis of the neutron is a time-irreversible process. It then follows that the Lie-isotopic isomechanical treatment presented in this section is an approximation of the broader treatment via the covering Lie-admissible genomechanics outlined in Section 1-2 (see Refs. [32] [33] for extensive treatments).

With reference to the geno-Schrödinger and geno-Heisenberg equations (I-10) and (I-11), respectively, the genomechanical treatment can be studied via the extension of isotopic element (19) into a time-dependent form characterizing the genotopic elements for motions forward and backward in time.

2.6.3. Relativistic representation of the neutron synthesis. The relativistic representation of *all* characteristics of the neutron in synthesis (66) was first achieved by Santilli in Refs. [35] [36] (see independent review [31], Section 6.3, page 342 on) via the isosymmetry of the irregular Dirac-Santilli isoequation (37), namely, the isotopy $\hat{\mathcal{P}}(3.1)$ of the spinorial covering of the Poincaré symmetry (Section 2.5.11) and can be outlined as follows.

Recall that the non-potential hadronic representation of strong interactions via the isotopic element (1) (Section 2.3) implies the "absorption" of the Coulomb potential by the Hulten potential, as essentially implied by the charge independence of strong interactions.

This feature permits to ignore Coulomb binding energies in first approximation, resulting in a *weakly bounded* relativistic hadronic structure of the neutron, namely, a state with a small binding energy typical of all non-potential interactions.

Consider first synthesis (61) of the neutroid, and assume its representation in the iso-Minkowski isospace $\hat{M}(\hat{x}, \hat{\eta}, \hat{I})$ with isospacetime (II-19). Assume in first approximation that the proton is perfectly spherical for which $n_1 = n_2 = n_3 = 1$ and assume that the density of the region of neutroid-proton overlapping is close to that of the vacuum, thus implying the value $n_4 = 1$. These assumptions imply the simplified isometric

$$\hat{\eta} = Diag.(1, 1, 1, -1)e^{-K}, \ \ K > 0. \tag{71}$$

The relativistic version of the synthesis (61) with the representation of all characteristics of the neutroid then follows via a simply isotopy of the relativistic treatment of the hydrogen atom.

In the transition to the relativistic treatment of the structure of the neutron, Eq. (66), the isoproton cannot any longer be assumed to be perfectly

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spherical and the density of the overlapping region becomes dominant, resulting in values of the characteristic quantities $n_\mu \neq 1,\ \mu = 1, 2, 3, 4$.

In Refs. [35] [36], Santilli assumes that the value of the characteristic quantity n_4 representing the density of the neutron is equal to the density of the proton-antiproton fireball of the Bose-Einstein correlation [46] [47], resulting in the values

$$n_4 = 0.62,\ \ b_4 = \frac{1}{n_4} = 1.62, \tag{72}$$

where $b_4 = 1/n_4$ is the notation used in Ref. [36].

The Lorentz-Santilli isotransforms (II-42) then imply the following *isorenormalization of the rest energy of the electron,* namely a renormalization caused by non-potential interactions (Ref. [36], Eqs. (7.1), page 191)

$$E_e = m_e c^2 = 0.511\ MeV \ \ \rightarrow \ \ E_{\tilde{e}} = m_e \frac{c^2}{n_4^2} = 1.341\ MeV. \tag{73}$$

As one can see, the above isorenormalization removes the problem of the missing $0.782\ MeV$ energy in the neutron synthesis when represented on the iso-Minkowskian isospace over an isofield, thus rendering consistent the needed isorelativistic isoequations.

It should be stressed that the above isorenormalization continues to be based on the etherino mechanism for the delivery of the missing energy to the neutron [42].

The relativistic representation of the spin of the neutron in synthesis (66) was first achieved in Refs. [35] [36]. Recall from Figure 10 that the spin S_1 of isoelectron is opposite to the spin S_2 of the isoproton. The relativistic spin-orbit coupling implies the *constraint* that the orbital angular momentum L_1 of the isoelectron inside the isoproton be equal to the isoproton spin,

$$L_1 = S_2. \tag{74}$$

The above identity is manifestly impossible for the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1)$ and relativistic quantum mechanics, but it is indeed possible for the covering isosymmetry $\hat{\mathcal{P}}(3.1)$. In fact, with reference to Section II-3.5.3, Eqs. (II-122), identity (II- 241) implies the con-

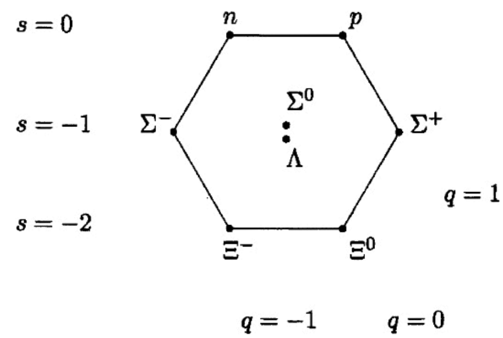

Figure 12: *An illustration of the compatibility of the quantum mechanical $SU(3)$ and subsequent classifications of baryons unto families, and the structure of individual baryons as a hadronic bound state with physical constituents generally emitted free in the spontaneous decays with the lowest mode, Eqs. (241). Said compatibility is achieved via the multi-valued hyperunit characterized by the isounits or isotopic elements of the individual baryons, Eqs. (242), and its use to build the hyperstructural image of the applicable Lie symmetry that turns out to be locally isomorphic to the original symmetry due to the positive-definite character of the isounit.*

ditions (Ref. [36], Eqs. (7.2), page 192)

$$\begin{gathered}\hat{L}_3 \star |\hat{\psi}>= \pm n_1 n_2 |\hat{\psi}>= \\ = \hat{S}_3 \star |\hat{\psi}>= \tfrac{1}{2}\tfrac{1}{n_1 n_2}|\hat{\psi}>, \\ \hat{L}^{\hat{2}} \star |\hat{\psi}>= (n_1^2 n_2^2 + n_2^2 n_3^2 + n_3^2 n_1^2)|\hat{\psi}>= \\ = \hat{S}^{\hat{2}} \star |\hat{\psi}>= \tfrac{1}{4}(n_1^{-2} n_2^{-2} + n_2^{-2} n_3^{-2} + n_3^{-2} n_1^{-2})|\hat{\psi}>,\end{gathered} \tag{75}$$

admitting the simple solution

$$\begin{gathered}n_k^2 = \tfrac{1}{\sqrt{2}} = 0.706, \\ b_k^2 = \tfrac{1}{n_k^2} = \sqrt{2} = 1.415, \quad k = 1, 2, 3,\end{gathered} \tag{76}$$

where $b_k^2 = 1/n_k^2$ in the notation of Ref. [36].

The relativistic representation of the anomalous magnetic moment of the neutron was also achieved for the first time in Ref. [36], Eqs. (7.4), page 192. The representation is again permitted by the contribution from

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the orbital motion of the isoelectron inside the isoproton, and it is given by non-relativistic expression (234) with a more accurate representation of the orbital contribution.

It should be indicated that the non-relativistic and relativistic structure models of the neutron outlined in this section are mere approximation of a much more complex reality in which all characteristics of the constituents are mutated, thus including the charge.

2.6.4. Remaining baryons. A central requirement for the consistency of model (66) is that *the excited states of the neutron are the conventional states of the hydrogen atom.* This illustrates again the hadronic suppression of quantum mechanical energy spectra, since the latter are typical for the classification, rather than the structure of hadrons.

By recalling the condition that the number of elementary constituents of hadrons increases with the increase of the rest energy, the hadronic structure of the remaining baryons is reducible to two isoparticle structures derived from the spontaneous decays generally those with the lowest mode, much along the structure of mesons [21] (see [31], Section 6.3.J, page 366 for an independent review)

$$\begin{aligned}
p^+(938\ MeV) &= stable,\\
n(940\ MeV) &= (\tilde{p}^+, \tilde{e}^-)_{hm},\\
\Lambda(1115\ MeV) &= (\tilde{p}^+, \tilde{\pi}^-)_{hm},\\
\Sigma^+(1189\ MeV) &= (\tilde{p}^+, \tilde{\pi}^0)_{hm},\\
\Sigma^0(1192\ MeV) &= (\tilde{n}, \tilde{\pi}^0)_{hm},\\
\Sigma^-(1197\ MeV) &= (\tilde{n}, \tilde{\pi}^-)_{hm},\\
\Sigma^0(1314\ MeV) &= (\tilde{\Lambda}, \tilde{\pi}^0,\\
\Xi^-(1321\ MeV) &= (\tilde{\Lambda}, \tilde{\pi}^-)_{hm},
\end{aligned} \tag{77}$$

by keeping in mind that numerous alternative internal exchanges of isoparticles do indeed occur while keeping constant the total characteristics.

It is an instructive exercise for the interested reader to see that all the above models verify condition (II-199) on the lack of hadronic excited states, as well as conditions (II-188) for the validity of isodeterministic Lemma II-3.7 and related rapid convergence of isoseries.

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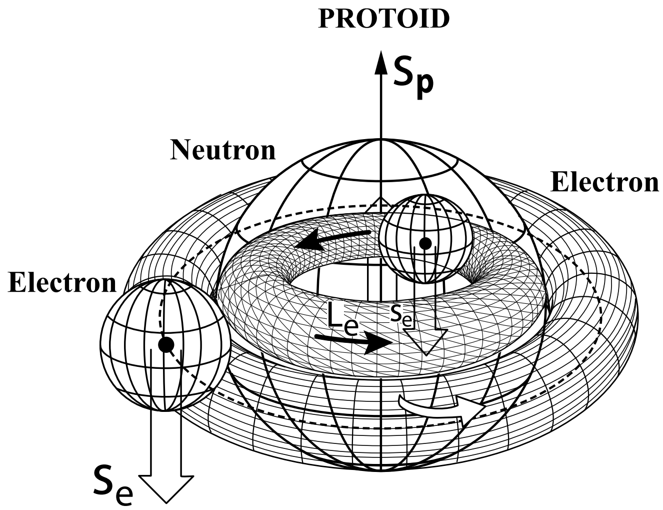

Figure 13: *A conceptual rendering of the protoid as weakly bounded electron and neutron in singlet contact coupling, resulting in a negatively charged hadron with spin $S = 0$ with intriguing applications.*

The compatibility of hadronic structure models (77) with the $SU(3)$-color or more recent classifications is achieved at the hyperstructural level [33] [37] with the following ordered hyperunit [21] (Figure 11)

$$\hat{I}_{tot} = \{\hat{I}_p, \hat{I}_n, \hat{I}_\Lambda, \hat{I}_{\Sigma^+}, \hat{I}_{\Sigma^0}, \hat{I}_{\sigma^-}, \hat{I}_{\Xi^0}, \hat{I}_{\Xi^-}\} \tag{78}$$

Said compatibility is ultimately due to the positive-definiteness of the individual isounits that, in turn, implies the local isomorphism between the hypersymmetry $\tilde{SU}(3)$ and the conventional $SU(3)$ symmetry.

2.6.5. Industrial applications. There is no doubt that the "completion" of quantum mechanics is, by far, Einstein's most important legacy because of its *basic* implications for mathematics, physics, chemistry and other quantitative sciences, with expected industrial applications generally beyond our expectation at this writing.

As an illustration, we outline in this section the industrial applications expected from the technology underlying the synthesis of the neutron from the hydrogen which technology has been possible thanks to the "completion" of quantum into hadronic mechanics.

Besides the synthesis of neutroids (Figure 10) and neutrons (11), the

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Directional Neutron Source (DNS) [39] (Figure 9) permits the study of the synthesis of a *negatively charged, strongly interacting particles* [48] (patent pending) with rest energy of about $940\ MeV$, spin 0, charge radius $R = 1\ fm$, and mean-life of about $3\ s$ which is called the *protoid*, and denoted $\hat{p}_1$ (Figure 13).

Recall that contact non-Hamiltonian interactions solely depend on wave-overlapping, thus being charge independent. The protoid is predicted to be given by a singlet coupling of an electron essentially at contact with a neutron, thus yielding total angular momentum 0, with the following structure model:

$$\hat{p}_1 = (e^-_\downarrow, n_\uparrow)_{hm}. \tag{79}$$

Hadronic mechanics predicts a second negatively charged proton called *pseudo-proton*, and denoted $\hat{p}_2$, with essentially the same rest energy, charge radius and mean-life of the neutroid but with spin 1 (Figure 14).

Recall that an electric arc submerged within a hydrogen gas creates a plasma in its surrounding comprising: protons, electrons, neutrons and valence electron pairs in singlet couplings known as *isoelectronia* with rest energy of about $1\ MeV$, null spin and magnetic moment, and a mean-life expected to be of the order of $1\ s$ or fractions thereof (Section 2.8.3).

Following the synthesis of the neutron via the "compression" of an electron inside the proton (Section 2.6.2), the pseudo-proton is predicted to be generated by a "compression" of an electron, this time, inside the neutron, or equivalently, by the "compression" of an isoelectronium inside the proton, with structure model

$$\hat{p}_2 = (\tilde{e}, \tilde{n})_{hm} \approx [\tilde{p}, (\tilde{e}^-_\uparrow, \tilde{e}^-_\downarrow)]_{hm}. \tag{80}$$

The spin $\S_{\hat{p}2} = 1$ is due to the sum of the proton spin $S_p = 1/2$ plus the constrained orbital motion of the electron pair along the proton spin $L_{1,2} = 1/2$.

Recall that protons are *repelled* by nuclei. Hence, the industrially significant feature of negatively charged hadrons is that they are *attracted* by nuclei, thus initiating a basically new approach toward the controlled nuclear fusions that resolves the extremely large Coulomb repulsion that has opposed nuclear fusions to date.

In view of the above features, the technologies that can be developed with the DNS are the following:

1. The detection of fissionable nuclear material that can be concealed in suitcases, containers or underground, since the neutron irradiation of fissionable material under controlled directionality, energy and flux, triggers the decay of some of the nuclei with a shower of clearly detectable

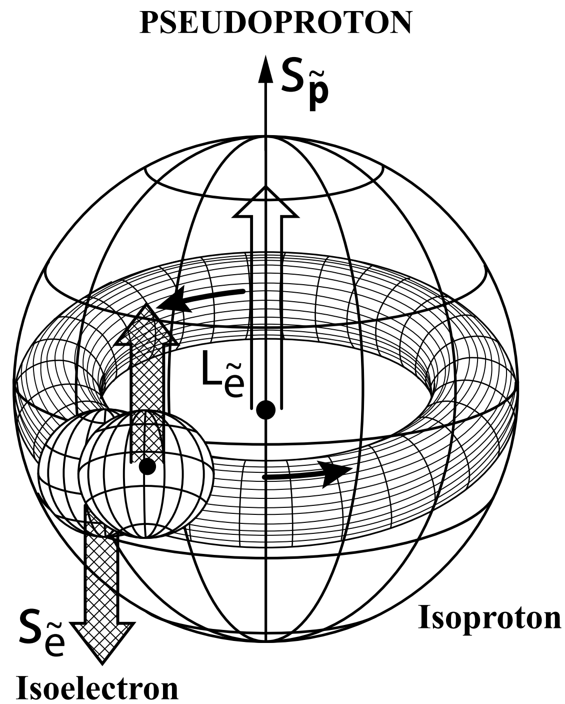

Figure 14: *A conceptual rendering of the pseudo-proton created by the "compression" of an electron, this time, inside a neutron, or equivalently, of a valence electron pair in singlet coupling, resulting in a negatively charged hadron with spin* 1.

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radiations. By comparison, most nuclear material are detected as ordinary metals under currently available X-ray, microwave and other scans.

2. The detection of the presence and concentration of precious metals in mining operations, since the neutron irradiation, for instance, of the walls of mining tunnel triggers known nuclear transmutations with the emission of sharp, clearly detectable photons, while statistical data identify the concentration.

3. The control of large metal welds in civilian and military naval construction, which control is rendered particularly effective by the easy mobility of the DNS.

4. Cancer treatments via pseudo-proton irradiation which is expected to be less invasive and more localized than currently available proton irradiation since, in the former case, pseudo-protons are *attracted* by cancerous nuclei, while, in the latter case, protons are *repelled*.

5. Initiation of the much overdue research and development toward the recycling of nuclear waste via its stimulated decay while delivering energy, which recycling is possible in a number of ways, including the irradiation of nuclear waste pellets with a sufficiently intense pseudo-proton flux resulting in a deficiency of the nuclear charge Z for a given atomic number A and ensuing decay.

Note that the above applications are possible thanks to the *mean-life of DNS synthesized particle of the order of seconds or fractions thereof,* which are quite large for particle standard.

By comparison, negatively charged strongly interacting particles synthesized in contemporary physics laboratory with mean-lives of the order of 10^{-30} s or so have no known industrial or medical applications.

2.7. Einstein's determinism in nuclear structures.

2.7.1. Historical notes. As it is well known, nuclei were first assumed to be bound states of protons and electrons under their mutual Coulomb attraction which force, being inversely proportional to the square of the distance, is extremely big at nuclear distances.

This assumption was soon dismissed because it was considered inconsistent with quantum mechanics and unable to represent experimental data.

Consequently, nuclei were assumed to be bound states of protons and neutrons under a strong nuclear force that remains unknown to this day.

In reality, nuclei provide additional, rather clear evidence on the "lack of completeness" of quantum mechanics beyond scientific doubt, due to the well known inability by quantum mechanics to achieve an exact representation of nuclear experimental data.

As an illustration, in about one century of efforts and the use of large public funds, quantum mechanics has been unable to achieve a representation of the characteristics of the simplest nucleus, the *deuteron,* with embarrassing deviations between the prediction of the theory and the experimental data for heavier nuclei such as the Zirconium [25] (see also K. R. Popper [15], J. Dunning-Davies [16], J. Horgan [17] and quoted literature):

1. Quantum mechanics has been unable to represent the spin $J = 1$ of the deuteron in its ground state $L = 0$. This is due to the fact that the sole stable quantum mechanical bound state of two particles with spin $1/2$ is the singlet state, in which case the spin of the deuteron should be $J = 0$. For the intent of preserving the validity of quantum mechanics in nuclear physics, the spin $J = 1$ of the deuteron is represented via the assumption of a *combination of excited orbital spaces,* $L = 0, 1, 2$ contrary to the evidence that isolated deuterons are in their ground state, with the consequential inability to represent the deuteron positive parity $P = (-1)^L - 0$, and other insufficiencies.

2. Quantum mechanics has been unable to achieve an exact representation of the deuteron magnetic moment, because the representation of about 1% of the experimental value is still missing despite all possible relativistic (as well as quark-inspired) corrections, with large deviations occurring for heavier nuclei.

3. Quantum mechanics has been unable to represent the stability of the deuteron, since neutrons are well know to be naturally unstable, thus mandating a quantitative representation of the mechanism turning the neutron into a permanently stable particle when bonded to a proton.

4. Quantum mechanics has been unable to represent the proton-neutron exchange in the deuteron structure.

5. Quantum mechanics has been unable to achieve an explicit and concrete representation of the attractive strong nuclear force bonding together the proton and the neutron. Due to the intent of preserving the exact validity of quantum mechanics in nuclear physics, with the consequential sole representation of strong interactions via a potential in a Hamiltonian, the representation of strong nuclear forces has been attempted by adding potentials in the Hamiltonian, up to fifty additive potentials (sic) without the achievement of the needed representation of experimental data.

In a variety of works outlined in these papers (see monograph [25] and

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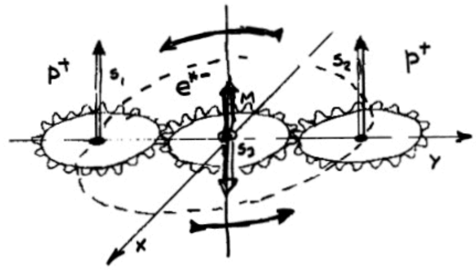

Figure 15: *The "gear model" used by Santilli [25], page 180, to illustrate the prediction from the spin $J_D = 1$ that the deuteron is a three-body hadronic bound state of one isoelectron and two isoprotons, with the individual couplings $p - e$ and $e - p$ being in singlet as necessary for stability. The spin $J_D = 1$ is then achieved from the "constrained angular momentum" of the bound state studied in Sections II-3.5.3 and II-3.5.4.*

papers quoted therein), Santilli has expressed the view that the "completion" of quantum into hadronic mechanics permits the apparent resolution of the above insufficiencies or sheer inconsistencies via a return to the original conception of nuclei as bound state of electrons and protons, although under non-linear, non-local and non-potential interactions caused by deep mutual overlapping.

2.7.2. Deuteron three-body model. Following the achievement of mathematical, theoretical, experimental and industrial advances in structure model (66) of the neutron as a hadronic bound state of one isoelectron and one isoproton, Santilli proposed in Section IV of Ref. [25], page 143 on (see also independent reviews [31] and [49]), the structure model of the deuteron D as a *restricted three-body hadronic bound state of two isoprotons and one isoelectron* (Figures 13, 14) which can be reduced in first approximation to a two-body hadronic bound state of one isoproton and one isoneutron

$$D = (\tilde{p}^+_\uparrow, \tilde{e}_\downarrow, \tilde{p}^+_\uparrow)_{hm} \approx (\tilde{p}^\uparrow, \tilde{n}^\uparrow)_{hm}. \tag{81}$$

Ref. [25] then presented various arguments showing that model (81) apparently resolves quantum mechanical insufficiencies (1 - 5) of the preceding section.

The analysis of model (81) was conducted in the 1999 monograph [25] following the publication in 1998 of the first proof of the EPR argument [9], as well as its detailed study in the 1995 monograph [33], Chapter 4, particularly Appendix 4C, page 166. Therefore, Santilli was aware that, unlike the case for atomic structures, deuteron model (81) does admit a classical counterpart (Section II-3.7) due to the inapplicability of Bell's inequality (9) [3] in favor of isoidentity (10).

However, at the time of writing the 1999 monograph [25], Santilli was unaware of the additional apparent proof of the progressive validity of Einstein's determinism in the interior of hadrons, nuclei and stars [10], and its apparent full achievement in the interior of gravitational collapse.

It is, therefore, important to review the main aspects of model (81) to indicate, apparently for the first time, its verification of Einstein's determinism we have called isodeterministic according to Lemma II-3.7 and Corollary II-3.7.1, and verify the apparent resolution of historical problems (1 - 5) of the preceding section.

1. NONRELATIVISTIC REPRESENTATION OF THE SPIN $J_D = 1$ IN THE GROUND STATE $L_D = 0$. With reference to Figure 24, model (81) includes *five* angular momenta: 1) The parallel spins S_1 and S_2 of the isoprotons; 2) The angular momentum L_{1-2} of the two isoprotons; 3) The spin of the isoelectron S_3; and 4) The angular momentum L_3 of the isoelectron inside one of the two isoprotons.

Recall that, according to experimental data on nuclear dimensions, the isoproton and the isoneutron are expected to have a diameter of $1.73\ fm$, while the diameter of the deuteron is $4,26\ fm$.

Consequently, the two isoprotons are separated by about $0.886\ fm$. In model (81), this space is occupied by the isoelectron acting like a "gluon" of the two isoprotons, while permitting their triplet alignment necessary for the spin $J_D = 1$.

In fact, the isoelectron is in the singlet coupling with each of the two isoprotons necessary for stability according to hadronic mechanics (Figure 14).

Recall also that, in semiclassical approximation, the wavepacket of the electron is of the order of $2.2\ fm$.

The above data provide not only a considerable synchronicity in model (81), but also a rather strong bond caused by the nuclear forces, the Coulomb attraction between the isoelectron and the isoprotons, as well as the strong attractive force originating from deep wave-overlapping in singlet coupling.

Recall finally that the diameter of the horizon for the full applicability of hadronic mechanics has been selected in Paper I to be $d = 2 \times 10^{-13}\ cm = 2 fm$, while the deuteron is about double that size.

The above data on dimensions suggest the use of the following isosymmetries for the characterization of the angular momentum and spin of the deuteron:

Use the regular isosymmetries $\hat{O}(3)$ and $\hat{SU}(2)$ for the characterization of the angular momentum and spin outside the hadronic horizon. We are here referring to isosymmetries which can be constructed via a non-unitary transform of the corresponding Lie symmetries, thus preserving conventional eigenvalues for angular momentum and spin (Section I-3.8 and I-3.9).

Use the irrreegular isosymmetries $\hat{O}(3)$ and $\hat{SU}(2)$ for the characterization of the angular momentum and spin inside the hadronic horizon. We are here referring to isosymmetries which cannot be constructed via non-unitary transforms o the corresponding Lie symmetries, by therefore having anomalous values of the angular momentum and spin (Section II-3.4).

It is then easy to see that the diameter $D = 2.59\ fm$ of the orbital motion of the two isoprotons is close to, but bigger than the hadronic horizon $d = 2\ fm$. We can therefore see the regular isorotational symmetry $\hat{O}3)$ admitting conventional angular momentum eigenvalues with ground state $L_{1-2} = 0$.

By contrast we have to use the irregular $\hat{O}(3)$ isosymmetry for the rotation of the isoelectron in the interior of the isoproton, resulting in the constrained value $L_3 = 1/2$ for the neutron (Sections 2.6.2 and 2.6.3).

In this way, Ref. [25] achieved the first known representation of the spin of the deuteron in its true ground state according to values:

$$J_D = S_1 + S_2 + S_3 + L_{1-2} + L_3 = \\ = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + 0 + \frac{1}{2} = 1. \tag{82}$$

By comparing the hadronic structure model of the neutron, Figure 10, and that of the deuteron, Figure 15, it is evident that the isoelectron is entirely compressed inside the isoproton in the former case, but only partially compressed in the latter case.

This occurrence is expected due to the presence in model (81) of the second isoproton with ensuing strongly attractive Coulomb force responsible for the partial extraction of the isoelectron from the isoproton, although this possibility has not been investigated to date.

Similarly, structure (81) appears to be particularly suited for the representation of the proton-neutron exchange in the deuteron structure, although this possibility has also not been subjected to a quantitative study to our best knowledge.

2. RELATIVISTIC REPRESENTATION OF THE SPIN $J_D = 1$ IN THE GROUND STATE $L_D = 0$. Ref. [25] was written following the 1995 publication of *Elements* *of Hadronic Mechanics,* Vols. I [32] and II [33]. Therefore, Ref. [25] studied model (81) with the entire sue of said Volumes whose knowledge, particularly that reviewed in Section II-3, is herein tacitly assumed.

The study of model (81) was done in Ref. [25], Section IV-2.3.4, page 161, via the use of the irregular iso-Dirac equation (II-201) defined on iso-semi-product (II-200) of an iso-Minkowskian space $\hat{M}(\hat{x}, \hat{\eta}, \hat{I}_{orb})$ for the relativistic description of the hadronic orbital motion, multiplied by a three-dimensional complex-valued isounitary isospace for the characterization of the hadronic spin $\hat{E}(\hat{z}, \hat{\eta}, \hat{I}_{spin}$.

It should be recalled that, in principle, *all* conventional intrinsic characteristics of the proton and the electron, including their charges, are expected to be mutated when said particles are in conditions of total mutual penetration. These mutations are quantitatively represented by the Lorentz-Poincaré-Santilli isosymmetry of the irregular iso-Dirac equation (Section II-2.5.11) and its irregular Pauli-Santilli isomatrices (Section II-3.4).

However, Ref. [25], Section 2.3.5, page 162, has shown that, for the case of the deuteron (but not necessarily so for heavier nuclei), the spins and changes of the isoprotons and of the isoelectron can be assumed in first approximation to have conventional values. In this case, the sole mutations achieving the representation of experimental data of the deuteron are those for shapes and constrained angular momenta.

The first known relativistic representation of the spin $J_D = 1$ of the deuteron in its ground state $L_D - 0$ was achieved in Ref. [25], Section 3.5.6, thanks to constrained spin-orbit coupling inside the isoneutron of model (81).

The latter coupling has been reviewed at the non-relativistic level in Section II-3.5.3 and at the relativistic level in Section II-3.5.4, see in particular Eqs. (II-118), (II-119), with a detailed description available in Ref. [33], Chapter 6.

The resulting relativistic characterization of hadronic angular momenta and spins is given by

$$\begin{aligned} S_{k\alpha} &= \epsilon_{kij} \hat{\gamma}_{i\alpha} \star \hat{\gamma}_{j\alpha}, \quad \alpha = \tilde{p}1, \tilde{p}2, \tilde{e}, \\ L_{k,p1-p2} &= \epsilon_{kij} r_{i,p1-p2} \star p_{j,p1-p2}, \end{aligned} \tag{83}$$

where $\hat{\gamma}_k$ are the irregular Dirac-Santilli isomatrices.

We then have the irregular isocommutation rules Ref. [36], Eqs. (6.4),

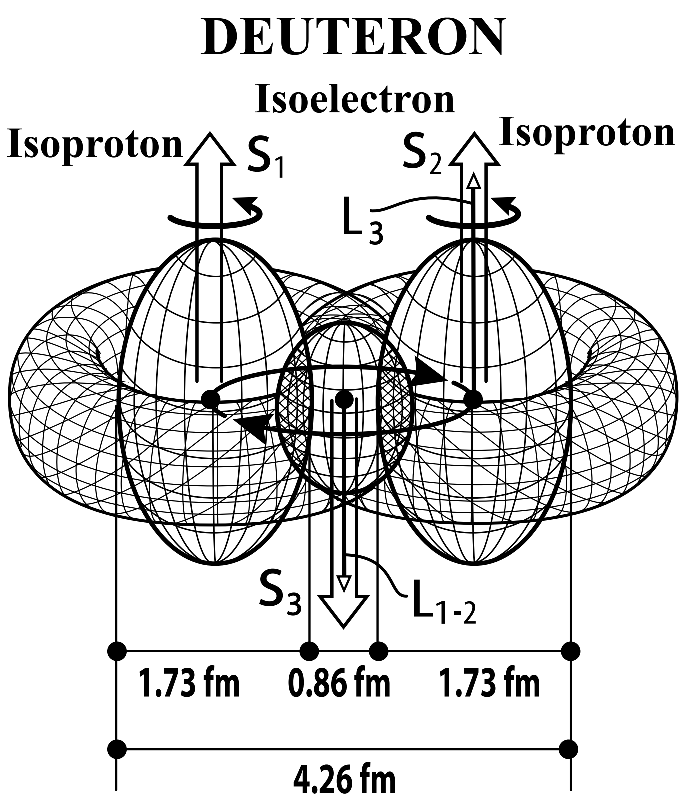

Figure 16: *A conceptual rendering of the deuteron model (91) illustrating the representation for the first time of spin $J_D = 1$ with angular momentum $L_D = 0$, where: the two isoprotons have the charge diameter $D_{\tilde{p}} = 1.73\ fm$; the deuteron has the charge diameter $D_D = 4.26\ fm$; the interspace between the isoprotons is then of $0.86\ fm$; the isoelectron has a point-like charge, the hadronic diameter $D^{hm}_{\tilde{e}} = 1\ fm$, and a semi-classical wavepacket with $D^{cl}_{\tilde{e}} = 2.2\ fm$. The above data illustrate the stability of model (81) with the central isoelectron allowing the isoprotons in the triplet coupling needed to achieve the spin $J_D = 1$ in the true ground state (Figure 14). The doughnuts around the two isoprotons are used to represent the proton-neutron exchange forces.*

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page 190,

$$[S_{i,\alpha}\hat{,}S_{j\alpha}] = 2\epsilon_{kij} m^2_{k\alpha} \hat{S}_{k\alpha},$$
$$[L_{i,1-2}\hat{,}L_{j,1-2}] = \epsilon_{ijk} m^2_{k,1-2} L_{k,1-2}, \tag{84}$$

and isoeigenvalues, Eqs, [36], Eqs. (6.4d), page 190,

$$\hat{S}_{3\alpha} \star |\hat{\psi}> = \pm \tfrac{1}{m_{1\alpha} m_{2\alpha}} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}}_{\alpha} \star |\hat{\psi}> = (m^{-2}_{1\alpha} m^{-2}_{2\alpha} + m^{-2}_{2\alpha} m^{-2}_{3\alpha} + m^{-2}_{3\alpha} m^{-2}_{1\alpha}) |\hat{\psi}>,$$
$$\hat{L}_{3,1-2} \star |\hat{\psi}> = \pm m_{1,1-2} m_{2,1-2} |\hat{\psi}>,$$
$$\hat{L}^{\hat{2}}_{\beta} 1-2 \star |\hat{\psi}> = (m^2_{1,1-2} m^2_{2,1-2} + m^2_{2,1-2} m^2_{3,1-2} + m^2_{3,1-2} m^2_{1,1-2}) |\hat{\psi}> . \tag{85}$$

Under the assumption that the hadronic medium in the interior of the proton is homogeneous, we can assume the values [36]

$$m_{1\alpha} = m_{1,1-2} = m_{2\alpha} = m_{2,1-2} = m_{3\alpha} = m_{3,1-2} = \frac{1}{\sqrt{2}} = 0.842, \tag{86}$$

under which the isoeigenvalues become

$$\hat{S}_{3\alpha} \star |\hat{\psi}> = \pm \tfrac{1}{2} |\hat{\psi}>,$$
$$\hat{S}^{\hat{2}}_{\alpha} \star |\hat{\psi}> = \tfrac{3}{4} |\hat{\psi}>,$$
$$\hat{L}_{3,1-2} \star |\hat{\psi}> = \pm \tfrac{1}{2} |\hat{\psi}>,$$
$$\hat{L}^{\hat{2}}_{1-2} \star |\hat{\psi}> = \tfrac{3}{4} |\hat{\psi}> . \tag{87}$$

It then follows that the total angular momentum of the isoneutron is given by

$$J_{\tilde{n}} = S_2 + S^{spin}_{\tilde{e}} + L^{orb}_{\tilde{e}} = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2}. \tag{88}$$

The total value of the deuteron spin $J_D = 1$ in its ground state $L_D = 0$ then follows from the fact indicated above that the orbital diameter is bigger than the hadronic horizon.

3. REPRESENTATION OF THE DEUTERON REST ENERGY, STABILITY AND SIZE. The hadronic representation of the rest energy, stability and charge radius of the deuteron were first achieved in Ref. [25], Eqs. (5.2.16), page 179,

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via the use of structural equations (29) (see also independent reviews [31], [49])

$$\left[\frac{1}{r^2}(\frac{d}{dr}r^2\frac{d}{dr}) + \bar{m}(E \pm W_1 e^{\frac{2}{r}} + W_2 \frac{e^{-br}}{1-e^{-br}})\right] = 0,$$

$$E_{tot} = E_1 + E_2 - E = 1875.7\ MeV, \ \ \bar{m} = \frac{m}{I^2}$$

$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_1}{\hbar} = \infty, \tag{89}$$

$$R = b^{-1} = 2.13 \times 10^{-13}\ cm = 2.13\ fm,$$

with solutions for the k-parameters

$$k_1 = 1 \ \ k_2 = 2.5. \tag{90}$$

The total mass/rest energy of the deuteron is then given by

$$M_D = M_p + M_n - E = 1875.7\ MeV, \tag{91}$$

where the binding energy is given by $E = 2.2\ MeV$.

Value (90) should be compared to the corresponding values for the neutron, Eqs. (69). As one can see, values (90) verify conditions (35) on the validity of the isodeterminism of Lemma II-3.7 as well as with the increase of the rapid convergence of isopperturbative series of Corollary II-3.7.1.

Note that, according to our assumption, values (81) verify the crucial condition (35) for the lack of existence of excited states, because excited states would imply the existing of the deuteron constituents out of the hadronic horizon with its consequential disintegration.

Note also that the stability of the deuteron is intrinsic in model (81) since the deuteron is reduced to the only known, massive, permanently stable particles existing in the universe, the proton and the electron.

Needless to say, the above representation should be considered as a first approximation, with a number of improvements being possible such as the inclusion of Coulomb interactions with possible excited states within the hadronic horizon.

4. REPRESENTATION OF THE DEUTERON MAGNETIC MOMENT. As it is well known, the experimental value of the deuteron magnetic moment is given by [13] [14]

$$\mu_D^{exp} = 0.857\ \mu_N. \tag{92}$$

For the quantum mechanical ground state with $L_D = 0$, in case consistent, one would obtains

$$\mu_D^{qm} = \frac{1}{2}(g_p^S + g_n^S) = 2.792 - 1.913 = 0.879\ \mu_N, \tag{93}$$

(where the g's are tabulated g-factors), by therefore missing about 3% of experimental value (256), which deviation cannot be entirely resolved via relativistic or other corrections.

The first, exact, non-relativistic representation of the deuteron magnetic moment was achieved by Santilli for the approximate two-body version of model (81) in paper [50] of 1994 (written at the Joint Institute for Nuclear Research, Dubna, Russia) under the sole assumption that the charge distributions of protons and neutrons are deformed under strong nuclear forces, resulting in a corresponding mutation of their intrinsic magnetic moment as predicted by E. Fermi [13], J.M. Blatt and V. F. Weisskopf [14], and other founders of nuclear physics (see their recollection in Ref. [25], page 158-159, Section IV-2.3.1, under the title *The Historical Hypothesis*).

Santilli achieved the first, exact, relativistic representation of magnetic moment of the two-body deuteron model in paper [51] of 1996 via the use of the iso-Dirac equation.

The 1998 Ref. [25], Section IV-2.3.6, page 163, presents the first known exact representation of the deuteron magnetic moment for the full three-body model (81).

It is important to review and upgrade the latter representation to show its compatibility with Einstein's determinism according to Lemma II-3.7 and rapid convergence according to Corollary II-3.7.1.

Assume, in first approximation, that the isoparticles of model (81) have the same spheroidal shape with semiaxes $n_1^2,\ n_2^2,\ n_3^2$ under the condition of preserving the volume of the original particles assumed for simplicity to be normalized to 1

$$n_1^2 \times n_2^2 \times n_3^2 = 1. \tag{94}$$

Represents with n_4^2 the density of the two isoprotons (denoted with the subindeces 1 and 2), resulting in the isotopic element for the two isoprotons

$$\hat{T}_{1,2} = Diag.(n_1^2, n_2^2, n_3^2, n_4^2). \tag{95}$$

Then, the Lorentz-Santilli isotransforms (II-42) imply the following mutation of the intrinsic magnetic moment of the proton (see Ref. [36], page 190 for a detailed derivation with the notation $b_\mu = 1/n_\mu,\ \mu = 1, 2, 3, 4$)

$$\begin{gathered} \mu_{1,2}^{qm} \ \rightarrow \ \tilde{\mu}_{1,2}^{hm} = \tfrac{n_4}{n_3}\mu_{1,2}^{qm} = \tilde{g}^S \mu_N, \\ \tilde{g} = \tfrac{n_4}{n_3} g. \end{gathered} \tag{96}$$

Recall that the isoprotons have Spin $S_{1,2} = 1/2$ and $L_D = 0$.

In order to compute the magnetic moment of the three-body model (81), we first compute *the intrinsic and orbital contributions of the isoprotons*

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via a simple isotopy of the corresponding quantum mechanical expression [45]

$$\tilde{\mu}_k^{hm} =$$

$$\frac{1}{J+1} < \hat{Y}_{L,S}|\tfrac{1}{2}\tilde{g}_k^S S + \tfrac{1}{2}\tilde{g}_k^L L)|\hat{Y}(L,S) >= \tag{97}$$

$$= \frac{1}{2(J+1)}\tilde{g}_k^S\left[J(J+1) - L(L+1) + S(S+1),\right], \quad k = \tilde{p}1, \tilde{p}2,$$

where $\hat{Y}_{L,S}$ are the isospherical isoharmonics (Ref. [33], Section 6.4, page 240) and $J_k = S_k + L_k, \ k = 1, 2$ is the total angular momentum.

Simple calculations show that, for $L_k = 0$ and $\mu_p^{exp} = 2.79\ \mu_N$, expression (261) yields the value

$$\tilde{\mu}_k^{hm} =$$

$$= \frac{1}{2}\tilde{g}_k^s = \frac{n_4}{2n_3}g_k^S = \frac{n_4}{n_3}2.79\ \mu_N. \tag{98}$$

To compute *the intrinsic and orbital contribution from the isoelectron,* we recall from Eq. (69) that its intrinsic value is ignorable in first approximation, the sole expected contribution being that from its constrained orbital motion inside the proton.

By recalling that the electron has a point-like charge which cannot be mutated by isotopies, then the density of the electron remains the conventional one for the vacuum with $n_{4,\tilde{e}} = 1$.

Under the approximation that the deformation of the isoelectron is the same as those of the two isoprotons, we have the mutation of the value (70) for the neutron,

$$\mu_{\tilde{e},orb}^{hm} = -\frac{1}{n_3}\mu_{\tilde{e}}^{orb} = -\frac{1}{n_3}4.6\ \mu_N. \tag{99}$$

The quantum mechanical magnetic moment of the three-body model (81) of the deuteron, in case consistent, would be given by

$$\mu_D^{qm} = 2\mu_p + \mu_e = 2 \times 2.79 - 4.6\ \mu_N = 0.98\mu_N, \tag{100}$$

thus lacking an exact representation of value (92) because in *excess* for about 8%.

By comparison, the magnetic moment for model (81) according to hadronic mechanics is given by

$$\mu_D^{hm} = 2\frac{n_4}{n_3}\mu_{\tilde{p}} + \frac{1}{n_3}\mu_{\tilde{e}} = \frac{n_4}{n_3}5.78 - \frac{1}{n_3}4.6\ \mu_N = 0.857\ \mu_N. \tag{101}$$

By assuming the value $n_4 = 0.62$ from the density of the proton-antiproton fireball in the Bose-Einstein correlation [46] [47], we can write the *hadronic representation of the deuteron magnetic moment*

$$\mu_D = \frac{0.62 \times 5.78}{n_3} - \frac{4.6}{n_3} = 0.857\ \mu_N, \tag{102}$$

which is verified for

$$n_3 = 1.20. \tag{103}$$

The use of the volume-preserving condition (94) then yields the remaining ellipsoidal values

$$n_1 = n_2 = 0.91. \tag{104}$$

The above values imply that the shape deformation of the deuteron constituents caused by the internal strong nuclear force turns charge distributions and/or wavepackets from a spherical to a *prolate* spheroidal ellipsoid with semiaxes

$$n_1^2 = n_2^2 = 0.83, \;\; n_3^2 = 1.44, \tag{105}$$

which prolate deformation was predicted by all preceding works by Santilli in the field.

Note that *the magnetic moment of the deuteron is represented exactly not only in its value, but also in its sign,* that of being parallel to the spins of the isoprotons.

It then follows that *representation (101) can be construed as evidence on the presence of two protons with parallel spins inside the deuteron,* which presence mandates three-body model (81) with a central exchanged electron for stability (Figures 14, 15).

Note also that the *prolate* deformation of shape is necessary because the quantum mechanical expression (93) is in *excess* of experimental value (92). By comparison, an *oblate* deformation would *increase* said excess value due to the increase of the rotating charge distribution at the equator.

Almost needless to say, representation (101) is intended to illustrate the *capability* by hadronic mechanics to achieve an exact representation of the deuteron magnetic moment.

A number of improvement of representation (101) are possible, among which we mention a more accurate value of the density n_4 of the proton, a more accurate value of the orbital contribution of the isoelectron due to its geometric difference with that for the neutron indicated above, and other improvements.

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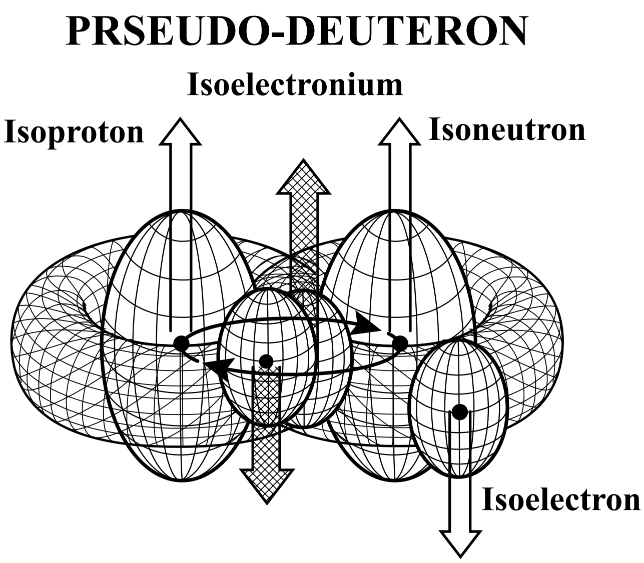

Figure 17: *An illustration of the negatively charged pseudo-deuteron predicted by hadronic mechanics via the use of a deuterium gas in the Directional Neutron Source of Figure 8. Due to its negative charge, the pseudo-deuteron can potentially initiate a basically new type of nuclear fusions, here called hyperfusions, based on the natural attraction of the pseudo-deuteron by natural deuteron resulting in the helium without production of harmful radiations or release of radioactive waste.*

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5. REPRESENTATION OF NUCLEAR FORCES. Hadronic structure model (81) achieves the first known explicit concrete representation of strong nuclear forces thanks to the structural lifting of the mathematics underlying quantum mechanics into the covering isomathematics.

Such a generalization resulted to be necessary due to the non-Hamiltonian character of strong interactions, with particular reference to the generalization of the associative product into isoproduct (1) and the representation of strong nuclear forces via isotopic element (4), with ensuing lifting of the Schrödinger equation, from its historical form $H\psi = E\psi$, into the isotopic form

$$H \star \psi = H(r,p)\hat{T}(\psi, ...)\hat{\psi} = E\hat{\psi}. \tag{106}$$

6. REDUCTION OF STABLE NUCLEI TO ISOELECTRONS AND ISOPROTONS. The reduction of neutrons to isoprotons and isoelectrons clearly implies the possible reduction of all stable matter in the universe to the only known, massive and stable particles, protons and electrons [52].

This important possibility has been confirmed by the reduction of the deuteron to isoprotons and isoelectrons [25], and it is under study by A. A. Bhalekar and R. M. Santilli [54].

2.7.3. Industrial applications. In Section I-1.4 we have indicated the societal duty of seeking basically *new* forms of clean energies, while continuing research along conventional lines.

This is due to the inability of identifying industrially viable nuclear fusions in about three quarters of a century and the investment of billions of dollars of public funds.

Recall that the primary obstacle opposing the achievement of the controlled nuclear fusion of two deuterons $D(1,2,1)$ into the Helium $He(2.4.0)$ is the *extreme repulsive Coulomb force between nuclei* that, the distance of $1\ fm = 10^{-15}\ m$ needed to activate strong nuclear forces is given by

$$\begin{aligned} F = k\tfrac{e^2}{r^2} = \\ = (8.99 \times 10^9)\tfrac{(1.60\times10^{-19})^2}{(10^{-15})^2} = 230\ N, \end{aligned} \tag{107}$$

which force is extremely large for nuclear standards.

As it is well known, the efforts done to date for overcoming such a large repulsive force have been the use of high energies resulting, as expected, in uncontrollable instabilities at the initiation of the fusion itself.

Another problem that has prevented the achievement of the controlled

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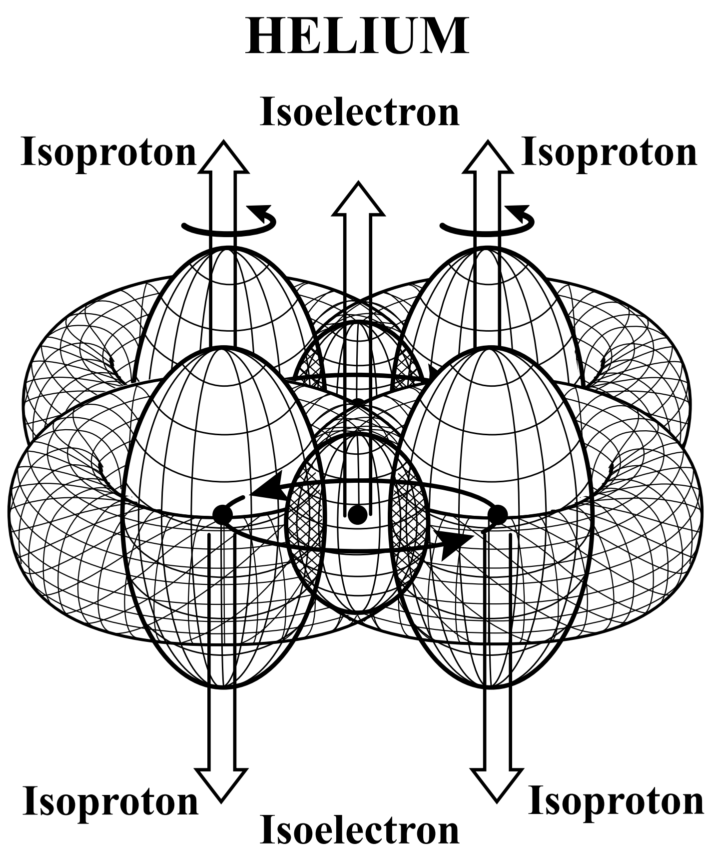

Figure 18: *An illustration of the helium nucleus as predicted by hadronic mechanics and consisting of four isoprotons and two isoelectrons. Note that the illustration depicts the conventional conception of the Helium as being composed by two protons and two neutrons, with the sole replacement of the neutrons with their physical constituents permitted by hadronic mechanics, the proton and the electron. Note finally that the uncertainties of the constituents of the helium follow Einstein's determinism per Lemma II-3.7 and Corollary II-3.7.1.*

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fusion of deuterons into helium is the need for their *singlet coupling*

$$D(1,2,1_{\downarrow}) + D(1,2,1_{\uparrow}) \rightarrow He(2,4,0), \tag{108}$$

which is of extremely difficult engineering realization at low energies, and virtually impossible at high energies, resulting in nuclear fusions that can at best be at random.

For additional technical requirements needed to achieve the fusion of deuterons into helium, interested readers may consult the *hadronic laws for nuclear fusions,* which have been systematically studied in Ref. [25], Chapter 4 and Section 4.2, page 188, in particular.

The Directional Neutron Source (DNS, Figure 8) is produced and sold by Thunder Energies Corporation, now called Hadronic Technologies Corporation, for the synthesis of neutrons (Section 2.6.3) and pseudo-protons (Section 2.6.5) from a commercially available hydrogen gas.

However, the same DNS is produced for the use, without any modification, of a commercially available *deuterium gas* as a basic feedstock, in which case, hadronic mechanics predicts the synthesis of a *negatively charged deuteron* called *pseudo-deuteron* [48] (patent pending) which is predicted to have a structure of the type (Figure 17)

$$\hat{D} = \left[\tilde{p}_{\uparrow}, (\tilde{e}^{-}_{\uparrow}, \tilde{e}^{-}_{\downarrow}), \tilde{n}_{\uparrow})\right]_{hm}. \tag{109}$$

The mechanism for the synthesis of the pseudo-deuteron (which we write with nuclear symbols $D(-1,2,1)$) from a deuterium gas stems from the main feature of the DNS, that of actually "compressing" electrons in the interior hadrons (Figure 9) thanks to a specially conceived electric arc and other engineering features. Consequently, pseudo-deuteria are expected to be synthesized in the DNF operating on deuterium gas, following molecular separation with the ensuing presence of valence electron pairs called *isoelectronia* (Section 2.8.2 and Figure 20), of course, in a way dependent on power, pressure, flow, and other factors.

Preliminary calculations via hadronic mechanics indicate that the pseudo-deuteron should have a mean-life of the order of a second, thus being sufficient for industrial applications.

The importance of the research under consideration is that, contrary to current trends, *pseudo-deuterons are attracted, rather than being repelled by natural deuterons.*

This basic feature provides plausible means to search for basically *new* nuclear fusions called *hyper-fusions,* here referred to fusions of natural, positively charged nuclei and synthesized negatively charged nuclei, such as the hyper-fusion of a pseudo-deuteron and a deuteron into the helium

$$\hat{D}(-1,2,1_{\uparrow}) + D(1,2,1_{\downarrow}) \ \rightarrow \ He(2,4,0) + 2e^{-}, \tag{110}$$

where the singlet coupling is naturally achieved because the pseudo-deuteron has not only the charge, but also the magnetic moment opposite that of the natural deuteron.

The energy expected to be released by each hyperfusion (110) is given by

$$\Delta E = E_{He} - (E_{\hat{D}} + E_D) == 23.8\ MeV = 3.81 \times 10^{-12}\ J, \tag{111}$$

and occurs *without any emission of harmful radiation (since electrons can be stopped with a metal shield), and without the release of radioactive waste.*

As an illustration, the possible achievement of the rather realistic number of 10^{18} controlled fusions (110) per hour would yield the significant release of about $10^6\ J$ of clean energy per hour without harmful radiations or waste.

In regard to the possible achievement of controlled hyper-fusions, we should mention that the DNS is considered in this section for the mere intent of achieving experimental measurements on the sole *existence* of pseudo-deuterons. Their production in the needed number and energy, and the hadronic reactor needed for the utilization of the produced heat, evidently require specialized equipment under proper funding.

By recalling that conventional nuclear fusions have requested billions of dollars of investments of public funds without any industrially viable result to date, readers should be warned against any expectation prior to the investment of equally large funds

2.8. Einstein's determinism in molecular structures.

2.8.1. Insufficiencies of quantum chemistry. There is no doubt that quantum chemistry has permitted truly historical advances in the 20th century. However, as it is the case for nuclear physics, serious scholars are expected to admit that quantum chemistry cannot be *exactly valid* for chemical structures and processes because of a number of insufficiencies, such as [30] (for dissident views, see also Refs. [15] [16] [17]):

1. Even though possessing excellent practical values, *the quantum chemical notion of valence is a 'nomenclature' without quantitative treatment because, due to their equal charge, the Schrödinger equation predicts that valence electron pairs 'repel' (rather than attract) each other,* due to the necessary sign + of the Coulomb potential in the equation for the electron pair

$$i\frac{\partial}{\partial t}\psi(t,r) = \left[-\frac{\hbar^2}{m}\Delta_r + \frac{e^2}{r}\right]\psi(t,r). \tag{112}$$

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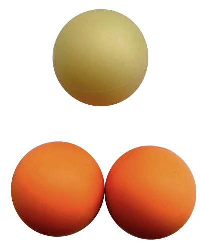

Figure 19: *Lectures in hadronic chemistry are generally initiated by showing a ball in representation of a hydrogen atom with the recollection that quantum mechanics achieved an exact representation of all its experimental data. The speaker then shows two joined balls representing a hydrogen molecule, with the recollection that, this time, quantum mechanics and chemistry have not achieved an exact representation of molecular data. The reason indicated for this dichotomy is that mutual distances in the atomic structure, that are of the order of* 10^{-8} cm*, are such to allow an effective point-like approximation of the constituents. By contrast, mutual distances for valence electron pairs in molecular structures, which are of the order of* 10^{-13} cm*, are smaller than the size of the electron wavepackets, which is of about* 2.2×10^{-13} cm*, thus prohibiting an effective point-like approximation of particles in favor of the representation of their actual size and as well as of the ensuing contact non-potential interactions due to deep wave overlapping.*

In particular, the *repulsive* force between the two identical valence electrons has the extremely big value of $230\ N$ at mutual distances of $1\ fm$, Eq. (107), with repulsive values remaining big at atomic distances. Consequently, *quantum chemistry misses a quantitative model of molecular structures.*

2. According to quantum chemistry (see, e.g., Ref. [55]), valence electron bonds between two atoms are created by the overlap of atomic orbitals. Even though approximately valid, such a model is "incomplete" because, due to the point-like approximation of the electrons, said orbitals remain mostly independent, thus allowing their polarizations under a sufficiently strong electric or magnetic field, with the consequential prediction that substances are generally ferromagnetic.

3. Quantum mechanics has achieved an exact representation of all experimental data of the *hydrogen atom* H, but in the transition to the *hydrogen molecule* $H_2 = H - H$ (where $-$ represents valence bond), quantum mechanics and chemistry still miss $1\ \%$ of the H_2 binding energy which corresponds to the rather significant value of $950\ BTU$.

4. Quantum chemistry still misses to this day the quantitative identi-

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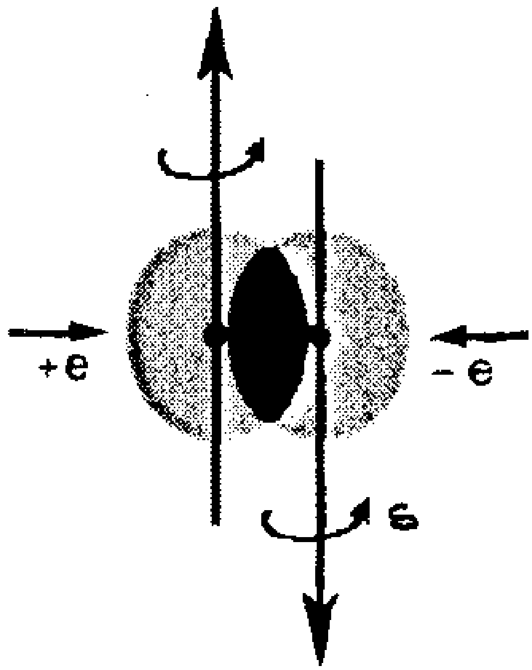

Figure 20: *According to the basic principles of quantum mechanics and chemistry, electron valence pairs in molecular structures should "repel" (rather than "attract") each other with a force of the order of* 230 *N, Eq. (107), which is extremely big for particle standards. Hadronic mathematics, mechanics and chemistry have achieved an attractive force between valence electron pairs so strong to turn them into a quasi-stable particle called the isoelectronium [30], that allowed the first known achievement of exact representation of experimental data on the hydrogen [52] and water [53] molecules that are not possible with quantum chemistry. In Santilli's view, the impossibility in quantum chemistry to achieve an attractive force between valence electron pairs as occurring in nature, is additional evidence supporting the need for the "completion" of quantum principles into covering vistas according to Einstein's legacy.*

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fication of the *attractive force* bonding together neutral, dielectric and diamagnetic molecules in their liquid state, with ensuing lack of quantitative representation of the liquid state for fuels.

5. The basic axioms of quantum chemistry are reversible over time. Consequently, quantum chemistry is structurally unable to provide an exact representation of combustion, as well as all energy-releasing processes with ensuing implications for the solution of problems of large societal values, such as the improvement of fossil fuel combustion.

We cannot possibly review in this paper the ongoing process toward the solution of the above insufficiencies of quantum chemistry via the covering hadronic chemistry [30]. Nevertheless, we believe it is important to indicate the main evidence supporting the validity for molecular structures of the "completion" of quantum chemistry into a suitable covering theory according to Einstein-Podolsky-Rosen argument [1].

Insufficiencies (1 - 5) above are known to chemists, as attested by the widespread distribution of works in the field by Santilli and other scholars. For this reason, said insufficiencies are referred to as *the best kept secrets in the best Ph. D. courses in chemistry around the world.*

2.8.2. Attractive force in valence electron bonds. Recall that in the 1978 paper [19], Santilli: 1) Identified the non-potential, strongly attractive force created by the deep overlapping of the wavepackets of particles in singlet coupling; 2) Constructed the foundations of hadronic mechanics with particular reference to the iso-Schrödinger equation for quantitative treatments of non-potential interactions; and 3) Applied the emerging new methods to achieve an exact representation of all characteristics of mesons as bound states of electrons and positrons (Sections 2.5.3 to 2.5.5).

In the 1995 paper [95], A. O. E. Animalu and R. M. Santilli showed that the indicated non-potential force is essentially charge independent since it remains strongly attractive also for electron-electron pairs in superconductivity thanks to the "absorption" of the Coulomb potential by the Hulten potential irrespective of its sign, Eq. (5.1.15), page 836, Ref. [19] (reviewed in Eq. (31) above).

Thanks to the "completion" of quantum into hadronic chemistry (hc), in the 2001 monograph [30] Santilli worked out a new notion of valence bonds, today known as *strong valence bond,* which consists of valence isoelectron pairs, known as *isoelectronium* and denoted $\mathcal{I}$, with a fully identified *attractive* force with the structure (see the independent reviews [31]

[96] [97])

$$\mathcal{I} = (\tilde{e}_{\uparrow}, \tilde{e}_{\downarrow})_{hc}. \quad (113)$$

The achievement of the attractive force between the identical valence electrons, apparently done for the first time in Ref. [30], Chapter 4, is described in detail in Section 2.3, and it is essentially based on the "completion" of all products into isoproducts (1) with consequential "completion" of the local Newton's differential calculus into the non-local isodifferential calculus (7) (8), resulting in the iso-Schrödinger equation

$$\begin{gathered} i\frac{\partial}{\partial t}\psi(t,r) = \left[-\frac{\hbar^2}{m}\Delta_r + \frac{e^2}{r}\right]\psi(t,r) \ \rightarrow \\ \rightarrow i\frac{\hat{\partial}}{\hat{\partial} t}\psi(t,r) = \left[-\frac{\hbar^2}{m}\hat{\Delta}_r + \frac{e^2}{r}\right]\hat{T}(\psi,\hat{\psi},...)\hat{\psi}(t,r) \approx \\ \approx i\frac{\hat{\partial}}{\hat{\partial} t}\hat{\psi}(t,r) = \left[-\frac{\hbar^2}{m}\hat{\Delta}_r - W\frac{e^{-br}}{1-e^{-br}}\right]\hat{T}(\psi,\hat{\psi},...)\hat{\psi}(t,r), \end{gathered} \quad (114)$$

with a strong attraction characterized by the Hulten potential.

In Santilli's view, *the achievement of an attractive force in valence electron pairs via the "completion" of the quantum wavefunction $\psi(t,r)$ into the hadronic isowavefunction $\hat{\psi}(t,r)$ is a strong confirmation of the final statement by Einstein, Podolski and Rosen [1] according to which quantum wavefunctions do not represent the entire physical or chemical reality.*

The strength of the internal bond allows the reduction of four-body molecules, such as the hydrogen $H_2 = H - H$, down to a *restricted three-body system* that, as such, admit full analytic solutions with major simplification of otherwise notoriously complex elaborations.

It should be noted that, besides the existence of molecules, the biggest evidence on the existence of the isoelectronium as a particle is provided by experimental data on the photoionization of the hydrogen and helium molecules (see, e.g., Ref. [56]) in which bonded electron pairs in singlet have been systematically detected to survive molecular separation.

For comparison with preceding hadronic structures, let us recall that Ref. [30], Eqs. (4.5) to (4.23), pages 169 to 173, provided the following representation via model (29) of the rest energy $E = 1\ MeV$ and change radius $R = 6.84 \times 10^{-11}\ cm$ of the isoelectronium for the case of full stability

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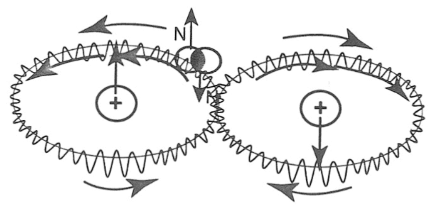

Figure 21: *An illustration of the isochemical model of the hydrogen molecule achieved by R. M. Santilli and D. D. Shillady [52] with the following main features: 1) Valence electron pairs are bonded into isoelectronia according to the principles of hadronic chemistry; 2) Due to their strong internal forces, isoelectronia are forced to have oo orbits around corresponding nuclei resulting in opposite angular momenta that represent the diamagnetic character of the hydrogen and other molecules; 3) The conventional four-body equations for the hydrogen molecule are reduced to a restricted three-body form admitting a full analytic solution.*

$\tau = \infty$

$$\left[\frac{1}{r^2}\left(\frac{d}{dr}r^2\frac{d}{dr}\right) + \bar{m}\left(E \pm W_1 e^{\frac{2}{r}} + W_2\frac{e^{-br}}{1-e^{-br}}\right)\right] = 0,$$

$$E_{tot} = E_1 + E_2 - E = 1\ MeV,\ \ \bar{m} = \frac{m}{I^2} \tag{115}$$

$$\tau^{-1} = 2\pi\lambda^2|\hat{\psi}(0)|^2\frac{\alpha^2 E_1}{\hbar} = \infty,$$

$$R = b^{-1} = 6.84 \times 10^{-11}\ cm,$$

with solutions for the k-parameters

$$k_1 = 0.19\ \ k_2 = 1, \tag{116}$$

for which the binding energy is null (in first approximation) as expected from the sole use of contact non-potential interactions, resulting in the total energy of the isoelectronium $E = E_1 + E_2 = 1.022\ MeV$.

Note that, as expected from the charge independence of contact interactions, values (114) for the isoelectronium $(\tilde{e}_1, \tilde{e}_2)_{hc}$ are rather close to the corresponding values (50) for f the π^0 meson $(\tilde{e}^-, \tilde{e}^+)_{hm}$.

Interested readers should meditate a moment on the capability of contact interactions to achieve a strongly attractive force by overcoming the extremely big repulsive Coulomb force of $230\ N$.

This occurrence emphasizes the need in hadronic structure models such as (112) of the notion of *isoparticles* in which *all* intrinsic characteristics are mutated, thus including the charge.

This occurrence also suggests the *study of structure models of the elementary charge,* such as that of Ref. [44], capable of such a mutation in condition of deep mutual overlapping to represent the physical evidence of a strong attraction in singlet valence couplings.

In 1999, D. D. Shillady and R. M. Santilli proved that the isotopic branch of hadronic chemistry achieves the first known, numerically exact representation of the binding energy and other characteristics of the hydrogen [52] (Figure 23) and water [53] (Figure 24) molecules via isoperturbative isoseries (namely isoseries based on isoproduct $A\hat{T}B$ with $\hat{T} \ll 1$) whose convergence is at least *one thousand times faster* than the convergence of conventional quantum chemical series, thus providing an important experimental confirmation in molecular structures of both Einstein's determinism per Lemma II-3.7 as well as Corollary II-3.7.1.

In regard to the rather complex case in which the isoelectronium is partially unstable, the numerically exact representation of H_2 and H_2O data, and other aspects, we refer the interested reader to the above quoted literature.

The strong valence bond of isoelectronia evidently resolves Insufficiencies 1) and 2) of the preceding section.

Insufficiency 3) is resolved by the fact that, due to the strength of its bond, isoelectronium (112) is forced to have an oo-orbit around the respective nuclei (Figure 22) with ensuing opposite angular momenta and opposite magnetic polarities that permit a quantitative representation of the diamagnetic character of the hydrogen, water and other molecules.

The apparent resolution of Insufficiency 4) was provided by the new chemical species of *magnecules* [30] (see the U. S. patent [57], the latest experimental verification [58] and Figure 25) that allowed the identification of an actual attractive force between water molecules in their liquid state [98].

The resolution of Insufficiency 5) can be studied via the *Lie-admissible branch of hadronic chemistry,* also called *genochemistry* [30], although it does not appear that, at this writing, a consistent representation of energy releasing processes is of interest to contemporary chemists at large.

2.8.3. Industrial applications. It is a truism to state that the combustion of fossil fuels occurring in the ongoing disproportionate number of civilian, industrial and military vehicles is the same as it was at the dawn of our civilization some $50,000$ years ago because, in all cases, we strike a spark and lit the fuel.

To be "True Researchers" in Einstein's words, it is our societal duty to

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Species	H_2	H_2O	HF
SCF-energy (DH) (a.u.)	-1.132800^a	-76.051524	-100.057186
Hartree-Fock^d (a.u.)			-100.07185^d
Iso-energy (a.u.)	-1.174441^c	-76.398229^c	-100.459500^c
Horizon R_c (Å)	0.00671	0.00038	0.00030
QMC energy^{d,e} (a.u.)	-1.17447	-76.430020^e	-100.44296^d
Exact non-rel. (a.u.)	-1.174474^f		-100.4595^d
Corellation (%)	99.9^b	91.6^b	103.8
SCF-dipole (D)	0.0	1.996828	1.946698
Iso-dipole (D)	0.0	1.847437	1.841378
Exp. dipole (D)	0.0	1.85^g	1.82^g
Time^h (min:s)	0:15.49	10:08.31	6:28.48

(DH^+) Dunning-Huzinaga (10S/6P), [6,2,1,1,1/4,1,1]+H_2P_1+3D1.
^{a}LEAO-6G1S + optimized GLO-2S and GLO-2P.
^{b}Relative to the basis set used here, not quite HF-limit.
^{c}Iso-energy calibrated to give exact energy for HF.
^{d}Hartree-Fock and QMC energies from Luchow and Anderson [33].
^{e}QMC energies from Hammond *et al.* [30].
^{f}First 7 sig. fig. from Kolos and Wolniewicz [34].
^{g}Data from Chemical Rubber Handbook, 61st ed., p. E60.
^{h}Run times on an O2 Silicon Graphics workstation (100 MFLOPS max.).

Figure 22: *A reproduction of Table 4.2 of Ref. [52] presenting the first known exact representation of the experimental data of the hydrogen molecule via isomathematics and isochemistry, by providing an experimental confirmation of the validity in molecular structres of Einstein's determinism according Lemma II-3.7, with a convergence of the isoperturbative series at least one thousand times faster than conventional series, by therefore providing an experimental confirmation of Corollary II-3.7.1.*

initiate the laborious process of trial and errors in seeking a clean combustion of fossil fuels that, to be as such, as to be *new,* namely based on new mathematical, physical and chemical principles.

In the hope of initiating the expectedly long and laborious process of trials and errors, Santilli has submitted in Ref. [59] the study of the new form of combustion of carbon and oxygen, known as *hypercombustion,* intended to achieve the full combustion of fossil fuels via a small percentage of nuclear fusions of the novel magnecular forms of $6-C-12$ and $8-O-16$ into the stable $14-Si-28$, thus without the emission of harmful radiations and without the release of radioactive waste [69] [70] (Figure 24).

We should mention in this respect the gaseous fuel *magnegas* [30] which is synthesized by hadronic reactors from a mixture of oil and water, in production and sale world wide by Magnegas Corporation, now called Taronis Corporation, whose combustion in air shows no appreciable carbon monoxide CO and hydrocarbons (HC) (Figure 25).

This environmental result was achieved via a combustion temperature more than double that of commercially available fuels, which temperature is permitted by magnecular bonds that are *weaker* than molecular bonds, and at which temperature CO and HC cannot remain unburned.

HyerCombustion aims at achieving the needed higher combustion temperature of petroleum fuels via the use of the indicated nuclear processes that are solely possible under the"completion" of quantum into hadronic mechanics.

It appears advisable to indicate that, besides the expected environmental advances, the new chemical species of magnecules appears to have significant medical applications, such as the possible killing of the corona virus in lungs (and other organs) of patients via ventilators releasing the new polarized species of *magneoxygen,* rather than conventional oxygen [62] - [65].

2.9. Einstein's determinism in gravitational collapse.

In the recent paper [10], Santilli recalls that iso-space-time metrics contain as particular cases all possible symmetric metrics in $(3+1)$-dimensions, thus including the Riemannian metric.

Ref. [10] then factorizes the space component of the Schwartzchild metric $g_s(r)$ according to isotopic rule introduced in Refs. [66] [67]

$$g_s(r) = \hat{T}(r)\delta, \tag{117}$$

where δ is the Euclidean metric.

In this way, Santilli reaches the following realization of the isotopic

	OH^+	OH^-	H_2O	HF
SCF-Energy[a]	-74.860377	-75.396624	-76.058000	-100.060379
Hartree-Fock[b]				-100.07185[b]
Iso-Energy[c]	-75.056678	-75.554299	-76.388340	-100.448029
Horizon R_c (Å)	0.00038	0.00038	0.00038	0.00030
QMC Energy[b,d]	-76.430020[d]			-100.44296[b]
Exact non-rel.				-100.4595
Iso-Dipole (D)	5.552581	8.638473	1.847437	1.8413778
Exper. Dipole			1.84	1.82

[a] Dunning-Huzinaga ($10s/6p$), (6,2,1,1,1/4,1,1)+H2s1+H2p1+3d1.
[b] Iso-Energy calibrated to give maximum correlation for HF.
[c] Hartree-Fock and QMC energies from Luchow and Anderson [22].
[d] QMC energies from Hammond, Lester and Reynolds [21].

Figure 23: *A reproduction of Table 5.1. of Ref. [53] presenting the first exact representation of the experimental data of the water molecule, including its electric and magnetic moments, via isomathematics and isochemistry, by therefore providing a second experimental confirmation of the validity in molecular structures of Einstein's determinism according to Lemma II-3.7, with a convergence of the isoperturbative series at least one thousand time faster than conventional series, thus providing an experimental confirmation of Corollary II-3.7.1.*

element

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}, \tag{118}$$

where M is the gravitational mass of the body considered with ensuing isodeterministic isoprinciple, Ref. [10], Eq. (46), page 16,

$$\Delta\hat{r}\Delta\hat{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r\to 0} = 0, \tag{119}$$

which confirms the possible recovering of full classical determinism in the interior of gravitational collapse essentially as predicted by Einstein (see Ref. [16], Chapter 6 in particular, for a critical analysis of black holes).

It should perhaps be indicated that the 1993 paper [100] identified the universal isosymmetry of all possible (non-singular) Riemannian line elements in $3 + 1$)-dimensions formulated on iso-Minkowskian isospaces [78] over isofields. Papers [66] [67] introduced the factorization of a full Riemannian metric $g(x),\ x = (r, t)$ in $(3 + 1)$-dimensions

$$g(x) = \hat{T}_{gr}(x)\eta, \tag{120}$$

where $\hat{T}_{gr}$ is the *gravitational isotopic element*, and η is the Minkowski metric $\eta = Diag.(1, 1, 1, -1)$.

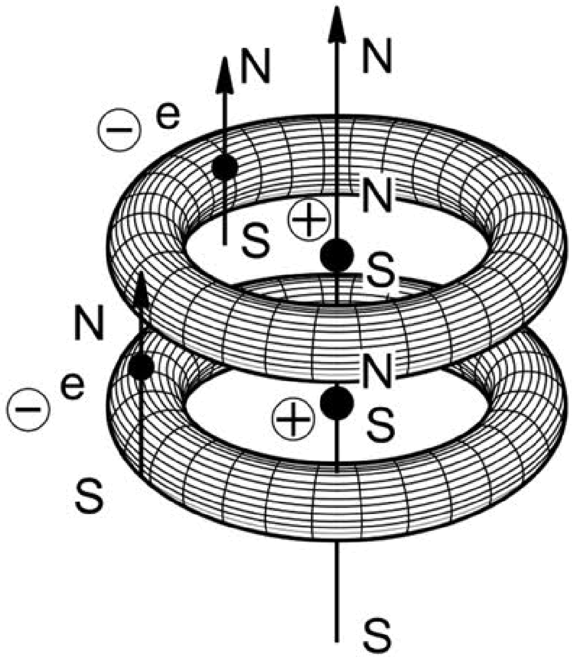

Figure 24: *In this figure, we present a conceptual rendering of the new chemical species of Santilli magnecules [30] for the case of the elementary hydrogen magnecules* $\tilde{H}_2 = H \times H$ *(where* $\times$ *denotes magnecular bond). The species is obtained via the use of special electric arcs producing electric and magnetic fields over certain minimum values [**?**], which arcs ionize the hydrogen molecule and polarize the electron orbits into toroids by therefore creating a new magnetic dipole moment which does not exist in the natural state. Polarized atoms bond together in the configuration of the figure which is stable at ambient temperatures by creating the new species of magnehydrogen (see Refs. [62] to [65]). It should be indicated that elementary magnecules between other selected atoms verify all hadronic laws for controlled nuclear fusions ([25], Section 4), for instance, of* $\tilde{CO} = 6 - C - 12 \times 8 - O - 16$ *into* $14 - Si - 28$ *without the emission of harmful radiations and without the release of radioactive waste [69] [70].*

Refs. [66] [67] then reformulated the Riemannian geometry via the transition from a formulation over the field of real numbers $\mathcal{R}$ to that over the isofield of isoreal isonumbers $\hat{\mathcal{R}}$ where the *gravitational isounit* is evidently given by

$$\hat{I}_{gr}(x) = 1/\hat{T}_{gr}(x). \tag{121}$$

The above reformulation turns the Riemannian geometry into a new geometry called iso-Minkowskian isogeometry [78] which is locally isomorphic to the *Minkowskian* geometry, while maintaining the mathematical machinery of the Riemannian geometry (covariant derivative, connection, geodesics, etc.) although reformulated in terms of the isodifferential isocalculus.

The following advantages should be mentioned for the *identical* iso-Minkowskian reformulation of general relativity, including Einstein's field equations, known as *isogravitation* [83]:

1) The achievement of a consistent operator form of gravity via the axiom-preserving embedding of the gravitational isounit $\hat{I}_{gr}(x)$ in the *unit* of relativistic quantum mechanics [33] [21];

2) The achievement of the universal *LPS isosymmetry* of *all* non-singular Riemannian metrics [100], which symmetry is locally isomorphic to the LP symmetry, (Section II-2), while being notoriously impossible in a conventional Riemannian space over the reals;

3) The achievement of clear compatibility, actually a *true isounification,* of general and special relativity since the latter can be identically recovered with the simple limit

$$\hat{I}_{gr} \rightarrow I = Diag.(1,1,1,1), \tag{122}$$

implying the transition from the universal LPS isosymmetry to the LP symmetry of special relativity with ensuing recovering of conservation and other special relativity laws;

4) The possibility of initiating systematic studies on *interior gravitational problems* along the forgotten Schwartzchild's second paper [73];

5) The achievement of axiomatic compatibility between gravitation and electroweak interactions thanks to the replacement of curvature into the covering notion of *isoflatness,* while offering realistic hopes to achieve a grand unification [38]; and other intriguing advances.

3. CONCLUDING REMARKS.

At the end of their historical paper [1], Einstein, Podolsky and Rosen state:

While we have thus shown that the wavefunction [of quantum mechanic] does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

In the preceding Papers I and II we have provided a review and upgrade of the apparent proof of the existence of classical counterparts for

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Figure 25: *A picture of the 300 kW mobile magnegas refinery built by Santilli when Chief Scientist of Magnegas Corporation (now Taronis Corporation) to process a mixture of fossil oil and water into magnegas whose combustion with the proper stochiometric oxygen has no detectable CO or HC [57]. The oxygen content of magnegas appears to have special polarizations [62] - [65] deserving tests for its possible use in ventilators to kill Corona viruses due to the emission of strong UV light in its depolarization in human lungs.*

extended particles within physical media [9] (Section II-3.7), and of the apparent progressive verification of Einstein determinism in the interior of hadrons, nuclei and stars [10] (Section II-3.8).

In this third paper, it appears we have additionally proved, apparently for the first time, the above quoted, concluding EPR statement. In fact, the "completion" of the quantum wavefunction $\psi(t, r)$ of the Schrödinger equation into the hadronic isowavefunction $\hat{\psi}(\hat{t}, \hat{r})$ of the Schrödinger-Santilli isoequation allows the achievement, otherwise impossible via quantum mechanics, of an exact representation of all characteristics of: the neutron in its synthesis from the hydrogen atom (Section 2.6); the deuteron (Section 2.7); the attractive force between the identical electrons in valence couplings (Section2.8); and the progressive achievement of Einstein's determinism in interior dynamical conditions (Section 2.9).

Acknowledgments. The writing of these papers has been possible thanks to pioneering contributions by numerous scientists we regret not having been able to quote in this paper (see Vol. I, Refs. [**?**] for comprehensive literature), including:

The mathematicians: H. C. Myung, for the initiation of iso-Hilbert spaces; S. Okubo, for pioneering work in non-associative algebras; N. Kamiya, for basic advances on isofields; M. Tomber, for an important bibliography on non-associative algebras; R. Ohemke, for advances in Lie-admissible algebtas; J. V. Kadeisvili, for the initiation of isofunctional isoanalysis; Chun-Xuan Jiang, for advances in isofield theory; Gr. Tsagas, for advances in the Lie-Santilli theory; A. U. Klimyk, for the initiation of Lie-admissible deformations; Raul M. Falcon Ganfornina and Juan Nunez Valdes, for advances in isomanifolds and isotopology; Svetlin Georgiev, for advances in the isodifferential isocalculus; A. S. Muktibodh, for advances in the isorepresentations of Lie-Santilli isoalgebras; Thomas Vougiouklis, for advances on Lie-admissible hypermathematics; and numerous other mathematicians.

The physicists and chemist: A. Jannussis, for the initiation of isocreation and isoannihilation operators; A. O. E. Animalu, for advances in the Lie-admissible formulation of hadronic mechanics; J. Fronteau, for advances in the connection between Lie-admissible mechanics and statistical mechanics; A. K. Aringazin for basic physical and chemical advances; R. Mignani for the initiation of the isoscattering theory; P. Caldirola, for advances in isotime; T. Gill, for advances in the isorelativity; J. Dunning Davies, for the initiation of the connection between irreversible Lie-admissible mechanics and thermodynamics; A. Kalnay, for the connection between Lie-admissible mechanics and Nambu mechanics; Yu. Arestov, for the experimental verification of the iso-Minkowskian structure of hadrons; A. Ahmar, for experimental verification of the iso-Minkowskian character of Earth's atmosphere; L. Ying, for the experimental verification of nuclear fusions of light elements without harmful radiation or waste; Y. Yang, for the experimental confirmation of magnecules; D. D. Shillady, for the isorepresentation of molecular binding energies; R. L. Norman, for advances in the laboratory synthesis of the neutron from the hydrogen; I. Gandzha for important advances on isorelativities; A. A. Bhalekar, for the exact representation of nuclear spins; Simone Beghella Bartoli important contributions in experimental verifications of hadronic mechanics; and numerous other physicists.

I would like to express my deepest appreciation for all the above courageous contributions written at a point in time of the history of science dominated by organized oppositions against the pursuit of new scientific knowledge due to the trillions of dollars of public funds released by governmental agencies for research on pre-existing theories.

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Finally, I would like to express my deepest appreciation and gratitude to my wife Carla Gandiglio Santilli for decades of continued support of my research amidst a documented international chorus of voices opposing the study of the most important vision by Albert Einstein.

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INTERNATIONAL TELECONFERENCE ON EINSTEIN'S ARGUMENT THAT
"QUANTUM MECHANICS IS NOT A COMPLETE THEORY."
September 1 to 5, 2020 http://eprdebates.org/epr-conference-2020.php

NONUNITARY LIE-ISOTOPIC AND LIE-ADMISSIBLE SCATTERING THEORIES OF HADRONIC MECHANICS:

Irreversible Deep-Inelastic Electron-Positron and Electron-Proton Scattering

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Abstract:

In this teleconference, we have debated whether high energy scattering regions should be re-inspected under the historical 1935 argument by A. Einstein, B. Podolsky and N. Rosen [10] that "quantum mechanics is not a complete theory" in view of the apparent proofs by R. M. Santilli, especifically, for high density regions suggesting the "completion" the quantum mechanical scattering theory into the iso- and geno-scattering theories of hadronic mechanics. The non-unitary Lie isotopic and Lie-admissible scattering theories of hadronic mechanics, also known as iso- and geno-scattering theories, respectively, were first systematically studied in R. M.Santilli's volumes [11]-[14], particularly in Chapter 12 of Vol. II. In Ref. [27], A. O. E. Animalu and R.M. Santilli continued these studies by developing a generalization of the Feynman graph method for the computation of the S matrix for high density scattering regions that cannot be consistently decomposed into a finite number of isolated point-particles according to various "no-decomposition theorems" [22]. More recently, by using Myung's nonlinear-Riccati differential realization of Santilli's Lie-admissible equation of motion to characterize the generalized structure functions, A.O. E. Animalu [32] has extended the results of Paper [27] to the geno-scattering theory of deep inelastic, thus irreversible electron-positron and electron-proton scattering processes, and obtained a good agreement between the geno-theory and the experimental data, which are presented and discussed in relation to the EPR argument..

Key words: Nonunitary theories, Irreversible deep-inelastic scattering theories,

https://doi.org/10.52202/059404-0001

TABLE OF CONTENT

1. IMPLICATIONS OF THE EPR ARGUMENT ON HIGH ENERGY SCATTERING EXPERIMENTS

1.1 The EPR argument.

A most mysterious experimental evidence in nature is the capability of particles to influence each other instantly at a distance. The scientific community of the early 1900 assumed that such an effect is predicted by quantum mechanics, for which reason the effect was called and continue to be called to this day *quantum entanglement as charaterized* in Fig.1 below. By contrast, Albert Einstein noted that quantum mechanics can only represent point-like particles isolated in vacuum, thus being unable to predict their entanglement, in which case the sole possible representation of the entanglement is that via superluminal communications that would violate special relativity. To avoid such a violation, Einstein published in 1935 a historical paper jointly with his graduate students, Boris Podolsky and Nathan Rosen arguing that "quantum mechanics is not a complete theory" (EPR argument) [1], in the sense that quantum mechanics is valid for the atomic structure and other systems, but there may exist more complex systems in nature requiring a suitable "completion" for their consistent treatment, as it is the case for particle entanglement, thus including the possible recovering of classical determinism at least under limit conditions.

Immediately following the appearance of paper [1], Niels Bohr [2] voiced strong opposition to the EPR Argument essentially on grounds that complex systems can be

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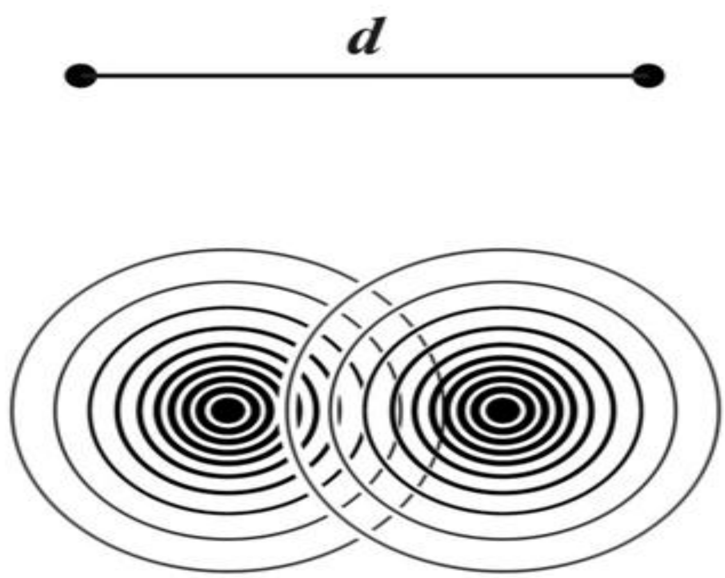
reduced to their elementary constituents, thus being fully treatable with quantum mechanics. Subsequently, J. S. Bell [3] proved a theorem essentially stating that a quantum mechanical system of particles with spin$\frac{1}{2}$ cannot admit a classical counterpart, and therefore prevents the achievement of Einstein's determinism, and *de facto* confirms the view that quantum mechanics is valid for all possible conditions existing in the universe (see, Ref.[4] for a vast bibliography).

Fig.1; Schematic elaboration of quantum entanglement



Since coordinates, potentials, differential calculus and wavefunctions can only be defined at isolated points, quantum mechanics can only represent particles as being isolated in vacuum (top view), thus requiring superluminal speeds for their instantaneous entanglement at a distance, as pointed out by A. Einstein, B. Podolsky and N. Rosen (EPR argument) [1]. Beginning with his Ph. D. studies in the 1960s, R. M. Santilli and a number of scholars [8-31] have: pointed out that the wavepacket of particles fills up the entire universe; the interactions caused by their overlapping/entanglement (bottom view) is not representable via a Hamiltonian; constructed an axiom-preserving completion of quantum mechanics into hadronic mechanics which represents particles as extended in permanent and continuous overlapping by therefore avoiding any need for superluminal speeds; provided a number of proofs, applications and experimental verifications of the EPR argument

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1.2 Earlier EPR verifications.

Despite authoritative opposing views, the search for a completion of quantum mechanics was continued by several authors, among whom we recall:

1) A completion of quantum mechanics by W. Heisenberg [5], hereon referred to as *Heisenberg's non-linear theory*, includes interactions non-linear in the wave functions as expected, for instance, in nuclear structures. Heisenberg's non-linear theory is based on the only possible quantum mechanical representation, in the Hamiltonian form $H(r,p)\psi(r) = E\psi(r)$ that, unfortunately, violates the superposition principle $\psi(r) = \sum_{k=1,2,\ldots,N}\psi_k(r)$, and, therefore, prevents a quantitative representation of the individual constituents $\psi_k(r)$.

2) A non-local completion of quantum mechanics by D. Bohm and L. de Broglie [6], hereon referred to as the *Bohm-de Broglie non-local theory,* includes non-local interactions that are also expected in the nuclear and other structures, which completion appears to be the first attempt of achieving Einstein's determinism in scientific records. Unfortunately, the Bohm-de Broglie non-local theory is semi-classical and, as such, cannot be considered a completion of quantum mechanics according to the ERP argument [1]. Additionally, interactions occurring over a volume cannot be reduced to a finite number of isolated points and, therefore, they cannot be represented with a Hamiltonian.

3) D. Bohm hypothesis of *hidden variables* [6] has realization expected to void Bell's theorem[3], with ensuing genuine broadening of quantum mechanics. Unfortunately, under the use of 20th century applied mathematics, Bell's inequality [3] prevents hidden variables to have concrete and explicit realizations because the said inequality applies for the infinite family of unitary equivalence of quantum mechanics.

1.3. EPR verification with energy-releasing processes.

An international teleconference was held on September 1 to 5, 2020 [7], to discuss available studies on the completion of quantum mechanics according to the EPR argument, including the studies of the preceding section and the lifetime research by R. M. Santilli on the proof of the EPR argument [8-27], and others.

In his Ph. D. thesis in the mid-1960s [8,9] at the University of Torino, Italy, , R. M. Santilli noted that Heisenberg's equations for the time evolution of an observable A in the infinitesimal and finite forms,

$$i\frac{dA}{dt} = [A,H] = AH - HA \qquad (1)$$

$$A(t) = e^{Hti}A(0)e^{-itH} \qquad (2)$$

are invariant under anti-Hermiticity, thus are unable to represent time-irreversible processes such as combustion or nuclear fusions. Additionally, Santilli proved a theorem establishing that a macroscopic time-irreversible system cannot be consistently decomposed into a finite number of quantum mechanical particles, thus establishing that macroscopic irreversibility originates at the level of elementary particles, as confirmed experimentally by the clear irreversibility of high energy scattering experiments at CERN,. FER- MILAB and other particle physics laboratories. Hence, Santilli concluded that *the inability to provide a consistent representation of irreversible processes is a first clear evidence of its lack of completion of quantum mechanics according to the EPR argument [1].*

To initiate the identification of an appropriate completion for irreversible processes, Santilli proposed in 1967 a completion of quantum mechanics for time-irreversible processes characterized by brackets (A,H) that are no longer invariant under anti-Hermiticity. Following an extensive search in mathematical libraries, Santilli selected brackets (A,H) that are Lie-admissible according to A. A. Albert [10] in the sense that the attached anti-symmetric brackets $[A,B]^* \equiv (A,B)-(B,A)$ are Lie. Therefore, Santilli proposed in 1967 the completion of Heisenberg's time evolution into a Lie-admissible form for the consistent representation of irreversible processes [8,9], which he subsequently finalized in the infinitesimal and finite form [11,12]

$$i\frac{dA}{dt} = (A,H) \equiv ARH - HSA = A < H - H > A \tag{3}$$

$$A(t) = e^{HSti} A(0) e^{-itRH} \tag{4}$$

with corresponding completion of Schrödinger equation [13-15]

$$\langle \hat{\psi} | < H \equiv \langle \hat{\psi} | RH = \langle \hat{\psi} | ER \tag{5}$$

$$H > | \hat{\psi} \rangle \equiv HS | \hat{\psi} \rangle = ES | \hat{\psi} \rangle \tag{6}$$

where: R is generally assumed to have unit value, $R=1; S=1-F/H$ with ensuing Eq. (3) $idA/dt = AH - HA + AF$, is an operator representing the external terms F in Lagrange's and Hamilton's analytic equations; and $\hat{\psi}$ is the completed wavefunction under irreversibility. Note that irreversibility is assured because $R \neq S$.

The mathematics underlying the Lie-admissible dynamical equations, known as *Lie-admissible mathematics* or *genomathematics* for short, was subsequently developed in collaboration with a number of mathematicians (see the general review [13-15]) and Tutoring Lecture IV of Ref. [7]). Applications of the Lie-admissible treatment of irreversible processes, including nuclear fusions, are reviewed in Ref. [15].

1.4. EPR verification with the neutron synthesis.

late 1977, when at Harvard University under DOE support, Santilli [16] was requested to study the synthesis of the neutron from the hydrogen atom in the core of stars. He discovered in this way that quantum mechanics is completely inapplicable to the moist fundamental synthesis in nature because of various technical reasons, including the fact that the mass of the neutron is 0.782 *MeV bigger* than the sum of the masses of the proton and of the electron. Under these conditions the Schrödinger equation of quantum mechanics would require a *positive potential energy* resulting in a *mass excess* that are outside scientific boundaries. Santilli noted that, despite the clear evidence of the synthesis of the neutron in the core of stars, quantum mechanics admits no bound state at short distance between a pro- ton and an electron despite the fact that , at 1 fm mutual distance, they experience the *Coulomb attraction*

$$F = k\frac{e^2}{r^2} = (8.99\times10^9)\frac{(1.60\times10^{-19})^2}{(10^{-15})^2} = 230N \tag{7}$$

which is astronomically big for particle standards. Consequently, *the synthesis of the*

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neutron in the core of stars is a second clear evidence of the lack of completion of quantum mechanics according to the EPR argument [1]. In Table V of Ref. [16] of 1978, Santilli noted that, when mesons were assumed to be bound states of elementary particles produced free in their spontaneous decays, there was a systematic appearance of a 'mass excess' similar to that of the neutron.

To initiate these studies, Santilli suggested the completion of quantum mechanics into the covering *hadronic mechanics* [11-12] with general Lie-admissible dynamical equations (3)-(5) and their particularization for $R = S = \hat{T}$ in the infinitesimal form and finite form

$$i\frac{dA}{dt} = [A,H]^* \equiv ATH - HTA = A*H - H*A \tag{8}$$

and finite form $$A(t) = e^{H\hat{T}ti}A(0)e^{-it\hat{T}H} \tag{9}$$

with completion of the Schrödinger equation

$$H*|\hat{\psi}\rangle \equiv HT|\hat{\psi}\rangle = ET|\hat{\psi}\rangle \tag{10}$$

The mathematics and mechanics underlying the above equations are known as *Lie-isotopic mathematics and mechanics,* and characterize branches of hadronic mechanics also known as *isomathematics and isomechanics,* (see Refs. [13-15] and Tutoring Lecture I of Refs. [7]).

It should be noted that isoeigenvalue equation (10) verifies the superposition principle, by therefore admits the decomposition of the isowavefunction into those of the constituents of the hadronic bound state $\hat{\psi} = \sum_{k=1,2,\dots,N}\hat{\psi}_k$, Hence, isomathematics resolves the limitation of Heisenberg's non-linear theory [5] indicated in Section 1.2 via the embedding of all non-linear terms in the isotopic element $\hat{T} = \hat{T}(\hat{\psi}_1, \hat{\psi}_2, \dots, \hat{\psi}_N)$ and reconstructing linearity on isospaces.

Note that Eqs. (8)-(10) are also invariant under anti-Hermiticity as it is the case for Eqs.(1), (2), thus solely able to represent time reversal invariant systems. However, the representation of systems requires the knowledge of *two* operators, namely the conventional Hamiltonian H and the isotopic element $\hat{T}$ representing extended, thus deformable and hyperdense constituents in conditions of mutual penetration/entanglement with ensuing non-linear, non-local and non-potential interactions. Hence, Eqs.(8)-(10)) are particularly suitable for the representation of the entanglement of particles (see Figure 1 for details) and its application to stable nuclei as established by experimental evidence, namely, composed of extended nucleons in partial mutual penetration.

Under the approximation that the considered two-body hadronic bound states are stable, the neutron synthesis was studied with realizations of the isotopic operator $\hat{T}$ of the type [15]

$$\hat{T} = \prod_{k=1,2} Diag.(1/n_{1k}^2, 1/n_{2k}^2, 1/n_{3k}^2, 1/n_{4k}^2)\exp[-\Gamma(\psi,\hat{\psi},\dots)] > 0, \tag{11}$$

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normalized to the values $n_{k\alpha}^2 = 1$, for the sphere; $n_{k\alpha}^2$ represents the density of the exponent is a positive-normalized to the value $n_{4\alpha}^2 = 1$ for the vacuum; and the exponent is a positive definite function representing non-linear, non-local and non-potential interactions caused by mutual overlapping/entanglement of the particles considered with realizations of the type

$$\exp\left[-\Gamma(\psi,\hat{\psi},...)\right] = \exp\left[-\left|\psi/\hat{\psi}\right|\int\hat{\psi}_1\hat{\psi}_2^+ d^3r\right] << 1 \quad 2 \tag{12}$$

where ψ is a quantum mechanical wave function, and $\hat{\psi}$ is the completed wave function under isotopy. Note the recovering of quantum mechanics identically and uniquely whenever the overlapping of the wavepackets is ignorable and the integral of realization (12) can be assumed to be null.

It should be note that the representation (11), (12) of non-local interactions characterizes a full operator theory, by therefore resolving the semi-classical limitation of the Bohm-de Broglie non-local theory [6] indicated in Section 1.2.

In order to achieve a consistent representation of the neutron synthesis, Santilli assumed that the integral in Eq. (12) is a constant and introduced the following simple realization of the isotopic element [16]

$$\hat{T} = e^{-\frac{|V_H|}{V_Q}} \approx 1 - \frac{|V_H|}{V_Q} \tag{13}$$

where V_Q is the quantum mechanical Coulomb potential $V_Q = +e^2/r$ for the proton-electron system at short distances, and V_H is the strongly attractive hadronic potential caused by the mutual penetration of the wavepackets which can be represented by the Hulten potential,

$$V_H = W\frac{e^{-kr}}{1-e^{-kr}} \tag{14}$$

But the Hulten potential behaves like the Coulomb potential at short distances. Therefore, Santilli absorbed the latter in the former (except for a renormalization of W which is here ignored), and reached the following eigenvalue equation for the two-body hadronic bound state of a proton and an electron at 1 fm mutual distances (Table V of Ref. [16])

$$\left[-\frac{1}{\overline{m}}\Delta - W\frac{e^{-kr}}{1-e^{-kr}}\right]\hat{\psi}(r) = \mathrm{E}\hat{\psi}(r) \tag{15}$$ i.

where $\overline{m}$ is renormalization of the reuced mass caused by wave overlapping identified more details in Section 1.9

Under the above formalism, the Lie-isotopic branch of hadronic mechanics was able to achieve an *exact* representation of all characteristics of the neutron in its synthesis from the hydrogen at the non-relativistic and relativistic levels (see the review in Ref. [15] and Ref. [25]).

In paper [17] of 1995, A. O. E. Animalu and R. M. Santilli noted that hadronic bound state (15) holds with a strongly attractive Hulten potential irrespective of whether the Coulomb potential is attractive or repulsive, thereby reaching in this way the first known *attractive force between the identical electrons of the Cooper pair in superconductivity* and, therefore, confirming *Animalu's isosuperconductivity* [18].

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1.5 EPR verification with classical images.

In Ref. [19] of 1998, Santilli confirmed the validity of Bell's inequality for point-like particles with spin 1⁄2 under potential interactions, but indicated its inapplicability (rather than violation) un- der extended, therefore deformable particles in conditions of mutual entanglement with ensuing non-linear, non-local and non-potential interactions.

By using the Lie-isotopic $SU(2)$-spin symmetry (see the review in Ref. [24]) with the explicit and concrete realization of Bohm's hidden variables

$$\hat{T} = Diag.(\lambda, 1/\lambda), \; Det\hat{T} = 1, \tag{16}$$

Santilli proved that systems of extended, deformable and hyperdense particles with spin $\frac{1}{2}$ in conditions of deep mutual entanglement do indeed admit classical counterparts, and provided specific examples (see the review in Tutoring Lecture II of Ref. [7]).

Thanks to the deformability of neutrons according to Eq. (11), with consequential alteration of their magnetic moments, the above verification if the EPR argument was then used in Ref. [19] for the first known numerically exact and time invariant representation of nuclear magnetic moments (see the review in Ref.]25]).

1.6. EPR verification with Einstein's determinism.

The recovering of classical images in Ref. [19] evidently established the foundations for the achievement of Einstein's determinism [1].
In fact, in paper [20] of 2019, Santilli proved that, under the standard isonormalization

$$\langle \hat{\psi} | * \hat{T} * | \hat{\psi} \rangle = \hat{T} \tag{17}$$

Heisenberg's uncertainties are completed into the form

$$\Lambda r \Lambda p = \frac{1}{2} \left| \langle \hat{\psi}(r) | * [\bar{r}, \bar{p}] * | \hat{\psi}(r) \rangle \right| \approx \frac{1}{2} \hat{T} << 1 \tag{18}$$

(18) where the very small value of $\hat{T}$ is established by structure (12) as well as by all fits of experimental data to date [15] [23].

Isodeterministic principle (18) establishes the progressive validity of Einstein's determinism in the interior of hadrons, nuclei and stars, and its full achievement in the interior of gravitational collapse. The latter result is due to the fact that the isotopic element admits a realization in terms of the space component of Schwartzchild's metric with

$$\hat{T} = \frac{1}{1 - \frac{2M}{r}} = \frac{r}{r - 2M}$$

in ensuing full achievement of Einstein's determinism,

$$\Lambda \bar{r} \Lambda \bar{p} \approx \hat{T} = \frac{r}{r - 2M} \Rightarrow_{r \to 0} = \hat{U} \tag{19}$$

in the interior of a black hole whose center of gravity verifies full classical determinism.

1.7. EPR verification with valence bonds.

One of the biggest insufficiencies of quantum mechanics and chemistry discussed at the teleconference [7] is the lack of a consistent representation of valence electron bonds in molecular structures. In fact, quantum mechanics and chemistry predict that, due to their equal elementary charge, the identical electrons in valence bonds experience a *Coulomb repulsion* that, ac- cording to Eq. (7), has the value of 230 N which is so enormous for particle standards to prevent any possibility that current quantum chemical models if valence boinds may achieve the needed attraction.

The lack if a quantitative model of molecular structure has then implied the inabil- ity by quantum chemistry to achieve an exact representation of molecular binding ener- gies and other experimental data from unadulterated first principles, with deviations for the binding energies of about 2% that, rather than being small, are equivalent to about 950 BTU.

Following the joint work with A. O. E. Animalu for the identical electrons of the Cooper pairs in superconductivity [17], Santilli confronted the above limitations with systematic studies presented in monograph [24] and, via a procedure similar to that of Eqs. (11)-(15), did achieve the first and only known *attractive force between valence electron pairs in molecular structures.* In particular, the hadronic force resulted to be so strong that valence electron pairs bond into a quasi-particle called *isoelectronium* (see Chapter 4 of Ref. [24]). The alteration of the structure of the valence electrons to achieve an attraction when in total mutual overlapping is studied in Ref. [23] via the notion of *isoparticle* as a representation of the applicable symmetry indicated in Section 1.9.

In joint works with the chemist D. D. Shillady, Santilli proved that the notion of strong valence bond of hadronic chemistry achieves exact representations from unadulterated first principles of experimental data of the Hydrogen [25] and water [26] molecules.

1.8. EPR verification with the removal of quantum divergencies.

Recall the necessary condition for the completion of quantum mechan- ics and chemistry into isomechanics and isochemistry, respectively, according to which the isotopic product $A * B = A\hat{T}B$ must be applied to the *totality* of the products, thus including all products appearing in perturbative series. But the isotopic element ˆ $\hat{T}$ has very small values in all known applications. Consequently, perturbative series that are generally divergent in quantum mechanics and chemistry are turned under isotopy into strongly convergent series, as illustrated by the strong convergence of the series

$$A(t) = A(0) + (ATH - HTA)/1! + \ldots = K << \infty, \quad T << 1. \tag{20}$$

Consequently, *the validity of Einstein's determinism per Section 1.6 implies the removal of quantum mechanical divergencies* (see Corollary 3.7.1, page 128, Ref. [22]).
The actual verification of the above important property has been provided by D. D. Shillady and R. M. Santilli in papers [25] [26] with the proof that *the perturbative series of hadronic chemistry converge at least one thousand times faster than the corresponding quantum chemical series.*

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1.9. Implications for high energy scattering experiments

Note that the validity of relativistic quantum mechanics for particles in the vacuum of accelerators is beyond scientific doubt because particles can be well approximated as being point-like, the sole possible interactions being at a distance, thus derivable from a potential. Equally beyond scientific doubts are the numeric values of quantities actually measured, such as scattering angles, cross sections, etc.

The verifications of the EPR argument outlined in Sections 1.2 to 1.8 imply that *rel- ativistic quantum mechanics is inapplicable (rather than being violated) for the interior of the scattering region due to its density so big to approach that of black holes, with ensuing general lack of validity of the characteristics of intermediate particles predicted by quantum mechanics, such as mass, spin, etc.,.*

In the authors' view, the most important experimental evidence is that *high energy inelastic scattering events are strictly irreversible over time, thus requiring the Lie-admissible branch of hadronic mechanics and its related scattering theory (Section 1.3), while high energy elastic, thus time- reversible scattering events require the use of the Lie-isotopic branch of hadronic mechanics and related scattering theory (Section 1.4)*, see Refs. [15] [27] and more detailed presentation in the subsequent sections.

To illustrate the alteration, originally called *mutation* [16] and now called *isorenormal- ization* [14] of the rest energy and spin of intermediate particles for high energy elastic scattering events, recall that the scattering region is represented by the iso-Minkowskian isospace $\hat{M}(\hat{x},\hat{\eta},\hat{I})$ with isounit $\hat{I}=1/\hat{T}$, isometric $\hat{\eta}=\hat{I}\eta$, were η is the Minkowskian metric, $\hat{x}=x\hat{I}$ where x represents the Minkowskian spacetime coordinates, and isotopic line element [28]

$$(\hat{x}-\hat{y})^2=\left[\frac{(x_1-y_1)^2}{n_1^2}+\frac{(x_2-y_2)^2}{n_2^2}+\frac{(x_3-y_3)^2}{n_3^2}-(t_1-t_2)^2\frac{c^2}{n_4^2}\right]\hat{I}, \qquad (21)$$

with isounit

$$\hat{I}=1/\hat{T}=Diag\left(n_1^2,n_2^2,n_3^2,n_4^2\right)>0 \qquad (22)$$

where exponential (12) is incorporated in the *n*- characteristic quantities. The universal symmetry of line element (21) is the Lorentz-Poincaré-Santilli isosymmetry $_{I}{}^{\wedge}(3.1)$ [29]. The isorelativistic equation characterized by the second order iso-Casimir invariant (Eq. (39), page 91, Ref. [22]) is given by

$$\hat{P}^{\alpha}=\hat{\eta}_{\mu\nu}P^{\mu}P^{\nu}=0, \qquad (23)$$

,

which, under the approximation of a spherical scattering region $n_1^2=n_2^2=n_3^2=1$ and the time isounit $\hat{I}_t=1$, yields the iso-Klein-Gordon equation (for $\hbar=1$)

$$(\Lambda+\overline{m}c^2)\hat{\psi}=0 \qquad (24)$$

where $\overline{m}=m/n_4^2$ and m is the value predicted by relativistic quantum mechanics. To appreciate the value of the isorenormalization, $m\rightarrow\overline{m}$, we assume that the density of the scattering region is the same as that of the fireball of the Bose-Einstein correlation, in which case the fit of the experimental data yields the value $n_4^2 = 0.429$ [30] [31], resulting in the isorenormalized mass

$$m \to \overline{m} = \frac{m}{0.429} = 2.331m \tag{25}$$

Note the general increase of the rest energies of intermediate particles compared to the value predicted by quantum mechanics.

2. OUTLINE OF THE PRESENTATION

The presentation is based on A.O.E. Animalu's_ personal knowledge of P.A.M. Dirac (as a post-graduate student at Cavendish Lab, Cambridge, UK) and R. P. Feynman (briefly at UK and subsequently through interaction with his former student, R.Oakes, as post-doctoral research associate staff at W.W.Hansen Lab, Stanford University, Calif. USA) in the 1960s. Thereafter, Animalu was involved at Chapel Hill, NC, as theoretician in experimental measurement of angular distribution of photons from positron annihilation with valence electrons in metals using [*geno-dual* model] pseudo-potential theory...

The foundation for comparison of numerical values predicted by the iso- and/or geno-scattering theory of HM with those of the conventional QM scattering theory was laid while Animalu was working at M.I.T. Lincoln Lab and later (with R.M.Santilli & others) in a paper entitled[32], *"Iso-Feynman Propagator and Iso-Matrix of Hadronic Mechanics"* , published in the Hadronic Journal Vol.31, p.317-350(2008). with the following partial abstract :

In this paper, we present in the framework of the Feynman space-time picture of quantum electrodynamics (QED) a systemic method, based on non-linear, non-unitary transformation of Feynman's propagator and S-matrix into the corresponding iso-propagator and iso-S-matrix in hadronic mechanics (HM) for eliminating three basis of divergences in contemporary physical theories [of quantum mechanics (QM)]. (1) Arbitrariness of the (external) boundary conditions on the quantization volume for normalizations and associated differences in the quantum [Bose-Einstein(BE) and Fermi-Dirac (FD) statistics;(ii) the singularities in the Lorentz transformation and the theorem of addition of addition of two velocities, and (3) the singularities arising from the assumption that particles are point-like and the interaction between them long-range in character which calls for arbitrary cut-off of divergent integrals..... We wish, in this paper, to investigate the scattering processes of such *non-conservative time-irreversible* systems, under the name *genoscattering* theory, as exemplified by deep-inelastic electron-proton scattering, and electron-positron annihilation into two photons.

At Chapel-Hill, North Carolina, Animalu had reported (see, page 51 of Animalu's 1977 Prentice-Hall published book)[33] that experimental measurement of angular distribution of photons from positron annihilation with valence electrons into two γ -rays in metals, is given, for a single plane wave state, by the distribution function(at $0^0 K$):

$$n_k = \begin{cases} 1 & (k \le k_F) \\ 0 & (k > k_F) \end{cases}$$

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where k_F is the Fermi wave number that is determined, in accordance with Fermi-Dirac statistics, by the electron density (ρ) via the relation, $k_F = (3\pi^2\rho)^{1/3}$. The quantity measured by positron annihilation is

$$N(k_z) = \int_{-\infty}^{+\infty}\!\!\int n_k \, dk_x dk_y = \begin{cases} \pi(k_F^2 - k_z^2) & (\text{for } k_z \le k_F) \\ 0 & (\text{for } k_z > k_F) \end{cases}$$

which represents the area of a cross-section of the Fermi surface at $k = k_z$ as shown in Fig. 2.(a) and $N(k_z)$ versus k_z is an inverted parabola as shown in Fig2(b).

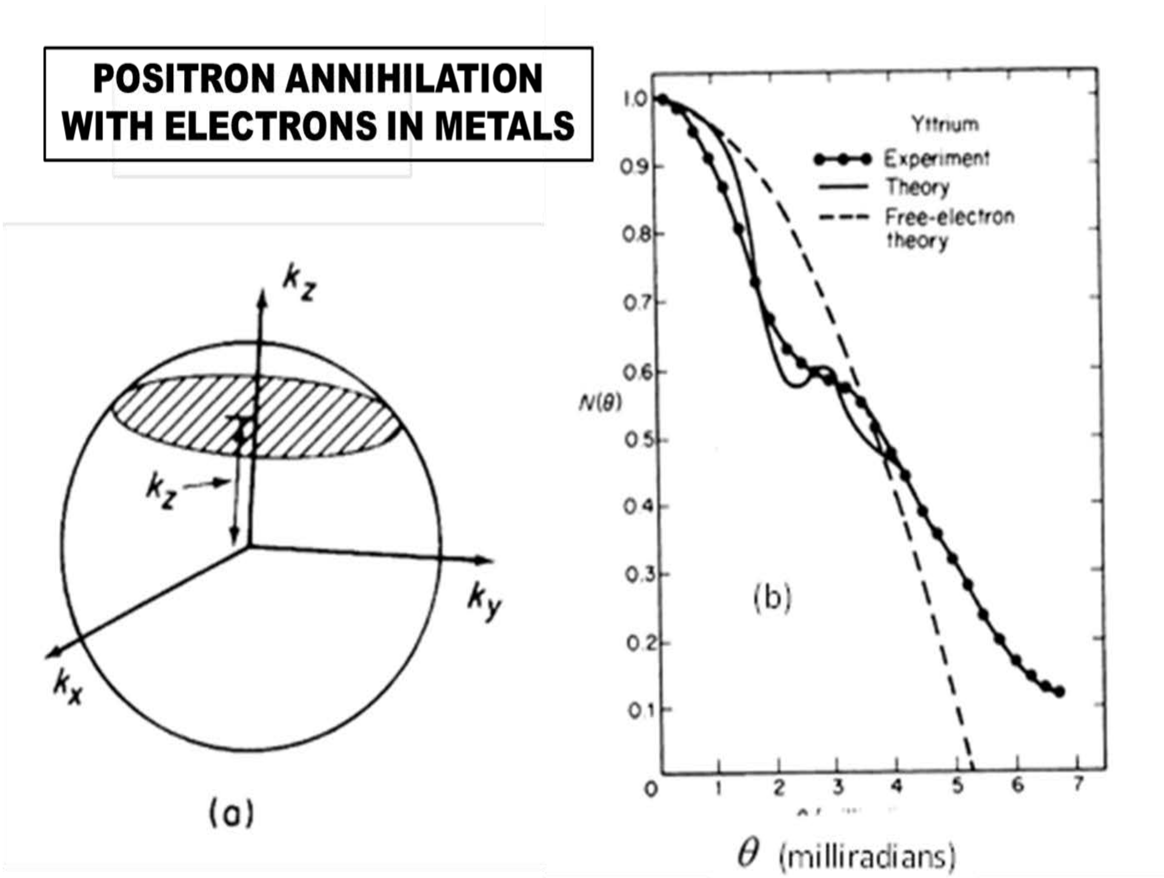


Fig. 2 (a) Cross-sectional area of the Fermi sphere represented by $N(\theta), (\theta = hk_x / mc)$; (b) Experimental and theoretical ([*geno-dual model*] pseudopotential calculation) angular distribution of photons from positron annihilation in a c-axis Yttrium crystal [From R.W. Williams, T.L. Loucks and R. Mackintosh, Phys. Rev. Letters, 16, p.168 (1966) & sourced as Fig.(2.14) from A.O.E. Animalu, *Intermediate Quantum Theory of Crystalline Solids*, Prentice-Hall, (1977) p.51.

Bearing the insight from the above results in mind, let us turn next to the use of the Feynman space-time picture of quantum electrodynamics (QED) in terms of the Feynman propagator and the S-matrix method for characterizing the following unitary time-reversible processes:

$$p+\bar{p}\rightarrow\gamma,\ \mathbf{e}^{-}+\mathbf{e}^{+}\rightarrow\boldsymbol{\gamma},$$

$$\mathbf{e}^{-}+\mathbf{e}^{+}\rightarrow\boldsymbol{\pi}^{\mathbf{o}}\rightarrow\mathbf{e}^{-}+\mathbf{e}^{+}$$

and non-unitary time-irreversible processes:

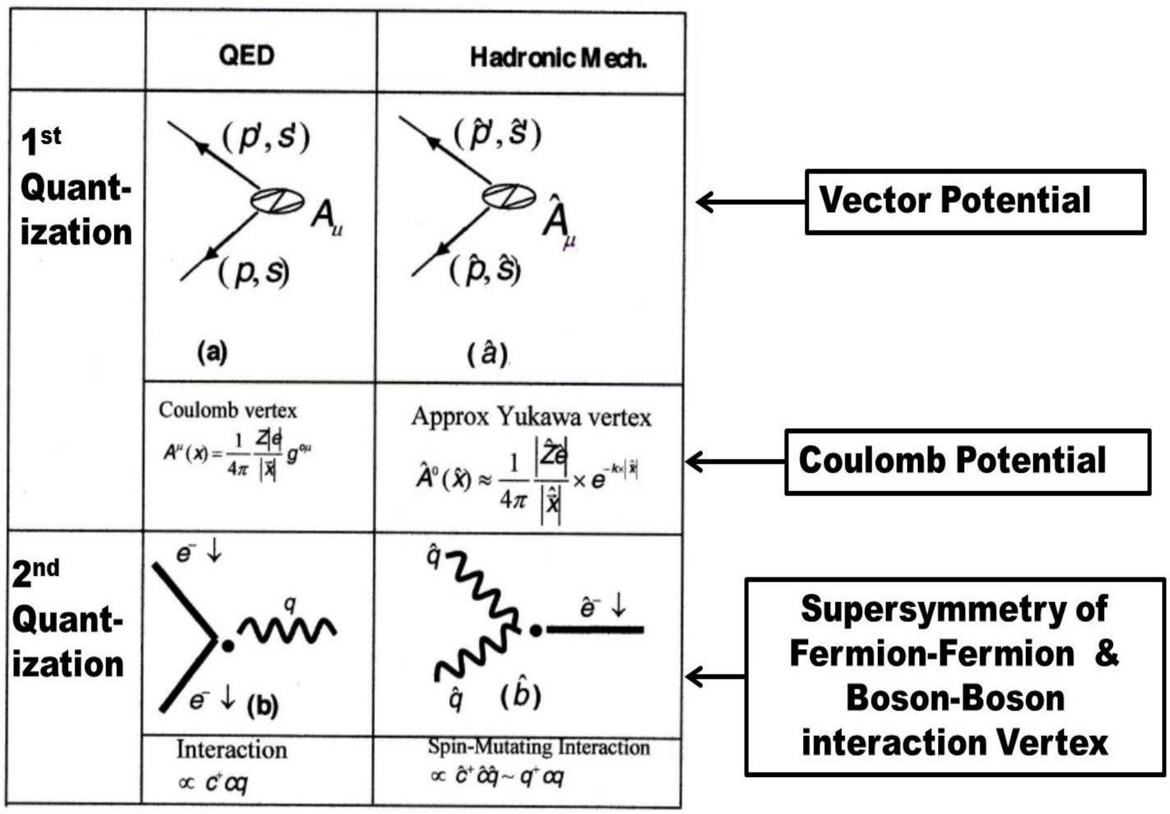
$$\mathbf{e}^{-}+\mathbf{p}\rightarrow\mathbf{e}^{-}+\mathbf{p}_{\mathbf{resonance}}\rightarrow\mathbf{e}^{-}+\mathbf{X}$$

$$\mathbf{e}^{-}+\mathbf{e}^{+}\rightarrow\boldsymbol{\gamma}+\boldsymbol{\gamma}^{\mathbf{d}}$$

Before going to detailed review in Sec.2, we summarize in Fig.3a the three sources of divergences in contemporary physical theories of quantum mechanics (QM) and how they are eliminated by "lifting" (non-unitary transformation) into the corresponding iso-propagator and iso-S-matrix of hadronic mechanics (HM).

..

Fig.3a: **ISOTOPIES OF FEYNMAN' DIAGRAMS WITHOUT DIVERGENCES**



Correspondence Principle between the Feynman graphs for fermion-fermion interaction vertex with boson emission in 1st & 2nd Quantization schemes in QED & the isotopic images for the boson-boson interaction vertex with single fermion spin emission in HM.

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The characterization of the transition from the classical (Coulomb vertex) (A_0) to quantum (approximately Yukawa vertex, $\hat{A}_0$) is shown in Fig. 3b. Further generalization involves non-linear processes typified by the existence of current loops as shown in Fig. 4, and the progressive characterization of the Lie-isotopic and Lie-admissible regimes of scattering processes in Fig. 5; the characterization of the internal cube-hexagon (geno-dual) symmetry and existence of current loop in the Lie-Admissible geno-scattering processes in Fig.6; and the hyperspace geometry of current loop(~Kepler vortex) system in Fig. 7.

Fig.3b: The classical (Coulomb vertex) (A_0) and its quantum (approximately Yukawa vertex,)

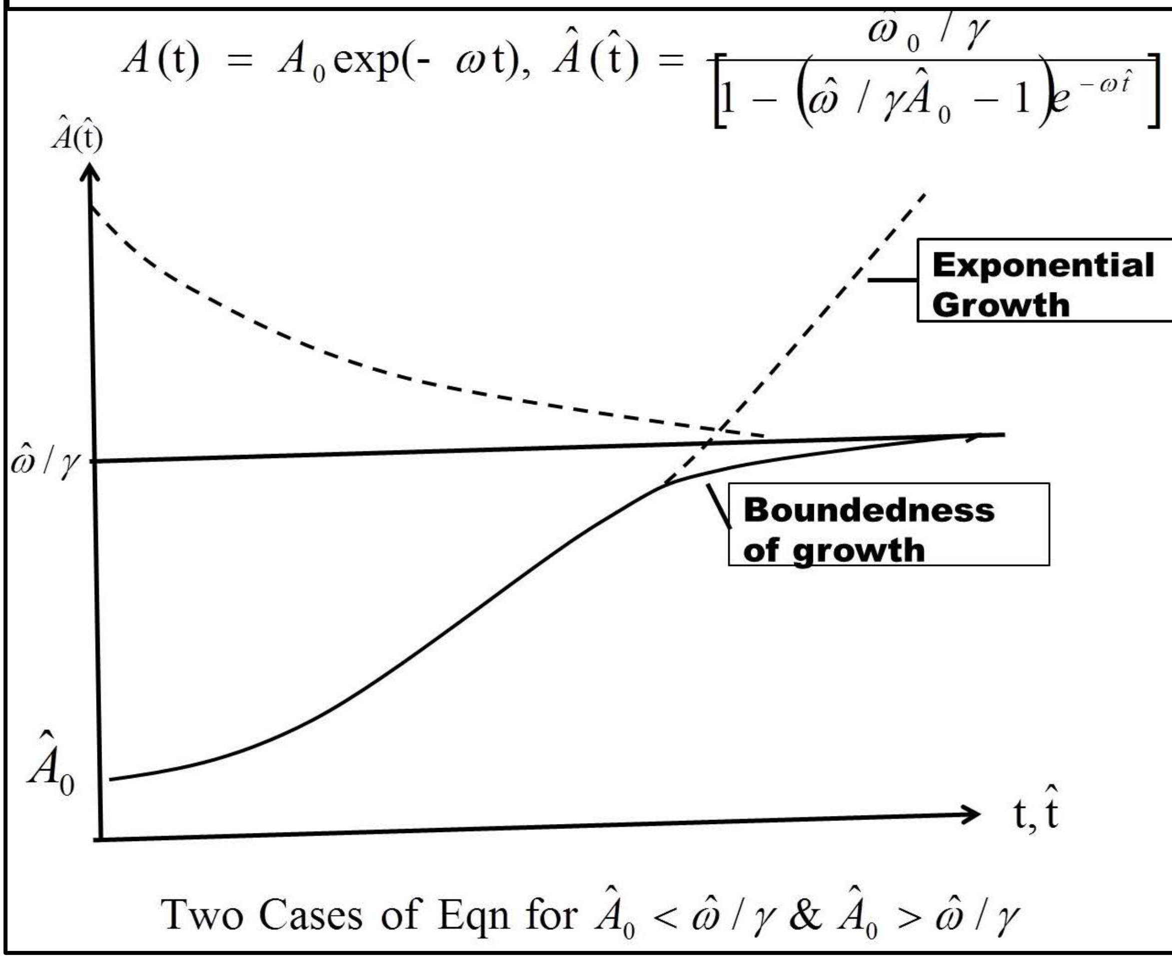
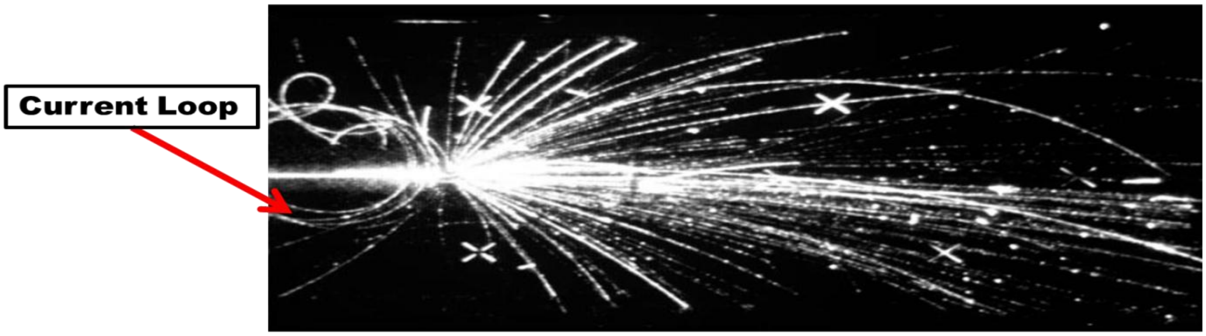


Fig. 4: Examples of Typically Irreversible Inelastic Scattering Processes with Current Loops



$$e^- + e^+ \rightarrow 2\gamma\gamma \quad \text{or} \quad e^- + e^d \rightarrow \gamma + \gamma^d$$

$$p + \overline{p} \rightarrow 2\gamma \quad \text{or} \quad p + p^d \rightarrow \gamma + \gamma^d$$

$$e^- + p \rightarrow e^- + p_{resonance} \rightarrow e^- + X$$

Fig. 5: Progressive Characterization of the Lie-Isotopic & Lie-Admissible Regimes of Scattering Processes

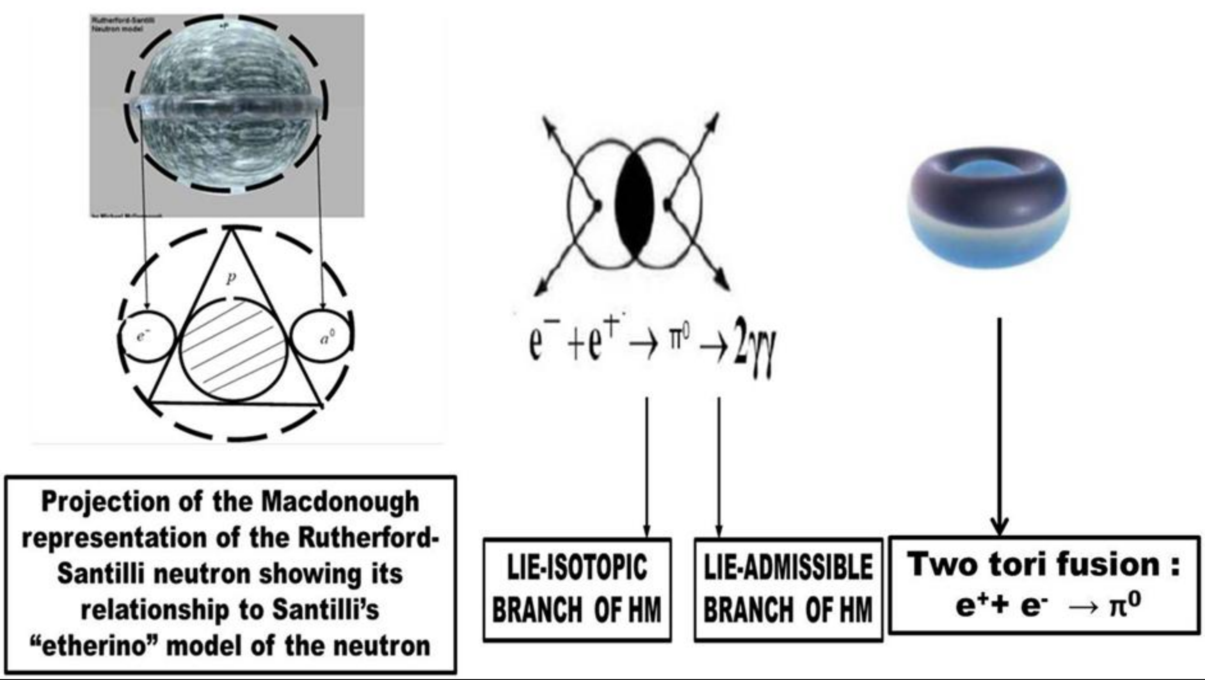


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Fig. 6: Characterization of the Internal Cube-Hexagon (geno-dual) Symmetry & Existence of Current loop in Lie-Admissible Geno-scattering Processes

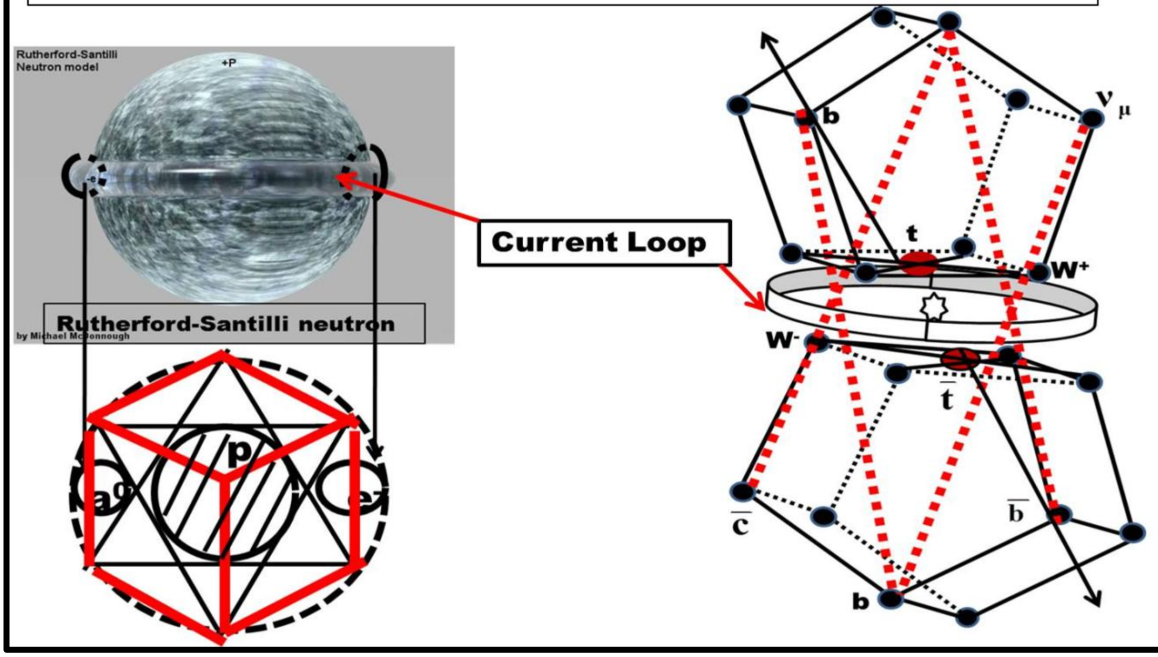
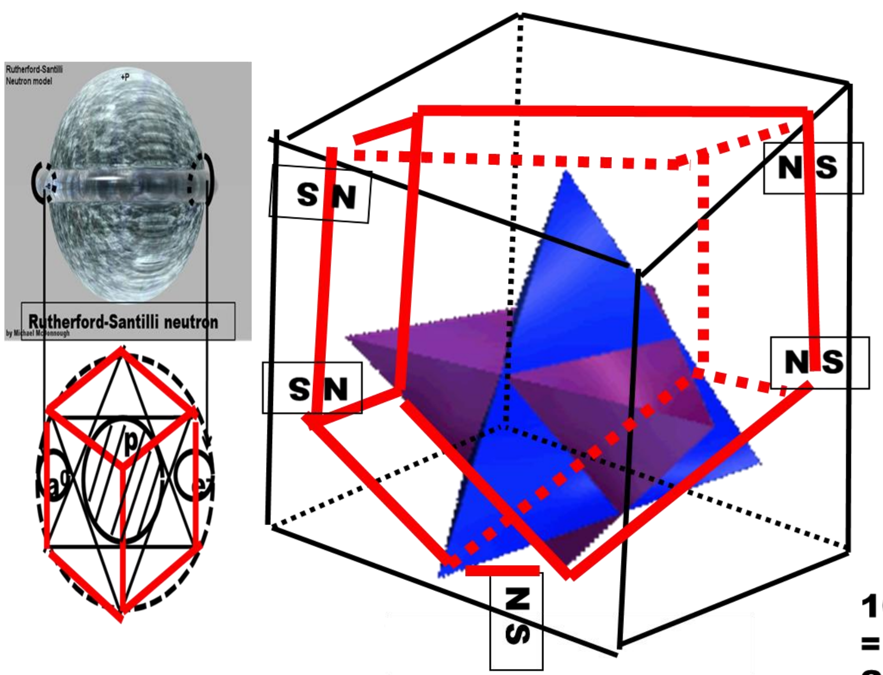


Fig. 7: Hyperspace Geometry of Current Loop (~ Kepler's Vortex) System



Geno-Dual (4x4) Metric Tensor

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	±1

16+10+7-1 = 32 = 2^5 =Dim of $2^N x 2^N$ rep of SO(2N)=SO(10) (N=5).

No. of (points+lines+planes) are Cube(26)+(Kepler) Vortex(7x6=42) +Pentagon(32): 26+42+32=100

Let me underscore the question that led the presenter to the above scenario of cube-hexagon (geno-dual) metric tensor principle in three-dimensional projective geometry in the era of search for a unified field theory of gravity and electromagnetism from my interaction in 1968 with P.A.M. Dirac while he was searching for a Maxwell-like (**dual) gauge principle** to relate electric charge and magnetic charge in projective geometric terms. Following treatment of the Lorentz force and the linear Dirac (negative) energy relativistic wave equation for electric charge on the same footing as a corresponding dual of Lorentz force and a positive energy relativistic wave equation for magnetic charge and non-negative mass (represented by a current loop). Dirac's way of thinking led me (as summarized below) to a correspondence principle

Particle ⟷ Point, Field ⟷ Line, Current ⟷ Plane,

Such a correspondence principle implies geometric characterization of conventional electrodynamics and an analogous so-called e-magnetodynamics of current loop and to the Feynman space-time diagrammatic approach based on 10x10 representations of the dynamical group unifying strong interaction with electromagnetism and space-time geometry and violations of the discrete symmetries – parity, charge conjugation, time-reversal and spin-parity:

$$SU^{>}(3)\times SU^{<}(3)\times U^{>}(1)U^{<}(1)\times O^{>}(4,2)\times O^{<}(4,2)$$

This defines the SO(2N)xSO(2N) group-theoretic approach to scattering of N-particle (N=5) systems which we proceed next to review and compare with experiment .in section.3. Discussion of results and conclusions will be presented in Sec.4.

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Conventional Electrodynamics	**E-magnetodynamics of Current Loop**
$\left\|\mathbf{F}_{\mu\nu}\right\| \equiv \begin{pmatrix} 0 & \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ -\mathbf{E}_1 & 0 & \mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{E}_2 & -\mathbf{B}_3 & 0 & \mathbf{B}_1 \\ -\mathbf{E}_3 & \mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$	$\left\|G_{\mu\nu}\right\| \equiv \begin{pmatrix} 0 & \mathbf{J}_1 & \mathbf{J}_2 & \mathbf{J}_3 \\ -\mathbf{J}_1 & 0 & \mathbf{B}_3 & -\mathbf{B}_2 \\ -\mathbf{J}_2 & -\mathbf{B}_3 & 0 & \mathbf{B}_1 \\ -\mathbf{J}_3 & \mathbf{B}_2 & -\mathbf{B}_1 & 0 \end{pmatrix}$
Maxwell's eqn & Lorentz force	$(\mathbf{J}^m \to \mathbf{B},\ \mathbf{B} \to -\mathbf{J}^m)$
$\partial^\nu F_{\mu\nu} = J^e_\mu ; \mathbf{f}^e = e\mathbf{E} + \mathbf{J}^e \times \mathbf{B}.$	$\mathbf{J}^m = -(2e\hbar/mc)\mathbf{A},\quad \mathbf{B} = \mathrm{curl}\mathbf{A} \equiv \mathrm{curl}(\zeta\mathbf{J}^m),$
$\mathbf{E} \to \mathbf{B},\ \mathbf{B} \to -\mathbf{E}.\ \mathbf{J}^e \to \mathbf{J}^m$	$\zeta = (mc/2\|e\|\hbar)$

$$\mathrm{Det}\left\|F_{\mu\nu} - \lambda\eta_{\mu\nu}\right\| \equiv \lambda^4 - (R_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma})\lambda^2 + (\in_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma})^2 = 0,$$

$$\mathrm{Det}\left\|G_{\mu\nu} - \lambda\eta_{\mu\nu}\right\| \equiv \lambda^4 - (R_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma})\lambda^2 + (\in_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma})^2 = 0,$$

$$\left\|\eta_{\mu\nu}\right\| \equiv diag(+1,-1,-1,-1), R_{\mu\nu\rho\sigma} \equiv (\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\sigma\nu}),$$

$$R_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma} - 2\in_{\mu\nu\rho\sigma}F^{\mu\rho}F^{\nu\sigma} = \begin{cases}\mathbf{E}^2 \\ \mathbf{B}^2\end{cases} \Rightarrow \begin{cases}\mathbf{E}^2 - \mathbf{B}^2 \pm 2\mathbf{B.E} = 0 \\ \mathbf{B}^2 - \mathbf{E}^2 \pm 2\mathbf{E.B} = 0\end{cases}$$

$$R_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma} - 2\in_{\mu\nu\rho\sigma}G^{\mu\rho}G^{\nu\sigma} = \begin{cases}\mathbf{J}^{m2} \\ B^2\end{cases} \Rightarrow \begin{cases}\mathbf{J}^{m2} - \mathbf{B}^2 \pm 2\mathbf{B.J}^m = 0 \\ \mathbf{B}^2 - \mathbf{J}^{m2} \pm 2\mathbf{J}^m.\mathbf{B} = 0\end{cases}$$

The significance of this algebraic 4x4-matrix unification scheme of Maxwell's equation with Lorentz force on one hand, and e-magnetodynamics with current loop on the other hand, is that we are now able to unify the gravitational field tensor with Maxwell's field tensor in an algebraic relation exhibiting dual symmetry with respect to interchange of electric and magnetic fields, as well as electric and mass/magnetic charges, and hence SO(2N) (N=5) group symmetry.

3. PRESENTATION OF RESULTS

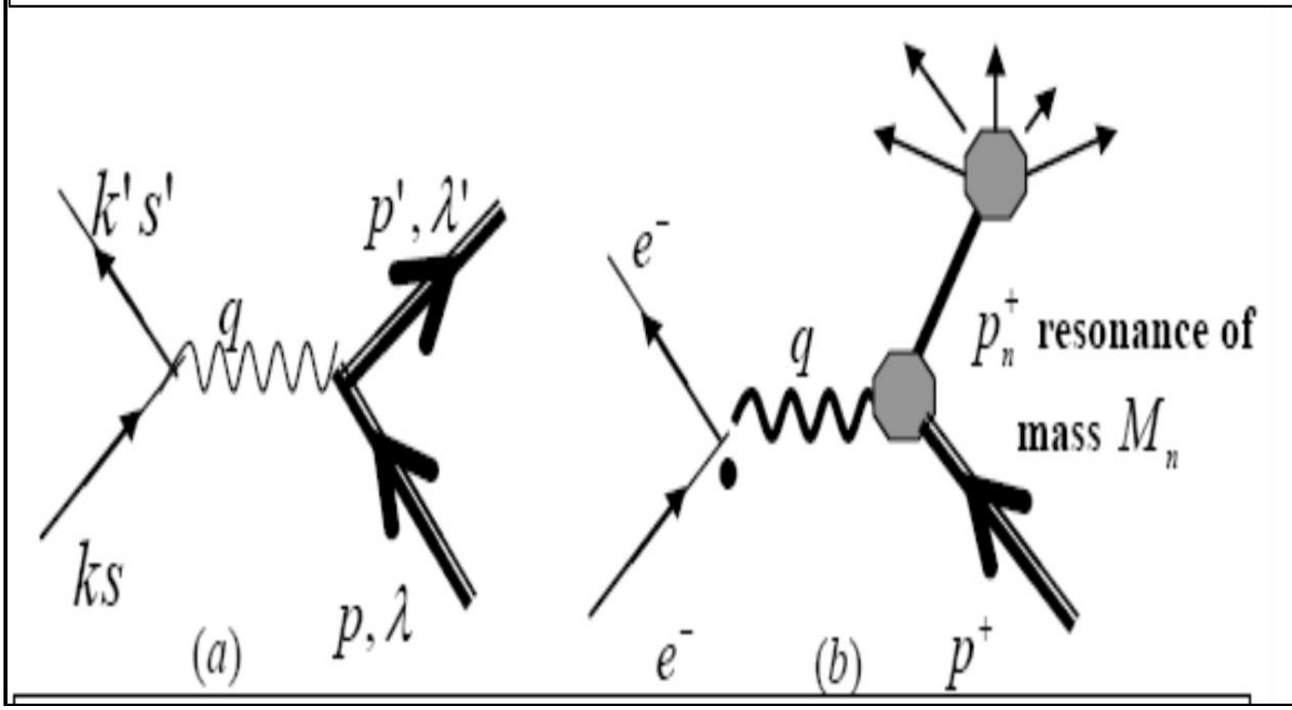
3.1 Feynman Graphs for Scattering Cross-Section

In an article entitled: *Review of Deep-inelastic e-p Scattering: A hadronic mechanics Viewpoint* by A.O.E. Animalu & C.E. Ekuma published in *African Journal of Physics* Vol.1, p.133-153 (2008) cited in 2011 ref. [27]:

www.santilli-foundation.org/docs/Isoscattering-V.pdf

we have exhibited the Feynman graph (shown below in Fig. 8) and summarized the conventional quantum mechanical (QM) and the corresponding hadronic mechanics (HM) expressions for the electron-proton scattering cross-section.

Fig. 8: Feynman graph for (a) electron-point-proton (Mott) scattering & (b) resonance model of deep-inelastic electron-proton scattering



 https://doi.org/10.52202/059404-0001

CONVENTIONAL QM EXPRESSIONS FOR THE SCATTRING CROSS-SECTION

A resonance of mass M_n is produced. $M_n^2 = M^2 + 2M\nu + q^2$, with $M^2 \equiv p^2,\ \nu \equiv q.p$,

$$\frac{d\sigma_{e^- - po\,\mathrm{int} - p^+}}{d\Omega} = \left.\frac{(Z\alpha)^2 E^2 \left(1 - \beta \sin^2(\theta/2)\right)}{4p^4 \sin^4(\theta/2)}\right|_{\beta=1}$$

$$= \frac{(Z\alpha)^2 \cos^2(\theta/2)}{E^2 \sin^4(\theta/2)} \equiv \frac{d\sigma_{Mott}}{d\Omega}$$

$$\beta = \frac{|\vec{\mathrm{p}}|}{\mathrm{E}},\ \tfrac{1}{2}(\vec{p}\,'-\vec{p})^2 = \vec{q}^{\,2} = 2p^2(1-\cos\theta) = 4p^2 \sin^2(\theta/2),$$

$$\frac{d^2\sigma}{d\Omega dE} = \frac{d\sigma_{Mott}}{d\Omega}\left[W_2(\nu, q^2) + 2W_1(\nu, q^2)\tan^2\frac{\theta}{2}\right]$$

$$M^2 = -q^2 \to \infty, \nu = q.p \to \infty,$$

$$2MW_2(\nu, q^2) = F_1(\omega),\ \nu W_2(\nu, q^2) = F_2(\omega)$$

$$\int_{q^2/2M}^{\infty} d\nu \left[W_2^p(\nu, q^2) + W_2^n(\nu, q^2)\right] \geq \frac{1}{2} \qquad \int_1^{20} \frac{d\omega}{\omega} \nu W_2^p = 0.78 \pm 0.04$$

R.M. Santilli has presented the predictions for the rest energy of particles within the scattering region

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HADRONIC MECHANICS EXPRESSIONS FOR THE SCATTRING CROSS-SECTION

Our characterization of the HM viewpoint is anchored on an association of the two Lorentz scalars, $(q^2 - p^2)$ and $q.p$, involved in the Bjorken scale:

$-q^2 \to \infty,\ \nu = q.p \to \infty,$ such that $x \equiv (p^2 - q^2)/2p.q \to -q^2/2M\nu \equiv 1/\omega$,

with the ***self-corresponding points*** of a non-unitary, nonlinear ("lifting") transformation, $(q,p) \to (\hat{q}, \hat{p})$ of a rectangular hyperboloid into a torus,

$$q.p = (1/2)(\hat{q}^2 - \hat{p}^2).$$

$$\frac{d\sigma}{d\Omega} \to \frac{d\hat{\sigma}}{d\hat{\Omega}} \equiv \frac{d\hat{\sigma}_{Mott}}{d\hat{\Omega}}\left[\hat{F}_2(\hat{\omega}) + 2\hat{F}_1(\hat{\omega})\tan^2(\hat{\theta}/2)\right].$$

$$\begin{cases} d\hat{F}_1 / d\hat{\omega} = \gamma\hat{F}_1(\hat{F}_0 - \hat{F}_1) \equiv \hat{x}_0\hat{F}_1 - \gamma\hat{F}_1^{\,2} \\ d\hat{F}_2 / d\hat{\omega} = -\gamma\hat{F}_2(\hat{F}_0 - \hat{F}_2) \equiv -\hat{x}_0\hat{F}_2 + \gamma\hat{F}_2^{\,2} \end{cases}$$

$$\Rightarrow \begin{cases} \hat{F}_1(\hat{\omega}) = \dfrac{\hat{x}_0/\gamma}{\left[1 - \left(\hat{x}_0/\gamma\hat{F}_{10} - 1\right)e^{+\hat{x}_0\hat{\omega}}\right]} \\ \hat{F}_2(\hat{\omega}) = \dfrac{\hat{x}_0/\gamma}{\left[1 - \left(\hat{x}_0/\gamma\hat{F}_{20} - 1\right)e^{-\hat{x}_0\hat{\omega}}\right]} \end{cases}$$

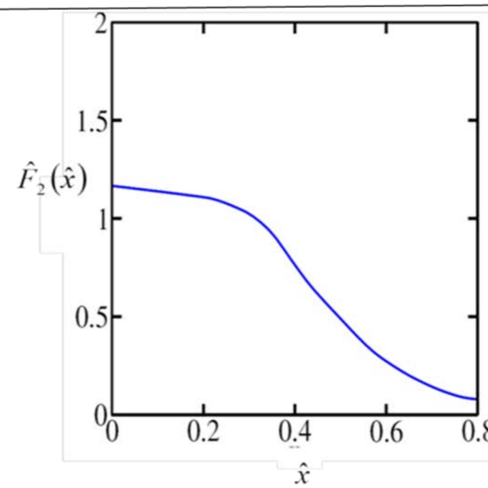
In terms of the reciprocal Bjorken variable, $\hat{x}$ the corresponding curve $\hat{F}_2(\hat{x})$ turns out to be an image of $\hat{F}_2(\hat{\omega})$ and has the form

$$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0/\xi}{\left[1 + \left(\hat{\omega}_0/\xi\hat{F}_{20} - 1\right)e^{+\hat{x}\hat{\omega}_0}\right]}$$

https://doi.org/10.52202/059404-0001

Fig. 9:
$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0 / \xi}{\left[1+\left(\hat{\omega}_0 / \xi\hat{F}_{20} - 1\right)e^{+\hat{x}\hat{\omega}_0}\right]}$ is tabulated for $\hat{F}_{20} > \hat{\omega}_0 / \xi$ & sketched below

#	
0	1.1672308
0.088888889	1.1424205
0.17777778	1.1164058
0.26666667	1.0621161
0.35555556	0.91052375
0.44444444	0.63087342
0.53333333	0.40818084
0.62222222	0.23984199
0.71111111	0.13305208
0.8	0.080026037



We summarize next in Fig.10 the characterization of isoscattering and genoscattering theories without divergences in hadronic mechanics.

Fig. 10: A schematic view of the main assumptions of (a) Lie-isotopic scattering theory[8] and (b) the Lie-admissible (genoscattering) theory[9], namely the exact validity of conventional quantum mechanics (QM) for the *EXTERIOR* conditions and the validity of the covering hadronic mechanics (HM) for the *INTERIOR* conditions of the scattering region, generally of arbitrary shape, but represented by spherical "extended" particle with a radius of about 10^{-13} cm in isoscattering theory, and by a torus with outer radius of ($\ell_0 = 10^{-13} cm$) and an inner (hole) radius ($\ell_i < \ell_0$) defining two "fundamental lengths" that ensure *ab initio* convergence of possible corrections.

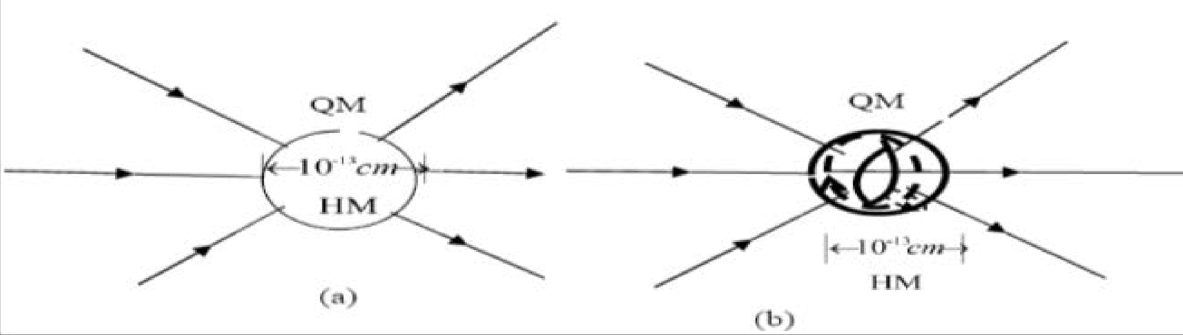


Fig. 11: Four times characterized in Santilli's iso-selfdual symmetry[7] of the theory of ANTI-MATTER

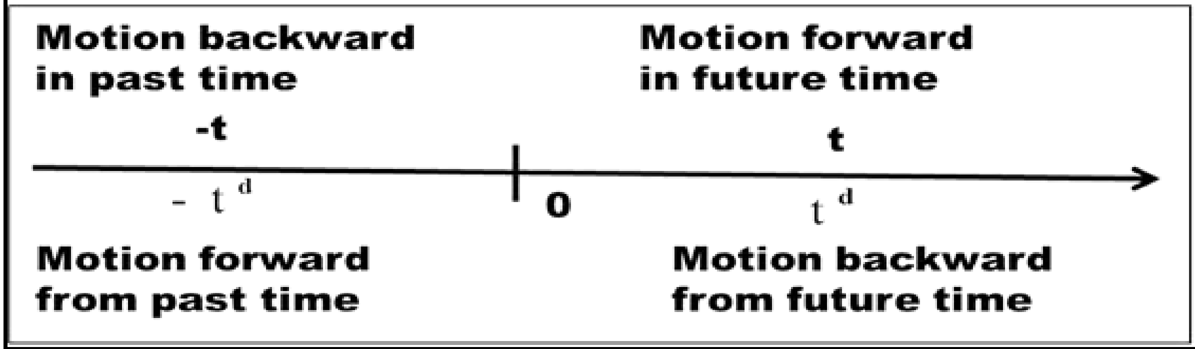
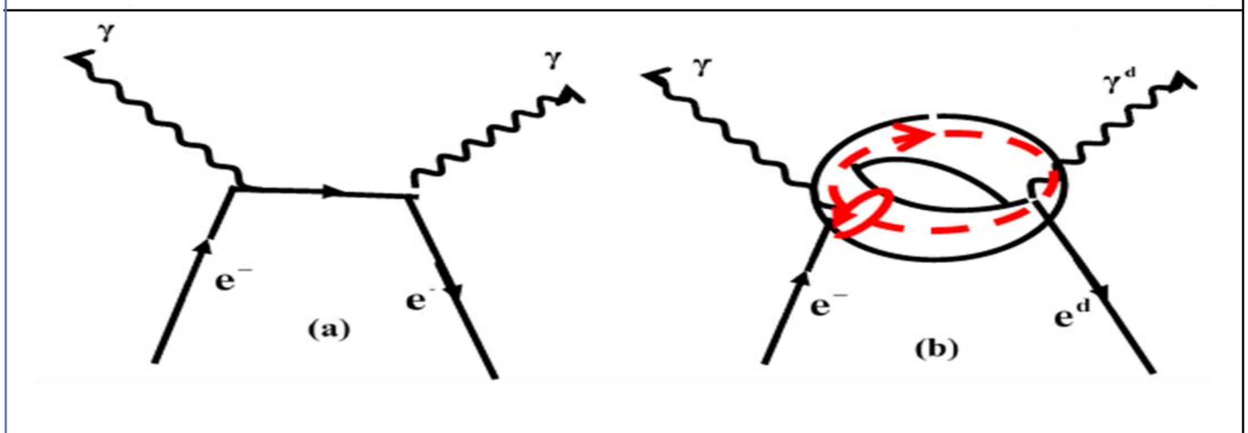


Fig.12: **Corresponding Feynman SPACE-TIME graphs[8] for:**

(a) $e^{+}+e^{-}\rightarrow 2\gamma$; **(b)** $e^{-}+e^{+}\rightarrow \gamma+\gamma^{d}$.



https://doi.org/10.52202/059404-0001

Fig.13. Conventional Feynman Graph/Rule for e-p Coulomb Scattering in : www.santilli-foundation.org/docs/Isoscattering-IV.pdf

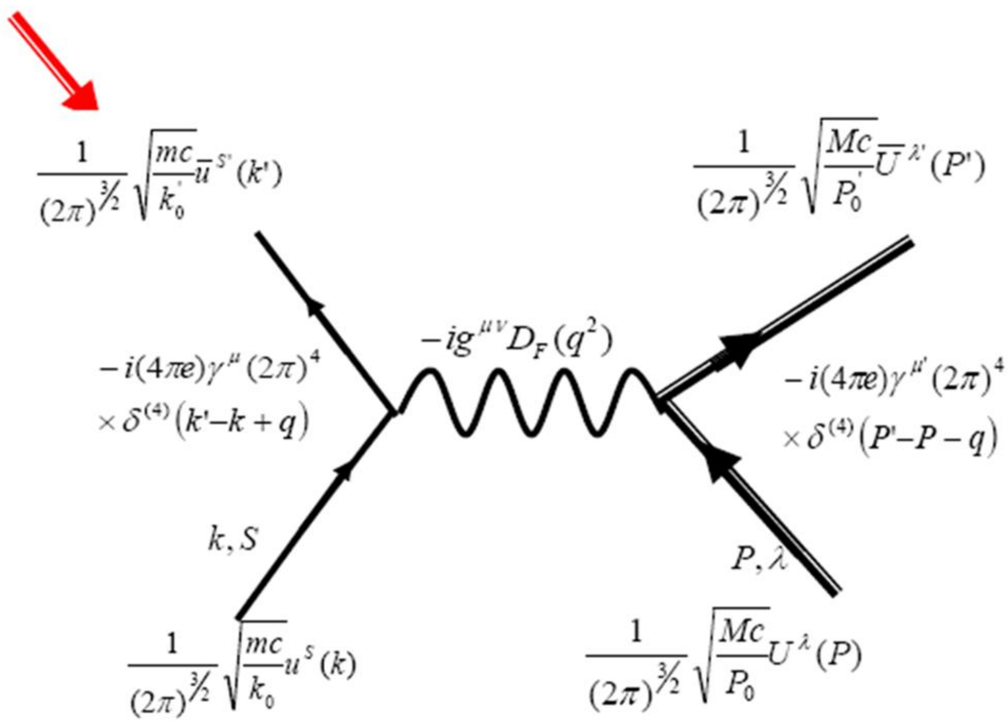
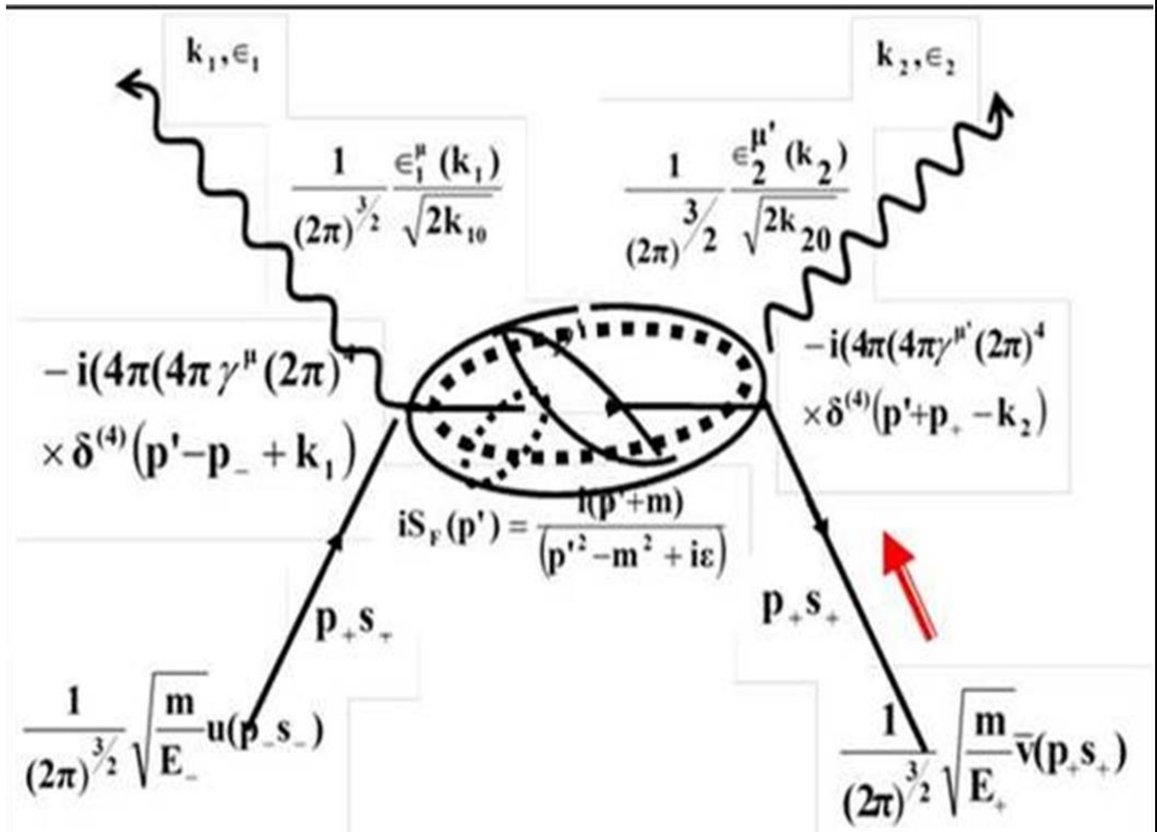


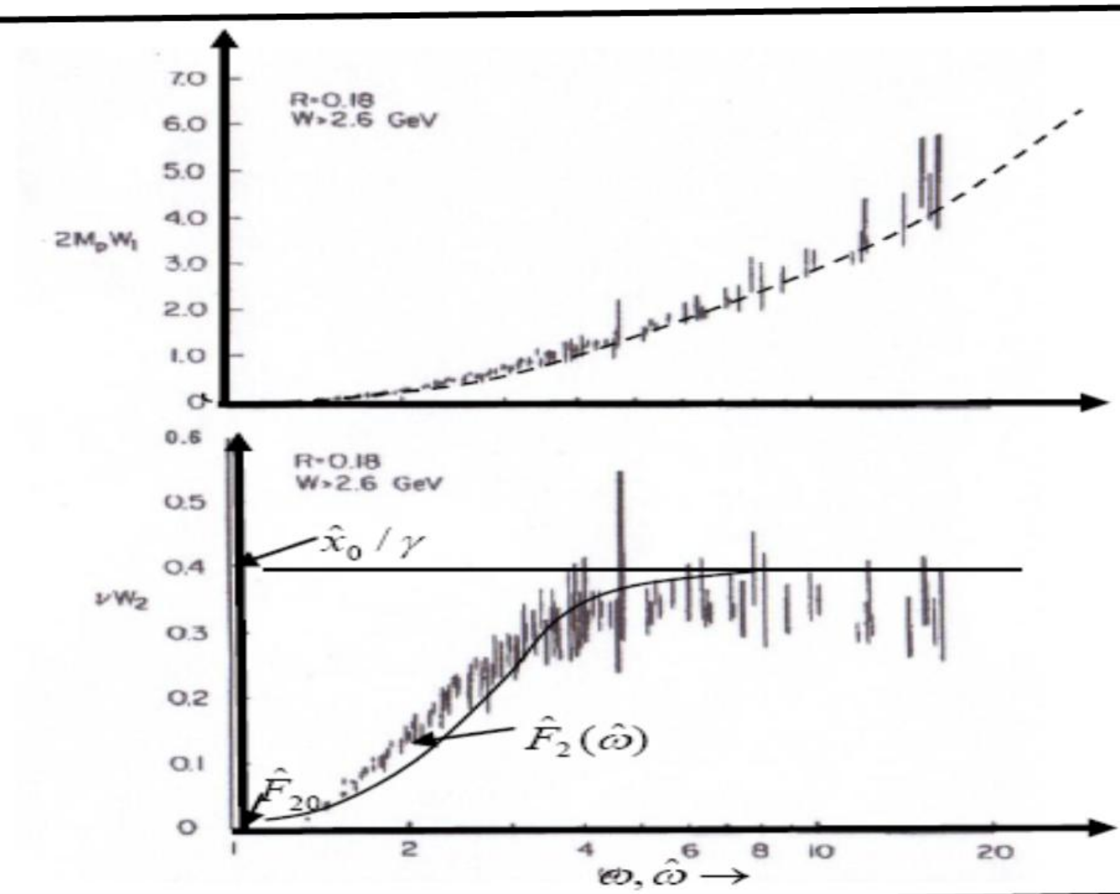
Fig.14: Conventional Feynman Graph/Rules for computation of the S-matrix for the annihilation process $e^{+}+e^{-}\rightarrow 2\gamma$ and its elaboration for the isoselfdual process $e^{+}+e^{-}\rightarrow \gamma+\gamma^{d}$



https://doi.org/10.52202/059404-0001

3.2 Comparison of Non-Unitary Geno-Sacttering Theory with Experiment (ref.[27])

Fig. 15 : Comparison of $\hat{F}_1(\hat{\omega})$ & $\hat{F}_2(\hat{\omega})$ for $\hat{F}_{20} < \hat{x}_0$ with $2MW_1 = \hat{F}_1(\hat{\omega})$ & $\nu W_2 = F_2(\hat{\omega})$ where $\omega = 2M\nu / q^2$ for proton; $W > 2.6\,GeV$, $q^2 > 1(GeV / c_0^2$ and R=0.18. Data from G.Miller *et al* Phys. Rev. D5, 528(1972).



As stated above, in terms of the reciprocal Bjorken variable, $\hat{x}$ the corresponding curve $\hat{F}_2(\hat{x})$ turns out to be an image of $\hat{F}_2(\hat{\omega})$ and has the form

$$\hat{F}_2(\hat{x}) = \frac{\hat{\omega}_0 / \xi}{\left[1 + \left(\hat{\omega}_0 / \xi \hat{F}_{20} - 1\right) e^{+\hat{x}\hat{\omega}_0}\right]}$$

whose feature is as shown for $\hat{F}_{20} < \hat{\omega}_0 / \xi$ and $\hat{F}_{20} \geq \hat{\omega}_0 / \xi$ in Figs 16-18

from A.O.E. Animalu and E. C. Ekuma, *Review of Deep-Inelastic e-p Scattering: A Hadronic Mechanics Viewpoint*, in African J. Phys. Vol.1, p.133-153 (2008) which was cited as Ref. [27] in the Proc. ICLATIP-3 Kathmandu Univ. Nepal (2011) as Santilli-Animalu-Isoscattering-V.pdf

https://doi.org/10.52202/059404-0001

Fig. 16 : Comparison of $\hat{F}_1(\hat{\omega})$ & $\hat{F}_2(\hat{\omega})$ for $\hat{F}_{20} < \hat{x}_0$ with $2MW_1 = \hat{F}_1(\hat{\omega})$ & $\nu W_2 = F_2(\hat{\omega})$ where $\omega = 2M\nu / q^2$ for proton; $W > 2.6\ GeV$, $q^2 > 1(GeV / c_0^2$ and R=0.18. Data from G.Miller *et al* Phys. Rev. D5, 528(1972).

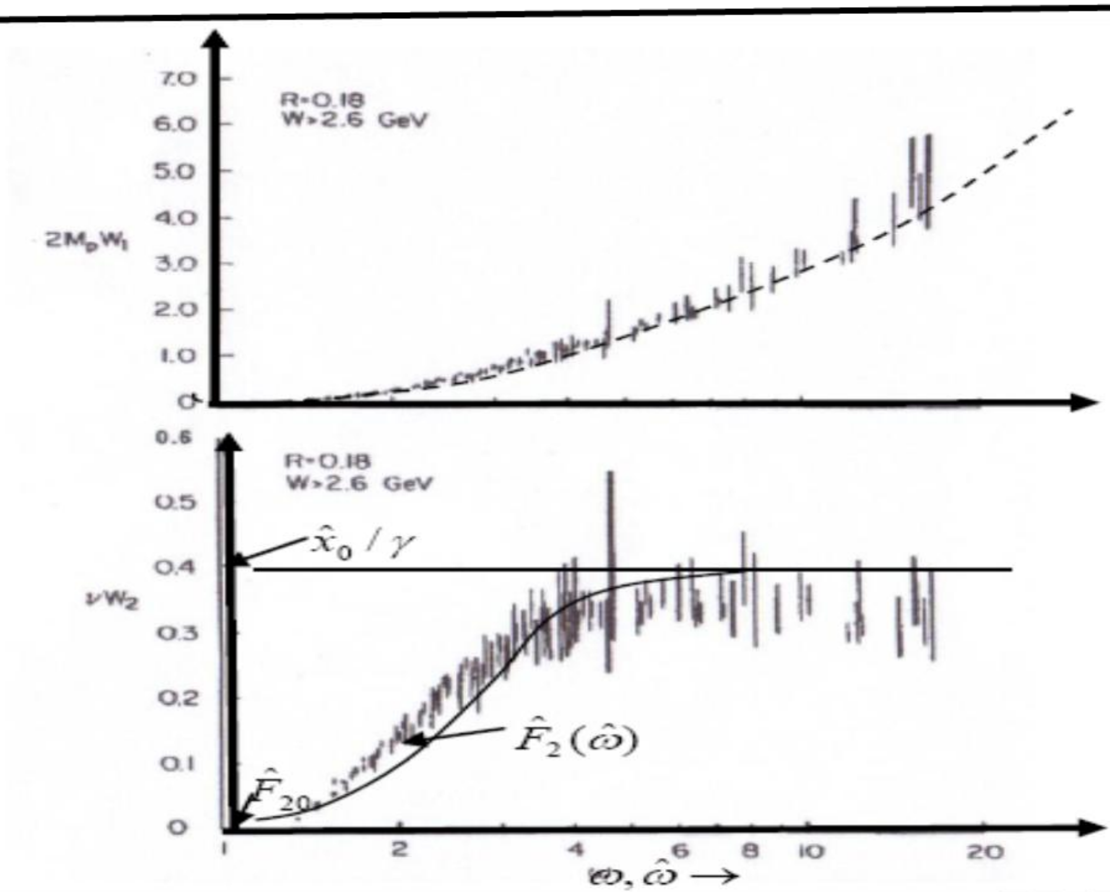
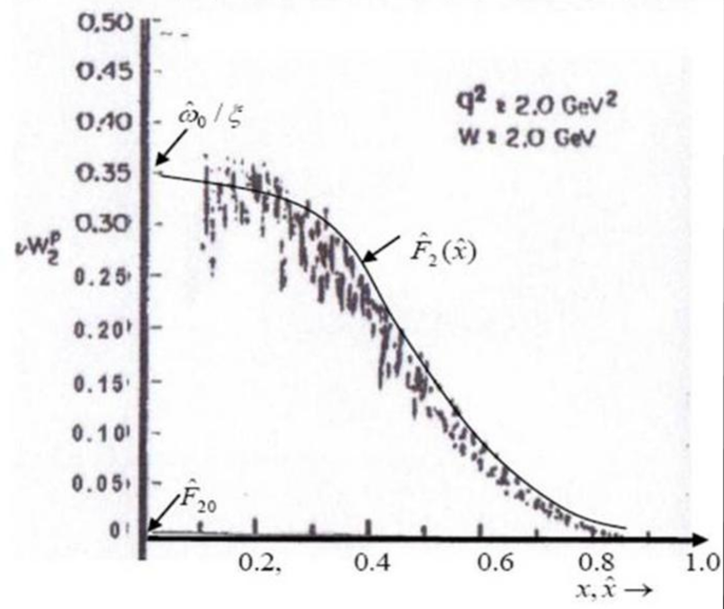


Fig. 17 : Comparison of $\hat{F}_2(\hat{x})$ for $\hat{F}_{20} < \hat{\omega}_0/\xi$ (solid line) with νW_2 versus $x = q^2/2M\nu$ for the proton for $W > 2.0\ GeV$, $q^2 > 2(GeV/c_0^2$ Data from Bodek *et al* Phys. Rev. Lett/ 30, 1087(1973); Phys. Lett. 51B, 417(1974); Phys. Rev. D20, 1471 (1979)

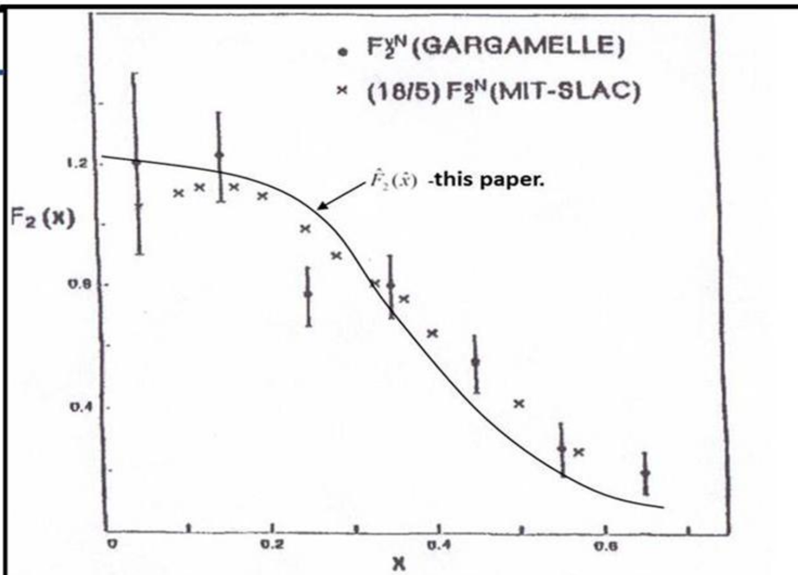
$\hat{x}$	$\hat{F}_2(\hat{x})$
0.0475727	0.334991
0.127295	0.325612
0.204184	0.331821
0.28397	0.310725
0.355572	0.279105
0.449308	0.219128
0.537629	0.144757
0.623138	0.0820683
0.697531	0.0426724
0.782875	0.00992946
0.86254	0.0109668
0.892858	-0.00686764



 https://doi.org/10.52202/059404-0001

Fig.18: Elaborated from A.O.E. Animalu and E. C. Ekuma, *Review of Deep-Inelastic e-p Scattering: A Hadronic Mechanics Viewpoint*, in African J. Phys. Vol.1, p.133-153 (2008) which was cited as Ref. [27] :
www.santilli-foundation.org/docs/Isoscattering-V.pdf

# $\hat{x}$	$\hat{F}_2(\hat{x})$
0	1.1672308
0.088888889	1.1424205
0.17777778	1.1164058
0.26666667	1.0621161
0.35555556	0.91052375
0.44444444	0.63087342
0.53333333	0.40818084
0.62222222	0.23984199
0.71111111	0.13305208
0.8	0.080026037



Early Gargamelle measurements of $F_2^{\nu N}$ compared with $(18/5)F_2^{eN}$ calculated from the MIT-SLAC results) and with HM model $\hat{F}_2^{\nu N}(\hat{x})$. (Source of data J.I. Friedman Nobel Lecture 1990, *Physics 1990 p. 715*

A question may be raised as to the fact that the data we have presented are consistent with hadronic mechanics, whereas the proton deep inelastic scattering data are expected to have 2-3 bumps due to high density situations with two bumps for e-p and three bumps for quarks in the standard model and similarly for mesons. An answer is that in the (Bjorken) limit of scale-invariance, the good agreement can only be considered preliminary, in view of the vast amount of available data that need to be analyzed based on the theory.

4 .DISCUSSION AND CONCLUSION

In A PubRelco Interview of R.M. Santilli with Scientific and Industrial implications New York, N.Y., April 15, 2019 In response to the question: "Prof. Santilli, could you please **review in a language** accessible to the general audience **Einstein's 1935 historical prediction that quantum mechanics and, therefore, quantum chemistry are incomplete theories",** the following answer was given: "Einstein did not accept the **uncertainty of quantum mechanics**, including the impossibility to identify the position of a particle with classical precision. For that reason, he made his famous quote **"God does not play dice with the universe."**

Einstein **accepted quantum mechanics for atomic structures**, **but believed that quantum mechanics is an "incomplete theory,"** in the sense that it could be broadened into such a form to **recover classical determinism** at least under special conditions. The same argument also applies to **quantum chemistry**". It admits interior entanglement/Lie-admissible penetration which is characterized by progressive generalization of the **Lie-algebraic product** of quantum mechanics (and hence quantum chemistry) in **"hadronic mechanics"** as follows:

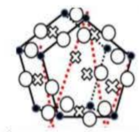
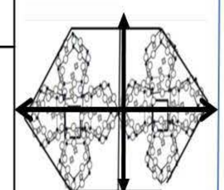
$$AB-BA\rightarrow\begin{cases} APB-BP^{+}A,(P^{+}=P=I)\rightarrow \text{Conventional Lie algebraic product} \\ APB-BP^{+}A,(P^{+}=P\neq I)\rightarrow \text{Lie isotopic algebraic product} \\ APB-BP^{+}A,(P^{+}\neq P\neq I)\rightarrow \text{Lie admissible algebraic product,} \end{cases}$$

This progressive Lie-admissible algebraic structure has been realized since (2011) in **geometric terms as a deformation of a point sphere into a torus** characterized in the framework of **"Non-unitary scattering theory of hadronic mechanics"** by R.M. Santilli and A.O.E. Animalu

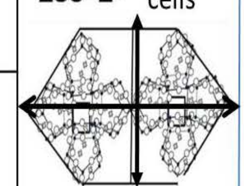
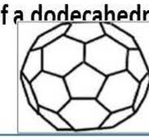
In view of the linguistic challenges of hadronic mechanics, and for ease in translation into other languages, we have done linguistic geometric elaboration of the EPR argument in analogue/digital SO(10) characterization as shown below

https://doi.org/10.52202/059404-0001

LINGUISTICS GEOMETRIC ELABORATION OF THE EPR ARGUMENT FOR ANALOGUE/DIGITAL SO(1O) CHARACTERIZATION OF HADRONIC ENERGY & CORONAVIRUS

Analogue	Digital (GENO-ASCII Code, A ~ 1,B~2,C~3,…,Z~26, so that HADRON ~ H+A+D+R+O+N=8+1+4+18+15+14=60; & (A,B,C,…Z)~(a,b,c,…,z), such that in SO(10),1+2+3+….+36=666.	(*) Remarks :
*Life; Cell; Bit	12+9+6+5= **32 = 2⁵**; Cell= 3+5+12+12=**32** ; Bit= 2+9+20=**31**	In SO(10) group representation of ***Life ("cell") by** 32 = 10 points + 15 lines + 7 planes.  & representation of *Lie-admissible **Penetration by** **256=2⁸** cells  **& *Classical Mechanics is given by 5 Bits (5x31=155) Forming Platonic Icosahedron dual of a dodecahedron.** 
*Classical Mechanics	(**3+12+1+19+19+9+3+1+12**)+(13+5+3+8+1+14+9+3+19)**=79+75=**155	
A Quantum Mechanics **Particle Physics** A Nuclear Dodecahedron	1+(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=75) =1+107+75 = **183;** (16+1+18+20+9+3+12+5=84)+(16+8+25+19+9+2+19=99) = 84+99=**183.** 1+(14+21+3+12+5+1+18=84)+(4+15+4+5+3+1+8+5+4+18+15+14=**98)=183**	
Wave Particle Duality **Quantum Chemistry** **A Dichotomy Exterior**	(23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25=92) =**227** (17+21+1+14+20+21+13=**107**)+(3+8+5+13+9+19+20+18+25=**120**)=**227** (1+4+9+3+8+15+20+15+13+25=113)+(5+24+20+5+18+9+15+18=**114**)=**227**	
Quantum Mechanics **Is not Complete** Wave-Particle Duality **Theory**	(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=**75**)+ (9+19)=28)+(14+15+20=49)+(3+15+13+16+12+5+20+5=**89**)+ (23+1+22+5=**51**)+(16+1+18+20+9+3+12+5=**84**)+(4+21+1+12+9+20+25=**92**)+ (20+8+5+15+18+25=91) **= 107+75+28+49+89+51+84+92+91 = 666**	
God Does Not Play A Dice with the Universe **Entanglement Interior**	**(7+15+4=26)+(4+15+5+19=43)+(14+15+20=49)+(16+12+1+25=54)+** **(1+4+9+3+5=22)+(23+9+20+8=60)+(20+8+5=33)+(21+14+9+22+5+18+9+5=113)+** (5+14+20+1+14+7+14+5+13+5+14+20=148)+(9+14+20+5+18+9+15+1=108) **=** **26+43+49+54+22+60+33+113+148+108 = 666**	
A Complete Time **Determinism**	(1+3+15+13+16+12+5+20+5=80)**+(20+9+13+5=47)+** **4+5+20+5+18+13+9+14+9+19+13=129)= 80+47+129=256**	
*Lie-Admissible Penetration	(12+9+5=26)+(1+4+13+9+19+19+9+2+12+5=93) + +(16+5+14+5+20+18+1+20+9+15+14=137) = **26+93+137=256**	
Hadronic Energy **Corona Virus** ***Corona Virus iso-spin**	(8+1+4+18+15+14+9+3=72)+(5+14+5+18+7+25=74) = **72+74 = 146** (3+15+18+15+14+1=66)+(22+9+18+21+19=89); 66+89= 155 **155+** [(9+19+15=43)+(19+16+9+14=58)=155+43+58=155+101 = 256	

LINGUISTICS GEOMETRIC ELABORATION OF THE EPR ARGUMENT FOR ANALOGUE/DIGITAL SO(2N) CHARACTERIZATION OF QUANTUM CHEMISTRY AND PARTICLE PHYSICS SCATTERING

Analogue	Digital (GENO-ASCII Code, A ~ 1,B~2,C~3,…,Z~26, so that WORD ~ W+O+R+D=23+15+18+4=60; & (A,B,C,…Z)~(a,b,c,…,z).	(*) Remarks : In SO(10) group
*Life; Cell; Bit	12+9+6+5= **32 = 2^5** ; Cell= 3+5+12+12=**32** ; Bit= 2+9+20=**31**	representation of
*Classical Mechanics	(**3+12+1+19+19+9+3+1+12**)+(13+5+3+8+1+14+9+3+19)**=79+75=**155	***Life ("cell") by**
A Quantum Mechanics **Particle Physics** **A Nuclear Dodecahedron**	1+(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=75) =1+107+75 = **183;** (16+1+18+20+9+3+12+5=84)+(16+8+25+19+9+2+19=99) = 84+99=**183.** 1+(14+21+3+12+5+1+18=84)+(4+15+4+5+3+1+8+5+4+18+15+14=**98)=183**	32 = 10 points + 15 lines + 7 planes. 
Wave Particle Duality **Quantum Chemistry** **A Dichotomy Exterior**	(23+1+22+5=51)+(16+1+18+20+9+3+12+5=84)+(4+21+1+12+9+20+25=92) =**227** (17+21+1+14+20+21+13=**107**)+(3+8+5+13+9+19+20+18+25=**120**)=**227** (1+4+9+3+8+15+20+15+13+25=113)+(5+24+20+5+18+9+15+18=**114**)=**227**	& representation
Quantum Mechanics **Is not Complete** **Wave-Particle Duality** **Theory**	(17+21+1+14+20+21+13=**107**)+(13+5+3+8+1+14+9+3+19=**75**)+ (9+19)=28)+(14+15+20=49)+(3+15+13+16+12+5+20+5=**89**)+ (23+1+22+5=**51**)+(16+1+18+20+9+3+12+5=**84**)+(4+21+1+12+9+20+25= **92**) + (20+8+5+15+18+25=91)**= 107+75+28+49+89+51+84+92+91 = 666**	of *A Complete **Time Determinism by** **$256=2^8$** cells 
God Does Not Play **A Dice with the Universe** **Entanglement Interior**	**(7+15+4=26)+(4+15+5+19=43)+(14+15+20=49)+(16+12+1+25=54)+** **(1+4+9+3+5=22)+(23+9+20+8=60)+(20+8+5=33)+(21+14+9+22+5+18+9+5=113)+** (5+14+20+1+14+7+14+5+13+5+14+20=148)+(9+14+20+5+18+9+15+1=108) **=** **26+43+49+54+22+60+33+113+148+108 = 666**	
***A Complete Time** **Determinism**	(1+3+15+13+16+12+5+20+5=80)**+(20+9+13+5=47)+** **4+5+20+5+18+13+9+14+9+19+13=129) = 80+47+129=256**	**& *Classical Mechanics** is given by **5 Bits**
Scattering of 2 Fermions **into 1 Boson has** **Incomplete** **Missing Particle**	(19+3+1+20+20+5+18+9+14+7=116)**+(15+6=21)+2+(**6+5+18+13+9+15+14+19=99)+ **(9+14+20+15=58)+1+(**2+15+19+15+14=65)**+(8+1+19=28)+** (9+14+3+15+13+16+12+5+20+5=102)+ (**13+9+19+19+9+14+7=90**)+(16+1+18+20+9+3+12+5=84) = **116+21+2+99+58+1+65+28+102 +90+84= 666**	**(5x31=155)** **Forming Platonic Icosahedron dual of a dodecahedron.** 
Lie-Admissible **Penetration**	(12+9+5=26)+(1+4+13+9+19+19+9+2+12+5)=93) + +(16+5+14+5+20+18+1+20+9+15+14=137) = **26+93+137=256**	

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SIGNIFICANCE FOR THE EPR ARGUMENT OF THE NEUTRON SYNTHESIS FROM HYDROGEN AND OF A NEW CONTROLLED NUCLEAR FUSION WITHOUT COULOMB BARRIER

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Abstract

In this paper, we present studies on the most fundamental nuclear fusion in nature, the synthesis of the neutron from the hydrogen atom in the core of stars which literally allows stars to "turn on". Such a synthesis is an important confirmation of the EPR argument on the lack of completeness of quantum mechanics (and of its wave-function) due to impossibility by quantum mechanics to describe the synthesis of the neutron, since the mass of the neutron is "bigger" than the sum of the masses of the proton and of the electron, as well as for other technical insufficiencies. For this reason, R. M. Santilli [1] proposed in April 1978 the "completion" of quantum mechanics into an axiom-preserving but non-unitary form which he called hadronic mechanics. Via hadronic mechanics it was possible to achieve a numerically exact representation of "*all*" characteristics of the neutron in its synthesis from the hydrogen at the non-relativistic and relativistic levels. This representation is a fundamental starting point for the description of other syntheses which are again impossible to be correctly represented by quantum mechanics, such as the compression of an electron inside a neutron to synthesize a "pseudo-proton" and of an electron pair in singlet coupling inside a natural deuteron to synthesize a "pseudo-deuteron". The importance of such particles is given by their natural ability to win the Coulomb barrier and be attracted instead of being repelled by positively charged nuclei. Considering that at the mutual distance of 1 fm two deuterons experience a repulsive force of 230 N, turning this repulsive force into an attractive one has enormous implications, such as the realization of the fusion of two deuterons into a Helium nucleus without the need to supply any energy in order to win the Coulomb barrier. Santilli has additionally shown that, as it is the case for the neutron, negatively charged pseudo-proton and pseudo-deuteron are evidently unstable, yet they have mean lives of the order of seconds, thus being sufficient for industrial applications. It is evident that that the synthesis of pseudo-protons and pseudo-deuterons is prohibited by quantum mechanics, while being allowed by its "completion" into hadronic mechanics, thus showing the importance of the original argument by Einstein, Podolsky and Rosen. See [2], [3] and [4] for an extensive treatment of the problem.

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1. Historical notes

Stars initiate their lives as being composed of hydrogen. They grow in time via the accretion of interstellar-intergalactic hydrogen, and eventually reach such pressures and temperatures in their core to synthesize the neutron as a “compressed” hydrogen atom according to the model proposed by H. Rutherford.

In Santilli’s view, the synthesis of the neutron from the hydrogen in the core of stars is one of the best illustrations of the validity of the EPR argument because said synthesis cannot be represented with quantum mechanics for numerous reasons.

First of all, there is the positive binding energy required by the proton and electron in order to synthesize a neutron, that has an excess mass with respect to the sum of the masses of proton and electron. Second, the contact interaction of the two particles implies a situation of mutual overlapping of their wavepackets, which is deeply non-local and non-Hamiltonian, therefore can NOT be represented by a potential, making Quantum Mechanics inapplicable in these conditions.

Dr. Ernest J. Sternglass was the first known performer of a synthesis of the neutron from hydrogen gas, during his Master’s thesis at Cornell University, NY, in 1951 using an electric arc running through an X-ray tube filled with hydrogen. He was in contact with the late Einstein, who showed great interest in this kind of experiments, and encouraged him to pursue his research. The experiment was repeated independently by Edward Trounson, a physicist at the Naval Ordinance Laboratory in 1952 with similar results. Then again the Italian priest physicist don Carlo Borghi repeated similar experiments in Brazil in the 60's. [5]

These early tests on the neutron synthesis were rejected by the scientific community, namely, for the impossibility of deriving strong interactions from a bound state between the electron and the proton. Those researchers experienced indifference and ostracism from the academic community, and none of them ever tried to publish papers on their experiments, we only have private correspondence or diary entries as an historical proof of their existence. The lack of clear neutron detections, due to the lack of proper technical instruments at the time, contributed to the dismissal of the experiments.

2. Hadronic Mechanics

In September 1977 soon after joining Harvard University, R M Santilli was requested by the DOE to study possible new approaches to the controlled nuclear fusion. He accepted under the condition that he could study first the most important fusion in nature, that of the neutron from the hydrogen in the core of stars.

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He soon discovered the impossibility of describing this phenomenon via conventional Quantum Mechanics, for the above stated reasons, and he understood that he needed to get back to the bases if he wanted to treat this problem seriously.

He understood that the above insufficiencies originate from the theory at the foundation of Quantum Mechanics, Lie's theory, because said theory solely admits Hamiltonian linear interactions, which cannot possibly happen in conditions of mutual overlapping.

By recalling the fundamental character of Lie's theory, it follows that the mathematics itself underlying quantum mechanics does not allow a consistent representation of nuclear fusions and other physical or chemical energy releasing processes, due to their known irreversibility over time.

Santilli had therefore to conceive a new mathematics, which he called isomathematics, that is able to achieve a "completion" of 20th century mathematical and physical methods for extended, deformable and hyperdense particles in interior dynamical conditions.

This allowed him to propose in April 1978 the "completion" of quantum mechanics into an axiom preserving but non unitary form which he called hadronic mechanics with the first known formulation of the "operator" from the Lie admissible "completion" of Heisenberg's equation [1].

The need to verify Pauli's principle in the interior of hadrons expressed in the title of Harvard's paper [1] originated from the prediction, implied by the EPR argument, that hyperdense hadronic matter alter the conventional spin of particles. In fact, the spin of an electron in the core of stars is expected to be different than its value in vacuum due to the extreme pressures in all directions, provided that the electrons is represented as an extended wavepackets via the isotopic element T of Eqs. (4.15.49) because, when particles are represented as point-like, no resistance can possibly be experienced.
Note the suggestion, also in the title of paper [1] to verify the validity of special relativity in the interior of hadrons because a "completion" of relativistic quantum mechanics for the structure of hadrons implies the "necessary" completion of special relativity into a covering form which were studied in the monographs written at Harvard University [6], [7].

Ref. [1] achieved a representation of all characteristics of mesons as hadronic bound states electrons and positrons, but in 1978 Santilli could not achieve a quantitative representation of the spin of the neutron in its synthesis from the hydrogen due to the need for a detailed study of the isotopies of spin 1/2 which he did in subsequent years.

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Following the study of the isotopies of spin and angular momentum in various papers, [8], [9], [10], Santilli was finally able to achieve an exact representation of all characteristics of the neutron in its synthesis from the hydrogen both at the nonrelativistic [11], [12] and at the relativistic level [13], [14].

3. **The synthesis of the neutron**

Following, and only following a theoretical understanding of the neutron synthesis, Santilli initiated systematic experimental tests, which achieved the first known actual detection of neutrons synthesized from the hydrogen, thanks to the availability of various neutron detectors. [15], [16], [17]

The synthesis of the neutron in a submerged electric arc, according to Hadronic Mechanics, starts with the separation of the hydrogen molecule into H atoms, followed by the ionization of the H atoms and the consequential creation of a plasma composed by protons and electrons. Then said protons and electrons are aligned along the tangent to a local magnetic line with opposite charges, opposing magnetic polarities and opposing spins. This condition is followed by Rutherford's "compression" of protons and electrons, one against the other, caused by the disconnection of the rapid DC discharge that generated the indicated plasma of protons and electrons. If the energy provided by the arc is enough to supply the missing 0.782 MeV, we obtain an actual Neutron. A specific reactor was built by Thunder Energies Corporation (now Hadronic Technologies Inc.) for industrial applications of this process. See [18] for more details.

More systematic tests were then conducted, with the participation of other scientists, that confirmed the possibility of achieving this synthesis in a laboratory, and produced more interesting data [19]. See also lecture [20] for more details.

4. The pseudoproton and its applications

With the same mechanism as the synthesis of the neutron, Hadronic Mechanics predicts the possibility of synthesizing other particles. One of the most interesting is the negatively charged pseudoproton. The pseudoproton is predicted to be generated by the "compression" of an electron, this time, inside the neutron.

The particle so generated has, among its most noticeable characteristics, similar rest energy as the neutron, spin 1/2 and of course negative charge. The industrially significant feature of negatively charged hadrons is that they are attracted by normal nuclei, instead of being repelled. Although it is not possible at the present moment to have a direct experimental verification of the existence of the pseudoproton, mainly because there are no specifically designed detectors available, it is still possible to

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have a very significative indirect verification via nuclear transmutations of elements. After many tests, using a modified General Motors 14 kW Electric Generator (in order to reduce the neutron interferences and achieve higher pressures in the ignition chambers), Santilli managed to achieve such a verification from two different laboratories that analyzed irradiated and non-irradiated silver samples. [21]

Besides the great scientific interest, these pseudoprotons have also possible industrial applications, for example in the medical field for the treatment of cancer. Currently some tumors are treated with proton irradiation, but the protons are rejected by the atomic nuclei, so this treatment is invasive and with a low efficiency. The irradiation of tumors with pseudoproton rays would have clear advantages because, unlike protons, pseudoprotons are attracted by tumor nuclei, thus requiring low irradiation energies as well as focusing the treatment only in the tumor area reducing collateral damage.

5. The pseudodeuteron and its applications

Another even more interesting product of these syntheses is the pseudo-deuteron. It can be obtained from a deuterium gas, with the "compression" of two electrons inside a normal deuterium nucleus, or deuteron. When the electric discharge happens, a local plasma of deuterium nuclei and electrons is created, especially in singlet coupling form. When the spark is disconnected, some of the deuterons and of the electron valence pairs are compressed one inside the other, forming an isodeuteron nucleus with expected mean life around 1 s. This isodeuteron nucleus is then naturally attracted by the other normal deuterons, with an attractive force that is inversely proportional to the square of the distance, making nuclear fusion simply unavoidable. This process is known as *Santilli Hyperfusion*.

Calculations show that at 1 fm distance two deuterium nuclei experience a repulsive force of about 230 N. This repulsion is called Coulomb barrier, and it's the main reason why, after decades of studies (and billions of dollars of investments) using Quantum Mechanics and the Standard Model to find an efficient way to bring two deuterons one close to the other and make fusion happen, we still have no clear results of a reaction with positive energetic balance, meaning that the energy spent to make the fusion happen has always been bigger than the energy obtained by the fusion itself. The isodeuteron, being negatively charged, is able to turn that repulsive force into an attractive one, with consequences of enormous scientific and industrial importance.

Besides, having opposite magnetic moments with the deuterons, pseudodeuterons are predicted to be able to naturally achieve a singlet coupling configuration, fundamental for the fusion to happen properly, while with normal deuterons this condition happens

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at random, limiting even more the efficiency of the process. See lecture [22] for more details

6. Conclusions

The synthesis of the neutron achieved by the Directional Neutron Source (DNS) developed by Hadronic Technologies and R. M. Santilli is by itself a big proof of the inability of Quantum Mechanics to provide a complete description of the physical reality, along the objections posed by A. Einstein B. Podolsky and N. Rosen.

The synthesis of negatively charged particles is yet another confirmation of the EPR argument, that has enormous implications for mankind and shows the importance of looking beyond the applicability conditions of QM and investigating reality with basically new perspectives.

Hadronic Mechanics proves to be a valid completion of the quantum mechanical wavefunction into the isowavefunction permitted by the admission of contact non-Hamiltonian interactions due to deep wave overlapping.

This completion, which is only one of the possible completions of commonly accepted Physics, opens the door to applications beyond our present imagination and deserves to be studied for all the possible benefits that it could bring to our society.

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Completeness is Unfalsifiable: Gödel and Popper for the EPR Debate / Kuhn and the Standard Model

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We first discuss the relevance of Gödelian incompleteness to the standard of completeness in the EPR argument, following similar arguments as prior work. We note that Einstein was not a dispassionate observer of the developments in Quantum Mechanics when he made his pronouncement about God and dice: he had tiffs with Schrödinger and Hilbert in the development of General Relativity.

We suggest that completeness is unfalsifiable, while incompleteness is falsifiable. Thus while new variables can be considered within varied theoretical frameworks, searching for 'completeness' itself would always suggest additional variables, even when data suggest none. We consider the ramifications for ideas of completeness in Popperian epistemology.

Then turning to Kuhn, we consider the nature of the paradigm shift about to take place in the Standard Model of quantum physics by examining current failings from within the paradigm. We suggest the solution can be found in negative mass antimatter, following prior work. We note that negative mass antimatter (with positive Energy and no symmetry breaking) resolves many of the concerns with the Standard Model, most notably dark matter and dark energy, but including also a reformulation of the CKM model of quarks and an update to the arrow of time for antimatter in Feynman diagrams. We examine several candidates for the crucial experiment that will cause this paradigm shift, primarily at CERN.

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I. INTRODUCTION

Completeness is at issue in the EPR paper, indeed from the very title.[1] However, it is unclear the motivation one has for completeness.

One has the historically well-known result about Heisenberg uncertainty,[2] to which most of the completionist arguments have been posed, but we explore in this paper two more important challenges to Einstein's famous concerns: what about Gödelian incompleteness, and is completeness falsifiable?

II. GÖDELIAN INCOMPLETENESS

One imagines along with Einstein a situation where, through additional variables, it is possible to 'complete' quantum mechanics, i.e. to eliminate Heisenberg uncertainty. This still leaves the greater concern of Gödelian incompleteness.[3] If we have used both algebra and Boolean algebra, we have the possibility of statements such as 'this statement is false', which are undecidable, rendering the space of statements incomplete (not all of them are true or false).

It appears this issue is being name-checked by Einstein in the paper, but is never seriously addressed. Even his supposed completion of quantum mechanics would remain incomplete in its logic, unless this concern is addressed.

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III. COMPLETENESS IS UNFALSIFIABLE

Falsifiability is the bedrock of scientific epistemology, particularly as understood by Popper.[4] This is the main concern with completeness as a motivation for further examination: it never ends.

There is no amount of evident incompleteness that would satisfy a person ideologically committed to completeness. Every failed attempt at completion only changes the parameters necessary. For instance, the Standard Model currently predicts a neutron electric dipole moment (EDM) too small to measured.[5] However, this was not the case initially, with 90 percent of theoreticans predicting it should have been measured,[6] and then shifting their theories when it was not observed. Expectations shifting past experimental goalposts is the trademark of an unfalsifiable theory, and yet most people persist in the pretense that the Standard Model is falsifiable, despite its history.

One wonders where Einstein has not: is completeness falsifiable? What would such evidence look like, that could not be handled by shifting goalposts, or inventing new ones? Perhaps as much as we as humans enjoy control over every variable, completeness is not a part of nature. It seems this is suggested by the Gödelian nature of logic systems.

But moreover, we must consider the Popperian view of completeness.

Theories can only be part of the scientific universe if they pass this basic muster. As unfalsifiable, completeness is a philosophical longing of humankind, not a scientific hypothesis. A muse for our most creative minds. An inspiration. But not a hypothesis. Not a theory.

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IV. INCOMPLETENESS IS FALSIFIABLE

This point is considerably more subtle, and we had not thought of it by the time of the lecture, initially granting that my point of view (incompleteness is ok) was also unfalsifiable, rendering completeness a matter of philosophical choice.

However, we have come to the conclusion that this is not correct. Incompleteness can be disproven by a correct and complete theory.

Take for instance Gödelian Incompleteness. One could imagine building a logic system without such concerns, with a series of rules perhaps. That would be a falsification of Gödel, and a limiting of its domain of application. However such constraints are unknown to the author.

Similarly for uncertainty in position, momentum, energy or time. We have precise lab equipment. We are able to measure these well. If the uncertainty principle has been breached, it would be easy to tell.

That's part of what made uncertainty such a strong hypothesis in the first place: it was not expected.

Incompleteness is both falsifiable and generally where we have been forced because complete theories, with predictions within the realm of experiment, have been falsified, morphing into complete theories with predictions just out of reach of experiment (since the core program of completeness-insistence is beyond question).

We suggest that, because of its falsifiability status, incompleteness is the only hypothesis (between completeness and incompleteness) that meets the criteria of scientific. That this is so because completeness disproves incompleteness seems trivial until one considers that incom-

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pleteness does not disprove completeness. Perhaps it seems unfair, like we are holding completeness to a higher standard, but falsifiability is the standard we accept regardless of modern epistemology choice.

V. AFTER FALSIFICATION: PARADIGMS SHIFTING

Another way one can view the falsification of a major theory is through the lens of Kuhn.[7] As the theory is falsified, the paradigm must shift. This proceeds eventually through a crucial experiment, but before that, through the breakdown of the paradigm.

The crucial experiment is meant to be the one that no one can explain and forces the paradigm to shift. But here we are a decade after the Wilkinson Microwave Anisotropy Probe (WMAP) first suggested it,[8] and instead of a new theory, we are just making up dark matter and dark energy to plug our 96 percent empty understanding of the universe. But when a theory does as poorly as ours does, we suggest even naming these things relative to the current theory can be a mistake. We have suggested calling these 'two pieces of evidence of the failure of the Standard Model', since the vacuum catastrophe renders the current model ridiculous for even the dark energy / cosmological constant piece.

Beyond dark matter and dark energy, we have more evidence that the Standard Model has failed, but it has all been bent by human willpower into one standard theory (whose claim to fame is not being falsifiable, but being 'consistent with observation', an unscientific measure). Other evidence of failure:

The lack of symmetry breaking in nearly all of observed re-

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ality. With the exception of meson CP violation, we do not observe the preferential creation of matter over antimatter. Symmetry breaking is another unfalsifiable hypothesis designed to patch the Standard Model. And why exactly are we making the entire universe match our local matter antimatter ratio? We have called this the Anthropic Principle, but we would suggest instead calling it the Myopic Principle (or Heliocentric Universe). We do not need to insist the entire universe be like it is here in order for us to exist!

Hyperinflation. Besides dark energy, besides dark matter, we need still another patch to deal with the very early universe. That patch is neat, but unmotivated. Three strikes... and the Standard Model somehow remains at bat!

CP violation / strong CP problem. Even if we wanted to cling to the wild idea that all the matter in the universe were created by the CP violation asymmetry in mesons, we would then be confronted with a huge issue: the lack of quark oscillations in the nucleus. Another falsification requiring another patch to remain intact.

Antimatter moving backwards in time. Feynman had a beautiful mind. Nobody would question this, least of all me. But the idea of antimatter moving backwards in time is wild and does not pass basic Ockam's razor muster. It would be quite surprising to a layman that the Standard Model contains lots of stuff moving backwards in time. No, there's never been an experimental observation of this, nor is there a falsifiable aspect of the proposal, why do you ask? (Remember that the requirement is a result of assumptions about the energy and mass of antimatter, which just may not hold).[9]

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Gravitons Not Normallizable. Unlike some, we are not comfortable living with infinites.[10] Wavefunctions must be normallized, including that of the graviton.

VI. NEW PARADIGM: NEGATIVE MASS ANTIMATTER

As a brief aside, we think it's incredible that the Standard Model predicts the exact same energy levels for antimatter molecules as for matter molecules (and this has been experimentally confirmed for anti-Hydrogen). [11] Completely incredible.

However, this suggests the light from far away stars cannot be taken as evidence of the matter or antimatter content of those stars without further corroborating evidence. This suggests we can only take our material adventures as a species for this evidence of matter antimatter ratio, which are needless to say much more limited. To be specific, we've been about 70 or so Astronomical Units from here (69 times the distance from the Earth to the Sun!). And immediately after leaving our solar system... Voyager terminated transmission.

Let's take a step back and marvel at the heft of this assumption! We have only been less than 100 times as far from Earth as the Earth is from the Sun, but what is that in real astronomically relevant units? Well, galaxy clusters are typically 1-10 Mpc diameter (and, given the clustering, might be taken as the largest gravitationally bound units). Otherwise our galaxy is about 30 kpc in diameter, so beyond there we can't really say for sure.

So, how far is the 70 Astronomical Units travelled by Voyager 1. rela-

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tive to these galactic and cluster diameters? 0.00000034 kpc. And from that experience, we have chosen to extrapolate, not just to the 30 kpc of our galaxy, not just to the Mpc of our cluster, but to the hundreds of Mpc of the entire universe?! (An aside on neutrinos from nearby supernovae, which one could argue extend this knowledge to around 1Mpc,[12] but this would require them to be Dirac not Majorana, in which case they also would be indistinguishable.)

And so we have found the culprit at the core of our assumptions: that symmetry must be broken and antimatter must have positive gravitational mass like matter. It is simply ridiculous to require all of the universe to be like it is so close to here.

This then suggests a pretty obvious candidate for dark matter and dark energy: negative mass antimatter. By this we mean that, relative to our commonly observed matter, antimatter repulses, rather than attracts, matter. For this author, this idea was suggested by combining the above idea about light from antimatter galaxies being indistinguishable with the suggestion from Feynman that this is the behavior expected to be mediated by the Graviton (where its spin-2 nature suggests opposites repel).[13]

It is quite remarkable that such a small change to the Standard Model can account for ALL of the concerns in the prior section! Let us review them one by one:

Dark Energy This is no longer some mysterious cosmological constant or contemptible absolute field, but rather a result of the gravitational repulsion of the rest of the universe which, since it is far away, is roughly constant over galactic distances. Notably, no vacuum catastrophe either.

Dark Matter This is the result of misinterpretation of current data assuming only positive mass allowed. The lumpiness is roughly the projection of the rest of the universe onto our galaxy.

Symmetry Unbroken We no longer have the problem of pair creation and annihilation, because we instead suppose no net Mass to the universe. Symmetry forever unbroken. Every galaxy cluster a hole in the universe compensated by the rest.

Hyperinflation This is the result of protoclusters being closer in the past than they are now. So in the past they were flung apart at first very rapidly (as soon as whatever force holding them together was overwhelmed), since the force goes inversely with distance apart.

CP Violation / strong CP problem This is particularly interesting, since we have a paper from ML Good that actually predicted CP violation in the K meson if its antimatter quark had negative mass![14] Years before it was observed! So why did the hypothesis never receive serious consideration, instead leading to the development of CKM theory? ML Good did not have the astronomical model, and so was forced to introduce an absolute gravitational field, which had not been en vogue since Einstein called it his greatest embarassment. Of course we do not form it as an absolute field, and note that the ultrafine splitting suggested by ML Good would be due instead to the sign of the mass of the rest of the universe outside our cluster (which is the same as the antiquark, but opposite of the quark).

Antimatter moves forward in time Since negative mass changes the relationship between energy and time,[9] Feynman diagrams no longer

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require antimatter to move backwards in time for a consistent propagator.

Gravitons Normallizable. Now that we have added the assumption that there is no net mass to the universe, we can define for any galaxy or cluster a radius of zero net mass. The integral around this surface then allows the normallization of the graviton, which in turn makes Quantum Field Theory a Grand Unified Theory. Hooray!

Unfortunately so far few are celebrating. But we do have a surprising number of experiments that should be able to distinguish the matter in the next few years,[15–17] and some dozens of theorists that have independently introduced models containing the idea (see my talk slides for a more comprehensive list).[18–23]

Even though WMAP failed to be a crucial experiment despite finding the Standard Model 96 percent wrong, we suggest that antimatter falling up is too far outside the Standard Model to receive the kid glove treatment of dark energy (which has led a renaissance of interest into the cosmological constant, despite Einstein's own embarrassment at the idea). One notes Einstein even originally described his cosmological constant as a uniform background of negative mass density, in response to a concern of Schrödinger's... almost 20 years before the EPR paper and decade before the famous God and dice comment!

Maybe Einstein wasn't mad at God playing dice, so much as who he was playing with.

VII. CONCLUSIONS

We have examined completeness, the idea at the core of the EPR concern, and suggested that as a hypothesis it is unfalsifiable. We have noted that incompleteness and uncertainty seem to be the empirical result as well, in addition to being plainly falsifiable.

We have then considered the ramifications of a paradigm shift in the Standard Model, and why we think negative mass antimatter will be the new paradigm. I am currently placing an open wager on the result of these CERN experiments. When at least one reports a gravitational mass of antimatter to 5 sigmas, I would like to have some nice scotch, or owe somebody a nice scotch. Any takers, just email boneye at alum dot mit dot edu .

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NONLOCALITY, ENTANGLED FIELD AND IIS PREDICTIONS, SUPERLUMINAL COMMUNICATION

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Abstract

From the EPR prediction, the nonlocality and entangled state become the frontline in modern physics. First, we introduce briefly some researches on EPR, which include Santilli's studies. Next, based on the generalized Lorentz transformation (GLT) with superluminal in the complete special relativity, we propose that the entangled states must obey GLT because of they possess the superluminal and some characters as the spacelike vectors. Further, it changes the phase of the entangled field, whose phase particle (phason) has some characters and corresponding equations. It is tachyon, and assume that it is similar to photon and J=1 and m=0 or mass is very small as similar neutrino, and may show the action at a distance. We research that this field as wave has some characters. Third, we discuss the superluminal quantum communication by a pair of entangled states is generated on both positions, or by preparing and transmitting a pair of entangled instruments, so the superluminal quantum communication. Manipulation for one position can pass the same message or information to the other, so we may implement the superluminal communication. Finally, assume that the entangled field has a similar magnetic theory, which may be a quantum cosmic field, or be the extensive quantum theory, or God or the Buddha-fields and so on. These are all macroscopic fields, which correspond to de Broglie-Bohm nonlinear "hidden variable" theory, but it is microscopic. In a word, study and application of nonlocality and entangled field have important scientific and social significance.

Keywords: nonlocality, entangled field, superluminal, special relativity, generalized Lorentz transformation, quantum communication, prediction.
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I. Introduction

Based on the Einstein-Podolsky-Rosen (EPR) correlations and Bell inequalities, current new experiments validated that quantum mechanics possesses the nonlocality and entangled state, etc.

First, Aspect, et al., realized EPR experiment by the measure on the linear-polarization correlation of pairs of photons emitted in a radiative cascade of calcium and time-varying analyzers, and it agrees with the quantum mechanical predictions and the greatest violation of generalized Bell inequalities [1,2]. Ghosh and Mandel demonstrated the existence of nonclassical effects in the interference of two photons [3]. Further, the entangled state evolves a great hotspot in physics. Kavassalis and Noolandi discussed a new view of entanglements in dense polymer systems, which predict a geometrical transition from the entangled to the unentangled state in agreement with experimental data [4]. Horne, et al., discussed two-particle interferometry, which employs spatially separated, quantum mechanically entangled two-particle states [5]. Mermin discussed extreme quantum entanglement in a superposition of macroscopically distinct states [6]. Hardy investigated nonlocality for two particles without using inequalities for all entangled states except maximally entangled states such as the singlet state [7]. Goldstein provided a proof on Hardy theorem [8]. Kwiat, et al., reported new high-intensity source of polarization-entangled photon pairs with high momentum definition [9]. Strekalov, et al., reported a two-photon interference experiment that realizes a postselection-free test of Bell inequality based on energy-time entanglement [10].

In 1998 Santilli published a paper showing that the objections against the EPR argument are valid for point-like particles in vacuum (*exterior dynamical systems*), but the same objections are inapplicable (rather than being violated) for extended particles within hyperdense physical media (*interior dynamical systems*) because the latter systems appear to admit an identical classical counterpart when treated with the isotopic branch of hadronic mathematics and mechanics. Now Santilli reviewed, upgraded and specialized the basic mathematical, physical and chemical methods for the study of the EPR prediction that quantum mechanics is not a complete theory. This includes basic methods [11], apparent proof of the EPR argument [12], and examples and applications, in which the validity of the EPR final statement is the effect that the wavefunction of quantum mechanics does not provide a complete description of the physical reality. The axiom-preserving "completion" of the quantum mechanical wavefunction due to deep wave-overlapping when represented via isomathematics, and shown that it permits an otherwise impossible representation of the attractive force between identical electrons pairs in valence coupling, as well as the representation of *all* characteristics of various physical and chemical systems existing in nature [13]. Moreover, Santill studied the classical determinism of EPR prediction by isomathematics [14].

Pan, et al., reported experimental test of quantum nonlocality in three-photon Greenberger-Horne-Zeilinger (GHZ) entanglement, and three specific experiments, involving measurements of polarization correlations between three photons, lead to predictions for a fourth experiment, and found the fourth experiment is agreement with the quantum prediction [15]. Stefanov, et al., investigated the quantum correlations with spacelike separated beam splitters in motion and experimental test of multisimultaneity [16]. Pan, et al., demonstrated experimental entanglement purification for general mixed states of polarization-entangled photons and arbitrary unknown states [17]. Yu, et al., discussed a test of entanglement for two-level systems via the indeterminacy relationship [18]. Zhao, et al., used two entangled photon pairs to generate a four-photon entangled state, which is then combined with a single-photon state, and reported experimental demonstration of five-photon entanglement and open- destination teleportation (for N = 3) [19].

Amico, et al., reviewed the properties of the entanglement in many-body systems [20]. Korbicz, et al., shown structural approximations of positive maps and entanglement-breaking channels [21]. Orus discussed geometric entanglement in a one-dimensional valence-bond solid state [22]. Schmidt, et al., detected entanglement of a mechanical resonator and a qubit in the nanoelectromechanical systems [23]. Thomale, et al., investigated the entanglement gap separating low-energy in the entanglement spectrum of fractional quantum Hall states, and a new principle of adiabatic continuity [24]. Salart, et al., reported the first experiment where single-photon entanglement is purified with a simple linear-optics based protocol [25]. Chavez, et al., observed entangled polymer melt dynamics [26]. Jungnitsch, et al., provided a way to develop entanglement tests with high statistical significance [27]. Huber, et al., detected high-dimensional genuine multipartite entanglement of mixed states [28]. Sponar, et al., discussed the geometric phase in entangled systems for a single-neutron interferometer experiment [29]. Friis, et al., investigated relativistic entanglement of two massive particles [30]. Jack, et al., measured correlations between arbitrary superpositions of orbital angular momentum states generated by spontaneous parametric down-conversion, and quantified the entanglement of modes within two-dimensional orbital angular momentum state spaces [31]. Bussieres, et al., tested nonlocality over 12.4 km of underground fiber with universal time-bin qubit analyzers [32]. Mazzola, et al., investigate the dynamical relations among entanglement, mixedness, and nonlocality in a dynamical context [33]. Miao, et al., discussed universal quantum entanglement between an oscillator and continuous fields [34]. Chitambar, et al., considered multipartite-to-bipartite entanglement

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transformations and polynomial identity testing [35]. Carmele, et al., discussed the formation dynamics of an entangled photon pair [36].

Many experiments on the quantum entangled state shown some new characteristics: 1). The coherency. 2). The nonlocality [7,13,16,32-34]. 3). The quantum teleportation [17,19]. 4). The superluminal [16]. In this paper, we propose that the entangled states must obey relativity with the superluminal and its quantum theory, and discuss some predictions and the superluminal quantum communication, etc.

2. Complete Special Relativity

Based on the basic principles of the special relativity, according to the constancy of the velocity of light in the vacuum principle, it implies the invariance of the squared interval:

$$s^2 = r_{mn}^2 - c^2 t_{mn}^2 . \qquad (1)$$

From this we derived necessarily two symmetrical types of topological separated structures, i.e., the classification of the timelike and the spacelike intervals, and obtained simultaneously the Lorentz transformation (LT) with smaller velocity v<c and the generalized Lorentz transformation (GLT) with larger velocity $\bar{v}$ >c [37-39].

It is well-known that the Lorentz transformation (LT) is:

$$x_1' = \gamma(x_1 - vt), t' = \gamma(t - vx_1 / c^2) , \qquad (2)$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$. GLT is [37-39]:

$$x_1' = \bar{\gamma}(x_1 - c^2 t / \bar{v}), t' = \bar{\gamma}(t - x_1 / \bar{v}) , \qquad (3)$$

where $\bar{\gamma} = 1/\sqrt{1-(c/\bar{v})^2}$.

In deriving LT, an additional independent hypothesis has been used, thus the values of velocity are restricted absolutely, and the spacelike interval is excluded. LT and GLT are connected by the de Broglie relation $v\bar{v} = c^2$.

Further, based on the special relativity principle, an invariant speed c_h is necessarily obtained. Therefore, the exact basic principles of the special relativity should be redefined as: I. The special relativity principle, which derives necessarily an invariant speed c_h. II. Suppose that the invariant speed c_h in the theory is the speed of light in the vacuum *c.* If the second principle does not hold, for example, the superluminal motions exist, the theory will be still the extensive special relativity, in which the formulations are the same, only c is replaced by the invariant speed c→c_h. The fundamental properties of any four-vector and the strange characteristic of these tachyons are described. We discussed various other superluminal transformations and their mistakes.

We think that LT is unsuitable for photon and neutrino, the photon transformation (PT) is unified for space $x'=r+ct$ and time $t'=t+(r/c)$. It may reasonably overcome some existing difficulties, and cannot restrict that the rest mass of photon and neutrino must be zero. LT, GLT and PT together form a complete structure of the Lorentz group. If the invariant speed c_h are various invariant velocities, the diversity of space-time will correspond to many worlds [37,39]. Moreover, it may prove [39] that the local Lorentz transformations for different systems cannot derive the varying speed of light (VSL) theory searched warmly [40-42]. VSL is probably connected only with the general relativity.

3. Entangled Field and Its Predictions

The earliest entangled state originated from the induction and unification between men and nature in the Chinese traditional culture. Now we research some possible theories of the entangled states and corresponding predictions.

3.1. Entangled Relativity and Predictions

Since the quantum entangled state possesses some characters, for example, coherency, nonlocality and superluminal, etc., we propose that it may and must apply the complete special relativity (CSR) and GLT.

From Eq.(1) we derive $s^2=r_{mn}^2-c^2t_{mn}^2$>0 for the spacelike interval, the speed defined as $|v|=|r_{mn}/t_{mn}|$>c is always superluminal. We may choose an inertial frame that $t'_{mn}=0$ (the simultaneity), so that calibration time. In this case $|\bar{v}|=\infty$, i.e., the action at a distance. But $r'_{mn}=0$ (at the same space position) cannot be obtained [37-39]. Its mass should be $m=m_0/\sqrt{1-(c/\bar{v})^2}$.

Further, the quantum entangled states are related with space-time, which form the entangled fields. Their chance are the superluminal phase velocities, and as the spacelike vectors possess some fundamental characteristics in any four-vectors [37,39]:

(P; E/c)	$(j;c\rho)$	(A;φ)	(k;ω/c)	(dk; dω/c)	$(w_\alpha=\frac{\gamma}{c^2}\frac{d(\gamma v)}{dt};w_0=\frac{\gamma}{c}\frac{d\gamma}{dt})$
P=mv>E/c	$j=\rho v>c\rho$	$A>\varphi$	$k>\omega/c$	dk>dω/c	$w_\alpha<w_0$

Here P is momentum, and E is energy, etc. Because $\omega/k=\bar{v}$ is phase velocity, (k;ω/c) is usually a timelike vector with $\bar{v}$>c. While dω/d k=v is group velocity, so (dk; dω/c) is usually a spacelike vector with c>v. Usual the timelike and spacelike intervals are two topological separated parts by the light-cone.

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In the timelike vectors, only x, p, j, A, k, dk and w_α can be zero, and then LT is derived. In the spacelike vectors, only t, E, $\rho, \varphi, \omega, d\omega$ and w_0 can be zero, then GLT is derived. In this case φ=0, but A≠0; $\rho = 0$, but $j \neq 0$, etc. These are some predictions based on CSR. Mariwalla [43] let E=0, so GLT of the four-vector (p; E/c) was derived.

For any four-vector ($\vec{A}; A_0$), its LT is

$$A_1' = \gamma(A_1 - vA_0 / c), A_0' = \gamma(A_0 - vA_1 / c), \qquad (4)$$

and GLT is

$$A_1' = \bar{\gamma}(A_1 - cA_0 / \bar{v}), A_0' = \bar{\gamma}(A_0 - cA_1 / \bar{v}). \qquad (5)$$

Both possess the most perfect symmetrical form. Only A_1, A_0 interchange each other between A_1 and A_0 representations, and LT (4) and GLT (5) also interchange from v/c to c/$\bar{v}$.

The entangled field is in the spacelike interval, so the area is larger and the possibility is more.

3.2. Entangled Quantum Theory and Predictions

The quantum representations on the entangled fields are known, in which two basic spin states are quantized $|\frac{1}{2}>$ and $|-\frac{1}{2}>$, and group of two particles are (s^A, s^B), so there are four eigen-states:

$$|\frac{1}{2}>_A|\frac{1}{2}>_B,\ |\frac{1}{2}>_A|-\frac{1}{2}>_B,\ |-\frac{1}{2}>_A|\frac{1}{2}>_B,\ |-\frac{1}{2}>_A|-\frac{1}{2}>_B. \qquad (6)$$

While the coupling represents are:

$$SX_{SM} = MX_{SM}(S = s^A + s^B). \qquad (7)$$

For S=0 and M=0, $X_{00} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B - |-\frac{1}{2}>_A|\frac{1}{2}>_B]$ is single state.

For S=1 and M=0, ± 1, $X_{10} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B + |-\frac{1}{2}>_A|\frac{1}{2}>_B]$, $X_{11} = |\frac{1}{2}>_A|\frac{1}{2}>_B$ and $X_{1-1} = |-\frac{1}{2}>_A|-\frac{1}{2}>_B$ are threefold states. Here X_{00}, X_{10} are two entangled states. X_{11}, X_{1-1} carry through equal weight superposition, it may compose four entangled states:

$$|\psi^{\pm}>_{AB} = \frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|-\frac{1}{2}>_B \pm |-\frac{1}{2}>_A|\frac{1}{2}>_B], \qquad (8)$$

$$|\phi^{\pm}>_{AB}=\frac{1}{\sqrt{2}}[|\frac{1}{2}>_A|\frac{1}{2}>_B \pm|-\frac{1}{2}>_A|-\frac{1}{2}>_B].\qquad(9)$$

This derives the non-locality, and is similar to the Bell basis in quantum mechanics. It is the entangled state of quantum theory, and may describe quantum teleportation [44].

The entanglement is probably a new field, and exchanges tachyon or phase particle (phason), which corresponds to change of phase. They each other are the phase velocities. Its character is tachyon, and assume that it is similar to photon and J=1 and m=0 or mass is very small as similar neutrino. It shows the action at a distance.

Assume that the entangled field has the wave-particle duality, from which we propose its quantum theory: They are bosons, and based on the same energy-momentum relation:

$$E^2=p^2c^2+m^2c^4,\qquad(10)$$

we derive Klein-Gordon equation of quantum mechanics:

$$\frac{\partial^2\psi}{\partial x_\mu^2}-\frac{m^2c^2}{\hbar^2}\psi=0.\qquad(11)$$

When m_0 =0, it is Maxwell equation. The Schrödinger equation is:

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{1}{2m}\hbar^2\nabla^2\psi+V\psi.\qquad(12)$$

It agrees with the complete relativity, and is a quantum theory of the complete relativity.

Quantum entanglement corresponds possibly to the nonlinear superposition principle [35,45]. These are also some predictions based on the similar quantum theory of entangled fields.

Maudlin researched the quantum nonlocal entanglement and special relativity in modern physics [46,47].

3.3. Entangled Waves and Predictions

The entangled fields possess the wave property. It is known that two entangled fields can interference each other. They should diffract from each other, or even reflect. Especially combined quantum theory it will have the barrier penetration.

At the same time, wave may further develop to the field. It may combine the mechanical wave theory [48]. Such Schrödinger equation may develop to following nonlinear equations:

$$\frac{\partial\psi}{\partial t}+\alpha\psi\nabla\psi=\frac{i\hbar}{2m}\nabla^2\psi;\qquad(13)$$

and a similar KdV equation:

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$$\frac{\partial\psi}{\partial t}+\sigma\psi\frac{\partial\psi}{\partial x}=\frac{\partial^3\psi}{\partial x^3}. \qquad (14)$$

From Boussinesq equation and Klein-Gordon equation we may develop similarly to the nonlinear equation:

$$\phi_{tt}-\phi_{xx}=\sigma(\phi^2)_{xx}+\phi_{xxxx}. \qquad (15)$$

Further, many phase spaces exist probably in our world, for example, body-mind-spirit and they are entangled each other. The extensive quantum entanglement is a real "spooky action at a distance" (Einstein). Entangled states in parapsychology can explain synchronization, telepathy as resonance of the thought field [37,49]., and the unification of men and nature, etc. In quantum mechanics, the participant in the Wheeler interpretation is the unification of men and nature. Combined animism, it can be further explained prediction, premonition and other phenomena of parapsychology. The same frequency of the thought field is easy to synchronize. Quantum entanglement among living things produces their synchronization and magic special functions.

4. Superluminal Quantum Communication

First, Bernett, et al., proposed the quantum teleportion via dual classical information and nonclassical EPR channels [50]. Then Bouwmeester, et al., investigated experimental quantum teleportation [44]. Pan, et al., realized experimentally entangled freely propagating particles that never physically interacted with one another or which have never been dynamically coupled by any other means. It demonstrates that quantum entanglement requires the entangled particles neither to come from a common source nor to have interacted in the past [51].

Raimond, et al., performed manipulating quantum entanglement experiments with Rydberg atoms and microwave photons in a cavity, and investigated entanglement as a resource for the processing of quantum information, and operated a quantum gate and applied it to the generation of a complex three-particle entangled state [52]. Pan, et al., experimentally demonstrated observation of highly pure four-photon GHZ entanglement. Their technique can, in principle, be used to produce entanglement of arbitrarily high order or, equivalently, teleportation and entanglement swapping over multiple stages [53]. Zbinden, et al., reported an experimental test of nonlocal quantum correlation in relativistic configurations, in which entangled photons are sent via an optical fiber network to two villages near Geneva, separated by more than 10 km where they are analyzed by interferometers [54].

It already is widely applied, for example, quantum information [52], quantum swapping [53], quantum non-cloning and so on. Energy-time entangled photon pairs violate Bell inequalities by photons more than 10.9 km

[54] and 12.4 km [51]. At present, some physicists proposed that their entangled distance is infinite, and even is an action at a distance. I think, the quantum entangled state is probably a new fifth interaction [39]. Its strength seems to obey neither the Newtonian long-range gravitational law nor the short-range strong-weak interactions.

Cocciaro, et al., searched superluminal quantum communications on recent experiments and possible improvements. Some physicists (Bell, Eberhard, Bohm and Hiley) suggested that quantum correlations could be due to superluminal communications (tachyons) that propagate isotropically with velocity v>c. For finite values of v, quantum mechanics and superluminal models lead to different predictions. Some years ago a Geneva group and Cocciaro group did experiments on entangled photons to evidence possible discrepancies between experimental results and quantum predictions [55]. But, so far, no evidence for these superluminal communications has been obtained and only lower bounds for the superluminal velocities have been established. Cocciaro, et al., described an improved experiment that increases by about two orders of magnitude the maximum detectable superluminal velocities. No evidence for superluminal communications has been found and a higher lower bound for their velocities has been established [56].

Gao Shan analyzed the relation between quantum collapse, consciousness and superluminal communication. Quantum collapse as result of quantum nonlocality may permit the realization of quantum superluminal communication (QSC). He demonstrated that the combination of quantum collapse and the consciousness of the observer will permit the observer to distinguish nonorthogonal states in principle. This provides a possible way to realize QSC [57]. He introduced a possible mechanism of nonlinear quantum evolution and investigated its implications for quantum communication, so it is shown that the distinguishability of nonorthogonal states can be used to achieve quantum superluminal communication, which must exist based on the quantum nonlocal influence [58]. Reversely, Zhang analyzed the relation and the difference between the quantum correlation of two points in space and the communication between them, and proved the impossibility of the superluminal quantum communication from statistical separability [59].

Walleczek, et al., discussed the apparent conflict between quantum mechanics and the theory of special relativity, and nonlocal quantum information transfer without superluminal signalling and communication [60].

So far, it is generally believed that the entangled states do not transmit information, but affect each other instantly. We suppose that it is similar to electromagnetic field, and may apply to the superluminal communication. In a word, entanglement seems to be a particular synchronism.

A pair of entangled states is generated on both positions, or preparing and transmitting a pair of entangled instruments, so the superluminal quantum

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communication can be realized. Thus encoding the states (different phases) to form two codes: yes or no, ↑ (positive) or ↓ (opposite), both correspond to 1 or 0. Manipulation for one position can pass the same message or information to the other, so we may implement the superluminal communication. Further, it develops to the corresponding information theory.

Its important basis is that Enrique Galvez, et al., proposed a complete set of instruments to generate entangled photons in the laboratory, and the experimental process into a manual placed on the network [61].

Moreover, the exact quantum communication seems to be inconsistencies with the quantum non-cloning theorem [62]. Further, Barnum, et al., proposed the quantum non-broadcasting theorem [63].

5. Applications, Tests and Other Predictions

Musser searched the spooky action at a distance as the phenomenon that reimagines space and time, and what it means for black holes, the Big Bang and theories of everything [64]. He proposed that the nonlocality exists widely in black holes, the cosmic macrostructure and particle collisions [64]. Jacques, Kaiser and Peruzzo, et al., realized delayed-choice experiments [65-67].

Quantum theory is reversible and localized. Reversible black holes have radiation, so entropy decrease [68-71]. The nonlocal black holes have information overflow. General black holes are only come into and no leave to all v<c material. But, for $\bar{v}$ >c black holes may generate information. Both are linked by $v\bar{v}=c^2$. Giddings, et al., discussed black holes, quantum information, unitary evolution and observables in effective gravity [72,73].

We propose that the entangled fields may be developed by a similar magnetic field.

First, the magnetic induction seems to be transient transmission, in which A≠0, but φ =0. It presupposes that there should be a large external field similar to the geomagnetic field. This may be a quantum cosmic field, whose wave function of Universe obeys the Wheeler-de Witt equation:

$$(\hbar^2 G_{ijkl}\frac{\delta}{\delta g_{ij}}\frac{\delta}{\delta g_{kl}}+\sqrt{G^3}R)\psi(g)=0\,. \qquad (16)$$

Here $G_{ijkl}=\frac{1}{2\sqrt{G}}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl})$. This may also be the extensive quantum theory [74-76], the mysterious natural field on the induction between men and nature, God or the Buddha-fields [77,78] which correspond to real world is computer simulation or computational universe [79,80], three dimensional truth-goodness-beauty space [81] or body-mind-spirit space [82] and so on. These are all macroscopic fields, which corresponds to the Gaia hypothesis as a intertwined complex whole, and to de Broglie-Bohm nonlinear "hidden variable" theory [83,84], but it is microscopic. In fact, any order society can form a gauge field with customs and law to regulate standards of conduct for all.

If a similar magnetic field exists, it will be the rotation field, whose equations are:

$$\oint_S BdS = 0 \text{, and } \nabla B = 0 . \tag{17}$$

Such it may be used and analogous to communication. Probably, these fields are the origin of the entangled field, the entanglement is only a result.

Further, we should try to find a magnetic monopole. If we find a similar charge, the theory will be developed to the similar electrodynamics, in which Maxwell equations are:

$$\frac{\partial F_{lm}}{\partial x_k} + \frac{\partial F_{mk}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_m} = 0 , \tag{18}$$

$$\frac{\partial F_{ik}}{\partial x_k} = \frac{4\pi}{c} j_i . \tag{19}$$

Lorentz equation is:

$$\frac{dp}{dt} = e(-grad\varphi - \frac{\partial A}{c\partial t} + \frac{v}{c} \times rotA) = e(E + \frac{v}{c} \times H) . \tag{20}$$

Third, we may develop a similar magnetohydrodynamics. Fourth, combining quantum mechanics, it derives a similar quantum electrodynamics (QED). Fifth, combining general relativity, it derives the electromagnetic general relativity [85].

Discover these fields and their corollaries are also our predictions. The general predictions include the telepathy and the induction between men and nature, etc.

In a word, study and application of nonlocality and entangled field have important scientific and social significance.

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Comparison of various nuclear fusion reactions and ICNF

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Abstract.

Modern day demand of clean, cheap and abundant energy gets fulfilled by the novel fuels that have been developed through hadronic mechanics / chemistry. In the present paper, a short review of Hadronic nuclear energy by intermediate controlled nuclear synthesis and comparison with other fusion reactions has been presented.

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Introduction

Atomic nucleus and sub-nuclear particles have always been considered an unlimited source of energy. The discovery of nuclear fission by Otto Hahn and Fritz Strassmann paved the way for conventional nuclear energy. However, nuclear fission generates large amount of nuclear waste that risks ecosystem whereas nuclear synthesis is known to create much less pollution, thus is green. It is also comparatively more inexhaustible energy source. Hence, harnessing energy through nuclear synthesis reactions has been so far the Holy Grail. With the discovery of stellar nucleo-synthesis by Hans Bethe paved the way for nuclear synthesis of two or more light nuclei into a heavier nucleus. Of course, the energy released in this process could be harnessed.

The energy conversion from thermonuclear fusion reaction is marred by very low energy gains of the thermonuclear reactions. The energy input was larger than output obtained, hence was not economically feasible. Cold fusion on the other hand does not have sufficient energy to bring about fusion reaction in a sustained way.

With advent of ultra-short pulse laser technology, low temperature initiation of fusion even at the high plasma density can be materialized. This technology has allowed fusion of hydrogen-boron for low-cost fusion energy. However, amongst them ICNF does have an upper-edge as with the Hadronic mechanics, the processes taking place are easier to understand and hence more reproducible. Hadronic mechanics is of paramount importance for understanding nuclear synthesis as in this case nucleus cannot be considered as point mass.

Intermediate Controlled Nuclear Fusion (ICNF)

Intermediate Controlled Nuclear Fusion (ICNF) as proposed by Prof. Santilli are systematic energy releasing nuclear syntheses. The reaction rate is controllable via one or more mechanisms capable of performing the engineering optimization of the applicable laws.

Basic assumptions of Hadronic mechanics as proposed by Prof. Santilli are-

i) Nuclear force: Nuclear force is partly represented by a Hamiltonian and partly by the non-potential type terms that is the latter cannot be represented with a Hamiltonian.
ii) Stable nuclei: According to Heisenberg-Santilli Lie-isotopic equations the sub-nuclear particles are in contact with each other (technically, in conditions of mutual penetration of about 10^{-3} of their charge

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distributions). Consequently, the nuclear force is expected to be partially of potential and partially of nonpotential type, with ensuing nonunitary character of the theory, and related applicability of hadronic mechanics.

iii) Unstable nuclei and nuclear fusion: In case of Heisenberg-Santilli Lie-admissible equation (1) for the time evolution of a Hermitean operator A, in their infinitesimal and finite forms

$$i\frac{dA}{dt} = (A \hat{,} H) = ARH - HSA \tag{1}$$

where Hermitean, H represents non-conserved total energy; the genotopic elements R and S represent non-potential interactions. Thus, irreversibility is assured.

Irreversibility is assured in this case by the different values of the genounit for forward (f) and backward (b) motions in time by equation (2)

$$I^{>} = 1/R \neq {}^{<}I = 1/S \tag{2}$$

Lie-admissible branch of hadronic mechanics is ideally suited to represent the decay of unstable nuclei and also nuclear synthesis, since both are irreversible over time.

iv) Neutron synthesis: Neutron is assumed to be compressed hydrogen atom (as originally conjectured by Rutherford) as shown by reaction (i).

$$p^{+} + a + e^{-} \rightarrow n \qquad \text{(i)}$$

where 'a' is Santilli's etherino (It represents in a conventional Hilbert space transfer of 0.782 MeV and spin ½ missing in the synthesis of neutron from the environment to the neutron structure.)

Don Borghi's experiment and Santilli's hadronic mechanics appropriately explains the Rutherford's conjecture of neutron as a compressed hydrogen atom.

Thus, the CNF is governed by Santilli's laws of controlled nuclear synthesis:

- The orbitals of peripheral atomic electrons are controlled such that nuclei are systematically exposed.
- CNF occurs when nuclei spins are either in singlet planar coupling or triplet axial coupling.

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- The most probable CNF is those occurring at threshold energies and without the release of massive particles.
- CNF requires trigger, an external mechanism that forces exposed nuclei to come in femto-meter range.

The CNF has been realized using magnecules. The magnecules have systematic and controlled exposure of nuclei which have singlet planar or triplet axial coupling. In case of ICNF, proposed by Prof. Santilli energy supplied is of threshold value just sufficient to expose the atomic nuclei from within the electron cloud. Since the energy is not very high the production of ionizing radiations or sub-nuclear particles are avoided. The reaction is carried out in sealed tanks called hadronic reactors.

HyperCombustion

Hypercombustion is combination of Magnecular Combustion and controlled nuclear fusion. Initially the fossil fuels are ignited with a series of rapid DC discharges, each having at least 100 kV and at least 100 J known as HyperSparks. This convert fossil fuels from their natural molecular to a magnecular form which enhances the combustion temperature, magnecular conversion and the energy output via the fusion of Carbon-12 and Oxygen-16 into Silicon-28. This reduces emission of carbondioxide (green house gas) and also enhances energy output due to additional fusion reaction as compared to only the chemical energy released in case of molecular combustion. Since the fusion is ICNF there is release of no radioactive contaminants either, making the process green.

HyperFusion

Pseudo-protoid, an intermediate state prior to the full synthesis of the pseudo-proton is given by a bound state in singlet coupling of an electron and a neutron under the strongly attractive contact interactions of iso-mechanics, this is given by the equation-

$$n_{\uparrow} + e^{-}_{\downarrow} \rightarrow \tilde{p}$$

HyperFusions are fusions of natural, positively charged nuclei and synthesized negatively charged nuclei.

E.g. pseudo-deuteron and a deuteron into the helium

$$\widehat{D}\ (-1, 2, 1_{\uparrow}) + D\ (1, 2, 1_{\downarrow}) \rightarrow He\ (2, 4, 0) + 2e^{-}$$

The energy released by each hyperfusion is

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$\Delta E = E_{He} - (E\,\widehat{D} + E_D) = 23.8$ MeV $= 3.81 \times 10^{-12}$ J

It is estimated that 10^{18} controlled fusions per hour would yield the significant release of about 10^6 J of clean energy per hour without harmful radiations or waste.

Low Energy Fusion

It was first reported by Fleishmann, Pons and Hawkins in 1989, popularly called as cold fusion as it takes place at room temperature. The major drawback was non-reproducibility by other laboratories. This could be due to insufficient energy required to expose the atomic nuclei from within the covering atomic electron cloud. Difficulty in obtaining required triggering mechanism within the lattice of the metal crystal structure may have been encountered.

High Energy Fusion

It is reported by various laboratories, basically trying to mimic thermonuclear reactions taking place in stars. Hence popularly called as hot fusion. The reactions are not self-sustaining and compound nucleus undergoes fission leading to formation radioactive wastes. Atomic electron clouds are completely stripped off. Kinetic energy of the nuclei is increased to overcome coulombic barrier and the energy attained by the compound nucleus is generally higher than the fission barrier which results in fission reaction or nuclear decay as prominent exit channels.

The advantages of Hadronic fusions are-

1. **Aneutronicity of the reaction:** Aneutronic fusion is a nuclear fusion reaction without formation of neutrons. The majority of the energy released is released in form of charged particles. The charged particles like protons or alpha particles are easy to handle and can be directly used to convert to electricity. This reduces problem related to neutron radiation such as ionizing damage, neutron activation and requirements for biological shielding, remote handling and safety.

   E.g:

$$C(12,6,0^+,12.0000) + D(2,1,1^+,2.0141) + TR \rightarrow N(14,7,1^+,14.0030) + \text{Heat}$$

$$\Delta E = 0.0111 \text{ amu} = 0.339 \text{ MeV}$$

where TR is trigger mechanism (high voltage DC arc).

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2. **Hybridization with conventional molecular combustion:** ICNF can be amalgamated with the conventional molecular combustion by using engineering innovations 'Hyper Combustion'. These can be realized by using 'Hyper Furnaces' which is advanced version of Hadronic Reactors. This would drastically decrease the environmental impact as the combustion would be more complete. The global warming is more problematic due to hydrocarbons formed on incomplete combustion of fossil fuels. Also the amount of CO_2 generated per unit energy output would be less. Moreover magnecular combustion by itself is known to have better energy output as compared to conventional molecular combustion.

Conclusions

ICNF and HyperFusions are more promising than hot or cold fusion in terms of reproducibility and energy input to output ratio. The successful achievement of ICNF with industrial relevance depends on the proper selection of the hadronic fuel. The original and final nuclides are light, natural and stable isotope. The nuclear fusion causes no emission of ionizing radiations. The energy produced ΔE is much bigger than the total energy used by the equipment for its production. ICNF relies on magnetic properties of the precursor where as HyperFusion relies on the overcoming the coulombic barrier by the opposite electrical charge. Production of negatively charged nuclei holds promising applications in other fields such as medicines, etc. Thus, it can be concluded that hadronic nuclear energy is truly green.

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Inaugural Lecture.

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Abstract.

This opening lecture for this very important conference is intended to describe very briefly the background to the major topic to be discussed, as well as both highlighting the possibly changing scientific times in which we find ourselves and raising one or two speculative thoughts which might lead to further work in the not too distant future. There can be little doubt in the minds of most open-minded scientists that science in general and physical science in particular face several problems which are normally hidden from view. Many of these, but by no means all, are involved with the issue of uncertainty and it is this which forms the basis for the larger part of what will follow in these proceedings. The well-known Einstein-Podolsky-Rosen article will be central as will the, until now, little known resolution of the problems raised by that article for the world of science. It will be shown how recent events might indicate a possible change in the attitude towards criticism of some widely accepted results as well as towards some slightly more unconventional explanations for phenomena which have not, in reality, been afforded truly watertight explanations up to the present. As far as this latter point is concerned, the possibly provocative idea of openly reintroducing an aether into the physical description of events will be mooted.

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Ladies and Gentlemen,

May I first express my sincere thanks for the generous invitation to present this opening talk for this somewhat unusual conference – unusual because of its format caused by the coronavirus outbreak. However, although the format may be unusual, that cannot detract from the extreme importance of the contents of what is to follow these general opening remarks. There can be little doubt to all open-minded people that science in general, and physical science in particular, faces several oft hidden problems. Many of these, although by no means all, are involved with issues of uncertainty and it is with this issue that the larger part of this conference will be concerned. In a totally unrelated area of physics, changes seem to have occurred recently which might indicate some hope that more open-mindedness has entered into that area and, therefore, the possibility of a change of attitude in other spheres of endeavour might have arisen also.

In 1988, together with a colleague, I published a letter in a well-respected journal in which the validity of the Bekenstein-Hawking expression for the entropy of a black hole was questioned. The follow-up article detailing the entire argument was, however, rejected and subsequently my colleague and I encountered real difficulties in having articles accepted for publication in front line journals. However, thirty years later, shortly after Hawking's death, I was contacted by that same original journal to referee an article. I did so, as much out of curiosity than anything and found it to be a piece of work dealing, amongst other things, with the aforementioned entropy expression. Consequently, I roundly criticised the submitted article in my report – not out of any sense of pique but because I genuinely believed it to be incorrect. The article was rejected for publication. Shortly afterwards I was asked, by the same journal, to referee another paper on a totally different topic. Again I did so and my recommendations were accepted and followed through exactly. The end result was that a few months later I received an award as a Referee of the Year! Was it a coincidence that, after thirty years, but following Hawking's death, I seemed to have been accepted back into the fold? I accept that all this could be an almost unbelievable coincidence but is it just possible that it is an indicator of a change in philosophy of at least some in the hierarchy that appears to control so much in the physics community? If so, now could well be exactly the right time to push for a true open-minded examination of at least some of the major problems facing modern day science and which are the fundamental topics of this conference.

The question of uncertainty affects many areas, including my own special interest of thermodynamics, although, in that case, the affect may be felt indirect. For a moment consider the situation in thermodynamics. In traditional classical thermodynamics there are no uncertainties; all the variables, for example the internal energy and total number of particles, possess definite values. However, when systems composed of a large number of particles are to be considered, the methods of statistical mechanics have to be employed due to our present state of knowledge. As a consequence, when incorporated into thermodynamics, the realm known as statistical thermodynamics is entered. This is, in some crucial ways, totally different from classical thermodynamics because the introduction of statistical techniques has introduced uncertainty into the picture. No longer are there definite values for the internal energy or total number of particles; rather average values are considered. These average values, as with the average values of other thermodynamic variables, can fluctuate in this new regime. Hence, a degree of uncertainty is introduced which leads to the derivation of thermodynamic uncertainty relations. It is important to note, though, that these relations have been introduced via the recourse to statistical methods to describe details of the system under consideration. They have been introduced because, in a system composed of a large number of particles, it is not possible to write down all the equations of motion of the individual particles, let alone solve the resulting set of simultaneous equations. The uncertainty, therefore, has been introduced as a result of our inability to solve the exact problem; there is no inherent uncertainty in the original system. This reasoning follows for all statistical thermodynamic theories and indicates a very real difference between classical and statistical thermodynamics.

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Indeed, the same reasoning may be seen to apply to many, if not all, problems considered utilising probability theory. For example, in introducing probability, it is popular to consider the tossing of a coin. If the coin is simply tossed, the outcome when it lands – head or tails – is totally uncertain. However, this is not so if someone is in possession of all the initial conditions pertaining to the toss. If the initial speed is known, the height to which the coin rises may be found, as may the time taken to reach that height. Similarly, the time taken to fall back to a given level may be found. If the rate of rotation is also known, that, together with the total time of flight, should enable the state of the coin on reaching the desired final level to be ascertained. Hence, the uncertainty associated with this problem really arises through a lack of knowledge of the initial conditions in the problem; it is not an inherent property of the actual system.

It may be seen, therefore, that neither statistical thermodynamics nor probability may be termed complete theories in the sense that neither provides exact solutions to problems. In both, uncertainty is introduced as a result of the inability to write down and solve a set of exact equations and/or a lack of knowledge of initial conditions.

Recent rereading of some books on quantum mechanics would seem to indicate a similar situation existing in that branch of physics as well. For example, in Heisenberg's well-known book *The Physical Principles of the Quantum Theory*, the initial derivation of the uncertainty relations relies on an obvious approximation which might raise a few minor queries but the slightly later, more rigorous, derivation draws on notions from probability. Indeed the ideas of probability are closely associated with the wave function as is seen from discussions of Schrodinger's equation and its wave function. Once probability enters any discussion an element of uncertainty must follow in the subsequent theory. Hence, one must wonder if the uncertainty relations of quantum mechanics are a product of the theory rather than a natural property of the systems the theory is purporting to portray? This, of course, is highly reminiscent of the situation already mentioned as occurring in statistical thermodynamics. However, the very fact that probabilistic ideas enter the subject at all must surely indicate that the theory cannot be

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complete? Here the idea of a theory being complete is intended to indicate that the theory is capable of describing any relevant physical system exactly without any degree, however slight, of uncertainty. That may, or may not, be the precise notion put forward in the famous Einstein-Podolsky-Rosen article but that is the meaning adopted here so far and, in that sense, neither statistical thermodynamics nor quantum theory may be adjudged complete.

Some might well feel that at least some of my comments so far – if not all of them – are a little naïve, even childlike. However, I would remind everyone that two quotations from the Bible might seem appropriate at this juncture. We might be reminded of the quote from St. Paul's Epistle to the Corinthians where he says 'when I became a man, I put away childish things'. However, it seems we might take note also of Jesus's comment that one needs to become like a little child if one wishes to enter the Kingdom of Heaven It seems to me that it's just possible that scientists could learn something from these two quotes if taken together. Over the years, science has become more and more dependent on more and more advanced abstruse mathematics and maybe all scientists should stand back a little and reflect, rather than rushing blindly on using methods and results authenticated by 'conventional wisdom' but not necessarily by common sense. Maybe we should return to some childlike thinking. I would say I don't feel this mild criticism applies to the methods of Hadronic Mechanics, although there is a huge amount of new mathematics to absorb in that field but, when you become used to the new notation, that mathematics is not too difficult to comprehend; - unlike some of the modern additions to accepted conventional theory, several of which seem to be attempting to transport us to some mythical land of make-believe!

As far as the Einstein-Podolsky-Rosen, or EPR, ideas are concerned, it is worth noting that questions about the completeness of quantum mechanics as a physical theory have been discussed at length ever since that famous, some might be tempted to say infamous, paper first appeared. Many experiments were carried out in attempts to both prove and disprove the assertions contained therein and a great deal of thought went into the theoretical investigations of such as Bell. All the references to this work

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may be found in the collected papers by Bell on quantum philosophy, which may found in *Speakable and Unspeakable in Quantum Mechanics* as published by Cambridge University Press. Less well-known is the resolution of the paradox advanced by Santilli in 1998 and it is the lack of publicity for this work which poses a significant question for the scientific community. Although, when you read even just the abstract for that paper, maybe some answers become apparent. With talk of such concepts as nonlinear, nonlocal, non-canonical, axiom-preserving isotopies and spin-isospin symmetry and iso-spaces, some will be put off by the implied effort to understand properly what follows in the body of the paper, while others will dismiss the work out-of-hand because it depends crucially on concepts unfamiliar to them. This may be a totally improper attitude towards proposed new science but many will have forged impressive curricula vitae based on what they regard as well-established concepts and procedures and will be reluctant to jeopardise their personal positions. Hence, the huge question for the scientific community - when do we agree to examine with a truly open mind, radical new proposals for help in solving age-old problems? It seems there was no difficulty in examining and accepting a wide range of results from Riemannian geometry, as well as the uncertainties introduced by quantum mechanics, into physics and chemistry some one hundred years ago, so why not afford the same respect to Hadronic Mechanics or are the fundamental results of quantum mechanics to remain sacrosanct even when they don't answer all the important questions facing the scientific community?

These are vitally important questions in general but are particularly apposite when considering the so-called EPR paradox and work related to it. Basically, the EPR claims that quantum mechanics is an incomplete theory because its description of physical reality does not include all elements of reality, while every element of physical reality should be precisely represented in a complete theory. Santilli's new approach has important consequences as far as the EPR argument is concerned. Traditionally, commuting quantities are believed to be independent but, in the so-called iso-topic completion of quantum mechanics, iso-commuting quantities can

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be mutually interacting, although it should be understood that such interactions are structurally different from those of action-at-a-distance/potential type. Fundamentally, quantum mechanics may be considered an incomplete theory in that it does not contain the element of reality given by the nonlocal structure of interactions expected from the mutual wave overlapping. Hadronic mechanics overcomes this problem.

It is important to realise though that, as Santilli himself points out, hadronic mechanics is not intended to represent all elements of reality; it is not meant to be a final theory. Physics is, after all, a discipline which will never admit final theories. Hadronic mechanics simply provides one type of completion of quantum mechanics – that of axiom preserving type. It might also be noted at this point that Santilli has also shown via his new mathematics that von Neumann's theorem on hidden variables is quite simply inapplicable under isotopies – note, not violated, but inapplicable! He has also established that the oft-quoted Bell's inequality is not valid universally but holds for the conventional form of quantum mechanics specifically.

Recently, of course, the matter has resurfaced with the announcement of experimental results supporting the EPR assertions at Basel. This has provoked further contemplation of this whole issue of completeness and just what it really means. The Basel team noted that the phenomenon dated back to thought experiment of 1935 and that it allowed measurement results to be predicted precisely but, of course, it must be remembered always that thought experiments are just that – thought experiments – and such are very difficult to interpret due to the assumptions made not always being totally clear, possibly not even to the originators themselves. In fact, in a purely thought experiment, it is easy to imagine a situation where a fundamental assumption is made with no-one realising that has occurred. Remember that we all indulge in thought experiments – some even when we are asleep – but their true validity only becomes apparent when we have ceased our contemplation and committed our thoughts to paper and resulting concrete scrutiny. Supposedly, the essence of a good practical experiment is that it should be readily repeatable. It is relatively easy to see how this could be true, but could equally well be untrue, of any thought experiment. Hence, in

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my personal view, important results derived via thought experiments should always be treated with extreme care. Nevertheless, as far as the thought experiment leading to the EPR paradox is concerned, it is one which has been viewed and examined over a large number of years and, seemingly, has always led to a genuine paradox in physics.

Basically, via a thought experiment, Einstein, Podolsky and Rosen showed that precise predictions are possible theoretically in certain circumstances. Briefly, such a notion may be explained as follows:- they considered two systems in an entangled state in which their properties are strongly correlated. In this case, the results of measurements on one system may be used to predict the results of corresponding measurements on the second system with arbitrary precision in principle. It was also the case that the two systems could be separated spatially. The resulting paradox is that an observer may use measurements on the first system to make more precise statements about the second system than an observer who has direct access to that second system but not the first.

The Basel team used lasers to cool atoms to a small fraction of a degree above the absolute zero of temperature. At such low temperatures, the atoms are thought to behave completely according to the rules of quantum mechanics and form a Bose-Einstein condensate. In this ultra-cold cloud, the atoms collide with one another constantly, causing their spins to become entangled. The researchers involved then took measurements of the spin in spatially separated regions of the condensate. By using high-resolution imaging, they were able to measure the spin correlations between the separate regions directly and simultaneously localise the atoms in precisely defined positions. Hence, in this experiment, the researchers seem to have succeeded in using measurements in a given region to predict precisely the results for another region.

Experimental physics is certainly not my forte; in fact, I've not been directly involved in that area since my undergraduate days. Hence, I don't know if any serious objections to this work by the Basel team have surfaced since I read of their claims. If such have emerged, the argument over the validity of

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the EPR paradox will, no doubt, rumble on. If none has, or does, emerge than it is conceivable that a new era for physics might be opening up since it is surely the case that applications will follow which we all hope will be of benefit to mankind rather than the opposite.

As a follow-up to these comments, it might be worth raising the question of the presumed boundary between classical and quantum mechanics. Precisely when is something small enough to warrant the use of quantum mechanics to describe it? Is this boundary clear cut or does the transition evolve over what might be thought of as a blurred region in which either or both apply? I confess this seemingly simple point is one I have never seen discussed but is one that has preyed on my mind for years with no apparent resolution in the offing. It might be wondered if the reintroduction of an aether could help in the resolution of this and possibly many other difficulties encountered in modern physics. For example, the uncertainty in the position and speed of a very small particle could be accounted for by the presence of a boundary layer between the said small particle and the aether. It is certain that, if the existence of an aether is true, then such a boundary layer must exist and, if the ideas put forward by C. Kenneth Thornhill concerning an aether are valid, then the size of aether particles would be extremely small and small in comparison with the size of recognised elementary particles. Obviously this situation would not apply so obviously to macroscopic bodies because their individual size would far outweigh that of the proposed aether particles. The notion of reintroducing the idea of an aether receives some support these days with the renewed interest in some quarters in the work of Nikola Tesla. His writings, as well as those of the myriad major scientists working on problems of, or at least involving, electromagnetic ideas towards the end of the nineteenth century, contain constant references to this medium. It seems we should all be approaching problems with much more open minds and not be guided too rigidly by conventional wisdom. As the saying goes – think outside the box!

These latter points are all speculative thoughts but, nevertheless, thoughts which have materialised over years and lead to questions, at least, which I feel need carefully considered answers in order to serve the cause of the

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advancement of scientific knowledge well. At this point, at the very beginning of this conference, it might be remembered also that this event, which could prove vitally important to the future in physical science, has come about due to one person – Ruggero Santilli. Most of us know the enormous contribution he has made, a contribution far too extensive to even begin to summarise here. However, one example which has been totally ignored by those in authority deserves mention and that is his proposal of a method for disposing of nuclear waste safely and on site. This proposal needed to be checked independently by approximately three relatively small experiments. Request after request was made for these to be carried out but to no avail. I myself drew attention to this in 2008 at a conference in Monza which included in the audience the then European Commissioner dealing with energy matters. Again nothing transpired. This is just one example of Ruggero Santilli's work but one which, since it has been ignored, could prove costly to mankind. Before closing, I would like to draw your attention to one other small, but I feelsignificant, point he raised many years ago and which serves to illustrate the point I was making earlier about the assumptions we all make in physics. I drew attention to the difficulty all must really experience in thought experiments in remaining totally aware of any, and all, assumptions made at the outset. As I said also, when one comes to write down thoughts on paper, the assumptions made and their consequences, become somewhat clearer but they may never be forgotten. When Einstein proposed his special theory of relativity many years ago, he made an assumption concerning the constancy of the speed of light. Today it is commonplace both in the media and, crucially, in scientific circles as well to hear people claim that 'Einstein said the speed of light is constant'; this is almost a basic statement of modern physics to some. All have forgotten that, as Ruggero Santilli pointed out so clearly several years ago, Einstein's assumption was that the speed of light remained constant in a vacuum. Here Ruggero stressed, via a popular example, a vitally important scientific truth – when you are using, or quoting, a previously derived result in science, check diligently to see what precise assumptions have been made in deriving the said result. Many errors could be avoided so easily if this simple procedure was adhered to strictly.

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May I close by expressing the sincere hope that this proves to be an enormously successful conference and one which leads to a more open-minded approach to the solution of the important problems facing twenty-first century science. Finally, on behalf of all participating in this event, I should like to thank Ruggero Santilli, his wife Carla and all his colleagues, working unnamed and unrecognised behind the scenes, for organising it!

Thank you.

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ZUR THEORIE DER q, ω-LIESCHEN MATRIXGRUPPEN

THOMAS ERNST

ABSTRACT. Based on the three papers by Hahn 1949, Annaby et. al. 2012 and Varma et. al. 2018, we introduce the matrix of multiplicative q, ω-polynomials of order $M \in \mathbb{Z}$ with the corresponding q-addition. We prove that this constitutes a so-called q, ω-Lie group with two dual q, ω-multiplications. We also show that the corresponding q, ω-Bernoulli and q, ω-Euler matrices form q, ω-Lie subgroups. In the limit $\omega \to 0$ we obtain corresponding formulas for q-Appell polynomial matrices.

Primary 17B37; Secondary 11B68, 33D15

Keywords— q, ω-Lie group; multiplicative q-Appell polynomial matrix; Hahn–Pascal matrix

ZUSAMMENFASSUNG. Basierend auf den drei Veröffentlichungen von Hahn 1949, Annaby et. al. 2012 und Varma et. al. 2018, führen wir die multiplikative q, ω-Polynommatrix der Ordnung $M \in \mathbb{Z}$ ein, mit der entsprechenden q-Addition. Wir beweisen, dass dies eine sogenannte q, ω-Liesche Gruppe mit zwei dualen q, ω-Multiplikationen darstellt. Wir zeigen auch, dass die entsprechenden q, ω-Bernoulli und q, ω-Euler Matrizen q, ω-Liesche Untergruppen bilden. Im Grenzwert $\omega \to 0$ erhalten wir entsprechende Formeln für q-Appell-Polynommatrizen.

INHALTSVERZEICHNIS

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1. Einführung

Wir stellen einige neue Konzepte für q, ω Polynommatrizen vor, von denen einige vorher nur im q-Fall aus den Artikeln des Autors bekannt waren. Durch die logarithmische Methode für q-Analysis erfolgt dieser Übergang fast automatisch, weil die Addition durch die q, ω-Addition ersetzt wird. In dem Artikel [8] wurden Matrixgruppen mit zwei dualen Multiplikationen eingeführt. Später in [10] wurde bewiesen, dass die sogenannte q-Appell-Polynommatrix-Gruppe ein erstes konkretes Beispiel von q-Lie-Gruppen war. Obwohl wir die q, ω-Addition verwenden, werden die q-Binomialkoeffizienten beibehalten. Stattdessen wird die Potenz von x zu den zwei Hauptfolgen geändert.

In diesem Artikel werden die vorherigen Formeln mit q-Pascal-Matrizen einfach zu sogenannten q, ω-Pascal-Matrizen erweitert. Summenformeln mit der neuen q, ω-Addition können dabei in Matrixform umgeschrieben werden.

Dieser Artikel ist wie folgt organisiert: Im Abschnitt 1 werden die Hauptdefinitionen angegeben.

Der Hauptzweck des Abschnitts 2 ist die Einführung der q, ω-Addition und der multiplikativen q, ω-Appellschen Polynome und Zahlen. Das Umbral-Kalkül wird immer implizit angenommen. Im Abschnitt 3 werden die relevanten Matrizen und die Hauptmatrix q, ω-Differenzengleichung eingeführt. Im Unterabschnitt 3.1 werden die multiplikativen q, ω-Polynommatrizen zur Vorbereitung für die q, ω-Liesche Gruppe, den Hauptzweck dieses Artikels, eingeführt.

Im Unterabschnitt 3.2 wiederholen wir zunächst die Matrixformen der q, ω-Bernoulli und q, ω-Eulerschen Polynome aus [9] zur Vorbereitung für die Berechnung ihrer Zahleninverse.

Sei $\omega \in \mathbb{R},\ \omega > 0$. Man setze $\omega_0 \equiv \frac{\omega}{1-q},\ 0 < q < 1$. Sei I ein Intervall, das ω_0 enthält. Wir gehen davon aus, dass die Variable x zu I gehört.

Definition 1. Der Endomorphismus ϵ im Vektorraum der Polynome wird definiert durch

$$\epsilon f(x) \equiv f(qx + \omega). \tag{1}$$

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Dieser Endomorphismus ist eine Verallgemeinerung des Operators mit demselben Namen im q-Kalkül [5]. In [3, S. 136] ist bewiesen, dass

$$\epsilon^k f(x) = f(q^k x + \omega\{k\}_q),\ k \in \mathbb{N}. \tag{2}$$

Definition 2. [11] Sei φ eine stetige reelle Funktion von x. Wir definieren den q,ω-Differenzenoperator $\mathrm{D}_{q,\omega}$ wie folgt:

$$\mathrm{D}_{q,\omega}\left(\varphi\right)(x) \equiv \begin{cases} \frac{\varphi(qx+\omega)-\varphi(x)}{(q-1)x+\omega}, & \text{if } x \neq \omega_0; \\ \frac{d\varphi}{dx}(x) & \text{if } x = \omega_0. \end{cases} \tag{3}$$

Eine Funktion $f(x)$ ist n Mal q,ω-differenzierbar, wenn $\mathrm{D}_{q,\omega}^n f(x)$ vorhanden ist. Wenn wir darauf hinweisen möchten, dass dieser Operator auf der Variable x operiert, werden wir $\mathrm{D}_{q,\omega,x}$ für den Operator schreiben. Weiterhin, $\mathrm{D}_{q,\omega}(K) = 0$, wie für die Ableitung.

Dieser Operator interpoliert zwischen zwei bekannten Operatoren, dem Nørlundschen Differenzenoperator

$$\triangle_\omega[f(x)] \equiv \frac{f(x+\omega)-f(x)}{\omega}, \tag{4}$$

und der Jacksonschen q-Ableitung

$$\left(\mathrm{D}_q\varphi\right)(x) \equiv \begin{cases} \frac{\varphi(x)-\varphi(qx)}{(1-q)x}, & \text{if } q \in \mathbb{C}\setminus\{1\},\ x \neq 0; \\ \frac{d\varphi}{dx}(x) & \text{if } q = 1; \end{cases} \tag{5}$$

Die folgende Definition erscheint zum ersten Mal.

Definition 3. Ein q,ω-Analogon des mathematischen Objekts G ist eine mathematische Funktion $F(q,\omega)$ mit der Eigenschaft $\lim_{\omega\to 0} F(q,\omega) = G_q$, das q-Analogon von G. Sowohl F als auch G können Funktionen von mehreren Variablen sein. Sie können auch Operatoren sein. Die Funktion $F(q,\omega)$ wird ω-Analogon von G_q genannt.

Satz 1.1. [3, (16), S. 137] *Der q,ω-Differenzenoperator für ein Produkt von Funktionen.*

$$\mathrm{D}_{q,\omega}\left(fg\right)(x) = \mathrm{D}_{q,\omega}\left(f\right)(x)g(x) + f(qx+\omega)\mathrm{D}_{q,\omega}\left(g\right)(x). \tag{6}$$

Bemerkung 1. Diese Formel wird zum Nachweis von (28) verwendet.

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Wir führen nun zwei Hauptfolgen ein, die die Ciglerschen Polynome in [5, 5.5] verallgemeinern.

Definition 4.

$$[13, (15)]\ [x]_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1}(q^m x + \omega\{m\}_q). \tag{7}$$

$$[13, (16)]\ (x)_{q,\omega}^{k} \equiv \prod_{m=0}^{k-1}(x - \omega\{m\}_q), \tag{8}$$

wobei $\{m\}_q$ das q-Analogon von m bezeichnet.

Die beiden folgenden Formeln entsprechen der Formel $\mathrm{D}x^n = nx^{n-1}$:

$$[12, 2.5], [13, (17)]\ \mathrm{D}_{q,\omega}(x)_{q,\omega}^{n} = \{n\}_q (x)_{q,\omega}^{n-1}. \tag{9}$$

$$[13, (18)]\ \mathrm{D}_{q,\omega}[x]_{q,\omega}^{n} = \{n\}_q [qx+\omega]_{q,\omega}^{n-1}. \tag{10}$$

Als nächstes führen wir zwei q,ω-Analoga der Exponentialfunktion ein:

Definition 5. Die q,ω-Exponentialfunktion $\mathrm{E}_{q,\omega}(z)$ [13, (21)] wird definiert durch

$$\mathrm{E}_{q,\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{(z)_{q,\omega}^{k}}{\{k\}_q!},\ |(1-q)z-\omega| < 1. \tag{11}$$

Die komplementäre q,ω-Exponentialfunktion $\mathrm{E}_{\frac{1}{q},\omega}(z)$ [13, (26)] wird definiert durch

$$\mathrm{E}_{\frac{1}{q},\omega}(z) \equiv \sum_{k=0}^{\infty} \frac{[z]_{q,\omega}^{k}}{\{k\}_q!},\ |\omega| < 1. \tag{12}$$

Wir haben den Namen geändert zu $\mathrm{E}_{\frac{1}{q},\omega}(z)$, weil $\mathrm{E}_{\frac{1}{q},0}(z) = \mathrm{E}_{\frac{1}{q}}(z)$ [5].

Satz 1.2. [13, (19)] *Die q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

$$\mathrm{D}_{q,\omega} f(z) = f(z),\ f(0) = 1. \tag{13}$$

[13, (24)] *Die komplementäre q,ω-Exponentialfunktion ist die einzigartige Lösung der q,ω-Differenzengleichung*

$$\mathrm{D}_{q,\omega} f(z) = f(qz+\omega),\ f(0) = 1. \tag{14}$$

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Satz 1.3. [13, (21)] *Die meromorphe Fortsetzung der q, ω-Exponentialfunktion* $\mathrm{E}_{q,\omega}(z)$ *ist gegeben durch*

$$\mathrm{E}_{q,\omega}(z) = \frac{(-\omega; q)_\infty}{((1-q)z - \omega; q)_\infty}. \tag{15}$$

[13, (26)] *Die meromorphe Fortsetzung der komplementären q, ω-Exponentialfunktion* $\mathrm{E}_{\frac{1}{q},\omega}(z)$ *ist gegeben durch*

$$\mathrm{E}_{\frac{1}{q},\omega}(z) = \frac{((q-1)z + \omega; q)_\infty}{(\omega; q)_\infty}. \tag{16}$$

Korollarium 1.4.

$$\mathrm{E}_{q,\omega}(z)\mathrm{E}_{\frac{1}{q},-\omega}(-z) = 1 \tag{17}$$

2. Die q, ω-Liesche Gruppe von q, ω-Appellschen Polynommatrizen

Wir erweitern zunächst einige Definitionen von [8].

Definition 6. Eine q, ω-Liesche Gruppe $(G_{n,q,\omega,\cdot,\cdot_{q,\omega}}, \mathrm{I}_g) \supseteq \mathrm{E}_{q,\omega}(\mathfrak{g}_q)$ ist eine möglicherweise unendliche Menge von Matrizen $\in \mathrm{GL}(n, \mathbb{R})$ und eine Mannigfaltigkeit mit zwei Multiplikationen: $\cdot$, der üblichen Matrixmultiplikation und der verdrehten Matrixmultiplikation $\cdot_{q,\omega}$, die separat definiert wird.

Jede q, ω-Liesche Gruppe hat eine Einheit, die für beide Multiplikationen mit I_g bezeichnet wird. Jedes Element $\Phi \in G_{n,q,\omega}$ hat eine Inverse Φ^{-1} mit der Eigenschaft $\Phi \cdot_{q,\omega} \Phi^{-1} = \mathrm{I}_g$.

Definition 7. Angenommen, $(G_1, \cdot_1, \cdot_{1:q,\omega})$ und $(G_2, \cdot_2, \cdot_{2:q,\omega})$ sind zwei q, ω-Liesche Gruppen, dann ist $(G_1 \times G_2, \cdot, \cdot_{q,\omega})$ eine q, ω-Liesche Gruppe mit dem Namen Produkt-q, ω-Liesche Gruppe. Diese Gruppe hat Gruppenoperationen definiert durch

$$(g_{11}, g_{21}) \cdot (g_{12}, g_{22}) = (g_{11} \cdot_1 g_{12}, g_{21} \cdot_2 g_{22}), \tag{18}$$

und

$$(g_{11}, g_{21}) \cdot_{q,\omega} (g_{12}, g_{22}) = (g_{11} \cdot_{1:q,\omega} g_{12}, g_{21} \cdot_{2:q,\omega} g_{22}). \tag{19}$$

Definition 8. Wenn $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q, ω-Liesche Gruppe ist und $H_{n,q,\omega}$ eine nichtleere Teilmenge von $G_{n,q,\omega}$ ist, dann wird $(H_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ eine q, ω-Liesche Untergruppe von $(G_{n,q,\omega}, \cdot, \cdot_{q,\omega})$ genannt, falls

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(1)

$$\Phi \cdot \Psi \in H_{n,q,\omega} \text{ und } \Phi \cdot_{q,\omega} \Psi \in H_{n,q,\omega} \text{ for all } \Phi, \Psi \in H_{n,q,\omega}. \tag{20}$$

(2)

$$\Phi^{-1} \in H_{n,q,\omega} \text{ for all } \Phi \in H_{n,q,\omega}. \tag{21}$$

(3) $H_{n,q,\omega}$ eine Untermannigfaltigkeit von $G_{n,q,\omega}$ ist.

Um die folgenden Polynome verwenden zu können, müssen wir die q-Addition verallgemeinern. Die gewöhnliche q-Addition ist der Sonderfall $\omega = 0$. Genau wie bei der q-Addition verwenden wir Buchstaben in einem Alphabet für die q, ω-Additionen. Die Gleichheit der Buchstaben wird mit $\sim$ bezeichnet. Man beachte im Folgenden die Tatsache, dass jeweils die Variable x in $(x)_{q,\omega}^{\nu}$ oder in $[x]_{q,\omega}^{\nu}$ durch die Konstante a multipliziert wird, müssen wir auch ω mit a multiplizieren.

Definition 9. Die NWA q, ω-Addition wird wie folgt definiert:

$$(x \oplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (y)_{q,\omega}^{k}. \tag{22}$$

Die NWA q, ω-Subtraktion wird wie folgt definiert:

$$(x \ominus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} (-y)_{q,-\omega}^{k}. \tag{23}$$

Die JHC q, ω-Addition wird wie folgt definiert:

$$(x \boxplus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [y]_{q,\omega}^{k}. \tag{24}$$

Die JHC q, ω-Subtraktion wird wie folgt definiert:

$$(x \boxminus_{q,\omega} y)^n \equiv \sum_{k=0}^{n} \binom{n}{k}_q (x)_{q,\omega}^{n-k} [-y]_{q,-\omega}^{k}. \tag{25}$$

Korollarium 2.1. *Eine Erweiterung der Formel* [5, 4.29]

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \{n\}_q (x \oplus_{q,\omega} y)^{n-1}, \; \oplus_{q,\omega} \equiv \oplus_{q,\omega} \vee \boxplus_{q,\omega}. \tag{26}$$

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Beweis.

$$\mathrm{D}_{q,\omega,x}(x \oplus_{q,\omega} y)^n = \sum_{k=0}^{n-1} \binom{n}{k}_q \{n-k\}_q (x)_{q,\omega}^{n-k-1} (y)_{q,\omega}^k = \mathrm{RS}. \tag{27}$$

□

Satz 2.2. *Die Kettenregel für den q,ω-Differenzenoperator.*

$$\mathrm{D}_{q,\omega}\left((ax)_{q,a\omega}^n\right) = a\{n\}_q (ax)_{q,a\omega}^{n-1}. \tag{28}$$

$$\mathrm{D}_{q,\omega}\left([ax]_{q,a\omega}^n\right) = a\{n\}_q [aqx + a\omega]_{q,a\omega}^{n-1}. \tag{29}$$

Beweis. Wir beweisen (28) durch Induktion. Die Formel (28) gilt für $n = 1, 2$. Angenommen, sie gilt für $n-1$. Dann haben wir

$$\begin{aligned} &\mathrm{D}_{q,\omega}\left[(ax)_{q,a\omega}^{n-1}(ax - \{n-1\}_q a\omega)\right] \\ &\overset{\text{durch}(6)}{=} a(ax)_{q,a\omega}^{n-1} + a^2\left[qx + \omega - \{n-1\}_q\right]\{n-1\}_q (ax)_{q,a\omega}^{n-2} \\ &= a(ax)_{q,a\omega}^{n-1}\left[1 + q\{n-1\}_q\right] = \mathrm{RS}. \end{aligned} \tag{30}$$

Die Formel (29) wird in ähnlicher Weise bewiesen. □

Korollarium 2.3. *Vier q,ω-Additionen für die q,ω-Exponentialfunktion.*

$$\mathrm{E}_{q,\omega}(x \oplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,\omega}(y). \tag{31}$$

$$\mathrm{E}_{q,\omega}(x \ominus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{q,-\omega}(-y). \tag{32}$$

$$\mathrm{E}_{q,\omega}(x \boxplus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},\omega}(y). \tag{33}$$

$$\mathrm{E}_{q,\omega}(x \boxminus_{q,\omega} y) \equiv \mathrm{E}_{q,\omega}(x)\mathrm{E}_{\frac{1}{q},-\omega}(-y). \tag{34}$$

Beweis. Man verwende die Formeln (22) und (24). □

Satz 2.4. *Die q,ω-Differenzen für die q,ω-Exponentialfunktionen sind:*

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{q,a\omega}(ax) = a\ \mathrm{E}_{q,a\omega}(ax), \tag{35}$$

$$\mathrm{D}_{q,\omega}\ \mathrm{E}_{\frac{1}{q},a\omega}(ax) = a\ \mathrm{E}_{\frac{1}{q},a\omega}(aqx + a\omega), \tag{36}$$

Beweis. Dies ergibt sich aus der Kettenregel (28) und (29). □

Satz 2.5. *Die NWA q,ω-Addition ist kommutativ und assoziativ.*

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Beweis. Ähnlich dem Nachweis für NWA q-Addition. □

Satz 2.6. *Die JHC q,ω-Addition ist assoziativ. Wir nehmen an, dass alle JHC q,ω-Additionen ganz rechts im Funktionsargument stehen.*

Beweis. Dies ergibt sich aus der Assoziativität der Multiplikation. □

Definition 10. Die Ward-ω-Zahl $\overline{n}_{q,\omega}$ wird definiert durch

$$\overline{n}_{q,\omega} \sim 1 \oplus_{q,\omega} 1 \oplus_{q,\omega} \ldots \oplus_{q,\omega} 1, \tag{37}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Definition 11. Die Jacksonsche ω-Zahl $\tilde{n}_{q,\omega}$ wird definiert durch

$$\tilde{n}_{q,\omega} \sim 1 \boxplus_{q,\omega} 1 \boxplus_{q,\omega} \ldots \boxplus_{q,\omega} 1, \tag{38}$$

wobei die Anzahl der Summanden auf der rechten Seite n ist.

Die allgemeinste Form von Polynom in diesem Artikel ist das q,ω-Appell-Polynom, das wir nun definieren werden.

Definition 12. Sei $\mathcal{A}_{q,\omega}$ die reelle Zahlenfolgen $\{u_{\nu,q}\}_{\nu=0}^{\infty}$, so dass

$$\sum_{\nu=0}^{\infty} |u_{\nu,q}| \frac{r^{\nu}}{\{\nu\}_q!} < \infty, \tag{39}$$

für einen q,ω-abhängiger Konvergenzradius $r = r(q) > 0$, wobei $0 < q < 1$.

Die q,ω-Appellsche Zahlenfolge wird mit $\{\Phi^{(n)}_{\nu,q,\omega}\}_{\nu=0}^{\infty}$ bezeichnet.

Definition 13. Sei $h(t,q,\omega),\ h(t,q,\omega)^{-1} \in \mathbb{R}[[t]]$. Für $f_n(t,q,\omega) = h(t,q,\omega)^n$ werden die multiplikativen q,ω-Appellschen Zahlen von Grad ν und Ordnung n, $\Phi_{\nu,q,\omega} \in \mathcal{A}_{q,\omega}$ durch die folgende erzeugende Funktion gegeben:

$$f_n(t,q,\omega) = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!} \Phi^{(n)}_{\nu,q,\omega},\ \ \Phi^{(n)}_{0,q,\omega} = 1. \tag{40}$$

Der Bequemlichkeit halber fixieren wir den Wert für $n = 0$ und $n = 1$:

$$\Phi^{(0)}_{\nu,q,\omega} \equiv \delta_{0,\nu},\ \ \Phi^{(1)}_{\nu,q,\omega} \equiv \Phi_{\nu,q,\omega}. \tag{41}$$

Definition 14. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die multiplikative q,ω-Appellsche Polynomfolge $\{\Phi^{(n)}_{\nu;q,\omega}(x)\}_{\nu=0}^{\infty}$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion gegeben:

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$$f_n(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu;q,\omega}(x). \tag{42}$$

Definition 15. Der Bequemlichkeit halber fixieren wir wieder den Wert für $n = 0$ und $n = 1$:

$$\Phi^{(0)}_{\nu,q,\omega}(x) = (x)^\nu_{q,\omega},\ \Phi^{(1)}_{\nu,q,\omega}(x) \equiv \Phi_{\nu,q,\omega}(x). \tag{43}$$

Motivation: Die erste Definition folgt, weil die zwei Hauptfolgen die Potenzfunktion ersetzen.

Als nächstes werden Verallgemeinerungen der beiden Formeln [5, 4.107, 4.111] vorgestellt.

Satz 2.7.

$$\mathrm{D}_{q,\omega}\Phi_{\nu;q,\omega}(x) = \{\nu\}_q \Phi_{\nu-1;q,\omega}(x). \tag{44}$$

Die Formel [13, (30)]*in Umbralform:*

$$\Phi_{\nu;q,\omega}(x) \doteq (\Phi_{q,\omega} \oplus_{q,\omega} x)^\nu. \tag{45}$$

Definition 16. Man vergleiche mit [13, (31)]. Für $f_n(t,q,\omega) \in \mathbb{R}[[t]]$ wie oben wird die komplementäre q,ω-Appellsche Polynomfolge $\{\Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x)\}_{\nu=0}^{\infty}$ von Grad ν und Ordnung n durch die folgende erzeugende Funktion definiert:

$$f_n(t,q,\omega)\mathrm{E}_{\frac{1}{q},\omega t}(xt) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \Phi^{(n)}_{\nu;\frac{1}{q},\omega}(x). \tag{46}$$

Bemerkung 2. Diese Definition wird in der Formel (49) verwendet.

Satz 2.8. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir ein ω-Analogon von* [10, (43)]*:*

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \tag{47}$$

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Beweis. Wir zeigen, dass beide Seiten von (47) dieselbe erzeugende Funktion haben.

$$\begin{aligned} &f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x \oplus_{q,\omega} y)t) \overset{\text{durch}(22)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) \\ &f_K(t,q,\omega)\mathrm{E}_{q,\omega t}(yt) \overset{\text{durch}(42)}{=} \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x) \sum_{l=0}^{\infty} \frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;q,\omega}(y) \\ &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,q,\omega}(y). \end{aligned} \tag{48}$$

□

Bemerkung 3. Die Formel (47) definiert $\Phi^{(M+K)}_{\nu,q,\omega}(x \oplus_{q,\omega} y)$ als rechte Seite der Formel. Es gibt keine andere Definition von dieser Funktion.

Satz 2.9. *Angenommen, M und K sind die x-Ordnung bzw. y-Ordnung. Dann haben wir:*

$$\Phi^{(M+K)}_{\nu,q,\omega}(x \boxplus_{q,\omega} y) = \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y). \tag{49}$$

Beweis. Wir zeigen, dass beide Seiten von (49) dieselbe erzeugende Funktion haben.

(50)

$$\begin{aligned} &f_{M+K}(t,q,\omega)\mathrm{E}_{q,\omega t}((x \boxplus_{q,\omega} y)t) \overset{\text{durch}(24)}{=} f_M(t,q,\omega)\mathrm{E}_{q,\omega t}(xt) f_K(t,\tfrac{1}{q},-\omega) \\ &\mathrm{E}_{\frac{1}{q},\omega t}(yt) \overset{\text{durch}(42),(46)}{=} \sum_{k=0}^{\infty} \frac{t^k}{\{k\}_q!}\Phi^{(M)}_{k;q,\omega}(x) \sum_{l=0}^{\infty} \frac{t^l}{\{l\}_q!}\Phi^{(K)}_{l;\frac{1}{q},\omega}(y) \\ &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\{\nu\}_q!} \sum_{k=0}^{\nu} \binom{\nu}{k}_q \Phi^{(M)}_{k,q,\omega}(x)\Phi^{(K)}_{\nu-k,\frac{1}{q},\omega}(y). \end{aligned}$$

□

3. Der Matrixansatz

In diesem Abschnitt verallgemeinern wir Resultate aus [9] und [10] durch die Einführung der Variable ω.

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Definition 17. [6], [9] Ein q-Analogon der Polya-Veinschen Matrix. Die $n \times n$ Matrix $\mathbf{D}_{n,q}$ ist gegeben durch

$$\begin{aligned} &\mathbf{D}_{n,q}(i, i-1) \equiv \{i\}_q, \; i = 1, \ldots, n-1, \\ &\mathbf{D}_{n,q}(i, j) \equiv 0, \; j \neq i-1. \end{aligned} \tag{51}$$

Die folgende Vektorform für q, ω-Appellsche Polynome und Zahlen wird in den Formeln (69), (89), (90), (108) und (109) verwendet.

$$\phi_{n,q,\omega}(x) \equiv (\Phi_{0,q,\omega}(x), \Phi_{1,q,\omega}(x), \ldots, \Phi_{n-1,q,\omega}(x))^T, \tag{52}$$

$$\phi_{n,q,\omega} \equiv \phi_{n,q,\omega}(0). \tag{53}$$

Definition 18. Die folgende Abkürzung wird verwendet.

$$\xi_{n,q,\omega}(x) \equiv ((x)_{q,\omega}^0, (x)_{q,\omega}^1, (x)_{q,\omega}^2, \ldots, (x)_{q,\omega}^{n-1})^T. \tag{54}$$

Definition 19. Man definiere die q, ω-Appellsche Polynommatrix durch

$$\overline{\Phi}_{n,q,\omega}(x)(i, j) \equiv \binom{i}{j}_q \Phi_{i-j,q,\omega}(x), \; 0 \leq i, j \leq n-1. \tag{55}$$

Definition 20. Die q, ω-Appellsche Zahlenmmatrix wird definiert durch

$$\overline{\Phi}_{n,q,\omega}(i, j) \equiv \overline{\Phi}_{n,q,\omega}(0)(i, j), \; 0 \leq i, j \leq n-1. \tag{56}$$

Satz 3.1. *Die Formel (44) kann in Matrixform geschrieben werden. Vergleiche mit* [6, (83)].

$$\mathrm{D}_{q,\omega}\phi_{n,q,\omega}(x) = \mathbf{D}_{n,q}\phi_{n,q,\omega}(x). \tag{57}$$

Angenommen, $y(t)$ ist ein Vektor der Länge n, ist die folgende q, ω-Differenzengleichung in $\mathbb{R}^n$ von grundlegender Bedeutung:

$$\mathrm{D}_{q,\omega}y(t) = \mathbf{D}_{n,q}y(t), y(0) = y_0, \; -\infty < t < \infty. \tag{58}$$

Gemäß der Formel (57), ist die allgemeine Lösung von (58) der q, ω-Appell-Polynomvektor von Grad ν und Ordnung m. Die Anfangswerte sind dann der Vektor von q, ω-Appellschen Zahlen der Ordnung m u.s.w.. Der Anfangswert kann auch die Vektorfunktion e_0 sein. Die Lösung ist dann die Vektorfunktion $\xi_{n,q,\omega}(x)$.

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Definition 21. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}(x)$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j)(x) \equiv \begin{cases} \binom{i}{j}_q (x)_{q,\omega}^{i-j}, i \geq j \\ 0, \text{ sonst} \end{cases} \tag{59}$$

Diese Matrix wird in der Formel (71) verwendet.

Satz 3.2. *Die allgemeine Lösung von (58) kann auch geschrieben werden:* $y(t) = \mathrm{E}_{q,\omega}(\mathbf{D}_{n,q}t)y_0$. *Dies ist eigentlich eine endliche Reihe, die sich in der folgenden Form ausdrücken lässt:*

$$\sum_{k=0}^{n-1} \frac{(\mathbf{D}_{n,q}t)_{q,\omega}^k}{\{k\}_q!} \equiv \mathrm{P}_{n,q,\omega}(t). \tag{60}$$

Der folgende Sonderfall wird häufig verwendet.

Definition 22. Die q,ω-Pascalsche Matrix $\mathrm{P}_{n,q,\omega}$ ist gegeben durch

$$\mathrm{P}_{n,q,\omega}(i,j) \equiv \mathrm{P}_{n,q}(i,j)(1) = \binom{i}{j}_q (1)_{q,\omega}^{i-j}, i,j = 0,\ldots,n-1. \tag{61}$$

Des Weiteren haben wir das folgende q,ω-Analogon von [1, S. 233 (7)], was daraus folgt, dass $\mathrm{P}_{n,q,\omega}(t)$ eine q,ω-Exponentialfunktion ist.

$$\mathrm{P}_{n,q,\omega}(s \oplus_{q,\omega} t) = \mathrm{P}_{n,q,\omega}(s)\mathrm{P}_{n,q,\omega}(t), \;\; s,t \in \mathbb{C}. \tag{62}$$

Das impliziert

$$\mathrm{P}_{n,q,\omega}^k = \mathrm{P}_{n,q,\omega}(\overline{k}_{q,\omega}). \tag{63}$$

Durch (62) erhalten wir viele kombinatorische Identitäten. Einige davon sind

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k} [-1]_{q,\omega}^{k-j} = \delta_{i,j} \tag{64}$$

und

$$\sum_{k=j}^{i} \binom{i}{k}_q \binom{k}{j}_q (1)_{q,\omega}^{i-k} (1)_{q,\omega}^{k-j} = (\overline{2_{q,\omega}})^{i-j} \binom{i}{j}_q, \;\; i \geq j. \tag{65}$$

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3.1. **Multiplikative q,ω-Appellsche Polynommatrizen.**

Definition 23. Ein ω-Analogon von [10, (47)]. Die multiplikativen q,ω-Appell-Polynommatrizen $(\mathcal{M}_{x,q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ der Ordnung $M \in \mathbb{Z}$ sind definiert durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \Phi^{(M)}_{i-j,q,\omega}(x),\ 0 \le i,j \le n-1. \tag{66}$$

Definition 24. Die multiplikativen q,ω-Appell Zahlenmatrizen oder die q,ω-Übertragung-Matrizen $(\mathcal{M}_{q,\omega})$ mit Elementen $\overline{\Phi}^{(M)}_{n,q,\omega}$ der Order $M \in \mathbb{Z}$ sind definiert durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(i,j) \equiv \overline{\Phi}^{(M)}_{n,q,\omega}(0)(i,j),\ 0 \le i,j \le n-1. \tag{67}$$

Satz 3.3. *Eine Formel für die q,ω-Übertragung-Matrix*

$$\overline{\Phi}_{n,q,\omega} = f_n(t,q,\omega)\mathbf{D}_{n,q}, \tag{68}$$

wobei $f_n(t,q,\omega)$ durch (42) definiert wird.

Beweis. Dies ist ähnlich wie bei der Formel [9, (79)]. □

Satz 3.4. *Der q,ω-Appell-Polynomvektor von x kann als Produkt der q,ω-Appellzahlen-Matrix mal $\xi_{n,q,\omega}(x)$ ausgedrückt werden. Ein q,ω-Analogon von* [2, (3.9), S. 432] *und ein ω-Analogon von* [10, (49)].

$$\phi_{n,q,\omega}(x) = \overline{\Phi}_{n,q,\omega}\xi_{n,q,\omega}(x). \tag{69}$$

Beweis. Dies ist die Formel (45) in Matrixform.

□

Satz 3.5. *Der q,ω-Appell-Polynomvektor von $x \oplus_{q,\omega} y$ kann als Produkt der q,ω-Appellschen Matrix von x mal der q,ω-Appellschen Vektor von y ausgedrückt werden. Ein q,ω-Analogon von* [2, (4.1), S. 436].

$$\phi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}_{n,q,\omega}(x)\phi_{n,q,\omega}(y). \tag{70}$$

Beweis. Dies ist die Formel (47) in Matrixform. □

Satz 3.6. *Ein q,ω-Analogon von* [2, S. 436].

$$\xi_{n,q,\omega}(x \oplus_q y) = \mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y). \tag{71}$$

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Beweis. Wir haben durch [12, 2.3]

$$\begin{aligned}\xi_{n,q,\omega}(x \oplus_q y)(i) &= (x \oplus_q y)^i_{q,\omega} \\ &= \sum_{k=0}^{i} \binom{i}{k}_q (x)^{i-k}_{q,\omega} (y)^k_{q,\omega} \overset{\text{durch}(59)}{=} (\mathrm{P}_{n,q,\omega}(x)\xi_{n,q,\omega}(y))(i).\end{aligned} \tag{72}$$

□

Satz 3.7. *Ein q,ω-Analogon von* [2, S. 436].

$$\overline{\Phi}^{(M+K)}_{n,q,\omega}\xi_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(K)}_{n,q,\omega}\overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y). \tag{73}$$

Beweis. Die beiden Matrizen $\overline{\Phi}^{(M)}_{n,q,\omega}(x)$ und $\overline{\Phi}^{(K)}_{n,q,\omega}$ sind Potenzreihen in $\mathbf{D}_{n,q}$ und wir haben

$$\begin{aligned}&\overline{\Phi}^{(M)}_{n,q,\omega}(x)\phi^{(K)}_{n,q,\omega}(y) \\ &\overset{\text{durch}(69)}{=} \overline{\Phi}^{(M)}_{n,q,\omega}(x)\overline{\Phi}^{(K)}_{n,q,\omega}\xi_{n,q,\omega}(y) = \overline{\Phi}^{(K)}_{n,q,\omega}\overline{\Phi}^{(M)}_{n,q,\omega}(x)\xi_{n,q,\omega}(y).\end{aligned} \tag{74}$$

Andererseits gemäß der Formel (47) ist dies gleich

$$\phi^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) \overset{\text{durch}(69)}{=} \overline{\Phi}^{(M+K)}_{n,q,\omega}\xi_{n,q,\omega}(x \oplus_{q,\omega} y). \tag{75}$$

Die Formel (73) folgt, indem die letzten Ausdrücke von (74) und (75) gleichgesetzt werden. □

Bemerkung 4. Die Formel (73) ergibt eine implizite Definition der Funktion $\xi_{n,q,\omega}(x \oplus_{q,\omega} y)$.

Satz 3.8. *Ein ω-Analogon von* [10, (55)]. *Wir gehen davon aus, dass M und K die x-Ordnung bzw. y-Ordnung sind. Die Formel (47) kann in der folgenden Matrixform umgeschrieben werden, wobei $\cdot$ auf der rechten Seite eine Matrixmultiplikation bezeichnet.*

$$\overline{\Phi}^{(M+K)}_{n,q,\omega}(x \oplus_{q,\omega} y) = \overline{\Phi}^{(M)}_{n,q,\omega}(x) \cdot \overline{\Phi}^{(K)}_{n,q,\omega}(y). \tag{76}$$

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Beweis. Wir berechnen das Matrixelement (i,j) der Matrixmultiplikation auf der rechten Seite.

$$\begin{aligned}
&\sum_{k=j}^{i}\binom{i}{k}_q \Phi^{(M)}_{i-k,q,\omega}(x)\binom{k}{j}_q \Phi^{(K)}_{k-j,q,\omega}(y)\\
&=\binom{i}{j}_q \sum_{k=j}^{i}\binom{i-j}{k-j}_q \Phi^{(M)}_{i-k,q,\omega}(x)\Phi^{(K)}_{k-j,q,\omega}(y)\\
&=\binom{i}{j}_q \sum_{k=0}^{i-j}\binom{i-j}{k}_q \Phi^{(M)}_{i-j-k,q,\omega}(x)\Phi^{(K)}_{k,q,\omega}(y)\\
&\overset{\text{durch}(47)}{=}\binom{i}{j}_q \Phi^{(M+K)}_{i-j,q,\omega}(x\oplus_{q,\omega} y)=\text{LS}.
\end{aligned} \tag{77}$$

□

Durch die Formel (66) sind die $\overline{\Phi}^{(M)}_{n,q}(x)$-Matrizen mit Matrixelementen q,ω-Appellsche Polynome multipliziert mit q-Binomial-Koeffizienten, und wir gelangen zur nächsten wichtigen Definition.

Definition 25. Ein ω-Analogon von [10, (57)]. Wir definieren die zweite Matrixmultiplikation $\cdot_{q,\omega}$ durch

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot_{q,\omega}\overline{\Phi}^{(K)}_{n,q,\omega}(y)\equiv\overline{\Phi}^{(M+K)}_{n,q,\omega}(x\boxplus_{q,\omega} y), \tag{78}$$

wobei $\overline{\Phi}^{(M+K)}_{n,q,\omega}(x\boxplus_q y)$ durch die Formel (49) definiert wird.

Satz 3.9. *Die Menge* $(\mathcal{M}_{x,q,\omega},\cdot,\cdot_{q,\omega},\mathrm{I}_n)$ *mit Multiplikationen gegeben durch (76) und (78), und Inverse* $\overline{\Phi}^{(-M)}_{n,q,\omega}(-x)$ *ist eine q,ω-Liesche Gruppe. Das Einheitselement ist die Einheitsmatrix* I_n *und das assoziative Gesetz gilt wie für Gruppen.*

Wir geben eine vereinfachte Version des entsprechenden Nachweises.

Beweis. $\mathcal{M}_{x,q,\omega}$ ist geschlossen unter den beiden Operationen durch (76) und (78). Durch (78) haben wir

$$\overline{\Phi}^{(M)}_{n,q,\omega}(x)\cdot_q\overline{\Phi}^{(-M)}_{n,q,\omega}(-x)=\overline{\Phi}^{(0)}_{n,q,\omega}(\theta)=\mathrm{I}_n, \tag{79}$$

das die Existenz eines inversen Elements und einer Einheit beweist.

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Zur Vereinfachung der Notation wird der letzte Teil in einem Spezialfall angegeben, das leicht zu verallgemeinern ist. Das assoziative Gesetz lautet:

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x)\cdot\overline{\Phi}_{n,q,\omega}^{(K)}(y)\right)\cdot_q\overline{\Phi}_{n,q,\omega}^{(J)}(z)=\overline{\Phi}_{n,q,\omega}^{(M)}(x)\cdot\left(\overline{\Phi}_{n,q,\omega}^{(K)}(y)\cdot_q\overline{\Phi}_{n,q,\omega}^{(J)}(z)\right), \tag{80}$$

das äquivalent zu

$$\overline{\Phi}_{n,q,\omega}^{(M+K+J)}((x\oplus_{q,\omega}y)\boxplus_{q,\omega}z)=\overline{\Phi}_{n,q,\omega}^{(M+K+J)}(x\oplus_{q,\omega}(y\boxplus_{q,\omega}z)) \tag{81}$$

ist. Die Formel (81) folgt jedoch aus der Assoziativität der beiden q,ω-Additionen. □

Sei

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x)\right)^k\equiv\overline{\Phi}_{n,q,\omega}^{(M)}(x)\cdot\overline{\Phi}_{n,q,\omega}^{(M)}(x)\cdot\cdots\cdot\overline{\Phi}_{n,q,\omega}^{(M)}(x). \tag{82}$$

Dabei steht auf der rechten Seite das Produkt von k gleichen Matrizen $\overline{\Phi}_{n,q}^{(M)}(x)$.

Die Formel (63) kann zu

$$\left(\overline{\Phi}_{n,q,\omega}^{(M)}(x)\right)^k=\overline{\Phi}_{n,q,\omega}^{(kM)}(\overline{k}_{q,\omega}x) \tag{83}$$

verallgemeinert werden.

3.2. q,ω-Bernoulli und q,ω-Eulersche Polynome.

Wir betrachten auch die besonderen Fälle q,ω-Bernoulli- und q,ω-Eulersche Polynome.

Definition 26. Es gibt zwei q,ω-Bernoulli-Polynome $\mathrm{B}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q,ω-Bernoulli-Polynome, und $\mathrm{B}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q,ω-Bernoulli-Polynome. Sie sind definiert durch die beiden erzeugenden Funktionen

$$\frac{t}{(\mathrm{E}_{q,\omega}(t)-1)}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{B}_{\mathrm{NWA},\nu,q}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{84}$$

und

$$\frac{t}{(\mathrm{E}_{\frac{1}{q},\omega}(t)-1)}\mathrm{E}_{q,\omega t}(xt)=\sum_{\nu=0}^{\infty}\frac{t^\nu\mathrm{B}_{\mathrm{JHC},\nu,q}(x)}{\{\nu\}_q!},\ |t|<2\pi. \tag{85}$$

Definition 27. Die Ward q,ω-Bernoullischen Zahlen sind gegeben durch

$$\mathrm{B}_{\mathrm{NWA},n,q,\omega}\equiv\mathrm{B}_{\mathrm{NWA},n,q,\omega}(0). \tag{86}$$

Die Jackson q,ω-Bernoullischen Zahlen sind gegeben durch

$$\mathrm{B}_{\mathrm{JHC},n,q,\omega} \equiv \mathrm{B}_{\mathrm{JHC},n,q,\omega}(0). \tag{87}$$

Um Platz zu sparen, verwenden wir die folgende Abkürzung in den Gleichungen (89) - (93), (96), (97), (100), (104), (105), (108)-(112), (115)-(116), (119)-(120). Für den JHC-Fall ändern wir gegebenenfalls $\oplus_{q,\omega}$ zu $\boxplus_{q,\omega}$.

$$\mathrm{NWA} = \mathrm{NWA} \vee \mathrm{JHC}. \tag{88}$$

Wir werden die folgenden Vektorformen für die q,ω-Bernoulli-Polynome verwenden, die q,ω-Analoga von [1, S. 239] entsprechen.

$$b_{\mathrm{NWA},n,q,\omega}(x) \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}(x), \mathrm{B}_{\mathrm{NWA},1,q,\omega}(x), \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega}(x))^T. \tag{89}$$

Die entsprechenden Vektorformen für Zahlen sind

$$b_{\mathrm{NWA},n,q,\omega} \equiv (\mathrm{B}_{\mathrm{NWA},0,q,\omega}, \mathrm{B}_{\mathrm{NWA},1,q,\omega}, \ldots, \mathrm{B}_{\mathrm{NWA},n-1,q,\omega})^T. \tag{90}$$

Wir stellen die NWA und JHC verschobenen, q,ω-Bernoulli-Matrizen vor.

Definition 28. Ein ω-Analogon von [9, (54)].

$$\begin{aligned} &\mathcal{B}_{\mathrm{NWA},n,q,\omega}(x) \\ \equiv&(b_{\mathrm{NWA},q,\omega}(x) \;\; \mathrm{E}(\oplus_{q,\omega})b_{\mathrm{NWA},q,\omega}(x) \;\; \cdots \;\; \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}}b_{\mathrm{NWA},q,\omega}(x)), \end{aligned} \tag{91}$$

wobei $\mathrm{E}(\oplus_{q,\omega})^{\overline{k}_{q,\omega}}((x)^n_{q,\omega}) \equiv (x \oplus_{q,\omega} \overline{k}_{q,\omega})^n,\; 0 \leq k \leq n-1$.

Wir benötigen zwei ähnliche Matrizen basierend auf den $\mathrm{B}_{\mathrm{NWA}}$ und $\mathrm{B}_{\mathrm{JHC}}$-Polynomen und Zahlen.

Definition 29. Zwei ω-Analoga von [9, (58),(67)] und zwei q,ω-Analoga von [4, S. 193]. Die NWA und JHC q,ω-Bernoulli-Matrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}(x), 0 \leq i,j \leq n-1. \tag{92}$$

Definition 30. Ein ω-Analogon von [10, 84]. Die NWA und JHC q,ω-Bernoulli-Zahlenmatrizen sind definiert durch

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{B}_{\mathrm{NWA},i-j,q,\omega}, 0 \leq i,j \leq n-1. \tag{93}$$

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Definition 31. Ein ω-Analogon von [10, 85]. Die Matrix $\mathcal{D}_{\mathrm{NWA},n,q,\omega}$ hat Matrixelemente

$$d_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q}\binom{i}{j}_q (1)_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{94}$$

Definition 32. Ein ω-Analogon von [10, 86]. Die Matrix $\mathcal{D}_{\mathrm{JHC},n,q,\omega}$ hat Matrixelemente

$$d_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{1}{\{i-j+1\}_q}\binom{i}{j}_q [1]_{q,\omega}^{i-j+1} \text{ if } i \geq j, \\ 0 \text{ sonst.} \end{cases} \tag{95}$$

Satz 3.10. *Ein ω-Analogon von* [10, 87]. *Die Inversen der q,ω-Bernoulli-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}, \ \left(\overline{\mathrm{B}}_{\mathrm{JHC},n,q,\omega}\right)^{-1} = \mathcal{D}_{\mathrm{JHC},n,q,\omega}. \tag{96}$$

Dies impliziert, dass

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}^{-k} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}^{k}. \tag{97}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\mathcal{D}_{\mathrm{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n} \frac{1}{\{k+1\}_q}\binom{n}{k}_q \mathrm{B}_{\mathrm{NWA},n-k,q,\omega}(1)_{q,\omega}^{k+1} = \delta_{n,0}. \tag{98}$$

Dann haben wir

$$\begin{aligned} &\sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q}\binom{i}{k}_q \mathrm{B}_{\mathrm{NWA},i-k,q,\omega}\binom{k}{j}_q (1)_{q,\omega}^{k+1-j} \\ &= \binom{i}{j}_q \sum_{k=j}^{i} \frac{1}{\{k+1-j\}_q}\binom{i-j}{k-j}_q \mathrm{B}_{\mathrm{NWA},i-k,q,\omega}(1)_{q,\omega}^{k+1-j} \\ &= \binom{i}{j}_q \sum_{k=0}^{i-j} \frac{1}{\{k+1\}_q}\binom{i-j}{k}_q \mathrm{B}_{\mathrm{NWA},i-j-k,q,\omega}(1)_{q,\omega}^{k+1} \overset{\text{durch}(98)}{=} \binom{i}{j}_q \delta_{i-j,0}. \end{aligned} \tag{99}$$

□

In [9] haben wir die folgenden Formeln betrachtet.

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(x \oplus_q y) = \mathrm{P}_{n,q}(x)\overline{\mathrm{B}}_{\mathrm{NWA},n,q}(y). \tag{100}$$

Diese Formeln können verallgemeinert werden zu

Satz 3.11. *Ein ω-Analogon von* [10, 92].

$$\overline{\Phi}_{n,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}(y). \tag{101}$$

Insbesondere haben wir

$$\overline{\Phi}_{n,q,\omega}(x) = \mathrm{P}_{n,q,\omega}(x)\overline{\Phi}_{n,q,\omega}. \tag{102}$$

Beweis.

$$\begin{aligned}\mathrm{RS} &= \sum_{k=j}^{i} \binom{i}{k}_q (x)_{q,\omega}^{i-k} \binom{k}{j}_q \Phi_{k-j,q,\omega}(y) \\ &= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q (x)_{q,\omega}^{i-k} \Phi_{k-j,q,\omega}(y) \\ &= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q (x)_{q,\omega}^{i-j-k}(x) \Phi_{k,q,\omega}(y) \\ &\overset{\text{durch}(43),(47)}{=} \binom{i}{j}_q \Phi_{i-j,q,\omega}(x \oplus_{q,\omega} y) = \mathrm{LS}.\end{aligned} \tag{103}$$

□

Satz 3.12. *Ein ω-Analogon von* [10, 95]. *Die Inversen der q,ω-Bernoulli-Polynommatrizen sind gegeben durch*

$$\left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x)\right)^{-1} = \left(\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} \mathrm{P}_{n,q,\omega}(x)^{-1} = \mathcal{D}_{\mathrm{NWA},n,q,\omega}\mathrm{P}_{n,q,\omega}(x)^{-1}. \tag{104}$$

Wenn die Ordnung erhöht wird, für $y = 0$ in (76), multiplizieren wir die q,ω-Übertragung-Matrix mit $\overline{\Phi}_{n,q,\omega}^{(M)}(x)$. Wenn die Ordnung konstant ist, in (102), multiplizieren wir die q,ω-Übertragung-Matrix mit der q,ω-Pascal-Matrix.

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Satz 3.13. *Die Funktionen der q,ω-Bernoulli Polynommatrizen* $(\mathcal{B}_{\mathrm{NWA},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ *und* $(\mathcal{B}_{\mathrm{JHC},q,\omega},\cdot,\cdot_q,\mathrm{I}_n)$ *mit Elementen*

$$\overline{\mathrm{B}}_{\mathrm{NWA},n,q,\omega}(x) \tag{105}$$

sind q,ω-Liesche Untergruppen von $\mathcal{M}_{x,q}$.

Beweis. Die Mengen $\mathcal{B}$ sind geschlossen unter den beiden Operationen durch (76) und (78). Die Existenz von Inversen folgt wie für $\mathcal{M}_{x,q}$.

□

Wir wenden uns nun den q,ω-Euler-Polynomen zu.

Definition 33. Es gibt zwei Arten von q,ω-Euler-Polynomen, $\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x)$, NWA q,ω-Euler-Polynomen, und $\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x)$, JHC q,ω-Euler-Polynomen. Sie sind definiert durch die folgenden zwei erzeugenden Funktionen:

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{q,\omega}(t)+1} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}_{\mathrm{NWA},\nu,q,\omega}(x),\ |t|<\pi, \tag{106}$$

und

$$\frac{2\mathrm{E}_{q,\omega t}(xt)}{\mathrm{E}_{\frac{1}{q},\omega}(t)+1} = \sum_{\nu=0}^{\infty} \frac{t^{\nu}}{\{\nu\}_q!}\mathrm{F}_{\mathrm{JHC},\nu,q,\omega}(x),\ |t|<\pi. \tag{107}$$

Definition 34. Wir werden die folgenden Vektorformen für diese Polynome benutzen.

$$f_{\mathrm{NWA},n,q,\omega}(x) \equiv (\mathrm{F}_{\mathrm{NWA},0,q,\omega}(x),\mathrm{F}_{\mathrm{NWA},1,q,\omega}(x),\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega}(x))^T. \tag{108}$$

Die entsprechenden q,ω-Euler-Zahlvektoren sind

$$f_{\mathrm{NWA},n,q,\omega} \equiv (\mathrm{F}_{\mathrm{NWA},0,q,\omega},\mathrm{F}_{\mathrm{NWA},1,q,\omega},\ldots,\mathrm{F}_{\mathrm{NWA},n-1,q,\omega})^T. \tag{109}$$

Wir stellen die NWA und JHC verschobenen, q,ω-Eulerschen Matrizen vor.

Definition 35. Ein ω-Analogon von [10, 103].

$$\begin{aligned}&\mathcal{F}_{\mathrm{NWA},n,q,\omega}(x)\\ &\equiv (f_{\mathrm{NWA},q,\omega}(x)\ \ \mathrm{E}(\oplus_{q,\omega})f_{\mathrm{NWA},q,\omega}(x)\ \cdots\ \mathrm{E}(\oplus_{q,\omega})^{\overline{n-1}_{q,\omega}}f_{\mathrm{NWA},q,\omega}(x)).\end{aligned} \tag{110}$$

Wir benötigen zwei ähnliche Matrizen, basierend auf den $\mathrm{F}_{\mathrm{NWA}}$-Polynomen.

Definition 36. Die beiden q,ω-Eulerschen Polynommatrizen sind definiert durch

$$\overline{F}_{\mathrm{NWA},n,q,\omega}(x)(i,j) \equiv \binom{i}{j}_q f_{\mathrm{NWA},i-j,q,\omega}(x). \tag{111}$$

Definition 37. Ein ω-Analogon von [10, 105]. Die NWA und JHC q,ω-Eulerschen Matrizen sind definiert durch

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}(i,j) \equiv \binom{i}{j}_q \mathrm{F}_{\mathrm{NWA},i-j,q,\omega}, 0 \le i,j \le n-1. \tag{112}$$

Definition 38. Ein ω-Analogon von [10, 106]. Die Matrix $\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{NWA},i,j} \equiv \begin{cases} \frac{(1)_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \ge j, \\ 0 \text{ sonst.} \end{cases} \tag{113}$$

Definition 39. Ein ω-Analogon von [10, 107]. Die Matrix $\mathcal{C}_{\mathrm{JHC},n,q,\omega}$ hat Matrixelemente

$$c_{\mathrm{JHC},i,j} \equiv \begin{cases} \frac{[1]_{q,\omega}^{i-j+1}+\delta_{i-j,0}}{2}\binom{i}{j}_q \text{ if } i \ge j, \\ 0 \text{ sonst.} \end{cases} \tag{114}$$

Satz 3.14. *Ein ω-Analogon von* [10, 108]. *Die Inversen der q,ω-Euler-Zahlenmatrizen sind gegeben durch*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\right)^{-1} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}. \tag{115}$$

Dies impliziert, dass

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}^{-k} = \mathcal{C}_{\mathrm{NWA},n,q,\omega}^{k}. \tag{116}$$

Beweis. Wir betrachten nur den NWA-Fall, für JHC, ändere man zu $[1]_{q,\omega}^{i-j+1}$. Wir zeigen, dass $\overline{\mathrm{F}}_{\mathrm{NWA},n,q,\omega}\mathcal{C}_{\mathrm{NWA},n,q,\omega}$ gleich der Einheitsmatrix ist. Wir wissen, dass

$$\sum_{k=0}^{n}(1)_{q,\omega}^{k}\binom{n}{k}_q \mathrm{F}_{\mathrm{NWA},n-k,q} + \mathrm{F}_{\mathrm{NWA},n,q} = 2\delta_{n,0}. \tag{117}$$

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Man führe eine Funktion $G(k)$ ein. Dann haben wir

$$\begin{aligned}
&\sum_{k=j}^{i} \binom{i}{k}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega} G(k-j) \binom{k}{j}_q \\
&= \binom{i}{j}_q \sum_{k=j}^{i} \binom{i-j}{k-j}_q \mathrm{F}_{\mathrm{NWA},i-k,q,\omega} G(k-j) \qquad (118)\\
&= \binom{i}{j}_q \sum_{k=0}^{i-j} \binom{i-j}{k}_q \mathrm{F}_{\mathrm{NWA},i-j-k,q,\omega} G(k) \overset{\text{durch}(117)}{=} \binom{i}{j}_q \delta_{i-j,0}.
\end{aligned}$$

Es ist jetzt offensichtlich, dass $G(k) = \frac{1}{2}\left[(1)_{q,\omega}^{k} + \delta_{k,0}\right]$ diese Gleichung für NWA löst, und dass der JHC-Fall ähnlich gelöst wird. □

Die letzten beiden Sätze werden auf ähnliche Weise belegt.

Satz 3.15. *Die Inversen der q,ω-Eulerschen Polynommatrizen sind gegeben durch*

$$\left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x)\right)^{-1} = \left(\overline{\mathrm{F}}_{\mathrm{NWA},n,q}\right)^{-1} \mathrm{P}_{n,q,\omega}(x)^{-1} = \mathcal{C}_{\mathrm{NWA},n,q} \mathrm{P}_{n,q,\omega}(x)^{-1}. \qquad (119)$$

Satz 3.16. *Ein ω-Analogon von* [10, 114]. *Die q,ω-Eulerschen Polynommatrizen $(\mathcal{F}_{\mathrm{NWA},q}, \ \cdot, \ \cdot_q, \ \mathrm{I}_n)$ und $(\mathcal{F}_{\mathrm{JHC},q}, \ \cdot, \ \cdot_q, \ \mathrm{I}_n)$ mit Elementen*

$$\overline{\mathrm{F}}_{\mathrm{NWA},n,q}(x) \qquad (120)$$

sind q,ω-Lieschen Untergruppen von $\mathcal{M}_{x,q}$.

4. Schlussfolgerung

Wir haben Formeln aus Arbeiten von Arponen [4], Aceto et al. [2], Ernst [8], [9], [10], vereinigt und q,ω-deformiert, um eine erste Synthese von q,ω-Appell-Polynommatrizen vorzustellen. Einige Formeln für q-Pascal

-Matrizen sowie Formeln für q-Bernoulli- und q-Eulerschen Matrizen werden verallgemeinert. Wir haben die ersten konkreten Beispiele von q,ω-Lieschen Untergruppen angegeben. Wir glauben, dass es keine weitere Analoga gibt, die q,ω-Analoga sind trotzdem sehr interessant.

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Bi-α Iso-Differential Calculus

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Abstract

In this paper we define bi-α(multiplicative) iso-derivative for iso-functions of first, second, third, fourth and fifth kind. They are deducted the main properties of the multiplicative iso-derivative. They are deducted and proved mean value theorems for multiplicative iso-differentiable functions, criteria for increasing and decreasing of multiplicative iso-differentiable functions, criteria for concavity and convexity of multiplicative iso-differentiable functions. In the paper it is introduced the concept for multiplicative iso-integral and they are deducted the main properties. As applications of multiplicative iso-derivative and multiplicative iso-integral we consider some classes multiplicative iso-differential equations.

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1 Introduction

As it is well known, Isaac Newton had to develop the differential calculus, (jointly with Gottfried Leibniz), with particular reference to the historical definition of velocities as the time derivative of the coordinates, $v = dr/dt$, in order to write his celebrated equation $ma = F(t, r, v)$, where $a = dv/dt$ is the acceleration and $F(t, r, v)$ is the Newtonian force acting on the mass m. Being local, the differential calculus solely admitted the characterization of massive points. The differential calculus and the notion of massive points were adopted by Galileo Galilei and Albert Einstein for the formulation of their relativities, thus acquiring a fundamental role in 20th century sciences.

In his Ph. D. thesis of 1966 at the University of Turin, Italy, the Italian-American scientist Ruggero Maria Santilli[1] pointed out that Newtonian forces are the most widely known in dynamics, including action-at-a-distance forces derivable derivable from a potential, thus representable with a Hamiltonian, and other forces that are not derivable from a potential or a Hamiltonian, since they are contact dissipative and non-conservative forces caused by the motion of the mass m within a physical medium. Santilli pointed out that, due to their lack of dimensions, massive points can solely experience action-at-a-distance Hamiltonian forces.

On this ground, Santilli initiated a long scientific journey for the generalization of Newton's equation into a form permitting the representation of the actual extended character of massive bodies whenever moving within physical media, as a condition to admit non-Hamiltonian forces. Being a theoretical physicist, Santilli had a number of severe physical conditions for the needed representation. One of them was the need for a representation of extended bodies and their non-Hamiltonian forces to be invariant over time as a condition to predict the same numerical values under the same conditions but at different times.

The resulting new calculus, today known as Santilli IsoDifferential Calculus, or IDC for short, stimulated a further layer of studies that finally signaled the achievement of mathematical and physical maturity. In particular, we note: the isotopies of Euclidean, Minkowskian, Riemannian and symplectic geometries; the isotopies of classical Hamiltonian mechanics, today known as the Hamilton-Santilli isomechanics;and the isotopies of quantum

[1]Prof. Santilli's curriculum is available in from the link http://www.world-lecture-series.org/santilli-cv

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mechanics, today known as the isotopic branch of Hadronic mechanics.

In this paper we define bi-α(multiplicative) iso-derivative for iso-functions of first, second, third, fourth and fifth kind. They are deducted the main properties of the multiplicative iso-derivative. They are deducted and proved mean value theorems for multiplicative iso-differentiable functions, criteria for increasing and decreasing of multiplicative iso-differentiable functions, criteria for concavity and convexity of multiplicative iso-differentiable functions. In the paper it is introduced the concept for multiplicative iso-integral and they are deducted the main properties. As applications of multiplicative iso-derivative and multiplicative iso-integral we consider some classes multiplicative iso-differential equations.

2 Definition for Multiplicative Iso-Derivative

Suppose that $A \subset \mathbb{R}$, $f, \hat{T} : A \to (0, \infty)$ be enough times differentiable functions. If it is necessary, we suppose that $\frac{x}{\hat{T}(x)} \in A$ or $x\hat{T}(x) \in A$ for any $x \in A$ so that to be defined the iso-functions of the second, third, fourth and fifth kind. With $\widetilde{f}$ we will denote the corresponding iso-function of the first, second, third, fourth and fifth kind.

Definition 2.1. *Define the multiplicative iso-derivative of $\widetilde{f}$ as*

$$\widetilde{f}^{*\circledast}(x) = e^{\widetilde{f}^{\circledast}(x) \curlywedge \widetilde{f}(x)}, \quad x \in A.$$

1. Iso-functions of the first kind.

$$\hat{f}^{\wedge\wedge*\circledast}(x) = e^{\frac{1}{f(x)(\hat{T}(x))^2} \frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

2. Iso-functions of the second kind.

$$\hat{f}^{\wedge*\circledast}(x) = e^{\frac{1}{f(x\hat{T}(x))(\hat{T}(x))^2} \frac{f'(x\hat{T}(x))(\hat{T}^2(x) + x\hat{T}(x)\hat{T}'(x)) - f(x\hat{T}(x))\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

3. Iso-functions of the third kind.

$$\hat{\hat{f}}^{*\circledast}(x) = e^{\frac{1}{f\left(\frac{x}{\hat{T}(x)}\right)(\hat{T}(x))^2} \frac{f'\left(\frac{x}{\hat{T}(x)}\right)\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} - f\left(\frac{x}{\hat{T}(x)}\right)\hat{T}'(x)}{1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}}}.$$

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4. Iso-functions of the fourth kind.

$$f^{\wedge *\circledast}(x) = e^{\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x)+x\hat{T}'(x))}{\hat{T}(x)f(x\hat{T}(x))\left(1-x\frac{\hat{T}'(x)}{\hat{T}(x)}\right)}}.$$

5. Iso-functions of the fifth kind.

$$f^{\vee *\circledast}(x) = e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x)f\left(\frac{x}{\hat{T}(x)}\right)}}.$$

Example 2.2. *Let $A = [0, \infty)$, $\hat{T}(x) = 1 + x$, $f(x) = 1 + x^2$, $x \in A$. Then*

$$\begin{aligned} \hat{f}^{\wedge\wedge}(x) &= \frac{f(x)}{\hat{T}(x)} = \frac{1+x^2}{1+x}, \\ f'(x) &= 2x, \\ \hat{T}'(x) &= 1, \quad x \in A. \end{aligned}$$

Hence, the iso-derivative of the iso-function $\hat{f}^{\wedge\wedge}$ is given by

$$\hat{f}^{\wedge\wedge\circledast}(x) = \frac{-1+2x+x^2}{1+x}, \quad x \in A,$$

and its multiplicative iso-derivative is given by

$$\hat{f}^{\wedge\wedge *\circledast}(x) = e^{\frac{-1+2x+x^2}{(1+x)(1+x^2)}}, \quad x \in A.$$

3 Properties of the Multiplicative Iso-Derivative

In this section, we will deduct some of the properties of the multiplicative iso-derivative.

Theorem 3.1. *Let $\hat{f}, \hat{g} : A \to \mathbb{R}$ be iso-differentiable functions. Then for any $a, b \in \mathbb{R}$, we have*

$$\left(a\hat{f} + b\hat{g}\right)^{*\circledast} = \left(\left(\hat{f}^{*\circledast}\right)^{\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right)\left((\hat{g}^{*\circledast})^{\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right).$$

Proof. We have

$$\begin{aligned}
\left(a\hat{f}+b\hat{g}\right)^{*\circledast} &= e^{(a\hat{f}+b\hat{g})^{*\circledast}\hat{\times}(a\hat{f}+b\hat{g})} \\
&= e^{\frac{a\hat{f}^{\circledast}+b\hat{g}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}} \\
&= e^{a\frac{\hat{f}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}+b\frac{\hat{g}^{\circledast}}{\hat{T}(a\hat{f}+b\hat{g})}} \\
&= \left(e^{\left(\hat{f}^{\circledast}\hat{\times}\hat{f}\right)\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right) \\
&\quad \left(e^{\left(\hat{g}^{\circledast}\hat{\times}\hat{g}\right)\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right) \\
&= \left(\left(\hat{f}^{*\circledast}\right)^{\frac{a\hat{f}}{a\hat{f}+b\hat{g}}}\right)\left(\left(\hat{g}^{*\circledast}\right)^{\frac{b\hat{g}}{a\hat{f}+b\hat{g}}}\right).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let* $\hat{f}, \hat{g} : A \to (0, \infty)$ *be iso-differentiable functions. Then*

$$\left(\hat{f}\hat{\times}\hat{g}\right)^{*\circledast} = \hat{f}^{*\circledast}\hat{g}^{*\circledast}e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}}.$$

Proof. By the properties of the iso-derivative, we have

$$\begin{aligned}
\left(\hat{f}\hat{\times}\hat{g}\right)^{\circledast} &= \hat{g}\hat{\times}\hat{f}^{\circledast}+\hat{f}\hat{\times}\hat{g}^{\circledast} \\
&\quad +\hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T}-x\hat{T}'}.
\end{aligned}$$

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Then

$$
\begin{aligned}
(\hat{f}\hat{\times}\hat{g})^{\circledast} \curlywedge (\hat{f}\hat{\times}\hat{g}) &= \frac{1}{\hat{f}\hat{g}\hat{T}^2}\Bigg(\hat{g}\hat{T}\hat{f}^{\circledast} + \hat{f}\hat{T}\hat{g}^{\circledast} \\
&\quad + \hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T}-x\hat{T}'}\Bigg) \\
&= \left(\hat{f}^{\circledast} \curlywedge \hat{f}\right) + (\hat{g}^{\circledast} \curlywedge \hat{g}) \\
&\quad + \frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left(\hat{f}\hat{\times}\hat{g}\right)^{*\circledast} &= e^{(\hat{f}\hat{\times}\hat{g})^{\circledast} \curlywedge (\hat{f}\hat{\times}\hat{g})} \\
&= e^{\left(\hat{f}^{\circledast} \curlywedge \hat{f}\right) + \left(\hat{g}^{\circledast} \curlywedge \hat{g}\right) + \frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}} \\
&= e^{\hat{f}^{\circledast} \curlywedge \hat{f}} e^{\hat{g}^{\circledast} \curlywedge \hat{g}} e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}} \\
&= \hat{f}^{*\circledast}\hat{g}^{*\circledast} e^{\frac{\hat{T}'}{\hat{T}(\hat{T}-x\hat{T}')}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.3. *Let $\hat{f}, \hat{g} : A \to (0, \infty)$ be iso-differentiable functions. Then*

$$(\hat{f}\hat{g})^{*\circledast} = \hat{f}^{*\circledast}\hat{g}^{*\circledast}.$$

Proof. By the properties of the iso-derivative, we have

$$(\hat{f}\hat{g})^{\circledast} = \hat{f}^{\circledast}\hat{g} + \hat{g}\hat{f}^{\circledast}.$$

Hence,

$$
\begin{aligned}
(\hat{f}\hat{g})^{\circledast} \curlywedge (\hat{f}\hat{g}) &= \frac{1}{\hat{T}\hat{f}\hat{g}}\left(\hat{f}^{\circledast}\hat{g} + \hat{f}\hat{g}^{\circledast}\right) \\
&= \hat{f}^{\circledast} \curlywedge \hat{f} + \hat{g}^{\circledast} \curlywedge \hat{g}.
\end{aligned}
$$

Therefore

$$\begin{aligned}(\hat{f}\hat{g})^{*\circledast} &= e^{(\hat{f}\hat{g})^{\circledast}\leftthreetimes(\hat{f}\hat{g})}\\ &= e^{\hat{f}^{\circledast}\leftthreetimes\hat{f}+\hat{g}^{\circledast}\leftthreetimes\hat{g}}\\ &= e^{\hat{f}^{\circledast}\leftthreetimes\hat{f}}e^{\hat{g}^{\circledast}\leftthreetimes\hat{g}}\\ &= \hat{f}^{*\circledast}\hat{g}^{*\circledast}.\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $\hat{f},\hat{g}:A\to(0,\infty)$ be iso-differentiable functions and $\hat{g}\neq 0$ on A. Then*

$$\left(\hat{f}\leftthreetimes\hat{g}\right)^{*\circledast}=\frac{\hat{f}^{*\circledast}}{\left(\hat{g}^{*\circledast}\right)^{\frac{1}{\hat{T}^2}}}e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}}.$$

Proof. We have

$$\hat{f}\leftthreetimes\hat{g}=\frac{\hat{f}}{\hat{T}\hat{g}}$$

and by the properties of the iso-derivative, we get

$$\begin{aligned}\left(\hat{f}(x)\leftthreetimes\hat{g}(x)\right)^{\circledast} &= \frac{1}{\left(\hat{T}(x)\right)^2(\hat{g}(x))^2}\Bigg(\hat{g}(x)\hat{\times}\left(\hat{f}(x)\right)^{\circledast}-\hat{f}(x)\hat{\times}(\hat{g}(x))^{\circledast}\\ &\quad -\hat{f}(x)\hat{g}(x)\frac{\hat{T}(x)\hat{T}'(x)}{\hat{T}(x)-x\hat{T}'(x)}\Bigg)\\ &= \frac{1}{\hat{T}^2\hat{g}^2}\Bigg(\hat{T}\hat{g}\hat{f}^{\circledast}-\hat{T}\hat{f}\hat{g}^{\circledast}\\ &\quad -\hat{f}\hat{g}\frac{\hat{T}\hat{T}'}{\hat{T}-x\hat{T}'}\Bigg)\\ &= \frac{1}{\hat{g}\hat{T}}\hat{f}^{\circledast}-\frac{1}{\hat{T}\hat{f}\hat{g}}\left(\frac{\hat{g}^{\circledast}}{\hat{g}}\right)\\ &\quad -\frac{\hat{f}}{\hat{T}^2\hat{g}}\cdot\frac{\hat{T}'}{\hat{T}-x\hat{T}'}.\end{aligned}$$

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Hence,

$$\begin{aligned}\left(\hat{f}\curlywedge\hat{g}\right)^{\circledast}\curlywedge(\hat{f}\curlywedge\hat{g}) &= \frac{\hat{g}}{\hat{f}}\Bigg(\frac{1}{\hat{g}\hat{T}}\hat{f}^{\circledast}-\frac{1}{\hat{T}\hat{f}\hat{g}}\left(\frac{\hat{g}^{\circledast}}{\hat{g}}\right)\\ &\quad -\frac{\hat{f}}{\hat{T}^2\hat{g}}\cdot\frac{\hat{T}'}{\hat{T}-x\hat{T}'}\Bigg)\\ &= \hat{f}^{\circledast}\curlywedge\hat{f}-\frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast}\curlywedge\hat{g}\right)\\ &\quad -\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}\end{aligned}$$

and

$$\begin{aligned}e^{\left(\hat{f}\curlywedge\hat{g}\right)^{\circledast}\curlywedge(\hat{f}\curlywedge\hat{g})} &= e^{\hat{f}^{\circledast}\curlywedge\hat{f}-\frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast}\curlywedge\hat{g}\right)-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}}\\ &= e^{\hat{f}^{\circledast}\curlywedge\hat{f}}e^{-\frac{1}{\hat{f}^2}\left(\hat{g}^{\circledast}\curlywedge\hat{g}\right)}e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}}\\ &= \frac{\hat{f}^{*\circledast}}{\left(\hat{g}^{*\circledast}\right)^{\frac{1}{\hat{T}^2}}}e^{-\frac{\hat{f}\hat{T}'}{\hat{T}^2\hat{g}(\hat{T}-x\hat{T}')}}.\end{aligned}$$

This completes the proof. □

4 Monotonicity

Theorem 4.1. *Let $\hat{f}: A\to(0,\infty)$ be iso-differentiable function and*

$$\hat{T}'(x)>0,\quad \hat{T}(x)-x\hat{T}'(x)>0,\quad x\in A. \tag{4.1}$$

If $\hat{f}^{\wedge\wedge*\circledast}>1$ on A, then $f^{\vee}$ is an increasing function.

Proof. Since $\hat{f}^{\wedge\wedge*\circledast}>1$ on A, we have that

$$\frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{\hat{T}(x)-x\hat{T}'(x)}>0,\quad x\in A.$$

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By the last inequality and by the second inequality of (4.1), we get

$$f'(x)\hat{T}(x) - f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Hence, using the first inequality of (4.1), we arrive at

$$f'(x)\hat{T}(x) > f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Therefore $f'(x) > 0$, $x \in A$. Because $\frac{Id}{\hat{T}}$ is an increasing function, we find that $f^{\vee}$ is an increasing function. This completes the proof. □

Theorem 4.2. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) < 0, \quad x \in A. \tag{4.2}$$

If $\hat{f}^{\wedge\wedge*\circledast} < 1$ on A, then $f^{\vee}$ is a decreasing function.

Proof. By the definition of multiplicative iso-derivative and $\hat{f}^{\wedge\wedge*\circledast} < 1$, it follows that

$$\frac{f'(x)\hat{T}(x) - f(x)\hat{T}'(x)}{\hat{T}(x) - x\hat{T}'(x)} < 0, \quad x \in A.$$

Hence and the second inequality of (4.2), we conclude that

$$f'(x)\hat{T}(x) - f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Now, applying the first inequality of (4.2), we get

$$f'(x)\hat{T}(x) > f(x)\hat{T}'(x) > 0, \quad x \in A.$$

Therefore $f'(x) > 0$ for $x \in A$ and f is an increasing function on A. By the second inequality of (4.2), we find

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x > y.$$

Hence,

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &< f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x > y. \end{aligned}$$

Thus, $f^{\vee}$ is a decreasing function on A. This completes the proof. □

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Theorem 4.3. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) > 0, \quad x \in A. \tag{4.3}$$

If $\hat{f}^{\wedge *\circledast} > 1$ on A, then $f^{\wedge}$ is an increasing function on A.

Proof. By the definition of the multiplicative iso-derivative and by the condition $f^{\vee *\circledast} > 1$ on A, we get

$$\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x) + x\hat{T}'(x))}{f(x\hat{T}(x))(\hat{T}(x) - x\hat{T}'(x))} > 0, \quad x \in A,$$

Applying (4.4), we find

$$f'(x\hat{T}(x)) > 0, \quad x \in A.$$

Thus, f is an increasing function on A. Since $\hat{T}'(x) > 0$, $x \in A$, we get that

$$x\hat{T}(x) > y\hat{T}(y), \quad x, y \in A, \quad x > y.$$

Hence, using that f is an increasing function on A, we find

$$\begin{aligned} f^{\wedge}(x) &= f(x\hat{T}(x)) \\ &< f(y\hat{T}(y)) \\ &= f^{\wedge}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Therefore $f^{\wedge}$ is an increasing function on A. This completes the proof. $\square$

Theorem 4.4. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function and*

$$\hat{T}'(x) > 0, \quad \hat{T}(x) - x\hat{T}'(x) > 0, \quad x \in A. \tag{4.4}$$

If $\hat{f}^{\wedge *\circledast} < 1$ on A, then $f^{\wedge}$ is a decreasing function on A.

Proof. Applying the multiplicative iso-derivative and the condition $f^{\vee *\circledast} >$ 1 on A, we find

$$\frac{f'(x\hat{T}(x))\hat{T}(x)(\hat{T}(x) + x\hat{T}'(x))}{f(x\hat{T}(x))(\hat{T}(x) - x\hat{T}'(x))} <, \quad x \in A,$$

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Employing (**??**), we arrive at

$$f'(x\hat{T}(x)) < 0, \quad x \in A.$$

Thus, f is a decreasing function on A. Since $\hat{T}'(x) > 0$, $x \in A$, we find that

$$x\hat{T}(x) > y\hat{T}(y), \quad x, y \in A, \quad x > y.$$

From here and from the fact that f is an increasing function on A, we find

$$\begin{aligned} f^{\wedge}(x) &= f(x\hat{T}(x)) \\ &> f(y\hat{T}(y)) \\ &= f^{\wedge}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\wedge}$ is a decreasing function on A. This completes the proof. □

Theorem 4.5. *Let* $\hat{f} : A \to (0, \infty)$ *be an iso-differentiable function on* A*and* $\hat{T}'(x) < 0$, $x \in A$. *If* $f^{\vee *\circledast}(x) > 1$, $x \in A$, *then* $f^{\vee}$ *is an increasing function on* A.

Proof. By the definition of the multiplicative iso-derivative and by the condition $f^{\vee *\circledast}(x) > 1$, $x \in A$, we find

$$\begin{aligned} f^{\vee *\circledast}(x) &= e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)}} \\ &> 1, \quad x \in A. \end{aligned}$$

Hence,

$$\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)} > 0, \quad x \in A.$$

Therefore

$$f'\left(\frac{x}{\hat{T}(x)}\right) > 0, \quad x \in A,$$

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and f is an increasing function on A. Since $\hat{T}'(x) < 0$, $x \in A$, we get

$$\hat{T}(x) > \hat{T}(y), \quad x, y \in A, \quad x < y.$$

Then

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x < y.$$

From here, we arrive at

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &< f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\vee}$ is an increasing function on A. This completes the proof. □

Theorem 4.6. *Let $\hat{f} : A \to (0, \infty)$ be an iso-differentiable function on Aand $\hat{T}'(x) < 0$, $x \in A$. If $f^{\vee * \circledast}(x) < 1$, $x \in A$, then $f^{\vee}$ is a decreasing function on A.*

Proof. Applying the definition of the multiplicative iso-derivative and the condition $f^{\vee * \circledast}(x) > 1$, $x \in A$, we arrive at

$$\begin{aligned} f^{\vee * \circledast}(x) &= e^{\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)}} \\ &< 1, \quad x \in A, \end{aligned}$$

whereupon

$$\frac{f'\left(\frac{x}{\hat{T}(x)}\right)}{\hat{T}(x) f\left(\frac{x}{\hat{T}(x)}\right)} < 0, \quad x \in A,$$

and

$$f'\left(\frac{x}{\hat{T}(x)}\right) < 0, \quad x \in A.$$

So, f is a decreasing function on A. Because $\hat{T}'(x) < 0$, $x \in A$, we find

$$\hat{T}(x) > \hat{T}(y), \quad x, y \in A, \quad x < y.$$

Thus,

$$\frac{x}{\hat{T}(x)} < \frac{y}{\hat{T}(y)}, \quad x, y \in A, \quad x < y,$$

and

$$\begin{aligned} f^{\vee}(x) &= f\left(\frac{x}{\hat{T}(x)}\right) \\ &> f\left(\frac{y}{\hat{T}(y)}\right) \\ &= f^{\vee}(y), \quad x, y \in A, \quad x < y. \end{aligned}$$

Consequently $f^{\vee}$ is a decreasing function on A. This completes the proof. $\square$

5 Definition for Multiplicative Iso-Integral. Properties

Definition 5.1. *Suppose that $\hat{T}(x) - x\hat{T}'(x) > 0$, $x \in A$. Define indefinite multiplicative iso-integral for the iso-function of the first kind $\hat{f}^{\wedge\wedge}$ as follows*

$$\hat{\int} \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int} \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x) - x\hat{T}'(x)\right)\hat{\times} dx}. \tag{5.1}$$

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By (5.1), it follows

$$\hat{\int} \hat{f}^{\wedge\wedge *\circledast}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int} \log\left(\hat{f}^{\wedge\wedge *\circledast}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times} dx}$$

$$\begin{aligned}
&= e^{\int \frac{1}{f(x)(\hat{T}(x))^2} \frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}}\left(\hat{T}(x)-x\hat{T}'(x)\right)dx} \\
&= e^{\int \frac{1}{\frac{f(x)}{\hat{T}(x)}} \frac{f'(x)\hat{T}(x)-f(x)\hat{T}'(x)}{(\hat{T}(x))^2}dx} \\
&= e^{\int \frac{\left(\frac{f}{\hat{T}}\right)'(x)}{\frac{f(x)}{\hat{T}(x)}}dx} \\
&= e^{\log \frac{f(x)}{\hat{T}(x)}} \\
&= \frac{f(x)}{\hat{T}(x)} \\
&= \hat{f}^{\wedge\wedge}(x).
\end{aligned}$$

Suppose that $[a, b] \subset \mathbb{R}$.

Definition 5.2. *Suppose that $\hat{T}(x) - x\hat{T}'(x) > 0$, $x \in A$. Define indefinite multiplicative iso-integral for the iso-function of the first kind $\hat{f}^{\wedge\wedge}$ as follows*

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times} dx}. \tag{5.2}$$

In this case, we say that $\hat{f}^{\wedge\wedge}$ is multiplicative iso-integrable on $[a, b]$

Below, we will deduct some of the properties of the multiplicative iso-integral.

Theorem 5.3. *Let $\hat{f}^{\wedge\wedge}$ is multiplicative iso-integrable on $[a, b]$. Then*

$$\hat{\int}_a^a \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = 1.$$

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Proof. We have

$$\hat{\int}_a^a \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^a \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$
$$= 1.$$

This completes the proof. □

Theorem 5.4. *Let $c \in [a,b]$ and $\hat{f}^{\wedge\wedge}$ is multiplicative iso-integrable on $[a,b]$. Then*

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = \left(\hat{\int}_a^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_b^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

Proof. We have

$$\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$
$$= e^{\hat{\int}_a^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx+\hat{\int}_b^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$
$$= e^{\hat{\int}_a^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx} e^{\hat{\int}_b^c \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$
$$= \left(\hat{\int}_a^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_b^c \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

This completes the proof. □

Theorem 5.5. *Let $\hat{f}^{\wedge\wedge}$, $\hat{g}^{\wedge\wedge}$ be multiplicative iso-integrable on $[a,b]$. Then*

$$\hat{\int}_a^b \left(\hat{f}^{\wedge\wedge}(x)\hat{g}^{\wedge\wedge}\right)^{\hat{\times}\hat{d}\hat{x}} = \left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right).$$

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Proof. By the definition for multiplicative iso-integral, we get

$$\hat{\int}_a^b \left(\hat{f}^{\wedge\wedge}(x)\hat{g}^{\wedge\wedge}(x)\right)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx+\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx} e^{\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= \left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)..$$

This completes the proof. □

Theorem 5.6. *Let* $\hat{f}^{\wedge\wedge}$, $\hat{g}^{\wedge\wedge}$ *be multiplicative iso-integrable on* $[a, b]$ *and*

$$\hat{g}^{\wedge\wedge}(x) \neq 0, \quad \hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}} \neq 0.$$

Then

$$\hat{\int}_a^b \left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)^{\hat{\times}\hat{d}\hat{x}} = \frac{\left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}{\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}.$$

Proof. Using the definition for multiplicative iso-integral, we find

$$\hat{\int}_a^b \left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)^{\hat{\times}\hat{d}\hat{x}} = e^{\hat{\int}_a^b \log\left(\frac{\hat{f}^{\wedge\wedge}(x)}{\hat{g}^{\wedge\wedge}(x)}\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx-\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}$$

$$= \frac{e^{\hat{\int}_a^b \log\left(\hat{f}^{\wedge\wedge}(x)\right)\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}}{e^{\hat{\int}_a^b \log(\hat{g}^{\wedge\wedge}(x))\left(\hat{T}(x)-x\hat{T}'(x)\right)\hat{\times}dx}}$$

$$= \frac{\left(\hat{\int}_a^b \hat{f}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}{\left(\hat{\int}_a^b \hat{g}^{\wedge\wedge}(x)^{\hat{\times}\hat{d}\hat{x}}\right)}.$$

This completes the proof. □

6 Linear Homogeneous Multiplicative Iso-Differential Equations

Consider the equation

$$\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x) = a(x)\hat{\times}\hat{f}^{\wedge\wedge}(x).$$

Hence,

$$\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x) \leftthreetimes \hat{f}^{\wedge\wedge}(x) = a(x)$$

and

$$\begin{aligned}\hat{f}^{\wedge\wedge*\circledast}(x) &= e^{\left(\hat{f}^{\wedge\wedge}\right)^{\circledast}(x)\leftthreetimes\hat{f}^{\wedge\wedge}(x)}\\ &= e^{a(x)}.\end{aligned}$$

Then the solution of the considered iso-differential equation can be represented in the form

$$\hat{f}^{\wedge\wedge}(x) = \hat{\int} e^{a(x)}\hat{\times}\hat{d}\hat{x}.$$

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Bi-α Iso-Differential Inequalities and Applications

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Abstract

In this lecture, firstly we deduct some multiplicative iso-differential inequalities for multiplicative iso-functions of first, second, third, fourth and fifth kind. Then they are deducted and proved some bi-α-multiplicative iso-differential inequalities. As applications, in the lecture are deducted some uniqueness results for some classes multiplicative iso-differential equations and bi-α-multiplicative iso-differential equations.

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1 Multiplicative iso-Differential Inequalities

Let D is a domain in $\mathbb{R}^2$, $a > 0$, $x_0 \in \mathbb{R}$, $J = [x_0, x_0 + a)$, $\hat{T} \in \mathcal{C}^1(J)$, $\hat{T}(x) > 0$ in J, $f \in \mathcal{C}(D)$.

Definition 1.1. *(solution of multiplicative iso-differential inequality) A function $y(x)$ is said to be a solution of the multiplicative iso-differential inequality*

$$(1') \qquad \left(\hat{y}^\wedge(\hat{x})\right)^{\circledast} > \hat{f}^\wedge(\hat{x}, \hat{y}^\wedge(\hat{x}))$$

or

$$(1) \qquad y'(x) > y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

in J if

1. *$y'(x)$ exists for all $x \in J$,*

2. *for all $x \in J$ the points $(x, y(x)) \in D$,*

3. *$y'(x) > y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$ for all $x \in J$.*

The solutions of the multiplicative iso-differential inequalities

$$y'(x) \geq y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y'(x) < y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y'(x) \leq y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

are defined analogously. Our first result for multiplicative iso-differential inequalities is stated in the following theorem.

Theorem 1.2. *(basic theorem for the multiplicative iso-differential inequalities) Let $\hat{T}(x) - x\hat{T}'(x) \geq 0$ for every $x \in J$, $y_1(x)$ and $y_2(x)$ be the solutions of the multiplicative iso-differential inequalities*

$$(2) \qquad y_1'(x) \leq y_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

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(3) $\qquad y_2'(x) > y_2(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$

on J, respectively. Then the inequality

$$y_1(x_0) < y_2(x_0)$$

implies that

(4) $\qquad y_1(x) < y_2(x) \qquad \text{for} \qquad \forall x \in J.$

Proof. We suppose that (4) is not true. Then we define the set

$$A = \{x : \qquad x \in J, \qquad y_1(x) \geq y_2(x)\}.$$

From our assumption it follows that $A \neq \emptyset$.

Let x^* be the greatest lower bound of the set A. Then $x_0 < x^*$ and

$$y_1(x^*) \geq y_2(x^*).$$

Let us assume that

$$y_1(x^*) > y_2(x^*).$$

Because $y_1(x)$ and $y_2(x)$ are continuous functions in J then there exists a $\epsilon > 0$ such that

$$y_1(x^* - \epsilon) \geq y_2(x^* - \epsilon),$$

which is a contradiction with the definition of x^*. Consequently

$$y_1(x^*) = y_2(x^*).$$

Let $h < 0$. We have

$$y_1(x^* + h) < y_2(x^* + h),$$

and hence

$$y_1'(x^* - 0) = \lim_{h \longrightarrow 0} \frac{y_1(x^*+h) - y_1(x^*)}{h}$$

$$= \lim_{h \longrightarrow 0} \frac{y_1(x^*+h) - y_2(x^*)}{h}$$

$$\geq \lim_{h \longrightarrow 0} \frac{y_2(x^*+h) - y_2(x^*)}{h}$$

$$= y_2'(x^* - 0),$$

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i.e.,

(5) $\qquad y_1'(x^* - 0) \geq y_2'(x^* - 0).$

From (2) we get

$$y_1'(x^*) \leq y_1(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\frac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)},$$

from where, using (5),

(6) $\qquad y_1(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\frac{\hat{T}(x^*)-x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \geq y_2'(x^* - 0).$

On the other hand, from (3) we have

$$y_2'(x^* - 0) > y_2(x^*)\frac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\frac{\hat{T}(x^*) - x^*\hat{T}'(x^*)}{\hat{T}(x^*)},$$

whereupon, using (6),

$$\begin{aligned} &y_2(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\tfrac{\hat{T}(x^*)-x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \\ &< y_1(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_1(x^*))\tfrac{\hat{T}(x^*)-x^*\hat{T}'(x^*)}{\hat{T}(x^*)} \\ &= y_2(x^*)\tfrac{\hat{T}'(x^*)}{\hat{T}(x^*)} + f(x^*, y_2(x^*))\tfrac{\hat{T}(x^*)-x^*\hat{T}'(x^*)}{\hat{T}(x^*)}, \end{aligned}$$

and since $\hat{T}(x^*) - x^*\hat{T}'(x^*) \geq 0$ we get the contradiction

$$f(x^*, y_2(x^*)) < f(x^*, y_2(x^*)).$$

Consequently

$$A = \varnothing,$$

from where we conclude that

$$y_1(x) < y_2(x) \qquad \text{in} \qquad J.$$

□

Corollary 1.3. *Let* $\hat{T}(x) - x\hat{T}'(x) \geq 0$ *in the interval* J. *Let also,*

(i) $y(x)$ *be a solution of the initial value problem*

$$(6)\qquad \begin{aligned} &y'(x) = -y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \qquad \text{in} \qquad (x_0, x_0 + a),\\ &y(x_0) = y_0, \end{aligned}$$

(ii) $y_1(x)$ *and* $y_2(x)$ *be the solutions of the multiplicative iso-differential inequalities*

$$(7)\qquad y_1'(x) < y_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$$

$$(8)\qquad y_2'(x) > y_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

in J, *respectively,*

(iii) $y_1(x_0) \leq y_0 \leq y_2(x_0)$.

Then

$$y_1(x) < y(x) < y_2(x)$$

for all $x \in (x_0, x_0 + a)$.

Proof. We shall prove that

$$y(x) < y_2(x) \qquad \text{for} \qquad \forall x \in (x_0, x_0 + a).$$

1. case $y_0 < y_2(x_0)$. Then from the last theorem we have that

$$y(x) < y_2(x) \qquad \text{in} \qquad (x_0, x_0 + a).$$

2. case $y_0 = y_2(x_0)$.

Let

$$z(x) = y_2(x) - y(x).$$

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Then

$$z(x_0) = y_2(x_0) - y(x_0) = 0,$$

$$z'(x) = y_2'(x) - y'(x),$$

$$z'(x_0) = y_2'(x_0) - y'(x_0)$$

$$> y_2(x_0)\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_2(x_0))\frac{\hat{T}(x_0)-x_0\hat{T}'(x_0)}{\hat{T}(x_0)}$$

$$-y(x_0)\frac{\hat{T}'(x_0)}{\hat{T}(x_0)} - f(x_0, y(x_0))\frac{\hat{T}(x_0)-x_0\hat{T}'(x_0)}{\hat{T}(x_0)}$$

$$= 0,$$

therefore the function z is an increasing function to the right of x_0 in a sufficiently small interval $[x_0, x_0+\delta]$. Consequently $y(x) < y_2(x)$ for all $x \in (x_0, x_0+\delta]$, from where

$$y(x_0+\delta) < y_2(x_0+\delta).$$

Now the last theorem gives that

$$y(x) < y_2(x) \qquad \text{in} \qquad [x_0+\delta, x_0+a).$$

Since δ can be chosen sufficiently small, then

$$y(x) < y_2(x) \qquad \text{in} \qquad (x_0, x_0+a).$$

□

Theorem 1.4. *Let $\hat{T}(x) - x\hat{T}'(x) \geq 0$, $\hat{T}'(x) \leq 0$, $\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \leq P$ in J for some positive constant P, and for all (x, y), $(x, z) \in D$ such that $x \geq x_0$, $y \geq z$, we have*

$$f(x, y) - f(x, z) \leq L(y - z),$$

for some positive constant L. Let also,

(i) *$y(x)$ be a solution to the initial value problem (6),*

(ii) *$y_1(x)$ and $y_2(x)$ be solutions to the multiplicative iso-differential inequalities (2) and (3) on J, respectively.*

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(iii) $y_1(x_0) \le y_0 \le y_2(x_0)$.

Then

$$y_1(x) \le y(x) \le y_2(x) \qquad \text{for} \qquad \forall x \in J.$$

Proof. Let $\epsilon > 0$, $\lambda > LP$. Let also,

$$z_1(x) = y_1(x) - \epsilon e^{\lambda(x-x_0)}, \qquad x \in J.$$

Then

$$z_1(x_0) = y_1(x_0) - \epsilon < y_1(x_0)$$

and

$$\begin{aligned} &z_1'(x) = y_1'(x) - \epsilon\lambda e^{\lambda(x-x_0)} \\ (9)\qquad &\le y_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}. \end{aligned}$$

On the other hand, from the definition of the function $z_1(x)$ we have

$$z_1(x) \le y_1(x) \qquad \text{in} \qquad J.$$

Then

$$f(x, y_1(x)) - f(x, z_1(x)) \le L(y_1(x) - z_1(x))$$

or

$$f(x, y_1(x)) \le f(x, z_1(x)) + L(y_1(x) - z_1(x)) \qquad \text{in} \qquad J.$$

From the last inequality and (9) we become

$$z_1'(x) \le z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, z_1(x)) + L(y_1(x) - z_1(x)))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$= z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + L\epsilon e^{\lambda(x-x_0)}\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$\le z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + LPe^{\epsilon(x-x_0)} - \epsilon\lambda e^{\lambda(x-x_0)}$$

$$< z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$$

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i.e.,

$$(10)\qquad \begin{aligned} & z_1'(x) < z_1(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_1(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \qquad \text{in} \qquad J, \\ & z_1(x_0) < y(x_0). \end{aligned}$$

Let now

$$z_2(x) = y_2(x) + \epsilon e^{\lambda(x-x_0)}, \qquad x \in J.$$

Then

$$z_2(x) > y_2(x) \qquad \text{in} \qquad J.$$

Therefore

$$f(x, z_2(x)) - f(x, y_2(x)) \le L(z_2(x) - y_2(x)) \qquad \text{in} \qquad J,$$

from where

$$f(x, y_2(x)) \ge f(x, z_2(x)) + L(y_2(x) - z_2(x)) \qquad \text{in} \qquad J.$$

Also, using the last inequality,

$$\begin{aligned} & z_2'(x) = y_2'(x) + \epsilon\lambda e^{\lambda(x-x_0)} \\ & > y_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \ge y_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, z_2(x)) + L(y_2(x) - z_2(x)))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \ge z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - L\epsilon e^{\lambda(x-x_0)}\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & \ge z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - L\epsilon P e^{\lambda(x-x_0)} + \epsilon\lambda e^{\lambda(x-x_0)} \\ & > z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}, \end{aligned}$$

i.e.,

$$(11)\qquad \begin{aligned} & z_2'(x) > z_2(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, z_2(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} \qquad \text{in} \qquad J, \\ & z_2(x_0) > y_2(x_0). \end{aligned}$$

From (10) and (11) it follows that the functions $z_1(x)$ and $z_2(x)$ satisfy all conditions of the basic theorem for the multiplicative iso-differential inequalities. Therefore

$$z_1(x) < y(x) < z_2(x) \qquad \text{in} \qquad (x_0, x_0 + a),$$

i.e.

$$y_1(x) - \epsilon e^{\lambda(x-x_0)} < y(x) < y_2(x) + \epsilon e^{\lambda(x-x_0)} \qquad \text{in} \qquad (x_0, x_0 + a),$$

from here, when $\epsilon \longrightarrow 0$,

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \text{in} \qquad J.$$

□

Corollary 1.5. *Let for every points (x, y), $(x, z) \in D$ such that $x \geq x_0$, we have*

$$(12) \qquad |f(x, y) - f(x, z)| \leq L|y - z|$$

for some positive constant L, $-P \leq \frac{\hat{T}'(x)}{\hat{T}(x)} \leq 0$, $0 \leq \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \leq P$ in J for some positive constant P.

Let also,

(i) *y be a solution to the initial value problem (6),*

(ii) *$y_1(x)$ and $y_2(x)$ be solutions to the multiplicative iso-differential inequalities (2) and (3) on J, respectively,*

(iii) *$y_1(x_0) = y_0 = y_2(x_0)$.*

Then for every $x_1 \in J$, $x_1 > x_0$, either $y_1(x_1) < y(x_1)$ ($y(x_1) < y_2(x_1)$) or $y_1(x) = y(x)$ ($y_2(x) = y(x)$) for $\forall x \in [x_0, x_1]$.

Proof. From (12) we have that if $y \geq z$ then

$$-L(y - z) \leq f(x, y) - f(x, z) \leq L(y - z).$$

Therefore all conditions of the last theorem are fulfilled. Consequently

$$y_1(x) \leq y(x) \leq y_2(x) \qquad \text{for} \qquad \forall x \in J.$$

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Also, we have

$$y'(x) - y_1'(x) = y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} - y_1'(x)$$

$$\geq y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

$$-y_1'(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y_1(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

$$= (y(x) - y_1(x))\tfrac{\hat{T}'(x)}{\hat{T}(x)} + (f(x, y(x)) - f(x, y_1(x)))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}$$

$$\geq -(y(x) - y_1(x))P - LP(y(x) - y_1(x))$$

$$= -P(1 + L)(y(x) - y_1(x)),$$

from where

$$(y(x) - y_1(x))' + P(1 + L)(y(x) - y_1(x)) \geq 0,$$

and

$$\Big(e^{P(1+L)x}(y(x)\Big)' \geq 0,$$

From the last inequality, when $x \leq x_1$, we get

$$\int_{x_1}^{x} \Big(e^{P(1+L)x}(y(x)\Big)' dx \leq 0,$$

or

$$(13) \qquad e^{P(1+L)x}(y(x) - y_1(x)) \leq e^{P(1+L)x_1}(y(x_1) - y_1(x_1)).$$

Then, if $y(x_1) = y_1(x_1)$, using (13), we have that for every $x \in [x_0, x_1]$

$$y(x) \leq y_1(x),$$

whereupon

$$y(x) = y_1(x) \qquad \text{for} \qquad \forall x \in [x_0, x_1].$$

□

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Definition 1.6. *A solution $r(x)$ $(\rho(x))$ of the initial value problem* (6) *which exists in $J = [x_0, x_0 + a)$ is said to be maximal(minimal) if for an arbitrary solution $y(x)$ of* (6) *existing in J, the inequality $y(x) \leq r(x)$ $(\rho(x) \leq y(x))$ holds for all $x \in J$.*

Theorem 1.7. *Let $f(x, y)$ be continuous in $\overline{S}_+ = \{(x, y) : \quad x_0 \leq x \leq x_0 + a, \quad |y - y_0| \leq b\}$ and hence there exists a $M > 0$ such that $|f(x, y)| \leq M$ for all $(x, y) \in \overline{S}_+$. Let also, $\hat{T}(x) - x\hat{T}'(x) \geq 0$ in $[x_0, x_0 + a)$, $\frac{|\hat{T}'(x)|}{\hat{T}(x)} \leq P$, $\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \leq P$ in $[x_0, x_0 + a)$. Then there exists a maximal solution $r(x)$ and a minimal solution $\rho(x)$ of the initial value problem* (6) *in the interval $[x_0, x_0 + \alpha)$, where*

$$\alpha = \left\{a, \frac{2b}{2P(b + |y_0| + M) + b}\right\}.$$

Proof. We will prove the existence of a maximal solution.

Let

$$0 < \epsilon \leq \frac{b}{2}.$$

Let us consider the initial value problem

$$y'(x) = y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} + \epsilon \qquad \text{in} \qquad [x_0, x_0 + a), \tag{14}$$

$$y(x_0) = y_0.$$

We define

$$\overline{S}_\epsilon = \{(x, y) \in \mathbb{R}^2 : \quad x_0 \leq x \leq x_0 + a, \quad |y - (y_0 + \epsilon)| \leq \frac{b}{2}\}.$$

We have that

$$\overline{S}_\epsilon \subset \overline{S}_+,$$

because

$$\begin{aligned} \tfrac{b}{2} &\geq |y - (y_0 + \epsilon)| \\ &= |y - y_0 - \epsilon| \\ &\geq |y - y_0| - \epsilon, \end{aligned}$$

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or

$$|y-y_0| \leq \tfrac{b}{2}+\epsilon$$

$$\leq \tfrac{b}{2}+\tfrac{b}{2}$$

$$= b.$$

Also, for every $(x,y) \in \overline{S}_+$ we have

$$\left|y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon\right| \leq |y(x)|\tfrac{|\hat{T}'(x)|}{\hat{T}(x)} + |f(x,y)|\tfrac{|\hat{T}(x)-x\hat{T}'(x)|}{\hat{T}(x)} + \epsilon$$

$$\leq P(b+|y_0|) + MP + \epsilon$$

$$\leq P(b+|y_0|+M) + \tfrac{b}{2}.$$

From here and from the multiplicative iso-Cauchy-Peano's existence theorem it follows that the problem (14) has a solution $y(x,\epsilon)$ which is defined in $[x_0, x_0+\alpha)$.

Let now

$$0 < \epsilon_2 < \epsilon_1 < \epsilon.$$

We have

$$y(x_0,\epsilon_2) = y_0 + \epsilon_2 < y_0 + \epsilon_1 = y(x_0,\epsilon_1),$$

$$y'(x,\epsilon_2) = y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_2,$$

$$y'(x,\epsilon_1) = y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_1$$

$$> y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_2.$$

From here and from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x,\epsilon_2) < y(x,\epsilon_1) \qquad \text{for} \qquad \forall x \in [x_0, x_0+\alpha).$$

Using the proof of the multiplicative iso-Cauchy-Peano's existence theorem we have that the sequence $\{y(x,\epsilon)\}_{\epsilon>0}$ is equip-continuous and uniformly bounded.

Let $\{\epsilon_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that

$$\lim_{n\longrightarrow\infty} \epsilon_n = 0$$

and the corresponding sequence $\{y(x,\epsilon_n)\}_{n=1}^{\infty}$ of solutions of (14) is defined in $[x_0, x_0+\alpha)$.

We have

$$y(x,\epsilon_n) = y_0 + \epsilon_n + \int_{x_0}^{x}\left(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t,y(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt,$$

$$y_0 = y(x_0,0) < y_0 + \epsilon_n,$$

$$y'(x,\epsilon_n) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon_n$$

$$> y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}.$$

From here and from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x) < y(x,\epsilon_n) \qquad \text{in} \qquad [x_0, x_0+\alpha).$$

Consequently

$$y(x) \leq \lim_{n\longrightarrow\infty} y(x,\epsilon_n) := r(x) \qquad \text{for} \qquad \forall x \in [x_0, x_0+\alpha).$$

□

Theorem 1.8. *Let $r(x)$ be a maximal solution to the initial value problem* (6) *in J, $J = [x_0, x_0+a)$. Let also, $y(x)$ be a solution to the multiplicative iso-differential inequality* (2) *in J. If*

$$y(x_0) \leq y_0$$

then

$$y(x) \leq r(x) \qquad \text{in} \qquad J.$$

Proof. Let $x_1 \in [x_0, x_0+a)$. Let also $\epsilon > 0$ be chosen enough small. We consider the problem

(15)
$$y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x)) + \epsilon \qquad \text{in} \qquad J,$$

$$y(x_0) = y_0.$$

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Let $r(x,\epsilon)$ be a maximal solution of the problem (15) in the interval J. We have that

$$\lim_{\epsilon\longrightarrow 0} r(x,\epsilon) = r(x)$$

uniformly in $[x_0, x_1]$.

Since

$$y(x_0) \leq y_0 < y_0 + \epsilon,$$

$$y'(x) \leq y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$< y(x)\tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\tfrac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} + \epsilon$$

and

$$r'(x,\epsilon) = r(x,\epsilon)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,r(x,\epsilon))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)},$$

then from the basic theorem for the multiplicative iso-differential inequalities it follows that

$$y(x) < r(x,\epsilon) \qquad \text{in} \qquad [x_0, x_1],$$

whereupon

$$y(x) \leq \lim_{\epsilon\longrightarrow 0} r(x,\epsilon) = r(x).$$

$\square$

2 Existence and Uniqueness of Solutions

In this chapter $(x_0, y_0) \in \mathbb{R}^2$, D is a domain in $\mathbb{R}^2$ containing the point (x_0, y_0), J is an interval in $\mathbb{R}$ containing x_0, $\hat{T}(x) \in \mathcal{C}^1(J)$, $\hat{T}(x) > 0$ for every $x \in J$.

We begin to develop the theory of existence and uniqueness of solutions of the initial value problem

$$(1') \qquad \Big(\hat{y}^\wedge(\hat{x})\Big)^{\circledast} = \hat{f}^\wedge(\hat{x}, \hat{y}^\wedge(\hat{x})), \qquad x \in J,$$

$$(2) \qquad y(x_0) = y_0,$$

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where f will be assumed to be continuous in the domain D.

The equation $(1')$ can be rewritten in the following form

$$\frac{y'(x)\hat{T}(x)-y(x)\hat{T}'(x)}{\hat{T}(x)\Big(\hat{T}(x)-x\hat{T}'(x)\Big)} = \frac{f(x,y(x))}{\hat{T}(x)}, \qquad x \in J,$$

or

$$y'(x)\hat{T}(x) - y(x)\hat{T}'(x) = f(x,y(x))\Big(\hat{T}(x) - x\hat{T}'(x)\Big), \qquad x \in J,$$

or

$$(1) \qquad y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}, \qquad x \in J.$$

Definition 2.1. *We will say that a function $y(x)$ is a solution to the initial value problem* (1), (2) *if*

1. $y(x_0) = y_0$,

2. $y'(x)$ *exists for all* $x \in J$,

3. *for all* $x \in J$ *the points* $(x, y(x)) \in D$,

4. $y'(x) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$ *for all* $x \in J$.

If $f(x, y(x))$ is not continuous, then the nature of the solutions of (1) is quite arbitrary. For example, let

$$f(x,y(x)) = \frac{4(y(x)-2)}{x(1-x)} - \frac{y(x)}{1-x}, \qquad \hat{T}(x) = e^x,$$

and $(x_0, y(x_0)) = (0, 0)$. Then the equation (1) admits the representation

$$y'(x) = y(x) + \Big(\frac{4(y(x)-2)}{x(1-x)} - \frac{y(x)}{1-x}\Big)(1-x)$$

$$= y(x) + 4\frac{y(x)-2}{x} - y(x)$$

$$= \frac{4}{x}(y(x)-2),$$

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its general solution is

(3) $\qquad y(x) = 2 + Cx^4,$

where C is a constant. From here, we conclude that

$$y(0) = 2 \neq 0,$$

therefore the considered initial value problem has no any solution. If we take $(x_0, y(x_0)) = (0, 2)$, then every function (2) will be a solution of the considered initial value problem.

We shall need the following result to prove existence, uniqueness, and several other properties of the solutions of the initial value problem (1), (2).

Theorem 2.2. *Let $f(x, y(x))$ be continuous function in the domain D, then any solution of the initial value problem* (1), (2) *is also a solution of the integral equation*

(4) $\qquad y(x) = y_0 + \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$

and conversely.

Proof. An integration of the equation (1) yields

$$y(x) - y(x_0) = \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Conversely, if $y(x)$ is any solution of (4), then

$$y(x_0) = y_0,$$

and since $f(x, y(x))$ is a continuous function in D and $\hat{T}$ is a continuous function in J, then $y(x)$ is a continuous function in J and we can differentiate (4), from where we find

$$y'(x) = y(x) \frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x)) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}.$$

□

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We shall solve the integral equation (4) by using the method of successive approximations due to Picard. For this reason, let $y_0(x)$ be any continuous function, we often take $y_0(x) \equiv y_0$, which we will suppose to be initial approximation of the unknown solution of (4), then we define $y_1(x)$ as follows

$$y_1(x) = y_0 + \int_{x_0}^{x} \Big(y_0(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

We pick this $y_1(x)$ as our next approximation and substitute this for $y(x)$ in the right side of (4) and call it $y_2(x)$,

$$y_2(x) = y_0 + \int_{x_0}^{x} \Big(y_1(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Continuing in this way, the $(m+1)$st approximation $y_{m+1}(x)$ is obtained from $y_m(x)$ by means of the relation

$$(5) \qquad y_{m+1}(x) = y_0 + \int_{x_0}^{x} \Big(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt, \qquad m = 0, 1, 2, \ldots.$$

If the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly to a continuous function $y(x)$ in the interval J and for all $x \in J$ the points $(x, y_m(x)) \in D$, then we may pass to the limit in both sides of (5), to obtain

$$\begin{aligned} y(x) &= \lim_{m \longrightarrow \infty} y_{m+1}(x) \\ &= y_0 + \lim_{m \longrightarrow \infty} \int_{x_0}^{x} \Big(y_m(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt \\ &= y_0 + \int_{x_0}^{x} \Big(y(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt, \end{aligned}$$

so that $y(x)$ is the desired solution. Below we will suppose that a and b are positive real numbers. Let P be positive real number such that

$$\frac{|\hat{T}'(x)|}{\hat{T}(x)} \leq P, \qquad \frac{|\hat{T}(x) - x\hat{T}'(x)|}{\hat{T}(x)} \leq P \qquad \text{for} \qquad \forall x \in [x_0 - a, x_0 + a].$$

Theorem 2.3. *Let the following conditions be satisfied*

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(i) *$f(x,y)$ is continuous in the closed rectangle $\overline{S}: |x-x_0| \le a$, $|y-y_0| \le b$ and hence there exists a $M > 0$ such that $|f(x,y)| \le M$ for all $(x,y) \in \overline{S}$,*

(ii) *$f(x,y)$ satisfies a uniform Lipschitz condition*

$$|f(x,y_1) - f(x,y_2)| \le L|y_1 - y_2|$$

for all (x,y_1), (x,y_2) in the closed rectangle $\overline{S}$,

(iii) *$y_0(x)$ is continuous in $|x-x_0| \le a$, and $|y_0(x) - y_0| \le b$.*

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ generated by Picard iterative scheme (5) converges to the unique solution $y(x)$ of the initial value problem (1), (2). This solution is valid in the interval $J_h : |x-x_0| \le h$, where $h = \min\Big\{a, \frac{b}{P(b+|y_0|+M)}\Big\}$. Further, for all $x \in J_h$ the following error estimate holds

$$|y(x) - y_m(x)| \le Ne^{(P+PL)h}\min\Big\{1, \frac{((P+PL)h)^m}{m!}\Big\}, \qquad m = 0,1,2,\ldots, \tag{6}$$

where

$$\max_{x\in J_h} |y_1(x) - y_0(x)| \le N.$$

Remark 2.4. *This Theorem is called a local existence theorem since it guarantees a solution only in the neighborhood of the point (x_0, y_0).*

Proof. We will show that the successive approximations $y_m(x)$ defined by (5) exist as continuous function in J_h and $(x, y_m(x)) \in \overline{S}$ for all $x \in J_h$. Since $y_0(x)$ is a continuous function for all x such that $|x - x_0| \le a$, the function $F_0(x) = f(x, y_0(x))$ is continuous function in J_h, and hence $y_1(x)$ is continuous in J_h.

Also,

$$|y_1(x) - y_0| = \left|\int_{x_0}^{x}\left(y_0(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_0(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_0(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\Big((b + |y_0|)P + MP\Big)dt\right|$$

$$= P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Assuming that the assertion is true for $y_m(x)$, $m \geq 1$, then it is sufficient to prove that it is also true for $y_{m+1}(x)$. For this, since $y_m(x)$ is continuous in J_h, the function $F_m(x) = f(x, y_m(x))$ is also continuous function in J_h. Moreover,

$$|y_{m+1}(x) - y_0| = \left|\int_{x_0}^{x}\left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\Big((b + |y_0|)P + MP\Big)dt\right|$$

$$\leq P(b + |y_0| + M)|x - x_0|$$

$$\leq P(b + |y_0| + M)h$$

$$\leq b.$$

Now we will prove that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ converges uniformly in J_h. Since $y_1(x)$ and $y_0(x)$ are continuous in J_h, there exists a constant $N > 0$ such that

$$|y_1(x) - y_0(x)| \leq N \qquad \text{for} \qquad \forall x \in J_h.$$

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Also, for every $x \in J_h$, we have

$$|y_2(x) - y_1(x)| = \left|\int_{x_0}^{x}\left((y_1(t) - y_0(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_0(t)))\frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_0(t))|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_1(t) - y_0(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_1(t) - y_0(t)|\frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}(NP + LNP)dt\right|$$

$$= NP(1 + L)|x - x_0|.$$

Supposing that

$$|y_m(x) - y_{m-1}(x)| \leq N\frac{\left((P+LP)|x-x_0|\right)^{m-1}}{(m-1)!}, \qquad x \in J_h, \tag{7}$$

for some $m \in \mathbb{N}$.

We will prove that

$$|y_{m+1}(x) - y_m(x)| \leq N\frac{\left((P + LP)|x - x_0|\right)^m}{m!}, \qquad x \in J_h.$$

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Really,

$$|y_{m+1}(x) - y_m(x)|$$

$$= \left|\int_{x_0}^{x}\left((y_m(t) - y_{m-1}(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_m(t)) - f(t, y_{m-1}(t)))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t)) - f(t, y_{m-1}(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}\left(|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + L|y_m(t) - y_{m-1}(t)|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt\right|$$

$$\leq \left|\int_{x_0}^{x}(P + PL)|y_m(t) - y_{m-1}(t)|dt\right|$$

$$\leq N(P + PL)^{m+1}\left|\int_{x_0}^{x}\frac{(t-x_0)^m}{m!}dt\right|$$

$$= N(P + PL)^{m+1}\frac{|x-x_0|^{m+1}}{(m+1)!}.$$

Thus inequality (7) is true for all $m \in \mathbb{N}$.

Next, since

$$N\sum_{m=1}^{\infty}\frac{\left((P+PL)|x-x_0|\right)^{m-1}}{(m-1)!} \leq N\sum_{m=0}^{\infty}\frac{\left((P+PL)h\right)^{m}}{m!}$$

$$= Ne^{(P+PL)h} < \infty,$$

we have that the series

$$y_0(x) + \sum_{m=1}^{\infty}(y_m(x) - y_{m-1}(x))$$

converges absolutely and uniformly in the interval J_h, and hence its partial sums

$$y_1(x), y_2(x), \ldots, y_m(x), \ldots$$

converge to a continuous function in this interval, i.e.,

$$y(x) = \lim_{m\longrightarrow\infty} y_m(x).$$

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As we have seen above we have that $y(x)$ is a solution to the problem (1), (2).

To prove that $y(x)$ is the only solution, we assume that $z(x)$ is also a solution to the initial value problem (1), (2) which exists in the interval J_h and $(x, z(x)) \in \overline{S}$ for all $x \in J_h$. The hypothesis (ii) is applicable and we have

$$|y(x) - z(x)| \leq \left| \int_{x_0}^{x} \left(|y(t) - z(t)| \frac{\hat{T}'(t)}{\hat{T}(t)} + |f(t, y(t)) - f(t, z(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_{x_0}^{x} \left(P|y(t) - z(t)| + LP|y(t) - z(t)| \right) dt \right|$$

$$= (P + LP) \left| \int_{x_0}^{x} |y(t) - z(t)| dt \right|, \qquad x \in J_h.$$

Consequently

$$|y(x) - z(x)| = 0$$

for all $x \in J_h$.

Finally, we will obtain the error bound (6).

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For $n > m$ the inequality (7) gives

$$|y_n(x) - y_m(x)| = |y_n(x) - y_{n-1}(x) + y_{n-1}(x) - y_{n-2}(x) + \cdots + y_{m+1}(x) - y_m(x)|$$

$$\le \sum_{k=m}^{n-1} |y_{k+1}(x) - y_k(x)|$$

$$\le N \sum_{k=m}^{n-1} \frac{\Big((P+LP)|x-x_0|\Big)^k}{k!}$$

$$\le N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!} \tag{8}$$

$$= N\Big((P+PL)h\Big)^m \sum_{k=0}^{n--m-1} \frac{\Big((P+PL)h\Big)^k}{(m+k)!} \qquad \left(\frac{1}{(m+k)!} \le \frac{1}{m!k!}\right)$$

$$\le N\frac{\Big((P+PL)h\Big)^m}{m!} \sum_{k=0}^{n-m-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\le N\frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h},$$

and hence as $n \longrightarrow \infty$, we get

$$|y(x) - y_m(x)| \le N\frac{\Big((P+PL)h\Big)^m}{m!} e^{(P+PL)h}$$

in J_h.

The inequality (8) provides

$$|y_n(x) - y_m(x)| \le N \sum_{k=m}^{n-1} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$\le N \sum_{k=0}^{\infty} \frac{\Big((P+PL)h\Big)^k}{k!}$$

$$= Ne^{(P+PL)h},$$

and as $n \longrightarrow \infty$, we find

$$|y(x) - y_m(x)| \le Ne^{(P+PL)h}$$

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in J_h. ☐

Definition 2.5. *If the solution of the initial value problem* (1), (2) *exists in the entire interval* $|x - x_0| \leq a$, *we say that the solution exists globally.*

The next result is called a global existence theorem.

Theorem 2.6. *Let the following conditions be satisfied*

(i) $f(x, y)$ *is continuous in the strip* $T : |x - x_0| \leq a, |y| < \infty$,

(ii) $f(x, y)$ *satisfies a uniform Lipschitz condition in* T,

(iii) $y_0(x)$ *is continuous in* $|x - x_0| \leq a$.

Then the sequence $\{y_m(x)\}_{m=1}^{\infty}$ *generated by Picard iterative scheme exists in the entire interval* $|x-x_0| \leq a$, *and converges to the unique solution* $y(x)$ *of the initial value problem* (1), (2).

Proof. For any continuous function $y_0(x)$ in $|x - x_0| \leq a$, as in the proof of the local existence Theorem, can be established the existence of each $y_m(x)$ in $|x - x_0| \leq a$ satisfying $|y_m(x)| < \infty$. Also, as in the proof of the previous Theorem we have that the sequence $\{y_m(x)(\}_{m=1}^{\infty}$ converges to $y(x)$ in $|x - x_0| \leq a$, replacing h by a throughout the proof and recalling that the function $f(x, y)$ satisfies the Lipschitz condition in the strip T. ☐

Corollary 2.7. *Let* $f(x, y)$ *be continuous in* $\mathbb{R}^2$ *and satisfies a uniform Lipschitz condition in each strip* $T_a : |x| \leq a, |y| < \infty$, *with the Lipschitz constant* L_a. *Then the initial value problem* (1), (2) *has a unique solution which exists for all* x.

Proof. For any x there exists an $a > 0$ such that $|x-x_0| \leq a$. From here and from $T \subset T_{a+|x_0|}$, it follows that the function $f(x, y)$ satisfies the conditions of the previous Theorem in the strip T. Hence, the result follows for any x. ☐

We will note that there exist positive constants M_1 and M_2 such that

$$\Big|\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \leq M_1, \quad \Big|1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big| \leq M_2 \quad \text{for} \quad x \in [x_0 - a, x_0 + a].$$

Theorem 2.8. *(multiplicative iso-Peano's existence theorem) Let f is defined, continuous and bounded function on the strip $T = \{(x, y) \in \mathbb{R}^2 : |x - x_0| \leq a, |y| < \infty\}$. Then the Cauchy problem* (1), (2) *has a bounded solution $y(x)$ which is defined on $|x - x_0| \leq a$ and*

$$|y(x)| \leq \Big(1 + e^{aM_1}\Big)(|y_0| + \sup_{(x,y)\in V} |f(x, y)|M_2 a) \quad \text{for} \quad \forall x \in [x_0 - a, x_0 + a].$$

Remark 2.9. *We can consider our main result as a continuation of the well - known Peano's Theorem.*

If we put

$$g(x, y) = y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x))\Big(1 - x\frac{\hat{T}'(x)}{\hat{T}(x)}\Big),$$

then g is unbounded function on the strip T. Therefore we can not apply the classical Peano's Theorem for the Cauchy problem (1), (2), *because g has to be bounded on T.*

Proof. Since f is a bounded function on T then there exists a positive constant M such that

$$|f(x, y)| \leq M \quad \text{for} \quad (x, y) \in T.$$

We will prove our main result for $x \in [x_0, x_0 + a]$. In the same way one can prove the main result for $x \in [x_0 - a, x_0]$.

For $x \in [x_0, x_0 + a]$ we define the sequence $\{y_m(x)\}_{m=1}^{\infty}$ as follows

$$y_m(x) = y_0 \quad \text{for} \quad x \in \Big[x_0, x_0 + \tfrac{a}{m}\Big],$$

$$y_m(x) = y_0 + \textstyle\int_{x_0}^{x-\frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt \quad \text{for}$$

$$x \in \Big[x_0 + k\tfrac{a}{m}, x_0 + (k + 1)\tfrac{a}{m}\Big], \quad k = 1, 2, \ldots, m - 1.$$

For this sequence we have

1. Let $m \in \mathbb{N}$ is arbitrary chosen.

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If $x \in \left[x_0, x_0 + \frac{a}{m}\right]$ then

$$|y_m(x)| = |y_0|.$$

If $x \notin \left[x_0, x_0 + \frac{a}{m}\right]$ and $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ for some $k = 1, 2, \ldots, m-1$, then

$$|y_m(x)| = \left|y_0 + \int_{x_0}^{x-\frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq |y_0| + \int_{x_0}^{x-\frac{a}{m}} (M_1|y_m(t)| + MM_2)dt$$

$$= |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt + MM_2\left(x_0 + (k+1)\frac{a}{m} - \frac{a}{m} - x_0\right)$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt,$$

i.e. for $x \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right]$ we have

$$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x-\frac{a}{m}} |y_m(t)|dt$$

$$\leq |y_0| + MM_2a + M_1 \int_{x_0}^{x} |y_m(t)|dt.$$

From here and the Gronwall's inequality we get

$|y_m(x)| \leq |y_0| + MM_2a + M_1 \int_{x_0}^{x} (|y_0| + MM_2a)e^{M_1(x-t)}dt$

$= |y_0| + MM_2a + e^{M_1x}M_1(|y_0| + MM_2a) \int_{x_0}^{x} e^{-M_1t}dt$

$= |y_0| + MM_2a + e^{M_1x}(|y_0| + MM_2a)\Big(e^{-M_1x_0} - e^{-M_1x}\Big)$

$\leq |y_0| + MM_2a + e^{M_1(x-x_0)}(|y_0| + MM_2a)$

$\leq |y_0| + MM_2a + e^{aM_1}(|y_0| + MM_2a)$

$= (1 + e^{aM_1})(|y_0| + MM_2a) =: M_3 \quad \text{for} \quad x \in \Big[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\Big],$

for some $k = 1, 2, \ldots, m-1$.

Consequently for every $x \in [x_0, x_0 + a]$ we have

(9) $\qquad |y_m(x)| \leq M_3$

for every $m \in \mathbb{N}$.

Therefore the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded on $[x_0, x_0 + a]$.

2. Let $x_1, x_2 \in [x_0, x_0 + a]$ and $m \in \mathbb{N}$ is arbitrarily chosen. Then

1. case. $x_1, x_2 \in \Big[x_0, x_0 + \frac{a}{m}\Big]$. Then

$$y_m(x_1) = y_m(x_2) = y_0,$$

and therefore

$$|y_m(x_2) - y_m(x_1)| = 0.$$

2. case. Let $x_1 \in \Big[x_0, x_0 + \frac{a}{m}\Big]$, $x_2 \notin \Big[x_0, x_0 + \frac{a}{m}\Big]$. Then there exists $k \in \{1, 2, \ldots,$
$m-1\}$, such that $x_2 \in \Big[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\Big]$ and

$y_m(x_1) = y_0,$

$y_m(x_2) = y_0 + \int_{x_0}^{x_2 - \frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt,$

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from here,

$$|y_m(x_2) - y_m(x_1)| = \left|\int_{x_0}^{x_2 - \frac{a}{m}} \left(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\left(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right)\right)dt\right|$$

$$\leq \int_{x_0}^{x_2 - \frac{a}{m}} \left(|y_m(t)|\left|\frac{\hat{T}'(t)}{\hat{T}(t)}\right| + |f(t, y_m(t))|\left|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\right|\right)dt$$

$$\leq MM_2\left(x_2 - \frac{a}{m} - x_0\right) + M_1 \int_{x_0}^{x_2 - \frac{a}{m}} |y_m(t)|dt$$

now we use that $x_1 \in \left[x_0, x_0 + \frac{a}{m}\right]$

$$\leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2 - \frac{a}{m}} |y_m(t)|dt,$$

i.e.

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1 \int_{x_0}^{x_2 - \frac{a}{m}} |y_m(t)|dt.$$

From here and (9) we obtain

$$|y_m(x_2) - y_m(x_1)| \leq MM_2(x_2 - x_1) + M_1M_3 \int_{x_0}^{x_2 - \frac{a}{m}} dt$$

$$= MM_2(x_2 - x_1) + M_1M_3\left(x_2 - \frac{a}{m} - x_0\right)$$

$$\leq (MM_2 + M_1M_3)(x_2 - x_1).$$

3. case. Let $x_1, x_2 \notin \left[x_0, x_0 + \frac{a}{m}\right]$. Without loss of generality we can suppose that $x_1 \leq x_2$. Let

$$x_1 \in \left[x_0 + k\frac{a}{m}, x_0 + (k+1)\frac{a}{m}\right], \quad x_2 \in \left[x_0 + i\frac{a}{m}, x_0 + (i+1)\frac{a}{m}\right], \quad k \leq i,$$

$k, i \in \{1, 2, \ldots, m-1\}$.

Then

$$y_m(x_2) = y_0 + \int_{x_0}^{x_2-\frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt,$$

$$y_m(x_1) = y_0 + \int_{x_0}^{x_1-\frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt,$$

$$y_m(x_2) - y_m(x_1) = \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt,$$

$$|y_m(x_2) - y_m(x_1)| = \Big|\int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt\Big|$$

$$\leq \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} \Big(|y_m(t)|\Big|\frac{\hat{T}'(t)}{\hat{T}(t)}\Big| + |f(t, y_m(t))|\Big|1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big|\Big)dt$$

$$\leq M_1 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} |y_m(t)|dt + MM_2 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} dt$$

$$= M_1 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} |y_m(t)|dt + MM_2(x_2 - x_1)$$

now we apply (9)

$$\leq M_1M_3 \int_{x_1-\frac{a}{m}}^{x_2-\frac{a}{m}} dt + MM_2(x_2 - x_1)$$

$$= (M_1M_3 + MM_2)(x_2 - x_1).$$

From 1, 2, and 3 cases follows that for every $x_1, x_2 \in [x_0, x_0 + a]$ we have

$$|y_m(x_2) - y_m(x_1)| \leq (M_1M_3 + MM_2)|x_2 - x_1| \quad \text{for} \quad \forall m \in \mathbb{N}. \tag{10}$$

Let $\epsilon > 0$ is arbitrary chosen and fixed. Let $\delta = \frac{\epsilon}{MM_2+M_1M_3}$. Then if $x_1, x_2 \in [x_0, x_0 + a]$, $|x_1 - x_2| < \delta$, using (10), we get

$$|y_m(x_2) - y_m(x_1)| \leq (M_1M_3 + MM_2)|x_2 - x_1|$$

$$< (M_1M_3 + MM_2)\delta = \epsilon.$$

Consequently $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous family on $[x_0, x_0 + a]$.

Therefore there exists a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ of the sequence $\{y_m(x)\}_{m=1}^{\infty}$ which is uniformly convergent to $y(x)$ on $[x_0, x_0 + a]$.

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For $y_{m_p}(x)$, $x \in [x_0, x_0 + a]$, we have

$$y_{m_p}(x) = y_0 + \int_{x_0}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$(11) \quad = y_0 + \int_{x_0}^{x} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

$$+ \int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt.$$

Since f is a continuous and bounded function on T we have

$$(12) \quad \begin{aligned} &\lim_{p\longrightarrow\infty} \int_{x_0}^{x} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt \\ &= \int_{x_0}^{x} \Big(y(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt. \end{aligned}$$

Also,

$$\Big|\int_{x}^{x-\frac{a}{m_p}} \Big(y_{m_p}(t)\tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_{m_p}(t))\Big(1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt\Big|$$

$$\leq \int_{x-\frac{a}{m_p}}^{x} \Big(|y_{m_p}(t)|\Big|\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big| + |f(t, y_{m_p}(t))|\Big|1 - t\tfrac{\hat{T}'(t)}{\hat{T}(t)}\Big|\Big)dt$$

$$\leq M_1 \int_{x-\frac{a}{m_p}}^{x} |y_{m_p}(t)|dt + MM_2\tfrac{a}{m_p}$$

now we use (9)

$$\leq (M_1M_3 + MM_2)\frac{a}{m_p} \longrightarrow_{p\longrightarrow\infty} 0.$$

From here and (11), (12), when $p \longrightarrow \infty$, we get

$$y(x) = y_0 + \int_{x_0}^{x} \Big(y(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y(t))\Big(1 - t\frac{\hat{T}'(t)}{\hat{T}(t)}\Big)\Big)dt$$

for every $x \in [x_0, x_0 + a]$. Therefore y is a solution of the Cauchy problem (1), (2) which is defined on $[x_0, x_0 + a]$. From (9) follows that $|y(x)| \leq M_3$ for every $x \in [x_0, x_0 + a]$. □

Corollary 2.10. *Let $f(x, y)$ be continuous in $\overline{S}$, and hence there exists a $M > 0$ such that $|f(x, y)| \leq M$ for all $(x, y) \in \overline{S}$. Then the initial value problem* (1), (2) *has at least one solution in J_h.*

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Proof. The proof is the same as that of the proof of multiplicative iso-Peano's existence theorem with some obvious changes. □

Definition 2.11. *(ϵ-approximate solution) A function $y(x)$ defined in J is said to be an ϵ-approximate solution of the multiplicative iso-differential equation* (1) *if*

1. *$y(x)$ is continuous for all $x \in J$,*

2. *for all $x \in J$ the points $(x, y(x)) \in D$,*

3. *$y(x)$ has piecewise continuous derivative in J which may fail to be defined only for a finite number of points, say x_1, x_2, ..., x_k,*

4. $\left|y'(x) - y(x)\frac{\hat{T}'(x)}{\hat{T}(x)} - f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\right| \leq \epsilon$ *for all $x \in J$, $x \neq x_i$, $i = 1, 2, \ldots, k$.*

The existence of an ϵ-approximate solution is provided in the following theorem.

Theorem 2.12. *Let $f(x,y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x,y)| \leq M$ for every $(x,y) \in \overline{S}$. Then for all $\epsilon > 0$, there exists an ϵ-approximate solution $y(x)$ of the multiplicative iso-differential equation* (1) *in the interval J_h such that $y(x_0) = y_0$.*

Proof. Because the function $f(x,y)$ is a continuous function in the closed rectangle $\overline{S}$, it is uniformly continuous in this rectangle. Thus, for a given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ so that

$$|f(x,y) - f(x_1,y_1)| \leq \epsilon,$$

$$\left|y\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x,y)\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)} - y_1\frac{\hat{T}'(x_1)}{\hat{T}(x_1)} - f(x_1,y_1)\frac{\hat{T}(x_1)-x_1\hat{T}'(x_1)}{\hat{T}(x_1)}\right| \leq \epsilon$$

for all (x,y), $(x_1,y_1) \in \overline{S}$ such that

$$|x - x_1| \leq \delta \qquad \text{and} \qquad |y - y_1| \leq \delta.$$

We shall construct an ϵ-approximate solution in the interval $[x_0, x_0 + h]$. A similar process will define it in the interval $[x_0 - h, x_0]$.

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For this aim, we divide the interval $[x_0, x_0+h]$ into m parts

$$x_0 < x_1 < x_2 \ldots < x_m = x_0 + h$$

such that

$$x_i - x_{i-1} \le \min\Big\{\delta, \tfrac{\delta}{P(|y_0|+b+M)}\Big\}, \qquad i = 1, 2, \ldots, m. \tag{13}$$

Now we define a function $y(x)$ in the interval $[x_0, x_0+h]$ in the following manner

$$\begin{aligned} &y(x) = y(x_{i-1}) + (x - x_{i-1})\Big(y(x_{i-1})\tfrac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1}))\tfrac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\Big), \\ &x_{i-1} \le x \le x_i, \qquad i = 1, 2, \ldots, m. \end{aligned} \tag{14}$$

Obviously, this function $y(x)$ is continuous and has a piecewise continuous derivative

$$y'(x) = y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})} + f(x_{i-1}, y(x_{i-1})\frac{\hat{T}(x_{i-1}) - x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})},$$

$x_{i-1} < x < x_i$, $i = 1, 2, \ldots, m$, which fails to be defined only at the points x_i, $i = 1, 2, \ldots, m-1$. Since in each subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, the function $y(x)$ is a straight line, to prove that $(x, y(x)) \in \overline{S}$ it suffices to show that

$$|y(x_i) - y_0| \le b$$

for all $i = 1, 2, \ldots, m$.

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For this reason, in (14) let $i = 1$ and $x = x_1$ to obtain

$$y(x_1) = y_0 + (x - x_1)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big),$$

$$|y(x_1) - y_0| = \Big|(x - x_1)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big)\Big|$$

$$\leq (x_1 - x_0)\Big(|y_0|\tfrac{|\hat{T}'(x_0)|}{\hat{T}(x_0)} + |f(x_0, y_0)|\tfrac{|\hat{T}(x_0) - x_0\hat{T}'(x_0)|}{\hat{T}(x_0)}\Big)$$

$$\leq (x_1 - x_0)(P|y_0| + MP)$$

$$\leq hP(M + |y_0|)$$

$$\leq hP(b + |y_0| + M)$$

$$\leq \tfrac{b}{P(b+|y_0|+M)} P(b + |y_0| + M)$$

$$= b.$$

Now let the assertion be true for $i = 1, 2, \ldots, k-1 < m-1$, then from (14)

$$y(x_1) - y_0 = (x_1 - x_0)\Big(y_0 \tfrac{\hat{T}'(x_0)}{\hat{T}(x_0)} + f(x_0, y_0)\tfrac{\hat{T}(x_0) - x_0\hat{T}'(x_0)}{\hat{T}(x_0)}\Big),$$

$$y(x_2) - y(x_1) = (x_2 - x_1)\Big(y(x_1)\tfrac{\hat{T}'(x_1)}{\hat{T}(x_1)} + f(x_1, y(x_1))\tfrac{\hat{T}(x_1) - x_1\hat{T}'(x_1)}{\hat{T}(x_1)},$$

$\ldots$

$$y(x_k) - y(x_{k-1}) = (x_k - x_{k-1})\Big(y(x_{k-1})\tfrac{\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})} + f(x_{k-1}, y(x_{k-1}))\tfrac{\hat{T}(x_{k-1}) - x_{k-1}\hat{T}'(x_{k-1})}{\hat{T}(x_{k-1})}\Big).$$

From here,

$$y(x_k) - y_0 = \textstyle\sum_{l=1}^{k}(x_l - x_{l-1})\Big(y(x_{l-1})\tfrac{\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})} + f(x_{l-1}, y(x_{l-1}))\tfrac{\hat{T}(x_{l-1}) - x_{l-1}\hat{T}'(x_{l-1})}{\hat{T}(x_{l-1})}\Big),$$

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which gives

$$|y(x_k)-y_0| \le \sum_{l=1}^{k}(x_l-x_{l-1})\Big(|y(x_{l-1})|\frac{|\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})}+|f(x_{l-1},y(x_{l-1}))|\frac{|\hat{T}(x_{l-1})-x_{l-1}\hat{T}'(x_{l-1})|}{\hat{T}(x_{l-1})}\Big)$$

$$\le \sum_{l=1}^{k}(x_l-x_{l-1})\Big((b+|y_0|)P+MP\Big)$$

$$= P(M+b+|y_0|)\sum_{l=1}^{k}(x_l-x_{l-1})$$

$$= P(M+b+|y_0|)(x_k-x_0)$$

$$\le P(M+b+|y_0|)h$$

$$\le P(M+b+|y_0|)\frac{b}{P(M+b+|y_0|)}$$

$$= b.$$

Finally, if $x_{i-1} < x < x_i$, then from (13) and (14)

$$|y(x)-y(x_{i-1})| = (x-x_i)\Big|y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}+f(x_{i-1}),y(x_{i-1}))\frac{\hat{T}(x_{i-1})-x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}\Big|$$

$$\le (x-x_i)\Big(|y(x_{i-1})|\frac{|\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})}+|f(x_{i-1}),y(x_{i-1}))|\frac{|\hat{T}(x_{i-1})-x_{i-1}\hat{T}'(x_{i-1})|}{\hat{T}(x_{i-1})}\Big)$$

$$\le (x_i-x_{i-1})\Big((|y_0|+b)P+MP\Big)$$

$$\le \frac{\delta}{P(|y_0|+b+M)}P(M+|y_0|+b)$$

$$= \delta.$$

Therefore

$$\Big|y'(x)-y(x)\frac{\hat{T}'(x)}{\hat{T}(x)}-f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\Big|$$

$$= \Big|y(x_{i-1})\frac{\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}+f(x_{i-1},y(x_{i-1}))\frac{\hat{T}(x_{i-1})-x_{i-1}\hat{T}'(x_{i-1})}{\hat{T}(x_{i-1})}-y(x)\frac{\hat{T}'(x)}{\hat{T}(x)}-f(x,y(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}\Big|$$

$$\le \epsilon$$

for all $x \in J_h$, $x \neq x_i$, $i = 1, 2, \ldots, m-1$. This completes the proof that $y(x)$ is an ϵ-approximate solution of the multiplicative iso-differential equation (1).

This method of constructing an approximate solution is said to be multiplicative iso-Cauchy-Euler method. □

Theorem 2.13. *(multiplicative iso-Cauchy-Peano's existence theorem) Let $f(x, y)$ be continuous in $\overline{S}$ and hence there exists a $M > 0$ such that $|f(x, y)| \leq M$ for every $(x, y) \in \overline{S}$. Then the initial value problem* (1), (2) *has at least one solution in J_h.*

Proof. We shall prove the assertion for the interval $[x_0, x_0 + h]$.

Let $\{\epsilon_m\}_{m=1}^{\infty}$ be a monotonically decreasing sequence of positive numbers such that

$$\lim_{m \longrightarrow \infty} \epsilon_m = 0.$$

For each ϵ_m we construct an ϵ_m-approximate solution $y_m(x)$.

As in the proof of the theorem for existence of ϵ-approximate solutions we have

$$|y_m(x)| \leq b + |y_0|$$

for every $m \in \mathbb{N}$ and for every $x \in J_h$. In other words, the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is uniformly bounded in J_h.

Let x, $x^* \in [x_0, x_0 + h]$. Then

$$|y_m(x) - y_m(x^*)| \leq \left| \int_x^{x^*} \left(|y_m(t)| \frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_m(t))| \frac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \right) dt \right|$$

$$\leq \left| \int_x^{x^*} \Big(P(|y_0| + b) + MP \Big) dt \right|$$

$$\leq P(M + b + |y_0|)|x - x^*|$$

and from this it follows that the sequence $\{y_m(x)\}_{m=1}^{\infty}$ is equip-continuous.

Consequently the sequence $\{y_m(x)\}_{m=1}^{\infty}$ contains a subsequence $\{y_{m_p}(x)\}_{p=1}^{\infty}$ which converges uniformly in $[x_0, x_0 + h]$ to a continuous function $y(x)$. We define

$$e_m(x) = \begin{cases} y_m'(x) - y_m(x) \frac{\hat{T}'(x)}{\hat{T}(x)} - f(x, y_m(x)) \frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)} \\ \text{at the points where } y_m'(x) \text{ exists} \\ 0 \quad \text{otherwise.} \end{cases}$$

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Then

(15) $$y_m(x) = y_0 + \int_{x_0}^{x}\Big(y_m(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_m(t)) + e_m(t)\Big)dt$$

and

$$|e_m(x)| \le \epsilon_m.$$

Since $f(x,y)$ is continuous in $\overline{S}$ and $y_{m_p}(x)$ converges to $y(x)$ uniformly in $[x_0, x_0+h]$, the function $f(x, y_{m_p}(x))$ converges to $f(x, y(x))$ uniformly in $[x_0, x_0+h]$. Thus, by replacing m by m_p in (15) and letting $p \longrightarrow \infty$, we become that $y(x)$ is a solution to the integral equation (4). □

Remark 2.14. *We suppose that all conditions of the multiplicative iso-Cauchy-Peano's existence theorem are satisfied. Further, let the initial value problem* (1), (2) *has a solution $y(x)$ in an interval $J = (\alpha, \beta)$. We have*

$$|y(x_2) - y(x_1)| \le P(M + |y_0| + b)|x_2 - x_1|$$

for every x_1, $x_2 \in J$. Therefore

$$y(x_2) - y(x_1) \longrightarrow 0$$

as $x_1, x_2 \longrightarrow \alpha^+$. Thus, by the Cauchy criterion of convergence we have that

$$\lim_{x \longrightarrow \alpha^+} y(x)$$

exists.

A similar argument holds for

$$\lim_{x \longrightarrow \beta^-} y(x).$$

Theorem 2.15. *Let all conditions of the multiplicative iso-Cauchy-Peano's existence theorem be satisfied. Let also, $y(x)$ be a solution of the initial value problem* (1), (2) *in the interval $J = (\alpha, \beta)$. Then $y(x)$ can be extended over the interval $(\alpha, \beta + \gamma]$ ($[\alpha - \gamma, \beta)$) for some $\gamma > 0$.*

Proof. We define the function $y_1(x)$ as follows.

$$y_1(x) = y(x) \qquad \text{for} \qquad x \in (\alpha, \beta),$$

$$y_1(\beta) = y(\beta - 0).$$

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We observe that for all $x \in (\alpha, \beta]$ we have

$$y_1(x) = y(\beta - 0) + \int_\beta^x \Big(y_1(t) \tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \tfrac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y(x_0) + \int_{x_0}^\beta \Big(y_1(t) \tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \tfrac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$+ \int_\beta^x \Big(y_1(t) \tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \tfrac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y(x_0) + \int_{x_0}^x \Big(y_1(t) \tfrac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t)) \tfrac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

Therefore the left-hand derivative $y_1'(\beta - 0)$ exists and

$$y_1'(\beta - 0) = y_1(\beta) \frac{\hat{T}'(\beta)}{\hat{T}(\beta)} + f(\beta, y_1(\beta)) \frac{\hat{T}(\beta) - \beta\hat{T}'(\beta)}{\hat{T}(\beta)}.$$

Thus, $y_1(x)$ is a continuation of $y(x)$ in the interval $(\alpha, \beta]$.

Let $y_2(x)$ be a solution to the problem

$$y'(x) = y(x) \tfrac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y(x)) \tfrac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)},$$

$$y(\beta) = y_1(\beta),$$

existing in the interval $[\beta, \beta + \gamma]$ for some $\gamma > 0$.

We define the function

$$y_3(x) = \begin{cases} y_1(x) & x \in (\alpha, \beta], \\ y_2(x) & x \in [\beta, \beta + \gamma], \end{cases}$$

which is a continuation of $y(x)$ in the interval $(\alpha, \beta + \gamma]$.

Also,

$$y_3(x) = y_0 + \int_{x_0}^x \Big(y_3(t) \frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t)) \frac{\hat{T}(t) - t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

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for every $x \in (\alpha, \beta+\gamma]$, because for all $x \in [\beta, \beta+\gamma]$ we have

$$y_3(x) = y(\beta-0) + \int_\beta^x \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y_0 + \int_{x_0}^\beta \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}x(t)} \Big) dt$$

$$+ \int_\beta^x \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt$$

$$= y_0 + \int_{x_0}^x \Big(y_3(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_3(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt.$$

□

Theorem 2.16. *(multiplicative iso-Lipschitz uniqueness theorem) Let $f(x, y)$ be continuous and satisfies a uniform Lipschitz condition in $\overline{S}$ with a Lipschitz constant L. Then the problem* (1), (2) *has at most one solution in $|x - x_0| \le a$.*

Proof. We suppose that the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$, $x \in [x_0 - a, x_0 + a]$. Then

$$y_1(x) = y_0 + \int_{x_0}^x \Big(y_1(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_1(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

$$y_2(x) = y_0 + \int_{x_0}^x \Big(y_2(t)\frac{\hat{T}'(t)}{\hat{T}(t)} + f(t, y_2(t))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

whereupon

$$y_1(x)-y_2(x) = \int_{x_0}^x \Big((y_1(t)-y_2(t))\frac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t))-f(t, y_2(t)))\frac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)} \Big) dt,$$

and

$$|y_1(x) - y_2(x)| \le \Big| \int_{x_0}^x \Big(|y_1(t) - y_2(t)|\frac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))|\frac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)} \Big) dt \Big|$$

$$\le \Big| \int_{x_0}^x \Big(P|y_1(t) - y_2(t)| + LP|y_1(t) - y_2(t)| \Big) dt$$

$$= P(1+L) \Big| \int_{x_0}^x |y_1(t) - y_2(t)| dt \Big|.$$

From the last inequality and Gronwall's type inequality we conclude that

$$|y_1(x) - y_2(x)| = 0 \qquad \text{in} \qquad [x_0 - a, x_0 + a].$$

□

Theorem 2.17. *(multiplicative iso-Peano's uniqueness theorem) Let $f(x, y)$ be continuous in*

$$\overline{S}_+ = \{(x, y) \in \mathbb{R}^2 : x_0 \le x \le x_0 + a, |y - y_0| \le b\}$$

and nonincreasing in y for all $[x_0, x_0 + a]$. Let also,

$$\hat{T}'(x) \le 0, \qquad \hat{T}(x) - x\hat{T}'(x) \ge 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) *has at most one solution in $[x_0, x_0 + a]$.*

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0 + a]$ which differ in $[x_0, x_0 + a]$. We assume that

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

while $y_1(x) = y_2(x)$ for $x \in [x_0, x_1]$, i.e., x_1 is the greatest lower bound of the set A consisting of those x for which $y_2(x) > y_1(x)$. This greatest lower bound of the set A exists because the set A is bounded below by x_0 at least. Thus for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_1(x)) \ge f(x, y_2(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \ge f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

$$\ge y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} + f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}}$$

for all $x \in (x_1, x_1 + \epsilon)$, and from here

$$y_1'(x) \ge y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon).$$

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Hence the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$.

Because

$$z(x_1) = y_2(x_1) - y_1(x_1) = 0$$

we obtain

$$z(x) \leq z(x_1) = 0 \qquad \text{in} \qquad (x_1, x_1 + \epsilon)$$

or

$$y_2(x) \leq y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon).$$

This contradiction proves that

$$y_1(x) = y_2(x) \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

□

Theorem 2.18. *(multiplicative iso-Peano's uniqueness theorem) Let $f(x, y)$ be continuous in $\overline{S}_+$ and nondecreasing in y for every $x \in [x_0, x_0 + a]$. Let also,*

$$\hat{T}'(x) \leq 0, \qquad \hat{T}(x) - x\hat{T}'(x) \leq 0 \qquad \text{for} \qquad \forall x \in [x_0, x_0 + a].$$

Then the problem (1), (2) *has at most one solution in* $[x_0, x_0 + a]$.

Proof. Let the problem (1), (2) has two solutions $y_1(x)$ and $y_2(x)$ in $[x_0, x_0 + a]$ which differ in $[x_0, x_0 + a]$. Let

$$y_2(x) > y_1(x) \qquad \text{in} \qquad (x_1, x_1 + \epsilon) \subset [x_0, x_0 + a],$$

and

$$y_2(x) = y_1(x) \qquad \text{for} \qquad \forall x \in [x_0, x_1].$$

Therefore for every $x \in (x_1, x_1 + \epsilon)$ we have

$$f(x, y_2(x)) \geq f(x, y_1(x)),$$

$$f(x, y_1(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}} \geq f(x, y_2(x))^{\frac{\hat{T}(x) - x\hat{T}'(x)}{\hat{T}(x)}},$$

$$y_1(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}} \geq y_2(x)^{\frac{\hat{T}'(x)}{\hat{T}(x)}},$$

whereupon

$$y_1(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_1(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

$$\geq y_2(x)\frac{\hat{T}'(x)}{\hat{T}(x)} + f(x, y_2(x))\frac{\hat{T}(x)-x\hat{T}'(x)}{\hat{T}(x)}$$

for every $x \in (x_1, x_1 + \epsilon)$. Consequently

$$y_1'(x) \geq y_2'(x) \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon)$$

and then the function

$$z(x) = y_2(x) - y_1(x)$$

is nonincreasing function in $(x_1, x_1 + \epsilon)$, therefore

$$y_2(x) - y_1(x) \leq y_2(x_1) - y_1(x_1) = 0 \qquad \text{for} \qquad \forall x \in (x_1, x_1 + \epsilon),$$

which is a contradiction. From here we conclude that $y_1(x) = y_2(x)$ for every $x \in [x_0, x_0 + a]$. □

Lemma 2.19. *(multiplicative iso-Osgood's lemma) Let $w(z)$ be continuous function in $[0, \infty)$, $w(0) = 0$, $z + w(z) > 0$ in $(0, \infty)$, $z + w(z)$ be increasing function in $[0, \infty)$, and*

(16) $\qquad \lim_{\epsilon \longrightarrow 0^+} \int_\epsilon^a \frac{dz}{z+w(z)} = \infty.$

Let $u(x)$ be a nonnegative continuous function in $[0, a]$. Then the inequality

(17) $\qquad u(x) \leq P\int_0^x (u(t) + w(u(t)))dt, \qquad 0 < x \leq a,$

implies that $u(x) \equiv 0$ in $[0, a]$.

Proof. We define the function

$$v(x) = \max_{0 \leq t \leq x} u(t)$$

and assume that $v(x) > 0$ for $0 < x \leq a$. Then

$$u(t) \leq v(x) \qquad \text{for} \qquad \forall t \in [0, x].$$

Because $u(x)$ is a continuous function in $[0, a]$ then there exists $x_1 \in [0, x]$ such that

$$v(x) = u(x_1).$$

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Therefore, using that $z + w(z)$ is an increasing function in $[0, \infty)$,

$$v(x) = u(x_1) \leq P \int_0^{x_1} (u(t) + w(u(t)))dt$$

$$\leq P \int_0^{x_1} (v(t) + w(v(t)))dt$$

$$\leq P \int_0^{x} (v(t) + w(v(t)))dt.$$

Let

$$\overline{v}(x) = P \int_0^x (v(t) + w(v(t)))dt.$$

We have

$$\overline{v}(x) \geq 0, \qquad v(x) \leq \overline{v}(x),$$

and

$$\overline{v}'(x) = P(v(x) + w(v(x)))$$

$$\leq P(\overline{v}(x) + w(\overline{v}(x))),$$

and since

$$\overline{v}(x) + w(\overline{v}(x)) \geq 0,$$

then

$$\frac{\overline{v}'(x)}{P(\overline{v}(x) + w(\overline{v}(x)))}.$$

Consequently for $0 < \delta < a$ we have

$$\int_\delta^a \frac{d\overline{v}(x)}{P(\overline{v}(x) + w(\overline{v}(x)))} \leq \int_\delta^a dx,$$

whereupon

$$\lim_{\delta \longrightarrow 0^+} \int_\delta^a \frac{d\overline{v}(x)}{P(\overline{v}(x)+w(\overline{v}(x)))} = \lim_{\delta \longrightarrow 0^+} \int_{\overline{v}(\delta)}^{\overline{v}(a)} \frac{dy}{P(y+w(y))}$$

$$\leq a,$$

which contradicts with (16). Consequently $u(x) \equiv 0$ in $[0, a]$. □

Theorem 2.20. *(multiplicative iso-Osgood's uniqueness theorem) Let $f(x, y)$ be continuous in $\overline{S}_+$ and for all (x, y_1), $(x, y_2) \in \overline{S}_+$ it satisfies*

$$|f(x, y_1) - f(x, y_2)| \leq w(|y_1 - y_2|),$$

where $w(z)$ satisfies all conditions of the multiplicative iso-Osgood's lemma. Then the problem (1), (2) has at most one solution in $[x_0, x_0 + a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0, x_0 + a]$. Then, if

$$z(x) = |y_1(x) - y_2(x)|, \qquad x \in [x_0, x_0 + a],$$

we have

$$z(x) = \left|\int_{x_0}^{x}\left((y_1(t) - y_2(t))\tfrac{\hat{T}'(t)}{\hat{T}(t)} + (f(t, y_1(t)) - f(t, y_2(t)))\tfrac{\hat{T}(t)-t\hat{T}'(t)}{\hat{T}(t)}\right)dt\right|$$

$$\leq \int_{x_0}^{x}\left(|y_1(t) - y_2(t)|\tfrac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))|\tfrac{|\hat{T}(t)-t\hat{T}'(t)|}{\hat{T}(t)}\right)dt$$

$$\leq \int_{x_0}(P|y_1(t) - y_2(t)| + Pw(|y_1(t) - y_2(t)|))dt$$

$$= P\int_{x_0}^{x}(z(t) + w(z(t)))dt.$$

Let

$$u(x) = z(x_0 + x).$$

Therefore

$$u(x) \leq P\int_{x_0}^{x_0+x}(z(t) + w(z(t)))dt$$

$$= P\int_{0}^{x}(z(x_0 + t) + w(z(x_0 + t)))dt$$

$$= P\int_{0}^{x}(u(t) + w(u(t)))dt.$$

Consequently $u(x)$ satisfies the multiplicative iso-Osgood's lemma, from where $u(x) \equiv 0$ in $[0, a]$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

Lemma 2.21. *(multiplicative iso-Nagumo's lemma) Let $u(x)$ be nonnegative continuous function in $[x_0, x_0 + a]$ and $u(x_0) = 0$, and let $u(x)$ be differentiable at $x = x_0$ with $u'(x_0) = 0$. Then*

$$\int_{x_0}^{x} u(t)dt \leq a\int_{x_0}^{x}\frac{u(t)}{t - x_0}dt, \qquad x \in [x_0, x_0 + a],$$

and the inequality

$$u(x) \leq \int_{x_0}^{x}\frac{u(t)}{t - x_0}dt, \qquad x \in [x_0, x_0 + a],$$

implies that $u(x) = 0$ in $[x_0, x_0 + a]$.

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Proof. Let

$$g(x) = \int_{x_0}^{x} u(t)dt - a\int_{x_0}^{x} \frac{u(t)}{t-x_0}dt, \qquad x \in [x_0, x_0+a].$$

Since

$$\lim_{x\longrightarrow x_0} \frac{u(x)}{x-x_0} = u'(x_0) = 0,$$

then the integral

$$\int_{x_0}^{x} \frac{u(t)}{t-x_0}dt$$

exists for $x \in [x_0, x_0+a]$.

Also,

$$g'(x) = u(x) - a\frac{u(x)}{x-x_0} = u(x)\frac{x-x_0-a}{x-x_0} \le 0$$

for every $x \in [x_0, x_0+a]$. Therefore g is a nonincreasing function in $[x_0, x_0+a]$, whereupon

$$g(x) \le g(x_0) \qquad \text{for} \qquad \forall x \in [x_0, x_0+a],$$

or

$$\int_{x_0}^{x} u(t)dt \le a\int_{x)}^{x} \frac{u(t)}{t-x_0}dt$$

for every $x \in [x_0, x_0+a]$.

Let now

$$v(x) = \int_{x_0}^{x} \frac{u(t)}{t-x_0}dt, \qquad x \in [x_0, x_0+a].$$

Then

$$u(x) \le v(x), \qquad x \in [x_0, x_0+a],$$

and

$$v'(x) = \tfrac{u(x)}{x-x_0}$$

$$\le \tfrac{v(x)}{x-x_0}, \qquad x \in [x_0, x_0+a].$$

Consequently

$$\tfrac{d}{dx}\left(\tfrac{v(x)}{x-x_0}\right) = \tfrac{v'(x)(x-x_0)-v(x)}{(x-x_0)^2}$$

$$\le 0$$

or the function

$$l(x) = \frac{v(x)}{x - x_0}$$

is a nonincreasing function in $[x_0, x_0 + a]$ and since $l(x_0) = 0$, we have that

$$v(x) \leq 0 \qquad \text{in} \qquad [x_0, x_0 + a],$$

from where

$$v(x) = 0 \qquad \text{in} \qquad [x_0, x_0 + a].$$

Consequently $u(x) = 0$ in $[x_0, x_0 + a]$. □

Theorem 2.22. *(multiplicative iso-Nagumo's theorem) Let $P(a+1) \leq 1$, $f(x, y)$ be continuous in $\overline{S}_+$ and for all (x, y_1), $(x, y_2) \in \overline{S}_+$ it satisfies*

$$|f(x, y_1) - f(x, y_2)| \leq k|x - x_0|^{-1}|y_1 - y_2|, \qquad x \neq x_0. \qquad k \leq 1.$$

Then the problem (1), (2) *has at most one solution in* $[x_0, x_0 + a]$.

Proof. Let $y_1(x)$ and $y_2(x)$ are two solutions of the problem (1), (2) in $[x_0, x_0 + a]$. Then for $x \in [x_0, x_0 + a]$ we have

$$|y_1(x) - y_2(x)| \leq \int_{x_0}^x \Big(|y_1(t) - y_2(t)| \tfrac{|\hat{T}'(t)|}{\hat{T}(t)} + |f(t, y_1(t)) - f(t, y_2(t))| \tfrac{|\hat{T}(t) - t\hat{T}'(t)|}{\hat{T}(t)} \Big) dt$$

$$\leq \int_{x_0}^x \Big(P|y_1(t) - y_2(t)| + k(t - y_0)^{-1}|y_1(t) - y_2(t)|P \Big) dt$$

$$\leq P \int_{x_0}^x |y_1(t) - y_2(t)| dt + P \int_{x_0}^x \tfrac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq aP \int_{x_0}^x \tfrac{|y_1(t) - y_2(t)|}{t - x_0} dt + P \int_{x_0}^x \tfrac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$= (a+1)P \int_{x_0}^x \tfrac{|y_1(t) - y_2(t)|}{t - x_0} dt$$

$$\leq \int_{x_0}^x \tfrac{|y_1(t) - y_2(t)|}{t - x_0} dt.$$

Let

$$u(x) = |y_1(x) - y_2(x)|, \qquad x \in [x_0, x_0 + a].$$

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Then $u(x_0) = 0$ and from the mean value theorem we have

$$u'(x_0) = \lim_{h\longrightarrow 0} \frac{u(x_0+h)-u(x_0)}{h}$$

$$= \lim_{h\longrightarrow 0} \frac{|y_1(x_0)+hy_1'(x_0+\theta_1 h)+y_2(x_0)-hy_2'(x_0+\theta_2 h)|}{h}$$

$$= (\text{sgn}h) \lim_{h\longrightarrow 0} |y_1'(x_0+\theta_1 h) - y_2'(x_0+\theta_2 h)|$$

$$= 0, \qquad 0 < \theta_1, \theta_2 < 1.$$

Then the conditions of multiplicative iso-Nagumo's lemma are satisfied and $u(x) = 0$, i.e., $y_1(x) = y_2(x)$ in $[x_0, x_0 + a]$. □

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PRINCIPLE(S) OF CAUSALITY AS *DE FACTO* FUNDAMENTAL IN MATHEMATICAL PHYSICS, INCLUDING FOR CHANCE CAUSALITY APPLIED IN QUANTUM MECHANICS

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Abstract

According to the causality theory presented in the differential ontology and epistemology of Johansen (2008), concepts of randomness and probability are (i) *built on fundamental* types of causality, and (ii) represent themselves *particular, elaborated* types of causality. Hence, it is argued that Einstein vas basically correct in insisting on preserving the causality principle against the Copenhagen interpretation of quantum mechanics which he considered to be an incomplete theory. Adequate philosophical interpretations of quantum mechanics and of further developments into hadronic mechanics require concise differentiations and combinations of various types of causality, including chance causality. This is argued to be in agreement with some crucial results from the mathematical physics of David Bohm as well as of Ruggero Maria Santilli in relation to upgraded discussions of the Einstein-Podolsky-Rosen paradox.

Keywords: Einstein; Bohr; Bohm; Santilli; Popper: Einstein-Podolsky-Rosen; differential ontology; differential epistemology; causality; probability; randomness; chance; hadronic mechanics; Fibonacci algorithm; qualitative informatics.

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I have mentioned Santilli, and I should like to say that he – one who belongs to a new generation – seems to me to move on a different path. Far be it from me to belittle the giants who founded quantum mechanics under the leadership of Planck, Einstein, Bohr, Born, Heisenberg, de Broglie, Schrodinger, and Dirac. Santilli too makes it very clear how greatly he appreciates the work of these men. But in his approach he distinguishes the region of the 'arena of incontrovertible applicability' of quantum mechanics (he calls it 'atomic mechanics') from nuclear mechanics and hadronics, and his most fascinating arguments in support of the view that quantum mechanics should not, without new tests, be regarded as valid in nuclear and Hadronic mechanics, seem to me to augur a return to sanity: to that realism and objectivism for which Einstein stood, and which had been abandoned by those two very great physicists, Heisenberg and Bohr. (Karl Popper 1982:14)

Die Quantenmechanik ist sehr achtung-gebietend. Aber eine innere Stimme sagt mir, daß das doch nicht der wahre Jakob ist. Die Theorie liefert viel, aber dem Geheimnis des Alten bringt sie uns kaum näher. Jedenfalls bin ich überzeugt, daß der *nicht würfelt.*

(Einstein 1926: Letter to Max Born)

> *(Quantum mechanics is very imposing. But an inner voice tells me that this is still not the true Jacob. The theory delivers much, but it barely brings us closer to the secret of The Old One. In any case, I am convinced that He does not throw dice.)*

What is in the notion of "throwing a dice" ?

Let us inspect and unfold what logical operations that reside *enfolded* in the notion of throwing dices as an exemplar of randomness and probability distributions. There are six classes of possible results for every event, the top face of the cube ending up as 1, 2, 3, 4, 5 or 6. When we consider the total result of many such events, we can group these results into six classes of results E_j where index j varies from 1 to 6.

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In each particular event the individual effect is uniquely determined by physical laws and the initial conditions for the particular throw of the dice, such as gravitation, the force and direction of the throw, inertia and texture of the table, shape and texture of the dice etc. Thus, each individual effect is de facto uniquely determined by its corresponding and preceding individual cause, with the effect resulting from the cause by *physical causality*. If we specify and compare the individual causes in sufficient detail to pinpoint the decisive physical differences between causes that result in the physical differences between effects, the six *classes* of effects are also to be regarded as the result of six corresponding and preceding *classes of causes* C_i, where index i varies from 1 to 6. Thus, also the six classes of effects would result from six *classes* of causes by physical causality.

In the case of throwing dice it is difficult to specify and compare the individual causes in sufficient detail to establish the six classes of causes. It is hard to see that any easy attempt to specify significant variation in attributes between individual causes would favor one class of effects towards the other classes of effects. This consideration becomes reinforced when our empirical experience indicates that the six classes of effects occur almost equally often, and the more equal the more events of throwing the dice we consider. Hence, we find it adequate to regard the result of throwing dice *as if* it was *random* which class of effect the dice ended up into. This does not imply that we *really* mean that each individual effect is *not* uniquely determined from each individual cause, or that the six classes of effects are not uniquely determined from six imagined classes of causes, or that we will deny that both these determinations happen by *physical* causality *if* we investigated the events in sufficient microscopic physical detail. It only means that such an investigation is *not worth the effort and trouble* for our purpose at hand. It represents a huge advantage in *thought economy* for description and explanation when we radically simplify the whole constellation of physical events by applying the *simile* category of randomness instead of remaining (solely) at the physical level for description and explanation.

When it is considered *random* which class of effect the throw of the dice ends up into, this implies that the result can be considered random in *relation* to both the *individual* physical cause and to the *classes* of physical causes. Compared to description and explanation by merely physical causality, this represents a radical simplification at the cause side of the logical expression.

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The concept of randomness ignores and deletes all *internal* distinctions at the side of the cause, regarding them *as if* they were irrelevant for the effect. However, this does not imply any *annihilation* of cause. Now the form of the logical expression becomes:

> ***Exp1***: IF [cause] a dice is thrown as an individual event, with a resulting individual effect belonging to six possible classes of E_j; THEN [effect] it is considered *random* which class of E_j that will be the effect of the individual event

Thus, introduction of the concept of randomness does no way contradict causality as such, but makes possible a *novel* type of causality which we denote *randomness causality*. By *adding* this novel type of causality, including the according type of simile, the universe of causal relations becomes *expanded*, not *restricted*.

We realize that randomness causality represents a certain, novel type of causality *built on* the preexisting causality type of physical causality, and – further – that the adequacy of randomness causality is *underpinned* by *certain* relations of physical causality. Thus, it represents a philosophical mistake of category, i.e. a mistake in consistent meta-thinking, to consider randomness causality to *contradict* or undermine physical causality.

This is with respect to the very *category* of randomness causality as regarded from conceptual logic. In order for this novel causality type to become *adequately* mobilized and applied, in partial substitution of underpinning relations of physical causality, certain requirements *from* said underpinning physical relations must be met. These requirements fall into two classes, depending on whether the physical requirements for randomness are considered *ad negativo* vs. *ad positivo*:

Ad negativo: The totality of *external* physical relations existing together with the cause in *Exp1* and with the physical underpinnings of individual causes, or during the time span from cause to effect, is considered *irrelevant* for the relation between cause and effect at the level of physical causality. When already having established the concept of randomness, this means that these external physical relations are regarded as *random* and as *cancelling out* in relation to the cause in *Exp1* and to the physical underpinnings of individual causes. Thus, the exclusion of these external physical relations expresses the relevance of the concept of randomness for excluding purposes. We denote this as *negative* randomness.

Ad positivo: *Exp1* expresses *positive* randomness, i.e. the application of randomness causality *after* irrelevant externalities have become excluded by negative randomness. In order for this (positive) causality type to become adequate, it is required, as already stated, that the differences in individual physical causes really do result in a an (approximately) equal – i.e. random – distribution between the six possible classes of physical *effects*. In order for this to happen, it must also be the case that the individual cases of physical causes are distributed (approximately) equally between six imagined classes of physical *causes* C_i preceding and corresponding to their respective six classes of E_j.

That these said (approximately) equal distributions *really* are the case we *discover* by solely investigating the distribution of individual effects among the six classes of E_j. It is this discovery from inspecting the distribution of physical effects from *physical* causality that makes it adequate to *ignore* any inspection of the differences between individual *physical* causes, and thus also to ignore the make-up and internal differences between the imagined six *classes* of physical causes C_i.

When we from our scientific *thinking* do not see any obvious reason for one E_j to occur more often than another one, it is adequate to consider it random which E_j an individual physical effect will show to belong to. In the next step, though, it makes a big difference whether our hypotheses from thinking becomes *supported* or not, through experimental evidence. In the case of throwing dices it *does* become supported from our observation of the distribution of individual physical effects. It is our discovery from observation of the individual *physical* effects that makes it adequate to regard any difference between physical causes *as if* they were random.

Thus, the positive randomness causality of *Exp1* does not contradict physical causality, but presupposes physical causality by observing a *certain* pattern in the distribution of physical effects, i.e. of *physical* effects by *physical* causality from *physical* causes.

Thus, *before* establishing *Exp1*, and especially after observation from experiment, there is another cognitive operator in place, stating that *because* it is regarded as random what E_j the dice ends up into, the application of *Exp1* is regarded as an adequate consequence.

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Obviously, observations of a *random* pattern in the distribution of individual physical effects between the classes E_j will *not* happen if the physical system is characterized by *deviations* from an idealized situation of throwing a dice, e.g. if the dice is thrown a very short distance, if the dice is not a regular cube, if the eyes of the dice are magnetic and the dice is thrown at a magnetic table, etc. Thus, there exist obvious constraints for which physical systems that can be adequately described or explained with good approximation by randomness causality. Only *certain* physical systems can be adequately approached by means of randomness causality.

When it is regarded as *random* which E_j a throw of a dice ends up with as physical effect, this means that the six different classes of effects are regarded as *non-differentiated* with respect to *probability*. However, in next steps of thought the considered randomness between each six classes, rather automatically leads to various cases of *non*-randomness and *differentiations* with respect to probability. As trivial examples, the probability of effect E_1 OR E_2 from one throw of the dice will be twice the probability of effect E_3, and the probability of effect E_1 to occur twice from throwing the dice two times will be 1/6 x 1/6 = 1/36. The laws of probability distributions and mathematical probability theory as a whole emerges from systematic unfoldments of what resides enfolded in the very *concepts* of randomness and probability.

Since the concept of probability both presupposes and follows rather directly from the concept of randomness, we realize that *probability causality* also presupposes and follows rather directly from randomness causality as a novel and somewhat more elaborated *type* of causality than randomness causality. *Because* classes of physical effects are regarded as random compared to each other, more elaborated regroupings and sequences of these classes of effects must be *non*-random and *differ in probability* by exact mathematical laws. Thus, explanation of a physical system by means of probability causality will essentially relate to physical causality the same way as explanation of a physical system by means of randomness causality relates to physical causality in manners we already have clarified. We apply the broader term *chance causality* to cover both randomness causality and probability causality.

To sum up we realize that:

(i) randomness does not *contradict* causality, but implies a certain *type* of causality as expressed by *Exp1*;

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(ii) randomness causality does not *eliminate physical* causality, but represents a more *elaborated* type of causality (by adding a certain simile) which *presupposes* physical causality;
(iii) randomness causality represents an *adequate wrapping* with (formally regarded) *partial* substitution of physical causality only when being *underpinned* by *certain* cases of physical systems already characterized by *physical* causality.

(From the above it should be clear that what here is stated with respect to randomness and randomness causality, also holds for probability causality and thus for chance causality as a whole.)

It follows from (i) that the very question of whether the universe is based on random events or causality does not make any good sense, and even less to claim the first alternative. It follows from (ii) that the very question of whether the universe is based on chance causality or physical causality does not make any good sense, and even less to claim the first alternative. These conclusions follow from strict philosophical reasons without respect to the scientific theory under consideration. For theories in mathematical physics to reach adequate and fully mature expressions, they should be consistent with points (i), (ii) and (iii), and theoretical developments might benefit from deeper and more detailed reflections on their scientific material in relation to these points.

In theories of mathematical physics interpretations and discussions with respect to the role of causality, tend to consider causality only in the sense of *physical* causality. In our philosophical treatise *Outline of Differential Epistemology* (Johansen 2008), we pretended to have presented a rather complete systematic development and exhibition of the whole *nexus of causality types* (cf. Johansen 2008: ch.3, 113-194; 248-9). It was disclosed and explained that the nexus included several types of causality, classified into ten fundamental types and ten elaborated types. *Chance causality* was presented as one among the ten elaborated types (cf. Johansen 2008: 165-175), while *physical causality* was presented as the *least* fundamental among the ten fundamental type (cf. Johansen 2008: 155-157). Above we have sought to clarify that chance causality for strict reasons of conceptual logic can not constitute any fundamental causality type on an equal footing with physical causality. However, even physical causality can not adequately be considered *that* fundamental as usually regarded in theoretical physics. In the following we will present *some* clarification of why this must be the case, in order to

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contribute to some according clarification when contemplating the role of causality in theoretical physics. (For further discussion of various aspects, see Johansen 2008a, 2008b, 2008c, 2013, 2017.)

Abstract causality vs. formal-logical causality

When throwing a dice is considered a physical event implying physical causality between individual cause and individual effect, there already reside more fundamental types of causality *enfolded* in the very notion of physical causality.

Before differentiating between *types* of causality there already must exist a *universal* and *abstract* concept of causality *as such* in cognition, namely the concept of the relation between two relata where a logically proceeding relatum denoted 'effect', *with logical necessity* follows from a logically preceding relatum denoted 'cause'. In conventional formal logic causality is approached by the notion *material implication* where binary *truth values* of cause and effect in a logical expression *first* are assumed (or determined) *independently* of each other, *whereafter* material implication is defined as a certain *truth function* of the four pairs of said truth values, more specifically that the material implication is decided as true iff the pair (cause is true; effect is false) does not show up. While highly useful for many purposes, e.g. computer electronics, this approach to define causality is too shallow to hit the mark of that which it attempts to target and catch the essence of.

From the definition of material implication the following expressions will be decided as true:

(1) $p \Rightarrow (q \Rightarrow p)$
$-p \Rightarrow (p \Rightarrow q)$
$(p \Rightarrow q) \vee (q \Rightarrow p)$

However, the claimed truths of such expressions are rather contra-intuitive and not aligned with the concept of the relation IF…THEN… that *tacitly* is *de facto operative* in our ordinary cognition. The modus operandi of our innate, subconscious (or rather *supra*conscious) category of 'causality' does not start out with first separating and establishing candidates to cause and effect, for next comparing pairs of their truth values, and thereafter deciding one pair as a causal relation in distinction to the three other pairs. Our innate category

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starts out with a logical entity of truth (cause) which *unfolds organically* into another logical entity of truth (effect) as its logically *necessary* result or fruit in a relation which we are not able to reflect upon before *after* the unfoldment has taken place. (And if performing such a reflection we will remobilize the *same* innate category of causality at a meta-level.)

To hit the mark of the implicate, innate cognitive category of causality an adequate approach has to be much more abstract, deeper and accurate than what was the case for establishing the concept of material implication in conventional formal logic. In our treatise (Johansen 2008) we presented a novel – and tentatively completed – theory in order to cover and solve core issues of philosophy. We denoted our philosophical theory *differential ontology,* including *differential epistemology* as the more sophisticated "head" unfolded from and (next) into the ontological "body". Our philosophy presented a systematic unfoldment of categories residing enfolded inside *information as such*, i.e. inside *something being,* whatever it might be, conceived in its most elementary, abstract and universal sense.

The starting point for our systematic philosophical exhibition and successive unfoldments of categories, was *information as such*. The concept of information in the most abstracted *qualitative* sense, was established as close to Gregory Bateson's famous definition of information as a *difference that makes a difference* for something/someone. (Our relatively minor and subtle deviations from the definition by Bateson do not matter much for the present text.) This definition can be reformulated as an input-difference *making* an output-difference for a *subject*. (If the subject is not a human, or not even a living being, when e.g. one billiard ball receives – and reacts to – an input-difference from being hit by another billiard ball, the category of subject is operative by a minimum of anthropomorphic projection applied as an adequate *simile*. Already our grammar, with classes of subjects, verbs and objects, applies a minimum of such a simile.) Thus, the category of subject, whether in the emphatic sense or in the simile sense, is with necessity implied in the very definition of information in the most abstract sense. One striking illustration of the *tacit* de facto inclusion of the subject may be the notion of "rock hard reality", not possible to deny for anyone with their senses intact, applying as exemplar the situation of a heavy stone falling on the toes of a human. Here we notice that a preferred exemplar of "rock hard reality" *depends* on the inclusion of the subject, i.e. of one emphatic subject showing (strong) *emotion*. In the most

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abstract concept of information the input-difference makes an output-difference for a *third* entity which is considered a *subject* by having *emotion* (in the most abstract sense), and – as already stated – with pseudo-subjects *not* considered as having emotions treated with a minimum of anthropomorphic projection from the analyzing human subject.

When starting out with this most abstract definition of information as such, the very act of the input-difference *making* (or better: *unfolding into*) the output-difference, can be adequately conceived as the *causal* relation between input-difference and output-difference, when this act of making, or unfoldment into, is cognitively regarded as *logically* necessary. In order to conceive this relation as *solely* logically necessary, we have to conceive the input-difference unfolding into an output-difference as *abstracted into* an imagined *pure and free-standing thought universe* of solely logical relations *without* regard to any connection to a physical input-difference and a physical output-difference.

It is *inside* this imagined pure and free-standing universe of thought, *without* regard to correlations to *physical* input-differences and output-differences, we can conceive causality, in its most abstract sense, as the relation from an input-difference unfolding into an output-difference. The cognitive category of causality must be grasped in its purest and most abstract sense before we can study *how* the category is universally *implied* in various *types* of causality such as e.g. physical causality.

In our philosophical treatise we explained how the most abstract notion of causality as organically unfolding the input-difference into an output-difference, could be adequately *back-reflected* by a certain *formal* representation achieved by means of set theory when placing and relating elements and classes at concisely differentiated *ontological levels of thought* inside a freestanding thought universe. The differentiations in ontological levels was presented as unfolding with necessity from consistent reflection from and upon the thought of information as such, in some distinction to modal logic which have tried to overcome – with some success – shortcomings in the notion of material implication by adding various logical operators while still not acknowledging the necessity of developing differentiations in ontological levels of thought in an organic, strict and systematic way quite different from "freely" playing around with voluntaristic constructions inside a logical toy universe.

In the present context it would take too much space to try to represent our formal expressions and according philosophical reasoning. The main point

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is that we found it possible and adequate to express the most abstract notion of causality by means of formal logic, so that abstract causality could be renamed *formal-logical causality* presupposing that this refers to the *particular* formal expression presented in our treatise.

Projective causality

Let us take a look at sensory perception. Neuroscience has shown that perception has a stepwise constitution, so that there is a lot going on from the subject receives an initial *recept,* defined as the first and most elementary kind of sensory information objectively possible to register for a subject, say a human, until the subject sense a *percept* available for its consciousness. The subject will consider its percept, say a flash of light, as residing *outside* the boundary represented by the skin of the subject, while in reality the percept occurs *inside* the boundary of the subject, with the preceding recept occurring at the *immediate* inside. Thus the subject performs an *outward and backward projection* of the real location of its perceived input-difference. Further, *what* sensory input the subject perceives, both in quality and in quantities of the quality, depends on the algorithms (including their semantics) constituting the sensory apparatus of the subject. During the perception these algorithms are hidden for the subject who de facto applies a projection outwards and backwards also of these subject-internal algorithms and their related subject-*internal* differences. Such projection of one causal relation between input-difference and output-difference to *another* causal relation between input-difference and output-difference, we denote as *projective causality*.

By applying technical instruments more sensitive than our (direct) perception, *another* subject can research the exact relations between input- and output-differences constituting sensory perception, e.g. when studying how inputs of volumes and frequencies of sound from an external source become perceived by a human subject as corresponding but different volumes and frequencies. The researcher will tend to find that such relations between input- and output-differences follow the Weber-Fechner law for sensory perception. The Weber-Fechner relation implies that:
(i) Input-differences are represented *logarithmically* as the output-differences registered by the subject. (If the volume of a sound increases with a factor of 8, the human ear will perceive this as increase with a factor of 3 due to $2^3=8$.)

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(ii) Input-differences below the lowest threshold and above the highest threshold for reception by the subject will *not* be represented as output-differences at all. (Frequencies too high or too low will not be heard by the human ear.) (iii) Since the *resolution* of input-differences is higher than the resolution of output-differences, any output-difference will cover *plural* preceding input-differences which hence becomes conflated into the *same* output-difference. (If the difference between two frequencies are too small, the human ear will not perceive any difference.)

With regard to perception we thus see that many input-differences do not unfold into any output-difference, and therefore they do not constitute any information for the perceiving subject (only for the external researcher). Also, we see that when the subject projects his percept (with implied subject-internal algorithms) outwards, there is implied a *quantitative* (logarithmic) transformation between the "real" external input-difference (as measured by the researcher) and the input-difference as perceived by the subject. The *qualitative* incongruence is even more radical since the perceiving subject does not have any *access* to the quality of any *external* difference. The first input-difference that constitutes information is the *recept* located at the immediate *inside* of the subject. Thus, the external input-difference is better considered as a *pre-input*-difference.

When we move from perception to proceeding information processing by the subject, projective causality must still be involved in every step of thought, although the implied incongruences (at least the qualitative ones) will be less radical in most cases. The tacit continued presence of projective causality is due to the fact that the subject can not process or reflect upon its distinctions *before* it has manifested them, and during the act of manifestation the distinction is *hidden* for the subject. Thus, the subject is always processing information *one step ahead* (when observed by the meta-subject of an external researcher) of what itself can be able to conceive.

If, say, you make a distinction between yellow and green in an observed rainbow, there is not inserted any *physical* border between yellow and green as when children draw a line between objects by a black pencil. The border between the two colors is invisible at the perceived physical level, while the border still has a real existence as a mental category in the inherent make-up of the subject. It is the tacit *projection* of the border category, residing in the mental domain, onto the conceived physical domain, that constitutes the difference between yellow and green in the object perceived by the subject. The subject perceives the difference in color *after* the category of border has become projected outwards. What the subject perceives as an input-difference from the physical domain is, when regarded from an external subject with imagined access to the inside of the subject, to be regarded as an output-difference where the inside category of border becomes projected. In general, since the criterion for classification always is hidden (while at the same time expressed) *in* the classification, the subject will always consider the *level* of being/reality it operates on (at least) *one step lower* than what is the case if regarded from an advanced external subject. Thus, projective causality, with this *objectively* implied *simile* of unidirectional level substitution, is with necessity tacitly present in all information processing and thus also tacitly enfolded in all (other) types of causality.

In sensory perception the internal classifications that are projected remain hidden for the subject itself. We denote such as *traceless classifications* which yields trackless representations and processing of information. In more conscious information processing the projected internal classifications can become preserved and reflected upon, and we denote such as *reflexive classifications*.

We may consider, as an example from more refined thought, a logician wrapping his head around whether the expression "I am always lying" (E) is true or false. This seems tricky to decide when he has only one out of two binary truth values to his disposition and the assumption of each of them leads to a contradiction. The contradictions arise when expression E is applied self-referentially to also include itself as something to be true vs. false *about*, as indicated by the term 'always' interpreted as expanded without contextual limitation. Thus, the logician realizes that his trouble originates from that E conflates two different *levels* or logical types of expressions. In order to seek clarification he has to add a meta-level where E can be regarded to also be about itself. The logician unfolds a differentiation already residing enfolded in expression E as soon as he experienced some trouble, so that his reflexive classification into two levels arrives after the more immediate manifestation of expression E in the mind of the logician.

The logician's adequate differentiation into two *levels* of ontological being residing inside a considered freestanding logical universe of logical categories of thought, must be considered to have *physical* correlates in his

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corresponding brain chemistry, and the logician will seek to express the two levels and the relation between them by physical means as drawings of classification schemes or written logical operators. In general, zero information can exist without being manifested by a *physical* expression and carrier ("physical" as regarded *relatively* to the ontological level of information) as the lower side of the "coin" of the information "atom", whatever minimal (as software or net bank money expressed by pixels at a computer screen). Here, information represents the *upper* side of the coin and the physical manifestation/carrier the *lower* side, as far as the physical manifestation is regarded as *expressing* the information. The information of the same amount of money can be expressed by *plural* alternative physical carriers (coins, bills, net bank pixels), as well as the same universal Turing machine can be expressed by *plural* kinds of computer hardware. Thus, it can not be a 1:1 relation between information and its physical manifestation and carrier, and in this relation information represents the upper and most significant side of the coin vis-à-vis the lower side of the coin that represents the *substance* which incarnate the information. There is no information without substance, and thus without a differentiation (and relation) between two different ontological levels; and the substance is significant only insofar it manifests and carries the information.

The triad of Truth, truth and false

Back to our logician contemplating expression E. *Before* starting out to decide the truth value of the expression, he has first to receive and get the *immediate meaning* of the expression from possessing ordinary skills of language. *Thereafter* he starts out to reflect upon the truth vs. not of the expression from applying formal logic. Thus, he can *not* question the truth of the verbal expression E as being exposed to and received by him in the *first* place, i.e. the reality (inside a thought universe) of his *initial thought object* which only *later on* becomes reflected upon by his logical contemplations. To judge by logic *whether* an expression is true or not, it is presupposed (somewhat pre-meta) that the logician in the first place *did* receive and conceive the expression itself, i.e. the very thought object for his logical reflections, as taken for true. This can be denoted the *prior* truth of expression E without which the following logical investigation of *whether* E is true vs. false can not happen or have any meaning.

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Thus, conventional formal logics is not restricted only be some shortcomings shortly indicated previously in this text, but also by *ignoring* the arrival of the initial expressions of the thought *objects* for exercises of formal logics; where these initial expressions are *a priori* to be regarded as prior truths *qua stated*. If we denote the prior truth with capital letter as *Truth*, the truth values of true and false become assigned afterwards as statements *about* the Truth expression. Without Truth there would not *exist* any thought object to think about as true vs. false, so that Truth must have logical priority to both true and false. It is not possible to start out with False as category, since the truth value false only can be *about* Truth and in this sense must be logically secondary and parasitic on the category of Truth. Conventional formal logic is constructed as if truth and non-truth are existing at (only) an equal footing, while from an extended contemplation there is always a *triad* involved in logical reflection, where Truth becomes differentiated into being true or false when Truth becomes reflected upon. In much human thinking Truth is *not* reflected upon, but *unfolds organically* into *another* Truth by mostly unconscious types of causality. It is due to ignoring this circumstance that conventional formal logic includes as true various expressions where claimed truthfulness appear contra-intuitively inadequate, as the examples we gave in (1).

When a newborn baby opens its eyes, it will be confused and perhaps start wondering whether it is true or false that it is not dreaming. But in the first place, before the baby starts wondering about truth values and strives to place its novel visual experience ontologically, the baby can not question the fact of its visual experience as such. Thus, the triad of Truth and truth vs. false is operative in real life phenomena also outside the free-standing thought universe of formal logics.

By analogy, it is mistaken to conceive the categories of *creation and destruction* as existing (only) at an equal ontological footing. It is not possible to destroy anything that is not *already* created; thus the prior category is *Creation* that next differentiates into being treated by (further) *creation* vs. *destruction*. It may be reason to question the adequacy of considering the category *entropy* that fundamental as in most theoretical physics. From a consistent triadic approach it may seem most reasonable to consider *Syntropy* as the prior category which becomes differentiated into *negentropy vs. entropy*. In the present context, however, it will lead way too far to attempt to lift and reinterpret the laws of thermodynamics from a triadic approach.

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(As an aside with respect to moral philosophy: From analogous triadic reflections we might find that nor do the categories of *good and evil* exist (only) at an equal ontological footing. The category *evil* (vs. *good)* has to be *about* something, and if this prior something was not *Good* it is hard to see how the category of evil can have any *meaning* as negation of anything.)

The profound ontological significance of the Fibonacci algorithm

Already from our reflections above concerning abstract causality and projective causality it is indicated that when information is constituted as an input-difference unfolding into an output-difference for a subject, this happens by (i) the subject *projecting* subject-internal difference onto an ontological level residing *below* the ontological level where the subject-internal differences reside themselves, so that this projection is implied in the constitution of the input-difference; and by (ii) the subject unfolding the input-difference by means of its inherent *causal* operator(s) into the output-difference. When regarded formally as abstract, universal and elementary as possible, this has the form of the *Fibonacci algorithm* where the subject processes its preceding state by tacitly combining it with a projection of its present state, whereafter the next, proceeding state of the same subject unfolds by causal necessity. This means that the Fibonacci algorithm is tacitly residing inside information as such, when information is considered in its most abstract *qualitative* sense. Consequently, the Fibonacci algorithm must constitute the fundamental bridge *between the qualitative and quantitative* aspects of nature (cf. also Johansen 2006, 2008a, 2014b).

A radical implication from this apparent philosophical result was that even the field of natural numbers, as a distinguished part of (cognitively conceived) nature, should be possible to *unfold* from systematic reflection on the Fibonacci algorithm. In our treatise *Fibonacci generation of natural numbers and prime numbers* (Johansen 2011) the field of natural numbers became established as a supra-structure generated uniquely from the Fibonacci algorithm by successive alternations between ordinal and cardinal aspects of Fibonacci entities/numbers. Thus, while the Fibonacci series trivially is a subset of natural numbers, from this deeper contemplation, representing some Copernican turn, the natural numbers themselves emerged as generated from the Fibonacci algorithm (cf. also Johansen 2014a).

Our mathematical results connected to this refoundation of number theory may suggest that profound and concise reflections on information and causality categories hold a potential for catalyzing clarification and progress also in topics of theoretical physics.

The nexus of causality types

In our philosophical treatise different ontological levels and dimensions are systematically developed from successive reflections on categories residing tacitly enfolded in the very concept of information as such. Our causality theory does not hold any *autonomous* position towards ontology in general, but are *anchored* in *this* differential ontology and epistemology. Our development and differentiations between various causality types are, more accurately expressed, presented as integral and crucial aspects, unfolding more organically, *inside* the development of our differential ontology.

As a whole this causality theory is too extensive and complex to become much presented in this text, but at least we can provide a condensed – and by necessity rather cryptical – description in order to give *some* idea about the nature of the *fundamental* causality types and of the relations between them: (cf. Johansen 2017)

Fig. I: Illustration of the causality nexus anchored in the three dimensions physical (horizontal in black; 3 + 1D compressed as 1D time), algorithmic (vertical in yellow), and transalgorithmic (depth in red). Description of first-order alternates between process (black) and transfiguration (yellow), second-order between blue and orange. Higher orders activate from emergence (red) and unfold as structural change in process (light blue) or innovative change in transfiguration (dark green), with the possibility of the last being retroactive (purple). Whatever degree of order and systemic complexity, the illustrated conglomerate of causality types and arrows constitutes a completed nexus of information flows.

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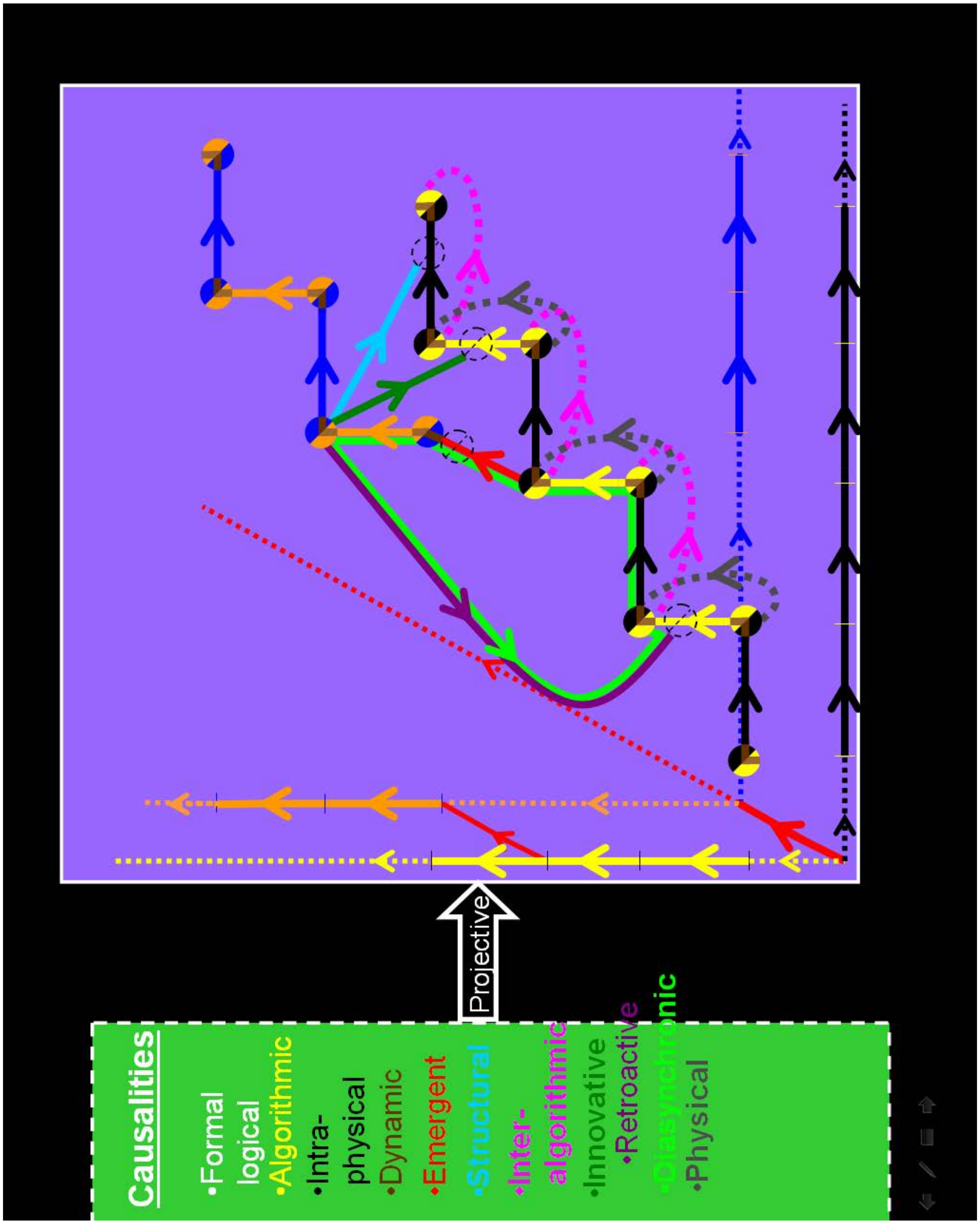
Causalities
•Formal logical
•Algorithmic
•Intra-physical
•Dynamic
•Emergent
•Structural
•Inter-algorithmic
•Innovative
•Retroactive
•Diasynchronic
•Physical
Projective

Formal logical causality: this category is universal for all thinkable information, i.e., for *any* information flow in *any* described information matrix, i.e., in the imagination of a pure and free-standing logical universe. Formal logical causality is deduced in its precise form from specified classification logic between the thinkable classes and elements from ontology differentiated vertically. All other causality types are subtypes and "clothes" of this abstract one, which is what qualify them as causality types. They unfold from specified additions of different *similes, necessary* in any dynamic system description, explicitly stated or not.

Algorithmic causality: this is the causal relation from an input-value to an output-value inside the algorithm.

Intra-physical causality: this is the causal relation from start point to end point of a process.

Dynamic causality: this is the causal relation with the two subclasses: a) from end point of a process to start point in an algorithm; b) from end point of an algorithm to start point in a process.

Projective causality: this is the causal relation from the meta-subject to the thought object as a whole; the potential inner classifications and causal relations being actualized in this projection (including formal logical causality). In fig. 1, the arrow of projective causality originates from the field (in green) of an enfolded nexus of causality types, denoting a segment *inside* the thinking meta-subject that makes the description, and manifests as the field (in indigo) of an unfolded nexus of causality types. The frame of the originating field is marked with broken white lines in order to distinguish its ontological status from the nexus projected into the derived field.

Structural causality: this is the meta-algorithmic causality relation directing the process-output from an algorithm to the process-input for another algorithm and hence positioning all algorithms in a structure.

Inter-algorithmic causality: this is the causal relation from an algorithmic output to the algorithmic input for another algorithm, hence ignoring the intermediary physical process by a projection to the vertical algorithmic axis.

Emergent causality: this is the causal relation from an algorithm to a meta-algorithm.

Innovative causality: this is the causal relation from a meta-algorithm to a first-order algorithm. An important subtype of innovative causality is the *retroactive* causal relation from a meta-algorithm to a first-order algorithm earlier connected to the meta-algorithm by emergent causality.

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Diasynchronic causality: this is the causal relation made up by a *circuit* of algorithmic, physical, intraphysical, dynamic, projective, emergent, structural, and retroactive innovative causality.

Physical causality: this is the physical relation from a process output to the process input of the next process; hence, ignoring all intermediary algorithmic and transalgorithmic transfigurations by a projection from the vertical axis or the depth axis to the horizontal axis.

It follows from the illustration of the causality nexus in fig.1, that, e.g., the conventional notion of *physical* causality is far from constituting the *most* fundamental causality type. It is also far from any *trivial* causality types, due to its condensation of many involved causality paths through plural shortcuts and similes. Thus, it follows from strict and consistent philosophical-ontological reflection on the nexus of causality types which *constitutes* the reality of information in the cosmos, that ideas about cosmos as *fundamentally* physical or—even worse—*only* physical, are basically *radically amputated and illusionary* as judged by strict standards of scientifically informed and informing philosophy/meta-science.

From these fundamental causality types, various *elaborated* causality types constituted by combinations of fundamental causality types were exhibited by Johansen (2008: ch. 3.2); among these are: randomness causality, probability causality, stochastic causality, intentional causality, selective causality, and imagined causality. Thus, more elaborated and epistemologically refined causality types, crucial in human and social systems, were understood *inside* the causality nexus anchored in the three ontological dimensions (see Johansen 2008a and 2008c for specified applications of this causality theory).

It follows from our philosophical work that without a sufficiently differentiated and concise ontology, it becomes difficult and in part impossible to discover, differentiate and adequately place and relate several types of causality. Far most of theoretical physics is not much sophisticated in ontological differentiations, which leads – more or less – to corresponding restrictions in reflections on causality in general and on various causality types. Still, the most common "folk ontology" among physicists is limited to the simple binary distinction between the physical world and the mental world, for next to consider the physical world as the primary world or even as the only "real" world. We may take even Einstein as an example expressing a rather naïve ontology subscribing to philosophical materialism, although without the conventional notion of 'matter', expressing support to the tradition from Hume and Mach (cf. Einstein 2000 [1954]:81).

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For many – not to say far most – purposes of physics, say, engineering by Newtonian mechanics, shortcomings in more or less subtle ontological differentiations do not matter much – if at all –, nor do shortcomings with regard to understanding the rather intricate relations between more fundamental causality types that reside enfolded in the conflated notion of physical causality. Extensive philosophical meta-reflections will in most cases show contra-productive with respect to solving the task at hand, and the required implied relations between cognitive categories are best delegated to the wisdom, speed and precision performed by unconscious algorithms.

However, in order to adequately approach and treat more *fundamental* issues in theoretical physics, which present crucial *paradigmatic* challenges, more abstract differentiations and meta-reflections may make a constructive difference. With respect to quantum physics more tricky philosophical issues became actualized as soon as the role of the *observer* had to be included into a broader perspective in order to understand *what* real entity that manifests through quantum measurements targeting the wave function.

Approaching the Einstein-Podolsky-Rosen paradox

When approaching the *Einstein-Podolsky-Rosen paradox* (Einstein et al. 1935) in theoretical physics it is not adequate to consider the (mathematical) *chance* distribution as an attribute by the (physical) wave function, as *opposed* to causality (which tacitly is considered as *physical* causality). We have clarified that as located inside our theory of causality, anchored in our ontological framework, this can *not* with logical consistency be considered as an *absolute* opposition, in the sense of representing two opposing categories in their ontological *basis*. Firstly, *physical causality* does not represent the *most* fundamental type of causality, but *enfolds* and is *internally built from* more fundamental types of causality. Secondly, the concept of *chance* (including the concepts of randomness and probability) does itself represent a certain *type* of causality. Thirdly, chance causality does not represent any *fundamental* causality type, but one among several *elaborated* types of causality. Fourthly, chance causality, as when applied in explanation of a physical system, *presupposes physical* causality as *one* among the causality types chance causality is made-up by and from by addition and inclusion of certain simile operators.

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When not being consistent with the four points above, theoretical discussion of EPR will contain some categorical conflations and inaccuracies. This does not implicate, however, that the discussion is without intellectual merit or importance, but it does implicate that more basic and consistent categorical differentiations and relations might catalyze further clarification of the issues discussed.

Our differential philosophy might be characterized as a systematic *qualitative* informatics, i.e. that unfoldment into qualitative differentiations and categories precedes related quantifications *of* the unfolded qualities, as already indicated by the significance of the Fibonacci *algorithm* for refoundation of number theory as a whole (and in next steps catalyzing certain novel mathematical results more technically regarded), or by the significance of the Weber-Fechner logarithm in constitution of recepts.

We may contrast this to the *quantitative* informatics presented by Shannon&Weaver (1949) where the concept of information was *defined* (1949: 103f) from the concept of probability when contemplating signal to noise ratios and applying entropy formulas from theoretical physics. Their approach was technically sophisticated and showed highly fruitful, e.g. for developments of telecommunications. However, their quantitative “definition” of information appears as a second-hand pseudo-definition, since it already *tacitly presupposed* the very quality of ‘information’ to have become established (and thereafter becoming differentiated qualitatively into the concept of ‘signal’ as input-difference at the sender side and an output-difference at the receiving side) before it became quantified for practical purposes.

Later on both Chaitin and Kolmogoroff presented theories of quantitative informatics which were basically complementary to Shannon&Weaver, and Zurek presented a reasonable synthesis of Shannon vs. Chaitin/Kolmogoroff. In any case these potent developments of quantitative informatics, *based* on i.a. the notion of chance causality, has to be regarded as second-hand as compared to the qualitative informatics represented by our differential philosophy where the very category of chance causality does not occur before rather late in our systematic unfoldment of causality types. Thus, from a more profound and basically qualitative approach it is not adequate to refer to the quantitative second-hand definition of information applied in the information theory of Shannon (or others), with the according fundamental role played by chance causality, in order to approach the deeper issues of theoretical physics as e.g. addressed by the EPR paradox.

The abstract, while concise, definition by Turing (1935) of information as computation, connected to his astonishing invention of the Universal Turing Machine, is definitely more qualitative (in the first place) and independent of probability reflections than the definitions of information referred above. Still though, our definition is more qualitative, abstract and universal than Turing's, with according possible robustness towards more fundamental progresses in informatics (e.g. Deutsch, Diaz/Rowlands, Bohm; cf. Johansen 2008:260f for a short discussion).

Einstein, Podolsky and Rosen (1935) argued quantum mechanics to not represent a *complete (physical) theory* because the *description* of (physical) reality by the wave function in quantum mechanics was judged as not being complete.

They stated as a *necessary* condition for a complete theory that "every element of the physical reality must have a counterpart in the physical theory" (ibid.: 777). (We apply symbol E_R to denote the first kind of element, and symbol E_T to denote the second kind of element.) They stated as a *sufficient* condition for the occurrence of $\mathrm{E_R}$ that the "value of a physical quantity" can be predicted with certainty, i.e. with probability=1 (ibid.). Next, from performing a certain *thought experiment*, consistent with quantum mechanics and its mathematical transformation theorems, they argued the occurrences of *certain* $\mathrm{E_R}$s that were *not* possible to describe by a corresponding counterpart of $\mathrm{E_T}$. Consequently, quantum mechanics could *not* be considered a *complete* theory.

It was concluded as an open question *whether* a complete theory, overcoming their argued limitations of quantum mechanics, could be achieved, but the authors stated their *belief* in such a more advanced and general theory to be possible.

Their thought experiment (ibid.: 779f) considered two (physical) systems interacting for some time, after which system I and system II are separated. The initial states of the two systems when they start to interact, are assumed as known. Then, from the Schrödinger equation with the wave function we can calculate the state of the *combined* system I&II at any time, including *after* the two systems separate. However, we can not calculate the state of *each* system after their interaction has become terminated. According to quantum

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mechanics, such calculation is only possible from a *measurement* executing *reduction* of the wave packet. It is assumed that measurement only takes place in system I, *either* of *coordinate(s)* (position), thus being considered as E_R, *or* of *momentum* considered as E_R. Depending on which of the two binary alternatives for measurement that is chosen, the inferred wave function for system II, *after* the two systems have been separated, will look different. It does not appear consistent that the *same* system II, after separation, can be assigned with two *different* wave functions.

Next, their thought experiment assumed two E_R candidates, namely two particles P and Q with corresponding eigenfunctions of two *non-commuting* operators with respective eigenvalues. They presented a technical proof, concluded in their eq. (18), that the two *different* eigenfunctions, depending on starting out with measurement of the *momentum* of P vs. with of the *coordinate(s*) of P, represent alternative expressions of the *same* reality, and thus that *both* of the two non-commuting P and Q should be considered *simultaneously* as E_R.

Since the measurement process in system I is considered to not have *any* possible influence on the state of system II *after* the two systems have become separated, so that the measurement process in system I is *irrelevant* for the state of system II, it does not seem to make sense that the objective *E_R status* of something residing in system II *after* the separation should depend on whether the measurement procedure in system I targeted position vs. momentum of a particle residing merely in system I, i.e. *excluded* from system II.

The authors concluded that quantum mechanics offers a non-complete description of objectively existing E_Rs residing in system II since a calculation of *both* E_Rs residing simultaneously in system II is *impossible* to achieve (predict) from a definite measurement in system I, because such a measurement *has to* exclude one calculation on behalf of the other.

Niels Bohr (1935) replied to this critique by pointing out that the very access to *receive any* experimental data about what was going on at the quantum level *required* experimental apparatus and procedures at the *classical* level. From the freely chosen *specifics* of the experimental design at the classical level it would be uniquely *determined whether* the position or the momentum of an elementary particle became targeted and measured. Consequently, it was not empirically *possible* to avoid that measurement of position vs. momentum of an elementary

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particle *had* to be binary. Thus, it did not make sense to *criticize* quantum mechanics for not being "complete" in its description since the very *access* to the quantum level *presupposed* such incompleteness. Further, *without* such access to the quantum level, with said according narrowing constraints, *zero* predictions or possible assignments of *values* of E_R, or indicating any *existence* of E_R at all, would not be possible. Thus, Bohr argued that you could not criticize quantum mechanics for shortcomings with respect to not achieving *complete* descriptions at the quantum level when these shortcomings with necessity was entailed in the classical apparatus and procedures to get any access to the quantum level *at all*.

Bohr presented the principle of *complementarity,* with respect to the quantum level, in order to account for the fact that even if it was not possible by experimental apparatus to measure (and calculate from) position and momentum of an elementary particle *simultaneously*, *both* approaches, corresponding to according measuring devices, should at the quantum level be regarded as contributing to physical knowledge at an equal footing. Although not referred to by Bohr, this appears basically similar to the *gestalt switch* in psychology of perception, where, say, the alternation between rabbit and duck gives more complete information about the whole object for perception than each of the two perspectives. With respect to natural philosophy, regarded more in general, Bohr's consideration implied that the complementarity principle had to be applied with more strict and basic necessity at the quantum level than at the classical level.

Further, Bohr addressed some possible self-referential inconsistencies in the argument by Einstein et al. by clarifying that their critical conclusion was *based on* applying the transformation theorems developed *inside* the mathematics of quantum mechanics – and thus, at least to some extent, subscribing to the paradigmatic framework of quantum theory. Bohr also pointed out that quantum mechanics involved exchange of energy at the quantum level so that *time and energy* variables should be regarded as conjugates, rather analogous to position vs. momentum, and that this conjugation had an interesting similarity to a certain paradox in Einstein's relativity theory. In order to perform experiments to *test* predictions from relativity theory, highly accurate assignments of time and space coordinates are required, as determined at the *classical* level, despite that relativity theory, especially the *general* theory of relativity, implies a novel theory about the very

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relation between the coordinates of space and time where these coordinates can *not* be determined independently of each other. Thus, one crucial point from Bohr's anti-critique is that we can not require from an advanced theory of physics that such paradoxes can be completely *avoided*; thus the question is how they are next treated and attempted reconciled from firstly acknowledging the *necessity* of the involved theoretical paradox.

In the proceeding discussions in theoretical physics about the EPR paradox, Bohr's anti-critique tended to be judged as satisfactory. There should be no doubt that Bohr clarified some key issues in a rather concise as well as in a creatively interesting manner. On the other hand, there might be that the arguments by Einstein et al. addressing possible *limitations* by quantum mechanics, compared to an imagined *more* advanced theory of physics, enfolded some rather deep and relevant points, despite that the philosophical clarity in presenting the argument was not that impressing and that the mathematical dressing of the argument did not support the basic argument *that* much, as much clarified by Bohr's anti-critique. The more subtle and challenging request might be to attempt to access the possibly brilliant *intuition* by Einstein, never mind shortcomings in the published presentation of 1935 *from* the intuition.

David Bohm (1951) supported the anti-critique by Bohr (cf. ibid.: 611), while on the other hand Bohm followed Einstein by arguing that quantum theory should not imply denial or downplaying of causal laws. Bohm's rather profound and constructive discussion of the EPR paradox might thus be said to represent some complementary *superposition* of both Bohr and Einstein.

With respect to causality Bohm pointed out "the role of causal laws in making possible the identification of an object, *whether it changes or not*" (ibid.: 163). In general "an object is identified by the way it *reacts* to forces of various kinds…Since the statement than an object reacts in a *definite* way to forces implies that it obeys causal laws, we conclude that no object can even be *identified* as such unless it obeys causal laws" (ibid.:163f; italics by me).

Elementary particles as protons and electrons do not represent any exception in this regard: "It is from the *reaction* to electric and magnetic forces, and from the ionization of other atoms by the electric forces produced by a charged particle, that an electron or proton is *identified*" (ibid.:163; italics by me).

Bohm noted “that this criterion also includes *seeing* the object with the aid of light” (ibid.: 163; italics by me). Since causal laws are involved in all perception (cf. our previous discussion of projective causality and constitution of percepts), and there are no observations without perception, all observations must obey causal laws.

According to Bohm the role of causal laws in order to identify an object was “certainly no less important” (ibid.: 163) in quantum physics than at the classical level. At the same time Bohm acknowledged Bohr’s complementarity principle for the quantum level, connected to Heisenberg’s uncertainty relation, and did not find it possible to interpret the non-commuting variables of momentum and position as separately and simultaneously existing and precisely defined elements of reality (cf. ibid.: 622f). And “*exact* causal laws would be meaningless in a context in which there were no *precisely* defined variables to which they could apply” (ibid.: 625; italics by me). Bohm interpreted the wave function as describing “the propagation of *correlated potentialities*” (ibid.: 621; italics by me), so that the quantum concept of a potentiality became more fundamental than the notions of momentum and position.

In his general ontology Bohm (1987, 1993) regarded borders distinguishing physical objects not as totally absolute, but more – or less – as dotted lines. While the ontological assumption of complete separation between independent physical objects obviously represented an adequate approximation for theoretical physics at the classical level, Bohm argued that this assumption had to be relaxed with respect to certain phenomena occurring from the quantum level.

Bohm (1951: 624-628) sought to clarify the *interrelation* between the classical and quantum level, as well as between their respective theoretical concepts. Rather than viewing the classical level as some special case from a generalized quantum theory, Bohm argued that the quantum world and the classical world should be understood *complementary* as mutually dependent. One of his points was that quantum theory *presupposes* the classical level because “the last stages of a measuring apparatus are always classically describable” (ibid.: 625). Without such measurements, quantum theory can hardly be said to have any meaning at all. And, if we look at the uncertainty relation between position and momentum, and the related complementarity between wave and particle, this relation does not manifest before in *interaction* with a *classical* system of measuring devices (cf. ibid.: 625, 627).

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If we reflect a bit on the very concept of a *physical wave*, it is implied that the form of a wave as a mathematical concept must have a physical manifestation and carrier, as e.g. sound waves carried by molecules in the air or ocean waves carried by water molecules. Thus, a physical wave must be carried by a huge number of physical particles (or at least by *something* physical as in contrast to the information of the wave pattern itself). Here, the form of the wave appears at a much *larger scale* than each of the physical particles that make up the physical wave. *If* we consider a physical wave to have a complementary state as a particle at the *same* scale as itself, the concept of a physical wave will then tacitly (by transitivity) imply a relation *between* the large-scale particle and the small-scale particles. The wave-particle duality at the quantum level is well known, but transformations between the physical state and the wave state have also been documented as possible at the molecular level where (more or less) the same *information* of the molecule is preserved during the transformation to its specific frequency constellation (cf. i.a. Gariaev et al. 2000, 2011; Montagnier et al. 2011, 2014; Marvi&Ghadiri 2020; Brand et al. 2020). The philosophical point here, of some possible relevance for theoretical physics, is that the conceptual contrast between a physical wave and a physical particle is not an *absolute* opposition, but *relative to the scales* considered adequate for description of the involved phenomena.

Then one may ask: When describing a quantum phenomenon as a wave, and consider the wave to be a *physical* wave and not only a pure *mathematical* notion, what are the physical *sub-entities* that make up the wave? *If* such sub-entities are imagined to exist, then the application of chance causality in describing measurement probabilities in quantum mechanics might not be that *completely* different, after all, from applying chance causality when throwing dices.

In his work *Causality and Chance in Modern Physics* (Bohm 1984; first ed. 1957), Bohm presented a sophisticated reflection on the philosophical categories of causality vs. chance. There are significant overlaps between some key points in Bohm's treatment and our own treatment of chance vs. causality (originally presented in a publication from 1991 and developed without knowledge about Bohm's work), especially where he discusses "chance and necessary causal interconnections" (ibid.: 139-146). In his last work, *The Undivided Universe. An Ontological Interpretation of Quantum Theory* (Bohm

1993), Bohm stated: “To sum up then…in no sense is probability being regarded as a fundamental concept. Rather the properties of the individual system are taken as primary, and probabilities are interpreted in terms of these”.

In his works Bohm presented various *causal* interpretations of physical phenomena and theories often opinioned *not* to be causal, with related discussions of causality in different aspects. In the present context we have focused mostly on his basic points about the role of causality as expressed in more direct relation to his discussion of the EPR paradox.

Bohm followed Einstein in demanding that principle(s) of causality should prevail also in interpretation of and further development of quantum theory. However, Bohm found Einstein’s requirement of one-to-one correspondence between any conceivable E_R with a counterpart E_T to be too strong, and he did not share Einstein’s optimistic belief that a *complete* (physical) theory should be possible to achieve. Bohm wrote:

> *A complete theory will always require concepts that are more general than that of analysis into precisely defined elements. We may probably expect that even the more general types of concepts provided by the present quantum theory will also ultimately be found to provide only a partial reflection of the infinitely complex and subtle structure of the world. As science develops, we may therefore look forward to the appearance of still never concepts, which are only faintly foreshadowed at present* (Bohm 1951: 622)

The last words by Bohm may be taken as rather prophetic when reflecting upon the immense contributions to progress in theories of physics, as well as to related progress in mathematics, chemistry, biology and technology, achieved by **Ruggero Maria Santilli**.

Some basics about the hadronic sciences initiated by Santilli

Santilli initiated the establishment of vast new fields of scientific theory and discovery denoted by the umbrella term *hadronic science(s)* covering hadronic mathematics, hadronic mechanics, hadronic chemistry, hadronic biology, and hadronic technology. The main reason for the choice of the term ‘hadronic’ was that Santilli initially approached the hadrons in the nucleus by

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regarding elementary particles as *extended* particles, in distinction to conventional quantum mechanics treating elementary particles more simplistically *as if* they were point particles, and found it necessary to develop novel mathematics in order to adequately analyze extended particles. Next, this mathematics and related development of novel physical theory showed both potent and rather necessary when also addressing plural issues of physics *outside* nuclear physics. Hadronic mathematics was structured by a layered architecture where a novel layer of *isonumbers* emerged as a "second floor" above natural numbers where the "elevator" (these metaphors are mine, not Santilli's) between the two floors was constituted by the *isotopic element* which indicated the transform of the conventional unit, represented by the natural number 1, to *another* (arbitrary) unit whereby a whole field of novel numbers emerged from the basic *relation* between the two units. A further layer of *genonumbers* emerged as a "third floor" of non-commuting numbers accounting more directly for irreversibility as category. An even further layer of *hyperstructural numbers* emerged as a "fourth floor" of numbers *themselves* having an intrinsic layered structuring, somewhat similar to one hand possessing plural fingers and thus being multi-valued.

In the architecture of hadronic number theory the numbers residing at each level were included as a *sub-set* of the numbers residing at the level above (that is, when taking the "elevator" down again and performing downwards *degeneration* as the opposite transform of upwards *lifting*). Further, in the architecture of hadronic number theory each level in the number landscape had a "mirrored twin landscape" of numbers, denoted its *isodual*.

In hadronic mathematics *hadronic geometry* corresponds isomorphically to the architecture of hadronic number theory. Although Santilli by far has been the most innovative and important scientist contributing to the development of the hadronic sciences, by a rough estimate some 2-300 scientists world wide have also published contributions in more or less specialized fields inside the hadronic sciences. Some obviously important contributions have been *T. Vougiouklis* creating much of the sufficiently abstract hyperstructures to inspire Santilli's mathematical inventions; *S. Georgiev* lifting the ordinary calculus to the more complex *isocalculus* published in voluminous detail; *J. Dunning-Davis* lifting the laws of thermodynamics to a more general formulation by isomathematics; *A. Animalu* pioneering the field of iso-superconductivity (along with Santilli himself); and *C. Illert* pioneering hadronic biology (along with Santilli himself).

With regard to hadronic geometry the achievements by Illert (1995; cf. also Johansen 2008a, 2008b, 2008c) hold extraordinary significance. Illert wanted to find a universal formula to describe the growth pattern of sea shells, with as few variables as possible, compared with a data base covering some 100 000 empirical cases of sea shell growth. This showed not to be possible by applying Euclidian geometry, nor with the geometries of Minkowski (applied in Einstein SR) or of Riemann (applied in Einstein GR), while it *did* show possible by applying hadronic geometry. Further, formulated by the mathematical concepts of hadronic geometry, the universal formula showed to be surprisingly simple, entailing only two basic variables, while at the same time, for certain *particular* species of sea shells, the growth pattern, as described by hadronic geometry, included non-trivial information flows jumping forwards and backwards as perceived from our ordinary experience of Euclidian time. This circumstance could be interpreted as *further* support for the adequacy and potency of hadronic geometry, since such non-trivial time flows were included as possible at the genotypic level of hadronic geometry, and also – before Illert's discovery – had been *predicted* by Santilli to later become discovered in empirical systems!

Santilli himself presented results in nuclear physics providing further support to the relevance and potency of hadronic geometry. The discovery by Illert stands out as rather spectacular since it provided crucial support to hadronic geometry from an extensive study at the *classical* level involving much more complex entities (sea shells) than elementary particles. More generally regarded, this was not that much of a surprise from hadronic mathematics, since higher and more complex levels of hadronic geometry, in this case: the level of *genotopic* geometry, were assumed to become more relevant for analysis the more complex the targeted empirical system was assumed to be.

When it showed not possible to find a universal formula for sea shell growth at the *classical* level by means of the Minkowski geometry of Einstein SR, nor by means of the Riemann geometry of Einstein GR, while it *did* show possible to find by *hadronic* geometry, it ought to suffice to give Santilli, the main *inventor* of hadronic mathematics and geometry, a very strong voice with regard to an adequate *hadronic* reconsideration of the EPR paradox and the implied relations between the classical and the quantum level.

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(For an introductory overview of the hadronic sciences, see Gandzha and Kadeisvili (2011). For a general bibliography per 2008, see Institute for Basic Research (2008). For some key publications, see Santilli 1994, 2001, 2003, 2006, 2008.)

Santilli's reconsideration of the EPR discussion from achievements in hadronic mechanics

From the very onset of developing hadronic mechanics the whole body of conventional quantum mechanics, addressing elementary particles as idealized point particles instead of as extended particles, had to be considered a *sub-set* of, and explained *from,* the lifted and broader theory of hadronic mechanics, due to being based on simplified assumptions and thereby scientifically limited. Quite recently, Santilli (2019, 2020) has directly addressed and provided a rather extensive reconsideration of the EPR discussion, based on achievements by hadronic mechanics. Although he also previously has given substantial comments to the EPR discussion (cf. Santilli 1998), the recent publications of Santilli offer much more and sharpened foods for thought.

In his publications Santilli has often displayed a humble attitude with respect to (anyone) ever achieving a complete or *final* theory about physical/empirical systems, much aligned with the attitude displayed by Bohm in the quotation we referred above before as a transition to introducing Santilli. Santilli (2019) states that " 'completion of quantum mechanics' is used in Einstein's sense for the intent of honoring his memory", and Santilli (2020) claims "there is no doubt that the 'completion' of quantum mechanics is, by far, Einstein's most important legacy". Taken together, we may interpret this as Santilli regarding development *towards* a complete theory in the sense of Einstein as adequate and highly important, and that various achievements of hadronic mechanics as a matter of fact *have* provided important results along that line.

Let us shortly address at least a few key points in Santilli's reconsideration of the EPR discussion from achievements by hadronic mechanics.

Reversible vs. irreversible time. Some of the objections against the EPR argument had as necessary condition the conventional axiom of quantum theory where *time* was considered invariant with respect to time-reversal, i.e. that time at the quantum level could flow backwards with the same probability as forwards as conceived in Euclidian time. The obvious incompatibility between this axiom and the arrow of thermodynamics at the classical level, became resolved by hadronic *mechanics* lifting *both* classical descriptions and quantum descriptions to a *genotypic* level of description which basically accounted for irreversibility of time *across* the distinction between the classical and quantum level (while at the same time allowing three novel and non-trivial categories of time as necessarily "attached", categorically more secondary, to this irreversibility). It may be of some interest to note this theoretical achievements being somewhat foreshadowed by Bohm's closing note in his thick book *Quantum Theory*:

> *We propose also that irreversible processes taking place in the large scale environment may also have to appear explicitly in the fundamental equations describing phenomena at the nuclear level.*
>
> (Bohm 1951: 628)

Radical shrinking of the span of the insecurity relation. The *isotopic elements* required for adequate descriptions by iso-mechanics (or geno-mechanics) of coordinates and momenta for particles within *hyperdense* media (as the interior of hadrons, nuclei, or stars), have showed to always be *very small*. This reduces rather radically, and proportional to the density of the non-empty medium, the *span* of the insecurity relation between position and momentum when an adequate description by means of hadronic mechanics are being applied. This shrinking could not be discovered by treating elementary particles as point particles instead of as extended particles. From this discovery Santilli provided a mathematical formulation of the so-called *iso-deterministic iso-principle*, implying that the product of (iso)standard (iso)deviations for (iso)coordinates and (iso)momenta progressively *approaches* a classical description for extended particles with the increase in density of the medium.

Generalized lifting and revision of the conventional wave function. By lifting the description of the conventional wave function to a more generalized description by iso-mechanics, Santilli argued that it was possible to include a representation of the *attractive force* between identical electron pairs in valence coupling (the so-called fifth force, or contact force, connected to the notion of

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the *iso-electron*, with related orbit and magnetic polarization, in hadronic chemistry). This more advanced description of the wave function in quantum mechanics gave support to Einstein's suspicion that the wave function as described in conventional quantum mechanics did not represent a final or complete description.

It should be indicated already from these few key points that a scientifically competent discussion of EPR today, both philosophically and more directly related to theoretical physics, needs to be upgraded to the present state of *de facto* forefront theoretical physics.

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A General relativistic theory of electromagnetic field and its connection with Planck's constant

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Abstract:

A general relativistic theory of electromagnetic (EM) field is developed by constructing an EM tensor which is an outer product of EM vector potentials. The Einstein's equations are modified using this EM tensor and the coupling constant is found to be inversely proportional to Planck's constant. Maxwell's equations, in their current form, do not provide equations of motion; equations of motions are provided by Lorentz force equations which do not follow from Maxwell's equations. However, with the proposed theory of EM field, the modified Maxwell's equations lead to Lorentz force equations. The derived wavefunction for photons may be interpreted deterministically as the slowly varying envelope of the EM potential or statistically as the absolute square of the wavefunction is the probability density of photons.

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1. Introduction

The problems of classical electrodynamics can be divided into two classes. (i) The charge and current distributions are known and the resulting electromagnetic (EM) fields are calculated, and (ii) the external EM fields are specified and the motion of charged particles under the influence of EM fields are calculated [1]. When these two problems are combined as in the case of bremsstrahlung, the classical treatment is a two-step process: (i) the motion of charged particle in the external field is determined ignoring the emission of radiation by the charged particles, and then (ii) Maxwell's equations are solved to find the EM fields taking into account the trajectory of the moving charges. As pointed by Jackson [1], this way of handling problems in electrodynamics is of approximate validity since the emitted radiation due to accelerating charges carries off energy and momentum, and so must influence the subsequent motion of charged particles. A correct treatment must include the reaction of radiation on the motion of sources. A classical treatment of reactive effects of the radiation does not exist [1]. However, a semiclassical theory in which the field is treated classically and the charged matter is treated quantum mechanically, contain the back-action of the radiation field on the charge [2]. In Maxwell's theory (classical theory), the field equations do not provide the equations of motion for charged particles; equations of motion are given separately by Lorentz force equations. In contrast, in the theory of gravitation, the equation of motion of mass points follow from Einstein's field equations [3]. Bergmann [4] attributed this to the fact that the field equations of gravitation satisfy four identities, while Maxwell's equations satisfy only one. Another important difference is that Maxwell's equations, in the current form, are linear for vacuum. If solutions are obtained by the linear combinations, charged particles will not interact with each other. In contrast, Einstein's field equations are nonlinear; even the classical interaction of mass points is brought about by the nonlinear terms in the field equations [5,6]. Bergmann pointed out that a field theory can lead to laws of motion only if the (i) field equations satisfy at least four identities, and (ii) they are nonlinear [4]. In this paper, four identities that are satisfied by the EM vector potentials are first derived and then a general relativistic theory of EM field is developed. The resulting EM field equations are nonlinear and the equations of motion that resemble the Lorentz force equations follow from these nonlinear EM field equations.

Tolman, Ehrenfest, and Podolsky [7] investigated the gravitational interaction between two electromagnetic waves in vacuum and showed that the "test rays of light" in the neighborhood of an intense electromagnetic pulse are not deflected when the test ray is propagating parallel with the intense pulse. Later, Scully [8] showed that when a probe pulse and an intense laser pulse are propagating parallel with their velocities less than the speed of light in vacuum, interesting gravitational interaction between them can occur. For example, when an intense laser pulse propagates in a dielectric waveguide and the probe pulse propagates in the bulk dielectric (i.e. outside the waveguide) in the same direction as that of intense pulse, the probe pulse undergoes a small shift (towards the intense pulse) due to gravitational interaction between them. There have been many attempts to combine the theory of gravitation and electromagnetesim, which is summarized by Santilli [9]. Kaluza combined electromagnetism with gravitation in 5 D [10] and Klein applied this idea to quantum theory [11], laying a basis for various versions of string theory [12]. As in the Kaluza-Klein theory, the tensor formed by the outer product of electromagnetic vector potentials plays an important role in our approach.

Although Einstein's general theory of relativity (GR) is well accepted, alternative theories and modified GR theories have drawn significant interest [10-14]. The GR has been verified for masses on length scales of the solar system, but it faces challenges on quantum and cosmological scales [14]. In this paper, we retain the structure of GR theory, but introduce a novel electromagnetic tensor as an additional source term in Einstein's equations. Without this term, the modified Maxwell-Einstein equations do not lead to Lorentz Force equations. The relation between matter and EM field can be interpreted from two different standpoints [13]. The first is the *unitarian* standpoint which assumes only one entity, the EM field. The particles of matter are considered as singularities of EM field and mass is a derived notion to be expressed by EM field energy (or EM mass). The second is the *dualistic* standpoint which takes particles and fields as two different entities. The particles are the sources of the field, but are not a part of the field [13]. In classical electrodynamics, charged particles (the cause) are distinguished from EM field (the effect). The charged particles are considered as the sources for the EM field corresponding to dualistic standpoint. In this paper, it is postulated that the cause and effect are inseparable and the charge is embedded in the field itself. Using this idea, an electromagnetic tensor which is an outer product of EM vector potentials is constructed and the divergence of this tensor satisfies four identities. In the theory of gravitation, the constant appearing in Einstein's field equations is connected to the gravitational constant. Similarly, in the proposed theory, the coupling constant (κ) is connected to Planck's constant. Under the slowly varying envelope approximations, it is shown that the nonlinear EM field equations reduce to Schrodinger equation with one of the potential terms being the self-trapping potential. It is also shown that the rate of change of mean momentum of the EM wave packet is given by Newton's laws with one of the forces being Lorentz force. Using a weak field approximation for the metric tensor, modified Manakov equations are derived for orthogonal polarization components of electromagnetic

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fields. Manakov equations have been used to describe the evolution of orthogonal polarization components in a nonlinear fiber [15].

The modified Maxwell-Einstein equations lead to a wavefunction $\tilde{\phi}$ for photons, which can be interpreted in two different ways: (i) deterministic interpretation - $\tilde{\phi}$ is the slowly varying envelope of EM potential and momentum density of the EM wave is $\omega \operatorname{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]/2$, where ω is the mean frequency of the EM field, (ii) statistical interpretation - $|\tilde{\phi}(x,y,z)|^2$dxdydz represents the probability that a photon is present in the region between (x,y,z) and $(x+dx, y+dy, z+dz)$, and the probability current density is $\mathbf{j}(r,t) = \frac{\hbar}{2m}\operatorname{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]$. In the classical limit, Schrodinger equation leads to Lorentz force equations.

Next, by treating the nonlinear effects (i.e. spacetime curvature) as a small perturbation on the linear fundamental mode of a rectangular cavity resonator, a dispersion relation is derived. It is found that the resonant frequency of the cavity is shifted by an amount proportional to the square of the EM energy stored in the cavity, due to spacetime curvature. The dispersion relation is expressed as a special relativistic equation describing the relation between the EM energy, the EM momentum and rest mass, from which it is found that the coupling constant is inversely proportional to Planck's constant.

This paper is organized as follows. In Section 2, an EM tensor which is an outer product of EM vector potentials is constructed and the EM energy-momentum tensor appearing in Einstein's equations is modified using this tensor. The field equations are solved under the weak field approximations in Section 3 and it is shown that the modified Maxwell's equations reduce to modified nonlinear Schrodinger equation or modified Manakov equations, which lead to Lorentz force equations. Section 4 deals with the analysis of rectangular cavity resonator under the weak field approximation and the impact of spacetime curvature on the resonant frequency of the cavity is investigated.

2. Electromagnetic Tensor and Field Equations

The electromagnetic potential A^{μ} may be written as

$$A^{\mu} = \eta \frac{dx^{\mu}}{d\lambda} = \eta U^{\mu}, \tag{1}$$

where η is a scalar similar to charge (although of different dimension), λ is a parameter along the world line, and U^{μ} are the components of 4-velocity. Consider a locally inertial frame *O*. We follow the notations of [16]. Let

$$\vec{\mathbf{U}} \underset{O}{\rightarrow} (U^0, U^1, U^2, U^3). \tag{2}$$

In the frame *O*, the components of $\vec{\mathbf{U}}$ are constants along the worldline at a point *P*, i.e.,

$$\frac{d\vec{\mathbf{U}}}{d\lambda} = 0, \quad \text{or} \quad \frac{dU^{\alpha}}{d\lambda} = U^{\alpha}{}_{,\beta} \frac{dx^{\beta}}{d\lambda} = 0. \tag{3}$$

In the frame *O*, the Lorentz gauge condition is

$$A^{\mu}{}_{,\mu} = 0. \tag{4}$$

With the definition of Eq. (1), the Lorentz condition is nothing but the conservation of η. Using Eq. (1) in Eq. (4) and using $U^{\mu}{}_{,\mu} = 0$, we find

$$\frac{d\eta}{d\lambda} = 0. \tag{5}$$

Using Eqs. (5) and (3), we find

$$\frac{dA^{\alpha}}{d\lambda} = \frac{d\eta}{d\lambda} U^{\alpha} + \eta \frac{dU^{\alpha}}{d\lambda} = 0. \tag{6}$$

From Eq. (6), we have

$$\frac{dA^{\alpha}}{d\lambda} = A^{\alpha}{}_{,\beta} U^{\beta} = 0, \quad \text{or} \quad A^{\alpha}{}_{,\beta} A^{\beta} = 0. \tag{7}$$

Using Eqs. (7) and (4), we find the conservation relation

$$(A^{\alpha} A^{\beta})_{,\beta} = A^{\alpha} A^{\beta}{}_{,\beta} + A^{\alpha}{}_{,\beta} A^{\beta} = 0. \tag{8}$$

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Since (8) is a tensor equation, it is valid in any coordinate frame. So, we have

$$(A^{\alpha} A^{\beta})_{;\beta} = 0, \tag{9}$$

where semicolon denotes the covariant derivative. We define an electromagnetic tensor

$$T^{\alpha\beta} = A^{\alpha} A^{\beta}, \tag{10}$$

with the conservation relation

$$T^{\alpha\beta}_{\;\;;\beta} = 0. \tag{11}$$

To the best of our knowledge, the conservation relation (8) is not known in the literature. We choose the unit of A^{α} as $\sqrt{J/m}$ so that the unit of energy density, $(E^2 + H^2)/2$ is J/m^3 and the dimension of $T^{\alpha\beta}$ is J/m. Hence, $T^{\alpha\beta}$ may be termed as power-force tensor.

In fact, the tensor $T^{\alpha\beta}$ is similar to stress-energy tensor for 'dust', which is given by [4,15,16]

$$T^{\alpha\beta}_{dust} = \rho U^{\alpha} U^{\beta}, \tag{12}$$

where ρ is the energy density. Eq. (10) may also be written as

$$T^{\alpha\beta} = \eta^2 U^{\alpha} U^{\beta}, \tag{13}$$

with η^2 playing the role of ρ, although their dimensions are different.

To verify the validity of Eq. (8), consider a forward propagating plane wave

$$A^{j} = D^{j} \exp[i(k_{\nu} x^{\nu})], \qquad j = 1, 2. \tag{14}$$

Using the Lorentz gauge conditions, we find

$$k_1 D^1 = -k_2 D^2. \tag{15}$$

Using Eq. (15), we find that Eq. (8) is automatically satisfied for a forward propagating plane wave. To verify if Eq. (8) is satisfied when the EM field is confined, we solved Maxwell's equations using the finite difference time domain (FDTD) technique for a rectangular cavity resonator, which is a rectangular metallic waveguide that is closed off at both ends by metallic walls (see Fig. 1).

The length of the cavity is L_j in x^j direction and for simplicity, we assumed that $L_j=L$. The walls of the cavity are assumed to be a perfect conductor so that the tangential component of the electric field is zero at the conducting walls. We excited this cavity on the left side with a propagating plane wave given by Eq. (14) and the constants D^j satisfy Eq. (15) (for example, there is an antenna on the left wall which emits the EM field of the form given by Eq. (14)). Numerical solution of the Maxwell's equations showed that the Lorentz gauge condition, Eq. (4) and conservation relations, Eq. (8) are satisfied at each point in the cavity for $t >= 0$.

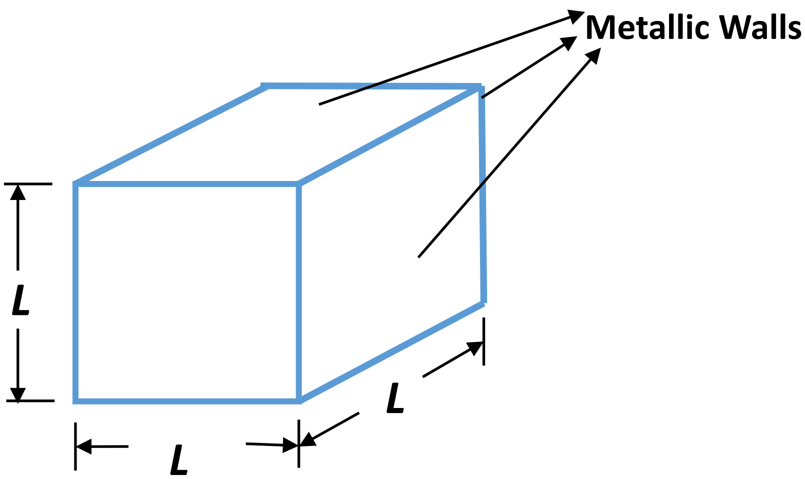

Figure 1. A rectangular cavity resonator.

2.1. Einstein's Field Equations

Einstein's field equations are given by [4,9,17]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} [T_{\mu\nu}^{mat} + T_{\mu\nu}^{em}], \quad (16)$$

where $T_{\mu\nu}^{mat}$ and $T_{\mu\nu}^{em}$ are the energy-momentum tensors of matter and electromagnetic field, respectively. For "dust", we have

$$T_{\mu\nu}^{mat} = \rho V_\mu V_\nu, \quad (17)$$

where ρ is the energy density of the matter and V_μ is its four-velocity. Santilli [9] has analyzed the gravitational field of partons under the assumptions that (i) gravitational field of any massive body is partially due to the EM fields of its charged basic constituents (weak assumption) and (ii) gravitation field is entirely due to the EM fields (strong assumption).

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When only the electromagnetic field is present (i.e. $\rho = 0$), Einstein's field equations are given by [4,17]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T^{em}_{\mu\nu}, \qquad (18)$$

where the EM energy-momentum tensor is given by

$$\begin{aligned} T^{em}_{\mu\nu} &= F_{\mu\gamma} F_{\nu}{}^{\gamma} - \frac{1}{4} g_{\mu\nu} F_{\gamma\delta} F^{\gamma\delta}, \\ F_{\gamma\delta} &= A_{\delta,\gamma} - A_{\gamma,\delta}, \end{aligned} \qquad (19)$$

and

$$(T^{\mu\nu})^{em}_{;\nu} = 0. \qquad (20)$$

According to the GR with the EM tensor given by Eq. (19), there is no gravitational interaction between two EM fields propagating in parallel in vacuum [7]. In this paper, we modify the EM tensor as

$$T'^{em}_{\mu\nu} = \alpha A_\mu A_\nu + T^{em}_{\mu\nu}, \qquad (21)$$

where α is a constant. Since $A_\mu A_\nu$ is the power-force tensor, α has a dimension of $1/m^2$.Using Eqs. (11) and (20), we find this new tensor to be divergence-free, i.e.

$$(T'^{\mu\nu})^{em}_{;\nu} = 0. \qquad (22)$$

Since $G^{\mu\nu}{}_{;\nu} = 0$, using $T'^{em}_{\mu\nu}$ instead of $T^{em}_{\mu\nu}$ in Eq. (18), we find

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa A_\mu A_\nu - \frac{8\pi G}{c^4} T^{em}_{\mu\nu}, \qquad (23)$$

where $\kappa = -8\pi\alpha G / c^4$. Since we chose the unit of A^μ to be $\sqrt{J/m}$, κ has the dimension of $1/(Jm)$, which is the same as that of $1/(hc)$, where h is Planck's constant. Santlli [18] discusses in detail the importance of the forgotten Freud identity [19] of Riemannian geometry that requires a first order source on the right hand side of Einstein's equations, as in Eq. (23).

Equation (23) may also be derived using the following Lagrangian density for the electromagnetic field,

$$\mathcal{L} = \left[2\kappa A^\mu A_\mu - \frac{8\pi G}{c^4} F^{\mu\nu} F_{\mu\nu} \right] \sqrt{|g|}, \qquad (24)$$

where g is the determinant of the matrix of metric components. The effect of the last term on the right hand side (RHS) of Eq. (23) on the evolution of electromagnetic field is already studied in Refs. [7-8]. While studying the evolution of electromagnetic field, we expect that the impact of the term with cosmological constant Λ is negligible and hence, we set $\Lambda = 0$. In this paper, we focus only on the impact of the first term on the right hand side of Eq. (23) on the spacetime curvature and the subsequent changes in the evolution of electromagnetic field.

2.2 Maxwell's Equations in Curved Spacetime

In a locally inertial frame, Maxwell's equations are given by

$$\eta^{\alpha\beta} \frac{\partial^2 A^\nu}{\partial \xi^\alpha \partial \xi^\beta} = 0, \tag{25}$$

where $\eta^{\alpha\beta}$ is the Minkowski metric. The Lorentz gauge conditions are

$$\frac{\partial A^\nu}{\partial \xi^\nu} = 0. \tag{26}$$

Using the transformation,

$$A'^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} A^\nu, \tag{27}$$

where A'^μ is the electromagnetic potential in the new coordinates $\{x^\mu\}$. Substituting Eq. (27) in Eqs. (25) and (26) and with $A'^\mu \to A^\mu$, we find

$$\begin{aligned} g^{\mu\nu} [A^\alpha_{,\mu,\nu} + \Gamma^\alpha_{\mu\sigma,\nu} A^\sigma + 2\Gamma^\alpha_{\mu\sigma} A^\sigma_{,\nu} \\ + \Gamma^\alpha_{\nu\sigma} \Gamma^\sigma_{\mu\rho} A^\rho - \Gamma^\sigma_{\mu\nu} (A^\alpha_{,\sigma} + \Gamma^\alpha_{\sigma\rho} A^\rho)] = 0, \end{aligned} \tag{28}$$

$$A^\nu_{,\nu} + \Gamma^\mu_{\mu\sigma} A^\sigma = 0, \tag{29}$$

$$g^{\mu\nu} = \eta^{\alpha\beta} \frac{\partial x^\mu}{\partial \xi^\alpha} \frac{\partial x^\nu}{\partial \xi^\beta}. \tag{30}$$

It may be noted that Eqs. (26) and (27) could as well be obtained using Einstein's principle of equivalence (comma-goes-to-semicolon rule [16]). In fact, Eqs. (28) and (29) describe the Maxwell's equations in curvilinear coordinates whether or not the spacetime is flat. For example, in a flat spacetime with spherical coordinates, we have

$$g_{00} = -1,\ g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2 \sin^2\theta, \tag{31}$$

and rest of the metric coefficients are zero. If $\vec{\mathbf{A}} \underset{O}{\to} (A^0, 0, 0, 0)$, Eq. (26) reduces to

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A^0}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A^0}{\partial \theta}\right) + \\ \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 A^0}{\partial \phi^2} - \frac{1}{c^2}\frac{\partial^2 A^0}{\partial t^2} = 0, \end{aligned} \tag{32}$$

which is nothing but Maxwell's equations in spherical coordinates. However, as the magnitude of $\vec{\mathbf{A}}$ increases, metric coefficients deviate from Eq. (31). Now, they are determined by Einstein's equation (23), and Eq. (28) provides the evolution of EM field. Equations (23) and (28) form a coupled system of equations that govern the

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evolution of metric coefficients and EM field, respectively, with the conservation relations (22) and (29). In the next Section, we solve this system of equations under the weak field approximations.

3. Weak field approximations

We use a first order perturbation theory and assume that

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{33}$$

where $|\kappa h_{\mu\nu}| << 1$, and $\eta_{\mu\nu}$ is the Minkowski metric. We use a harmonic coordinate system, for which

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0. \tag{34}$$

Using Eq. (33) and Eq. (34) in Eq. (23) and ignoring the terms proportional to κ^2 and higher, we obtain [17]

$$\Box h_{\mu\nu} = 2\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\mu}_{\ \mu}\right), \tag{35}$$

where $\Box$ is the four-dimensional Laplacian operator and $T_{\mu\nu} = A_\mu A_\nu$.

3.1 A^1 only:

We consider the case for which $A^2 = A^3 = A^0 = 0$, corresponding to an electromagnetic wave with the electric field in x^1-direction and the magnetic field in x^2-direction. For this case, Eq. (35) reduces to

$$\begin{aligned} &\Box h_{00} = (A_1)^2, \\ &h_{11} = h_{00},\ h_{33} = h_{22} = -h_{00}. \end{aligned} \tag{36}$$

Ignoring the terms proportional to κ^2 and higher, Maxwell's equations (26) and Lorentz gauge condition (29) become

$$\Box A^1 = -\kappa\left(\frac{(A^1)^3}{2} + \eta^{\mu\nu}h_{11,\mu}A^1_{\ ,\nu}\right), \tag{37}$$

$$A^1_{\ ,1} - h_{11,1}A^1 = 0. \tag{38}$$

where

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta},\ \text{and}\ A_\mu = \eta_{\mu\alpha}A^\alpha. \tag{39}$$

It may be noted that Eqs. (36)-(38) are Lorentz invariant. Using Eqs. (34) and (38), we find

$$A^1_{\ ,1} = 0 \quad \text{and}\ h_{11,1} = 0. \tag{40}$$

The first term on the right hand side of Eq. (37) has the form of third order nonlinear effect in nonlinear optics, which is responsible for Kerr effect or self-phase modulation (SPM) and four wave mixing (FWM) [20-21]. Hence, the effect of spacetime curvature may be interpreted as the nonlinear change in refractive index. Let

$$A^1 = \frac{1}{2}\left[\tilde{\phi}(\mathbf{r},t)e^{-i\omega t} + c.c.\right]. \tag{41}$$

Using Eq. (1), let the envelope $\tilde{\phi}$ be $\tilde{\eta}\tilde{U}^1$, where $\tilde{\eta}$ and $\tilde{U}^1$ are the envelopes of η and U^1, respectively. Using the slowly varying envelope approximation, we ignore the second order derivative of $\tilde{\phi}$ with respect to *t* and now, Eq. (37) reduces to a modified nonlinear Schrodinger equation (NLSE),

$$\frac{i\omega}{c^2}\frac{\partial\tilde{\phi}}{\partial t} + \frac{1}{2}\nabla^2\tilde{\phi} + \frac{3\kappa}{16}|\tilde{\phi}|^2\tilde{\phi} = -\frac{\kappa}{2}\left[\eta^{jk}h_{11,j}\tilde{\phi}_{,k} + i\omega h_{11,0}\tilde{\phi}\right]. \tag{42}$$

In deriving Eq. (42), we have ignored the third harmonic component proportional to $e^{-i3\omega t}$. In nonlinear optics, while deriving the nonlinear Schrodinger equation (NLSE) from the nonlinear wave equation, the third harmonic component is ignored [21]. Unless there is a special phase matching, the growth of third harmonic component is small. The second and third terms in Eq. (42) denote diffraction and Kerr effect, respectively. When the diffraction/dispersion balances the Kerr effect, a spatial/temporal pulse propagates without pulse broadening and such a pulse is called soliton [22,23]. Hence, κ may be interpreted as the nonlinear coefficient of vacuum. In the absence of the terms on the right hand side, Eq. (42) represents the three-dimensional (3-D) NLSE. In the 1-D case, NLSE admits the well-known soliton solutions [21-23]. Interestingly, in the 1-D case, the terms on the right hand side of Eq. (42) have the forms similar to self-steepening and Raman effects in nonlinear fiber optics [21].

3.1.1. *Lorentz Force*

Electric field intensity E_1 and magnetic field intensity H_2 are related to the vector potential A^1 by

$$E_1 = -A^1_{,0} \text{ and } H_2 = A^1_{,3}. \tag{43}$$

Let

$$E_1 = \frac{1}{2}\left[\tilde{E}_1 e^{-i\omega t} + c.c.\right],$$
$$H_2 = \frac{1}{2}\left[\tilde{H}_2 e^{-i\omega t} + c.c.\right]. \tag{44}$$

Using Eqs. (41), (43) and (44), and using the slowly varying envelope approximation, we find

$$\tilde{E}_1 = \frac{i\omega\tilde{\phi}}{c} \text{ and } \tilde{H}_2 = \tilde{\phi}_{,3}. \tag{45}$$

Using Eq. (44), electromagnetic momentum density may be written as [1]

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$$p^3 = \frac{1}{2}\mathrm{Re}\left[\tilde{E}_1\tilde{H}_2{}^*\right] = \frac{-\omega}{2c}\mathrm{Im}\left[\tilde{\phi}\tilde{\phi}^*{}_{,3}\right], \qquad (46)$$

and the energy density is [1]

$$\mathrm{E} = \tfrac{1}{4}\left[|\tilde{E}_1|^2 + |\tilde{H}_2|^2\right]. \qquad (47)$$

We multiply Eq. (40) by $\tilde{\phi}^*{}_{,3}$, add its complex conjugate and integrate over the volume $dV = dx^1dx^2dx^3$. First, consider the first term of Eq. (42):

$$\begin{aligned}\frac{i\omega}{c}\int[\tilde{\phi}_{,0}\tilde{\phi}^*_{,3} - \tilde{\phi}^*_{,0}\tilde{\phi}_{,3}]dV &= \frac{-\omega}{c}\frac{d}{dx^0}\int \mathrm{Im}(\tilde{\phi}\tilde{\phi}^*_{,3})dV, \\ &= 2\frac{d\left\langle p^3\right\rangle}{dx^0}.\end{aligned} \qquad (48)$$

Next consider the third term:

$$\begin{aligned}\frac{-6\kappa}{16}\mathrm{Re}\int|\tilde{\phi}|^2(\tilde{\phi}\tilde{\phi}^*_{,3})dV &= \frac{-6\kappa}{16}\mathrm{Re}\int|\tilde{\phi}|^2\,\tilde{\eta}\tilde{U}^1\tilde{H}^*_2 dV, \\ &= 2\,\mathrm{Re}\int\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3 dV,\end{aligned} \qquad (49)$$

where

$$\rho = \frac{-3\kappa}{16}|\tilde{\phi}|^2\,\tilde{\eta}, \qquad (50)$$

and $\tilde{\eta}$ is the complex envelope of η. ρ may be interpreted as the density of the embedded charge. Now Eq. (42) leads to

$$\frac{d\left\langle p^3\right\rangle}{dx^0} = \mathrm{Re}\left\langle\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\right\rangle - \left\langle h_{00,3}\mathrm{E}\right\rangle + \left\langle h_{33,0}\,p^3\right\rangle, \qquad (51)$$

where the subscript 3 on the first term on the RHS refers to the z-component of $\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*$. The first term on the RHS of Eq. (51) represents the Lorentz force on the embedded charge. It can be shown that in this simple case of transverse electromagnetic wave, the expectation of the Lorentz force is zero. Nevertheless, Eq. (51) shows that the equation of motion is built into Einstein-Maxwell's equations. In contrast, the conventional Maxwell's equations in vacuum do not provide the equations of motion for the charged particle; it has to be supplemented with Lorentz force equations to describe the interaction of charge and electromagnetic field. It may be possible that the time-independent solutions of Eqs. (23) and (28) correspond to elementary electric charges and their interactions would have the form similar to Eq. (51). The second and third terms on the right hand side of Eq. (51) are similar to those present in the Einstein's theory of gravitation under the weak field approximations, except that Eq. (51) has expectation operators.

3.1.2 Interpretation of $\tilde{\phi}$:

In Section 4.3, it will be shown that

$$\kappa = \frac{2(2\pi)^{3/2}}{\hbar c} \tag{52}$$

and using Eq. (52), Eq. (42) may be rewritten as

$$i\hbar\frac{\partial\tilde{\phi}}{\partial t} + \frac{\hbar^2}{2m_{eff}}\nabla^2\tilde{\phi} + V\tilde{\phi} = 0, \tag{53}$$

where

$$V = \frac{3}{8}|\tilde{\phi}|^2 + \left[\eta^{jk}h_{11,j}\tilde{\phi}_{,k}/\tilde{\phi} + i\omega h_{11,0}\right], \tag{54}$$

and

$$m_{eff} = \frac{\hbar\omega}{c^2} \tag{55}$$

is the effective mass of the wave packet. The potential V consists of self-trapping potential (the first term on the RHS of Eq. (54)) and the other terms are due to spacetime curvature. $\tilde{\phi}$

could be interpreted in two different ways.

(i)Deterministic interpretation : $\tilde{\phi}$ is the slowly varying envelope of EM potential A^1 and the momentum density is given by the Poynting vector, $\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^* = -\omega\,\mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]/(2c)$.

(ii)Statistical interpretation: $|\tilde{\phi}\,(x,y,z)|^2$dxdydz represents the probability that a photon is present in the region between (x,y,z) and $(x+dx, y+dy, z+dz)$, and the probability current density is

$$\mathbf{j}(r,t) = \frac{\hbar}{2m}\mathrm{Im}[\tilde{\phi}\nabla\tilde{\phi}^*]\,. \tag{56}$$

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Using Ehrenfest's theorem, Eq. (51) can be retrieved, i.e.

$$\begin{aligned}\frac{d\left\langle p^3\right\rangle}{dx^0} &= -\frac{(\nabla V)_3}{c} \\ &= \operatorname{Re}\left\langle \rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\right\rangle - \left\langle h_{00,3}\mathrm{E}\right\rangle + \left\langle h_{33,0}p^3\right\rangle.\end{aligned} \tag{57}$$

3.2 A^1 and A^2 only:

In this section, we assume that $A^3 = A^0 = 0$. Using $T_{11} = (A_1)^2, T_{22} = (A_2)^2, T_{12} = A_1A_2$ and the rest of $T_{\mu\nu}$ being zero, Equation (35) becomes

$$\Box h_{00} = (A_1)^2 + (A_2)^2, \tag{58}$$

$$\Box h_{11} = (A_1)^2 - (A_2)^2, \tag{59}$$

$$\Box h_{12} = 2A_1A_2, \tag{60}$$

$$h_{11} = -h_{22} \text{ and } h_{00} = -h_{33}, \tag{61}$$

and the rest of $h_{\mu\nu}$ are zero. The Maxwell's equations (28) take the following form

$$\begin{aligned}\Box A^1 = \kappa&\left[-\frac{(A^1)^2+(A^2)^2}{2}A^1 - \eta^{\mu\nu}h_{11,\mu}A^1{}_{,\nu} + h^{\mu\nu}A^1{}_{,\mu,\nu}\right] \\ &-\kappa\left(h_{11,2,1}A^2 + \frac{1}{2}(h_{12,2,2} - h_{12,1,1})\right)A^2 \\ &-\kappa\left[h_{11,2}A^2{}_{,1} + (h_{11,1} + 2h_{12,2})A^2{}_{,2} + h_{12,3}A^2{}_{,3} - h_{12,0}A^2{}_{,0}\right],\end{aligned} \tag{62}$$

$$\begin{aligned}\Box A^2 = \kappa&\left[-\frac{(A^1)^2+(A^2)^2}{2}A^2 - \eta^{\mu\nu}h_{22,\mu}A^2{}_{,\nu} + h^{\mu\nu}A^2{}_{,\mu,\nu}\right] \\ &-\kappa\left(h_{22,2,1}A^1 + \frac{1}{2}(h_{12,1,1} - h_{12,2,2})\right)A^1 \\ &-\kappa\left[h_{22,1}A^1{}_{,2} + (h_{22,2} + 2h_{12,1})A^1{}_{,1} + h_{12,3}A^1{}_{,3} - h_{12,0}A^1{}_{,0}\right],\end{aligned} \tag{63}$$

with the Lorentz gauge condition

$$A^1{}_{,1} + A^2{}_{,2} - h_{00,1}A^1 - h_{00,2}A^2 = 0. \tag{64}$$

The first term on the right hand side of Eq. (62) or Eq. (63) leads to the phase modulation proportional to the magnitude square of $\vec{\mathbf{A}}$. Eqs. (62) and (64) reduce to Eq. (37) when A^2 is zero. Let

$$A^j = \frac{1}{2}\left[\tilde{\phi}^j(\mathbf{r},t)e^{-i\omega t} + c.c.\right], \; j = 1,2. \tag{65}$$

Using the slowly varying envelope approximations and in the absence of terms with $h_{\mu\nu}$ and their first order derivatives, Eqs. (62) and (63) reduce to the three-dimensional Manakov equations

$$\frac{i\omega}{c^2}\frac{\partial\tilde{\phi}^j}{\partial t}+\frac{1}{2}\nabla^2\tilde{\phi}^j+\frac{3\kappa}{16}[|\tilde{\phi}^1|^2+|\tilde{\phi}^2|^2]\tilde{\phi}^j=0,\ j=1,2. \quad (66)$$

One-dimensional Manakov equations describe the evolutions of two polarization components in nonlinear optics [15,21]. From the third term of Eq. (66), it follows that the phase of the polarization component A^1 (or A^2) is modulated not only by its intensity, but also by the intensity of A^2 (or A^1), which is known as cross-phase modulation (XPM). Proceeding as in Section 3.1.1, it can be shown that the following Lorentz force equation can be obtained

$$\frac{d\left\langle p^3\right\rangle}{dx^0}=\mathrm{Re}\left\langle\rho(\tilde{\mathbf{U}}\times\tilde{\mathbf{H}}^*)_3\right\rangle, \quad (67)$$

where

$$p^3=\frac{1}{2}\mathrm{Re}\left[\tilde{E}_1\tilde{H}_2^{\ *}-\tilde{E}_2\tilde{H}_1^{\ *}\right]=\frac{-\omega}{2c}\mathrm{Im}\left[\tilde{\phi}^1(\tilde{\phi}^1)^*_{\ ,3}+\tilde{\phi}^2(\tilde{\phi}^2)^*_{\ ,3}\right], \quad (68)$$

$$\rho=\frac{-3\kappa}{16}[|\tilde{\phi}^1|^2+|\tilde{\phi}^2|^2]\tilde{\eta}, \quad (69)$$

$$E_j=-A^j_{,0},j=1,2, \quad (70)$$

$$H_2=A^1_{,3},\ \text{and}\ H_1=-A^2_{,3}. \quad (71)$$

3.3 Rectangular cavity resonator

We consider a closed cubicle cavity of dimension L^3 with perfectly conducting walls located at planes $\pm(L/2)\vec{j},\ j=x,y,z,$

where $\{x^1,x^2,x^3\}\to\{x,y,z\}$ (See Fig. 1). Without the loss of generality, we assume that z is the direction of propagation. The electromagnetic field in the cavity are divided into two types – (i) transverse electric (TE) for which the electrical field component $E_z = 0$ and (ii) transverse magnetic (TM) for which the magnetic field component $H_z = 0$. In this paper, we focus only the TE modes for which $A^3=A^0=0$. The linear modes (TE$_{mn}$) (i.e. the right hand sides of Eqs. (62) and (63) are zero) are given by [24,25]

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$$
\begin{aligned}
&A^j = g_j(x,y)\cos(\omega t)[D_j \exp(ik_z z) + c.c.]/2, \ \ j{=}1{,}2, \\
&g_1(x,y) = \cos[k_x(x+L/2)]\sin(k_y(y+L/2)], \\
&g_2(x,y) = \sin[k_x(x+L/2)]\cos(k_y(y+L/2)], \\
&D_1 k_x + D_2 k_y = 0,
\end{aligned} \tag{72}
$$

where

$k_x = m\pi/L,\ k_y = n\pi/L,\ m$ and n are integers, and $g_j(x,y)$ represents the transverse mode distributions and k_z is the propagation constant. The requirement that the tangential component of the electric field intensity should be zero at the planes *z=-L/2* and *L/2* leads to

$$
\begin{aligned}
&k_z = j\pi/L,\ j \text{ is an odd integer}, \\
&k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2/c^2.
\end{aligned} \tag{73}
$$

In this Section, we focus on the fundamental TE_{01} mode,

$$
\begin{aligned}
&A^1 = D_1 \cos(k_y y)\cos(k_z z)\cos(\omega t), \\
&A^2 = 0, \\
&k_y = k_z = \pi/L, k_x = 0.
\end{aligned} \tag{74}
$$

We wish to find the quasi-linear modes of the cavity under the weak field approximations satisfying the boundary condition that the tangential components of the electric field intensity are zero at the conducting walls. The evolution of A^1 and h_{00} in the cavity are given by Eqs. (37) and (36), respectively.

$$
\Box A^1 = -\kappa\left(\frac{(A^1)^3}{2} + \eta^{\mu\nu} h_{00,\mu} A^1{}_{,\nu}\right), \tag{75}
$$

$$
\Box h_{00} = (A^1)^2. \tag{76}
$$

Let

$$
A^1 = \psi(y,z)\exp(i\omega t) + c.c. \tag{77}
$$

Squaring Eq. (77) and substituting it in Eq. (76), we find that the excitation is at frequencies 0 and 2ω. Hence, let the response be

$$
h_{00} = [h_1(y,z)\exp(i2\omega t) + c.c.] + h_2(y,z). \tag{78}
$$

Substituting Eq. (78) in Eq. (75) and ignoring third harmonic components, we find

$$\begin{aligned}\psi_{,2,2}+\psi_{,3,3}+k^2\psi &= -\kappa\left(\frac{3|\psi|^2\psi}{2}-2k^2h_1\psi^*\right)\\ &\quad -\kappa(h_{2,2}\psi_{,2}+h_{2,3}\psi_{,3}+h_{1,2}\psi^*_{,2}+h_{1,3}\psi^*_{,3}),\end{aligned} \tag{79}$$

$$h_{1,2,2}+h_{1,3,3}+4k^2h_1=\psi^2, \tag{80}$$

$$h_{2,2,2}+h_{2,3,3}=2|\psi|^2\,. \tag{81}$$

To further simplify Eqs. (79)-(81), let

$$\psi(y,z)=\cos(k_y y)g(z), \tag{82}$$

We assume that $g(z)$ is real, which reduces to $\cos(k_z z)$ as $h_{\mu\nu}\to 0$. Substituting Eq. (82) in Eqs. (80) and (81), and separating components at the spatial frequencies 0 and $2k_y$, we find

$$\begin{aligned}h_1 &= B(z)+C(z)\cos(2k_y y),\\ h_2 &= D(z)+F(z)\cos(2k_y y),\end{aligned} \tag{83}$$

$$\begin{aligned}&4B(z)k^2+B_{,z,z}=g^2(z)/2,\\ &4(k^2-k_y^2)C(z)+C_{,z,z}=g^2(z)/2,\\ &D_{,z,z}=g^2(z),\\ &-4k_y^2F(z)+F_{,z,z}=g^2(z).\end{aligned} \tag{84}$$

Substituting Eqs. (83) and (84) in Eq. (79), we obtain

$$\begin{aligned}&[(\omega^2/c^2-k_y^2)g(z)+g_{,z,z}]\cos(k_y y)=-\kappa\left(\frac{3\cos^3(k_y y)g^3}{2}\right)\\ &-\frac{2\kappa\omega^2 g}{c^2}[B-C\cos(2k_y y)\cos(k_y y)]\\ &+\kappa g\left[2k_y^2\sin(2k_y y)\sin(k_y y)(C+F)\right]\\ &-\kappa g_{,z}\cos(k_y y)\left[(B+D)-(C+F)\cos(2k_y y)\right]_{,z}\,.\end{aligned} \tag{85}$$

In this Section, our objective is to find $g(z)$ which becomes zero at $z=\pm L/2$ so that the boundary condition is satisfied. In order to accomplish this, we follow the approach typically used to derive the nonlinear Schrodinger equation (NLSE) from the nonlinear Maxwell's equation [26,21]. In Refs. [26,21], fiber nonlinearity is treated as a small perturbation on the fundamental transverse mode (HE_{11}) and the NLSE is derived to describe the evolution of the mode weight of the fundamental transverse mode as a function of the propagation distance, z, by multiplying the nonlinear wave equation by the transverse mode distribution and integrating over the transverse dimensions x and y. Here, we follow the same approach. We assume that the field in the transverse direction is the same as that of a linear mode and the

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nonlinear terms appearing on the RHS of Eq. (85) cause a small perturbation to this linear mode. Multiplying Eq. (85) by $\cos(k_y y)$ and integrating from –*L*/2 to *L*/2 with respect to *y*, we obtain

$$g_{,z,z} + \beta^2 g = \kappa\left[-\frac{9}{8}g^3 + k^2(2B+C)\right] \\ -\kappa[k_y^2 g(C+F) + g_{,z}\left(B+D+(C+F)/2\right)_{,z}], \tag{86}$$

where $\beta = \sqrt{(\omega/c)^2 - k_y^2}$ is the eigenvalue to be determined under the condition that *g*(*z*) becomes zero at $\pm L/2$. For the linear case (i.e. κ = 0), from Eqs. (86) and (74), we have $\beta = k_z = \pi / L$. In a general case, Eq. (86) provides the evolution of a quasi-linear mode (in *z*-direction) with the transverse mode distribution being proportional to $\cos(\pi y / L)$. From the right hand side of Eq. (85), we see that there are excitations proportional to $\cos(3\pi y / L)$ and one should expect the generation of such higher order modes due to nonlinear effects. However, when Eq. (85) is multiplied by $\cos(\pi y / L)$ and integrated over *y*, higher order transverse modes do not contribute and Eq. (86) may be interpreted as the equation that provides the weight of the fundamental transverse mode distribution ($\cos(\pi y / L)$). As the amplitude of A^1 becomes larger, there could be a nonlinear coupling between the fundamental transverse mode ($\propto \cos(\pi y / L)$) and the higher order mode ($\propto \cos(3\pi y / L)$). However, such nonlinear interactions are not captured in Eq. (86).

Equations (86) and (84) form a coupled nonlinear differential system of equations which are solved using an explicit Runge-Kutta method (Matlab built-in function ode45). We look for a solution that is symmetric with respect to *z*=0. The problem can be formulated in two ways (i) For the given initial condition $g(0) = g_0$, and $g'(0) = 0$ where ' denotes differentiation with respect to z, the propagation constant β is found such that the boundary condition $g(L/2) = 0$ is satisfied (i.e. the tangential component of the electric field is zero at the walls) (ii) For the given β (or equivalently for the given ω), find $g(0)$ such that the boundary condition $g(L/2) = 0$ is satisfied. We follow the latter approach. Note that in the absence of nonlinearity (κ = 0), $\beta = k_z$, $\omega = ck$ and the amplitude $g(0)$ is arbitrary. Let

$$f_{res} = \frac{ck}{2\pi} = \frac{c}{\sqrt{2}L} \tag{87}$$

be the resonant frequency of the cavity when κ = 0 for the fundamental mode. In the presence of nonlinearity ($\kappa \neq 0$), as the frequency of the EM field deviates from the resonant frequency, the initial amplitude $g(0)$ (equivalently energy of the EM field) should be changed to satisfy the boundary condition at $z=L/2$. Due to nonlinear effects (which is the signature of spacetime curvature), let the frequency detuning be

$$\Delta f = f - f_{res} \tag{88}$$

where f is the frequency of the EM field. Figure 2 shows the evolution of the field $g(z)$ as a function of distance z. As the frequency detuning increases, the amplitude of the field at z=0 increases and hence, the EM energy stored in the cavity increases. If the frequency detuning Δf is negative, we found that the boundary condition that $g(L/2)=0$ cannot be satisfied. If Eq. (86) is solved with κ =0, one finds that Δf should be zero so as to satisfy the boundary condition $g(L/2)=0$ (unless Δf is so large that f coincides with the higher order resonant frequencies) and the amplitude of the field at z=0 is arbitrary.

To verify the validity of Eqs. (84) and (86), the coupled partial differential equations (73)-(75) are numerically solved using the FDTD technique with the boundary condition that the tangential components of the electric field intensity are zero at the metallic walls. In the numerical solution, the growth of higher order mode ($\propto \cos(3\pi y/L)$) was observed. To be consistent with the semi-analytical approach, the numerical solution of Eq. (79) is multiplied by $\cos(\pi y/L)$ and integrated from –L/2 to L/2 to obtain the mode weight of the fundamental mode. '+' in Fig. 2 show the numerical solutions obtained by the FDTD technique and as can be seen, the agreement between the semi-analytical approach and numerical approach is quite good.

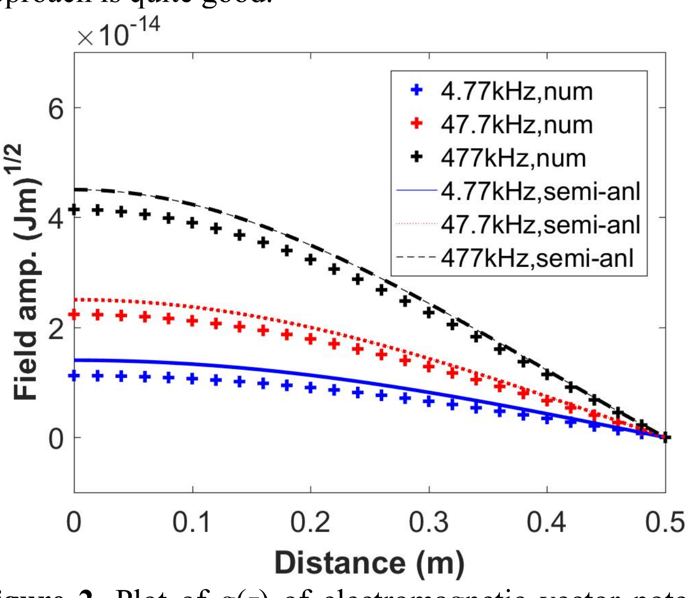

Figure 2. Plot of g(z) of electromagnetic vector potential A^1 vs distance, z for frequency detuning factors, Δf = 4.77 kHz, 47.7 kHz and 477 kHz. L = 1 m,

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resonant frequency, f_{res} = 211.98 MHz ; num = numerical, and semi-anl = semi-analytical.

The small discrepancy is attributed to the fact that semi-analytical approach does not take into account the coupling between the fundamental and higher order transverse mode.
We define the rest mass of the EM field confined to the cavity as

$$m = (\mathrm{E}_e + \mathrm{E}_m)/c^2, \qquad (89)$$

where E_e and E_m are the mean energy stored in electric and magnetic fields respectively,

$$\mathrm{E}_e = \left\langle \int E_x^2 dV \right\rangle, \mathrm{E}_m = \left\langle \int (H_y^2 + H_z^2) dV \right\rangle. \qquad (90)$$

$$E_x = -A^1_{,0},\ H_y = A^1_{,z},\ \text{and}\ H_z = -A^1_{,y}. \qquad (91)$$

For the given frequency *f*, the field $g(z)$ is calculated by solving Eqs. (86) and (84) numerically, and using Eq. (91), electric and magnetic field intensities are calculated. Using Eqs. (89) and (90), the rest mass is calculated for the given frequency *f* and plotted in Fig. 3. The line with '+' shows the mass calculated semi-analytically using the above procedure and the solid line shows the curve fitting. A good fit was found by using

$$m = \frac{8\pi^2\sqrt{2\pi(f^2 - f_{res}^2)}}{c^3\kappa}. \qquad (92)$$

Equation (92) may be rewritten as the following dispersion relation

$$\omega^2 = (kc)^2 + \frac{m^2c^6\kappa^2}{32\pi^3}, \qquad (93)$$

where the wave number at the resonant frequency is

$$k = \sqrt{k_y^2 + k_z^2} = (2\pi f_{res})/c. \qquad (94)$$

By setting $E = \hbar\omega$ and $p = \hbar k$, and if

$$\kappa = \frac{2(2\pi)^{3/2}}{\hbar c}, \qquad (95)$$

Equation (93) could be rewritten as a special relativistic relation relating the energy, momentum and rest mass of a particle,

$$E^2 = p^2c^2 + m^2c^4. \qquad (96)$$

As the amplitude of A^1 goes to zero, $m \to 0$, and hence, $\omega = kc$ is correct only for the EM field with vanishing amplitude. In the absence of spacetime curvature (κ =0), the EM field is governed by the linear Maxwell's equations and in this case, $E = pc$ even if the field is confined to a localized region. The relation between κ

and $\hbar$ given by Eq. (95) may be off by a scaling factor of $O(1)$ due to approximations made in the derivation of Eq. (86). We have considered the impact of spacetime curvature only on the fundamental mode TE_{01} and as the mode order increases, the dependence of the frequency detuning on the field intensity is expected to be given by a formula similar to Eq. (93), but there could be an additional constant in Eq. (93) that may depend on the mode order.

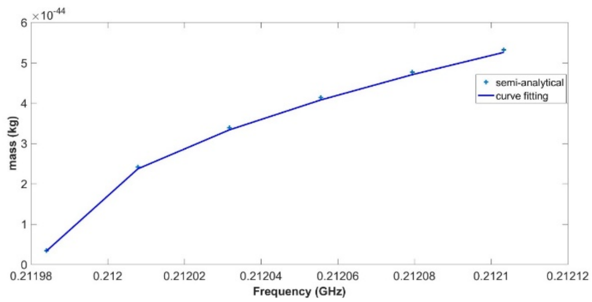

Figure 3. Plot of the rest mass as a function of the frequency of the electromagnetic field. L = 1 m, resonant frequency, f_{res} = 211.98 MHz.

Conclusions

An electromagnetic (EM) tensor which is an outer product of EM vector potential is used to modify Einstein-Maxwell equations. In Einstein's theory of gravitation, the coupling constant connecting Einstein tensor and stress-energy tensor is proportional to gravitational constant. Similarly, we find that the coupling constant connecting Einstein tensor and electromagnetic tensor is inversely proportional to Planck's constant. In classical electrodynamics, Maxwell's equations do not provide the equations of motion for charged particles; they are provided separately by Lorentz force equations. However, Einstein-Maxwell equations with the new EM tensor derived in this paper lead to equations of motion that resemble Lorentz force equations. Using slowly varying envelope approximation, these equations reduce to Schrodinger equation with a self-trapping potential.

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Role of the Lie-Santilli Isotheory for the proof of the EPR argument

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Abstract

In 1935, A. Einstein stated, in a historical paper with B. Podolsky and N. Rosen [1] that "quantum mechanics is not a complete theory" and that determinism could be recovered at least under limit conditions (EPR argument). In 1964, J. S. Bell [2] proved a theorem according to which a system of quantum mechanical particles with spin 1/2 with $SU(2)$ Lie algebra $[\sigma_i, \sigma_j] = 2\epsilon_{i,j,k}\sigma_k$, where the σs are the Pauli matrices, cannot admit a classical counterpart, thus appearing to disprove the EPR argument. In 1978, R. M. Santilli [3] discovered the axiom-preserving generalization-"completion" of the various branches of Lie's theory (universal enveloping algebras, Lie algebras, and Lie groups) based on the isoassociative product $X_i \star X_j = X_i\hat{T}X_j, \hat{T} > 0$, with Lie-Santilli isoalgebras $[X_i, X_j]^* = X_i \star X_j - X_j \star X_i = C_{i,j,k,}X_k$ classified into *regular* (*irregular*) when the C-quantities are constant (functions). In 1998 [4] Santilli proved that Bell's theorem is valid for point-particles, but it is inapplicable for systems of extended particles with spin 1/2 under deep mutual entanglement, and that said systems do admit classical counterparts when represented with the isotopic $SU(2)$ Lie-Santilli isoalgebars $[\Sigma_i, \Sigma_j]^* = 2\epsilon_{i,j,k}\Sigma_k$, where Σ_k are the new Pauli-Santilli isomatrices, with realization of the isotopic element $\hat{T} = Diag.(1/\lambda, \lambda), \quad \det \hat{T} = 1$ providing a concrete and explicit realization of "hidden variables" under the full validity of quantum

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axioms. Subsequently, Santilli [5] proved that Einstein's determinism is progressively approached in the structure of hadrons, nuclei and

stars and it is fully recovered at the limit of gravitational collapse (see Refs. [6] for a detailed presentation). In this lecture, by following our recent paper [7], we outline the aspects of the Lie-Santilli isotheory which are essential for Santilli's proofs of the EPR argument.

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1 Introduction

The well known *EPR argument* was proposed by A. Einstein, B. Podolsky and N.Rosen in 1935 [1] implies that; Quantum mechanics is not a complete theory but should be supplemented by additional variables. i.e. Quantum mechanics has to be deterministic. In this regard Einstein has made a famous statement that "God doesn't play dice with the universe."
In other words, Einstein believed that quantum mechanics is not a complete theory, in the sense that it could be broadened to recover classical determinism at least under limiting conditions.
Numerous objections against EPR argument have been raised by scholars including N.Bohr [8], S.Bell [2, 9], J.Von Neumann [10]. Till date, it is widely believed that Quantum mechanics is the final theory for all conceivable conditions existing in the universe.
Any Physical Theory operates with physical concepts which correspond with the objective reality. Success of a physical theory depends on;

- Correctness
- Completeness

Correctness is judged by the degree of agreement between theoretical conclusions and human experience.
Completeness of a Physical Theory Requires;

- Every element of the physical reality must have a corresponding concept in the physical theory.
- Elements of physical reality must be experiments and measurements.

Scientifically, a reasonable interpretation of physical reality would be; if, without in any way disturbing a system, we can predict with certainty (i.e.with probability unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.
If we start with the assumption that wave function does give a complete description of the physical reality, we arrive at the conclusion that two physical quantities with non commuting operators can have simultaneous reality. This implies that quantum mechanical description of physical reality given by wave function is not complete. i.e. quantum axioms do not admit hidden variables (Local Realism), [2]. So, quantum mechanics can not be described

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by local hidden variables. For that matter, assuming the validity of Bell's theorem, any deterministic hidden-variable theory that is consistent with quantum mechanics would be non-local. Hence, dismissal of EPR argument. Following decades of research since 1998, R.M.santilli, assuming the validity of quantum mechanics, with consequential validity of the objections against the EPR argument[11],[12] for point-like particles in empty space under linear, local and potential interactions (exterior dynamical problems); proved the inapplicability (and not their violation) of said objections for the broader class of extended, deformable and hyperdense particles within physical media under the most general known linear and nonlinear, local and non- local and potential as well as non- potential interactions (interior dynamical problems). Santilli's Contribution also provided the apparent proof that interior dynamical systems admit classical counter- parts in full accordance with the EPR argument via the representation of interior systems with of isomathematics, also called isotopic branch of hadronic mathematics, and isomechanics, also called isotopic branch of hadronic mechanics.
The main assumption of apparent proof of EPR argument is; particles can be represented as extended, deformable and hyperdense under the conditions of mutual overlapping/entanglement with ensuing contact at a distance. This eliminates objection 'quantum entanglement' regarding non-locality of quantum mechanics.

In the 2019 paper [5], *Santilli provided the apparent proof (of 'completion' of quantum mechanics* as isotopic/axiom-preserving type, being fully admitted by quantum mechanics merely subjected to a broader realization than that of Copenhegen school) that *Einstein's determinism is progressively approached in the interior of hadrons, nuclei and stars and it is fully achieved in the interior of gravitational collapse.*

Thus, inapplicability of 20th century 'applied mathematics' in general and of Lie's theory in particular to the interior dynamical systems led Santilli to construct a new mathematics, known as *isomathematics* exactly applicable to the interior dynamical systems. In particular, Lie-algebra structure required in quantum mechanics was lifted structurally to show that objections against EPR argument are inapplicable.

Appropriate lifting of conventional Lie theory applicable to exterior dynamical systems to Lie-Santilli isotheory applicable to interior dynamical systems was achieved by Santilli [13].
The Lie-Santilli isoalgebras and isogroups were elaborated with the conven-

tional mathematics of Lie's theory via conventional functional analysis and differential calculus on conventional space such as original Hilbert space H over conventional field $F(n, \times, 1)$.

2 Isomathematics

The basic multiplicative unit 1 is replaced by an arbitrary, positive definite quantity $\hat{I} = \frac{1}{\hat{T}}$ whether or not element of the original field. $\hat{T}$ is called the *isotopic element* and

$$\hat{I} = \frac{1}{\hat{T}} \tag{1}$$

is called the *isounit*, and all possible associative products are lifted via

$$X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j \tag{2}$$

with $\hat{I}$ being the correct left and right multiplicative unit for all the elements of the set considered such that

$$\hat{I} \hat{\times} X = X \hat{\times} \hat{I} = \hat{X} \tag{3}$$

for all X in the resulting new field called as *Santilli Isofield*. The new numbers $\hat{X}$ in the isofield are called as *isonumbers*
This new field is denoted by $\hat{F}(\hat{n}, \hat{\times}, \hat{I})$.

3 Lie-Santilli isotheory

It is well known that **Lie's theory** is at the true structural foundation of quantum mechanics via celebrated product;

$$[A, B] = A \times B - B \times A \tag{4}$$

where $A \times B = AB$ is the conventional associative product.

Today, by **Lie-Santilli isotheory** we mean the infinite family of isotopies of **Lie's theory** formulated on an iso-Hilbert space $\hat{H}$ defined over an isofield $\hat{F}$ generated by iso-Hremitean generators $X_k, k = 1, 2, 3, ..., N$ with all possible products lifted into the isoassociative form (2) and multiplicative

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isounit $\hat{I} = \frac{1}{\hat{T}}$, [14].

Generalization of Lie's theory by Santilli [15]in 1978 under the name *Lie-isotopic theory* with the basic product;

$$\begin{aligned}[A,\hat{} B] &= A \hat{\times} B - B \hat{\times} A \\ &= A \times \hat{T} \times B - B \times \hat{T} \times A \\ &= A\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, \eta, ...)B \\ &- B\hat{T}(t, x, \dot{x}, \ddot{x}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu, \tau, \eta, ...)A \end{aligned} \tag{5}$$

Lie-isotopic theory is also called as *Lie-Santilli isotheory.*

1. *Lie-Santilli isotheory* is based on isotopic product $[A,\hat{} B] = A\hat{T}B - B\hat{T}A$ where $\hat{T}$ is a hermitean matrix or operator with $\hat{T} = \hat{T}^\dagger$.

2. *Lie-admissible theory*, also called as *Lie-Santilli genotheory*, is based on the product $(A, B) = A\hat{T}B - B\hat{T}^\dagger A = A\hat{R}B - B\hat{S}A$ where $\hat{T}$ is a nonhermitean matrix or operator with $\hat{T} = \hat{R} \neq \hat{T}^\dagger = \hat{S}$.

3. *Hypertheory*, the most general formulation of hyperstructural character [16] is based on the product of type $A \otimes B = A\hat{R}B - B\hat{S}A$ where $\hat{R}$ and $\hat{S}$ are sets.

4 Lie Algebra

Let L **be an N-dimensional Lie algebra** over a field $F(n, \times, 1)$ of characteristic zero and associative product $nm = n \times m \in F$ and multiplicative unit 1.
Let the **generators of** L **are the Hermitean operators** $X_k, k = 1, 2, ...n$, on a Hilbert space $\mathcal{H}$ over F.
Let $\xi(L)$ **be the universal enveloping associative algebra** of ordered monomials based on the associative product;

$$X_i \times X_j \tag{6}$$

Let the Lie algebra L be isomorphic to the **anti-symmetric algebra attached to the enveloping algebra** $L \approx [\xi(L)]^-$ with ensuing Lie's theorems and and commutation rules;

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$$[X_i, X_j] = X_i \times X_j - X_j \times X_i = C_{ij}^k \times X_k \tag{7}$$

4.1 Isotopies of Lie Algebra

- the ***isotopy of the associative product***

$$X_i \hat{\times} X_j = X_i \times \hat{T} \times X_j \tag{8}$$

  where $\hat{T}$ (the *isotopic element*)is a fixed positive-definite operator with an arbitrary functional dependence on local variables;

- the ***isotopy of the enveloping algebra*** $\hat{\xi}(\hat{L})$ characterized by ordered monomials of the Poincare-Birkhoff-Witt-Santill is a theorem based on isoproduct (2);

- the ***isotopies of Lie algebras***, today called the ***Lie-Santilli isoalgebra*** $\hat{L}$ as the anti-symmetric algebra attached to the isoenvelope

$$\hat{L} \approx [\hat{\xi}(\hat{L})]^- \tag{9}$$

  with *Lie-Santilli isocommutation* rules

$$[X_i \hat{,} X_j] = X_i \hat{\times} X_j - X_j \hat{\times} X_i = \hat{C}_{ij}^k \hat{\times} X_k \tag{10}$$

- the *isotopies of Lie groups* today known as the ***Lie-Santilli isogroups***; and

- the ***isorepresentation theory***.

4.2 Lie-Santilli Isoalgebra

Definition 4.1 *A (finite-dimensional) isospace $\hat{L}$ over an isofield $\hat{F}(\hat{a}, +, \hat{\times})$ of isoreal numbers $\hat{R}(\hat{n}, +, \hat{\times})$, isocomplex numbers $\hat{C}(\hat{c}, +, \hat{\times})$ or isoquaternions $\hat{Q}(\hat{q}, +, \hat{\times})$ with isotopic element $\hat{T}$ and isounit $\hat{I} = \hat{T}^{-1}$ is called a "Lie-Santilli isoalgebra" over $\hat{F}$ when there is a composition $[\hat{A} \hat{,} \hat{B}]$ in $\hat{L}$, called "isocommutator", which verifies the following "isolinear and isodifferential rules" for all $\hat{a}, \hat{b} \in \hat{F}$ and $\hat{A}, \hat{B}, \hat{C} \in \hat{L}$*

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$$[\hat{a}\hat{\times}\hat{A}+\hat{b}\hat{\times}\hat{B},\hat{,}\hat{C}]=\hat{a}\hat{\times}[\hat{A},\hat{,}\hat{C}]+\hat{b}\hat{\times}[\hat{B},\hat{,}\hat{C}] \tag{11}$$

$$[\hat{A}\hat{\times}\hat{B},\hat{,}\hat{C}]=\hat{A}\hat{\times}[\hat{B},\hat{,}\hat{C}]+[\hat{A},\hat{,}\hat{C}]\hat{\times}\hat{B} \tag{12}$$

and "Lie-Santilli isoaxioms"

$$[\hat{A},\hat{,}\hat{B}]=-[\hat{B},\hat{,}\hat{A}] \tag{13}$$

$$[\hat{A},\hat{,}[\hat{B},\hat{,}\hat{C}]]+[\hat{B},\hat{,}[\hat{C},\hat{,}\hat{A}]]+[\hat{C},\hat{,}[\hat{A},\hat{,}\hat{B}]]=0 \tag{14}$$

It is important to note that the associative character of the underlying envelope is preserved while using isoreals, isocomplexes and isoquaternions. Consistent isotopic generalization of celebrated Lie's First, Second and Third theorems has been proved by Santilli in [17].

4.3 Isorepresentations of Lie-Santilli isoalgebras

Isorepresentations of Lie-Santilli isoalgebras is classified into;

1. **Regular isorepresentations** which occur due to C's of the rules (10) are constants; and

2. **Irregular isorepresentation** occurring when the C's of the rules (10) are functions of the local variables (an occurrence solely possible for the Lie-Santill isotheory).

5 Construction of Regular Isorepresentations

General Construction: Regular isorepresentation of Lie-Santilli isoalgebras $\hat{L}$ over an isofield of characteristic zero can be constructed via *non-unitary* transformations of the conventional representations of the conventional Lie algebra L.

The general rule for mapping Lie algebras into regular Lie-Santilli isoalgebras were identified for the first time by Santilli in [18] and then studied systematically in monographs [14]. They can be written as follows;

$$U \times U^{\dagger} = \hat{I} \neq I \tag{15}$$

This non-unitary transformation is **applied to the entire mathematics of Lie's theory leading to Santilli's isomathematics**. We get the following important fundamental transformations;

$$I \longrightarrow \hat{I} = U \times I \times U^{\dagger} = \frac{1}{\hat{T}}, \tag{16}$$

$$a \longrightarrow \hat{a} = U \times a \times U^{\dagger} = a \times U \times U^{\dagger} = a \times \hat{I} \in \hat{F}, a \in F \tag{17}$$

$$e^{A} \longrightarrow U \times e^{A} \times U^{\dagger} = \hat{I} \times e^{\hat{T} \times \hat{A}} = (e^{\hat{A} \times \hat{T}}) \times \hat{I} \tag{18}$$

$$\begin{aligned} A \times B \longrightarrow U \times (A \times B) \times U^{\dagger} &= (U \times A \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times B \times U^{\dagger}) \\ &= \hat{A} \hat{\times} \hat{B} \end{aligned} \tag{19}$$

$$\begin{aligned} [X_i, X_j] \longrightarrow U \times [X_i, X_j] \times U^{\dagger} &= [\hat{X}_i \hat{,} \hat{X}_j] = U \times (C_{ij}^{k} \times X_k) \times U^{\dagger} \\ &= C_{ij}^{k} \times \hat{X}_k \end{aligned} \tag{20}$$

$$\begin{aligned} <\psi \mid \times \mid \psi> \longrightarrow U \times <\psi \mid \times \mid \psi> \times U^{\dagger} &= <\psi \mid \times U^{\dagger} \times (U \times U^{\dagger})^{-1} \times U \times \mid \psi> \times (U \times U^{\dagger}) \\ &= <\hat{\psi} \mid \hat{\times} \mid \hat{\psi}> \times \hat{I}, \end{aligned} \tag{21}$$

$$\begin{aligned} H \times \mid \psi> \longrightarrow U \times (H \times \mid \psi>) U^{\dagger} &= (U \times H \times U^{\dagger}) \times (U \times U^{\dagger})^{-1} \times (U \times \mid \psi>) \\ &= \hat{H} \hat{\times} \mid \hat{\psi}> \end{aligned} \tag{22}$$

etc.

6 Classification of regular isounitary isoirreducible isorepresentations of the Lie-Santilli $\widehat{SU}(2)$ isoalgebras over isofields of characteristic zero

Santilli [21, 22] identified and **constructed the following regular isorepresentation of Lie-Santilli isoalgebra $\widehat{SU}(2)$, from the conventional**

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two-dimensional irreducible representation of the $SU(2)$ Lie algebra defined by the well known Pauli's matrices.
This Classification is merely given by either the nonunitary transform $U - Diag(n_1, n_2), n_k$ real > 0, or by $U - OffDiag(n_1, n_2)$.
Conventional Paulli matrices σ_k [19, 20] satisfy the rules $\sigma_i\sigma_j = i\varepsilon_{ijk}, i, j, k = 1, 2, 3$. We present the identification and classification ref.[21, 22] of these matrices due to isoalgebra $S\hat{U}_Q(2)$.

In general Lie-isotopic algebras are the image of Lie algebras under nonunitary transformations [23, 24]. **Under the transformation $UU^\dagger = \hat{I} \neq I$ a Lie commutator among the matrices acquires the Lie-isotopic form**

$$U(AB - BA)U^\dagger = A'QB' - B'QA',$$
$$A' = UAU^\dagger, B' = UBU^\dagger, Q = (UU^\dagger)^{-1} = Q^\dagger \qquad (23)$$

As a result, a first class of fundamental (adjoint) isorepresentations called as regular adjoint isorepresentations are characterized by the maps $J_k = \frac{1}{2}\sigma_k \to \hat{J}_k = UJ_kU^\dagger, UU^\dagger\hat{I} \neq I$ with isotopic contributions that are factorizable in the spectra, $\pm\frac{1}{2} \to +\frac{1}{2}f(\Delta), \frac{3}{4} \to (\frac{3}{4}f^2(\Delta))$ where $\Delta = detQ$ and $f(\Delta)$ is a smooth nowhere-null function such that $f(1) = 1$.

7 Iso-Pauli matrices

Santilli constructed the following example of regular iso-Pauli matrices.

$$\hat{\sigma_1} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \hat{\sigma_2} = \Delta^{-\frac{1}{2}} \begin{pmatrix} 0 & -ig_{11} \\ ig_{22} & 0 \end{pmatrix}$$
$$\hat{\sigma_3} = \Delta^{-\frac{1}{2}} \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} \qquad (24)$$

where $\Delta = detQ = g_{11}g_{22} > 0$.

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These representations verify the isotopic rules $\hat{\sigma_i} Q \hat{\sigma_j} = i\Delta^{\frac{1}{2}}\epsilon_{ijk}\hat{\sigma_k}$, and consequently the following isocommutator rules and generalized isoeigenvalues for $f(\Delta) = \Delta^{\frac{1}{2}}$ and

$$[\hat{\sigma_1}, \hat{\sigma_j}] = \hat{\sigma_i} Q \hat{\sigma_j} - \hat{\sigma_j} Q \hat{\sigma_i} = 2i\Delta^{\frac{1}{2}}_{ijk}\hat{\sigma_k} \quad (25)$$

$$\hat{\sigma_3} * \mid \hat{b_i^2}\rangle = \pm\Delta^{\frac{1}{2}} \mid \hat{b_i^2}\rangle \quad (26)$$

$$\hat{\sigma^2} * \mid \hat{b_i^2}\rangle = 3\Delta \mid \hat{b_i^2}\rangle, i = 1, 2 \quad (27)$$

This confirms the 'regular' character of the generalization considered here. The isonormalized isobasis is then given by a trivial extension of the conventional basis $\mid \hat{b}\rangle = Q^{-\frac{1}{2}} \mid b\rangle$.
In fact, regular iso-Pauli matrices (24) admit the conventional eigenvalues $\frac{1}{2}$ and $\frac{3}{4}$ for $\Delta = 1$ which can be verified by putting $g_{11} = g_{22}^{-1} = \lambda$.
It is important to emphasize the condition of isounitarity, i.e. $UU^\dagger = \hat{I} \neq I$ for which ${n_1}^2 = 1/{n_2}^2 = \lambda > 0$. Thus, realization of isotopic element $\hat{T} = Diag.(1/\lambda, \lambda)$ with $detQ$ ***provides a concrete and explicit realization of "hidden variable" under full validity of quantum axioms.***

Remarks:- This degree of freedom has major fundamental implications presented in [22] as well as for the spin component of the first known representation of nuclear magnetic moments presented in the papers [25, 26].

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Isodual Mathematics for Antimatter

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Abstract

Since the discovery of antimatter it has only been treated at the level of second quantization, where as, matter is treated at all levels of study, from Newtonian mechanics to quantum field theory. To resolve this scientific imbalance of 20th century, Santilli in 1993 [1],[2],[3] took up to study antimatter at all levels. In this paper we present the classical representation of antimatter at Newtonian level and emerging images at subsequent levels. The most appropriate theory of antimatter as proposed by Santilli [4] is based on a new map called isoduality which is applicable at the Newtonian level and all the subsequent levels of study of antimatter. Santilli also formulated the new anti-isomorphic isodual images of the Galilean, special and general relativities compatible with the experimental knowledge on electromagnetic interactions. Antigravity for antimatter [6] (and vice versa) is a natural consequence of this study and awaits validity due to lack of sufficient experimental evidence.

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1 Introduction

Scientific studies have come a long way from Newtons equations, Galilios relativity and Einsteins special and general relativities. Existence of antimatter asteroids and cosmic rays in the universe has already been suggested by phenomena like; 1) Catastrophic explosion in Tunguska in Siberia in 1908 of the power of thousand Hiroshima bombs with devastating effect and total absence of any debris and crater. Surprisingly, entire Earths atmosphere was charged for some days so much so that people in Sydney could read news papers without any artificial light. Such a large excitation of the atmosphere can only be explained by annihilation of matter by antimatter. 2) NASA has recently reported explosions in our upper atmosphere which can be caused only by small antimatter asteroids annihilating the upper portion of our atmosphere while coming in contact with it and 3) Astronauts and cosmonauts have observed flashes of light in the upper atmosphere which can only be interpreted as being due to antimatter cosmic rays coming in contact with our atmosphere. In short, the evidences of existence of antimatter asteroids hitting our earth has become a major threat to humanity and hence warrants a serious study of antimatter in general and antimatter asteroids, comets and galaxies in particular. We know that matter is described at all levels of study from Newtonian mechanics to Quantum field theory but antimatter is solely treated at the level of second quantization; as antimatter particles with negative-energy do not behave in a physical way. Thus, Newton, Galileo and Einstein's theories were solely describing matter and not antimatter. A. Schuster in 1898 conjectured existence of antimatter. It was discovered by Dirac [5] in 1920, fourteen years following the formulation of general relativity. He even submitted hole theory for the study of antimatter at the level of second quantization. Today, the stand adopted in general is that; As Einsteins special and general relativity do not provide a proper description of antimatter, it does not exist in the universe in appreciable amount; the sole exception being that of a man made antiparticles created in the laboratory. The above scientific imbalance was for the first time identified by Italian- American scientist Ruggero Maria Santilli who decided to ascertain whether a far away star or galaxy is made up of matter or antimatter. Santilli soon discovered the entire body of mathematical, theoretical and experimental formulation [6, 1, 7, 8] applicable to his aim as his previous knowledge at the graduate studies was insufficient. Santilli first took up to formulate the mathematics needed for classical and operator representation

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either neutral or charged antimatter. Secondly, a reformulation of Newton's, Galileo's and Einstein's theories suitable for the study of neutral or charged antimatter at all possible levels and thirdly, the formulation of experiments to ascertain, in due time, whether far away stars or galaxies are made up of matter or antimatter. Antimatter asteroids must be treated as as they are isolated in space. Also, they are too large for the treatment via operator theories. Hence, scientific studies in the detection of antimatter requires physical theories for classical treatment of antimatter. Santillis mathematical and theoretical studies in antimatter are unique in a way being capable of classical representation of neutral antimatter. In his writings, Santilli has specifically mentioned that A protracted lack of solution of physical problems is generally due to the use of insufficient or inadequate mathematics [6]. Moreover, he says that There can not exist a really new physical theory without a new mathematics, and there can not exist a really new mathematics without new numbers. Santilli spent decades of exhaustive research in developing new numbers; subsequently new mathemetics sufficient to treat neutral or charged antimatter. Santilli introduced new numbers called isodual numbers [9] where the prefix iso was introduced in the Greek sense meaning preserving the conventional axioms used for matter. The term dual indicates the map from matter to antimatter. Santilli's entire theory for of antimatter is called Isodual due to predominant role and importance of Santillis isodual numbers. Subsequently, in 1993 [1] Santilli constructed the isodualities of Euclidean and Minkowskian spaces, evidently needed for possible physical applications. He then proceeded to construct the isodual image of Lies theory [2] needed for the construction of basic symmetries for antimatter viz. isodual images of the Galileo and Lorentz symmetries. Second landmark discovery was a new formulation of differential calculus which was crucial for the achievement of the first ever known formulation of Newtons equation for neutral or charged antiparticles. Readers can find the complete formulation of isodual mathematics in the monograph [11] of 1994 and with more updation in [8]. Thereafter, Santill initiated his physical studies in the paper [12] of 1993 written on his original aim of possible detection of antimatter stars and galaxies. Subsequently, Santilli wrote an important paper [16] on all important classical representation of neutral antimatter; the submission of experiment in paper [7] of 1994 to test the gravity of positrons and paper [14] of 1994 on the causal space-time machine i.e. the capability of moving as desired in space and time without violation of causality which is an invertible consequence of gravitational repulsion between matter and

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antimatter. A.P.Mills, an experimentalist established that Santillis gravity experiment [15] is resolutory because displacement due to gravity of very low energy positrons on a scintillator at the end of a flight in a super-cooled, supervacuum tube is visible to the naked eye. Paper [16] of 1997 included the first isodualities of Galileo and Einstein's relativities; another basic physical discovery has been discussed in paper [17] of 1997 via the prediction that light emitted by antimatter is repelled by the matter gravitational field. This prediction is an invertible consequence of the main feature of the studies reviewed here, namely, the classical conjugation from neutral matter to neutral antimatter that evidently also applied to light. The prediction can mainly be used to ascertain whether a faraway galaxy is made up of matter or antimatter. One of the best papers of Santilli [1] in 1998 achieves the first ever known representation of the gravitational field of antimatter which serves in and sets the foundation for the first known grand unification of electroweak and gravitational integration including antigravity, developed in details in [6] of 2001. The first quantitative study of thermodynamics of antimatter is available in the paper [18] of 1999 written by J. Dunning Davis. Also, the treatment of matter and antimatter under the general conditions of irreversibility over time over classical operator level has been discussed in [19] of 2006 by Santilli. In his paper [20] in 2011, Santilli finally acquires the position to address his main objective namely To identify experimentally the existence of stars and galaxies, and detections of asteroids. The conformation that Santillis experiment in [7] on gravity of positrons in horizontal flight on earth is resolutory came via paper [21] of 2011 by the experimentalist V. de. Haan by confirming Mills results [15]. In this direction, the paper by Santilli with the mathematicians B. Davaas and T. Vougiouklis is completely the most advanced paper because it establishes that the universe is multi-valued and not multi-dimensional, as matter and antimatter co-exist in physically distinct space-times implies multivaluedness. Subsequently, Santillis studies on antimatter at three successive levels of study, including ; 1) Single-valued reversible 2) Single-valued irreversible and 3) Multi-valued irreversible conditions, are provided in monograph [22] of 2011 by theoretical physicists I. Gandzha and J.V. Kadieisvilli. Reference [23] in 2013 by A.A.Bhalekar is an excellent account of the basic mathematics behind the above subject matter.

2 Santillis Isodual mathematics

Inapplicability of 20th century mathematics for consistent representation of antimatter led to decades of rigorous studies of suitable formulation for quantitative representation of matter-antimatter annihilation. As such, application of the same (existing) mathematics for matter and antimatter proved to be incompatible due to matter-antimatter annihilation. Santilli found that the matter-antimatter annihilation could only be represented by the use of mathematics that is anti-homomorphic to each at all their levels. In fact, the mathematics anti-homomorphic to 20th century mathematics did not exist in 1980s. Physical theories describing antimatter at all the levels asked for construction of entirely new mathematics that would allow classical treatment of neutral or charged antimatter. While at Department of mathematics Harvard University in 1980s, under DOE support Santilli constructed the required new mathematics for the exact representation of antimatter, today known as Santilli isodual mathematics [2, 16, 10](monograph [6] for comprehensive presentation). This new mathematics is anti-homomorphic to the conventional mathematics. We outline the main branches of Santillis isodual mathematics;

2.1 Isodual Map

Note that the *term isodual denotes a conjugation* characterized by the word dual under the preservation of the axioms of conventional mathematics denoted by the Greek prefix "iso".

2.2 Isodual numbers

Isodual numbers are characterized via new basic isodual unit 1^d defined as;

$$1^d = (-1)^\dagger = -1 \tag{1}$$

with resulting isodual real, complex and quaternion numbers;

$$n^d = n1^d = (-n)^\dagger \tag{2}$$

and isodual multiplication defined as;

$$n^{dd}m^d = -n^d m^d \tag{3}$$

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with ensuing isodual operations of division power, square root e.t.c. under which 1^d is the basic unit of new theory. Also,

$$1^{dd} n^d = n^{dd} 1^d = n^d \tag{4}$$

As numeric field does not necessarily require that the basic unit be positive, it can indeed be taken as negative, and all the operations can be reformulated accordingly. This fact is the ultimate basis of the new theory of antimatter and the resulting new era of cosmology.

Lemma 2.1 *All quantities which are positive-definite when referred to fields(such as mass,energy, angular momentum,density, temperature, time etc.) became negative-definite when referred to isodual fields.*

Positive-definite quantities referred to positive-definite units characterize matter, and negative-definite quantities referred to negative definite units, characterize antimatter.

These characterizations lead to subsequent levels of representation of matter and antimatter.

Definition 2.1 *A quantity is called isoselfdual when it is invariant under isoduality.*

2.3 Isodual functional Analysis:

Functional analysis at large was subjected to isoduality with consistent applications of isodual theories resulting in a simple, unique and significant isodual functional analysis by Kadeisvili [24].

Isodual functions are defined as;

$$f^d(r^d) = -f^\dagger(-r)^\dagger \tag{5}$$

is called the isodual image of the conventional function,

2.4 Isodual differential calculus

This is the isodual image of the conventional differential calculus and related isodual derivative. *Isodual differential* coincides with the conventional differential by Santilli conception,

$$d^d r^d \equiv dr \tag{6}$$

Actually, because of this the new isodual calculus was not discovered since Newton's time till 1996.

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2.5 Isodual Lie theory

Let L be an *n-dimensional Lie algebra* with universal enveloping associative algebra $\xi(L), [\xi(L)]^{-} \approx L$ n-dimensional unit $I = diag(1, 1, ...1)$ for the regular representation, *ordered set of Hermitean generators* $X = X^{\dagger} = \{X_k\}$, conventional associative product $X_i \times X_j$, and familiar Lie's Theorems over a field $F(a, +, \times)$.

The *isodual universal associative algebra* $[\xi(L)]^d$ is characterized by the isodual unit I^d, isodual generators $X^d = -X$ and isodual associative product ;

$$X_i{}^d \times^d X_j{}^d = -X_i \times X_j \tag{7}$$

with corresponding infinite-dimensional basis(isodual version of conventional Poincare- Birkhoff-Witt theorem) characterizing the *isodual exponentiation of a generic quantity A*

$$e^{dA} = I^d + A^d/^d 1!^d + A^d \times^d A^d/^d 2!^d + \cdots = -e^{A^{\dagger}} \tag{8}$$

where e is the conventional exponentiation.

The attached *isodual Lie algebra* $L^d \approx (\xi^d)$ over the isodual field $F^d(a^d, +^d, \times^d)$ is characterized by the *isodual commutators*;

$$[X_i^d, X_j^d]^d = -[X_i, X_j] = C_{ij}^{kd} \times^d X_k^d \tag{9}$$

with a classical realization.

Let G be the conventional, connected, n-dimensional Lie transformation group on $S(x, g, F)$ admitting L as the Lie algebra in the neighbourhood of the identity, with generators X_k and parameters $\omega = \{\omega_k\}$. The *isodual Lie group* G^d admitting the isodual Lie algebra L^d in the neighborhood of the isodual identity I^d is the n-dimensional group with generators $X^d = \{-X_k\}$ and the parameters $\omega^d = \{\omega_k\}$ over the isodual field F^d with generic element

$$U^d(\omega^d) = e^{i^d \times^d \omega^d \times^d X^d} = -e^{i(-\omega)X} = -U(-\omega) \tag{10}$$

The *isodual symmetries* are then defined accordingly via the use of the isodual groups and they are ant-isomorphic to the corresponding conventional symmetries, as desired ref.[26] for additional details. *Conventional Lie symmetries are used for the characterization of matter where as Isodual Lie symmetries are used for the characterization of antimatter.*

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2.6 Isodual Euclidean Geometry

Let $S = S(x, g, R)$ be a conventional N-dimensional *metric space* with local coordinates $x = \{x_k\}, k = 1, 2, ...N$, nowhere degenerate, sufficiently smooth, real valued and symmetric metric $g(x, ...)$,and *related invariant*

$$x^2 = x^i g_{ij} x^j \tag{11}$$

over the reals R. The *isodual spaces* are the spaces $S^d(x^d, g^d, R^d)$ of $S(x, g, R)$ with isodual coordinates $x^d = x \times I^d$, *isodual metric*

$$g^d(x^d, ...) = -g^\dagger(-x, ...) = -g(-x, ...) \tag{12}$$

and *isodual interval*

$$(x-y)^{d2d} = [(x-y)^{id} \times^d g^d_{ij} \times^d (x-y)^{jd}]^d = [(x-y)^i \times g^d_{ij} \times (x-y)^{jd}] \times I^d \tag{13}$$

defined over the isodual field $R^d = R^d(n^d, +^d, \times^d)$ with the same isodual isounit I^d. The *three dimensional isodual Euclidean space* is defined as;

$$E^d(r^d, \delta^d, R^d) : r^d = \{r^{kd}\} = \{-r^k\} = \{-x, -y, -z\} \tag{14}$$

with

$$\delta^d = -\delta = diag(-1, -1, -1), I^d = -I = diag(-1, -1, -1) \tag{15}$$

Thus, the *isodual Euclidean geometry* is the geometry of the isodual space E^d over R^d which is given by step-by-step isoduality of the various aspects of conventional geometry.

Lemma 2.2 *The isoeuclidean geometry on E^d over R^d is anti-isomorphic to the conventional geometry on E over R.*

Isodual sphere is the perfect sphere in E^d over R^d with negative radius;

$$R^{d2d} = [x^{d2d} + y^{d2d} + z^{d2d}] \times I^d \tag{16}$$

2.7 Isodual Minkowski space

This new space $M^d(x^d, \eta^d, I^d)$ is characterized by the isodual image of the conventional *Minkowski space for matter* $M(x; \eta, I)$ where x denotes space-time coordinates, $\eta = Diag(1, 1, 1, 1)$ denotes the Minkowski metric [25], and

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$I = Diag(1,1,1,1)$ is the basic unit. Now the *isodual line element* is given by

$$x^{d2d} = (x^d \times^d \eta^d \times^d x^d)1^d \equiv x^2 \tag{17}$$

where the multiplication by 1^d is necessary for the isodual line element to have values in the isodual field. Note that the above isodual line element coincides with the conventional line element also by Santilli conception
It is important to note that Santilli's studies on antimatter requires a knowledge of the fact that **representation space of antimatter coexists with that of matter while being totally different from the latter**.

2.8 Isodual Riemannian geometry

Let $R(x, g, R)$ be a $3 + 1$ dimensional *Riemannian space* with basic unit $I = diag(1, 1, 1, 1)$ and *related Riemannian geometry* in local formulation. Then the *isodual Riemannian spaces* are given by

$$\begin{array}{rcl} x^d & = & \{-\widehat{x}^\mu\} \\ R^d(x^d, g^d, R^d) : g^d & = & -g = \{x\}, g \in R(x, g, R) \\ I^d & = & diag(-1, -1, -1, -1) \end{array} \tag{18}$$

with interval $x^{2d} = [x^{dt} \times^d g^d(x^d) \times^d x^d] \times I^d = [x^t \times g^d(x^d) \times x] \times I^d$ on R^d, where t stands for transposed.
The *isodual Riemannian geometry is the geometry of spaces R^d over R^d, and is also obtained by taking step-by-step isodualities of the conventional geometry, including, most importantly, the isoduality of the differential and exterior calculus.*

2.9 Isodual Lie theory and symmetries

These are *characterized by Hermitean generators* $X^\dagger = X$ verifying *Lie-Santilli isodual product*

$$[X^d, Y^d]^d = Y^d \times^d X^d - X^d \times^d Y^d \equiv [X, Y] \tag{19}$$

and related Lie-Santilli isodual theory formulated on isodual spaces over an isodual numeric field and elaborated via the isodual functional analysis and isodual differential calculus.

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It is important to note that the above isodual product coincides with the conventional Lie bracket also by Santillis conception; this identifies the deep meaning of the term isoduality. This new symmetry called as IsoSelfDuality (ISD) [3, 6, 1]is simply given by the invariance under isoduality. It can be verified that $P^d(3.1)$ does not verify ISD where as $P(3.1) \times P^d(3.1)$ does verify ISD as each symmetry is transformed in to other, resulting in a total invariance.

2.10 Isodual Lorentz-Poincare - Santilli symmetry

Santilli constructed *isoduality of the Lie theory*. He achieved this by way of the *isodual rotational symmetry* $SO^d(3)$, the isodual symmetry $SU^d(2)$, the *isodual Lorentz symmetry* $SO^d(3:1)$ and finally, the *isodual Lorentz-Poincare symmetry* $P^d(3:1)$ which is the fundamental symmetry of the new theory of antimatter. Here it is important to note that isodual mathematics is solely applicable to point-like abstraction of antimatter masses or particles.

Here it is important to note that *Isodual mathematics is solely applicable to point-like abstraction of antimatter masses or particles. Covering isodual isomathematics* is required for the representation of time reversal invariant systems of extended antimatter particles. Also, representation of their counterparts requires *isodual genomathematics*

The most general conceivable mathematics for antimatter is given by *Santilli isodual hypermathematics* which is particularly suited for multi-valued (rather than multi dimensional) formulations [27, 28].

2.11 Representation of antimatter at Newtonian level

As we know, *Newtonian treatment of antimatter consisting of N point-like particles is based on* a 7-dimensional representation space which is a Kronecker products of the Euclidean spaces of time t, coordinates r, and velocities v as;

$$S(t, r, v) = E(t, R_t) \times E(r, \delta, R_r) E(v, \delta, R_v) \tag{20}$$

where

$$r = (r_a^k) = (r_a 1, r_a^2, r_a^3) = (x_a, y_a, z_a) \tag{21}$$

$$v = (v_{ka}) = (v_{1a}, v_{2a}, v_{3a}) = (v_{xa}, v_{ya}, v_{za}) = \frac{dr}{dt} \tag{22}$$

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$$\delta = Diag : (1,1,1), k = 1,2,3, a = 1,2,3, ...N \tag{23}$$

where the base fields R_t, R_r and R_v are trivially identical all having trivial units $+1$, resulting in the trivial total unit

$$I_{Tot} = I_t \times I_r \times I_v = 1 \times 1 \times 1 = 1 \tag{24}$$

Newtons celebrated equations of motion for point-like particles are;

$$m_a \times \frac{dv_{ka}}{dt} = F_{ka}(t,r,v), k = 1,2,3, a = 1,2,3, ...N \tag{25}$$

For the isodual treatment of antiparticles basic space is $7N$*-dimensional isodual space given by,*

$$S^d(t^d, r^d, v^d) = E^d(t^d, R_t^d,) \times E^d(r^d; \delta^d, R^d) \times E^d(v^d, \delta^d, R^d) \tag{26}$$

with isodual unit and isodual metric

$$I^d_{Tot} = I^d_t \times I^d_r \times I^d_v \tag{27}$$

$$I^d_t = -1, I^d_r = I^d_v = Diag : (-1,-1,-1) \tag{28}$$

$$\delta^d = Diag(1^d, 1^d, 1^d) = Diag(-1,-1,-1) \tag{29}$$

This transformation results into celebrated Newton-Santilli isodual equations for point-like antiparticles first introduced by Santilli [6] as,

$$m^d_a \times^d dv_k a^d /^d d^d t^d = F^d_{ka}(t^d r^d v^d), k = x,y,z, a = 1,2,3, ...n \tag{30}$$

which has been experimentally verified. It is *important to note that the above isodual equations are anti-isomorphic to the conventional forms.*

2.12 Implications of Newton-Santilli isodual equations

Antimatter exists in a spacetime, co-existing, yet different than our own. As such, isodual Euclidean space $E^d(r^d, \delta^d, R^d)$ co-exist within, but is physically distinct from Euclidean space $E(r, \delta, R)$ and same occurs for full representation spaces $S^d(t^d, r^d, v^d)$ and $S(t, r, v)$.

Antimatter moves backward in time in a way as causal as the motion of matter forward. In fact, Newton-Santillis isodual equations provide the only known causal description of particles moving backward in time.

Antimatter is characterized by negative mass, negative energy and negative magnitudes of other physical quantities.

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3 Isodual Relativities

3.1 Isodual Galilean Relativity

First we introduce *isodual Galilean symmetry* $G^d(3.1)$ as *isodual image of the conventional symmetry* $G(3.1)$. For the Galilean symmetry of a system of N particles with non-null masses

$m_a, a = 1, 2, ...N, G^d(3.1)$ has i*sodual parameters and generators defined as;*

$$w^d = (\theta_k^d, r_0^{kd}, v_0^{kd}, t_0^{kd}) = -w \; j_k^d = \sum_{aijk} r_{ja}^d \times^d p_{ja}^k = -J_k, P_k^d = -P_k \tag{31}$$

and

$$G_k^d = \sum_a (m_a^d \times^d r_{ak}^d - t^d \times p_{ak}^d), H^d = \frac{1}{2}^d \times^d \sum_a p_{ak}^d \times^d p_a^{kd} + V^d(r^d) = -H \tag{32}$$

with *isodual commutator defined as*

$$[A^d, B^d]^d = \sum_{a,k} [(\partial^d A^d /^d \partial^d r_a^{kd}) \times^d (\partial^d B^d /^d \partial^d p_{ak}^d) - (\partial^d B^d /^d \partial^d r_a^{kd}) \times^d (\partial^d A^d /^d \partial^d p_{ak}^d) = -[A, B] \tag{33}$$

The structure constants and *Casimir invariants of the isodual Lie algebra* $G^d(3.1)$ *are negative-definite.* If $g(w)$ **is an element of the connected component** of the Galilei group $G(3.1)$ then *its isodual* is defined as

$$g^d(w^d) = e^{-i^d \times^d w^d \times^d X^d} = e^{i \times (-w) X} = -g(-w) \in G^d(3,1) \tag{34}$$

The *isodual Galilean transformations* are then given by the following;

$$t^d \to t'^d = t^d + t_0^d = -t, r^d \to r'^d = r^d + r_0^d = -r' \tag{35}$$

$$r^d \to r'^d = r^d + v_0^d \times^d t_0^d = -r', r^d \to r'^d = R^d(\theta^d) \times^d r^d = -R(-\theta) \tag{36}$$

where $R^d(\theta^d)$ is an element of the *isodual rotational symmetry.* The above isodual representation of antimatter is truly consistent with available classical experimental knowledge for matter [6]. *The situation in isodual space is described by the following Lemma.*

Lemma 3.1 *The trajectories under the same magnetic field of a charged particle in Euclidean space and the corresponding antiparticle in the isodual Euclidean space coincide.*

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Proof 3.1 *Consider a particle with charge $-e$ in the Euclidean space $E(r, \delta, R)$ i.e. the value $-e$ with respect to positive unit $+1$ of the underlying field of real numbers $R(n, +, \times)$. Suppose the particle is under the influence of the magnetic field B.*
The corresponding antiparticle via isoduality changes the sign (reversal of sign) of all physical quantities resulting in the charge $(-e)^d = +e$ in the corresponding isodual Euclidean space$E^d(r^d, \delta^d, R^d)$ simultaneously reversing the magnetic field $B^d = -B$ defined with respect to the negative unit $(+1)^d = -1$. This establishes the fact that the trajectory of the particle with charge $-e$ in the field B defined with respect to the unit $+1$ in the Euclidean space and that for the antiparticle in the field $-B$ defined with respect to the unit -1in the isodual Euclidean space coincide.

Corollary 3.1 *Antiparticles reverse their trajectories when projected from their own isodual space into our own space.*

3.2 Isodual Special Relativity

Classical relativistic treatment of point-like antiparticles can best be done via isodual special relativity . *Conventional special relativity [29] is constructed with the 4-dimensional unit of the Minkowski space,* $I = Diag.(1, 1, 1, 1)$ which represents dimensionless units of space $\{+1, +1, +1\}$ and the dimensionless unit of time $+1$, and is the unit of Poincare symmetry $P(3.1)$. *The isodual special relativity is defined by the map*

$$I = Diag(\{1, 1, 1\}, 1) > 0 \rightarrow I^d = -Diag(\{1, 1, 1\}, 1) < 0 \tag{37}$$

It is based on negative units of space and time.

The isodual special relativity is expressed by the isodual image of all mathematical and physical aspects of conventional relativity in such a way as to admit the negative definite unit I^d as the correct left and right unit,including: *the isodual Minkowski spacetime* $M^d(x^d, \eta^d, R^d)$ with isodual coordinates $x^d = x \times I^d$, isodual metric $\eta^d = -\eta$ and basic invariant over R^d

$$(x - y)^{d2d} = [(x^\mu - y^\mu) \times \eta^d_{\mu\nu} \times (x^\nu - y^\nu)] \times I^d \in R^d \tag{38}$$

and *fundamental isodual Poincare symmetry* [12]

$$P^d(3.1) = L^d(3.1) \times^d T^d(3.1) \tag{39}$$

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where $L^d(3.1)$ is the Lorentz-Santilli symmetry, $\times^d$ is the isodual direct product and $T^d(3,1)$ represents isodual translations. *The algebra of the connected component* $P_+^{\uparrow d}$ *of* $P^d(3,1)$ *can be constructed in terms of the isodual parameters* $w^d = \{-w_k\} = \{-\theta, -v, -a\}$ and isodual generators $X^d = -X = \{-M_{\mu\nu}, -P_\mu\}$, where the factorization by the 4-dimensional unit I is understood.

Also, the *isodual commutator rules* are given by;

$$[M^d_{\mu\nu}, M^d_{\alpha\beta}]^d = i^d \times^d (\eta^d_{\nu\alpha} \times^d M^d_{\mu\beta} - \eta^d_{\mu\alpha} \times^d M^d_{\nu\beta} - \eta^d_{\nu\beta} \times^d M^d_{\mu\alpha} + \eta^d_{\mu\beta}) \times^d \hat{M^d_{\alpha\nu}} \quad (40)$$

$$[M^d_{\mu\nu}, p^d_\alpha]^d = i^d \times^d (\eta^d_{\mu\alpha} \times^d p^d_\nu - \eta^d_{\nu\alpha} \times^d p^d_\mu)[p^d_\alpha, p^d_\beta] = 0 \quad (41)$$

The basic postulates of isodual special relativity are simple isodual image of the conventional postulates.
Isodual inversions and spacetime inversions are equivalent.

3.3 Isodual General Relativity

The most effective gravitational characterization of antimatter is isodual general relativity obtained by isodual map of all the aspects of conventional relativity. This is defined on the isodual Riemannian spaces $R^d(x^d, g^d, R^d)$. Isodual Riemannian geometry is defined on the isodual field of real numbers $R^d(n^d, +^d, \times^d)$ for which the norm is negative-definite. As a result, all quantities which are positive in Riemannian geometry become negative under isoduality, including the energy-momentum tensor. *Explicitly, the electromagnetic field follows the simple rule under isoduality*

$$F^d_{\mu\nu} = \partial^d A^d_\mu /^d \partial^d x^{\nu d} - \partial^d A^d_\nu /^d \partial^d x^{\mu d} = -F\mu\nu \quad (42)$$

and for the *energy-momentum tensor* we have

$$T^d_{\mu\nu} = (4m)^{-1d} \times^d (F^d_{\mu\alpha} \times^d F^d_{\alpha\nu} + (1/4)^{-1d} \times^d g^d_{\mu\nu} \times^d F^d_{\alpha\beta} \times^d F^{d\alpha\beta}) = -T_{\mu\nu} \quad (43)$$

In fact, isodual Riemannian geometry has negative-definite energy-momentum tensor and other physical quantities which open up new possibilities for attempting a grand unified theory.
Reader should note that the universal symmetry of the isodual general relativity, the isodual isoPoincare symmetry $\hat{P}^d(3.1)P^d(3.1)$ **has been introduced at the operator level in [10].**

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4 Antigravity

In the words of Santilli "*Isodual theory of antimatter predicts the existence of antigravity (here defined as the reversal of the sign of the curvature tensor in our space-time) for antimatter in the field of matter or vice-versa*" As such, the isodual theory of antimatter predicts in a consistent and systematic way at all levels of study, from Newtonian mechanics to Riemannian geometry, that matter and antimatter must experience gravitational repulsions ref [7, 30] and monograph [6]

We may summarize above results as; ***classical representation of antiparticles via isoduality renders gravitational interactions equivalent to the electromagnetic ones, in the sense that the Newtonian gravitational law becomes equivalent to the Coulombs law.***

These results could not have been achieved without isoduality.

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MINIMUM CONTRADICTIONS THEORY OF EVERYTHING

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Abstract

The purpose of this lecture is to show that no theory can be regarded as a complete one and this leads to indeterminism. This can be proved by means of a theorem which can show that the logical communication system, through which all theories are stated, is contradictory. According to the present point of view any uncertainty-incompleteness derives from the logical communication system itself and not from empirical principles which cannot be proven as valid. On this basis can be proved that space time is matter itself, Schrödinger's relativistic equation is valid and that the Ψ wave function is a complex space time function described in a Hypothetical Measuring Field (HMF). Thus, a Space Time Quantum Mechanics-Quantum Gravity can be stated and this is a Minimum Contradictions Theory of Everything. According to the present point of view, new phenomena and technologies, related to free energy and reactionless propulsion, can be explained and this reinforces the credibility of this theory.

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1. LOGIC ANALYSIS

If we denote by Λ a logic consisting of the Classical Logic denoted as P_I and the Sufficient Reason Principle regarded as a Complete Proof Principle denoted as P_{II}, we will have [1, 2]:

$$\Lambda \equiv P_I \cdot P_{II}$$

where P_{II} is defined as:

Complete Proof Principle - P_{II}: *"No statement is valid if there is not a complete logical proof of the statement, through valid statements different from it.*

*"*On this basis Theorem I can be derived i.e.:

Theorem I: *"Any system that includes logic* Λ *and a statement that is not theorem of logic* Λ *leads to contradiction"* [1, 2].

We name '0' the state before our communication and '1', '2', '3' the sequent states of this communication. '0' corresponds to the non-existence of any communication symbol while '1' to some symbol existence. From the non-existence of something cannot derive logically its existence. Working in the same way we have that a "posterior" does not derive logically from its "anterior". Therefore *the Anterior – Posterior Axiom is not theorem of* Λ. Applying Theorem I we obtain Statement I [1, 2]:

Statement I: *"Any system that includes logic* Λ *and the anterior-posterior axiom leads to contradiction."*

where the anterior – posterior axiom is stated as follows:

Anterior – Posterior Axiom (**A-P Axiom)***: "There is anterior-posterior everywhere in communication."*

Kantian 3rd antinomy has similarities with statement I [3, 4]; Kant's 3rd antinomy proof requires that a transcendental causality can exist a priori in contrast to theorem I.

Gödel and Rosser theorems [5, 6] could lead to same results as theorem I [1, 2]; however as H. Putnam [7] in his critic to R. Penrose's *Shadows of the Mind* [8] has noticed, they are restricted to computational well defined processes, not to logical communication in general [9].

Based on Theorem I and Statement I, we conclude that a system including the principles of logical communication leads to contradiction. This leads to the silence, and therefore when communicating logically it means that we decide to break the silence by avoiding contradictions on purpose [1, 2].

Despite statement I, we do communicate in a way we consider logical avoiding contradictions on purpose. Since contradictions are never vanished, we try to understand things through minimum possible contradictions. On this basis we can state [1, 2]:

Statement II - The Claim for Minimum Contradictions*: "What includes the minimum possible contradictions is accepted as valid."*

According to this claim we obtain a logical and an illogical dimension. In fact, through this claim we try to approach logic (minimum possible contradictions), but at the same time we expect something illogical since the contradictions cannot be vanished.

All axioms mentioned, the claim for minimum contradictions included, constitute the principles of the active logical language; when we speak we persist in logic despite of the existing contradictions.

Every theory includes at least the axioms mentioned; therefore no theory can be complete since it includes contradictions. On this basis, a *minimum contradictions physics* can derive where *the physical laws are the principles of the active logical language;* this physics is a stochastic matter-space-time QM implying a quantum gravity [1].

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COMMON ROOTS OF RELATIVITY THEORY AND QM

2.1 General

According to theorem I, further axioms beyond the ones of logical communication must be avoided since they can cause further contradictions. The systems of axioms we use in physics include the logical communication and, therefore, their contradictions are minimized when they are reduced to the logical communication itself.

At first sight, for a minimum contradictions physics we can make the following statement [1, 2]:

Statement III: *In a minimum contradictions physics everything is described in anterior–posterior terms.*

If there is space-time then there is anterior posterior so that space-time can be measured and denoted through the communication (language). Inversely, if there is anterior-posterior in communication then there is space-time. In fact, in order to write something we need space; also we need time since we cannot write in a simultaneous way. Thus, because of Statement III we can state the following [1, 2]:

Statement IV: *In a minimum contradictions physics everything is described in space-time terms.*

Since everywhere there is space-time and not something else, space-time can be regarded as the matter itself. A matter system, in general, has differences within its various areas. This means that a matter system, in general, is characterized by different rates of anterior - posterior (space-time) within its various points. This means that time can be regarded as a 4^{th} dimension which is compatible to Lorentz' transformations and in extension to a relativistic theory [1].

2.2. Hypothetical Measuring Field

Basic tool of this work is the Hypothetical Measuring Field (HMF) [1].

As Hypothetical Measuring Field (HMF) is defined a hypothetical field, which consists of a Euclidean reference space-time, in which at each *point A_0 the real characteristics of the corresponding, through the transformations of deformity, point A of the real field exist.*

In the HMF, it is defined as a relative space time magnitude sr, the ratio of the real infinitesimal space time magnitude ds to the corresponding, through the deformity transformations, infinitesimal magnitude ds_0 of the reference space- time, i.e. $sr = ds / ds_0$. This can apply to relative time $tr = dt / dt_0$, to relative length in a direction **n** $lr_n = dl_n / dl_{n0}$ and to a relative volume $vr = dv / dv_0$.

2.3 Equivalence of Energy and Time

In a space-time description we don't know a priori what energy is. We define energy dE of an infinitesimal space-time element its 'ability to exist'. We may notice that an infinitesimal space-time element with energy dE exists on condition that some corresponding 'anterior-posterior' exist too. With respect to the HMF a space-time element is observed during a time dt that is different from the time dt_0 of the corresponding reference space-time element. Various space-time elements in the HMF have different dt for the same dt_0. Thus, dt measures the duration *i.e*. the ability of a space-time element to exist; this ability, by definition is energy; when $dt = dt_0$, this ability is dE_0. Thus, we can write [1]:

$$dE \sim dt \quad \text{and} \quad dE / dE_0 = dt / dt_0 \tag{1}$$

which is a relativistic relation. *Relations (1) show the equivalence of a space-time element energy to the time flow rate within this element.* When dt is a constant period of time in the HMF, then Eq. (1) can be written in the form:

$$dE / dE_0 = dt / dt_0 = (f / v)/(f / v_0) = v_0 / v \tag{2}$$

where v is the frequency of a periodic phenomenon of comparison and f an arbitrarily constant factor through which we can change the scale of time.

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If $v = 1$, v_0 must be different in various points $(\mathbf{r},t)$ of the HMF. If this is the case Eq. (2) can be written in the form:

$$dE / dE_0 = v_0(\mathbf{r},t) \tag{3}$$

Thus, for the same equation we have the following versions [1]:

$$dE / dE_0 = dt / dt_0 \text{ observation (relativity theory)} \tag{4}$$

$$dE / dE_0 = v_0(\mathbf{r},t) \text{ action (quantum mechanics)} \tag{5}$$

Thus, at a first sight relativity theory and QM have common roots [1]. On this basis, we can reach the basic De Broglie's principle for a particle energy; in fact for $E_0 = h$ we have (arithmetically) that:

$$E = hv \tag{6}$$

3. STOHASTIC SPACE TIME

3.1 General

At second sight, taking into account the above mentioned and applying the claim of the minimum contradictions, we conclude that matter-space-time has logical and contradictory behavior at the same time; this can be valid when space time exists and not exists at the same time (illogical behavior) while it implies the existence of logic. This can be approached by the aid of a hypothetical measuring field HMF. If this is the case we can say that space-time has a probability to exist and to correspond to an infinitesimal area around a point $(\mathbf{r},t)$ of the HMF. Thus we can state the following:

Statement V: *Minimum Contradictions Physics can be described by Stochastic Space-Time.*

However physics describes any matter system i.e. matter, anti-matter, mass and charge. On this basis statement V has sense if there are various kinds of space time corresponding to the various forms of matter. Thus we can use signs +1, -1, +i, -i for various states. For the purposes of stochastic space-time description, the following definition is useful [1]:

In a HMF, we define as mean relative space time magnitude $\overline{sr}$ the ratio of the mean real infinitesimal space time magnitude $\overline{ds}$ to the corresponding infinitesimal magnitude ds_0 of the reference space- time i.e. $\overline{sr} = \overline{ds} / ds_0$. Thus, for mean relative time we obtain*:* $\overline{tr} = \overline{dt} / dt_0$. The relative space-time magnitudes mentioned above, are denoted by SR, TR ,.. when they refer to mean values of a particle space-time field.

4. EQUATIONS OF MINIMUM CONTRADICTIONS PHYSICS

The minimum contradictions equations are here mentioned in order that a general idea on the results of the minimum contradictions physics might be introduced.

The electromagnetic (*em*) space-time, is a space-time whose all magnitudes are considered imaginary and behave exactly like the gravitational (*g*) space-time. Electromagnetic (*em*) space-time is described by means of space-time wave functions such that [1]:

$$\Psi_{em}(\mathbf{r}_{em}, t_{em}) = \Psi^{g}_{em}(\mathbf{r}, t) \quad (7)$$

where Eq. (7) has meaning due to the coexistence of (*g*) and (*em*) space-time under a scale which appears to be equal to the fine structure constant α [1].

According to the spirit of this paper ***there is not potential acting at a distance*** **since space time is matter itself.** By the aid of Fourier analysis and *without any other physical principles* the following can be obtained [1]:

a. *Relative Space-Time Operators* (*Relative Time, Volume, Length in a direction* $\mathbf{n}$)

$$\hat{TR} = \frac{i}{2\pi}\frac{\partial}{\partial t}, \quad \hat{VR} = -2\pi i \frac{1}{\partial / \partial t}, \quad \hat{LR}_n = \left(1 - c^2 \frac{\partial^2 / \partial x_n{}^2}{\partial^2 / \partial t^2}\right)^{1/2} \frac{h}{m_0 c^2} \quad (8)$$

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b. *Particle Schrödinger's Relativistic Equation (Klein–Gordon) for (g) and for (em) Space-Time:*

$$\frac{\partial^2 \Psi}{\partial t^2}(\mathbf{r},t) - c^2 \nabla^2 \Psi(\mathbf{r},t) = -(m_0 c / \hbar)^2 \Psi(\mathbf{r},t) \tag{9}$$

c. *Many body Schrödinger's Relativistic Equations for (g) and for (em) space time:*

$$\frac{\partial}{\partial x_j} \frac{\Box \Psi_g(\mathbf{r},t)}{\Psi_g(\mathbf{r},t)} = 0 \qquad (j = 1,2,3,4) \tag{10}$$

$$\frac{\partial}{\partial x_j} \frac{\Box \Psi^g_{em}(\mathbf{r},t)}{\Psi^g_{em}(\mathbf{r},t)} = 0 \qquad (j = 1,2,3,4) \tag{11}$$

d. *Energy Conservation:*

$$\partial_t \left(\frac{\partial_t \Psi_g(\mathbf{r},t)}{\Psi_g(\mathbf{r},t)} + \frac{\partial_t \Psi^g_{em}(\mathbf{r},t)}{\Psi^g_{em}(\mathbf{r},t)} \right) = 0 \tag{12}$$

e. *Momentum Conservation:*

$$\partial_t \left(\frac{\nabla \Psi_g(\mathbf{r},t)}{\Psi_g(\mathbf{r},t)} + \alpha \frac{\nabla \Psi^g_{em}(\mathbf{r},t)}{\Psi^g_{em}(\mathbf{r},t)} \right) = 0 \tag{13}$$

f. *Geometry of (g) or (em) Space-Time i.e. Mean Relative Time and Mean Relative Length in a Direction* $\mathbf{n}$ *at a Point* $(\mathbf{r},t)$:

$$\overline{tr}(\mathbf{r},t) = \frac{ic}{2h} \frac{\partial_t \Psi}{(\Psi \Box \Psi)^{1/2}} \left(\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^* \right) \tag{14}$$

$$\overline{lrn}(\mathbf{r},t) = -\frac{ih}{2} \frac{\Psi}{\Box \Psi} \left(1 - c^2 \frac{\partial^2 \Psi / \partial x_n^2}{\partial^2 \Psi / \partial t^2} \right)^{1/2} \left(\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^* \right) \tag{15}$$

The above equations cannot describe a unified field on the basis of common boundary conditions; they need to be self determined [1]. This implies the existence of incompleteness-volition (Free Will) and Indeterminism. These equations imply a statistical interpretation and a distribution of matter space-time according to Schrödinger Relativistic Equation probability density

$$P(\mathbf{r},t) = (i\hbar / 2m_0 c^2) \left(\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^* \right) \tag{16}$$

In this case, Ψ function locally is described by an equivalent local space-time particle field wave function Ψi, where this field is regarded as extended to infinity. This can occur when equations (10 to 13) have constant values of m_{0g} or m_{0em} only in the vicinity of various $(\mathbf{r},t)$ of the (HMF).

5. QUANTUM GRAVITY

The gravitational acceleration $\mathbf{g}(\mathbf{r},t)$ represents the force that must be applied to a unit of mass at every point $(\mathbf{r},t)$ in order that energy will be distributed according to the probability density function $P(\mathbf{r},t)$. It can be proven that [1]:

$$\mathbf{g}(\mathbf{r},t) = \frac{c^2}{P(\mathbf{r},t)} \nabla P(\mathbf{r},t) = \frac{c^2}{\overline{tr}(\mathbf{r},t)} \nabla \overline{tr}(\mathbf{r},t) \tag{17}$$

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From equation (17) for a particle field, because of equation (16) it holds:

$$\mathbf{g}(\mathbf{r},t)=\frac{c^2\nabla\left(\Psi^*\partial_t\Psi-\Psi\partial_t\Psi^*\right)}{\left(\Psi^*\partial_t\Psi-\Psi\partial_t\Psi^*\right)} \tag{18}$$

Equations (17, 18) describe a unified relationship which is valid everywhere. Under certain simplifications it can be proven that equation (17) is compatible to Newton and to Coulomb law as well as to the relativistic formula for gravity [1]. This implies a convergence of a Deterministic with an Indeterministic point of view.

6. POSSIBILITY TO VERIFICATION OF (G) AND (EM) INTERACTION

6.1 General

According to the present point of view new phenomena and technologies, related to free energy and reactionless propulsion, can be explained and this reinforces its credibility. There have been made many devices to produce energy or to create propulsion through the vacuum [15-24]. In the present paper the effects - devices we use to possibly verify the (g) and (em) interaction are the following:

6.2 Santilli's Etherino

The *neutron* was conceived by H. Rutherford as a "compressed hydrogen atom" in the core of a star. Don Borghi claimed the laboratory synthesis of the neutron from protons and electrons; this experiment remained unverified for decades due to the lack of theoretical understanding of the results. R.M. Santilli has verified and theoretically explained this experiment by the aid of a particle process which he called as *etherino* (from ether) on the basis of Hadronic Mechanics [10, 11]. The *Etherino Process* has been found to be compatible to 'Minimum Contradictions Physics' which implies that space time is matter itself and consists of gravitational (g) and electromagnetic (em) space time which are interconnected and communicate through photons (particles with zero rest mass). A basic consequence of this, is the *Statement: "During the approach of an electron to a proton there is absorption of gravitational space time energy"* [1, 12].

6.3 U.S. Patent No. 8,952,773

The device of the U.S. Patent No. 8,952,773 [13] is illustrated in fig. 1; this device consists of a superconducting nozzle connected at its narrow end with a permanent magnet, which can create propulsion without any external energy source but only in the direction South to North. Experimental verification was carried out both at the Technological Institute of Thessaly (now the University of Thessaly) in the Laboratory of Renewable Energy and in the Solid State Physics Laboratory of the National Kapodistrian University of Athens.

6.4 PCT/GR2020/000040-Priority GR 20190100373 patent application

The device of the PCT/GR2020/000040 patent application [14] is illustrated in fig. 2 and fig.3 and consists of a soft iron core, of constant or changing axis of symmetry and constant or changing cross sectional area, which is surrounded by a REBCO tape solenoid producing DC or AC magnetic field and a magnetic shield which does not permit the magnetic field to penetrate it; this device can create a propulsive force upwards.

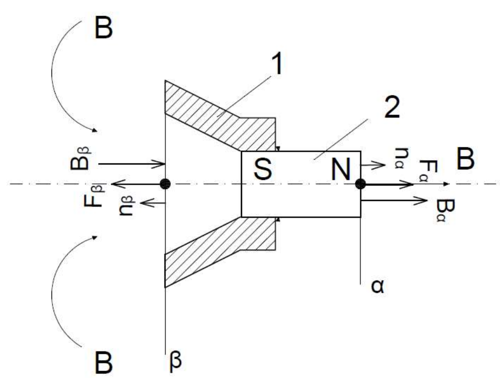

Figure1

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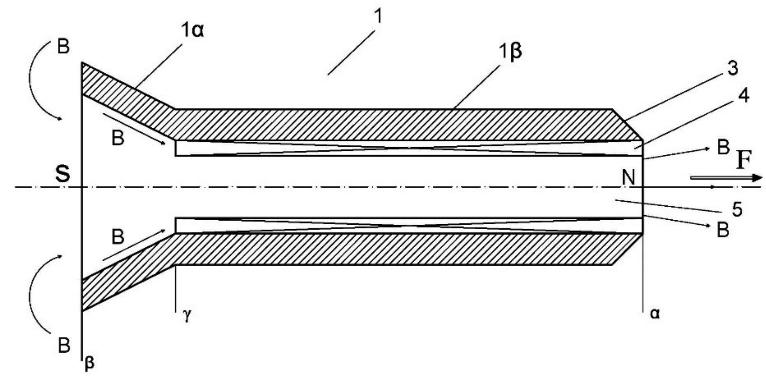

Figure 2

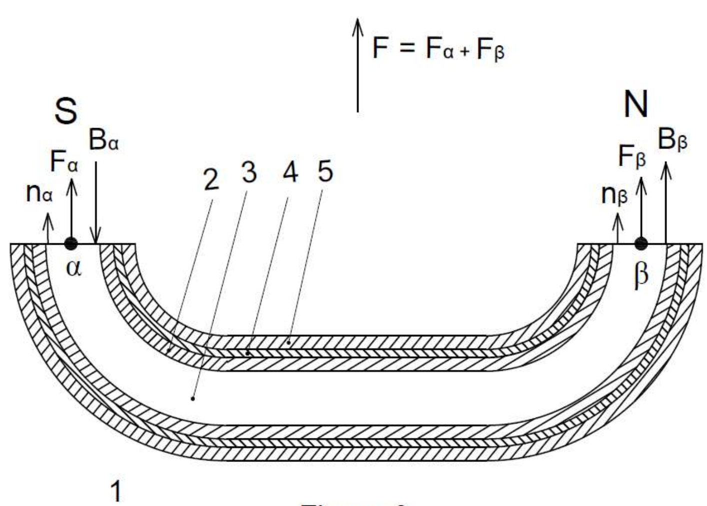

Figure 3

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A Proposed Physical Basis for Quantum Uncertainty Effects.

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Abstract:
Quantum scale "uncertainty" effects limiting measurement accuracy appear to reflect the actual properties of quantum particles as has been well substantiated in numerous experimental examples. However, the concept of uncertainty appears to lack any clear physical basis and stands as an effects descriptor, not as a causal description of actual particulate physical properties. The famous EPR paradox is examined, assessed and placed into current perspective then new theory is presented defining the functional causal basis of observed uncertainty effects. Lastly, experimental evidence will be presented in support of this new model.

"When we see probability we do not see causality, we see the limits placed upon our ability to observe overcome by way of an ingenious guess at the result. In this clever approach where cause is neglected *for the prediction of outcome*, we must not forget it is we who can not see. *Physical systems are not guessing at themselves."*
—R.N.

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Introduction.

To many the whole notion of uncertainty, as discussed in quantum mechanics, comes as something of anathema; the whole idea seems to contradict commonsense. It would appear, therefore, sensible to examine this apparent basis for much of modern physics again and in some detail. At the same time it would seem appropriate to examine other uncertainty relations which come into modern physics. Amongst these must be the idea of uncertainty relations in thermodynamics and it is an examination of these which could lead to an understanding of the entire issue, including possibly a further consideration of the question of the completeness of quantum mechanics as a theory and, therefore, of the validity of the claims of Einstein, Podolsky and Rosen[1]. There are no uncertainty relations in classical thermodynamics where, almost by definition, all physical quantities are taken to have quite definite values. However, for example when systems composed of a large number of particles are to be investigated, statistical methods have to be employed since it is impossible, at least at present, to evaluate exactly the behavior of each and every individual particle. Hence, the subject 'statistical mechanics' came into being. By its reliance on statistical methods and, therefore, the idea of probability, the outcome of investigations becomes less definite and uncertainties creep in. This is the source of the so-called thermodynamic uncertainty relations which are considered in many texts[2]. Note though that these uncertainty relations arise out of the introduction of uncertainty into the theory by investigators; they do not appear purely as a result of the physical situation being discussed. Hence, such relations and any deductions made using them must be viewed with a degree of skepticism and treated accordingly since it is not at all obvious that any such deductions are physically realistic. It might be wondered if the same could be true of the uncertainty relations of quantum mechanics. In his seminal book[3], Heisenberg first introduces his relations via a quite simple but definitely approximate method using a wave picture. He then proceeds to derive them also without explicit use of the wave picture but then obtains from them ***the mathematical scheme of quantum theory and its physical interpretation.*** However, at the basis of much of the mathematics associated with quantum theory is the wave equation with the so-called wave function associated with a probability. Once probability comes into things, uncertainties in measured quantities must necessarily follow. Hence, the question must be raised as to whether, or not, the uncertainties associated with quantum theory are real

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physical uncertainties or uncertainties introduced surreptitiously by theoreticians, just as occurs in statistical thermodynamics?

All the work that follows is really an extension of earlier work which appeared in the Hadronic Journal and is available online[4]. Any reader of the current work is encouraged to read the full work mentioned here first in order to grasp more easily that which follows.

Some Preliminaries.

To begin with, the original paper by Einstein, Podolsky and Rosen[1] should be examined. It may be noted that several important points concerning the thought experiment are proposed:

" ...every element of the physical reality must have a counterpart in the physical theory."

"The elements of the physical reality cannot be determined by a priori philosophical considerations, but must be found by an appeal to results of experiments and measurements."

"More generally, it is shown in quantum mechanics that, if the operators corresponding to two physical quantities, say A and B, do not commute, that is, if AB≠BA, then the precise knowledge of one of them precludes such a knowledge of the other. Furthermore, any attempt to determine the latter experimentally will alter the state of the system in such a way as to destroy the knowledge of the first. From this follows that either (1) the quantum mechanical description of reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. For if both of them had simultaneous reality-and thus definite values-these values would enter into the complete description, according to the condition of completeness."

"Thus, by measuring either A or B we are in a position to predict with certainty, and without in any way disturbing the second system, either the value of the quantity P (that is p_k) or the value of the quantity Q (that is q_r)."

"Previously we proved that either (1) the quantum-mechanical description of

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reality given by the wave function is not complete or (2) when the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality. Starting then with the assumption that the wave function does give a complete description of the physical reality, we arrived at the conclusion that two physical quantities, with noncommuting operators, can have simultaneous reality. Thus the negation of (1) leads to the negation of the only other alternative (2). We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete."

Two primary elements of the EPR argument may now be noted separately:

1. It is possible to define both position and momentum of two previously interacting quantum particles/systems.
2. Measurement may not (non locally) disturb system two if system one is measured, unless a hidden variable not yet defined within the context of *wave function* is identified.

Point two is clearly implied from the last sentence in the paper:

"We believe, however, that such a theory is possible."

and the aforementioned sentence:

"...every element of the physical reality must have a counterpart in the physical theory."

It is important to note at this juncture, the concerns of Heisenberg regarding such fanciful methods of deduction and exploration as thought experiment and human imagining alone, which appear to closely parallel Einstein's views of the same, as already noted above.

From Heisenberg's book[3], p. 15, concerning the reality of uncertainty as per his equations in physical systems, he states that

"*In this connection one should particularly remember that the human language permits the construction of sentences which do not involve any consequence and which therefore have no content at all—in spite of the fact that these sentences produce some kind of picture in our imagination*."

The reader of this present article should note this point as it is important in what follows.

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Analysis of EPR feasibility.

If the notion of the EPR argument is sound, one would expect the scheme to be used in some sort of demonstrable way. If the idea is good and leads to accurate measurement, some practical usage must have been made of it after all these years. Entangled science aside, is the basic notion in point one above actually demonstrable?

Let us bring forward the usual interpretations of the EPR ideas, and imagine two quantum particles which have interacted, and are now moving directly away from each other at a 180 degree relation. This is the interpretation most used, that akin to the thinking of Kumar[5] which defines the EPR idea as "two particles, A and B, [that] interact briefly and then move off in opposite directions."

Is this scheme actually able to measure anything, and is it used? Seemingly yes. Positron Emission Tomography scanning (the PET scan) appears to use this idea to measure biological processes and define the locations thereof. A PET scanner is essentially a gamma ray detector. In PET scans, Blood Oxygenation Level-Dependent relations indicative of tissue oxygen metabolism are detected through positron/electron annihilations created by way of an injected radioactive oxygen tracer such as ^{15}O, which has a half life of 123 seconds. As the unstable nucleus of a ^{15}O atom decays having been absorbed by dynamic oxygen using tissues such as neurons, it emits a positron. The positron annihilates when brought in contact with an electron, emitting 2 (gamma) annihilation photons which travel in exactly opposite directions, a 180 degree relation of two quantum particles moving at a constant mutual speed, allowing accurate measurement of the location of the source interaction in space, and also, inference could easily be drawn from one particle measurement to the values of the other.

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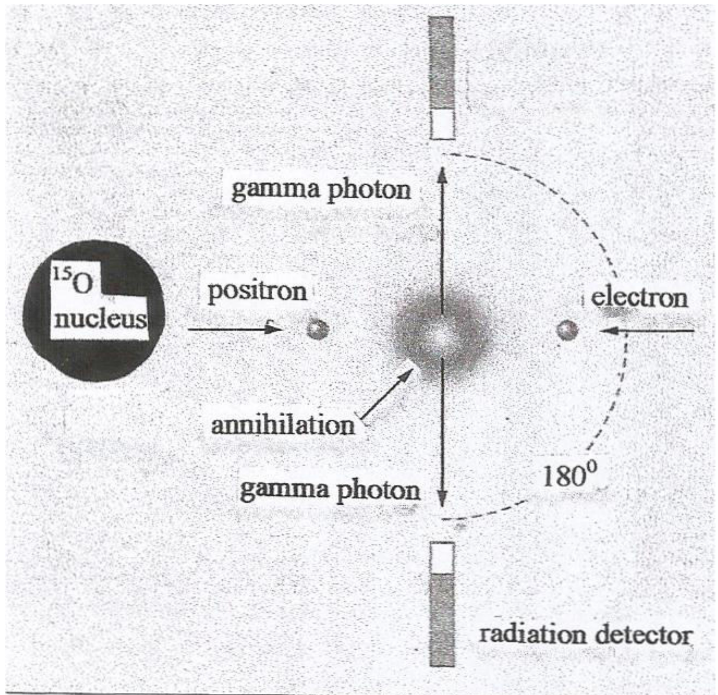

PET scan schematic representation.

It may be concluded that the basic notion is in fact quite functional as a system of measurement when used in a general way. It is clear also that scientific observers could easily infer the position and momentum of one particle from measurement of the other, which travels in mirror opposite, both at a known speed.

It seems the EPR scheme does allow actual measurement as it should in reality, and is not just a fanciful idea one may draw up to form a picture in one's head, and so, answers in this one aspect at least, Heisenberg's and also Einstein's standards of a workable theory as represented in good science.

Next, we move to the nonlocal aspects of the EPR theory and assess the outcome of experiments. Local realism insists that measurement of one separated system part could not ever superluminally affect the other separated parts of the system (presumably unless some missing, hidden variable is in play). Recall that, in the Copenhagen interpretation of QM, the wave function is entirely a probabilistic entity! However, it is found that

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nonlocal measurement effects moving well in excess of light speed are evidenced and those results then repeated in experiments involving entanglement.

In an article by Yin, et al.[6], it may be read

> "In the well-known EPR paper, Einstein et al. called the nonlocal correlation in quantum entanglement as `spooky action at a distance'. If the spooky action does exist, what is its speed? All previous experiments along this direction have locality and freedom-of-choice loopholes. Here, we strictly closed the loopholes by observing a 12-hour continuous violation of Bell inequality and concluded that the lower bound speed of `spooky action' was four orders of magnitude of the speed of light if the Earth's speed in any inertial reference frame was less than 10^{-3} times of the speed of light."

Here, the new theories come good and the matter may be resolved in favor of a hidden variable: the scalar wave within the aether. See reference 4. Of course, in any modern discussion of the EPR paradox, it must never be forgotten that a resolution was presented in 1998 by Ruggero Santilli[7] and this has, as far as is known, never been discredited. Hence, it appears that, when the whole question of the EPR paradox comes under discussion, reference should be made to this work.

Cause of quantum uncertainty effects.

Again the reader should remember of the cautionary words of Heisenberg:

> "In this connection one should particularly remember that the human language permits the construction of sentences which do not involve any consequence and which therefore have no content at all—in spite of the fact that these sentences produce some kind of picture in our imagination."

It might be postulated that the notion of "uncertainty" itself is exactly such an error as Heisenberg himself cautions against! This property is particulate anthropomorphism...we assign a human quality, *uncertainty*, a kind of affective and logical confusion, to a physical particle. Yes; humans can form this idea, an idea of a particle which is somehow confused as they are, but that is a human idea, not a physical idea. Although it may be pictured, it has no actual physical content.

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What could actually be causing the observed measurement results of quantum experiments? If not uncertainty, what is the physical cause of the measurement problem and seeming duality between particle and wave? Duality is always the mark of confused thinking, as are most if not all paradoxes. What could be causing the plainly available "uncertain" experimental effects. It must be a real physical object, and not some confused human imagining!

In truth, Heisenberg's uncertainty relation

$$\Delta x \Delta p_x \geq h$$

describes effects, not causes. There seems to be no physics in this! What actual object could cause these measurement and other "uncertain" quantum effects?

There is a hidden variable; that is, the aether and the longitudinal pressure waves (scalar waves) which form up "force carrier," entangled and gravitational effects. See reference 4.

Now recall boundary layer theory as applied to the aether; that is, the boundary between the aether itself and any body passing through it or over which it passes. Details of the theory of the boundary layer, due initially to Prandtl[8], may be found in most books on fluid mechanics such as that by Cole[9].

Imagine an aetherial boundary layer around a particle. The original uncertainty equation is missing the basis - there is no basis to the physics - it describes only effects. The boundary layer as a ***particle-surrounding scalar wave*** accounts for the causal mechanism of uncertainty effects, (as well as, possibly, nuclear decay and fusion as will be discussed in future work) - the measurement uncertainty is then caused by an actual wave surrounding the actual particle; not a wave-like particle duality. Physics has left out the aether and, hence, the wave around each quantum particle. The "uncertain" momentum and x component of velocity in the Heisenberg equations are themselves caused by this wave obscuring those aspects of the particle. The overall change as diffusion then refers to the heat within the scalar wave and hence its initial (quantum) size, ***delta in the Heisenberg equations then referring to the amount of change in temperature above absolute zero***, in a causal analysis and proper treatment[3]. That wave is the source of "diffusion"

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effects. Note how in the paper, ***Entropy of a column of gas under gravity***[10], heat first added to the system creates gravitational potential (in part) and not only increase in temperature. That gravitational potential is, by our present theories, the creation of the scalar waves which create a gravitational field. See reference 4.

If this is so, and our theory correct, a violation of measurement "uncertainty" should be observed in experiments if the scalar waves around the particles are deprived of heat. Indeed, this is exactly what is seen in experiments. The back action limit, the quantum limit on measurement precision bounded by uncertainty, is violated, and now, just as might be expected, absolute zero may be approached *arbitrarily close* to deprive the actual source of uncertainty effects of the heat needed to create them. As Clark and colleagues have pointed out recently[11]:

> "Here we propose and experimentally demonstrate that squeezed light can be used to cool the motion of a macroscopic mechanical object below the quantum backaction limit. We first cool a microwave cavity optomechanical system using a coherent state of light to within 15 per cent of this limit. We then cool the system to more than two decibels below the quantum backaction limit using a squeezed microwave field generated by a Josephson parametric amplifier."

Uncertainty is experimentally demonstrable as a function of heat instantiated within the boundary scalar wave surrounding the particle. It appears likely that, as heat is further reduced as absolute zero is approached more closely, *the cause of quantum uncertainty and fluctuation which is the omnidirectional motion of aether particles within the particle boundary scalar wave* is then reduced, perhaps by way of energy reduction of the aether particle itself and/or alignment of said omnidirectional particle motions, leading to the absence of any wave-forming particulate energy value at absolute zero temperature.

Quantum fluctuation effects and related uncertainty are caused by omnidirectional aether particle motion. Uncertainty itself within quantum particulate measurement dynamics is actually caused by the boundary wave, surrounding a quantum particle as a function of heat.

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Uncertainty effects emerge as a function of quantum scale, as the aether particle size is more closely approached.

Lastly, new experiments are seen where, as might be expected, heat is reduced to permit the proliferation of related condensate and EPR effects to emerge. Note, for example the paper by Fadel, et al[12] in which it is stated that

> "While spin-squeezed and other nonclassical states of atomic ensembles were used to enhance measurement precision in quantum metrology, the notion of entanglement in these systems remained controversial because the correlations between the indistinguishable atoms were witnessed by collective measurements only. Here we use highresolution imaging to directly measure the spin correlations between spatially separated parts of a spin-squeezed Bose-Einstein condensate. We observe entanglement that is strong enough for Einstein-Podolsky-Rosen steering: we can predict measurement outcomes for non-commuting observables in one spatial region based on a corresponding measurement in another region with an inferred uncertainty product below the Heisenberg relation."

Uncertainty within internal and external dynamical systems.

Clearly the ideas within this brief work refer only to uncertainty effects within the external dynamical problem, that of particulate interactions and not to the internal dynamical problem of hadronic construction which is that of non-potential contact interactions, meaning non-Hamiltonian systems (that is, variationally nonself-adjoint systems not representable with a Hamiltonian). Those hadronic and other like systems then, may be rightly understood without erroneous reference to uncertainty by way of the mathematics of Santilli.

These topics are discussed in detail at:
http://www.galileoprincipia.org/santilli-confirmation-of-the-epr-argument.php

Briefly, as derived from the web reference above:
Extended and hyperdense protons and neutrons in conditions of partial mutual penetration as occurring in a nuclear structure demonstrate

nonHamiltonian forces. The assumption of the exact validity of Heisenberg's uncertainty in the interior of a nucleus is non-scientific. The hadronic isomomentum is uniquely defined by

$$p' * \psi'(t', r') = - i\, \partial' \psi'(t', r') = - i\, U\, \partial\, \psi'(t', r') \qquad (1)$$

It is then plain that isolinear momenta isocommute on isospace over isofields by therefore confirming the principle of isotopies

$$[p'_i, p'_j]' = p'_i * p'_j - p'_j * p'_i = 0 \qquad (2)$$

This occurs because the isotopic element T of the isoproduct "*" , cancels out with its inverse, the isounit U =1/T. However, isomomenta no longer commute in our spacetime,

$$[p'_i, p'_j] = p'_i\, p'_j - p'_j\, p'_i \neq 0 \qquad (3)$$

because, in the absence of the isotopic product, the derivative does act non-trivially on the isounit U due to its general dependence on local coordinates, and this eliminates Heisenberg's uncertainty principle for the study of interior problems and actually replaces it with a much more general principle.

Some Final Thoughts on the Aether.

Before concluding, it might be appropriate to reflect on the demise of the aether theories over the last hundred years and more. In the intervening time, several people have doggedly pursued investigations into theories involving the aether concept, often at personal cost. Among those was Kenneth Thornhill and it might benefit many to read his work which is readily available on the internet.[13] In the cited article, he starts by showing

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that Planck's energy distribution for a black body radiation field may be derived for a gas-like aether with Maxwellian statistics. The gas consists of an infinite variety of particles whose masses are integral multiples of the mass of the unit particle. Also the frequency of electromagnetic waves correlates with the energy per unit mass of the particles, not with their energy, thus differing from Planck's quantum hypothesis. Identifying the special wave-speed, usually called the speed of light, with the wave-speed in the 2.7^0K background radiation field, leads to a mass of 0.5×10^{-39}kg for the unit aether particle. Interestingly, in this article he also shows that the speed of light should vary with the square root of the background temperature. It is not without interest to note that this suggestion by Thornhill would obviate any need for introducing theories of inflation to protect the Big Bang notion. More may be found on the whole question of the constancy, or otherwise, of the speed of light in the article by Farrell and Dunning-Davies[14].
Also, before ending this section, attention should be drawn to a companion paper by Thornhill[15] in which he discusses in detail the fact that, in a gas-like aether, the duality between the oscillating electric and magnetic fields, which are transverse to the direction of propagation of electromagnetic waves, becomes a triality with the longitudinal oscillations of the motion of the aether if electric field, magnetic field and motion are coexistent and mutually perpendicular. He points out that it must be shown that, if electromagnetic waves also comprise longitudinal condensational oscillations of a gas-like aether, analogous to sound waves in a material gas, then all three aspects of such waves must propagate together along identical wave fronts. This he shows to be the case. Further he finds that the equations governing the motion and the electric and magnetic field strengths in such an aether, together with their common characteristic hyperconoid, are all invariant under Galilean transformation.

Conclusion.

The notion of "uncertainty" within physical systems is only an anthropomorphic effects descriptor, not a causal description of physics. Fluctuation effects in quantum systems and uncertain measurement effects are in fact caused by a real object and not probability: the aether and the scalar waves within it. Quantum mechanics as interpreted by the Copenhagen interpretation is in point of fact: incomplete. The wave function must be augmented in its interpretation to represent aetherial and scalar wave dynamics, at which point the adjusted theory would in fact satisfy Einstein's highest standards as a physical theory.

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Possibility of geometrical interpretation of quantum mechanics and geometrical meaning of “hidden variables”

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Interpretation of wave function for free particle is suggested as a description of microscopic distortion of the space-time geometry, namely, as some closed topological 4-manifold. Such geometrical object looks in three-dimensional Euclidean space as its topological defect having stochastic and wave-corpuscular properties of quantum particle. All possible deformations (homeomorphisms) of closed topological manifold play the role of “hidden variables”, responsible for statistical character of the theory.

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1.Introduction Interpretation of quantum mechanics means here an explanation of strange features of its mathematical formalism ("copenhagen" interpretation) with the help of notions from everyday life (physical model). Attempts to find such explanation started just after the creation of quantum mechanics and this problem is still considered by many physicians as actual. For example, V.Ginsburg considered interpretation of quantum mechanics as the one of three great problems of modern physics (as the problem of appearance of life and the problem of irreversibility of time) [1]. The problem of interpretation of quantum mechanics was investigated for many years by t'Hooft [2] (here is a detailed list of references on the problem). . But why any interpretation is needed for mathematical formalism if it is in a good agreement with experiment? One of reasons is the fact that new physical models open new opportunities for development of theories. For example, many attempts (Einstein Weyl, Calutza and others) have been made for this reason to find geometrical interpretation of classical electrodynamics, although it is in a very good agreement with experiment [3,4]. In addition, the quantum theory cannot be considered as the final one. Another, more concrete, reason—the contradiction between Bohr and Einstein regarding the completeness of quantum mechanics which did not resolved until now [5,6]. In contrast to Bohr, Einstein thought that the quantum mechanics is not a complete theory because it says nothing about physical reality, responsible for statistical character of the theory (so called "hidden variables" [2,7,8]), and the answer to this question is, may be, the main result of this work. As for physical models, author knows two interpretations of quantum mechanics where mathematical formalism of quantum mechanics is not questioned. One is the Everett's "Many Universes Interpretation", where statistical character of quantum theory is explained by existence of infinite number of Universes, corresponding to various realizations of reality [9], This interpretation has its supporters in spite of exotic character and serious criticism [10]. Another interpretation is the 't Hooft's " The Cellular Automaton Interpretation of Quantum Mechanics", where a very special set of mutually orthogonal sates in Hilbert space is considered [2]. This approach is now under development. Among the works where the apparatus of quantum physics is undergoing serious changes we can mention the string theory (see, e.g. [11]) and Santlli's investigations [12]. The possibility is shown in this work to interpret the quantum mechanical wave function for free particle as a description of microscopic distortion of the space-time geometry. Some characteristics of this geometrical object play the role of "hidden variables" responsible for stochastic behavior of quantum particle, and these characteristics are the physical reality that exists before measurement. Other characteristics explain wave-corpuscular properties of the particle. It may be said that quantum mechanics within suggested interpretation satisfies the completeness criterion formulated by Einstein. Preliminary results see [13-17].

2. Quantum particle as the microscopic distortion of the space-tome geometry

Let's consider the free neutral particle with mass m and spin 0. It will be shown that wave function of such particle can be interpreted as a mathematical description of some geometrical object. This scalar wave function is the solution of the Klein-Fock-Gordon equation, and it has the form [18,19]

$$\Psi = const \cdot exp\left(-\frac{i}{\hbar}(Et - \boldsymbol{pr})\right). \tag{1}$$

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This function describes within existing interpretation the particle's state with definite energy E and definite momentum $\boldsymbol{p}$. The particle's position before measurements is unknown – it may be observed in any point with equal probability. This fact reflects statistical character of quantum mechanics – unusual property within classical representations. Another unusual property – wave-corpuscular dualism of quantum particles that is defined by phase of the wave function and by wavelength and frequency, connecting with the particle's energy and momentum by known relations [18,19]

$$\lambda_i = \frac{\hbar}{p_i},\ \omega = \frac{E}{\hbar},\ i = x, y, z. \quad (2)$$

Substituting (2) in (1), we have

$$\Psi = const \cdot exp(-i\omega t + i\boldsymbol{kr}),\ k_i = 2\pi\lambda_i \quad (3)$$

This type of functions (plane wave) is often used in classical physics (for example, for description of plane running sound wave). Within existing interpretation of quantum mechanics, the origin of periodical dependence of wave function is not discussing.

Let us rewrite the function (2) not with space-time coordinates x, y, z, t, but with only space coordinates x^1, x^2, x^3, x^4 of the space of events of the special theory of relativity – four dimensional pseudo Euclidean space of index 1(the Minkowski space [20]). Time, multiplied by light velocity, plays in this space the role of fourth space coordinate ($ct = x^4$). Let us rewrite (3), using relativistic designations where $\hbar = \mathrm{c} = 1$

$$\Psi = const \cdot exp\left(-ix^\mu p_\mu\right). \quad (4)$$

Here p_μ – the particle's 4-momentum ($p_1 = E, p_{2,3,4} = p_{x,y,z}$). Summation over repeating indexes is suggested in (4) with signature ($+ - - -$). In relativistic case [18.19]

$$p_1^2 - p_2^2 - p_3^2 - p_4^2 = m^2 \quad (5)$$

where m – the particle's mass. Let's write down (4) in such a way that it contains only values with dimensionality of length

$$\Psi = const \cdot exp\left(-2\pi i x^{\mu} \lambda_{\mu}^{-1}\right), \tag{6}$$

where

$$\lambda_1^{-2} - \lambda_2^{-2} - \lambda_3^{-2} - \lambda_4^{-2} = \lambda_m^{-2}, \quad \lambda_\mu = 2\pi p_\mu^{-1}, \quad \lambda_m = 2\pi m^{-1}. \tag{7}$$

In contrast to (1,3) function (6) does not look as a plane wave - it represent periodical function of four space coordinates in the Minkowski space.

Function (6) may be considered as a function realizing representation of the group whose elements are discrete translations along four coordinates axes in the Minkowski space. Indeed, function (6) goes into itself at translations

$$x^{\mu} \to x^{\mu'} + n_{\mu} \lambda_{\mu}, \tag{8}$$

where $n_{\mu} -$ integers $(\mu = 1{,}2.3.4)$. This group is isomorphic to the group $\mathbb{Z}^4$, whose elements are products of integers n_{μ} In turn, the group $\mathbb{Z}^4$ is isomorphic to the fundamental group of closed 4-manifold that is homeomorphic to the foir dimensional torus T^4 [21,22]. Now we nay formulate the main hypostasis: quantum particle, described by the wave function (6), can be considered as a closed space-time manifold that is homeomorphic to the four dimensional torus imbedded into five dimensional pseudo Euclidean space of index 1. Relation (7) imposes a metric restriction on the acceptable under deformations path lengths λ_i $(i = 1{,}2{,}3{,}4)$. Thus, the relation (7) defines also the geometrical interpretation of the particle's mass and 4-momentum. It will be shown in the next Section that such geometrical object looks in three dimensional Euclidian space as moving topological defect of this space having stochastic and wave-corpuscular properties of quantum particle.

Representation of particle as a closed manifold means that this particle before measurement may be considered as a "mixture" of its all possible

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geometrical representations (homeomorphisms), and only interaction with device fixes one of them. This means that wave function describes not an individual particle, but statistical ensemble of all its possible geometrical representations, and this explains statistical character of quantum mechanics. Thus, ensemble of all possible homeomorphisms plays the role of "hidden variables," responsible for stochastic behavior of particles.

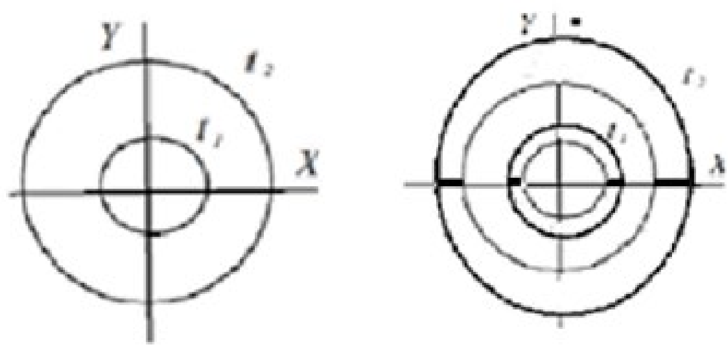
3. Quantum particle as a topological defect of Euclidean space.

Let's proceed to decoding of the representation of quantum particle as a closed 4-manifold, that is let's show how such object looks from the point of view of the observer in Euclidian space. But the important notice should be made before going to this problem. The geometry of four-dimensional closed manifolds is now under development: the full recognition algorithm is not now known even for three dimensional closed manifolds [22]. Therefore the only way to establish what the representation of quantum particle as a closed 4-manifold means from the point of view of the observer in Euclidtan space is to use low dimensional analogies. Having this in mind let's consider closed manifold homeomorphic to the two dimensional torus embedded into three dimensional pseudo Euclidean space of index 1. To obtain concrete result only one of infinite number of possible homeomorphisms of this manifold will be considered, namely usual two dimensional torus $T^2 = S^1 \times S^1$, where S^1 – a circle. Such torus may be considered in three-dimensional Euclidean space as a surface obtained by rotation of a circle around vertical axis lying in the plain of this circle (Fig.1a). In pseudo Euclidean three-dimensional space this circle is located in pseudo Euclidean plane and it looks on Euclidean plane of Fig1b as a isosceles hyperbola [23]. That is two dimensional torus, representing particle, looks in three dimensional Euclidean space as a hyperboloid (Fig.1b). Within considered low dimensional analogy physical space-time (space of events) is a two dimensional pseudo Euclidean space, and the particle's positions in different moments of time in the Euclidean (one dimensional) space are defined by points of intersection with this space of the projections of the hyperboloid's temporary cross-sections. These cross-sections look as expanding circles in two-dimensional Euclidean plane XY (Fig.2a).

a b

Fig 1.. Two-dimensional torus embedded into three-dimensional Euclidean and pseudo Euclidean spaces.

These circles can be considered as moving topological defect of one dimensional physical space. It is the fact that intersection point belongs to topological defect that distinguishes this point at Fig.2a from neighboring points of one dimensional Euclidean space, turning it into a physical “material point”.



a b

Fig.2. Topological defect of one dimensional Euclidean space (X-axis).

The particle’s positions in Euclidean (one dimensional) space are defined by points of its intersection with the circle, corresponding to the only one of the torus possible homeomorphisms. Accounting for all possible homeomorphisms leads, obviously, to “blurring” of this circle and so leads to transformation of the one intersection point in finite region of Euclidean space

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(this region is indicated at Fig.2b by a bold line segment on X-axis). This region has at every moment of time a finite size because the range of all possible homeomorphisms is limited by metric condition (7) that restrict the maximum possible dimensions of closed manifold. As a result, the observer in Euclidean space will detect the particle with equal probability in one of points of above mentioned region. This means that wave function describes not a position of separate particle but the ensemble of its possible positions, and this explains statistical character of quantum mechanics. It is obvious that all possible homeomorphisms of the closed manifold, representing this particle, play the role of "hidden variables", responsible for the particle's stochastic behavior: each homeomorphism corresponds to the one particle's possible position in Euclidean space. The points of the intersection region have different velocities. This means that the intersection region at Fig. 2b are moving expanding, and finally it will fill all Euclidean (one dimensional) space. In result the probability to observe the particle in any point of space will be the same, as it should be according to laws of quantum mechanics for free particle, described by wave function (1).

The fact that the particle can be represented in physical Euclidean space as a part of topological defect allows to explain the particle's wave properties. It is sufficient for this to suppose that the defect's position in the external five dimensional Euclidean space relative to the three dimensional space changes according to periodical low described by wave function (1) (a rigorous proof of this assumption is not possible within the framework of low dimensional analogy). It can be said that the phase of the defect's periodical movement is an additional degree of freedom on which the effect of the particle on the device depends. The particle's corpuscular properties (4-momentum) are defined through parameters of above periodical movement of defect by relations

$$p_\mu = 2\pi\lambda_{\mu.} \qquad (9)$$

These relations are identical to the definition (2) of the particle's wave length through its momentum within existing interpretation [19] ,but now they have the "reverse" meaning of definition of momentum through the wave length, as it should be in the consistent theory where less general concepts (classical momentum) are defined through more fundamental ones (wave length of the defect's periodical movement).

Conclusion. The wave function plays a dual role within suggested interpretation. First, it is a function, realizing the representation of the fundamental group for a closed 4-manifold, representing a free particle. Second, this function describes periodical movement of topological defect in the external space, and intersection of this defect with physical space defines the possible particle's positions. These properties of the wave function make it possible to explain the stochastic behavior of the particle and its wave-corpuscular dualism. The role of "hidden variables", responsible for the particle's stochastic behavior, is played by all possible homeomorphisms of the closed 4-manifold, representing the particle. Notice in conclusion that relation (7) defines geometrical interpretation of the particle's mass as a characteristic of some fundamental length $\lambda_{m.}$ Geometrical interpretation of elementary electrical charge and the particle's spin will be considered in subsequent publications.

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A NEW CONCEPTION OF LIVING ORGANISMS AND ITS REPRESENTATION VIA LIE-ADMISSIBLE H_v-HYPERSTRUCTURES

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Abstract

Recent studies have confirmed Einstein's 1935 legacy implying that quantum mechanics and chemistry are "incomplete" theories in the sense of being excellent for the description of systems composed by point-like constituents under potential interactions (such as the atomic structure), but said theories are "incomplete" for the description of complex time-irreversible systems of extended constituents with internal non-potential interactions (as expected in a cell). Sadi verifications were achieved thanks to the prior "completion" over the past half a century of quantum theories into the covering hadronic mechanics and chemistry with a time irreversible Lie-admissible structure. In this paper we present, apparently for the first time, a new conception of living organisms, solely permitted by the verifications of Einstein's legacy, composed by a very large number of extended wavepackets in conditions of continuous mutual penetration/entanglement and, therefore, of continuous communications via contact non-potential interactions. Due to the extremely large number of constituents and the extreme complexity of the multi-valued internal communications, in this paper we introduce, also apparently for the first time, the representation of the indicated new conception of living organisms via two hyperbimodular, Lie-admissible H_v-hyperstructures, the first with all hyperoperations ('hope') ordered to the right and the second with all hopes ordered to the left. The irreversibility of living organisms is represented by the inequivalence of the left and right hopes. The extremely large number of internal communications is represented by the extremely large number of solutions of the indicated hopes. We close the paper with the indication that new medical diagnostics and treatments are expected in the transition from the current quantum chemical conception of living organisms as collections of isolated point-like constituents to the indicated new conception.

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1. INTRODUCTION ON HYPERSTRUCTURES

The largest class of hyperstructures are called *H_v-structures* and introduced in 1990 [33], [34]. These satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions are the following:
In a set H equipped with a hyperoperation (abbreviation *hyperoperation*=***hope***)

$$\cdot: H\times H \rightarrow P(H)-\{\varnothing\},$$

we abbreviate by *WASS* the *weak associativity*: $(xy)z\cap x(yz) \neq \varnothing,\ \forall x,y,z\in H$ and by *COW* the *weak commutativity*: $xy\cap yx \neq \varnothing,\ \forall x,y\in H$.

The hyperstructure (H,·) is called ***H_v-semigroup*** if it is *WASS,* it is called ***H_v-group*** if it is reproductive H_v-semigroup, i.e. $xH = Hx = H,\ \forall x\in H$.
Motivation. I the classical theory the quotient of a group with respect to an invariant subgroup is a group. F. Marty from 1934, states that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition is an H_v-group [34].

The *powers* of an element $h\in H$ are: $h^1=\{h\},\ h^2=h\cdot h,\ \ldots,\ h^n=h\circ\ldots\circ h$, where (°) is the *n-ary circle hope*: the union of hyperproducts, n times, with all patterns of parentheses put on them. An H_v-semigroup (H,·) is a *cyclic of period s*, if there is a *generator* g, and a natural n, such that $H=h^1\cup\ldots\cup h^s$. If there is an h and s, such that $H=h^s$, then (H,·) is called *single-power cyclic of period s*.

In a similar way more complicated hyperstructures can be defined:
$(R,+,\cdot)$ is ***H_v-ring*** if (+) and (·) are *WASS*, the reproduction axiom is valid for (+) and (·) is *weak distributive* with respect to (+):

$$x(y+z)\cap(xy+xz)\neq\varnothing,\quad (x+y)z\cap(xz+yz)\neq\varnothing,\quad \forall x,y,z\in R.$$

Let $(R,+,\cdot)$ H_v-ring, $(M,+)$ *COW* H_v-group and there exists an external hope

$$\cdot:\ R\times M \rightarrow P(M):\ (a,x) \rightarrow ax$$

such that, $\forall a,b\in R$ and $\forall x,y\in M$, we have

$$a(x+y)\cap(ax+ay) \neq \varnothing,\quad (a+b)x\cap(ax+bx) \neq \varnothing,\quad (ab)x\cap a(bx) \neq \varnothing,$$

then *M* is an *H_v-module* over *F*. In the case of an H_v-field *F* instead of an H_v-ring *R*, then the *H_v-vector space* is defined.

For more definitions and applications on H_v-structures one can see the books [2], [4], [5], [8], [31], [32], [34], [35], [38], [40], [45], [52].

Definition 1.1 The ***fundamental relations*** β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [33], [34], [35], [50], [51].

The way to find the fundamental classes is given by the following:

Theorems 1.2 Let $(H,\cdot)$ be H_v-group and denote by U the set of finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.

Let $(R,+,\cdot)$ be H_v-ring. Denote by U the set of finite polynomials of elements of R. We define the relation γ in R as follows: $x\gamma y$ *iff* $\{x,y\}\subset u$ where $u\in U$. Then the relation γ^* is the transitive closure of the relation γ.

An element is called *single* if its fundamental class is singleton.

Fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field.

Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called ***smaller*** than $(*)$, and $(*)$ ***greater*** than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \quad \text{such that} \quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we say that *$(H,*)$ contains $(H,\cdot)$*. If $(H,\cdot)$ is a structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

Theorem 1.3 (The Little Theorem). Greater hopes than the ones which are *WASS* or *COW*, are also *WASS* or *COW*, respectively.

This Theorem leads to a partial order on H_v-structures.

A very interesting class of H_v-structures, is the following [9], [32]:

An H_v-structure is called ***very thin*** iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one.

A large class of H_v-structures is the following [9], [41]:

Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\to G$ be a map. We define a hope (∂), called *theta-hope,* we write ***∂-hope***, on G as follows

$$x\partial y = \{f(x)\cdot y,\ x\cdot f(y)\},\quad \forall x,y\in G.\quad (resp.\ \ x\partial y= (f(x)\cdot y)\cup(x\cdot f(y)),\ \forall x,y\in G)$$

If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

Let $(G,\cdot)$ be groupoid (or hypergroupoid) and $f:G\to P(G)-\{\varnothing\}$ be multivalued map. We define the (∂), on G as follows $x\partial y = (f(x)\cdot y)\cup(x\cdot f(y)),\ \ \forall x,y\in G.$

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Motivation for the ∂-hope is the map *derivative* where only the multiplication of functions can be used. Basic property: if $(G,\cdot)$ is a semigroup then $\forall f$, the (∂) is *WASS*.

Another well known and large class of hopes is given as follows [9], [31], [47]:

Let $(G,\cdot)$ be groupoid, then $\forall P\subset G,\ P\neq\emptyset$, we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}:\ x\underline{P}y=(xP)y\cup x(Py),\quad \underline{P}_r:\ x\underline{P}_ry=(xy)P\cup x(yP),\quad \underline{P}_l:\ x\underline{P}_ly=(Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. If $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or *WASS*.

A generalization of P-hopes is the following [6], [9]:

Construction 1.4 Let $(G,\cdot)$ be an abelian group and P any subset of G. We define the hope $\times_P$ as follows:

$$\begin{cases} x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \,/\, h\in P\} & \text{if } x\neq e \text{ and } c\neq e \\ x\cdot y & \text{if } x=e \text{ or } y=e \end{cases}$$

we call this hope P_e*-hope*. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

H_v-structures are used in Representation Theory of H_v-groups which can be achieved by generalized permutations or by H_v-matrices [34], [38], [49]. *H_v-matrix* is called a matrix if has entries from an H_v-ring. The hyperproduct of H_v-matrices is defined in a usual manner. The problem of the H_v-matrix representations is the following:

Definition 1.5 Let $(H,\cdot)$ be an H_v-group, find an H_v-ring $(R,+,\cdot)$, a set $M_R=\{(a_{ij})\,/\,a_{ij}\in R\}$ and a map

$$T:\ H\to M_R:\ h\mapsto T(h) \text{ such that } T(h_1h_2)\cap T(h_1)T(h_2)\neq\emptyset,\ \forall h_1,h_2\in H.$$

Then T is *H_v-matrix representation*. If $T(h_1h_2)\subset T(h_1)(h_2)$, $\forall h_1,h_2\in H$ is valid, then T is an *inclusion representation*. If $T(h_1h_2) = T(h_1)(h_2) = \{T(h)\,/\,h\in h_1h_2\}$, $\forall h_1,h_2\in H$, then T is a *good representation*.

Hopes on any type of ordinary matrices can be defined [8], [49], [53] they are called *helix hopes*.

Definition 1.6 Let $A=(a_{ij})\in M_{m\times n}$ be matrix and $s,t\in N$, with $1\leq s\leq m$, $1\leq t\leq n$. The helix-projection is a map $\underline{st}:M_{m\times n}\to M_{s\times t}:A\to A\underline{st}=(\underline{a}_{ij})$, where $A\underline{st}$ has entries

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$$\underline{a}_{ij} = \{ a_{i+\kappa s, j+\lambda t} \,/\, 1\leq i\leq s,\ 1\leq j\leq t \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n \}$$

Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ be matrices and $s=min(m,u)$, $t=min(n,v)$. We define a hyper-addition, called ***helix-sum***, by

$$\oplus : M_{m\times n}\times M_{u\times v}\to P(M_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset M_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})= \{(c_{ij})=(a_{ij}+b_{ij}) \,/\, a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

Let $A=(a_{ij})\in M_{m\times n}$, $B=(b_{ij})\in M_{u\times v}$ and $s=min(n,u)$. Define the ***helix-product***, by

$$\otimes: M_{m\times n}\times M_{u\times v}\to P(M_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset M_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \,/\, a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-sum is commutative, *WASS*, not associative. The helix-product is *WASS*, not associative and not distributive to the helix-addition.

Using several classes of H_v-structures one can face several representations [48].

Definition 1.7 Let $M=M_{m\times n}$ be module of $m\times n$ matrices over a ring R and $P=\{P_i:i\in I\}\subseteq M$. We define, a kind of, a P-hope $\underline{P}$ on M as follows

$$\underline{P} : M\times M \to P(M): (A,B) \to A\underline{P}B = \{ AP^t_iB: i\in I \}\subseteq M$$

where P^t denotes the transpose of the matrix P.

We present a proof for the fundamental relation analogous to Theorem 1.2 in the case of an H_v-module:

Theorem 1.8 Let $(M,+)$ be H_v-module over R. Denote U the set of expressions of finite hopes either on R and M or the external hope applied on finite sets of elements of R and M. We define the relation ε in M by: $x\boldsymbol{\varepsilon}y$ *iff* $\{x,y\}\subset u,\ u\in U$. Then the relation ε^* is the transitive closure of the relation ε.

Proof. Let $\underline{\varepsilon}$ be the transitive closure of ε, and denote by $\underline{\varepsilon}(x)$ the class of the element x. First, we prove that the quotient set $M/\underline{\varepsilon}$ is a module over R/γ^*.

In $M/\underline{\varepsilon}$ the sum ($\oplus$) and the external product ($\otimes$), using the γ^* classes in R, are defined in the usual manner:

$$\underline{\varepsilon}(x)\oplus\underline{\varepsilon}(y) = \{\underline{\varepsilon}(z):\ z\in \underline{\varepsilon}(x)+\underline{\varepsilon}(y)\},$$

$$\gamma^*(a)\otimes\underline{\varepsilon}(x) = \{\underline{\varepsilon}(z):\ z\in \gamma^*(a)\cdot\underline{\varepsilon}(x)\},\quad \forall a\in R,\ x,y\in M.$$

Take $x'\in\underline{\varepsilon}(x)$, $y'\in\underline{\varepsilon}(y)$. Then $x'\underline{\varepsilon}x$ *iff* $\exists x_1,\dots,x_{m+1}$ with $x_1=x'$, $x_{m+1}=x$, $u_1,\dots,u_m\in U$ such that $\{x_i, x_{i+1}\}\subset u_i$, $i=1,\dots,m$, and $y'\underline{\varepsilon}y$ *iff* $\exists y_1,\dots,y_{n+1}$ with $y_1=y'$, $y_{n+1}=y$ and $v_1,\dots,v_n\in U$ such that $\{y_j,y_{j+1}\}\subset v_j$, $j=1,\dots,n$. From the above we obtain

$$\{x_i, x_{i+1}\}+ y_1 \subset u_1+v_1,\ \ i=1,\dots,m-1,\quad x_{m+1} +\{y_j, y_{j+1}\} \subset u_m+v_j,\ \ j=1,\dots,n.$$

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The $u_i+v_1=t_i$, $i=1,..,m-1$, $u_m+v_j=t_{m+j-1}$, $j=1,..,n\in U$, so $t_k\in U$, $\forall k\in\{1,..,m+n-1\}$. Take $z_1,\dots,z_{m+n}$ with $z_i\in x_i+y_1$, $i=1,\dots,n$ and $z_{m+j}\in x_{m+1}+y_{j+1}$, $j=1,\dots,n$, thus, $\{z_k,z_{k+1}\}\subset t_k$, $k=1,\dots,m+n-1$. Therefore, $\forall z_1\in x_1+y_1=x'+y'$ is $\underline{\varepsilon}$ equivalent to $z_{m+n}\in x_{m+1}+y_{n+1}=x+y$. Thus, $\underline{\varepsilon}(x)\oplus\underline{\varepsilon}(y)$ is a singleton so we can write $\underline{\varepsilon}(x)\oplus\underline{\varepsilon}(y)=\underline{\varepsilon}(z)$, $\forall z\in\underline{\varepsilon}(x)+\underline{\varepsilon}(y)$. Similarly, using the properties of γ^* in R, we prove that $\gamma^*(a)\otimes\underline{\varepsilon}(x)=\underline{\varepsilon}(z)$, $\forall z\in\gamma^*(a)\cdot\underline{\varepsilon}(x)$.

The *WASS* and the weak distributivity on R and M guarantee that the associativities and the distributivity are valid for $M/\underline{\varepsilon}$ over R/γ^*. Therefore, $M/\underline{\varepsilon}$ is a module over R/γ^*.

Now let σ equivalence relation in M such that M/σ is module on R/γ^*. Denote $\sigma(x)$ the class of x. Then $\sigma(x)\oplus\sigma(y)$ and $\gamma^*(a)\otimes\sigma(x)$ are singletons $\forall a\in R$ and $x,y\in M$, i.e.

$$\sigma(x)\oplus\sigma(y)=\sigma(z),\quad \forall z\in\sigma(x)+\sigma(y),\quad \gamma^*(a)\otimes\sigma(x)=\sigma(z),\quad \forall z\in \gamma^*(a)\cdot\sigma(x).$$

Thus we write, $\forall a\in R$, $x,y\in M$ and $A\subset\gamma^*(a)$, $X\subset\sigma(x)$, $Y\subset\sigma(x)$

$$\sigma(x)\oplus\sigma(y)=\sigma(x+y)=\sigma(X+Y),\quad \gamma^*(a)\otimes\sigma(x)=\sigma(ax)=\sigma(A\cdot X).$$

By induction, extend these relations on finite sums and external products. Thus, $\forall u\in U$, we have $\sigma(x)=\sigma(u)$, $\forall x\in u$. Consequently $x'\in\varepsilon(x)$ implies $x'\in\sigma(x)$, $\forall x\in M$.

But σ is transitively closed, so we obtain: $x'\in\underline{\varepsilon}(x)$ implies $x'\in\sigma(x)$.

Thus, $\underline{\varepsilon}$ is the smallest equivalence on M such that $M/\underline{\varepsilon}$ is a module on R/γ^*, i.e. $\underline{\varepsilon}=\varepsilon^*$. ∎

The general definition of an H_v-Lie algebra was given as follows [30], [44]:

Definition 1.9 Let $(L,+)$ H_v-vector space on $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$ canonical, $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is zero of F/γ^*. Let ω_L the core of $\varphi':L\to L/\varepsilon^*$ and denote 0 the zero of L/ε^*. Consider the *bracket (commutator) hope*:

$$[\ ,\ ]: L\times L\to P(L): (x,y)\to [x,y]$$

then L is an *H_v-Lie algebra* over F if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y])\neq\varnothing$$

$$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2])\neq\varnothing,\quad \forall x,x_1,x_2,y,y_1,y_2\in L,\ \forall\lambda_1,\lambda_2\in F$$

(L2) $[x,x]\cap\omega_L\neq\varnothing$, $\forall x\in L$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L\neq\varnothing$, $\forall x,y\in L$

The enlargement or reduction of hyperstructures are examined in the sense that an extra element appears in one result or we take out an element. In both directions most useful in representation theory, are those H_v-structures with the same fundamental structure [36], [37]:

Let $(H,\cdot)$ be H_V-semigroup and $v\notin H$. Extend $(\cdot)$ into the $\underline{H}=H\cup\{v\}$ as follows: $x\cdot v=v\cdot x=v,\ \forall x\in H$, and $v\cdot v=H$. The $(\underline{H},\cdot)$ is an h/v-group where $(\underline{H},\cdot)/\beta^*\cong Z_2$ and v is a single element. We call $(\underline{H},\cdot)$ the *attach h/v-group* of $(H,\cdot)$.

Let $(G,\cdot)$ be semigroup and $v\notin G$ be an element appearing in a product ab, where $a,b\in G$, thus the result becomes a hyperproduct $a\otimes b=\{ab,v\}$. Then the minimal hope $(\otimes)$ extended in $G'=G\cup\{v\}$ such that $(\otimes)$ contains $(\cdot)$ in the restriction on G, and such that $(G',\otimes)$ is a minimal H_v-semigroup which has fundamental structure isomorphic to $(G,\cdot)$, is defined as follows:

$$a\otimes b=\{ab,v\},\quad x\otimes y=xy,\quad \forall(x,y)\in G^2-\{(a,b)\}$$

$$v\otimes v=abab,\quad x\otimes v=xab \quad \text{and} \quad v\otimes x=abx,\quad \forall x\in G.$$

$(G',\otimes)$ is very thin H_v-semigroup. If $(G,\cdot)$ is commutative then the $(G',\otimes)$ is strongly commutative.

Let $(H,\cdot)$ be hypergroupoid. We say that *remove* $h\in H$, if we consider the restriction of $(\cdot)$ on $H-\{h\}$. We say $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$. We say $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$ and consider h and $\underline{h}$ as one class.

The ***uniting elements*** method, introduced by Corsini & Vougiouklis [3], is the following: Let G be algebraic structure and let d be a property, which is not valid and it is described by a set of equations; then, consider the partition in G for which it is put together, in the same class, every pair that causes the non-validity of d. The quotient G/d is an H_v-structure. Then, quotient out the G/d by β^*, a stricter structure $(G/d)/\beta^*$ for which the property d is valid, is obtained.

An application of the uniting elements is when more than one property is desired. The following Theorem is valid [3], [34].

Theorem 1.10 Let $(G,\cdot)$ be a groupoid, $F=\{f_1,\dots,f_m,f_{m+1},\dots, f_{m+n}\}$ be system of equations on G consisting of two subsystems $F_m=\{f_1,\dots,f_m\}$, $F_n=\{f_{m+1},\dots, f_{m+n}\}$. Let σ, σ_m the equivalence relations defined by the uniting elements procedure using the systems F and F_m, and let σ_n be the equivalence relation defined using the induced equations of F_n on the grupoid $G_m=(G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^*\cong(G_m/\sigma_n)/\beta^*.$$

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In the paper [42], there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics can be appeared by Santilli's pioneer research.

Hyperstructures have applications in mathematics and in other sciences. These applications range from biomathematics -conchology, inheritance- and hadronic physics or on leptons, in the Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, can be widely applicable in linguistic, in sociology, in industry and production, too. For these applications the largest class of the hyperstructures, the class *H_v-structures*, is used, they satisfy the *weak axioms* where the non-empty intersection replaces the equality. The main tools of this theory are the fundamental relations which connect, by quotients, the H_v-structures with the corresponding classical ones. These relations are used to define hyperstructures as H_v-fields, H_v-vector spaces and so on, as well. The definition of the general hyperfield was not possible without the H_v-structures and their fundamental relations. *Hypernumbers or H_v-numbers* are called the elements of H_v-fields and they are important for the representation theory [6], [7], [29], [30], [39], [46].

The problem of enumeration and classification of hyperstructures, was started from the beginning, it is complicate in H_v-structures because we have very great numbers. The number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian; the 1.013.598 are cyclic. The partial order in H_v-structures and the Little Theorem, transfers and restrict the problem in finding the minimal, *up to isomorphisms*, H_v-structures.

2. LIE-SANTILLI ADMISSIBILITY IN HYPERSTRUCTURES

The *isofields* needed in the theory of *isotopies* correspond into the hyperstructures were introduced by Santilli & Vougiouklis in 1999 [6], [7], [29] and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology. We present in the following the main definitions and results restricted in the H_v-structures.

Definitions 2.1 A hyperstructure $(H,\cdot)$ which contain a unique scalar unit e, is called e-hyperstructure. In an e-hyperstructure, we assume that for every element x, there exists an inverse x^{-1}, *i.e.* $e \in x\cdot x^{-1} \cap x^{-1}\cdot x$. Remark that the inverses are not necessarily unique.

A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ is a hope, is called *e-hyperfield* if the following axioms are valid:

1. $(F,+)$ is an abelian group with the additive unit 0,

2. $(\cdot)$ is *WASS*,
3. $(\cdot)$ is weak distributive with respect to $(+)$,
4. 0 is absorbing element: $0\cdot x = x\cdot 0 = 0,\ \forall x\in F$,
5. exist a multiplicative scalar unit 1, i.e. $1\cdot x = x\cdot 1 = x,\ \forall x\in F$,
6. for every $x\in F$ there exists a unique inverse x^{-1}, such that $1\in x\cdot x^{-1}\cap x^{-1}\cdot x$.

The elements of an e-hyperfield are called *e-hypernumbers*. If the the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 2.2 [6], [7], [43]. *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$x\otimes y = \{xy, g_1, g_2, \ldots\},\ \forall x,y\in G\text{-}\{e\},\ \ g_1, g_2, \ldots \in G\text{-}\{e\}$$

$g_1, g_2, \ldots$ are not the same for each pair (x,y). Then $(G,\otimes)$ becomes an H_v-group, because it contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy=e$, so we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

Another important new field in hypermathematics comes straightforward from Santilli's Admissibility. We can transfer Santilli's theory in admissibility for representations in two ways: using either, the ordinary matrices and a hope on them, or using hypermatrices and ordinary operations on them [13], [15], [42], [43], [44], [47], [48].

The general definition is the following:

Definition 2.3 Let L be H_v-vector space over the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, the canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the fundamental field F/γ^*. Let ω_L be the core of the canonical map $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Take two subsets R, $S\subseteq L$ then a ***Lie-Santilli admissible hyperalgebra*** is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}= xRy-ySx= \{xry-ysx\ /\ r\in R,\ s\in S\}$$

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) When only S is considered, then $[x,y]_S= xy-ySx= \{xy-ysx\ /\ s\in S\}$

(b) When only R is considered, then $[x,y]_R= xRy-yx= \{xry-yx\ /\ r\in R\}$

(c) When $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS}= xRy-ySx= \{xr_1y-ys_1x,\ xr_1y-ys_2x,\ xr_2y-ys_1x,\ xr_2y-ys_2x\}.$$

(d) When $S=\{s_1,s_2\}$ then $[x,y]_S= xy-ySx= \{xy-ys_1x,\ xy-ys_2x\}$.

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(e) When $R=\{r_1,r_2\}$ then $[x,y]_R= xRy–yx= \{xr_1y–yx, xr_2y–yx\}$.

(f) We have one case which is 'like' P-hope for any subset $S\subseteq L$:

$$[x,y]_S = \{xsy–ysx \,/\, s\in S\}.$$

On non square matrices we can define admissibility, as well:

Construction 2.4 Let $(L=M_{m\times n},+)$ be H_v-vector space of m×n hyper-matrices on the H_v-field $(F,+,\cdot)$, $\varphi:F\to F/\gamma^*$, canonical map and $\omega_F=\{x\in F:\varphi(x)=0\}$, where 0 is the zero of the field F/γ^*. Similarly, let ω_L be the core of $\varphi':L\to L/\varepsilon^*$ and denote by the same symbol 0 the zero of L/ε^*. Take any two subsets $R,S\subseteq L$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{RS}: L\times L\to P(L): [x,y]_{RS}=xR^ty–yS^tx.$$

Notice that $[x,y]_{RS}=xR^ty–yS^tx=\{xr^ty–ys^tx \,/\, r\in R \text{ and } s\in S\}$.

Special cases, but not degenerate, are the 'small' and 'strict' ones:

(a) $R=\{e\}$ then $[x,y]_{RS} = xy–yS^tx = \{xy–ys^tx \,/\, s\in S\}$

(b) $S=\{e\}$ then $[x,y]_{RS} = xR^ty–yx = \{xr^ty–yx \,/\, r\in R\}$

(c) $R=\{r_1,r_2\}$ and $S=\{s_1,s_2\}$ then

$$[x,y]_{RS}=xR^ty–yS^tx=\{xr_1^ty–ys_1^tx,xr_1^ty–ys_2^tx,xr_2^ty–ys_1^tx,xr_2^ty–ys_2^tx\}$$

According to Santilli's iso-theory [9], [11], [13], [15], [22], [24], [25], [26], [27], [28], [39], [42], [46], [50], on a field $F=(F,+,\cdot)$, *a general isofield* $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\,\hat{\times}\,\hat{b}: = \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1} = \hat{T}^{-1}, \quad \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of F,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1} = \hat{a}, \ \forall\hat{a}\in\hat{F} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 2.5 *General enlargement*. On a field $F=(F,+,\cdot)$ and on the isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ we replace in the results of the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b}=\ \hat{a}\times\hat{T}\times\hat{b},\quad \text{with } \hat{1}=\hat{T}^{-1}$$

of the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}$, containing $\hat{T}$, for all $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\hat{F}-\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Thus, the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b}=\hat{a}\times\hat{H}_{ab}\times\hat{b}=\hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \ \forall\hat{a},\hat{b}\in\hat{F} \qquad \text{(iii)}$$

$\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ becomes *isoH_v-field.* The elements of $\hat{F}$ are called *isoH_v-numbers* or *isonumbers*.

Remarks 2.6 More important hopes, of the above construction, are the ones where only for few ordered pairs $(\hat{a},\hat{b})$ the result is enlarged, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b}=\ \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\quad \forall\hat{a},\hat{b}\in\hat{F}$$

and the rest are ordinary results, then we have a hyperstructure called *very thin isoH_v-field*.

The assumption that $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall\hat{a}\in\hat{F}$, has one inverse.

Construction 2.7 *The P-hope*. Consider an isofield $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-multiplication

$$\hat{a}\,\hat{\times}\,\hat{b}:=\ \hat{a}\times\hat{T}\times\hat{b},\ \ \text{with } \hat{1}=\hat{T}^{-1},\ \ \forall\hat{a},\hat{b}\in\hat{F}.$$

Take a set $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\hat{F}-\{\hat{0},\hat{1}\}$, define the ***isoP-$H_v$-field***, $\hat{F}=\hat{F}(\hat{a},\hat{+},\hat{\times}_P)$, where the hope $\hat{\times}_P$ as follows:

$$\hat{a}\,\hat{\times}_P\,\hat{b}:=\begin{cases}\hat{a}\times\hat{P}\hat{}\times\hat{b}=\{\hat{a}\times\hat{h}\hat{}\times\hat{b}\ /\ \hat{h}\hat{}\in\hat{P}\hat{}\} & \text{if } \hat{a}\neq\hat{1} \text{ and } \hat{b}\neq\hat{1}\\ \hat{a}\times\hat{T}\hat{}\times\hat{b} & \text{if } \hat{a}=\hat{1} \text{ or } \hat{b}=\hat{1}\end{cases} \qquad \text{(iv)}$$

The elements of $\hat{F}$ are called *isoP-H_v-numbers*.

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Remark. If $\widehat{P} = \{\hat{T},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{T}$. The inverses in isoP-H_v-fields, are not necessarily unique.

Example 2.8 In order to define a generalized P-hope on $\hat{Z}_7 = \hat{Z}_7(\underline{\hat{a}},\hat{+},\hat{\times})$, where we take $\hat{P}=\{\underline{\hat{1}},\underline{\hat{5}}\}$, the weak associative multiplicative hope is described by the table:

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4},\hat{6}}$	$\underline{\hat{6},\hat{2}}$	$\underline{\hat{1},\hat{5}}$	$\underline{\hat{3},\hat{1}}$	$\underline{\hat{5},\hat{3}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{6},\hat{2}}$	$\underline{\hat{2},\hat{3}}$	$\underline{\hat{5},\hat{4}}$	$\underline{\hat{1},\hat{5}}$	$\underline{\hat{4},\hat{6}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{1},\hat{5}}$	$\underline{\hat{5},\hat{4}}$	$\underline{\hat{2},\hat{3}}$	$\underline{\hat{6},\hat{2}}$	$\underline{\hat{3},\hat{1}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{3},\hat{1}}$	$\underline{\hat{1},\hat{5}}$	$\underline{\hat{6},\hat{2}}$	$\underline{\hat{4},\hat{6}}$	$\underline{\hat{2},\hat{3}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{5},\hat{3}}$	$\underline{\hat{4},\hat{6}}$	$\underline{\hat{3},\hat{1}}$	$\underline{\hat{2},\hat{3}}$	$\underline{\hat{1},\hat{5}}$

The hyperstructure $\hat{Z}_7= Z_7(\underline{\hat{a}},\hat{+},\hat{\times})$, is commutative and associative on the multiplication hope.

Consruction 2.9 The generalized P-construction can be applied on rings to obtain H_v-fields. Thus for, $\hat{Z}_{10} = Z_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, and if we take $\hat{P}=\{\underline{\hat{2}},\underline{\hat{7}}\}$, then we have the table

$\hat{\times}$	$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{\mathbf{9}}}$
$\underline{\hat{\mathbf{0}}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{\mathbf{1}}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{\mathbf{2}}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$
$\underline{\hat{\mathbf{3}}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{4},\hat{9}}$
$\underline{\hat{\mathbf{4}}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{\mathbf{5}}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$
$\underline{\hat{\mathbf{6}}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$
$\underline{\hat{\mathbf{7}}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{4}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{1},\hat{6}}$
$\underline{\hat{\mathbf{8}}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\hat{\mathbf{9}}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{4},\hat{9}}$	$\underline{\hat{2}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{8}}$	$\underline{\hat{1},\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2},\hat{7}}$

Then the fundamental classes are

$(0)=\{\hat{\underline{0}},\hat{\underline{5}}\}$, $(1)=\{\hat{\underline{1}},\hat{\underline{6}}\}$, $(2)=\{\hat{\underline{2}},\hat{\underline{7}}\}$, $(3)=\{\hat{\underline{3}},\hat{\underline{8}}\}$, $(4)=\{\hat{\underline{4}},\hat{\underline{9}}\}$,

and the multiplicative table is the following

×	**(0)**	**(1)**	**(2)**	**(3)**	**(4)**
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{Z}_{10} = Z_{10}(\hat{\underline{a}}, \hat{+}, \hat{\times})$, is an H_v-field.

3. APPLICATIONS TO A NEW CONCEPTION OF LIVING ORGANISMS

3.1 Einstein's argument that 'quantum mechanics is not a complete theory.'

As it is well know, Einstein accepted the validity of quantum mechanics for the representation of the atomic structure and other systems, but never accepted quantum mechanics as being a final theory capable of representing all possible elements of reality.

For this reason, Einstein expressed the view in 1935, jointly with his students Boris Podolsky and Nathan Rosen, that *'Quantum mechanics is not a complete theory'* (EPR argument) [10], in the sense that quantum mechanics (and we add nowadays quantum chemistry) could admit suitable enlargements for the representations of more complex systems.

Additionally, Einstein never accepted the uncertainties of quantum mechanics as being final in the sense that they are indeed valid for point-particles in vacuum but there could exist conditions in the universe recovering classical determinism. For this reason, Einstein's made his famous quote: *'God does not play dice with the universe.'*

3.2 Verification of Einstein's legacy by irreversible systems.

The most evident illustration if the validity of the lack of 'completeness' of quantum mechanics (and, therefore, of quantum chemistry) is given by the fact that *quantum mechanics and chemistry can only represent systems of point-like particles that are invariant under time-reversal*, such as the atomic structure. This is due to the invariance under anti-Hermiticity of the quantum mechanical Lie product between Hermitean operators

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$$[A,B] = AB - BA = -[A,B]^{\dagger},$$

where *AB* is the conventional classical associative product. In fact, the Lie product characterizes Heisenberg's time evolution of an observable A in terms of the Hamiltonian H,

$$idA/dt = [A,H] = AH - HA.$$

However, physical, chemical and biological processes such as nuclear fusion, combustion and living organisms, are irreversible over time.

The verification of Einstein's legacy via irreversible processes was first identified by R. M. Santilli during his Ph. D. Studies at the University of Torino, Italy, in the mid 1960s.

In fact, Santilli's Ph.D. thesis, published in the 1967 paper [13], provided the first known confirmation of the EPR argument (see also Ref. [13] of 1968) via the following Lie-admissible 'completion' of quantum mechanical Lie algebras for the representation of irreversible processes

$$(A,B) = ARB - BSA = (ATB - BTA) + (AJB + BJA), \quad R = T - J, \; S = T + J \neq 0,$$

where the new product *(A,B)* is Lie-admissible according to A.A. Albert [1] when the attached antisymmetric product

$$[A,B]^{*} = (A,B) - (B,A) = ATB - BTA$$

verifies the Lie axioms whenever *T* is nowhere singular. Also, according to Albert [1] the product (A,B) is called *Jordan-admissible* when the attached symmetric product

$$\{A,B\}^{*} = (A,B) + (B,A) = AJB + BJA$$

verifies the axioms of Jordan algebras.

Santilli called *hadronic mechanics* [19] and *hadronic chemistry* [22] the 'completion' of quantum mechanics and chemistry, respectively, with a Lie-admissible structure for the representation of irreversible structures and processes.

3.3 Lie-admissible genomathematics

The mathematics underlying Lie-admissible formulations, collectively known as *genomathematics* [16], [17], [20], [23], can be summarized as follows. A generally tacit assumption of conventional, classical, numeric fields underlying Lie's theory is that *the multiplication of two numbers to the right n−−>3 is equal to the multiplication of the same numbers to the left, 2<−−3 = 2−−>3.* Consequently, the indicated order of the multiplication is ignored in classical number theory, and we merely write 2×3 = 6.

In the transition from Lie theory to the covering Lie-admissible theory, the above ordering of the multiplication is no longer ignorable because the multiplication to the right 2>3 = 2S3 is no longer equal to the multiplication to the left 2<3 = 2R3 ≠ 2>3. This occurrence has permitted the identification of two, classical, numeric fields underlying Lie-admissible formulations [16]:

1) The *forward genofields* $F^>(n^>,>,I^>)$ with *forward genounit* $I^>=1/S$, *forward genonumbers* $n^>= nI^>$, and *forward genoproduct* $n^>>m^> = n^>Sm^>$, where *n, m* represent ordinary numbers; and

2) The *backward genofields* $F^<(n^<,<,I^<)$ with *backward genounit* $I^<=1/R$, *backward genonumbers* $n^< = I^<n$, and *backward genoproduct* $n^<< m^< = n^<Rm^<$.

Recall that Lie algebras can be constructed via the universal enveloping associative algebras ξ with classical, associative, modular product *AB*. The indicated inequivalence of the multiplications to the right and to the left implies the existence for Lie-admissible theories of *two universal, enveloping, genoassociative genoalgebars,* that to the right $\xi^>$ (left $\xi^<$) with genoassociative genoproduct to the right $A>B$ (left $A<B$), resulting in a non-trivial bimodular formulation.

The indicated bimodular formulations characterize the *time-irreversible, Lie-admissible, Heisenberg-Santilli genoequation* [13], [19]

$$idA/dt = (A, H) = ARH - HSA = A < H - H > A.$$

Recall that, in quantum mechanics, the modular associative multiplication to the right of an operator *H* to a Hilbert stat, $H_\psi(t,r) = E_\psi(t,r)$ yields the same eigenvalues *E* for the modular associative multiplication to the left $\psi(t,r)H = \psi(t,r)E$.

The Lie-admissible ‘completion’ of the above Schrödinger’s equation yields the non-trivial bimodular structure:

1) The modular genoassociative action to the right representing the time evolution forward in time via the *Schrödinger-Santilli genoequation to the right* [19]

$$H(r,p) > \psi^>(t^>,r^>) = H(r,p)S(\psi^>,\ldots)\psi(t,r) = E^>\psi^>(t^>,r^>)^>,$$

and

2) The modular genoassociative action to the left representing motion backward in time via *Schrödinger-Santilli genoequation to the left*

$$\psi^<(t^<,r^<) < H(r,p) = \psi^<R(\psi^<,\ldots)H(r,p) = \psi^<(t^<,r^<)E^<$$

where $E^> \neq E^<$.

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The representation of irreversible processes from first axiomatic principles is then evident whenever $R \neq S$.

3.4 Verifications of Einstein's legacy.

Following the above mathematical studies, Santilli dedicated decades to experimental and industrial verifications of hadronic mechanics and chemistry. Following the achieved of such a mathematical and applied maturity, Santilli proved Einstein's legacy that 'quantum mechanics is not a complete theory' as well as the progressive recovering of Einstein's determinism in the interior of hadrons, nuclei and stars and its full recovering in the interior of gravitational collapse [21], [25], [26], [27], [28]. These results were achieved via the representation of the *extended and overlapping character of the constituents of irreversible systems in terms of the forward genotopic element* with realizations of the type

$$\hat{T} = \Pi_{k=1,\ldots,N} Diag.(1/n_{1k}^2,\ 1/n_{2k}^2,\ 1/n_{3k}^2,\ 1/n_{4k}^2\ )e^{-\Gamma(\psi,\partial\psi,\ldots)}\ ,$$

where n_{1k}^2, n_{2k}^2, n_{3k}^2, (called *characteristic quantities*) represent the deformable semi-axes of the k-particle normalized to the values $n_{\mu k}^2$=1, μ=1, 2, 3 for the sphere; n_{4k}^2 represents the *density* of the k-particle considered normalized to the value n_{4k} = 1 for the vacuum; and $\Gamma(\psi,\partial\psi)$ represents non-linear, non-local and non-potential interactions caused by mutual overlapping/entanglement of the particles considered.

The aspects of studies [21], [25], [26], [27], [28] important for this paper are the following. Recall that particles originally in conditions of mutual overlapping/entanglement of their wave packets and then separated, have been experimentally proved to *instantly influence each other at a distance*, by therefore requiring superluminal communications that would violate special relativity. This is the very feature that prompted Einstein the argument that 'quantum mechanics is not a complete theory.' Santilli has achieved a quantitative representation of the indicated instantaneous communication at a distance via the representation of the extended character of the wavepacket of particles resulting in their continuous mutual penetration/entanglement at a distance of their center of mass, by therefore eliminating the need for superluminal communications. Above all, studies [21], [25], [26], [27], [28] have established that *the instantaneous communication of entangle particles at a distance occurs without any use of energy because the interaction are not derivable from a potential by basic assumptions.*

3.5 Application to a new conception of living organisms.

Note that all biological structures, including cells, viruses and large living organisms, are irreversible over time because they are born, grow and then die. Santilli introduced in monograph [18] of 1994 the representation of biological structures via *classical, multivalued, Lie-admissible formulations on a 3-dimensional Euclidean space,* namely, Lie-admissible formulations characterized by genounits, called *classical hyperunits*, with an ordered number of values all defined in the Euclidean space of our sensory perception

$$I^{>} = (I_1, I_2, ..., I_n) = 1/T = (1/T_1, 1/T_2, ...1/T_n) = 1/S.$$

Correspondently, the product of generic non-singular quantities *a, b* (such as numbers, functions, matrices, etc.), called *classical hyperproducts*, are equally multivalued, yet defined in our 3-dimensional Euclidean space

$$a > b = aSb = aT_1b + aT_2b + ... + aT_nb$$

in which all individual products are classical.

Correspondently, Ref. [18] introduced the notion of *classical hyperfields*, namely, sets of multi-valued elements, products and units which verify the axioms of numeric fields.

The transition from the classical, single-valued, Lie-admissible formulations outline in Section 3.3 to their multi-valued extension of was indicated in Ref. [18] as being necessary for the representation of the complexity of biological structures.

Note the fundamental character of the classical hyperunits and related hyperfields because the entire new formalism, including classical hyperalgebras, hyperspaces and hypertopology, are constructed via mere compatibility arguments with the base classical hyperfield.

In 1995, the Australian conchologist Chris Illert (see Part I of Ref. [12]) showed via computer simulations and direct calculations that the *growth of seashells over time cannot be consistently represented in a classical, 3-dimensional, single-valued Euclidean space* $E(r,\delta,1)$ with classical coordinates $r=(x,y,z)$ metric $\delta=Diag.(1,1,1)$ and unit 1 over the field of real numbers $(R,n,\times,1)$, because, in said space, seashell grow irregularly and then crack. Illert then showed a consistent representation of seashell growth via the use of a *3-dimensional, two-valued Euclidean space* $(\hat{E}, \hat{r}, \delta, 1)$ where

$$\hat{r} = \{(x_1,x_2), (y_1,y_2), (z_1,z_2)\}$$

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Santilli (see Part II of Ref. [18]) indicated that Illert's discovery confirms the need for hyperstructures in the representation of living organisms. In fact, the representation space used by Illert can be more accurately written as a *classical, 3-dimensional, two-valued, forward hyperspace* ($E^{>}, r^{>}, \delta^{>}, I^{>}$), over the forward, classical hyperfield ($R^{>}, n^{>}, >, I^{>}$) with classical forward hyperunit

$$I^{>} = \{(I_{1x}, I_{2x}), (I_{1y}, I_{2y}), (I_{1z}, I_{2z})\} = 1/T^{>} =$$

$$= \{(1/T_{1x}, 1/T_{2x}), (1/T_{1y}, 1/T_{2y}), (1/T_{1z}, 1/T_{2z})\}$$

and classical, 3-dimensional but two-valued products between arbitrary quantities *a, b* [18]

$$a > b = aT^{>}b = (a_xT_{1x}b_x + a_xT_{2x}b_x) + (a_yT_{1y}b_y + a_yT_{2y}b_y) + (a_zT_{1z}b_z + a_zT_{2z}b_z),$$

The Lie-admissible character of the representation and, therefore, its irreversibility, are assured when the backward hyperunit and, therefore, the hyperproducts, are different than the corresponding backward values.

A central notion of the above classical 2-valued, hyperstructural representation of seashells growth is the 3-dimensional character of the representation space, which is independent from the multi-valued character of each axis. Such a structure is necessary, on one side, to achieve compatibility of the mathematical representation with our sensory perceptions, while at the same time allowing an unlimited number if hidden degrees of freedom needed for a quantitative representation of the complexity of seashells. In fact, we inspect seashell growth with our three Eustachian tubes. Consequently, any *multi-dimensional representation, such as the use of a 6-dimensional space, would not be compatible with our sensory perception and, as such, not being experimentally verifiable.*

A major advance in the hyperstructural representation if biological structures was initiated by T. Vougiouklis in 1999 [39] with the lifting of the classical hyperstructures of 5 Ref. [18] to Vougiouklis H_v-structures (see also Ref. [42] and subsequent papers) which are formulated via hyperoperations (nicknamed 'hopes') including weak associativity (nicknamed '*WASS*'), weak commutativity (nicknamed '*COW*') and other hyperoperations.

The advantages of lifting the classical hyperstructures of Ref. [18] to Vougiouklis H_v-structures are several. The first advantage is a large increase of the representational capabilities which is necessary for a representation of biological structures such as the DNA, via a formulation that, at the abstract realization-free level, is compatible with the three-dimensional space of our sensory perception.

Other advantages are due to rather unique capabilities by Vougiouklis H_v-structures to characterize *bona fide* hyperfields on which the rest of the Lie-admissible formulation is expected to be built (see, e.g., Ref. [46]).

In this paper we introduce, apparently for the first time, a new conception of living organisms permitted by verifications [21], [25], [26], [27], [28] of the EPR argument according to which *a living organism, such as a cell, a virus or a human person, is composed by a very large number of extended constituents in conditions of continuous mutual entanglement of their wavepackets and, therefore, in continuous mutual communications.*

In view of the complexity and very large number of multi-valued internal communications, the best representation of the above conception of living organisms known to the authors, is given by two, hyperbimodular, Lie-admissible, H_v-structures, one for the representation of growth in time via hope, WASS and COW for ordered hypermodular hope to the right, and a second for the representation backward in time with hope, WASS and COW for ordered hypermodular hope to the left.

3.6 A specific hyperstructure formalism of living organisms.

As we present in section 3.3, in the transition from Lie theory to the covering Lie-admissible theory, we must specify an element S on the right and an element R to the left. In hyperstructure realization we can use as S and R, sets instead of elements. But in this case, we have hopes of constant length and the living organisms are not the case. Therefore, we suggest the use of a special case of the main e-construction to face the problem. Our construction equips the main product with an e-hope where the hyperproduct of two elements depend of those two elements. In fact, we keep the product and enlarge all the appropriate results.

Construction 3.1 *The Living Organism Construction.* In a set G equipped with several operations we take one product $(\cdot)$, where $(G,\cdot)$ is a group. Suppose that e is the unit, then we define in G, a large number of hopes $(\otimes)$ as follows:

$$e\otimes x = x\otimes e = x, \ \forall x\in G,$$

$$x\otimes y = \{xy, g_{xy1}, g_{xy2},\ldots\}, \ \forall x,y\in G\text{-}\{e\}, \text{ where } g_{xy1}, g_{xy1},\ldots\in G\text{-}\{e\}$$

$g_{xy1}, g_{xy2},\ldots$ depend on the pair (x,y). Then $(G,\otimes)$ becomes an H_v-group, because it contains the $(G,\cdot)$. The H_v-group $(G,\otimes)$ is an e-hypergroup. Moreover, if for each x,y such that $xy=e$, so we have $x\otimes y=xy$, then $(G,\otimes)$ becomes a strong e-hypergroup.

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Remarks. **1.** In the H_v-group $(G,\otimes)$ the hope $(\otimes)$ is *WASS* and if the $(G,\cdot)$ is commutative, then the hope $(\otimes)$ is *COW*.
2. The Living Organism Construction can be used as S or R in forward genofields or backward genofields, respectively, according to section 3.3.

Recall that, according to the Schrödinger equation of quantum chemistry, living organisms are composed by a collection of isolated points. By contrast, according to the Schrödienger-Santilli genoequation of hadronic chemistry, living organisms are composed by the indicated large number of extended constituents in conditions of continuous entanglement and communication.

It is hoped that the proposed new conception of living organisms may allow new diagnostics, e.g., via the identification of possible miscommunications between different constituents, as well as new treatments, e.g., via the disruption of selected communications.

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ENTANGLEMENT IS REAL IN 3-D 'GAME OF LIE' STRAIGHT LINE GEOMETRIC ALGEBRA CELLULAR AUTOMATON

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Abstract. The revolutionary tenet in Marius Sophus Lie's 1871 Norwegian Ph.D. dissertation *Over en Classe Geometriske Transformationer* - and nowhere else or since so clearly expressed - is that while "Descartes…has chosen the point as the element of the geometry of the plane", its "geometrical transformation…can be perceived as consisting of a transition from a point to a straight line as element", and more particularly "the straight line of length equal to zero". In that and its exploration he stands forth in the history of mathematics as the true founder of linear algebra (together with Grassmann), differential equations, continuous transformation groups, spherical geometry, and indeed the standard model by Gell-Mann's per se mistaken Lie supermultiplet adoption; and thus reaching over all dimensions and numerical systems, fundamentally including the real. This is the *terra firma* of my back to the future return to Lie and the ordinary three-dimensional Euclidean space in Cartesian extension that he primarily inhabited from nil by spanning there a tangential "line-complex" system instead of a swarm of particle points. The resulting straight "curve-net" is the infinitesimal generator realization of the instantaneous phase transition "between the Plücker line-geometry and a geometry whose elements are the space's spheres" in the form of a universally extending isotropic vector matrix (IVM) lattice embedded and distributed in chaperoning Cartesian coordinate cages filling space by hierarchical piling of the direct structural hybrid $R^3\times SO(3)$ wave-packets so constituted, and which by cellular automaton iterations in whole or parts exactly replicate the elementary particle, atomic and periodic table spectroscopy. Moreover, since the $R^3\times SO(3)$ curve-net is timeless, all lines there, e.g. those occupied by coherently superposed photon pairs, are everlasting, hence solving all dilemmas and paradoxes of entanglement.

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Introduction

Initially highly debated as such and analytically[1-3], entanglement with its from Schrödinger's *Verschrenkung*[3] imported bearings also on budding, branching, catenating, gluing etc., has over the years stood the tests to its now generally acknowledged designation as a special kind of nature's way of compounding physical reality and realization from the ground and onwards, namely, superposition. Elementary particles and waves, atoms, the periodic table, molecules and their combinations all submit to this fundamental both classical and quantum mechanical principle of joining together in matching constellations into larger states. These are then direct sum congregations of their irreducible ingredients and in subtraction thus release them in their accommodated postures so that, nothing else interfering, they still appear matched when measured along different paths after the separation.

But entangled states are coherent superpositions where each as a rule pair occupies one geodesics where it avoids forbidden collisions either by intertwining or more often diverging, with the counter-intuitive consequence of interdependence/identity of synchronized measurements between them over distances of any length; either according to the Einstein-Podolsky-Rosen (EPR) argument[1] by some shared internal mathematical algorithm ensuring equal evolution of the two, or according to the Copenhagen school[2] as a primary constitution of unknown kind.

As first resolved in the thought experiment Bell theorem in 1964[4], and 1999 in a laboratory replication by Aspect[5], and in both tested by a real world outcome, the evidence as accumulating in a legion of further investigations more and more favors the latter case. This has inspired a veritable panorama, not to say Pandora box of freely extrapolated macroworld realizations and applications which have reached cosmological dimensions with time travels, teleportation, wormholes, branes, multiverses and "connected spacetimes by entangling"[6] as almost everyday routine events, albeit so far totally unseen in practice. Yet, massive funding is directed to this novel utopia which of course has immense appeal and at least engenders a cornucopia of innovative mathematical, computer and animation techniques, theoretical insights and philosophical, epistemological, ontological and information science developments.

However, the alternative that there may be a an almost trivial, quite mundane combination of local gearing and remote coherence has not been investigated before but is an inherent feature of the three-dimensional space-filling lattice model in close compliance with Marius Sophus Lie's *geometriske transformationer*[7,8] here reported with special entanglement reference.

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Methods and Results

Marius Sophus Lie's groundbreaking Ph.D. thesis *Over en classe geometriske transformationer*[Ib.] from 1871 (and thus due for a most deserved 150-year anniversary) was in fact not well understood in the defence[9], and soon went down to oblivion in the faculty archives. When a hundred years later I as a budding cardiologist (see https://www.ncbi.nlm.nih.gov/pubmed/?term=trell+e) in search of clues on rotational transformations for electrocardiographic (ECG) applications obtained a photocopy of it from the Oslo University library I soon realized what a treasure it was; not directly to ECG of course but the very roots of linear algebra, differential equations and both the philosophical and structural "nature of Cartesian geometry".[7,8] With pivotal help and support from the inventor of Lie-admissible algebra[10,11], Professor Ruggero Maria Santilli, I made an English translation of it (now open access available)[8], expressing its crucial tenet that while "Descartes…has chosen the point as the element of the geometry of the plane", its "geometrical transformation…can be perceived as consisting of a transition from a point to a straight line as element", and more particularly "the straight line of length equal to zero".[7,8] Equally small as an infinitesimal point the difference is that as a linkable line it is in effect a partial derivative building unit filling both its space and path by its coherent steps as a virtual cellular automaton[12] (Figure 1) so as to crystalize a real form $R^3 \times SO(3)$

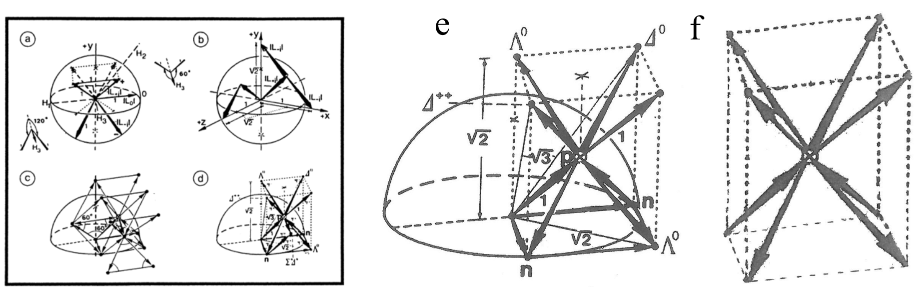

Figure 1. a-d) Killing A_3 root space diagram knits a hexagonal SO(3) infinitesimal generator lattice, e,f) globally distributed in space-filling parallelepiped R^3 chaperon enclosures to form a real space cellular automaton which in repeated steps of itself in parts or whole carries out a structural replication of the elementary particle, atomic and periodic table spectroscopies.

wave-packet compound which can structurally rewrite the elementary particle[13-21], atomic and periodic table spectroscopies. Previous reports on the latter[22-28] will here be complemented with the completion of the system with

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special emphasis on entanglement both in a classical and quantum mechanical classification, generator and computation regard.

Figure 2 summarizes how the charged *t* isospin root vectors can be linked to a coherent triple helix in a continuous cloverleaf 'singlet coil' sequence of 12 unit steps – equally many as the edges of the parallelepiped – into the only other space-filling regular solid convolution, namely, a complex of one octahedron and two tetrahedrons (Figure 2a), which in iteration (Figure 2b) forms an important nanotechnology structure, i.e. an ordinary space frame, alias

Figure 2. Linking of charged *t* isospin root vectors to an outwards connected lattice, iterating to space frame of isotropic vector matrix (IVM) constitution.

octahedron-tetrahedron, or octet truss[29] whose modular building blocks and their superposition can thus be realized from the ground of material organization. Figure 3 shows how the per se space-filling universal IVM lattice

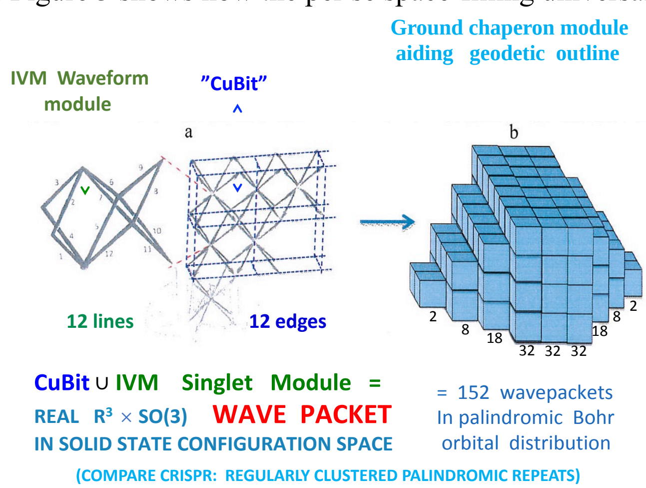

Figure 3. 2+8+18+32+32+32+18+8+2 = 152-step coherent chaperon sequence into a trapezoid bottom module of palindromic Bohr orbital distribution whose inner continuous 152×12-step SO(3) wave filament outlines the electron

is continuously distributed in Euclidean space by chaperoning R^3 encasements which in the first superposition period form a sideways tileable flat-bottomed neutron ground module which for vertical packing needs a reciprocal flat-roofed cap (Figure 4) so that each half of the combined ‘transition apparatus’ comprises 153×12 unit steps = 1836 = proton/electron relative inertial ratio. There is one square and one diagonal alternative of realization, of which the

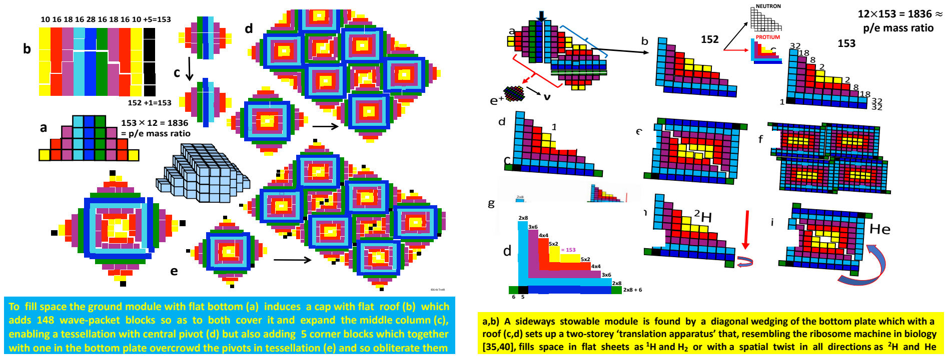

Figure 4. Hopefully self-explaining miniature illustrations (enlargeable in the screen) of square and twisted space-filling $R^3\times SO(3)$ wave-packet module alternatives. *Colors refer to the noble gas in each period*

former obliterates space by its extra step but the latter (Figure 5) can continue the tessellation over it in exact replication of the atomic nucleosynthesis

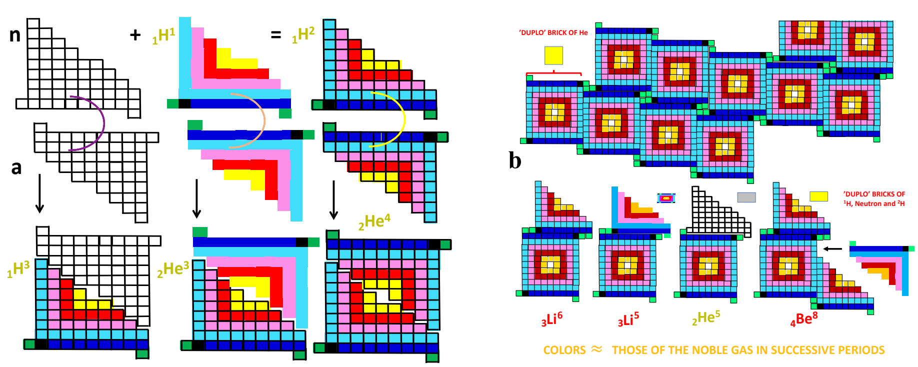

Figure 5. a) Nucleosynthesis of Deuterium from Neutron and Protium, and further to Tritium, $_2$Helium^3 and $_2$Helium^4. b) Same in Lithium to Beryllium.

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succession including the isotope spectrum, subshell filling, neutron excess etc. in total correspondence with the periodic table. This will be exemplified by increasingly simplified graphical representation in the further periods. Figure 6 shows the second and third, each concluded by the respective saturated noble

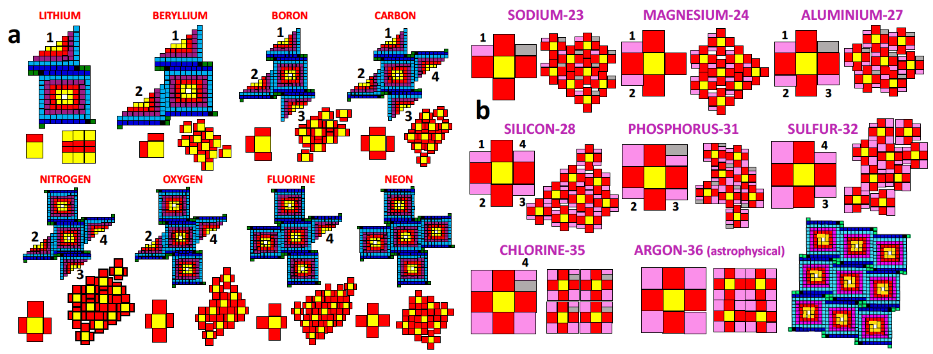

Figure 6. a) Fine-grained 'Lego' and coarse-grained 'Duplo'[30] representation of stable(st) second period atoms. b) Duplo representation of same in third period

gas with no remaining molecular binding sites. From the fourth period, when the neutron capture nucleosynthesis mechanism takes over in Iron and onwards, the 'triplo' representation (Figure 7a) is the most adequate for the marginal β decay superposition series demonstrated (Figure 7b) in the central isotopes of Iron to Zinc.

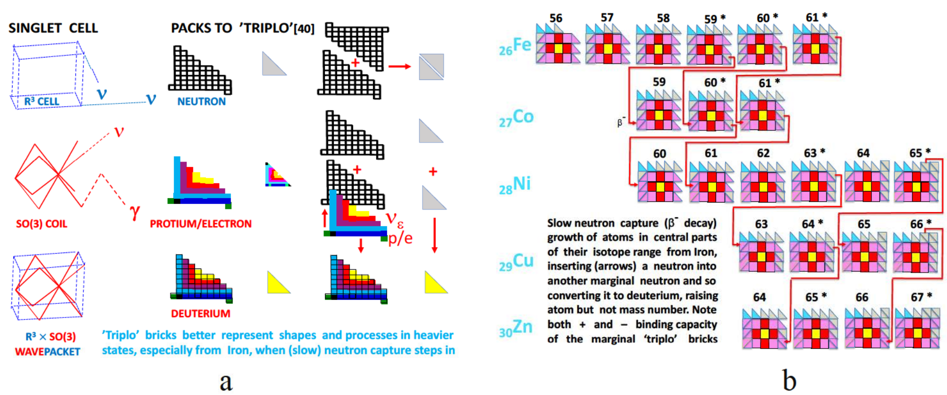

Figure 7. a) Triplo representation (note generation of photon and neutrinos from wave-packet edges) b) Triplo representation of central isotopes with slow neutron capture β decays from Fe to Zn

Also in the heaviest atoms this marginal accretion applies, but increasingly by the rapid neutron capture process and sometimes downwards by α decay. All of this will be exemplified in the forthcoming periods, and it is notable and a further support of the model that in their subshell filling stage the lanthanides and actinides, too, can be continuously included in the respective sixth and seventh period. With brief comments showing parts of the periodic table series to its conclusion in Oganesson and onwards, the further aim is a review of entanglement as reproducible in the system in close compliance with reality.
In Figure 8a the last part of the fourth period is overviewed, and summed up in Figure 8b, where the fulfilment with α/He blocks is demonstrated.

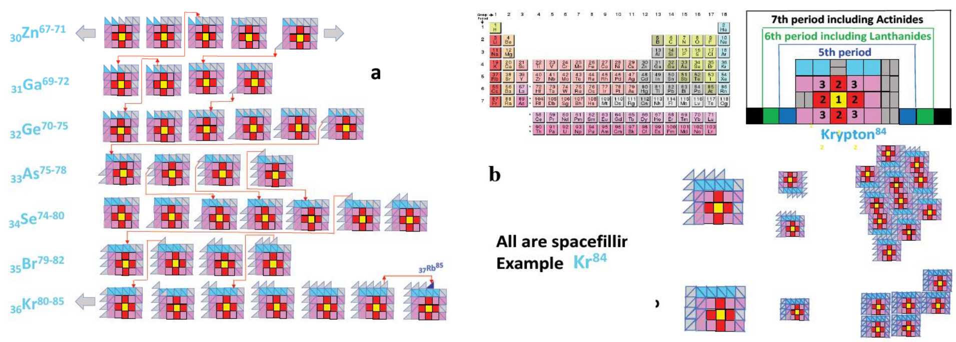

Figure 8. a) Zn to Kr. b) Summary of fourth period.

The conclusion of the fifth period from Rh to Xn is shown in Figure 9a, and the period is summarized in Figure 9b. Figure 10 comprises the same together with an example of spacefilling expansion in the sixth period.

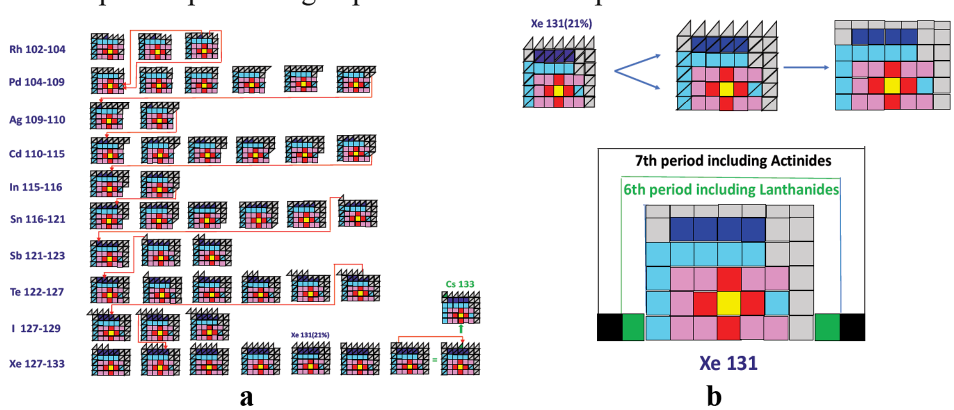

Figure 9. a) Rh to Xe. b) Summary of fifth period.

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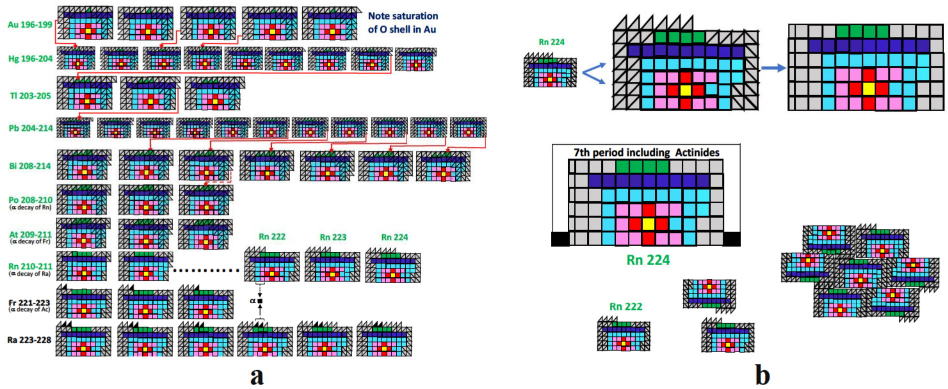

Figure 10. a) Conclusion of sixth period. Note saturation of subshells in same order as in reality and that all Lanthanides have been included in that subsurface saturation altering little of the binding sites in outer layer. Note also strong subshell filling and consequential stability in Au. b) Summary of sixth period.

Figures 11 and 12 show the initial and concluding stage of the seventh period.

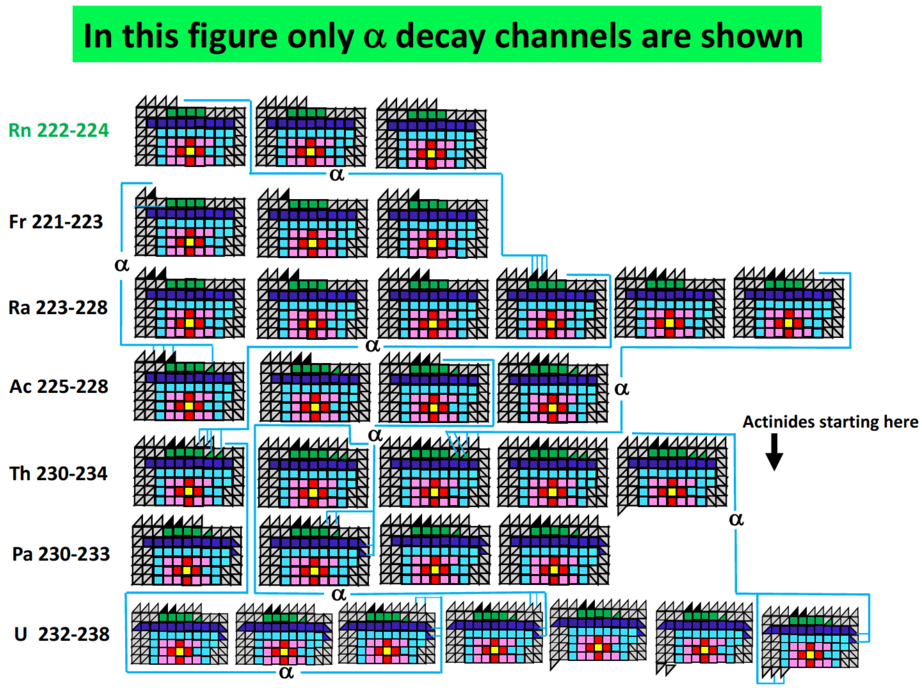

Figure 11. a) Start of seventh period. Here downward α decay nucleosynthesis is getting more frequent and is focused on in the figure.

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Figure 12. a) Conclusion of seventh period ending in full saturated Oganesson.

As demonstrated in Figure 13; a) Oganesson 294, which is the best documented isotope, is perfectly saturated and built in full α/He blocks with identical shell/subshell pattern and bolstering neutron excess as in reality, b) and in the 295 isotope providing first step into the eight period, for which, c) the model has good space in its tangential plane transferal of ground module comprisal.

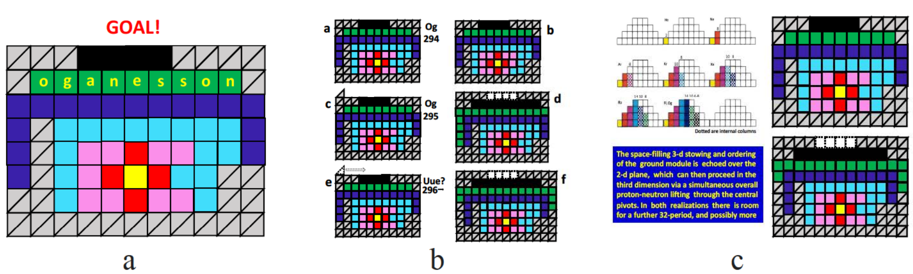
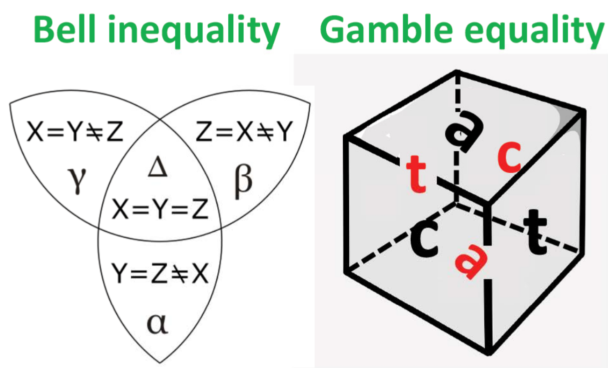


Figure 13. a) Oganesson 294. b) Isomers of Og294, Og 295, and possible Uue 296, and fulfilment of an eight period. c) Transferal of stowing pattern and potential in ground module to tangential plane distribution.

In conclusion so far, the reproducible descriptive results of the system should be sufficient evidence of its essential truth. It outlines, literally, a three-dimensional world which is atemporal but not ephemeral because its quantum second is eternity. In its permanence the entanglement phenomenon becomes natural and reciprocally a strong evidence of its framework. Can simultaneous action over distance have a local or global footing or both? The former is the EPR standpoint[1] and the latter is the Bell inequality analysis and its transferal to a still quantum mechanical instead of classical mechanism.[4]
It is important to realize that the search of inequality was to investigate and identify prospective quantum mechanical formulas that would enable locality, i.e., as surveyed in Figure 14, to endow a separating quantum superposition

In Bell's inequality re electron spin it is posed that X,Y,Z are the measurement variables with ± span in each. In reality this cannot evade that the entangled electrons must carry the spin with them; i.e. a carrier property before it can be measured as such. If e.g. an x is registered it must first be delivered. And when the outcome is ± there is in that minimal case a 2(x,-x, y,-y, z,-z) set of in- and outcomes created in the electron pair giving a 2/6 + 2/6 + 2/6 =1 total probability and no inequality. And if the only other option is a free mix between the variables there is the situation of two successive tosses of the CAT/cat dice with full equality



i	1	2	3	4	5	6	7	8
X_i	1	1	1	1	-1	-1	-1	-1
Y_i	1	1	-1	-1	1	1	-1	-1
Z_i	1	-1	1	-1	1	-1	1	-1
	Δ	γ	β	α	α	β	γ	Δ

P(X=Y) + P(Y=Z) + P(Z=X) ≥ 1

c	c a t	c a t
a	c a t	c a t
t	c a t	c a t
c	c a t	c a t
a	c a t	c a t
t	c a t	c a t

σ(cat)(cat) =1

Figure 14. Comparing local quantum mechanical prediction and real gambling outcome of Bell thought experiment on electron spin, showing that local prediction exceeds real outcome and therefore rules out locality (but not reality)

with a kind of DNA formula determining equal further development of the two or more offshoots, and then to compare with real observation violating the

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quantum mechanical prediction. Finding just one such example would solve the dilemma, and Bell chose electron ± spin over x,y, and z coordinate axes, so the analogy of a both dead and alive - and spinning - cat could be trebled and emulated in the form of a correspondingly signed dice to throw the real fallout by.

However, experimental tests are needed to evaluate whether the quantum mechanical expectation computations match synchronous measurements at arbitrarily faraway spaced observation sites, customarily named Alice (originally Albert) and Bob (originally Niels) and thus support local reality, or are inequal from them and therefore must imply instant action over distance. The majority of such experiments use polarization of photon pairs in set-ups originally devised by Aspect[5] (Figure 15) and evaluated by a similar protocol as

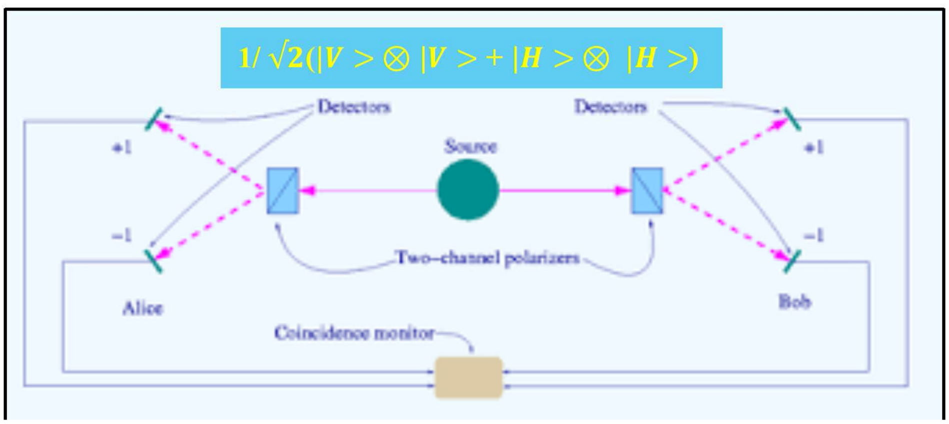

Figure 15. Experimental set-up of entanglement generation/analysis. Pairs of photons are discharged in opposite directions from the source through birefringent polarizers whose orientations are adjustable, and the split rays are individually detected, and + +, – –, + – and – + correlations under various angles counted in the coincidence monitor. Usually the A circuit is the local observer's, whereas B is arbitrarily far away. *V = vertical, H = horizontal*

Bell's, and overwhelmingly find that the real observations violate the predictions, no matter how long between each other they are, and by inference, then, over a beam between galaxies if so be. When using polarization of photons instead of spin of electrons, the quantum state of a pair of entangled photons is expressed as $1/\sqrt{2}(|V>\otimes|V> + |H>\otimes\ |H>)$, and is not the same as that of a spin-half particle, with different correspondence between angles and outcomes. Importantly, when the polarization of both photons is measured in equal direction both give identical, in 45º angle a random, and in 90º an anticorrelated outcome, but the general principle is the same. As mentioned the results excludes local quantum mechanical breeding of them, so the conclusion is that indeed instant interaction over distance is the case. However, there is one implication of its permanent observability at any site, namely that it must exist over the whole length of the trajectory, and this is the key to an entirely natural mechanism and explanation which is inherent in the present three-dimensional lattice system, as further explained in Figure 16. Every "geodetic curve"[7,8] there is lasting and in the quantum mechanical lingo could be likened to, let's say, a plank that lies extended, e.g. between the moon and asteroid B 612, weight-less and static in open space. But is it constant? Imagine that an astronaut tilts/measures it over any angle from whatever point along its span. By certainty; would that not instantly affect the whole plank and resolve whatever paradox and projection and exploitation of the EPR/Bell entanglement dilemma in the moment of a tilt? Well, now there is no such plank, but when instead there is a polarized, solely three-dimensional, i.e.

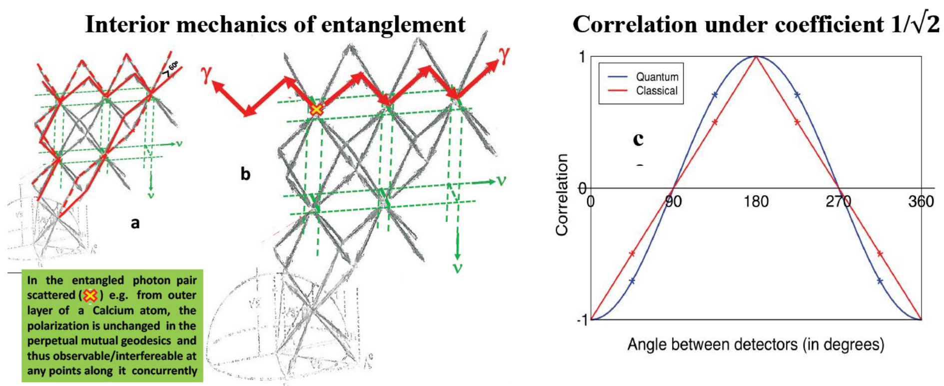

Figure 16. Interior mechanics of entanglement. a) In the SO(3) lattice there are both orthogonal and diagonal binary root vector sequences (red), all with a $\sqrt{0.5}/0.5 = \sqrt{2}/1$ vertical/horizontal span. b) when a photon pair is discharged from an intersection in the IVM lattice, it shares the same perpetual geodesics that lies there as a lopsided jagged plank which a polarizer at any point can by its inclination tilt or not tilt; in a single local moment turning the whole beam. c) When the QM correlation coefficient $1/\sqrt{2}$ is divided by the real $\sqrt{2}/1$ proportion the blue and red lines change place, supporting the classical case.

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timeless, worldline in which the mediation is the measure, there is an identical situation, offering a strong support of the 'game of Lie' cellular automaton plan as well as being a direct consequence of it.

Discussion

At least I have no problem with the scheme and dare to meet any still facepalming audience of the pre-critically inflating standard model creed with it. Especially the vitally needed explication of the electron as not an amorphous point but of already described alternating transition matrix or wave function form[30] is a fact that undermines the ivory tower from the ground. But still it is sky-rocketing, with entanglement as one of its strongest spellbinding thrusts. However, the actual potentials are limited. With ± in various angles effectively being the only spin-off options, their tentative Morse patterns cannot more than project a structure. And to retrieve them an observer must still travel to the site of recording, and cannot do it faster than the speed of light, and the same applies to any deciphering or other communication broadcast to a receiver maybe hundreds of light-years out there. And the alternative to blindly fish after unaddressed planks in the endless ocean of the present is equally futile on both sides.

However, and from now naming these tilted quanta Plancks, the real advancement is in the field of information and communication. In a three-dimensional world there is a temporal arrow but no separate axis, and all apparent motion in every hierarchical subsystem formed by the timeless progression of successive moments in its track is relative to how this approximates to or deviates from other per se immobile trajectories. Pair-produced photons peeled off in the Aspect constellation[5] (Figure 15), from the outmost layer in one of a Calcium isotope's surface electron modules (Figure 16 b) or a laser, thus form a rigid beam swept over in various angles by the effector/detector plane, and if generated in larger bundles collective interaction emerges and sustains in consecutive pulses enabling correspondingly expanded information load for afferent and efferent messages amplified by both input and output coding and complexification gates etc. The Plancks glide away there, under the bridge and may by extra polarizers be directed into side-channels where some of them align with the walls so as to seemingly disappear, only to be redirected (but also possibly intercepted) and reappear in yet another polarization up- or downstream like light in a prism, streaming yet still…Well, excuse for the panegyric, but the opportunities are copious, and surely under intense study and exploration in many places.

When once installed in, for example, a spacecraft, and the link established, there is instant communication reaching as far as the flight goes but not yet to the next destination until it is there. And transferring the method to organisms or larger objects is not possible because that would require that every smallest molecule and all atoms in it be tilted to be sieved through the filter, and when some are not but in counterposition, explosion will happen, and the passage is still local. The only way is by proxy as far as the link goes and the proxy may print the chart but not the things and to reach an intended extraterrestrial location the link has first to be carried there which takes as many lightyears and more as the distance. But there might be that higher civilizations have developed the technique and are sending out signals in its band, for which a search should therefore be made which could instate an on-line exchange but no material impacts; and everything by classical means and mechanisms.

So, even if appearing in *Science* the more grandiose top-down visions of cosmic wormholes and teleportation and multiverses and that "it is possible to build up very generic connected spacetimes by entangling discrete noninteracting systems"[6] remain unseen and out of scope with the still very fascinating subject at hand, which has its roots in the bottom-up local scale. The electrons in atomic shells are entangled and so determine the time-less Aufbau of the periodic system, and for that essential operation EPR was right, too; quantum mechanics, in its most sophisticated perturbation and variation methods is incomplete, even in solving the electronic shell structure of the Helium atom and onwards.[32]

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EXTENDING MATHEMATICAL MODELS FROM NUMBERS TO Hv-NUMBERS

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Abstract

Hypergoup is a set equipped with a hyperoperation which is associative and reproductive. The fundamental relation β* was introduced in 1970, which is the main tool in hyperstructures because it connects them with the corresponding classical structures. In 1990, Vougiouklis introduced the H_v-structures, by defining the *weak axioms* where the non-empty intersection replaces the equality. The quotient of a group by a partition is an H_v-group, so it is the largest class of hypergroups. The number of H_v-structures defined on a set is extremely greater than the number of the classical hyperstructures defined on the same set. Hyperstructures, especially the H_v-structures, have applications in many sciences including biomathematics, hadronic physics, lepton physics, and Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, it can be applied in linguistic, sociology, industry and manugfacturing. In this paper, we focus on Lie-Santilli's theory especially on the *Hypernumbers or H_v-numbers* needed for the mathematical representation. The e-hyperfields, can be used as isofields, in such way to cover additional properties. Large classes of H_v-structures can be used in the Lie-Santilli theory especially when multivalued problems appeared, in finite or infinite case.

Key words: hyperstructures, hope, H_v-structures, H_v-fields, Lie-Santilli iso-theory.

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1. HYPERSTRUCTURES

The largest class of hyperstructures, called H_v-*structures*, were introduced by the author in 1990 [19], [22]. These hyperstructures satisfy the *weak axioms* where the non-empty intersection replaces the equality. Some basic definitions are the following:

Definition 1.1 In a set H equipped with the hyperoperation (abbreviation: *hyperoperation*= ***hope***)

$$\cdot: H\times H \to P(H)-\{\varnothing\},$$

we abbreviate with *WASS* the *weak associativity*: $(xy)z\cap x(yz) \neq \varnothing$, $\forall x,y,z\in H$ and with *COW* the *weak commutativity*: $xy\cap yx \neq \varnothing$, $\forall x,y\in H$.

The hyperstructure $(H,\cdot)$ is called ***H_v-semigroup*** if it is *WASS,* it is called ***H_v-group*** if it is a reproductive H_v-semigroup, i.e. $xH = Hx = H$, $\forall x\in H$.

In the classical theory, the quotient of a group with respect to an invariant subgroup is a group. F. Marty stated in 1934 that, the quotient of a group with respect to any subgroup is a hypergroup. Finally, the quotient of a group with respect to any partition is an H_v-group [22].

The *powers* of an element $h\in H$ are: $h^1=\{h\}$, $h^2=h\cdot h$, …, $h^n=h^\circ\ldots^\circ h$, where $(^\circ)$ is the *n-ary circle hope*: the union of hyperproducts, n times, with all patterns of parentheses put on them. An H_v-semigroup $(H,\cdot)$ is a *cyclic of period s*, if there is a *generator* g, and a natural n, such that $H=h^1\cup\ldots\cup h^s$. If there is an h and s, such that $H=h^s$, then $(H,\cdot)$ is called *single-power cyclic of period s*.

In a similar way, more complicated hyperstructures can be defined:

Definitions 1.2 The $(\boldsymbol{R},+,\cdot)$ is an ***H_v-ring*** if $(+)$ and $(\cdot)$ are *WASS*, the reproduction axiom is valid for $(+)$ and $(\cdot)$ is *weak distributive* with respect to $(+)$:

$$x(y+z)\cap(xy+xz)\neq\varnothing, \quad (x+y)z\cap(xz+yz)\neq\varnothing, \quad \forall x,y,z\in R.$$

Let $(\boldsymbol{R},+,\cdot)$ be H_v-ring, $(\boldsymbol{M},+)$ be COW H_v-group and let there exist an external hope

$$\cdot : R\times M \to P(M): (a,x) \to ax$$

such that, $\forall a,b\in \boldsymbol{R}$ and $\forall x,y\in \boldsymbol{M}$, then we have

$$a(x+y)\cap(ax+ay) \neq \varnothing, \quad (a+b)x\cap(ax+bx) \neq \varnothing, \quad (ab)x\cap a(bx) \neq \varnothing,$$

then $\boldsymbol{M}$ is an ***H_v-module*** over $\boldsymbol{F}$. In the case of an H_v-field $\boldsymbol{F}$ instead of an H_v-ring R, then the ***H_v-vector space*** is defined.

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For more definitions and applications on H_v-structures one can see the books [1], [3], [4], [7], [9], [16], [22], [23], [28], [29], [32], [33].

Definition 1.3 The ***fundamental relations*** β^*, γ^* and ε^*, are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector spaces, respectively [19], [21], [22], [23], [31], [33].

The way to find the fundamental classes is given by the following:

Theorems 1.4 Let $(H,\cdot)$ be H_v-group and U be the set of finite products of elements of H. We define the relation β in H by setting $x\beta y$ iff $\{x,y\}\subset u$ where $u\in U$. Then β^* is the transitive closure of β.

Let $(R,+,\cdot)$ be H_v-ring and U the set of finite polynomials of elements of R. We define the relation γ in R by: $x\gamma y$ iff $\{x,y\}\subset u$ where $u\in U$. Then γ^* is the transitive closure of γ.

An element is called ***single*** if its fundamental class is singleton.

Definition 1.5 The fundamental relations are used for general definitions. Thus, an H_v-ring $(R,+,\cdot)$ is called ***H_v-field*** if R/γ^* is a field.

Definition 1.6 Let $(H,\cdot)$, $(H,*)$ be H_v-semigroups defined on the same set H. $(\cdot)$ is called ***smaller*** than $(*)$, and $(*)$ ***greater*** than $(\cdot)$, iff there exists an

$$f\in Aut(H,*) \quad \text{such that} \quad xy\subset f(x*y),\ \forall x,y\in H.$$

Then we say that $(H,*)$ *contains* $(H,\cdot)$. If $(H,\cdot)$ is a classical structure then it is called *basic structure* and $(H,*)$ is called *H_b-structure*.

The Little Theorem. Greater hopes than the ones which are WASS or COW, are WASS or COW, respectively.

This Theorem leads to a partial order, *posets*, on H_v-structures [28], [35], [9].

Definition 1.7 The H_v-semigroup $(H,\cdot)$ is called ***h/v-group*** if the quotient H/β^* is a group.

The h/v-groups are a generalization of the H_v-groups since in h/v-groups the *reproductivity of classes* is valid. This leads the quotient to be reproductive. In a similar way the *h/v-rings, h/v-fields, h/v-vector spaces* etc, are defined.

Definition 1.7 [18], [21], [22]. An H_v-structure is called ***very thin*** iff all hopes are operations except one, which has all hyperproducts singletons except only one, which is a subset of cardinality more than one.

Definition 1.8 [29], [32]. Let $(G,\cdot)$ be groupoid (resp., hypergroupoid) and $f:G\rightarrow G$ be a map. We define a hope (∂), called *theta-hope,* we write ***∂-hope***, on G as follows

$$x\partial y = \{f(x)\cdot y,\ x\cdot f(y)\},\ \forall x,y\in G. \ (\text{resp. } x\partial y= (f(x)\cdot y)\cup(x\cdot f(y), \forall x,y\in G)$$

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If $(\cdot)$ is commutative then ∂ is commutative. If $(\cdot)$ is *COW*, then ∂ is *COW*.

Let $(G,\cdot)$ be groupoid (or hypergroupoid) and $f:G\to\boldsymbol{P}(G)-\{\varnothing\}$ be multivalued map. We define the (∂), on G as follows

$$x\partial y = (f(x)\cdot y)\cup(x\cdot f(y), \ \forall x,y\in G.$$

Motivation for the theta-hope is the map *derivative* where only the multiplication of functions can be used. Basic property: if $(G,\cdot)$ is a semigroup then $\forall f$, the (∂) is *WASS*.

Definition 1.9 [17],[32]. Let $(G,\cdot)$ be groupoid, then $\forall P\subset G$, $P\neq\varnothing$ we define the following hopes called *P-hopes*: $\forall x,y\in G$

$$\underline{P}: x\underline{P}y= (xP)y\cup x(Py), \quad \underline{P}_r: x\underline{P}_ry= (xy)P\cup x(yP), \quad \underline{P}_l: x\underline{P}_ly= (Px)y\cup P(xy).$$

The $(G,\underline{P})$, $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ are called *P-hyperstructures*. If $(G,\cdot)$ is semigroup, then $x\underline{P}y=(xP)y\cup x(Py)=xPy$ and $(G,\underline{P})$ is a semihypergroup but we do not know about $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$. In some cases, depending on the choice of P, the $(G,\underline{P}_r)$ and $(G,\underline{P}_l)$ can be associative or WASS.

A generalization of P-hopes is the following [6], [9]:

Construction 1.10 Let $(G,\cdot)$ be abelian group and P, subset of G. Define the hope $\times_P$ as follows:`

$$\begin{cases} x\times_P y = x\cdot P\cdot y = \{x\cdot h\cdot y \mid h\in P\} & \text{if} \quad x\neq e \ \text{ and } \ c\neq e \\ x\cdot y & \text{if} \quad x=e \ \text{ or } \ y=e \end{cases}$$

we call this hope P_e-hope. The hyperstructure $(G,\times_P)$ is an abelian H_v-group.

The general definition of an H_v-Lie algebra was given as follows [31], [9]:

Definition 1.11 Let $(\boldsymbol{L},+)$ H_v-vector space on $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\to\boldsymbol{F}/\gamma^*$ canonical, $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is zero of $\boldsymbol{F}/\gamma^*$. Let ω_L the core of $\varphi':\boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Consider the *bracket (commutator) hope*:

$$[\ ,\ ]: \boldsymbol{L}\times\boldsymbol{L} \to P(L): (x,y) \to [x,y]$$

then $\boldsymbol{L}$ is an *H_v-Lie algebra* over $\boldsymbol{F}$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.

$$[\lambda_1x_1+\lambda_2x_2,y]\cap(\lambda_1[x_1,y]+\lambda_2[x_2,y]) \neq \varnothing$$

$$[x,\lambda_1y_1+\lambda_2y_2]\cap(\lambda_1[x,y_1]+\lambda_2[x,y_2]) \neq \varnothing, \ \forall x,x_1,x_2,y,y_1,y_2\in L, \ \forall\lambda_1,\lambda_2\in\boldsymbol{F}$$

(L2) $[x,x]\cap\omega_L \neq \varnothing$, $\forall x\in\boldsymbol{L}$

(L3) $([x,[y,z]]+[y,[z,x]]+[z,[x,y]])\cap\omega_L \neq \varnothing$, $\forall x,y\in\boldsymbol{L}$.

The enlargement or reduction of hyperstructures are examined in the sense that an extra element appears in one result or we take out an element. In both directions most useful are those H_v-structures with the same fundamental structure [25], [27]:

Let (H,·) be H_V-semigroup and $v \notin H$. Extend (·) into the $\underline{H}=H\cup\{v\}$ as follows: $x\cdot v=v\cdot x=v$, $\forall x\in H$, and $v\cdot v=H$. The $(\underline{H},\cdot)$ is an h/v-group where $(\underline{H},\cdot)/\beta^*\cong Z_2$ and v is a single element. We call $(\underline{H},\cdot)$ the *attach h/v-group* of (H,·).

Theorem 1.12 Let (G,·) be semigroup and $v\notin G$ be an element appearing in a product ab, where $a,b\in G$, thus the result becomes a hyperproduct $a\otimes b=\{ab,v\}$. Then the minimal hope (⊗) extended in $G'=G\cup\{v\}$ such that (⊗) contains (·) in the restriction on G, and such that $(G',\otimes)$ is a minimal H_V-semigroup which has fundamental structure isomorphic to (G,·), is defined as follows:

$$a\otimes b=\{ab,v\},\quad x\otimes y=xy,\ \ \forall (x,y)\in G^2-\{(a,b)\}$$

$$v\otimes v=abab,\quad x\otimes v=xab\ \ \text{and}\ \ v\otimes x=abx,\ \ \forall x\in G.$$

$(G',\otimes)$ is very thin H_v-semigroup. If (G,·) is commutative then the $(G',\otimes)$ is strongly commutative.

Let (H,·) be hypergroupoid. We say that *remove* $h\in H$, if we consider the restriction of (·) on H-{h}. We say $\underline{h}\in H$ *absorbs* $h\in H$ if we replace h by $\underline{h}$. We say $\underline{h}\in H$ *merges* with $h\in H$, if we take as product of $x\in H$ by $\underline{h}$, the union of the results of x with both h, $\underline{h}$ and consider h and $\underline{h}$ as one class.

Now we present some 'small' h/v-fields.

Constructions 1.13 On the rings $(\boldsymbol{Z}_4,+,\cdot)$ and $(\boldsymbol{Z}_6,+,\cdot)$ we will define all the multiplicative h/v-fields which have non-degenerate fundamental field and, moreover they are,

(a) very thin minimal, (b) COW, (c) they have 0 and 1, scalars.

I. On $(\boldsymbol{Z}_4,+,\cdot)$ we have the isomorphic cases: $2\otimes 3=\{0,2\}$ or $3\otimes 2=\{0,2\}$. The fundamental classes are [0]={0,2}, [1]={1,3} and we have $(\boldsymbol{Z}_4,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot)$. Thus it is isomorphic to $(\boldsymbol{Z}_2\times\boldsymbol{Z}_2,+)$. In this H_v-group there is only one unit and every element have a unique double inverse.

II. On $(\boldsymbol{Z}_6,+,\cdot)$, we have the only one hyperproduct,

(i) $2\otimes 3=\{0,3\}$, $2\otimes 4=\{2,5\}$, $3\otimes 4=\{0,3\}$, $3\otimes 5=\{0,3\}$, $4\otimes 5=\{2,5\}$

Fundamental classes: [0]={0,3}, [1]={1,4}, [2]={2,5} and we have

$$(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_3,+,\cdot).$$

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(ii) $2\otimes3=\{0,2\}$ or $2\otimes3=\{0,4\}$, $2\otimes4=\{0,2\}$ or $\{2,4\}$, $2\otimes5=\{0,4\}$ or $2\otimes5=\{2,4\}$, $3\otimes4=\{0,2\}$ or $\{0,4\}$, $3\otimes5=\{3,5\}$, $4\otimes5=\{0,2\}$ or $\{2,4\}$.

Fundamental classes: $[0]=\{0,2,4\}$, $[1]=\{1,3,5\}$ and we have

$$(\boldsymbol{Z}_6,+,\otimes)/\gamma^*\cong(\boldsymbol{Z}_2,+,\cdot).$$

Definition 1.14 The ***uniting elements*** method, introduced by Corsini & Vougiouklis in 1989 [2], is the following: Let G be algebraic structure and d be a property, which is not valid and described by a set of equations; then, consider the partition in G for which it is put together, in the same class, every pair that causes the non-validity of d. The quotient G/d is an h/v-structure. Then, quotient out the G/d by β^*, a stricter structure $(G/d)/\beta^*$ for which the property d is valid, is obtained.

A problem of the uniting elements occurs when more than one property is desired. The reason is that some of the properties lead straight to the classes than others. So, we apply the straightforward classes followed by the more complicated ones. The commutativity and reproductivity are easy applicable properties. One can do this because the following Theorem is valid [22], [25], [27].

Theorem 1.15 Let $(G,\cdot)$ be a groupoid, and $F=\{f_1,\ldots,f_m,f_{m+1},\ldots,f_{m+n}\}$ be system of equations on G consisting of subsystems $F_m=\{f_1,\ldots,f_m\}$, $F_n=\{f_{m+1},\ldots,f_{m+n}\}$. Let σ, σ_m be the equivalence relations defined by the uniting elements procedure on systems F and F_m, respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m=(G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^*\cong(G_m/\sigma_n)/\beta^*.$$

i.e. the following diagram is commutative

$$\begin{array}{ccccc} G & \xrightarrow{\rho_m} & G/\sigma_m & \xrightarrow{\varphi_m} & G_m \\ \rho\downarrow & & & & \downarrow\rho_n \\ G/\sigma & & & & G_m/\sigma_n \\ \varphi\downarrow & & & & \downarrow\varphi_n \\ (G/\sigma)/\beta^* & & \xrightarrow{\cong} & & (G_m/\sigma_n)/\beta^*. \end{array}$$

Where all maps ρ, φ, ρ_m, φ_m, ρ_n, φ_n, are the canonicals.

The problem of enumeration and classification of hyperstructures, started from the beginning, it is complicate in H_v-structures because we have very great numbers [10]. For example, the number of H_v-groups with three elements, up to isomorphism, is 1.026.462. There are 7.926 abelian; the 1.013.598 are cyclic. The partial order in H_v-structures and the Little Theorem, transfers and restrict the problem in finding *the minimal*, *up to isomorphisms*, H_v-structures.

2. REPRESENTATIONS

Representations (we abbreviate with ***rep***) of H_v-groups, can be considered either by H_v-matrices [21], [22], [23], [24], [27] or by generalized permutations [20].

Definition 2.1 ***H_v-matrix*** (or *h/v-matrix*) is called a matrix with entries elements of an H_v-ring or H_v-field (or h/v-ring or h/v-field). The hyperproduct of H_v-matrices $\mathbf{A}=(a_{ij})$ and $\mathbf{B}=(b_{ij})$, of type m×n and n×r, respectively, is a set of m×r H_v-matrices, defined in a usual manner:

$$\mathbf{A}\cdot\mathbf{B} = (a_{ij})\cdot(b_{ij}) = \{\mathbf{C}=(c_{ij}) \mid c_{ij}\in\oplus\mathbf{\Sigma} a_{ik}\cdot b_{kj}\},$$

where ($\oplus$) is the *n-ary circle hope* on the hyperaddition.

The rep problem by H_v-matrices is the following:

Definition 2.2 Let $(H,\cdot)$ be H_v-group, $(R,+,\cdot)$ be H_v-ring and $\mathbf{M}_R=\{(a_{ij}) \mid a_{ij}\in R\}$, then any

$$\mathbf{T}: H\to\mathbf{M}_R: h\to\mathbf{T}(h) \text{ with } \mathbf{T}(h_1h_2)\cap\mathbf{T}(h_1)\mathbf{T}(h_2)\neq\emptyset,\ \forall h_1,h_2\in H,$$

is called H_v-matrix rep. If $T(h_1h_2)\subset T(h_1)T(h_2)$, then T is an inclusion rep, if $T(h_1h_2)=T(h_1)T(h_2)$, then T is a good rep an induced rep T* for the hypergroup algebra is obtained. If T is one to one and good, then it is a faithful rep.

Theorem 2.3 A necessary condition in order to have an inclusion rep T of an H_v-group $(H,\cdot)$ by n×n H_v-matrices over the H_v-ring $(R,+,\cdot)$ is the following: For all $\beta^*(x)$, $x\in H$ there must exist elements $a_{ij}\in H$, $i,j\in\{1,\dots,n\}$ such that

$$T(\beta^*(a)) \subset \{A = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}),\ i,j\in\{1,\dots,n\}\}$$

Thus, every inclusion rep $T:H\to M_R: a\mapsto T(a)=(a_{ij})$ induces a homomorphic rep T* of H/β^* over R/γ^* by setting $T^*(\beta^*(a))=[\gamma^*(a_{ij})]$, $\forall\beta^*(a)\in H/\beta^*$, where the element $\gamma^*(a_{ij})\in R/\gamma^*$ is the ij entry of the matrix $T^*(\beta^*(a))$.

The rep problem by *Generalized Permutations* (write ***gp***), is described as follows [20]:

Definitions 2.4 Let X be a set, then a map $f:X\to P(X)-\{\emptyset\}$, is a *gp of X* if it is reproductive:

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$$\bigcup_{x \in X} f(x)=f(X)=X.$$

Denote by M_X the set of all gps on X. For an H_v-group $(X,\cdot)$ and $a \in X$, the gp f_a defined by $f_a(x)=ax$ is called *inner gp*. *Arrow* of f is any $(x,y) \in X^2$ with $y \in f(x)$. The $f_2 \in M_X$ *contains* $f_1 \in M_X$ or f_1 is a *sub-gp* of f_2, if $f_1(x) \subset f_2(x)$, $\forall x \in X$, then we write $f_1 \subset f_2$. If, moreover, $f_1 \neq f_2$, then f_1 is a *proper* sub-gp of f_2. An $f \in M_X$ is called *minimal* if it has no proper sub-gp. Denote $\underline{M}_X$ the set of all minimal gps of M_X. The *converse* of a gp f is the gp $\underline{f}$ defined by $\underline{f}(x)=\{z \in X: f(z) \ni x\}$, thus $\underline{f}$ is obtained by reversing arrows. We call *associated* to $f \in M_X$ the gp $f \circ \underline{f}$.

Theorem 2.5 Let $f \in M_X$, then $f \in \underline{M}_X$ iff, the following condition is valid:

If $a \neq b$ and $f(a) \cap f(b) \neq \emptyset$, then $f(a)=f(b)$ and $f(a)$ is singleton.

If $f \in \underline{M}_X$ then, $\underline{f} \in \underline{M}_X$. If $f \in \underline{M}_X$ then, $(f \circ \underline{f})(x)=\{y \in X: f(y)=f(x)\}$.

Several classes of H_v-structures can face special reps. Some of those classes are as follows [22], [24]:

Definition 2.6 Let $M=M_{m \times n}$, the set of m×n matrices on R and $P=\{P_i: i \in I\} \subseteq M$. We define, a kind of, P-hope $\underline{P}$ on M as follows

$$\underline{P}: M \times M \to P(M): (A,B)\ A\underline{P}B=\{AP^t_iB: i \in I\} \subseteq M$$

where P^t denotes the transpose of P. $\underline{P}$ is bilinear Rees' like operation where, instead of one sandwich matrix, a set is used. $\underline{P}$ is strong associative and the inclusion distributive to addition is valid:

$$A\underline{P}(B+C) \subseteq A\underline{P}B+A\underline{P}C,\ \forall A,B,C \in M$$

So $(M,+,\underline{P})$ defines a multiplicative hyperring on non-square matrices.

Let $M=M_{m \times n}$ be module of m×n matrices on R and take the sets

$$S=\{s_k: k \in K\} \subseteq R,\ \ Q=\{Q_i: j \in J\} \subseteq M,\ \ \ P=\{P_i: i \in I\} \subseteq M.$$

Define three hopes as follows

$$\underline{S}: R \times M \to P(M): (r,A) \to r\underline{S}A = \{(rs_k)A: k \in K\} \subseteq M$$

$$\underline{Q}_+: M \times M \to P(M): (A,B) \to A\underline{Q}_+B = \{A+Q_j+B: j \in J\} \subseteq M$$

$$\underline{P}: M \times M \to P(M): (A,B) \to A\underline{P}B = \{AP^t_iB: i \in I\} \subseteq M$$

Then $(M,\underline{S},\underline{Q}_+,\underline{P})$ is a hyperalgebra on R called *general matrix P-hyperalgebra*.

Hopes on any type of ordinary matrices can be defined [37], [8] they are called *helix hopes*.

Definition 2.7 Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$ be matrix and $s,t\in N$, with $1\leq s\leq m$, $1\leq t\leq n$. The *helix-projection* is a map

$$\underline{st}: \mathbf{M}_{m\times n}\to \mathbf{M}_{s\times t}: \ A\to A\underline{st}=(\underline{a}_{ij}),$$

where $A\underline{st}$ has entries

$$\underline{a}_{ij} = \{a_{i+\kappa s,j+\lambda t} \mid 1\leq i\leq s,\ 1\leq j\leq t \ \text{ and } \kappa,\lambda\in N,\ i+\kappa s\leq m,\ j+\lambda t\leq n\}$$

Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$, $s=\min(m,u)$, $t=\min(n,v)$. We define a hyper-addition, called ***helix-sum***, by

$$\oplus : \mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\mathbf{P}(\mathbf{M}_{s\times t}):(A,B)\to A\oplus B=A\underline{st}+B\underline{st}=(\underline{a}_{ij})+(\underline{b}_{ij})\subset\mathbf{M}_{s\times t}$$

where $(\underline{a}_{ij})+(\underline{b}_{ij})=\{(c_{ij})=(a_{ij}+b_{ij}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.
Let $A=(a_{ij})\in\mathbf{M}_{m\times n}$, $B=(b_{ij})\in\mathbf{M}_{u\times v}$ and $s=\min(n,u)$. Define the ***helix-product***, by

$$\otimes: \mathbf{M}_{m\times n}\times\mathbf{M}_{u\times v}\to\mathbf{P}(\mathbf{M}_{m\times v}): (A,B)\to A\otimes B=A\underline{ms}\cdot B\underline{sv}=(\underline{a}_{ij})\cdot(\underline{b}_{ij})\subset\mathbf{M}_{m\times v}$$

where $(\underline{a}_{ij})\cdot(\underline{b}_{ij})= \{(c_{ij})=(\sum a_{it}b_{tj}) \mid a_{ij}\in\underline{a}_{ij} \text{ and } b_{ij}\in\underline{b}_{ij})\}$.

The helix-sum is commutative, WASS, not associative. The helix-product is WASS, not associative and not distributive to the helix-addition.

We present a proof for the fundamental relation, analogous to Theorem 1.4, in the case of an H_v-module:

Theorem 2.8 Let $(\boldsymbol{M},+)$ be H_v-module over $\boldsymbol{R}$. Denote by U the set of expressions of finite hopes either on $\boldsymbol{R}$ and $\boldsymbol{M}$ or the external hope applied on finite sets. We define the relation $\boldsymbol{\varepsilon}$ in $\boldsymbol{M}$ by: $x\boldsymbol{\varepsilon}y$ iff $\{x,y\}\subset u$, $u\in U$. Then the relation $\boldsymbol{\varepsilon}^*$ is the transitive closure of the relation $\boldsymbol{\varepsilon}$.

Proof. Let $\underline{\boldsymbol{\varepsilon}}$ be the transitive closure of $\boldsymbol{\varepsilon}$, and denote by $\underline{\boldsymbol{\varepsilon}}(x)$ the class of the element x. First we prove that the quotient set $\boldsymbol{M}/\underline{\boldsymbol{\varepsilon}}$ is a module over $\boldsymbol{R}/\gamma^*$.
In $\boldsymbol{M}/\underline{\boldsymbol{\varepsilon}}$ the sum ($\oplus$) and the external product ($\otimes$), using the γ^* classes in $\boldsymbol{R}$, are defined in the usual manner:

$$\underline{\boldsymbol{\varepsilon}}(x)\oplus\underline{\boldsymbol{\varepsilon}}(y) = \{\underline{\boldsymbol{\varepsilon}}(z): z\in\underline{\boldsymbol{\varepsilon}}(x)+\underline{\boldsymbol{\varepsilon}}(y)\},\ \forall x,y\in\boldsymbol{M}.$$

$$\gamma^*(a)\otimes\underline{\boldsymbol{\varepsilon}}(x) = \{\underline{\boldsymbol{\varepsilon}}(z): z\in\gamma^*(a)\cdot\underline{\boldsymbol{\varepsilon}}(x)\},\ \forall a\in\boldsymbol{R},\ \forall x,y\in\boldsymbol{M}.$$

Take $x'\in\underline{\boldsymbol{\varepsilon}}(x)$, $y'\in\underline{\boldsymbol{\varepsilon}}(y)$. Then $x'\underline{\boldsymbol{\varepsilon}}x$ iff $\exists x_1,\dots,x_{m+1}$ with $x_1=x'$, $x_{m+1}=x$ and $u_1,\dots,u_m\in U$ such that $\{x_i, x_{i+1}\}\subset u_i$, $i=1,\dots,m$ and $y'\underline{\boldsymbol{\varepsilon}}y$ iff $\exists y_1,\dots,y_{n+1}$ with $y_1=y'$, $y_{n+1}=y$ and $v_1,\dots,v_n\in U$ such that $\{y_j,y_{j+1}\}\subset v_j, j=1,\dots,n$.

From the above we obtain

$$\{x_i, x_{i+1}\}+y_1\subset u_1+v_1,\ i=1,\dots,m-1,\quad x_{m+1}+\{y_j, y_{j+1}\}\subset u_m+v_j,\ j=1,\dots,n.$$

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The $u_i+v_1=t_i$, $i=1,\dots,m-1$, $u_m+v_j=t_{m+j-1}$, $j=1,\dots,n$ are elements of $\boldsymbol{U}$, therefore, $t_k\in\boldsymbol{U}$, $\forall k\in\{1,\dots,m+n-1\}$. Take elements $z_1,\dots,z_{m+n}$ with $z_i\in x_i+y_1$, $i=1,\dots,n$ and $z_{m+j}\in x_{m+1}+y_{j+1}$, $j=1,\dots,n$, thus, $\{z_k,z_{k+1}\}\subset t_k$, $k=1,\dots,m+n-1$.

Therefore, $\forall z_1\in x_1+y_1=x'+y'$ is $\underline{\boldsymbol{\varepsilon}}$ equivalent to $z_{m+n}\in x_{m+1}+y_{n+1}=x+y$. Thus $\underline{\boldsymbol{\varepsilon}}(x)\oplus\underline{\boldsymbol{\varepsilon}}(y)$ is a singleton so we can write, $\underline{\boldsymbol{\varepsilon}}(x)\oplus\underline{\boldsymbol{\varepsilon}}(y)=\underline{\boldsymbol{\varepsilon}}(z)$, $\forall z\in\underline{\boldsymbol{\varepsilon}}(x)+\underline{\boldsymbol{\varepsilon}}(y)$.

Similarly, using the properties of γ^* in $\boldsymbol{R}$, we prove that

$$\gamma^*(a)\otimes\underline{\boldsymbol{\varepsilon}}(x)=\underline{\boldsymbol{\varepsilon}}(z),\ \forall z\in\gamma^*(a)\cdot\underline{\boldsymbol{\varepsilon}}(x)$$

The WASS and the weak distributivity on $\boldsymbol{R}$ and $\boldsymbol{M}$ guarantee that the associativities and the distributivity are valid for $\boldsymbol{M}/\underline{\boldsymbol{\varepsilon}}$ over $\boldsymbol{R}/\gamma^*$. Therefore, $\boldsymbol{M}/\underline{\boldsymbol{\varepsilon}}$ is a module over $\boldsymbol{R}/\gamma^*$.

Now let σ equivalence relation in $\boldsymbol{M}$ such that $\boldsymbol{M}/\sigma$ is module on $\boldsymbol{R}/\gamma^*$. Denote $\sigma(x)$ the class of x. Then, $\sigma(x)\oplus\sigma(y)$ and $\gamma^*(a)\otimes\sigma(x)$ are singletons $\forall a\in\boldsymbol{R}$ and $x,y\in\boldsymbol{M}$, i.e.

$$\sigma(x)\oplus\sigma(y)=\sigma(z),\ \forall z\in\sigma(x)+\sigma(y),\ \gamma^*(a)\otimes\sigma(x)=\sigma(z),\ \forall z\in\gamma^*(a)\cdot\sigma(x).$$

Thus we write, $\forall a\in\boldsymbol{R}$, $x,y\in\boldsymbol{M}$ and $\boldsymbol{A}\subset\gamma^*(a)$, $\boldsymbol{X}\subset\sigma(x)$, $\boldsymbol{Y}\subset\sigma(x)$

$$\sigma(x)\oplus\sigma(y)=\sigma(x+y)=\sigma(\boldsymbol{X}+\boldsymbol{Y}),\quad \gamma^*(a)\otimes\sigma(x)=\sigma(ax)=\sigma(\boldsymbol{A}\cdot\boldsymbol{X}).$$

By induction, extend these relations on finite sums and products. Thus, $\forall u\in\boldsymbol{U}$, we have $\sigma(x)=\sigma(u)$, $\forall x\in u$. Consequently $x'\in\boldsymbol{\varepsilon}(x)$ implies $x'\in\sigma(x)$, $\forall x\in\boldsymbol{M}$. But σ is transitively closed, so we obtain: $x'\in\underline{\boldsymbol{\varepsilon}}(x)$ implies $x'\in\sigma(x)$.

Therefore, $\underline{\boldsymbol{\varepsilon}}$ is the smallest equivalence on $\boldsymbol{M}$ such that $\boldsymbol{M}/\underline{\boldsymbol{\varepsilon}}$ is a module on $\boldsymbol{R}/\gamma^*$, i.e. $\underline{\boldsymbol{\varepsilon}}=\boldsymbol{\varepsilon}^*$. ∎

Recall that, an element is *single* if its fundamental class is singleton, so, in an H_v-group if s is single then $\beta^*(s)=\{s\}$. Denote S_H the set of singles. If $S_H\neq\varnothing$, then we can answer to the very hard problem, that is to find the fundamental classes. The following theorems are proved [22], [24], [35]:

Theorem 2.9 Let $(H,\cdot)$ be H_v-group and $s\in S_H\neq\varnothing$. Let $a\in H$, take an element $v\in H$ such that $s\in av$, then $\beta^*(a)=\{h\in H{:}hv=s\}$, and the core of H is $\omega_H=\{u\in H{:}us=s\}=\{u\in H{:}\ su=s\}$. Moreover, $sx=\beta^*(sx)$ and $xs=\beta^*(xs)$, $\forall x\in H$.

Important Conlusion. Two elements a,b are in the fundamental relation β if there are two elements x,y who bring a,b in the relation β. That means that the fundamental relation β^* 'depend' on the results. This fact leads to a special proof where we need to discover the 'reason' to have the results. Every relation needs even the last one result to characterize its classes. However, if there are special elements, as the singles, which are strictly formed and carry inside them the relation, then these elements form the fundamental classes.

3. LIE-SANTILLI HYPER-ADMISSIBILITY

Last decade hyperstructures have applications in mathematics and in other sciences. These applications include biomathematics – conchology and inheritance - hadronic physics, lepton physics, and Santilli's iso-theory, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; consequently, it can be widely applied in linguistic, sociology, industry and production [1], [3], [4], [7], [9], [18], [28], [29], [31], [34], [36].

In [30], with '*The Santilli's theory 'invasion' in hyperstructures',* there is a first description on how Santilli's theories effect in hyperstructures and how new theories in Mathematics appeared by Santilli's pioneer research. In 1996 Santilli & Vougiouklis [14], point out that in physics the most interesting hyperstructures are the one called e-hyperstructures. These hyperstructures contain a unique left ant right scalar unit, which is an important tool in Lie-Santilli theory. One can see the books and related papers for more definitions and results related topics: [5], [6], [9], [11], [12], [13], [14], [15], [26], [30].

Definition 3.1 A hyperstructure $(F,+,\cdot)$, where $(+)$ is an operation and $(\cdot)$ a hope, is called *e-hyperfield* if the following axioms are valid: $(F,+)$ is an abelian group with the additive unit 0, $(\cdot)$ is WASS, $(\cdot)$ is weak distributive with respect to $(+)$, 0 is absorbing element: $0\cdot x=x\cdot 0=0$, $\forall x\in F$, there exist a multiplicative scalar unit 1, i.e. $1\cdot x=x\cdot 1=x$, $\forall x\in F$, and $\forall x\in F$ there exists a unique inverse x^{-1}, such that

$$1\in x\cdot x^{-1}\cap x^{-1}\cdot x.$$

The elements of an e-hyperfield are called *e-hypernumbers*. If the relation: $1=x\cdot x^{-1}=x^{-1}\cdot x$, is valid, then we say that we have a *strong e-hyperfield.*

Definition 3.2 *The Main e-Construction.* Given a group $(G,\cdot)$, where e is the unit, then we define in G, a huge number of hopes $(\otimes)$ as follows:

$$x\otimes y=\{xy, g_1, g_2,\ldots\},\ \forall x,y\in G-\{e\},\text{ and } g_1, g_2,\ldots\in G-\{e\}.$$

The $(G,\otimes)$ is an e-hypergroup, which is an H_b-group because it contains $(G,\cdot)$. Moreover, if $\forall x,y$ with $xy=e$, so $x\otimes y=xy$, then $(G,\otimes)$ is strong e-hypergroup.

The proof is immediate since we enlarge the results of the group by putting elements from G and applying the Little Theorem. Moreover, the unit e is unique scalar and $\forall x\in G$, there exists a unique inverse x^{-1}, such that $e\in x\cdot x^{-1}\cap x^{-1}\cdot x$. Finally, if the last condition of the difinition is valid, then $e=x\cdot x^{1}=x^{-1}\cdot x$, so the $(G,\otimes)$ is a strong e-hypergroup.

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Remark that the main e-construction gives an extremely large class of e-hopes. The most useful are the ones where only few products are enlarged and, even more, the extra elements are one or two.

Example 3.3 Consider the quaternion group $\boldsymbol{Q}=\{1,-1, i,-i, j,-j, k,-k\}$ with $i^2=j^2=k^2=-1$, $ij=k, jk=i, ki=j$. Denoting $\underline{i}=\{i,-i\}$, $\underline{j}=\{j,-j\}$, $\underline{k}=\{k,-k\}$ we may define a very large number ($*$) of hopes by enlarging only few products. For example, $(-1)*k=\underline{k}$, $k*i=\underline{j}$ and $i*j=\underline{k}$. Then the $(Q,*)$ is a strong e-hypergroup.

An important new field in hypermathematics comes straightforward from Santilli's Admissibility. We can transfer Santilli's theory in admissibility for representations in two ways: using either, the ordinary matrices and a hope on them, or using hypermatrices and ordinary operations on them [11], [12], [13], [15], [30].

The general definition is the following:

Definition 3.4 Let $\boldsymbol{L}$ be H_v-vector space over the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\to\boldsymbol{F}/\gamma^*$, the canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of the fundamental field $\boldsymbol{F}/\gamma^*$. Let ω_L be the core of the canonical map $\varphi':\boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Take two subsets $\boldsymbol{R}, \boldsymbol{S}\subseteq\boldsymbol{L}$ then a ***Lie-Santilli admissible hyperalgebra*** is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{RS}: \boldsymbol{L}\times\boldsymbol{L}\to\boldsymbol{P}(\boldsymbol{L}): [x,y]_{RS}= x\boldsymbol{R}y-y\boldsymbol{S}x= \{xry-ysx \mid r\in\boldsymbol{R},\ s\in\boldsymbol{S}\}$$

On non square matrices we can define admissibility as follow:

Construction 3.5 Let $(\boldsymbol{L}=\boldsymbol{M}_{m\times n},+)$ be H_v-vector space of m×n hyper-matrices on the H_v-field $(\boldsymbol{F},+,\cdot)$, $\varphi:\boldsymbol{F}\to\boldsymbol{F}/\gamma^*$, canonical map and $\omega_F=\{x\in\boldsymbol{F}:\varphi(x)=0\}$, where 0 is the zero of $\boldsymbol{F}/\gamma^*$. Similarly, let ω_L be the core of $\varphi':\boldsymbol{L}\to\boldsymbol{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\boldsymbol{L}/\varepsilon^*$. Take any two subsets $\boldsymbol{R},\boldsymbol{S}\subseteq\boldsymbol{L}$ then a *Santilli's Lie-admissible hyperalgebra* is obtained by taking the Lie bracket, which is a hope:

$$[\ ,\ ]_{RS}: \boldsymbol{L}\times\boldsymbol{L}\to\boldsymbol{P}(\boldsymbol{L}): [x,y]_{RS}=x\boldsymbol{R}^t y-y\boldsymbol{S}^t x.$$

Notice that $[x,y]_{RS}=x\boldsymbol{R}^t y-y\boldsymbol{S}^t x=\{xr^t y-ys^t x \mid r\in\boldsymbol{R} \text{ and } s\in\boldsymbol{S}\}$

Definition 3.6 According to Santilli's iso-theory [9], [13], [14], [15], [26], [30], on a field $F=(F,+,\cdot)$, *a general isofield* $\hat{\boldsymbol{F}}=\hat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is defined to be a field with elements $\hat{a}=a\times\hat{1}$, called *isonumbers*, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, equipped with two operations $\hat{+}$ and $\hat{\times}$ where $\hat{+}$ is the sum with the conventional additive unit 0, and $\hat{\times}$ is a new multiplication

$$\hat{a}\ \hat{\times}\ \hat{b} := \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\hat{\boldsymbol{F}} \qquad \text{(i)}$$

called *iso-multiplication*, for which $\hat{1}$ is the left and right unit of F⌈,

$$\hat{1}\,\hat{\times}\,\hat{a} = \hat{a}\times\hat{1} = \hat{a}\ ,\ \forall\hat{a}\in\widehat{\boldsymbol{F}} \qquad \text{(ii)}$$

called *iso-unit*. The rest properties of a field are reformulated analogously.

In order to transfer this theory into the hyperstructure case we generalize only the new multiplication $\hat{\times}$ from (i), by replacing with a hope including the old one. We introduce two general constructions on this direction as follows:

Construction 3.7 *General enlargement*. On a field $\boldsymbol{F}=(\boldsymbol{F},+,\cdot)$ and on the isofield $\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ we replace in the iso-product

$$\hat{a}\,\hat{\times}\,\hat{b}=\ \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1}$$

the element $\hat{T}$ by a set of elements $\hat{H}_{ab}=\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}$ where $\hat{x}_1,\hat{x}_2,\ldots\in\widehat{\boldsymbol{F}}$, containing $\hat{T}$, for all $\hat{a}\,\hat{\times}\,\hat{b}$ for which $\hat{a},\hat{b}\notin\{\hat{0},\hat{1}\}$ and $\hat{x}_1,\hat{x}_2,\ldots\in\widehat{\boldsymbol{F}}\text{-}\{\hat{0},\hat{1}\}$. If one of $\hat{a}$, $\hat{b}$, or both, is equal to $\hat{0}$ or $\hat{1}$, then $\hat{H}_{ab}=\{\hat{T}\}$. Therefore, the new iso-hope is

$$\hat{a}\,\hat{\times}\,\hat{b} = \hat{a}\times\hat{H}_{ab}\times\hat{b}= \hat{a}\times\{\hat{T},\hat{x}_1,\hat{x}_2,\ldots\}\times\hat{b},\ \forall\hat{a},\hat{b}\in\widehat{\boldsymbol{F}} \qquad \text{(iii)}$$

$\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ is an *iso*H_v*-field.* The elements of $\boldsymbol{F}$ are called *iso*H_v*-numbers* or *isonumbers*.

Remarks 3.8 Important hopes of this construction are those where the result is enlarged only for few ordered pairs $(\hat{a},\hat{b})$, even more, the extra elements $\hat{x}_i$, are only few, preferable exactly one. Thus, this special case is if there exists only one pair $(\hat{a},\hat{b})$ for which

$$\hat{a}\,\hat{\times}\,\hat{b}=\ \hat{a}\times\{\hat{T},\hat{x}\}\times\hat{b},\quad \forall\hat{a},\hat{b}\in\widehat{\boldsymbol{F}}$$

and the rest are ordinary results, then we have a so called *very thin iso*H_v*-field*.

The assumption that $\hat{H}_{ab}=\{\hat{T}\}$, $\hat{a}$ or $\hat{b}$, is equal to $\hat{0}$ or $\hat{1}$, with that $\hat{x}_i$, are not $\hat{0}$ or $\hat{1}$, give that the isoH_v-field has one scalar absorbing $\hat{0}$, one scalar $\hat{1}$, and $\forall\hat{a}\in\widehat{\boldsymbol{F}}$, has one inverse.

Construction 3.9 *The P-hope*. Consider an isofield $\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times})$ with $\hat{a}=a\times\hat{1}$, the isonumbers, where $a\in F$, and $\hat{1}$ is a positive-defined element generally outside F, with two operations $\hat{+}$ and $\hat{\times}$, where $\hat{+}$ is the sum with the conventional unit 0, and $\hat{\times}$ is the iso-multiplication

$$\hat{a}\,\hat{\times}\,\hat{b} := \ \hat{a}\times\hat{T}\times\hat{b}, \quad \text{with } \hat{1}=\hat{T}^{-1},\ \forall\hat{a},\hat{b}\in\widehat{\boldsymbol{F}}$$

Take a set $\hat{P}=\{\hat{T},\hat{p}_1,\ldots,\hat{p}_s\}$, with $\hat{p}_1,\ldots,\hat{p}_s\in\widehat{\boldsymbol{F}}\text{-}\{\hat{0},\hat{1}\}$, define the ***isoP-H$_v$-field***, $\widehat{\boldsymbol{F}}=\widehat{\boldsymbol{F}}(\hat{a},\hat{+},\hat{\times}_P)$, where the hope $\hat{\times}_P$ as follows:

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$$\hat{a}\ \hat{\times}_P\ \hat{b} := \begin{cases} \hat{a}\times\hat{P}\times\hat{b} = \{\hat{a}\times\hat{h}\times\hat{b} \mid \hat{h}\in\hat{P}\} & \text{if } \ \hat{a}\neq\hat{1} \ \text{ and } \ \hat{b}\neq\hat{1} \\ \hat{a}\times\hat{1}\times\hat{b} & \text{if } \ \hat{a}=\hat{1} \ \text{ or } \ \hat{b}=\hat{1} \end{cases} \qquad \text{(iv)}$$

The elements of $\hat{\boldsymbol{F}}$ are called *isoP-H_v-numbers*.

Remark. If $\hat{P} = \{\hat{1},\hat{p}\}$, that is that $\hat{P}$ contains only one $\hat{p}$ except $\hat{1}$. The inverses in isoP-H_v-fields, are not necessarily unique.

Consruction 3.10 The generalized P-construction can be applied on rings to obtain H_v-fields. Thus for, $\hat{\boldsymbol{Z}}_{10} = \boldsymbol{Z}_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, and if we take $\hat{P}=\{\underline{\hat{2}},\underline{\hat{7}}\}$, then we have the table

$\hat{\times}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$	$\underline{\hat{0}}$
$\underline{\hat{1}}$	$\underline{\hat{0}}$	$\underline{\hat{1}}$	$\underline{\hat{2}}$	$\underline{\hat{3}}$	$\underline{\hat{4}}$	$\underline{\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{9}}$
$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{6}}$
$\underline{\hat{3}}$	$\underline{\hat{0}}$	$\underline{\hat{3}}$	$\underline{\hat{2}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{6}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{4},\hat{9}}$
$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2}}$
$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{0}}$	$\underline{\hat{0},\hat{5}}$
$\underline{\hat{6}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{2}}$	$\underline{\hat{4}}$	$\underline{\hat{6}}$	$\underline{\hat{8}}$
$\underline{\hat{7}}$	$\underline{\hat{0}}$	$\underline{\hat{7}}$	$\underline{\hat{8}}$	$\underline{\hat{2},\hat{7}}$	$\underline{\hat{6}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{4}}$	$\underline{\hat{3},\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{1},\hat{6}}$
$\underline{\hat{8}}$	$\underline{\hat{0}}$	$\underline{\hat{8}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$	$\underline{\hat{0}}$	$\underline{\hat{6}}$	$\underline{\hat{2}}$	$\underline{\hat{8}}$	$\underline{\hat{4}}$
$\underline{\hat{9}}$	$\underline{\hat{0}}$	$\underline{\hat{9}}$	$\underline{\hat{6}}$	$\underline{\hat{4},\hat{9}}$	$\underline{\hat{2}}$	$\underline{\hat{0},\hat{5}}$	$\underline{\hat{8}}$	$\underline{\hat{1},\hat{6}}$	$\underline{\hat{4}}$	$\underline{\hat{2},\hat{7}}$

Then the fundamental classes are

$$(0)=\{\underline{\hat{0}},\underline{\hat{5}}\},\quad (1)=\{\underline{\hat{1}},\underline{\hat{6}}\},\quad (2)=\{\underline{\hat{2}},\underline{\hat{7}}\},\quad (3)=\{\underline{\hat{3}},\underline{\hat{8}}\},\quad (4)=\{\underline{\hat{4}},\underline{\hat{9}}\},$$

and the multiplicative table is the following

×	**(0)**	**(1)**	**(2)**	**(3)**	**(4)**
(0)	(0)	(0)	(0)	(0)	(0)
(1)	(0)	(1),(2)	(2),(4)	(3),(1)	(4),(3)
(2)	(0)	(2),(4)	(3)	(2)	(1)
(3)	(0)	(3),(1)	(2)	(3)	(4)
(4)	(0)	(4),(3)	(1)	(4)	(2)

Consequently, $\hat{\boldsymbol{Z}}_{10} = \boldsymbol{Z}_{10}(\underline{\hat{a}},\hat{+},\hat{\times})$, is an H_v-field.

4. CONCUSIONS

Anytime applied sciences ask from mathematics to have a model to express a new theory, then mathematicians search for existing mathematics if there is an appropriate one. If there is none, then they try to create a new mathematics to represent the required axioms. The H_v-structures can offer to the Lie-Santilli theory some of the models needed, because they are multivalued and there is a huge number of H_v-structures defined on the same set. Moreover, the appropriate H_v-fields can sometimes answer the basic question of Santilli's theory: What are the hypernumbers on which the entire theory is constructed via mere compatibility arguments.

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EPR argument and mystery of the reduced Planck's constant

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Abstract: According to Einstein, Podolsky and Rosen, 'quantum mechanics is incomplete' and it is popularly known as "EPR argument". R.M.Santilli is seriously working in this direction and trying to prove it. In this context, we would like to appeal that, when mass of any elementary particle is extremely small/negligible compared to macroscopic bodies, highly curved microscopic space-time can be addressed with large gravitational constants. Following this kind of approach, it is possible to show that, Reduced Planck's constant is a compactified coupling constant of electroweak gravity.

Keywords: Four gravitational constants; Electro weak Fermion; Reduced Planck's constant;

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Nomenclature	
1) Newtonian gravitational constant = G_N	18) Neutron life time= t_n
2) Electromagnetic gravitational constant= G_e	19) Weak interaction string tension= F_w
3) Nuclear gravitational constant = G_s	20) Strong interaction string tension= F_s
4) Weak gravitational constant = G_W	21) Electromagnetic interaction string tension= F_e
5) Fermi's weak coupling constant = G_F	22) Gravitational interaction string tension= F_g
6) Strong coupling constant = α_s	23) Weak interaction string potential= E_w
7) Mass of electroweak fermion = M_w	24) Strong interaction string potential = E_s
8) Reduced Planck's constant = $\hbar$	25) Electromagnetic interaction string potential = E_e
9) Speed of light = c	26) Gravitational interaction string potential = E_g
10) Elementary charge = e	27) Fine structure ratio = α
11) Strong nuclear charge = e_s	28) Nuclear fine structure ratio = α_n
12) Mass of proton = m_p	29) Mass of pions = $(m_\pi)^0, (m_\pi)^\pm$
13) Mass of neutron = m_n	30) Mass of weak bosons = $(m_z)^0, (m_W)^\pm$
14) Mass of electron = m_e	
15) Charge radius of nucleus= R_0	
16) Root mean square radius of proton = R_p	
17) Magnetic moment of proton = μ_p	

1. Introduction

As it is well known, Albert Einstein did not accept quantum mechanical uncertainties as being final, for which reason he made his famous quote "God does not play dice with the universe." More particularly, Einstein believed that "quantum mechanics is not a complete theory," in the sense that it could be broadened into such a form to recover classical determinism at least under limit conditions. Einstein communicated his views to B. Podolsky and N. Rosen and they jointly published in 1935 the historical paper [1] that became known as the EPR argument. In view of the rather widespread belief that quantum mechanics is a final theory valid for all conceivable conditions existing in the universe, objections against the EPR argument have been voiced by numerous scholars, including by N. Bohr [2], J. S. Bell [3,4], J. von Neumann [5] and others (see Ref. [6] for a review and comprehensive literature). The field became known as local realism and included the dismissal of the EPR argument based on claims that quantum axioms do not admit hidden variables λ [7, 8]. In this context, R.M.Santilli is seriously working and proving the 'EPR' argument based on 'isosymmetries' [9-13].

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Even though our approach is different, we would like to emphasize the point that, 'quantum mechanics' is certainly an incomplete theory because of its poor background associated with the mysterious origin of the 'reduced Planck's constant' and it is the root cause of failure of unification of 'quantum mechanics' and 'gravity'.

Subject of final unification is very interesting. But, unifying gravity and quantum mechanics (QM) is very much complicated and scientists are trying their level best in different ways. As gravitational effects are negligible at quantum level, standard model of particle physics attempts to explore the secrets of elementary particles. On the other hand, as quantum effects are negligible at macroscopic level, General theory of relativity (GTR) attempts to explore the secrets of the universe. The most complicated question to be answered is – If celestial objects are confirmed to be made up of various kinds of atoms, whether 'gravity' is causing the atoms to form into celestial spheres or quantum rules are causing the atoms to form into celestial spheres that show gravity?

Astrophysics point of view or 'Planck scale' point of view, there is a possibility of observing the combined effects of GTR and QM at intermediate energy scales. In between GTR and QM, there exist fascinating and most complicated astrophysical objects, i.e. Black holes. Even though their detection is a great mystery, one can see the best possibility of understanding QM and GTR at extreme energy scales. Here, we would like to emphasize the point that, astrophysical observations pertaining to Black holes and various other compact stellar objects just reveal the combined effects of GTR and QM but no way indicate the secrets of unification of QM and GTR. One most common point of QM and GTR is "mass". By understanding the massive origin of elementary particles, it may be possible to probe the secrets of QM and GTR.

The primary goal of quantum gravity is to join the laws of quantum mechanics with the laws of general relativity into a single mathematically consistent framework. Many scientists believe that, String theory [14,15,16] is one best candidate of quantum gravity. It is embedded with beautiful physical concepts like open strings, closed strings, string vibrations, string length, string tension and 'fermion-boson super symmetry'. Scientists strongly believe that, String theory is empowered with good mathematics and smartly fits gravity in unification program. Point to be noted is that, by considering the Planck length as characteristic amplitude associated with strings, String theory advances its ideological representation. Very unfortunate thing is that, even

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though, originally, String theory was proposed for understanding 'strong interaction, as Planck length is 20 orders of magnitude less than nuclear size, it is badly failing in explaining and predicting nuclear scale physical phenomena. Here we would like to stress the point that, the main reason for its fatal failure is – "implementation of the two famous physical constants H-bar and big G as-they-are". We would like to say that, without addressing the roots of H-Bar and big G, it is impossible to construct a workable model of final unification.

2. Three large atomic gravitational coupling constants

When mass of any elementary particle is extremely small/negligible compared to macroscopic bodies, highly curved microscopic space-time can be addressed with large gravitational constants and magnitude of elementary gravitational constant seems to increase with decreasing mass and increasing interaction range. Based on this logic, we consider the possibility of existence of three large gravitational constants assumed to be associated with the electromagnetic, strong and weak interactions [17-32]. Compared to multi-dimensions and unproved maths of any String theory model, our proposal can be given some positive consideration. Following the notion of string theory, compactification of un-observable spatial dimensions might be playing a key role in hiding the large magnitudes of the three atomic gravitational constants. If multi dimensional physics is having a real sense, then, compactification of large magnitudes of atomic gravitational constants can also be possible.

By following our idea, in analogy with Planck scale, as an immediate result, it seems possible to have three different string amplitudes corresponding to electromagnetic, strong and weak interactions. In this way, String theory can be shaped to a model of elementary particle physics associated with 3+1 dimensions. Another advantage is that, considering the combined effect of the three atomic gravitational constants, origins of H-Bar and big G can be understood. Including the Newtonian gravitational constant, as the subject under consideration deals with 4 different gravitational constants, our model can be called as 4G model of final unification or Microscopic Quantum Gravity. With further study, Planck scale and electroweak scale can be studied in a unified manner. During cosmic evolution, if one is willing to give equal importance to Higgs boson and Planck mass in understanding the massive origin of elementary particles, then

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it seems quite logical to expect a common relation between Planck scale and Electroweak scale.

3. Basic assumptions

1) There exists a characteristic electroweak fermion of rest energy, $M_w c^2 \cong 584.725$ GeV. It can be considered as the zygote of all elementary particles.
2) There exists a strong interaction elementary charge (e_s) in such a way that, its squared ratio with normal elementary charge is close to the reciprocal of the strong coupling constant.
3) Each atomic interaction is associated with a characteristic large gravitational coupling constant.

4. To validate assumption-1

To validate assumption-1, we argue with the following nuclear and particle level observations.

1) It is generally believed that, $(m_\pi)^0, (m_\pi)^\pm$ are the force carriers of strong interaction and $(m_z)^0, (m_w)^\pm$ are the force carriers of weak interaction. Considering Pions and electroweak bosons, to a great surprise, we noticed that,

$$\left(\frac{\sqrt{m_p m_n}}{M_w}\right) \cong 0.001606 \cong \left(\frac{\sqrt{\left(m_\pi c^2\right)^0 \left(m_\pi c^2\right)^\pm}}{\sqrt{\left(m_z c^2\right)^0 \left(m_w c^2\right)^\pm}}\right)$$
$$\cong \left(\frac{\sqrt{134.98 \times 139.57}\ \text{MeV}}{\sqrt{80379.0 \times 91187.6}\ \text{MeV}}\right) \cong 0.0016032. \tag{1}$$

2) It is also very interesting to note that,

$$\frac{\sqrt{m_p m_n}}{\sqrt{(m_\pi)^0 (m_\pi)^\pm}} \cong 6.84 \cong \frac{M_w}{\sqrt{(m_z)^0 (m_w)^\pm}} \cong 6.83. \tag{2}$$

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3) As neutron's weak decay is directly responsible for nuclear stability associated with beta emission, based on the two numerical coincidences, i.e. 0.0016 and 6.83, existence of our assumed 584.725 GeV weak fermion can be confirmed and it is also possible to have a relation of the form,

$$M_w \cong \left(\frac{\sqrt{(m_z)^0 (m_w)^{\pm}}}{\sqrt{(m_\pi)^0 (m_\pi)^{\pm}}} \right) m_p \cong 585.244 \text{ GeV}/c^2 . \quad (3)$$

4) With reference to nucleons and pions, it is reasonable to argue that, if one is willing to consider $\left(m_z c^2\right)^0 \& \left(m_w c^2\right)^{\pm}$ as the force carriers of weak interaction [33,34,35,36], one should not ignore the possibility of considering the proposed weak fermion of rest energy 584.725 GeV as the characteristic field generator of weak interaction.
5) Weak force carriers cannot exist without the existence of their weak field generating fermion.
6) We would like to emphasize that, independent of the famous semi empirical mass formula (SEMF), and using the ratio 0.0016, nuclear stability and nuclear binding energy can be understood with four simple terms and one energy coefficient. See section 7.

5. Semi empirical derivations

This section has been divided into 4 sub sections [26,29]. Based on the proposed second and third assumptions, in section 5.1, relations (3), (7) and (11) have been defined.

In Section 5.2 important numerical fitting relations (13), (14), (15) and (16) have been proposed (pertaining to nuclear charge radius, Planck size and Fermi's weak coupling constant) and an attempt has been made to infer an expression for weak gravitational constant.

In Section 5.3, based on the results obtained from Section 5.1 and Section 5.2, an important inference i.e. relation (21) has been made.

Section 5.4 includes simplified relations pertaining to elementary mass ratios, Newtonian gravitational constant and strong coupling constant.

5.1 Defined basic relations and their consequences

5.1.1 Ratio of Newtonian and electromagnetic gravitational constants

Considering the similarities in between gravitational and electromagnetic interactions, relation (3) has been defined to understand the role and to estimate the approximate magnitudes of the electromagnetic and Newtonian gravitational constants [37].

$$\left.\begin{aligned} m_p &\cong \left(\frac{G_N}{G_e}\right)^{\frac{1}{6}} \sqrt{M_{pl} \times m_e} \cong \left(\frac{G_N}{G_e}\right)^{\frac{1}{6}} \left(\frac{\hbar c m_e^2}{G_N}\right)^{\frac{1}{4}} \\ \text{where, } M_{pl} &\cong \sqrt{\frac{\hbar c}{G_N}} \cong \text{Planck mass} \end{aligned}\right\} \quad (4)$$

On rearranging relation (4),

$$M_{pl} \cong \sqrt{\frac{\hbar c}{G_N}} \cong \left(\frac{G_e}{G_N}\right)^{\frac{2}{3}} \left(\frac{m_p^2}{m_e}\right) \quad (5)$$

5.1.2 Proton – electron mass ratio

Pertaining to proton-electron mass ratio, relations (6) and (7) have been defined in the following way.

$$\frac{m_p}{m_e} \cong \left(\frac{G_e m_e^2}{\hbar c}\right)\left(\frac{G_s m_p^2}{\hbar c}\right) \quad (6)$$

$$\frac{m_p}{m_e} \cong \left(\frac{e_s^2}{4\pi\varepsilon_0 G_s m_p^2}\right) \div \left(\frac{e^2}{4\pi\varepsilon_0 G_e m_e^2}\right) \quad (7)$$

Based on the second assumption and relations (6) and (7),

$$\left(\frac{e_s}{e}\right)^2 \cong \frac{1}{\alpha_s} \cong \frac{G_s m_p^3}{G_e m_e^3} \cong \frac{G_s^2 m_p^4}{\hbar^2 c^2} \quad (8)$$

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$$\left(\frac{e_s}{e}\right) \cong \sqrt{\frac{1}{\alpha_s}} \cong \sqrt{\frac{G_s m_p^3}{G_e m_e^3}} \cong \frac{G_s m_p^2}{\hbar c} \tag{9}$$

Based on relation (8), quantitatively, it can be inferred that,

$$\sqrt{\frac{e_s^2}{4\pi\varepsilon_0 G_s m_p m_e}} \cong 2\pi \tag{10}$$

Based on relation (8), substituting $e_s^2 \cong \left(\frac{G_s m_p^3}{G_e m_e^3}\right) e^2$ in relation (10),

$$\frac{m_p}{m_e} \cong 2\pi\sqrt{\frac{4\pi\varepsilon_0 G_e m_e^2}{e^2}} \tag{11}$$

Based on relation (4) and (6), on eliminating $\hbar c$,

$$\frac{m_p}{m_e} \cong \left(\frac{G_s}{G_e^{1/3} G_N^{2/3}}\right)^{1/7} \tag{12}$$

5.2 Numerical fits and their consequences pertaining to nuclear charge radius, Planck size and Fermi's weak coupling constant

With reference to nuclear gravitational constant, nuclear charge radius can be fitted with,

$$R_0 \cong \left(\frac{2G_s m_p}{c^2}\right) \cong 1.2393 \text{ fm} \tag{13}$$

With reference to Planck size, it has been noticed that,

$$\left(\frac{G_s m_p}{c^2}\right) \div \sqrt{\frac{G_N \hbar}{c^3}} \cong \left(\frac{m_p}{m_e}\right)^6 \tag{14}$$

Based on relation (13), Fermi's weak coupling constant can be fitted with,

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$$G_F \cong \left(\frac{m_e}{m_p}\right)^2 \hbar c R_0^2 \cong \frac{4\hbar G_s^2 m_e^2}{c^3} \tag{15}$$
$$\cong 1.4402105\times10^{-62}\ \text{J.m}^3$$

Based on relations (13), (14) and (15),

$$G_F \cong \left(\frac{m_p}{m_e}\right)^{10} \frac{4\hbar^2 G_N}{c^2} \tag{16}$$

Based on the magnitude of weak gravitational constant proposed by Roberto Onofrio [21] and based on relation (16), it has been inferred that,

$$G_w \cong \left(\frac{m_p}{m_e}\right)^{10} G_N \tag{17}$$

Based on relations (16) and (17)

$$G_F \cong \frac{4\hbar^2 G_w}{c^2} \tag{18}$$

Based on relations (15) and (18),

$$\sqrt{\frac{G_F}{\hbar c}} \cong \frac{2G_s m_e}{c^2} \cong \sqrt{\frac{4G_w \hbar}{c^3}} \tag{19}$$

5.3 Important inference and its implications pertaining to first assumption

Based on the above relations (4) to (19), on eliminating the three proposed atomic gravitational constants, one can get the following relation.

$$\alpha \cong 4\pi^2 \left(\frac{m_e}{m_p}\right)^7 \sqrt{\frac{\hbar c}{G_N m_p^2}} \tag{20}$$

If one is willing infer that,

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$$\hbar c \cong G_w M_w^2 \tag{21}$$

Based on relations (19) and (21),

$$G_s m_e \cong G_w M_w \tag{22}$$

Based on relations (20), (21) and (22), the following relations can be obtained.

$$m_p \cong \left(\frac{4\pi^2}{\alpha}\right)^{\frac{1}{3}} \left(M_w m_e^2\right)^{\frac{1}{3}} \tag{23}$$

$$G_F \cong G_w M_w^2 R_w^2 \quad \text{where, } R_w \cong \frac{2G_w M_w}{c^2} \tag{24}$$

$$m_e \cong \left(\frac{G_w}{G_s}\right) M_w \tag{25}$$

$$m_p \cong \left(\frac{G_s}{G_w}\right)\left(\frac{G_s}{G_e}\right) M_w \cong \left(\frac{G_s^2}{G_w G_e}\right) M_w \tag{26}$$

$$\frac{m_p}{m_e} \cong \frac{G_s^3}{G_w^2 G_e} \tag{27}$$

$$\hbar c \cong \left(\frac{G_e G_w}{G_s}\right) m_p m_e \cong G_s M_w m_e \tag{28}$$

5.4 Simplified relations for elementary mass ratios, Newtonian gravitational constant and strong coupling constant

On eliminating proton and electron rest masses, Newtonian gravitational constant and strong coupling constant take the following simplified forms.

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$$G_N \cong \frac{G_w^{21} G_e^{10}}{G_s^{30}} \tag{29}$$

$$\frac{1}{\alpha_s} \cong \frac{G_s^{10}}{G_e^4 G_w^6} \tag{30}$$

Based on relations (29) and (30),

$$\alpha_s \cong \frac{G_N^{1/3} G_e^{2/3}}{G_w} \tag{31}$$

6. Characteristic unified relations pertaining to estimation of (G_e, G_s, G_w, G_N)

a) With the following relation, magnitude of G_e can be estimated.

$$G_e \cong \frac{e^2 m_p^2}{16\pi^3 \varepsilon_0 m_e^4} \tag{32}$$

b) After finding the value of G_e ,with the following relation, magnitude of G_s can be estimated.

$$G_s \cong \frac{G_w^2 M_w^4}{G_e m_p m_e^3} \cong \frac{\hbar^2 c^2}{G_e m_p m_e^3} \tag{33}$$

c) After finding the value of G_s , weak gravitational constant can be estimated with a relation of the form,

$$G_w \cong \sqrt{\left(\frac{m_e}{m_p}\right)\frac{G_s^3}{G_e}} \tag{34}$$

d) Thus, quantitatively,

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$$\left.\begin{array}{l} G_e \cong 2.374335\times10^{37}\ \text{m}^3\text{kg}^{-1}\text{sec}^{-2} \\ G_s \cong 3.329561\times10^{28}\text{m}^3\text{kg}^{-1}\text{sec}^{-2} \\ G_w \cong 2.909745\times10^{22}\ \text{m}^3\text{kg}^{-1}\text{sec}^{-2} \\ G_N \cong 6.679855\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{sec}^{-2} \\ G_F \cong 1.4402105\times10^{-62}\ \text{J.m}^3 \\ \alpha_s \cong 0.1151937 \text{ and } e_s \cong 2.9463591e \end{array}\right\}$$

e) Based on relation (25), we are working on developing procedures for estimating the magnitude of strong gravitational constant and weak gravitational constant independent of the reduced Planck's constant. Appropriate relations seem to be associated with the experimental values of strong coupling constant [38,39], nuclear charge radius [40,41,42,43], magnetic moment of proton and neutron life time [44].

$$\alpha_s \cong \frac{G_e m_e^3}{G_s m_p^3} \approx 0.1152 \tag{35}$$

$$R_0 \cong \frac{2G_s m_p}{c^2} \approx 1.24 \text{ fm.} \tag{36}$$

$$\mu_p \cong \frac{eG_s m_p}{2c} \approx 1.487\times10^{-26}\ \text{J.Tesla}^{-1} \tag{37}$$

$$t_n \cong \left(\frac{G_e^2 m_n^2}{G_w\left(m_n - m_p\right)c^3}\right) \approx 874.94 \text{ sec} \tag{38}$$

7. Understanding nuclear binding energy with single energy coefficient and four simple terms

We would like to emphasize the fact that, physics and mathematics associated with fixing of the energy coefficients of semi empirical mass formula (SEMF) [45,46,47] are neither connected with residual strong nuclear force nor connected with strong coupling constant α_s. Since nuclear force is mediated via quarks and gluons, it is necessary and compulsory to study the nuclear binding energy scheme in terms of nuclear coupling constants. In this direction, N. Ghahramany and team members have taken a great initiative in exploring the secrets of nuclear binding energy and magic numbers [48,49] with reference to quarks. Very interesting point of their study is that - nuclear binding energy can be understood with two or three terms having single variable energy coefficient. In this direction, based on three unified assumptions connected with gravity and atomic interactions, in a semi empirical approach, we have developed a very simple formula for nuclear binding energy with single energy coefficient having four simple terms [27]. Corresponding relations can be expressed in the following way. Starting from Z=3 to 118,

$$A_s \cong 2Z + 0.0016(2Z)^2 \cong 2Z + 0.0064Z^2 \tag{39}$$

$\cong$ Estimated mass number close to proton-neutron mean stability line.

$$BE \cong \left\{ A - A_{fg} - A^{1/3} - \frac{(A_s - A)^2}{A_s} \right\} (B_0 \cong 10.1 \text{ MeV}) \tag{40}$$

$\cong$ Estimated nuclear binding energy

Here, we would like to appeal that,

1) $A_{fg} \cong \left(1 + 0.0019 A\sqrt{ZN}\right)$ can be called as the geometric number of free or unbound nucleons.
2) $A^{1/3}$ can be called as radial factor associated with nucleons.
3) $\frac{(A_s - A)^2}{A_s}$ can be called as isotopic asymmetric term associated with mean stable mass number.
4) Binding energy coefficient, $B_0 \cong \frac{1}{\alpha_s}\left(\frac{e^2}{4\pi\varepsilon_0 R_0}\right) \cong 10.1 \text{ MeV}$ seems to be associated with nuclear radius, strong coupling constant and fine structure ratio.
5) Proceeding further, by considering electroweak interaction and eliminating the number 0.0019, it is possible to show that,

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$$BE \cong \left(A - A^{1/3} - \frac{(A_s - A)^2}{A_s} \right) 10.1 \text{ MeV} - \left[1 + \left(0.00161 A \sqrt{ZN} \right) \right] 11.9 \text{ MeV} \tag{41}$$

8. Estimating nuclear charge radii independent of quantum concepts

Without considering quantum concepts, nuclear charge radii can be estimated with the following expression. For medium and heavy atomic nuclides,

$$R_{(Z,N)} \cong \left[Z^{1/3} + \left(\sqrt{ZN} \right)^{1/3} \right] \left(\frac{G_s \sqrt{m_p m_n}}{c^2} \right) \tag{42}$$

This relation can be compared the currently believed various relations pertaining to the estimation of nuclear charge radii [50,51]

9. Discussion

We would like to emphasise the following points:

1) Even though quantum mechanics is successful in understanding the quantum effects of microscopic systems, origin of the reduced Planck's constant is still a mystery at the microscopic level.
2) String theory is silent on the universal applicability of the reduced Planck's constant.
3) During cosmic evolution, if one is willing to give equal importance to Higgs Boson and Planck mass in understanding the massive origin of elementary particles, it seems quite logical to expect a common relation between Planck scale and Electroweak scale.
4) When microscopic space time is more curved than the macroscopic space time curvature, it is natural to assign a large value to microscopic gravitational constant.
5) Compared to particles having a structure, for point particles the magnitude of gravitational constant can be much higher.
6) Magnitude of the elementary gravitational constant seems to increase with the decreasing mass of the elementary particle under consideration.
7) According to the String theory, the real world is a compact manifold and out of 10 dimensions, 6 spatial dimensions get compressed and will not allow any observer to identify their existence. Applying this idea to our

proposal, compactification of 6 unobservable space dimensions might be playing a key role in hiding the large magnitudes of the three atomic gravitational constants.

8) Using the strong nuclear charge, proton magnetic moment $\left(e_s\hbar/2m_p\right)$, nuclear fine structure ratio $\alpha_n \cong \left(e_s^2/4\pi\varepsilon_0\hbar c\right)$, unified nuclear binding energy coefficient $B_0 \cong \frac{1}{2}\sqrt{\alpha\times\alpha_n}\left(m_p c^2\right)$ and Fermi gas model of nuclear potential $E_F \cong \sqrt{\alpha\times\alpha_n}\left(m_p c^2 + m_n c^2\right)$ can be fitted. Another interesting application is that, based on strong charge conservation, electromagnetic charge conservation and super symmetry, fractional charge quarks can be understood with generation of quark fermions and quark bosons [29,30,31,32].

9) 'String Tension' is a practical aspect of String Theory [52]. Considering the proposed three atomic gravitational constants and following the universal applicability of 'speed of light', approximate tensions associated with weak, strong, electromagnetic and gravitational interactions can be represented by,

$$\left\{\begin{array}{l} F_w \cong \left(\dfrac{c^4}{4G_w}\right) \cong 6.94\times10^{10}\,\text{N} \\ F_s \cong \left(\dfrac{c^4}{4G_s}\right) \cong 6.065\times10^{4}\,\text{N} \\ F_e \cong \left(\dfrac{c^4}{4G_e}\right) \cong 8.505\times10^{-5}\,\text{N} \\ F_g \cong \left(\dfrac{c^4}{4G_N}\right) \cong 3.026\times10^{43}\,\text{N} \end{array}\right\} \tag{43}$$

10) Following the universal applicability of 'elementary charge', approximate (operating) energy potentials associated with the above string tensions can be represented by,

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$$\left\{\begin{array}{l} E_w \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{c^4}{4G_w}\right)} \cong 25.0 \text{ GeV} \\ E_s \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{c^4}{4G_s}\right)} \cong 23.3 \text{ MeV} \\ E_e \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{c^4}{4G_e}\right)} \cong 874 \text{ eV} \\ E_g \cong \sqrt{\frac{e^2}{4\pi\varepsilon_0}\left(\frac{c^4}{4G_N}\right)} \cong 8.355\times 10^7 \text{ J} \end{array}\right\} \tag{44}$$

11) These estimated weak, strong and electromagnetic energy potentials seem to be close to experimental values.
12) Relation (21) needs in depth discussion at fundamental level.
13) With reference to the current experimental values of root mean square radius of proton, (0.833 ± 0.01) fm and $(0.831 \pm 0.007_{stat} \pm 0.012_{syst})$ fm, we noticed one interesting relation. It can be expressed as,

$$R_p \cong \sqrt{\left(\frac{4\pi\varepsilon_0\hbar^2}{e_s^2 m_p}\right)\left(\frac{\hbar}{m_p c}\right)} \cong \sqrt{\frac{4\pi\varepsilon_0\hbar^3}{e_s^2 m_p^2 c}} \cong 0.835 \text{ fm} \tag{45}$$

In this relation,

a) $\left(\frac{4\pi\varepsilon_0\hbar^2}{e_s^2 m_p}\right) \cong 3.32$ fm can be inferred as the Bohr's model of probable distance of finding proton in the nuclear well where the operating charge is $e_s \cong 2.946e$.

b) $\left(\frac{\hbar}{m_p c}\right) \cong 0.21$ fm can be considered as the reduced Compton wavelength of proton.

Based on relation (45) and assumption(2),

$$\hbar \cong \left[\left(\frac{e_s^2}{4\pi\varepsilon_0 c}\right)\left(m_p R_p c\right)^2\right]^{\frac{1}{3}} \tag{46}$$

10. Conclusion

With reference to pions and electroweak bosons, it is possible to confirm the existence of $M_w c^2 \cong 584.725$ GeV. Proceeding further, based on the famous EPR argument and with further research, actual essence of final unification and mystery of the reduced Planck's constant can be understood.

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Measurements of the Polarization Correlation of the Two-Photon System Produced in Positron-Electron Annihilation

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Abstract

We present measurements of the polarization correlation of the two-photonsystem produced in positron-electron annihilation of ^{64}Cu that were conducted at Freiburg University 1976-1980. Our experiment was motivated by two contradicting results. An experiment conducted at Columbia University observed a quantum mechanical correlation whereas an experiment conducted in Catania, Sicily observed a correlation that was about 40% lower. The two back-to-back photons are either both right-handed or left-handed polarized. The polarization of each photon is measured with a Compton polarimeter on each side of the source consisting of a plastic scintillator as scatterer and a NaI detector, which records the Compton-scattered photons.The polarization correlation is measured by the difference in azimuth angle ϕ between the two polarimeters. We performed the measurements for different scatterer shapes, different scattering angles and different distances between the source and the polarimeters. We developed a detailed MonteCarlo program for simulating the quantum mechanical expectation for each measurement setup. All measurements agree rather well with quantum mechanics. We further reproduced the results of all conducted experiments with our simulation. The reduced Polarization correlation observed by the Sicilian experiment originated from a large fraction of double scattering in which the original polarization correlation is diminished. We further performed a test of Bells inequality with our results and those measured by other experiments.

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1 Introduction

In the early 1970s two experiments were conducted that measured the polarization correlation of the two annihilation photons from electron-positron annihilation. The motivation came from the EPR article [1] and Bell's investigation into hidden variables [2, 3]. The experiment by Kasday-Ulman-Wu (Columbia U., 1975) [4] measured a polarization correlation that confirmed the quantum mechanical prediction. The experiment by Faraci et al. (Catania U., 1974) [5] measured a value that was 30% lower than the quantum mechanical expectation. The discrepancy could not be explained. Therefore, K. Meisenheimer and myself under supervision of professor Runge built a new experiment for measuring the polarization correlation of the two annihilation photons from at the University of Freiburg. The experiment was conducted like a modern high-energy physics experiment with respect to data taking, analysis and simulation.

2 Measurement Principle

Positrons from a ^{64}Cu source annihilate with electrons in the conduction band typically in an S=0 state into two nearly back-to-back photons (HWHM=5.9$\pm$ 0.1 mrad), which are either both left-handed or right-handed polarized. Thus, the state vector in terms of momentum $\vec{k}_{\rm i}$ and polarization $\epsilon_{\rm i} = R_{\rm i},\ L_{\rm i}$ is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big\{\ |k_1, R_1\rangle\, |k_2, R_2\rangle - |k_1, L_1\rangle\, |k_2, L_2\rangle\ \Big\}. \tag{1}$$

In terms of linear polarization we get

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big\{\ |k_1, X_1\rangle\, |k_2, Y_2\rangle - |k_1, Y_1\rangle\, |k_2, X_2\rangle\ \Big\}. \tag{2}$$

The polarization is detected via Compton scattering. The probability that a photon polarized in the X direction is scattered under the angles θ and ϕ is

$$dP_{\rm i} = \frac{1}{2\pi} f(\theta_{\rm i})\Big[1 - m(\theta_{\rm i})\Big]\cos(2\phi_{\rm i})d\Omega \tag{3}$$

where $f(\theta_{\rm i})$ is the differential cross section for unpolarized photons $f(\theta_{\rm i})$

$$f(\theta_{\rm i}) = \frac{1}{C_{\rm N}}\Big(\frac{k'_{\rm i}}{k_{\rm i}}\Big)^2\Big[\frac{k_{\rm i}}{k'_{\rm i}} + \frac{k'_{\rm i}}{k_{\rm i}} - \sin^2\theta\Big] \tag{4}$$

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and $m(\theta_\mathrm{i})$ is the θ_i-dependent amplitude that describes the strength of the ϕ_i dependence and, therefore, the polarization sensitivity,

$$m(\theta_\mathrm{i}) = \sin^2\theta_\mathrm{i}\Big[\frac{k_\mathrm{i}}{k'_\mathrm{i}} + \frac{k'_\mathrm{i}}{k_\mathrm{i}} - \sin^2\theta\Big]^{-1}. \tag{5}$$

The parameter C_N is a normalization constant. The maximum polarization sensitivity is obtained for $\theta = 82°$ yielding $m(82°) = 0.691$ while $f(82°)/f(0°) = 0.203$ for $\theta = 82°$. So, $f(\theta)$ is nearly at the minimum, For fixed θ, $dP/d\Omega$ becomes maximum if the initial polarization is orthogonal to the scattering plane ($\phi = 90°$) and minimum if it is in the scattering plane ($\phi = 0°$). The analyzing power of a polarimeter is defined by

$$\epsilon_\mathrm{P} = \frac{P_\mathrm{max} - P_\mathrm{min}}{P_\mathrm{max} + P_\mathrm{min}} \tag{6}$$

where P_max is the maximum of $dP/d\Omega$ if $\vec{\epsilon}$ is parallel to the polarimeter axis and P_min is the minimum of $dP/d\Omega$ if $\vec{\epsilon}$ is orthogonal to the polarimeter axis. For a Compton polarimeter we get

$$\epsilon_\mathrm{CP} = M(\theta, \Delta\theta)\cdot N(\Delta\phi) \tag{7}$$

with

$$M(\theta,\Delta\theta) = \frac{\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} f(\theta)m(\theta)\sin\theta d\theta}{\int_{\theta-\Delta\theta}^{\theta+\Delta\theta} f(\theta)\sin\theta d\theta} \tag{8}$$

and

$$N(\Delta\phi) = \frac{\int_{\phi-\Delta\phi}^{\phi+\Delta\phi}\cos(2\phi)d\phi}{\int_{\phi-\Delta\phi}^{\phi+\Delta\phi} d\phi} = \frac{\sin(2\Delta\phi)}{2\Delta\phi} \tag{9}$$

where $\Delta\theta$ and $\Delta\phi$ are the polar angle and azimuth angle acceptances of the detector.

For example, for $\theta = 82°$, $\Delta\theta = 14°$ and $\Delta\phi = 10°$ the analyzing power is $\epsilon_\mathrm{CP} = 0.648 \pm 0.002$, which is 94% of the maximum analyzing power $\epsilon_\mathrm{CP}^\mathrm{max}$.

The detection efficiency of a polarimeter is defined by

$$\eta_\mathrm{P} = \frac{\text{number of detected particles}}{\text{number of particles hitting the polarimeter}} \tag{10}$$

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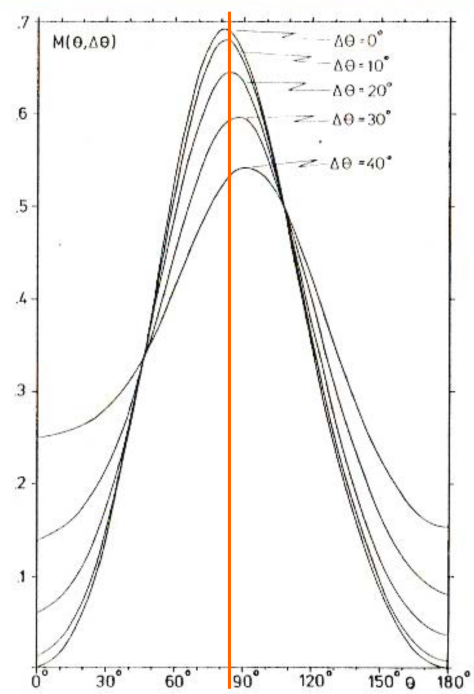

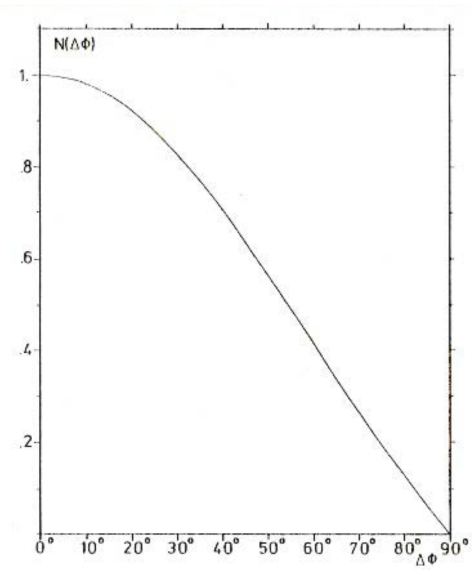

Figure 1: Left: The function $M(\theta, \Delta\theta)$ as a function of θ for different values of $\Delta\theta$. Right: The function $N(\Delta\phi)$ as a function of $\Delta\phi$.

For an ideal polarimeter $\eta_{\rm P} = 1$. For a Compton polarimeter

$$\eta_{\rm CP} = \eta_{\rm S} \cdot \delta_{\rm S} \cdot \eta_{\rm D} \cdot \delta_{\rm D} \cdot F(\theta, \Delta\Omega) \qquad (11)$$

where $\eta_{\rm S}$ is the scattering probability in the scatterer, $\eta_{\rm D}$ is the probability to record a signal in the detector, $\delta_{\rm S}$ is the detection limit of the scatterer, $\delta_{\rm D}$ is the detection limit of the detector and $F(\theta, \Delta\Omega) = (1/2\pi)\int_{\Delta\Omega} f(\theta) d\Omega$. For the $\theta = 82°$, $\Delta\theta = 14°$ and $\Delta\phi = 10°$ we get $\eta_{\rm CP} = (1.46 \pm 0.02) \times 10^{-3}$.

The probability that two photons represented by the state $|\psi\rangle$ are scattered under the polar angles θ_1 and θ_2 and azimuth angles ϕ_1 and ϕ_2 is given by

$$dP_{12} = \frac{1}{4\pi} f(\theta_1) f(\theta_2)\{1 - m(\theta_1)m(\theta_2)] \cos 2(\phi_1 - \phi_2) d\Omega_1 d\Omega_2\} \qquad (12)$$

The flight directions of the two photons after scattering are correlated with respect to the azimuth angles with a strength of $m(\theta_1)m(\theta_2)$, which has a maximum value of 0.478. The two scattering angles are completely independent.

The product $f(\theta_1)f(\theta_2)$ denotes the probability for scattering two arbitrary polarized photons under the angles θ_1 and θ_2.

3 Experimental Setup

Figure 2 shows the experimental setup. In the center a ^{64}Cu is placed inside a collimator. The two Compton polarimeters are placed on both sides of the collimator. Each polarimeter consists of a plastic scintillator as scatterer and a NaI counter covered with a lead shield as detector. The plastic scintillator and the NaI crystal are both read out by a photomultiplier. For most measurements the NaI detector is placed at the optimal scattering angle of $\theta = 82°$. We measured four coincidence numbers: N_2 the coincidence between the two scatterers, N_{31}, N_{32} the coincidence between the two scatterers and detector 1, 2 and N_4 the coincidence between all four counters. The three- and four-fold coincidences have azimuth angle dependences.

Figure 3 (top) shows a photograph of the experiment. We chose the gravitational axis as symmetry axis. Figure 3 (bottom) shows a close up view of the Compton polarimeter. The lead shield in front of the detector permits to define the acceptance in $\Delta\theta$ and $\Delta\phi$.

4 Polarization Correlation

We define the polarization correlation $R(\theta_1, \theta_2)$ in terms of coincidence numbers to reduce systematic uncertainties from the source intensity, alignment and calibration issues. Thus

$$R(\theta_1, \theta_2) = \frac{N_4(\phi_1, \phi_2)/N_2}{N_{31}(\phi_1)/N_2 \cdot N_{32}(\phi_2)/N_2} = \frac{N_4(\phi_1, \phi_2) \cdot N_2}{N_{31}(\phi_1) \cdot N_{32}(\phi_2)} \tag{13}$$

In the laboratory system there is no preferred polarization direction. The 3-fold coincidences $N_{31}(\phi_1)$ and $N_{32}(\phi_2)$ are isotropic in ϕ_1 and ϕ_2. The angular dependence comes from $N_4(\phi_1, \phi_2)$, yielding

$$R(\theta_1, \theta_2) = A[1 - \beta \cos 2(\phi_1 - \phi_2)] \tag{14}$$

where $A = 1$ is the normalization and β is the polarization correlation parameter. Deviations from $A = 1$ may come from systematic effects like non-perfect

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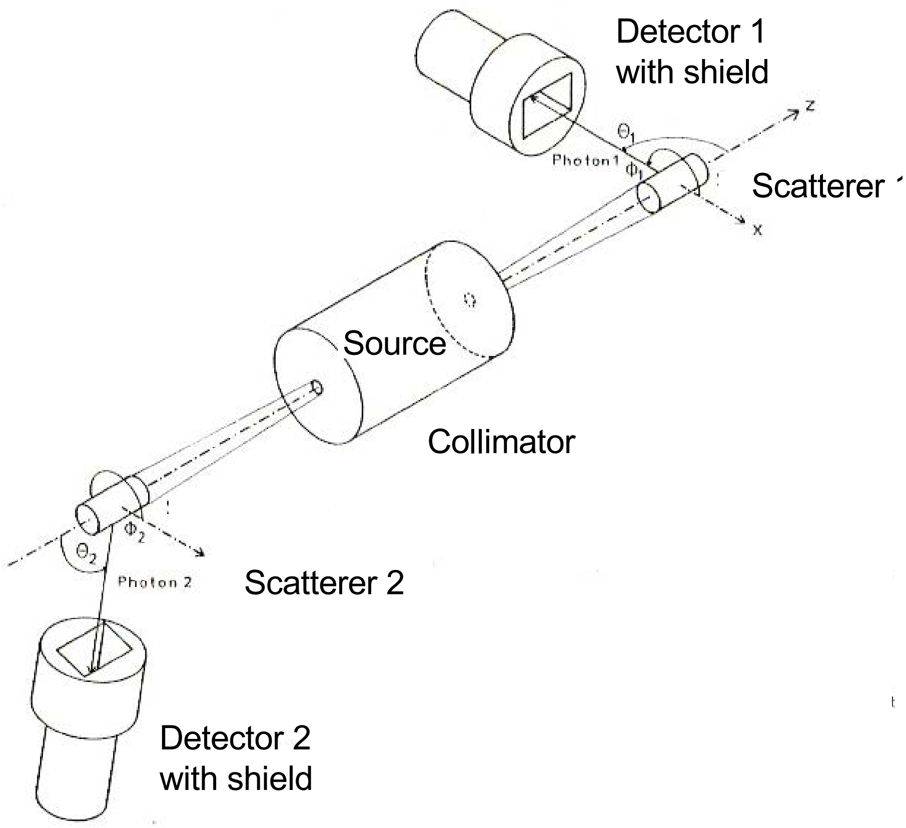

Figure 2: The experimental set up showing the source inside a collimator and the two Compton polarimeters, each consisting of a scatterer and a detector with a shield.

alignments. For uncorrelated photons $R(\phi_1, \phi_2) = 1$. In a quantum mechanical system, β is given by

$$\beta \leq m(\theta_1) \cdot m(\theta_2) \leq 0.478. \tag{15}$$

Several effects reduce the polarization correlation $R(\phi_1, \phi_2)$, such as the analyzing power of the Compton polarimeter, changes in the two-photon quantum state and backgrounds, which include accidental coincidences, uncorrelated coincident photons from a calibration source, secondary scattering in the

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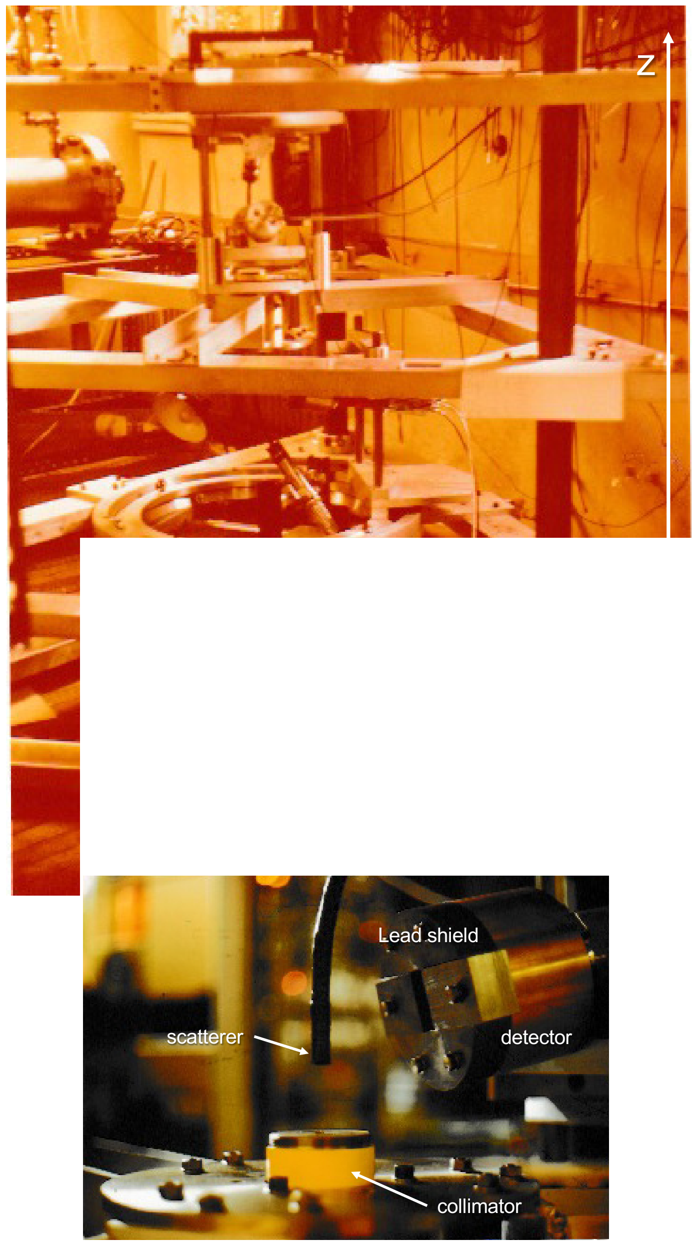

Figure 3: Top: Photograph of the experimental setup. Bottom: Photograph of a Compton polarimeter.

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light guide and lead shield and double scattering in the scintillator. Furthermore, systematic effects like misalignment of the scatterers may change the analyzing power or backgrounds. We determined the polarization correlation parameter β_{exp} from $R(\phi_1, \phi_2)$ by measuring the $\phi_2 - \phi_1$ dependence in four data sets M_{i}.

1. Data set M_1: cylindrical shaped scatterers placed at 16 cm from the source for 82° scattering angles.
2. Data set M_2: conical shaped scatterers placed at 16 cm from the source for 82° scattering angles.
3. Data set M_3: conical shaped scatterers placed at 16 cm from the source for 68° scattering angles.
4. Data set M_4: conical shaped scatterers placed at 42 cm from the source for 82° scattering angles.

For all measurements the acceptances were set to $\Delta\theta = 13.5°$ and $\Delta\phi = 9.5°$. For each measurement we recorded the following observables:

- Coincidence numbers N_2, $N_{31}(\phi_1)$, $N_{32}(\phi_2)$ and $N_4(\phi_1, \phi_2)$ and the corresponding accidental coincidences A_2, $A_{31}(\phi_1)$, $A_{32}(\phi_2)$, and $A_4(\phi_1, \phi_2)$.
- Single rates in the scatterers and detectors, N_{S1}, N_{S2}, N_{D1} and N_{D2}.
- Response times T_{S1}, T_{S2}, T_{D1} and T_{D2} and the TDC spectra for all three-fold and four-fold coincidences.
- The measured energies E_{S1}, E_{S2}, E_{D1} and E_{D2} for all three-fold and four-fold coincidences.

For a four-fold coincidence the response times have to be consistent with expectations within errors and the energies of the scattered photon E_γ and the recoil electron E_{e} have to satisfy $E_{\text{e}} + E_\gamma = E_{\text{S}} + E_{\text{D}} = 511$ keV within errors, where E_{S} and E_{D} are the energies measured in a scatterer and corresponding detector, respectively.

5 Simulation of the Experiment

The goal of the Monte Carlo simulation is to determine the quantum mechanical expectation value for the polarization correlation parameter β_{QM} for the four data sets. To obtain precise predictions we have simulated the experiment in very great detail, including the form and position of the source, the effect of the collimator, the shape and position of the scatterers, the opening of the lead slits and the absorption in the NaI detectors. The simulations allowed us to determine the dependence of β_{QM} on the scattering angle θ and the angular acceptances $\Delta\theta$ and $\Delta\phi$ and to study the effect of multiple scattering in scatterers on β_{QM}. We further determined the efficiencies of scatterers and detectors, energy distributions in the scatterers and detectors and ratios of coincidence numbers.

Note that due to the event-by-event simulation, β_{QM} cannot be determined from dP_{12} directly because for the generation of Compton scattering we need to make concrete assumption about the polarization state of the two-photon system. The pure state $|\psi >$, represented by the density matrix

$$\rho_{\mathrm{QM}} = |\psi\rangle\langle\psi| = \frac{1}{2}\Big(|x_1y_2\rangle\langle y_2x_1| + |y_1x_2\rangle\langle x_2y_1| + |x_1y_2\rangle\langle x_2y_1| + |y_1x_2\rangle\langle y_2x_1| \Big) \tag{16}$$

cannot be factorized into the individual polarization states, *i.e.* dP_{12} cannot be written as a product of dP_1 and dP_2, We have two possibilities to determine the quantum mechanical expectation value, either determine the analyzing power of each Compton polarimeter and then multiply their values or determine the expectation value for a symmetric mixture.

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\Big(|x_1y_2\rangle + |y_1x_2\rangle \Big) \tag{17}$$

for which we get

$$dP_{12}^{\mathrm{sm}} = \frac{1}{4\pi} f(\theta_1 0 f(\theta_2)\Big[1 - \frac{1}{2} m(\theta_1)m(\theta_2)\Big] \cos 2(\phi_2 - \phi_1). \tag{18}$$

This has the same form as dP_{12} except for the extra factor 1/2. We use both methods to simulate β_{QM} for the four data sets, yielding either $\beta_{\mathrm{QM}} = \epsilon_1^{\mathrm{CP}} \cdot \epsilon_2^{\mathrm{CP}}$ or $\beta_{\mathrm{QM}} = 2\beta^{\mathrm{sm}}$ for the symmetric mixture.

6 Analysis Requirements

Figure 4 shows the measured energy spectrum $E/m_{\mathrm{e}}c^2$ in the NaI detector. The signal region is marked by A. Region B results from events that are scattered in the scatterer and scattered again in the lead shield, while region C

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results from double scattered events. Region D is caused by the Sn calibration source. To reduce backgrounds from accidental coincidences and multiple scattering we apply different selection criteria and define four event categories.

- Category 0: measured coincidences corrected for accidental coincidences.
- Category T: add time requirements.
  * $t_{\mathrm{S2}} - t_{\mathrm{S1}} \leq 1.6$ ns for N_2.
  * $t_{\mathrm{D1}} - t_{\mathrm{S1}} \leq 4.6$ ns for N_{31}.
  * $t_{\mathrm{D2}} - t_{\mathrm{S2}} \leq 3.6$ ns for N_{32}.
- Category S: add energy sums.
  * $370 \leq E_{\mathrm{S1}} + E_{\mathrm{D1}} \leq 650$ keV.
  * $385 \leq E_{\mathrm{S2}} + E_{\mathrm{D2}} \leq 635$ keV.
- Category E: add detector energy constraints.
  * $200 \leq E_{\mathrm{D1}} \leq 325$ keV.
  * $195 \leq E_{\mathrm{D2}} \leq 320$ keV.

These selection criteria are applied successively.

7 Quantum Mechanical Expectations

We have simulated the quantum mechanical expectation value β_{QM} under different assumptions. Figure 5 (left) shows β_{QM} for single-scattered events for category 0 and category E as a function of the scattering angle for $\Delta\theta = 14.5°$ and $\Delta\phi = 10°$. The peak value is at $\theta = 82°$ and is not very different for the two categories. The solid curve shows the theoretical calculation of the polarization correlation for an ideal Compton polarimeter by Snyder *et al*. [6]. Our simulations confirm that the polarization correlation parameter for our setup is rather close to that of an ideal Compton polarimeter. Figure 5 (right) shows β_{QM} for single and multiple scattered events. The quantum mechanical expectation value β_{QM} is lowered also for category E even though the energy selection removes some multiple-scattered events. So, compared to the prediction by Snyder *et al*. the measured polarization correlation parameter for category E is reduced by about 10%. Figures 6 (left, right) respectively show

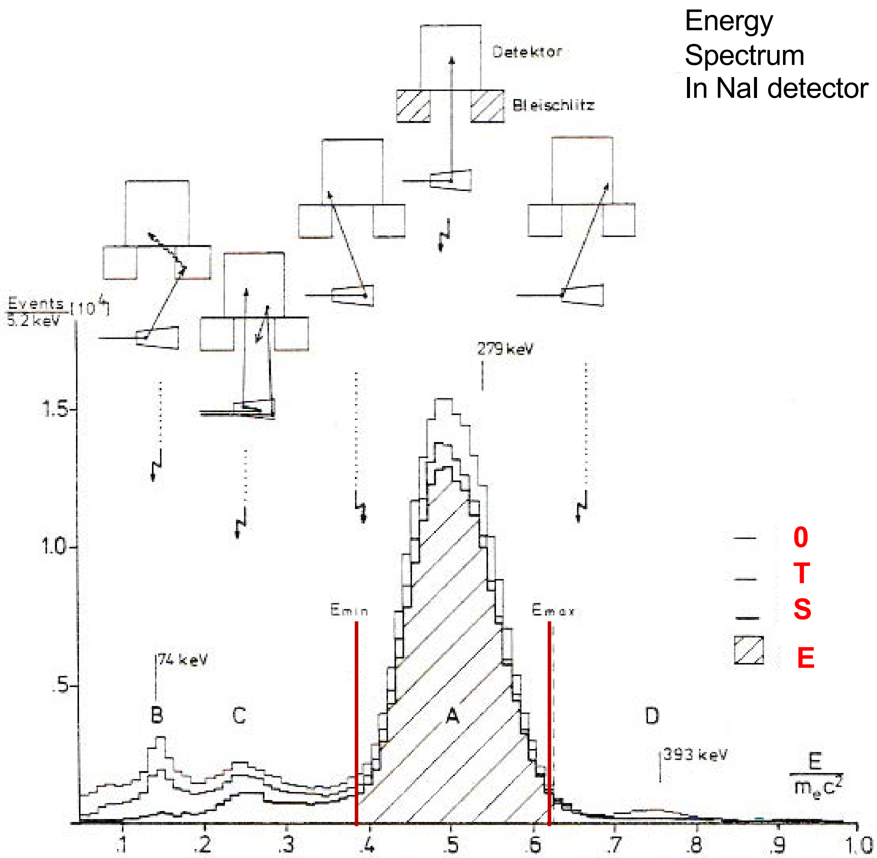

Figure 4: Energy spectrum in the NaI detector.

β_{QM} for single-scattered and single- plus multiple-scattered events as a function of $\Delta\theta$ for a scattering angle of $\theta = 82°$ for three values of $\Delta\phi$. Note that multiple scattering and large acceptances $\Delta\theta$ and $\Delta\phi$ reduce the correlation parameter β_{QM} significantly.

8 Systematic Effects

Multiple scattering in the scatterer affects β_{exp}. In our setup the amount of double-scattered four-fold coincidences is 17.8% for category 0 and 10% for category E. The effect of accidental coincidences on β_{exp} is rather small.

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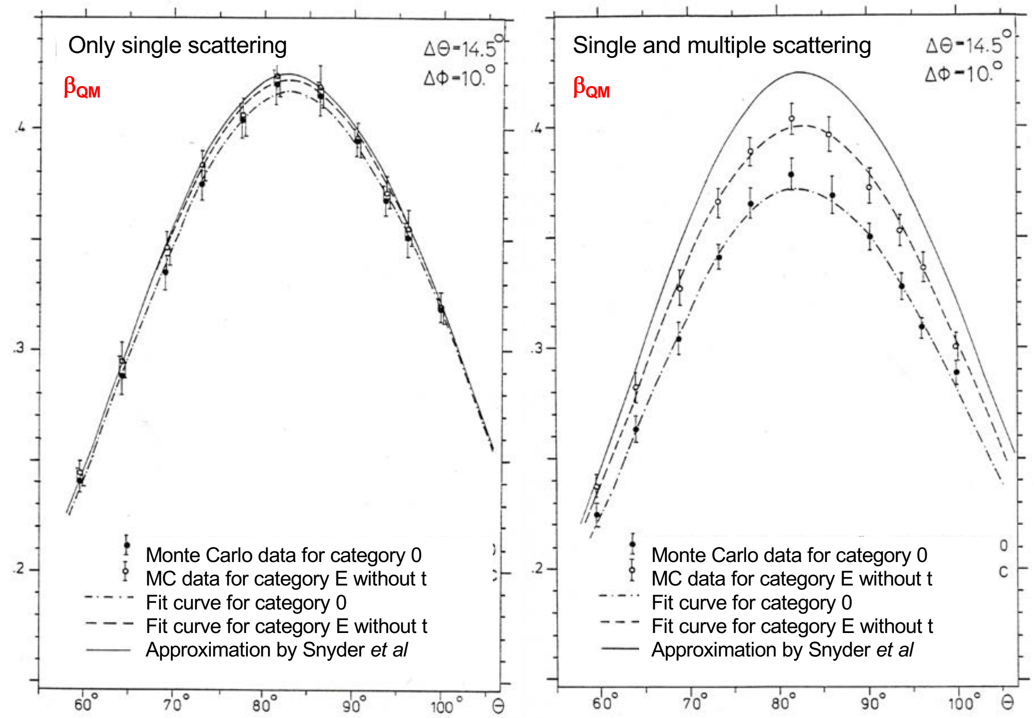

Figure 5: Left: Quantum mechanical expectation value $\beta_{\rm QM}$ as a function of θ for $\Delta\theta = 14.5°$ and $\Delta\phi = 10°$ for categories 0 and E. The curves show fits to the data. Right: The corresponding plots for single- and multiple-scattered events. The solid lines show the theoretical calculation of the polarization correlation parameter for an ideal Compton polarimeter by Snyder *et al*.

For category E the fraction of accidental four-fold coincidences is less 0.15%. Coincidences with a 1.34 MeV photon from ^{64}Cu source are negligible and events having secondary scatterings in other parts of the detector are removed by criteria S. The accuracy in the azimuth angle setting is better than 1° leading to a systematic error on $\beta_{\rm exp}$ less than 1%. Furthermore, deviations from rotation symmetry yields a negligible error on $\beta_{\rm exp}$.

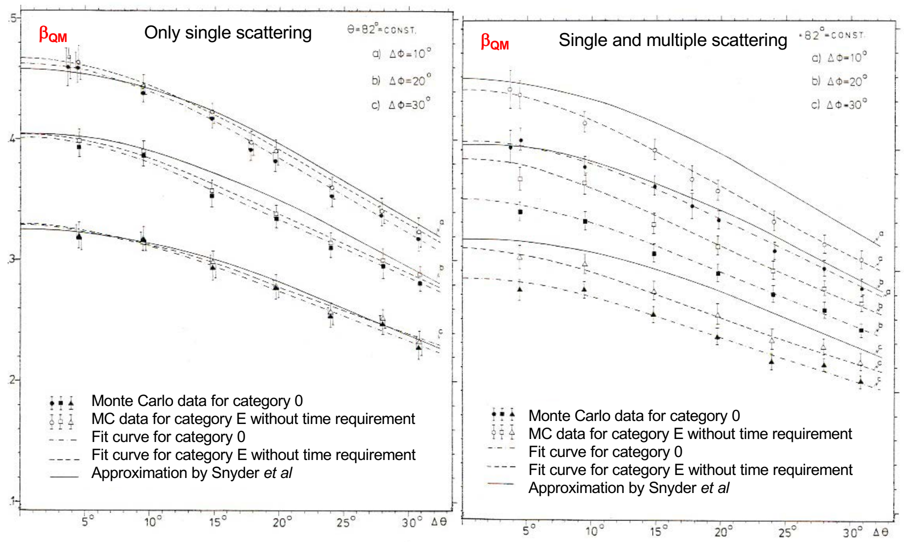

Figure 6: Left: Quantum mechanical expectation value β_{QM} as a function of $\Delta\theta$ for $\theta = 82^\circ$ and three values of $\Delta\phi = 10^\circ$ for categories 0 and E. The curves show fits to the data. Right: The corresponding plot for single- and multiple-scattered events. The solid lines show the theoretical calculation of the polarization correlation parameter for an ideal Compton polarimeter by Snyder *et al.*

9 Results

Figure 7 shows the measured polarization correlation $R_{12}(\phi_1, \phi_2)$ as a function of the azimuth angle $\phi = \phi_2 - \phi_1$ for data set II for categories 0 (top left), T (bottom left), S (top right) and E (bottom right). We fit each distributions to the function $A[1+\beta^{\text{cat}}_{\text{exp}} \cos 2(\phi_2 - \phi_1)]$ and extract A and $\beta^{\text{cat}}_{\text{exp}}$ for each category. We apply the same procedure to the other three data sets. Table 1 summarizes our measurements for the four data sets evaluated for each category. In addition, we list the quantum mechanical expectation value, the ratio of observed-to-expected polarization correlation parameters and the scaled value $\hat{\beta} = \beta_{\text{exp}}/\beta_{\text{QM}} \cdot \beta^{\text{ideal}}_{\text{QM}}$ where $\beta^{\text{ideal}}_{\text{QM}} = 0.478$ is the polarization correlation parameter for ideal Compton polarimeters expected for the representation of the two-photon system by the quantum mechanical state vector $|\psi\rangle$. The latter value will be used for comparison with other experiments. For all data sets

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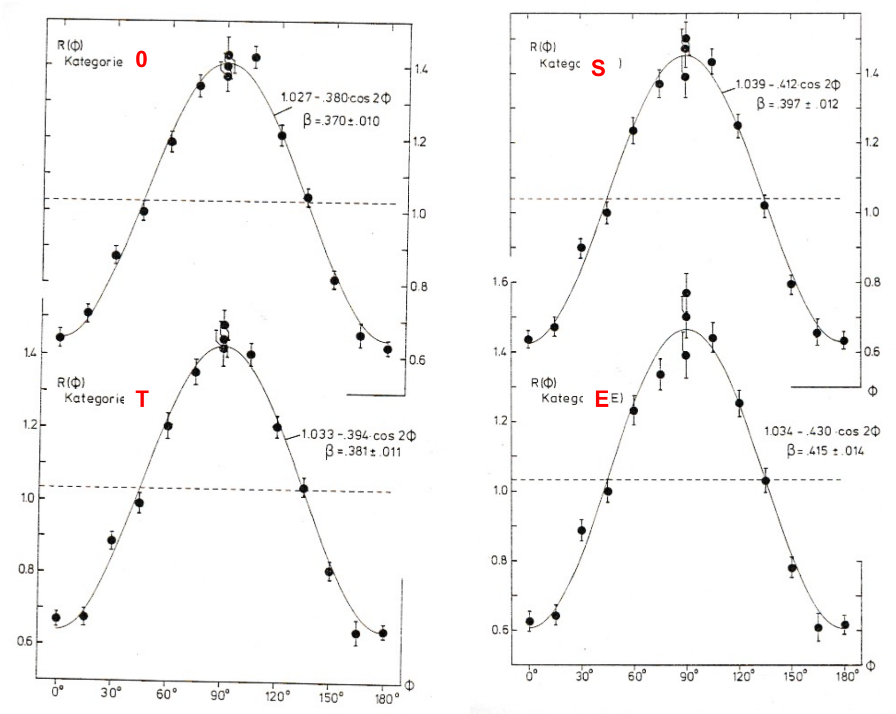

Figure 7: :Simulated polarization correlation for data set II. Top left: For category 0. Top right: For category T. Bottom left: For category S. Bottom right: For category E.

and categories the quantum mechanical correlation parameter is determined by $\beta^{\mathrm{QM}} = \epsilon_1 \cdot \epsilon_2 \cdot q_{\mathrm{corr}}$ where $q_{\mathrm{corr}} = 1.026 \pm 0.023$ is a correction factor. For data set II β_{QM} is evaluated also with method 2. The two methods yield the same results. Note that for category E $\beta_{\mathrm{exp}}/\beta_{\mathrm{QM}}$ is one. So for category E we use $\hat{\beta}^{\mathrm{E}} = \beta^{\mathrm{E}}_{\mathrm{exp}}/\beta^{\mathrm{E}}_{\mathrm{QM}} \cdot 0.478$, while for all other categories k we normalize to $\beta^{\mathrm{T}}_{\mathrm{QM}}$ yielding $\hat{\beta}^{\mathrm{k}} = \beta^{\mathrm{k}}_{\mathrm{exp}}/\beta^{\mathrm{T}}_{\mathrm{QM}} \cdot 0.478$.

Figure 8 shows our results for the four data sets and four categories scaled to the polarization correlation parameter of an ideal Compton polarimeter. For mixing of the first kind $\hat{\beta} = 0.239$ and for uncorrelated photons it is zero. The average over the four data sets yields $\langle\hat{\beta}^{0}\rangle = 0.454 \pm 0.009$, $\langle\hat{\beta}^{\mathrm{T}}\rangle = 0.463 \pm 0.009$, $\langle\hat{\beta}^{\mathrm{T}}\rangle = 0.483 \pm 0.010$ and $\langle\hat{\beta}^{\mathrm{E}}\rangle = 0.479 \pm 0.011$. For categories S and E, the measurements are in excellent agreement with the quantum mechanical

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Table 1: Measured polarization correlation parameter β_{exp}, the expected polarization correlation parameter β_{QM}, their ratio and the computed value for an ideal Compton polarimeter $\hat{\beta}$ for all four data sets and all categories.

data set	Category	β_{exp}	β_{QM}	$\beta_{\text{exp}}/\beta_{\text{QM}}$	$\hat{\beta}$
I	0	0.374 ± 0.009		0.959 ± 0.03	0.458 ± 0.014
	T	0.382 ± 0.01	0.390 ± 0.008	0.979 ± 0.033	0.468 ± 0.016
	S	0.395 ± 0.011		1.013 ± 0.035	0.484 ± 0.017
	E	0.418 ± 0.012	0.416 ± 0.009	1.005 ± 0.036	0.480 ± 0.017
II	0	0.370 ± 0.01		0.941 ± 0.03	0.450 ± 0.015
	T	0.381 ± 0.011	0.393 ± 0.007	0.969 ± 0.033	0.463 ± 0.016
	S	0.397 ± 0.012		1.01 ± 0.035	0.483 ± 0.017
	E	0.415 ± 0.014	0.413 ± 0.009	1.005 ± 0.04	0.480 ± 0.19
III	0	0.315 ± 0.01		0.940 ± 0.041	0.449 ± 0.02
	T	0.316 ± 0.011	0.335 ± 0.01	0.943 ± 0.043	0.451 ± 0.021
	S	0.337 ± 0.013		1.006 ± 0.049	0.481 ± 0.023
	E	0.368 ± 0.015	0.369 ± 0.011	0.997 ± 0.052	0.477 ± 0.025
IV	0	0.373 ± 0.019		0.969 ± 0.052	0.463 ± 0.025
	T	0.377 ± 0.02	0.385 ± 0.008	0.979 ± 0.055	0.468 ± 0.026
	S	0.387 ± 0.023		1.005 ± 0.062	0.480 ± 0.03
	E	0.405 ± 0.026	0.405 ± 0.009	1.00 ± 0.068	0.478 ± 0.032

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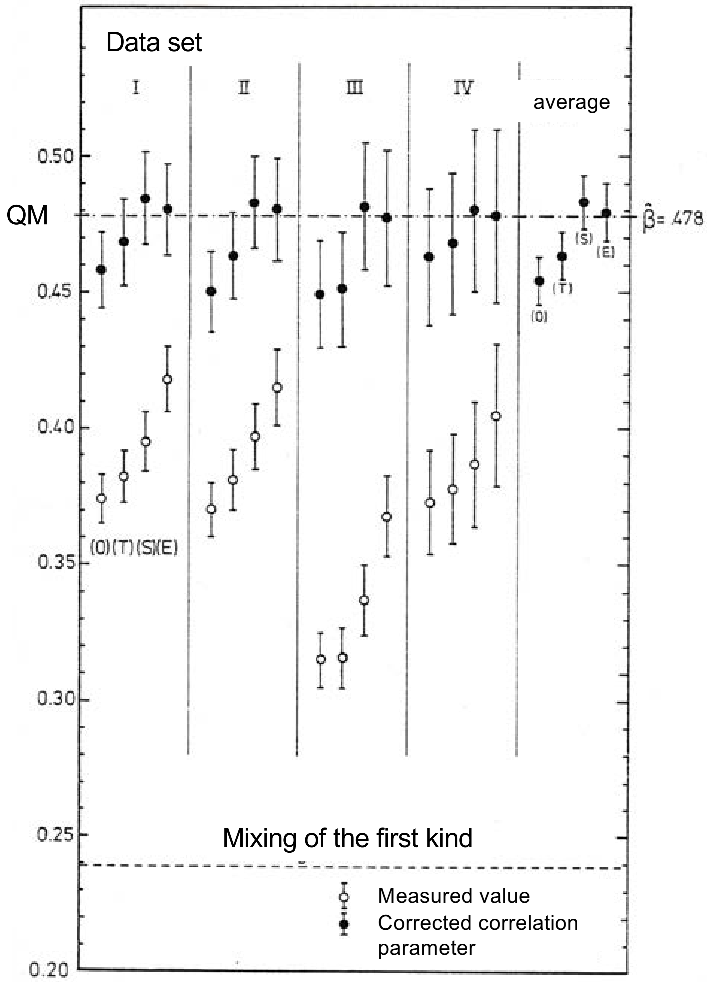

Figure 8: Measured polarization correlation parameters for the four data sets and four categories scaled to ideal Compton polarimeters. The last column shows the average values.

expectation.

The average values $\langle\hat{\beta}^0\rangle$ and $\langle\hat{\beta}^{\mathrm{T}}\rangle$ respectively are $5.0\pm1.9\%$ and $3.2\pm2.0\%$ below the quantum mechanical expectation value. This systematic deviation is caused by events that are scattered in other parts of the detector, which are removed by the selection criteria in category S. Assuming the azimuth angle distribution for such events to be similar to that of multiple scattered events we can use the estimate

$$\left\langle \beta_{\mathrm{app}}^{\mathrm{T}} \right\rangle \simeq \left(1 + 2\tilde{n}_{\mathrm{m}}\left[1 - \epsilon_{\mathrm{m}}^{0}/\epsilon_{\mathrm{s}}^{0}\right]\right)\langle \beta_{\beta_{\mathrm{s}}}^{0} \rangle, \tag{19}$$

where $\tilde{n}_{\mathrm{m}}$ is the amount of photons that are scattered a second time outside the scatterer ($\tilde{n}_{\mathrm{m}} = 41\%$), $\langle \beta_{\mathrm{s}} \rangle = 0.483$ is the quantum mechanical polarization correlation parameter for single-scattered events and $\epsilon_{\mathrm{m}}^{0}/\epsilon_{\mathrm{s}}^{0} = 0.37$ is the ratio of analyzing powers for multiple-scattered and single-scattered events. We get $\left\langle \beta_{\mathrm{app}}^{\mathrm{T}} \right\rangle = 0.459 \pm 0.006$, which is consistent with the average values $\langle \hat{\beta}^{0} \rangle$ and $\langle \hat{\beta}^{\mathrm{T}} \rangle$. The selection criteria in categories S and E remove most of this background. Note that for categories S and E the deviation from the quantum mechanical expectation value are $1.0\pm2.0\%$ and $0.2\pm2.3\%$, respectively. Thus, mixing of the first kind [7] is clearly ruled out as well as a value of $0.478/\sqrt{2}$ [3].

10 Comparison with other Experiments

Though the discrepancy between the results of Kasday [4] and Faraci [5] initiated this work, we found two other experiments by Langhoff [9] and Bruno [8]. Table 2 list the parameters of the four experiments. Using the information provided in the publications we simulated the polarization correlation in the four experiments. Figure 10 shows the results in comparison to our results. We have sorted them according to our four selection categories. The Faraci experiments has not incorporated any timing requirements and energy selections except for a loose time requirement on four-fold coincidences whereas the Kasday experiment applied similar selections as we did. The Bruno experiment did not give any time requirements but just imposed requirements on the energy sums, which varied slightly for the different data sets. The Langhoff experiment applied time constraints and requirements of the energy detected in the NaI detectors.

The final measurements of the Langhoff, Bruno and Kasday experiments and the those adjusted by our simulations agree well with quantum mechanics for categories S and E. With our simulation the results of the Faraci experiment are consistent with quantum mechanics while their own corrections yield values consistent with $\beta_{\mathrm{QM}}/\sqrt{2}$. Note that their corrections did not account for multiple scattering effects.

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Table 2: Parameters used in other polarization correlation experiments.

Parameter/Experiment	Faraci [5]	Kasday [4]	Bruno [8]	Langhoff [9]
β^+ source	^{22}Na	^{64}Cu	^{22}Na	^{22}Na, ^{64}Cu
Absorber	plexiglass	brass	Cu-plexiglass	plexiglass-aluminum
collimator $[\Omega/4\pi]$	-	3.13×10^{-4}	1.54×10^{-4}	3.28×10^{-4}
Scatterer:	cylindrical	conical	cylindrical	cylindrical
Radius [cm]	1.27	0.319/0.956	1.0	1.0
Length [cm]	2.55	0.383	3.0	2.0
Opening angle	-	9.46°	-	-
Detector: shield depth	-	1.39	1,2	-
Nai crystal ⊘ [cm]	5.1	5.1	7.0	3.82
Length [cm]	5.1	5.1	7.0	2.55
Source-scatterer [cm]	5.5/5.5	15.8/15.8	10.0/10.0	25.0/25.0
Scatterer-detector [cm]	20.0	5.1	10	10
Average scattering angle	60°, 80°	82°	60°, 82°, 98°	82°
θ acceptance [FWHM]	5.6°	23.8°	11.7°	8.5°
ϕ acceptance [FWHM]	6.0°	9.6°	11.4°	8.4°
Coincidence times t_2 [ns]	-	21	-	5
$t_{31},\ t_{32}$ [ns]	-	95	-	-
t_4 [ns]	30	-	-	30
S selection [keV]	-	427, 595	434, 588	-
	-	-	427, 595	-
	-	-	413, 609	-
E selection [keV]	-	256, 307	-	205, 320
	-	307, 358	-	-

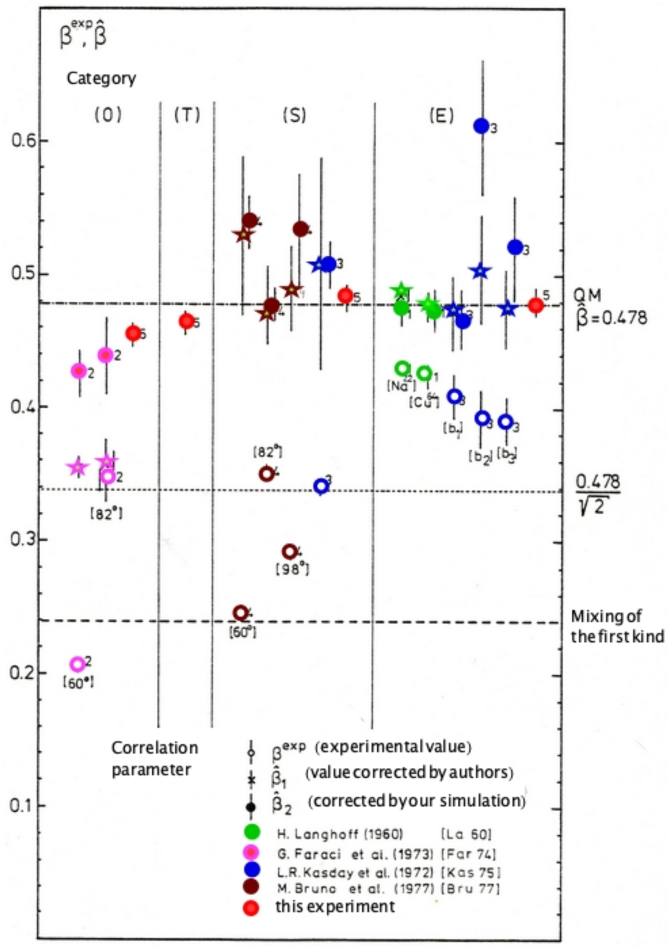

Figure 9: Measured polarization correlation parameters scaled to ideal Compton polarimeters of four experiments, Faraci (magenta points) [5], Kasday (blue points) [4], Bruno (brown point) [8] and Langhoff (green points) [9] in comparison to our results (red points). Open points show the measured values without correction, stars show the results after corrections by the authors and solid points depict the results with our corrections.

11 Test of Bell's Inequality

Bell's inequality states that

$$F(a,b,a',b') = |P(a,b) \mp P(a,b')| + |P(a',b) \mp P(a',b')| \leq 2 \qquad (20)$$

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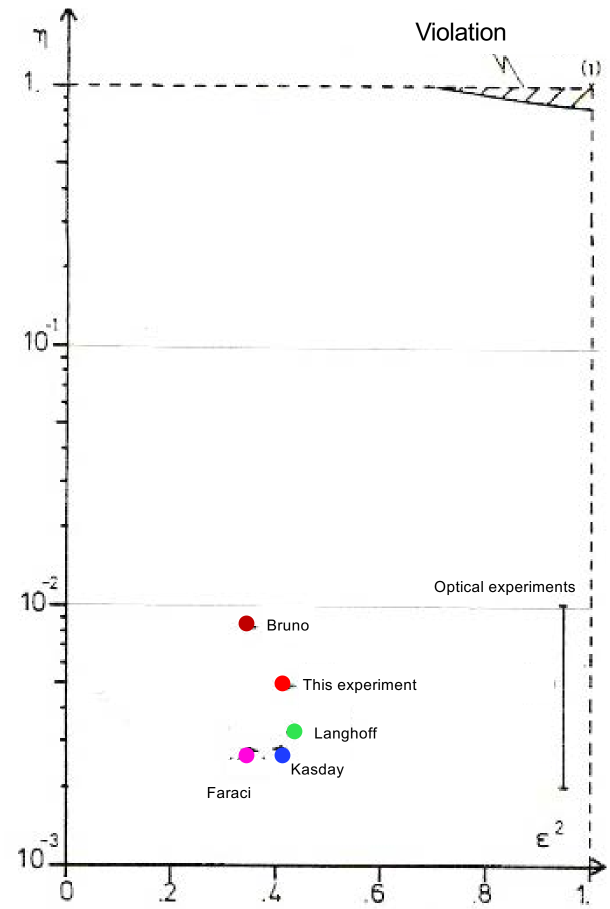

Figure 10: Graphical presentation of the violation of Bell's inequality in the $\eta - \epsilon^2$ plane. In addition, the detection efficiencies and squares of analyzing power are shown for the five experiments.

where $P(a, b) = \langle A(a) \cdot B(b) \rangle$. For a real experiment the quantum mechanical correlation is

$$P_{\mathrm{QM}}(\phi_1, \phi_2) = 1 - \frac{\Delta\Omega}{4\pi} p_2 \eta \{2 - \eta[1 - \epsilon^2 \cos 2(\phi_2 - \phi_1)]\} \tag{21}$$

where $p_2 = N_2/N_0 \frac{2\pi}{\Delta\Omega}$, ϵ is the analyzing power of a Compton polarimeter and η is the detection efficiency. Using the values $2\phi_1 = \pi/4$, $2\phi_1' = 3\pi/4$, $2\phi_2 = 0$

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and $2\phi'_2 = -\pi/2$ yields the inequality

$$F_{\rm QM} - 2 = 2\frac{\Delta\Omega}{2\pi} p_2 \eta \{\eta[1 + \epsilon^2\sqrt{2}] - 2\} \leq 2 \tag{22}$$

So, we find violation if $\eta[1 + \epsilon^2\sqrt{2}] > 2$. However, for present experiments $\eta[1 + \epsilon^2\sqrt{2}] \leq 1.68\eta \ll 2$. Thus, there is no violation of Bell's inequality since both the analyzing power and efficiency of present experiments are too small.

12 Conclusions

We measured the polarization correlation of the two-photon system from positron-electron annihilation. We varied the shape of the scatterer, scattering angle and distance source-scatterer. We simulated the quantum mechanical expectation value with a very sophisticated Monte Carlo program. All results are in good agreement with the quantum mechanical expectation value. Furthermore, we simulated the QM expectation value of four other experiment. The measurements either agree rather well or are consistent with the QM expectation value. The discrepancy of the Faraci with the QM expectation value is due to the neglect of multiple scattering, which reduces the polarization correlation parameter and needs to be corrected for.

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